The numerical range of banded biperiodic Toeplitz operators

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Abstract. The numerical range of an operator is a well studied concept with many applications in several areas of mathematics. In this paper, the numerical range of banded biperiodic Toeplitz operators is investigated, performing a reduction to the 2×2 case. Namely, the parametric equations of the boundary generating curves are deduced and two algorithms for the numerical generation of the numerical range are presented.

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1 Introduction

Let M_n be the algebra of $n \times n$ complex matrices. A matrix $T_n = (t_{kj}) \in M_n$ is said to be a *biperiodic Toeplitz* matrix if $t_{kj} := a_{k-j}$, for k odd, and $t_{kj} := b_{k-j}$, for k even, $k, j = 1, \ldots, n$. If there exists an integer $m \in \mathbb{N}$, $m < n$, such that $a_{k-j} = 0$ and $b_{k-j} = 0$, for $|k-j| > m$, $k, j = 1, \ldots, n$, then T_n is said to be a *banded biperiodic Toeplitz matrix* with *bandwidth* $2m+1$. Let l^2 be the Hilbert space of complex valued sequences $\{x_n\}_{n=0}^{+\infty}$, such that the series $\sum_{n=0}^{+\infty} |x_n|^2$ converge, endowed with the usual inner product $\langle x, y\rangle = \sum_{k=0}^{+\infty} x_k \overline{y}_k$. An infinite biperiodic Toeplitz matrix T with bandwidth $2m+1$ is completely determined by its entries in the $(m+1)$th and $(m+2)$th rows, that is, by the sequences

$$\begin{aligned} \{t_{m+1,k}\}_{k=1}^{\infty} &= \{a_m, a_{m-1}, \ldots, a_0, a_{-1}, a_{-2}, \ldots, a_{-m}, 0, \ldots\}, \\ \{t_{m+2,k}\}_{k=1}^{\infty} &= \{0, b_m, b_{m-1}, \ldots, b_0, b_{-1}, b_{-2}, \ldots, b_{-m}, 0, \ldots\}, \end{aligned} \tag{1}$$

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if m is even, and by

$$\{t_{m+1,k}\}_{k=1}^{\infty} = \{b_m, b_{m-1}, \ldots, b_0, b_{-1}, b_{-2}, \ldots, b_{-m}, 0, \ldots\},$$
$$\{t_{m+2,k}\}_{k=1}^{\infty} = \{0, a_m, a_{m-1}, \ldots, a_0, a_{-1}, a_{-2}, \ldots, a_{-m}, 0, \ldots\}, \tag{2}$$

if m is odd. The infinite matrix T induces a bounded linear operator $T : l^2 \times l^2 \to l^2 \times l^2$, which acts by the rule $Y = TX$, where X is the vector $\left(\{x_n\}_{n=0}^{+\infty}, \{y_n\}_{n=0}^{+\infty}\right) \in l^2 \times l^2$ written in the form of a column vector $X = [x_0, y_0, x_1, y_1, \ldots]^T$. In the sequel, we use T to represent interchangeably the infinite matrix T and the linear operator induced by T on $l^2 \times l^2$. These operators occur in many problems in mathematics and physics, such as the chain model of electronic structure [14] or the normal mode theory of one dimensional crystal [1].

Our investigation concerns the numerical range of banded biperiodic Toeplitz operators. The *numerical range* of a bounded linear operator A defined on a Hilbert space $\mathcal{H}$ with an inner product $\langle ., . \rangle_{\mathcal{H}}$ is the subset of the complex plane defined as

$$W(A) := \left\{ \frac{\langle Ax, x\rangle_{\mathcal{H}}}{\langle x, x\rangle_{\mathcal{H}}} : x \in \mathcal{H},\ \langle x, x\rangle_{\mathcal{H}} \neq 0 \right\}.$$

The Toeplitz-Hausdorff theorem asserts that $W(A)$ is convex and its closure contains the spectrum of A, $\sigma(A)$. If A is normal, then $\overline{W(A)}$ is the convex hull of $\sigma(A)$, throughout denoted by $\operatorname{Co}\sigma(A)$. Further, every extreme point (corner) of $W(A)$ is an eigenvalue of A. If $A \in M_n$ is *unitarily reducible*, that is, $A = U^*(A_1 \oplus \cdots \oplus A_n)U$ for some unitary matrix U and $n \geq 2$, then $W(A) = \operatorname{Co}\{W(A_1) \cup \cdots \cup W(A_n)\}$. The Elliptical Range Theorem states that the numerical range of $A \in M_2$ is an elliptical disk with foci at λ_1 and λ_2, the eigenvalues of A, and minor axis of length $\left(\operatorname{Tr}(A^*A) - |\lambda_1|^2 - |\lambda_2|^2\right)^{1/2}$. If $A \in M_n$, then $W(A)$ is the convex hull of a finite number of algebraic curves [12]. Nevertheless, for certain types of matrices the numerical range is still an elliptical disk, independently of the size of the matrices [4, 6]. For more details on the basic properties of $W(A)$ see e.g. [11, Chapter 1], [10] or [15].

This paper is organized as follows. In Section 2, asymptotic equivalence of biperiodic circulant and banded biperiodic sequences of Toeplitz matrices is investigated. In Section 3, the numerical range of banded biperiodic Toeplitz operators is characterized, following an approach used in the first algorithm provided in the last section, which relies on the reduction to the 2×2 case. In Section 4, we investigate the numerical range of biperiodic tridiagonal Toeplitz operators, identifying a class with an elliptical range. In Section 5, two algorithms for the generation of the numerical range of banded biperiodic Toeplitz operators are presented.

2 Asymptotic equivalence of biperiodic circulant and banded biperiodic sequences of Toeplitz matrices

In this section we compute the eigenvalues of biperiodic circulant matrices and approximate a sequence of banded biperiodic Toeplitz matrices by an asymptotically equivalent sequence of biperiodic circulant matrices [3]. For this purpose, we introduce the convenient notation and terminology.

A matrix $C_n = (c_{kj})$ such that $c_{kj} = c_{k-j}$, $k, j = 1, \cdots, n$, with $c_k = c_{k-n}$, $k = 1, \cdots, n-1$, is called a *circulant matrix*. Circulant matrices arise in many applications, such as problems involving the discrete Fourier transform [9]. Our study concerns *biperiodic circulant* matrices, that is, matrices $C_n = (c_{kj}) \in M_n$ of even size, such that $c_{kj} = a_{k-j}$, if k is odd, and $c_{kj} = b_{k-j}$, if k is even, $k, j = 1, \ldots, n$, with $a_k = a_{k-n}$, $k = 1, \ldots, n-2$, and $b_k = b_{k-n}$, $k = 2, \ldots, n-1$. We can define a natural embedding of a banded biperiodic Toeplitz matrix T_n with bandwidth $2m+1$ such that $2m+1 < n$, into a biperiodic circulant matrix C_{n+m} (C_{n+m+1}) adding m $(m+1)$ rows and m $(m+1)$ columns if $m+n$ is even (odd), and filling in the upper right and lower left corners of T_n with appropriate entries. Under these conditions, T_n is the principal submatrix in the first n rows and columns of a biperiodic circulant matrix C_{n+m} (C_{n+m+1}) and we say that T_n is a *compression* of C_{n+m} (C_{n+m+1}). It is clear that, for n even, there is a biperiodic circulant matrix C_n with the same size as T_n such that T_n and C_n only differ in the upper right and lower left corners, for $n > 2m+1$. For brevity, C_n will be called the biperiodic circulant matrix *associated* with T_n.

Let Γ be the complex unit circle and consider the function $f : \Gamma \to M_2$ defined as follows

$$f(\phi) = T_\phi = \begin{bmatrix} a_e(\phi) & a_o(\phi) \\ b_o(\phi) & b_e(\phi) \end{bmatrix}, \quad 0 \leq \phi < 2\pi, \tag{3}$$

where, for $k \in \mathbb{Z}$ and $m \in \mathbb{N}$,

$$a_e(\phi) := \sum_{-m \leq 2k \leq m} a_{2k}\, \mathrm{e}^{ik\phi}, \qquad a_o(\phi) := \sum_{-m \leq 2k-1 \leq m} a_{2k-1}\, \mathrm{e}^{ik\phi} \tag{4}$$

and

$$b_o(\phi) := \sum_{-m \leq 2k+1 \leq m} b_{2k+1}\, \mathrm{e}^{ik\phi}, \qquad b_e(\phi) := \sum_{-m \leq 2k \leq m} b_{2k}\, \mathrm{e}^{ik\phi}. \tag{5}$$

The function f is called the *symbol* of the Toeplitz operator T.

For n even, the $\frac{n}{2}$th complex roots of unity are throughout denoted by

$$\rho_k := \mathrm{e}^{-i\phi_k}, \quad \text{where } \phi_k = \frac{4k\pi}{n}, \ \ k = 0, \ldots, \frac{n}{2} - 1. \tag{6}$$

Theorem 2.1 *For n even, let $C_n \in M_n$ be the biperiodic circulant matrix associated with the biperiodic Toeplitz matrix T_n of bandwidth $2m+1$. Then there exists a unitary matrix $U \in M_n$ such that*

$$C_n = U\left(T_{\phi_0} \oplus \cdots \oplus T_{\phi_{\frac{n}{2}-1}}\right)U^*,$$

where $T_{\phi_k} \in M_2$, are defined in (3). The eigenvalues of C_n are

$$\lambda_{\pm}(\phi_k) = \frac{1}{2}\left(a_e(\phi_k) + b_e(\phi_k)\right) \pm \frac{1}{2}\sqrt{\left(a_e(\phi_k) - b_e(\phi_k)\right)^2 + 4a_o(\phi_k)b_o(\phi_k)}, \tag{7}$$

with associated eigenvectors $z_{\pm} = \left(x_k^{\pm}, y_k^{\pm}, x_k^{\pm}\rho_k, y_k^{\pm}\rho_k, \ldots, x_k^{\pm}\rho_k^{\frac{n}{2}-1}, y_k^{\pm}\rho_k^{\frac{n}{2}-1}\right)^T \in \mathbb{C}^n$:

$$T_{\phi_k}\begin{pmatrix} x_k^{\pm} \\ y_k^{\pm} \end{pmatrix} = \lambda_{\pm}(\phi_k)\begin{pmatrix} x_k^{\pm} \\ y_k^{\pm} \end{pmatrix}, \qquad k = 0, \ldots, \frac{n}{2} - 1. \tag{8}$$

Proof. Let $u_k = \left(1, 0, \rho_k, 0, \ldots, \rho_k^{\frac{n}{2}-1}, 0\right)^T$ and $v_k = \left(0, 1, 0, \rho_k, 0, \ldots, \rho_k^{\frac{n}{2}-1}\right)^T$, $k = 0, \ldots, n/2 - 1$. Having in mind that $\rho_k^{\frac{n}{2}+1} = \rho_k$, we can easily confirm that

$$C_n u_k = a_e(\phi_k)u_k + b_o(\phi_k)v_k \qquad \text{and} \qquad C_n v_k = a_o(\phi_k)u_k + b_e(\phi_k)v_k,$$

for a_e, a_o, b_e, b_o defined in (4) and (5). Considering the unitary matrix

$$U = \sqrt{\frac{2}{n}}\left[u_0, v_0, \ldots, u_{\frac{n}{2}-1}, v_{\frac{n}{2}-1}\right] \in M_n,$$

then $C_n = U\left(T_{\phi_0} \oplus \cdots \oplus T_{\phi_{\frac{n}{2}-1}}\right)U^*$. Hence, the spectrum of C_n is the union of the spectra of the 2×2 blocks T_{ϕ_k}, and so (7) follows. The rest of the proof is clear.

■

As a consequence of Theorem 2.1 and of the Elliptical Range Theorem [11, p.23], we have the following.

Corollary 2.1 *For n even and $C_n \in M_n$ under the conditions of Theorem 2.1, $W(C_n)$ is the convex hull of the $n/2$ ellipses (possibly degenerate) with center at $\frac{1}{2}\left(a_e(\phi_k) + b_e(\phi_k)\right)$, foci at (7) and minor axis of length*

$$\sqrt{|a_e(\phi_k)|^2 + |a_o(\phi_k)|^2 + |b_o(\phi_k)|^2 + |b_e(\phi_k)|^2 - |\lambda_+(\phi_k)|^2 - |\lambda_-(\phi_k)|^2},$$

for $k = 0, \ldots, n/2 - 1$.

We next derive an inclusion region for $W(T_n)$, where T_n is a banded biperiodic Toeplitz matrix with bandwidth $2m+1$, which generalizes [8, Lemma 5]. Estimations for $W(T_n)$ can be easily derived by embedding T_n into a circulant matrix of order $n+m$ $\;(n+m+1)$ if $n+m$ is even (odd). For $k \in \mathbb{R}$, we denote by $[k]$ the largest integer r with $r \leq k$.

Corollary 2.2 *Let T_n be a banded biperiodic Toeplitz matrix with bandwidth $2m+1$. Then*

$$W\left(T_n\right) \subseteq \mathrm{Co}\left(\bigcup_{k=0}^{[\frac{n+m+1}{2}]-1} W\left(T_{\phi_k}\right)\right), \tag{9}$$

where $T_{\phi_k} \in M_2$, and $e^{i\phi_k}$ are the $[\frac{n+m+1}{2}]$th roots of the unity.

Proof. Suppose that $T_n \in M_n$ has bandwidth $2m+1$. For $n+m$ even (odd), consider the biperiodic circulant matrix C_{n+m} (C_{n+m+1}) such that T_n is a principal submatrix of C_{n+m} (C_{n+m+1}). We find $W(T_n) \subset W(C_{n+m})$ ($W(T_n) \subset W(C_{n+m+1})$). Bearing in mind Corollary 2.1, the result follows. ■

Theorem 2.2 *Let $\lambda_{n,k}$, $k=0,\ldots,n-1$, be the eigenvalues of a biperiodic Hermitian Toeplitz matrix T_n. Assume that $\lambda_1(\phi_k) \leq \lambda_2(\phi_k)$ are the eigenvalues of the Hermitian matrix T_{ϕ_k}. Then*

$$m_f := \operatorname{ess} \inf_{\phi \in [0,2\pi[} \lambda_1(\phi) \leq \lambda_{n,k} \leq M_f =: \operatorname{ess} \sup_{\phi \in [0,2\pi[} \lambda_2(\phi). \tag{10}$$

Proof. From the Rayleigh-Ritz theorem [9, Lemma 2.1], it follows that

$$\max_{x \in \mathbb{C}^n \setminus \{0\}} \frac{x^* T_n x}{x^* x} = \max_k \lambda_{n,k} \quad \text{and} \quad \min_{x \in \mathbb{C}^n \setminus \{0\}} \frac{x^* T_n x}{x^* x} = \min_k \lambda_{n,k}.$$

By Corollary 2.2, we have

$$\max_k \lambda_{n,k} \leq \max_k \lambda_2(\phi_k) \leq \operatorname{ess} \sup_{\phi \in [0,2\pi[} \lambda_2(\phi)$$

and

$$\operatorname{ess} \inf_{\phi \in [0,2\pi[} \lambda_1(\phi) \leq \min_k \lambda_1(\phi_k) \leq \min_k \lambda_{n,k},$$

because, since T_n is Hermitian, the elliptical disks in (9) degenerate into line segments whose endpoints are the eigenvalues of T_{ϕ_k}. Hence, $\operatorname{ess} \inf_{\phi \in [0,2\pi[} \lambda_1(\phi) \leq \lambda_{n,k} \leq \operatorname{ess} \sup_{\phi \in [0,2\pi[} \lambda_2(\phi)$. ■

We consider two norms in M_n, namely, the *operator* (or *strong norm*) and the *Hilbert-Schmidt* (or *weak norm*) defined and denoted by $||A||^2 = \max_{x^*x=1} x^* A^* A x$ and $|A|^2 = \frac{1}{n} \mathrm{Tr}\,(A^* A)$, respectively.

Two sequences of $n \times n$ matrices $\{A_n\}$ and $\{B_n\}$ are said to be *asymptotically equivalent* if A_n and B_n are uniformly bounded in strong norm, i. e., $||A_n|| \leq M < \infty$ and $||B_n|| \leq L < \infty$, $n = 1, 2, \ldots,$ and $\lim_{n\to\infty} |A_n - B_n| = 0$ (see [9, p.17]).

Theorem 2.3 *For n even, let T_n be a banded biperiodic Toeplitz matrix and let C_n be the associated biperiodic circulant matrix. Then the sequences $\{T_n\}$ and $\{C_n\}$ are asymptotically equivalent.*

Proof. Firstly, suppose that T_n is Hermitian with eigenvalues $\lambda_{n,k}$, $k = 0, \dots, n-1$. Let $\lambda_1(\phi_k) \leq \lambda_2(\phi_k)$ be the eigenvalues of the Hermitian matrix $T(\phi_k)$, and let $m_f = \operatorname{ess} \inf_{\phi \in [0,2\pi[} \lambda_1(\phi)$ and $M_f = \operatorname{ess} \sup_{\phi \in [0,2\pi[} \lambda_2(\phi)$. Since T_n is Hermitian, then $||T_n|| = \max_k |\lambda_{n,k}|$. By Theorem 2.2, we obtain

$$\|T_n\| \leq \max(|m_f|, |M_f|).$$

Suppose, now, that T_n is non-Hermitian. Then consider $T_n = \operatorname{Re} T_n + i \operatorname{Im} T_n$ where $\operatorname{Re} T := (T + T^*)/2$ and $\operatorname{Im} T := (T - T^*)/(2i)$. Let $T_\phi = \operatorname{Re} T_\phi + i \operatorname{Im} T_\phi$, where $\operatorname{Re} T_\phi := (T_\phi + T_\phi^*)/2$ and $\operatorname{Im} T_\phi := (T_\phi - T_\phi^*)/(2i)$. Using the triangle inequality we get

$$\|T_n\| \leq \|\operatorname{Re} T_n\| + \|\operatorname{Im} T_n\| \leq M_{\operatorname{Re} T_\phi} + M_{\operatorname{Im} T_\phi}, \tag{11}$$

for

$$M_{\operatorname{Re} T_\phi} = \operatorname{ess} \sup_{\phi \in [0,2\pi[} |\mu_2(\phi)| \quad \text{and} \quad M_{\operatorname{Im} T_\phi} = \operatorname{ess} \sup_{\phi \in [0,2\pi[} |\psi_2(\phi)|,$$

where $\mu_1(\phi), \mu_2(\phi)$ ($|\mu_1(\phi)| \leq |\mu_2(\phi)|$) and $\psi_1(\phi), \psi_2(\phi)$ ($|\psi_1(\phi)| \leq |\psi_2(\phi)|$) are the eigenvalues of $\operatorname{Re} T_\phi$ and $\operatorname{Im} T_\phi$, respectively. Since T_n is banded, then $M_{\operatorname{Re} T_\phi}, M_{\operatorname{Im} T_\phi} < \infty$. We have $\|T_n\| \leq M_{\operatorname{Re} T_\phi} + M_{\operatorname{Im} T_\phi}$, so T_n is uniformly bounded in strong norm.

Since the matrix C_n is biperiodic circulant, there exists a unitary matrix U such that $C_n = U \left(T_{\phi_0} \oplus \cdots \oplus T_{\phi_{\frac{n}{2}-1}} \right) U^*$ and so $C_n^* = U \left(T_{\phi_0}^* \oplus \cdots \oplus T_{\phi_{\frac{n}{2}-1}}^* \right) U^*$. Hence, the eigenvalues of $C_n C_n^*$, denoted by $\alpha_{n,k}$, $k = 0, \dots, n-1$, are the eigenvalues of $T_{\phi_i} T_{\phi_i}^*$, $i = 0, \dots, n/2 - 1$. Then $\|C_n\|^2 = \max_k \alpha_{n,k}$, and C_n is uniformly bounded in strong norm.

Finally, $\lim_{n \to \infty} |T_n - C_n| = 0$, because T_n and C_n differ only in the upper right and lower left corners, for $n > 2m + 1$, and have fewer than $2 \sum_{k=0}^{m-1} m - k = m(m+1)$ non-zero entries, and so the factor $1/n$ in the weak norm drives $|T_n - C_n|$ to zero. ∎

3 The numerical range of banded biperiodic Toeplitz operators

Theorem 3.1 is in the same vein as Theorem 1 in [13].

Theorem 3.1 *Let $\lambda_1(\phi) \leq \lambda_2(\phi)$ be the eigenvalues of the Hermitian matrix $T_\phi \in M_2$, and let T be the banded biperiodic selfadjoint Toeplitz operator in (1) and (2). Then, $\overline{W(T)} = [m_f, M_f]$, where*

$$m_f := \operatorname{ess} \inf_{\phi \in [0,2\pi[} \lambda_1(\phi) \quad \textit{and} \quad M_f := \operatorname{ess} \sup_{\phi \in [0,2\pi[} \lambda_2(\phi).$$

Proof. Let T_n be the principal submatrix of T in the first n rows and columns and let C_n be the biperiodic circulant matrix associated with T_n. We have

$$\begin{aligned} x_k^* T_n x_k &= x_k^* C_n x_k + x_k^*(T_n - C_n) x_k \\ &= \alpha_{n,k} + x_k^*(T_n - C_n) x_k, \end{aligned} \tag{12}$$

being $\alpha_{n,k}$ the eigenvalues of C_n and x_k the respective normalized eigenvectors, $1 \leq k \leq n$. Let k', k'' be such that $\alpha_{n,k'} = \min_k \alpha_{n,k}$ and $\alpha_{n,k''} = \max_k \alpha_{n,k}$. From (12) it follows that

$$\left[\alpha_{n,k'} + x_{k'}^*(T_n - C_n)x_{k'} \ , \ \alpha_{n,k''} + x_{k''}^*(T_n - C_n)x_{k''}\right] \subseteq W(T_n). \tag{13}$$

Next, we claim that

$$|x_k^*(T_n - C_n)x_k| \xrightarrow[n\to\infty]{} 0.$$

In fact, let $S = [s_{ij}] = T_n - C_n$. By simple computations, we get

$$\begin{aligned} |x_k^* S x_k| &= \left| \sum_{i,j=1}^{n} \bar{x}_{ki} s_{ij} x_{kj} \right| \\ &\leq \sum_{i,j=1}^{n} |\bar{x}_{ki}|\, |s_{ij}|\, |x_{kj}| = \frac{1}{n} \sum_{i,j=1}^{n} |s_{ij}|\, \beta_{ki}\beta_{kj}, \end{aligned}$$

where $\beta_{ki} \geq 0$ does not depend on n. Since $\displaystyle\sum_{i,j=1}^{n} |s_{ij}|\beta_{ki}\beta_{kj}$ does not depend on n, the claim follows. Moreover, it is clear that $m_f = \lim\limits_{n\to+\infty} \alpha_{n,k'}$, $M_f = \lim\limits_{n\to+\infty} \alpha_{n,k''}$. Considering in (13) the limit for $n \to +\infty$, we conclude that

$$[m_f, M_f] \subseteq \overline{W(T)}.$$

We show that the reverse inclusion holds. Since T_n is selfadjoint, $W(T_n)$ is a line segment whose endpoints are eigenvalues of T_n, and according to Theorem 2.2, $W\left(T_n\right) \subseteq [m_f, M_f]$. Taking the limit as $n \to \infty$ and having in mind that $[m_f, M_f]$ is a closed interval, the result follows.

■

We recall that a *supporting line* of a convex set $S \subset \mathbb{C}$ at a boundary point z of S, is a line passing through z and defining two half-planes such that one of them does not contain S. The next result is a particular case of Theorem 1 in [2]. We present an alternative simple proof for the sake of completeness and as a support for the first algorithm provided in the last section.

Theorem 3.2 *Let T be a banded biperiodic Toeplitz operator and let $T_\phi \in M_2$, $0 \leq \phi < 2\pi$, be its symbol. Then*

$$\overline{W(T)} = \mathrm{Co}\left(\bigcup_{\phi\in[0,2\pi[} W(T_\phi) \right). \tag{14}$$

Proof. By Corollary 2.2, we obtain

$$W(T_n) \subseteq W(T_{n+1}) \subseteq W(T_{n+2}) \subseteq \cdots \subseteq \mathrm{Co}\left(\bigcup_{0\leq\phi<2\pi} W(T_\phi) \right). \tag{15}$$

Thus,

$$\overline{W(T)} \subseteq \mathrm{Co}\left(\bigcup_{\phi\in[0,2\pi[} W(T_\phi) \right).$$

We prove that the reversed inclusion holds. For $\theta \in [0, 2\pi[$, consider a supporting line of $W(T)$ perpendicular to the direction of slope θ and take its orthogonal projection on this direction, $W\left(\mathrm{Re}\left(\mathrm{e}^{-i\theta}T\right)\right)$. The symbol of $\mathrm{Re}\left(\mathrm{e}^{-i\theta}T\right)$ is the 2×2 Hermitian matrix $\mathrm{Re}\left(\mathrm{e}^{-i\theta}T_\phi\right)$, $0 \leq \phi < 2\pi$. Let $\lambda_1(\theta,\phi) \leq \lambda_2(\theta,\phi)$ denote its eigenvalues. By Theorem 3.1, $\overline{W\left(\mathrm{Re}\left(\mathrm{e}^{-i\theta}T\right)\right)} = [m_\theta, M_\theta]$ where $m_\theta = \mathrm{ess}\inf_{\phi\in[0,2\pi[} \lambda_1(\theta,\phi)$ and $M_\theta = \mathrm{ess}\sup_{\phi\in[0,2\pi[} \lambda_2(\theta,\phi)$. So,

$$\overline{W(\mathrm{Re}(\mathrm{e}^{-i\theta}T))} \subseteq \mathrm{Co}\left(\bigcup_{\phi\in[0,2\pi[} W(\mathrm{Re}(\mathrm{e}^{-i\theta}T_\phi)) \right).$$

Now, letting θ vary in the interval $[0, 2\pi[$, and having in mind the convexity of $W(T)$, we conclude that

$$\overline{W(T)} = \mathrm{Co}\left(\bigcup_{\phi\in[0,2\pi[} W\left(T_\phi\right) \right).$$

■

4 The numerical range of tridiagonal biperiodic Toeplitz operators

Tridiagonal matrices with elliptical numerical range have been investigated by some authors [4, 5, 6, 7]. Here, we consider a biperiodic real tridiagonal Toeplitz operator. The complex case may be treated following analogous arguments. However, the computations become more involved and cumbersome.

Theorem 4.1 *Let*

$$T = \begin{bmatrix} a_0 & a_{-1} & 0 & 0 & 0 & \cdots \\ b_1 & b_0 & b_{-1} & 0 & 0 & \cdots \\ 0 & a_1 & a_0 & a_{-1} & 0 & \cdots \\ 0 & 0 & b_1 & b_0 & b_{-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad a_{-1}, a_0, a_1, b_{-1}, b_0, b_1 \in \mathbb{R}. \tag{16}$$

Then

$$\overline{W(T)} = \mathrm{Co}\,(\mathscr{C}_1 \cup \mathscr{C}_2),$$

where

$$\mathscr{C}_j = \{(f_j(\theta)\cos\theta - f_j'(\theta)\sin\theta) + i(f_j(\theta)\sin\theta + f_j'(\theta)\cos\theta) : 0 \le \theta < 2\pi\}, \quad j = 1, 2, \tag{17}$$

with

$$f_j(\theta) = \frac{a_0 + b_0}{2}\cos\theta + \frac{1}{2}\Big[a_1^2 + a_{-1}^2 + b_1^2 + b_{-1}^2 + (a_0 - b_0)^2\cos^2\theta + 2(a_{-1}b_1 + a_1b_{-1})\cos(2\theta)$$

$$\pm 2\sqrt{\left(a_1^2 + b_{-1}^2 + 2a_1b_{-1}\cos(2\theta)\right)\left(a_{-1}^2 + b_1^2 + 2a_{-1}b_1\cos(2\theta)\right)}\,\Big]^{1/2}, \quad j = 1, 2. \tag{18}$$

Proof. Under the hypothesis, the symbol of T equals

$$T_\phi = \begin{bmatrix} a_0 & a_{-1} + a_1 e^{i\phi} \\ b_{-1}e^{-i\phi} + b_1 & b_0 \end{bmatrix}, \quad 0 \le \phi < 2\pi.$$

The eigenvalues of $\mathrm{Re}(e^{-i\theta}T_\phi)$ are

$$\lambda_\pm(\theta,\phi) = \frac{a_0 + b_0}{2}\cos\theta \pm \frac{1}{2}\Big[a_1^2 + a_{-1}^2 + b_1^2 + b_{-1}^2 + (a_0 - b_0)^2\cos^2\theta + 2(a_{-1}b_1 + a_1b_{-1})\cos(2\theta)$$

$$+2(a_{-1}a_1 + b_{-1}b_1)\cos\phi + 2a_{-1}b_{-1}\cos(\phi - 2\theta) + 2a_1b_1\cos(\phi + 2\theta)\Big]^{1/2}.$$

The extreme values of $\lambda_+(\theta,\phi)$ with respect to ϕ can be easily determined and are given in (18). The equations of the supporting lines of $W(T)$ perpendicular to the direction of slope θ and at the distance $f_j(\theta)$ to the origin may be written as

$$x\cos\theta + y\sin\theta = f_j(\theta), \quad j = 1, 2.$$

The envelope of the above family of supporting lines, whose parametric equations are in (17), gives rise to a so called "boundary generating curve" of $W(T)$. Due to the convexity of $W(T)$, $\partial W(T)$ is the convex hull of this curve. ■

The following result characterizes the numerical range of a class of biperiodic tridiagonal Toeplitz operators with elliptic shape.

Theorem 4.2 *Let T be the tridiagonal operator defined in (16), with $a_j, b_j \in \mathbb{C}$, $j = -1, 1$, satisfying $\overline{b}_{-1} = k\,a_1$ and $\overline{b}_1 = k\,a_{-1}$, for some $k \in \mathbb{C}$. Let*

$$\gamma_+ = \left(\frac{a_0 - b_0}{2}\right)^2 + (|a_1| + |a_{-1}|)^2\overline{k}.$$

Then $\overline{W(T)}$ *is the elliptical disc (possibly degenerate) centered at* $\dfrac{a_0+b_0}{2}$ *with foci* $\dfrac{a_0+b_0}{2} \pm \overline{\gamma}_+{}^{1/2}$ *and minor axis of length*

$$\sqrt{\frac{|a_0-b_0|^2}{2}+(|a_1|+|a_{-1}|)^2(1+|k|^2)-2|\gamma_+|}.$$

Proof. For $\theta \in [0,2\pi[$, consider the Hermitian matrix $\mathrm{Re}\left(\mathrm{e}^{-i\theta}T_\phi\right)$, whose eigenvalues are

$$\lambda_\pm(\theta,\phi)=\frac{\mathrm{Re}\left((a_0+b_0)\,\mathrm{e}^{-i\theta}\right)}{2}\pm\sqrt{\frac{\left(\mathrm{Re}\,(a_0-b_0)\,\mathrm{e}^{-i\theta}\right)^2}{4}+|z|^2},$$

where $z=\dfrac{1}{2}\left(\mathrm{e}^{-i\theta}+k\,\mathrm{e}^{i\theta}\right)\left(a_1\mathrm{e}^{i\phi}+a_{-1}\right)$. It can be easily checked that

$$\max_{0\le\phi<2\pi}\left|a_1\mathrm{e}^{i\phi}+a_{-1}\right|^2=(|a_1|+|a_{-1}|)^2$$

and

$$\min_{0\le\phi<2\pi}\left|a_1\mathrm{e}^{i\phi}+a_{-1}\right|^2=(|a_1|-|a_{-1}|)^2.$$

Then, the extreme values of $\lambda_+(\theta,\phi)$, with respect to ϕ read

$$f_{1,2}(\theta):=\frac{\mathrm{Re}\left((a_0+b_0)\,\mathrm{e}^{-i\theta}\right)}{2}+\sqrt{P_\pm+Q_\pm\cos(2\theta)+R_\pm\sin(2\theta)},$$

where

$$\begin{aligned}
P_\pm &= \frac{1}{8}\,|a_0-b_0|^2+\frac{1}{4}(|a_1|\pm|a_{-1}|)^2(1+|k|^2)\\
Q_\pm &= \frac{1}{8}\,\mathrm{Re}\,(a_0-b_0)^2+\frac{1}{2}(|a_1|\pm|a_{-1}|)^2\mathrm{Re}\,\bar{k}\\
R_\pm &= \frac{1}{8}\,\mathrm{Im}(a_0-b_0)^2+\frac{1}{2}(|a_1|\pm|a_{-1}|)^2\mathrm{Im}\,\bar{k}.
\end{aligned}$$

Let

$$\alpha_\pm:=P_\pm+\frac{1}{2}|\gamma_\pm|\quad\text{ and }\quad\beta_\pm:=P_\pm-\frac{1}{2}|\gamma_\pm|, \tag{19}$$

where

$$\gamma_\pm=\frac{(a_0-b_0)^2}{4}+(|a_1|\pm|a_{-1}|)^2\bar{k}.$$

If $Q_\pm^2+R_\pm^2=0$, then $\alpha_\pm=\beta_\pm=P_\pm$ and $W(T)$ is a circular disc. If $Q_\pm^2+R_\pm^2\neq 0$, for $\cos(\Gamma_\pm)=\frac{Q_\pm}{\sqrt{Q_\pm^2+R_\pm^2}}$ and $\sin(\Gamma_\pm)=\frac{R_\pm}{\sqrt{Q_\pm^2+R_\pm^2}}$, so that $\Gamma_\pm=\arg(\gamma_\pm)$, we have

$$P_\pm+Q_\pm\cos(2\theta)+R_\pm\sin(2\theta)=P_\pm+\sqrt{Q_\pm^2+R_\pm^2}\cos(\Gamma_\pm-2\theta)=\alpha_\pm\cos^2\left(\frac{\Gamma_\pm}{2}-\theta\right)+\beta_\pm\sin^2\left(\frac{\Gamma_\pm}{2}-\theta\right),$$

with $\alpha_\pm$ and $\beta_\pm$ in (19). Thus, the above mentioned extreme values are

$$f_{1,2}(\theta)=\frac{\mathrm{Re}\left((a_0+b_0)\mathrm{e}^{-i\theta}\right)}{2}+\sqrt{\alpha_\pm\cos^2\left(\frac{\Gamma_\pm}{2}-\theta\right)+\beta_\pm\sin^2\left(\frac{\Gamma_\pm}{2}-\theta\right)}.$$

The envelope of the family of supporting lines perpendicular to the direction with slope θ and at the distances $f_{1,2}(\theta)$ to the origin, is the curve with parametric equations

$$\left\{ \begin{array}{ll} x\cos\theta + y\sin\theta & = f_{1,2}(\theta) \\ -x\sin\theta + y\cos\theta & = f'_{1,2}(\theta) \ . \end{array} \right. \tag{20}$$

Eliminating θ in (20), we get

$$\frac{\left(X_\pm - \mathrm{Re}\left(\frac{a_0+b_0}{2}\right)\right)^2}{\alpha_\pm} + \frac{\left(Y_\pm - \mathrm{Im}\left(\frac{a_0+b_0}{2}\right)\right)^2}{\beta_\pm} = 1,$$

where $X_\pm = x\cos(\Gamma_\pm/2) + y\sin(\Gamma_\pm/2)$, $Y_\pm = -x\sin(\Gamma_\pm/2) + y\cos(\Gamma_\pm/2)$. Since $P_\pm + Q_\pm\cos(2\theta) + R_\pm\sin(2\theta) \geq 0$, these curves are ellipses because $\alpha_\pm, \beta_\pm \geq 0$. The semi-focal distances are given by $\sqrt{\alpha_\pm - \beta_\pm} = \sqrt{|\gamma_\pm|}$. Hence, the foci of the ellipses are $(a_0+b_0)/2 \pm \sqrt{|\gamma_+|}\mathrm{e}^{i\Gamma_+/2}$, and $(a_0+b_0)/2 \pm \sqrt{|\gamma_-|}\mathrm{e}^{i\Gamma_-/2}$. The result follows observing that one of the elliptical discs is contained in the other one. ∎

Remark 1. The numerical range of a tridiagonal matrix remains unchanged if, for some i, the entries $(i, i+1)$ and $(i+1, i)$ are interchanged [4]. Thus, the previous conclusion also holds if $\bar{a}_{-1} = kb_1$ and $\bar{b}_{-1} = ka_1$.

Corollary 4.1 *For T under the conditions of Theorem 4.2, $\overline{W(T)}$ is a circular disc (possibly degenerate) with center $(a_0+b_0)/2$ if and only if $(a_0 - b_0)/\left(2|a_1| + 2|a_{-1}|\right)$ is a square root of $-\bar{k}$.*

Proof. Observe that $|a_1| + |a_{-1}| = 0$ if and only if the infinite matrix associated to T is diagonal (see (1) or (2)), being then $\overline{W(T)}$ the line segment with endpoints a_0 and b_0. When $|a_1| + |a_{-1}| \neq 0$, the condition

$$\left(\frac{a_0 - b_0}{2(|a_1| + |a_{-1}|)}\right)^2 + \bar{k} = 0$$

is necessary and sufficient for $\alpha = \beta$ defined in (19). ∎

Example 4.1 *(Cf. [7, Theorem 8]) Consider the tridiagonal biperiodic Toeplitz operator T such that $a_0 = b_0 = 0$ and $a_{-1} = b_1 = a_1 = -b_{-1} = 1$. The eigenvalues of $\mathrm{Re}\left(\mathrm{e}^{-i\theta}T_\phi\right)$ satisfy $\lambda^2_\pm(\theta,\phi) = 1 + \sin(2\theta)\sin\phi$. Therefore,*

$$\lambda_\pm(\theta,\phi) = \pm\sqrt{1 + \sin(2\theta)\sin\phi}.$$

The extreme values of $\lambda_+(\theta,\phi)$, with respect to ϕ, are given by

$$f_1(\theta) = \sqrt{1 + |\sin(2\theta)|} \quad \text{and} \quad f_2(\theta) = \sqrt{1 - |\sin(2\theta)|}.$$

The equations of the supporting lines of $W(T)$ may be written as $x\cos\theta + y\sin\theta = f_{1,2}(\theta)$. By (17), *for* $0<\theta<\pi/2$, $f_1(\theta)=\sin\theta+\cos\theta$ *leading to the point* $1+i$;

for $\pi/2<\theta<\pi$, $f_1(\theta)=\sin\theta-\cos\theta$ *leading to the point* $-1+i$;

for $\pi<\theta<3\pi/2$, $f_1(\theta)=-\sin\theta-\cos\theta$ *leading to the point* $-1-i$;

for $3\pi/2<\theta<2\pi$, $f_1(\theta)=-\sin\theta+\cos\theta$ *leading to the point* $1-i$.

Thus $\mathscr{C}_1=\{1+i,1-i,-1+i,-1-i\}$. *A similar argument is valid considering* $f_2(\theta)$ *instead of* $f_1(\theta)$, *and we easily conclude that* $\mathscr{C}_1=\mathscr{C}_2$. *Thus,* $\overline{W(T)}$ *is the square*

$$\mathrm{Co}\{1+i,1-i,-1+i,-1-i\}.$$

5 Algorithms

We present two algorithms to compute the numerical range of a banded biperiodic Toeplitz operator $W(T)$. According to (14), we can reduce the determination of $W(T)$ to the 2×2 case, taking the convex hull of a union of elliptical discs (cf. Theorem 3.2). The first algorithm uses this approach taking reasonably finite discretizations of the angles θ,ϕ. The second algorithm computes the extreme of the eigenvalues $\lambda_+(\theta,\phi)$ of $\mathrm{Re}\left(\mathrm{e}^{-i\theta}A_\phi\right)$ for a fixed θ, in order to determine a family of supporting lines of $W(T)$. Then the envelope of the supporting lines is plotted, computing numerically the derivative $\mathrm{d}f/\mathrm{d}\theta$, where f as usual denotes the distance of the supporting line to the origin. Matlab programs based on these algorithms are implemented, with Figure 1 illustrating their use.

Algorithm 1

Step 1. Compute $\phi_k=\dfrac{2(k-1)\pi}{N}$, $k=1,\dots,N$, for some positive integer N;

Step 2. For each ϕ_k make the plot of $W(T_{\phi_k})$ in the following way:

- Compute $\theta_l=\dfrac{2l\pi}{M}$, $l=1,\dots,M$, for some positive integer M;
- For each θ_l described above, compute the eigenvalues of the matrix $\mathrm{Re}\left(\mathrm{e}^{-i\theta_l}T_{\phi_k}\right)$, and the associated eigenvectors $v_j(l,k)$, $j=1,2$. Evaluate

$$\rho_j(l,k):=\frac{v_j(l,k)^*T_{\phi_k}v_j(l,k)}{v_j(l,k)^*v_j(l,k)},\quad j=1,2,\quad k=1,\dots,N,\quad l=1,\dots,M.$$

  For each k, these points belong to the boundary of $W(T_{\phi_k})$.

Step 3. For each k ($k=1,\dots,N$) determine the convex hull of the points $\rho_j(l,k)$, $j=1,2$, $l=1,\dots,M$ so obtaining the collection of ellipses $\partial W(T_{\phi_k})$. Compute the convex hull of these ellipses.

An alternative algorithm to determine the boundary of $W(T)$ is the following one.

Algorithm 2

Step 1. For $\theta \in [0, 2\pi[$, compute the eigenvalues $\lambda_\pm(\theta, \phi)$ of $\mathrm{Re}\left(\mathrm{e}^{-i\theta} A_\phi\right)$.

Step 2. Consider $\theta_1, \ldots, \theta_N \in [-\frac{2\pi}{N}, 2\pi + \frac{2\pi}{N}]$ with $\theta_k = \dfrac{2\pi k}{N}, k = 0, \ldots, N+1$. For each θ_k, compute the extreme values of $\lambda_+(\theta_k, \phi)$ with respect to ϕ. Suppose that they are attained at $\phi_1(\theta_j)$ and $\phi_2(\theta_j)$, respectively, and are given by $f_j(\theta_k) = \lambda_+(\theta_k, \phi_j(\theta_k))$, $j = 1, 2,\ k = 0, 1, \ldots, N+1$.

Step 3. Compute

$$f_j'(\theta_k) = \frac{f(\theta_{k+1}) - f(\theta_{k-1})}{\theta_{k+1} - \theta_{k-1}}, \quad j = 1, 2,\ k = 1, \ldots, N.$$

Step 4. Plot the points $x_j(\theta_k) + iy_j(\theta_k),\ j = 1, 2,\ k = 1, \ldots, N$ such that

$$\left\{ \begin{array}{l} x_j(\theta_k) = \cos\theta_k f_j(\theta_k) - y \sin\theta_k f_j'(\theta_k), \\ y_j(\theta_k) = \sin\theta_k f_j(\theta_k) + y \cos\theta_k f_j'(\theta_k)\ . \end{array} \right. \tag{21}$$

Step 5. Take the convex hull of the above points, which describe the boundary of $W(T)$.

The next example illustrates the above presented algorithms.

Example 5.1 *Consider the pentadiagonal Toeplitz matrix T such that $a_0 = b_0 = a_1 = a_2 = a_{-2} = b_{-2} = 0$, $a_{-1} = 1$, $b_{-1} = -1$, $b_1 = 1$ and $b_2 = 1$. The symbol of T equals*

$$T_\phi = \begin{bmatrix} 0 & 1 \\ 1 - \mathrm{e}^{-i\phi} & \mathrm{e}^{i\phi} \end{bmatrix}, \quad 0 \leq \phi < 2\pi.$$

Algorithm 1 is illustrated in Fig.1(a) and Algorithm 2 in Fig.1(b). In this case, $\partial W(T) = \mathscr{C}_1$.

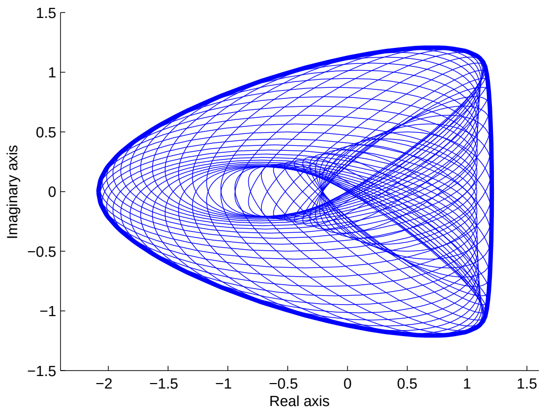
Finally, we give an example of the determination of $W(T)$, by analytical techniques.

Example 5.2 *Consider the biperiodic Toeplitz operator T with bandwidth $2m+1 = 7$ such that $b_{-3} = b_{-2} = b_{-1} = b_0 = b_2 = b_3 = 0$, $b_1 = -1$, $a_{-2} = a_0 = a_1 = a_2 = a_3 = 0$, $a_{-1} = 1$ and $a_{-3} = 2$. The symbol of T equals*

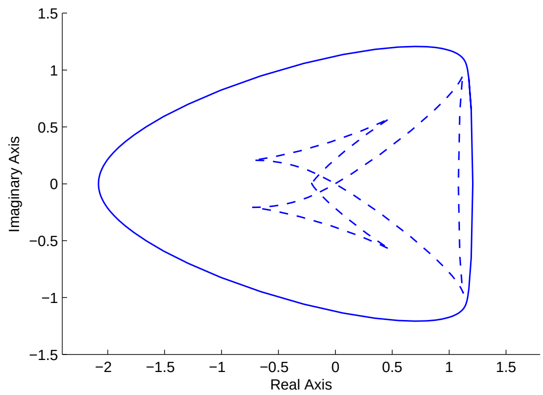
$$T_\phi = \begin{bmatrix} 0 & 1 + 2\mathrm{e}^{-i\phi} \\ -1 & 0 \end{bmatrix}, \quad 0 \leq \phi < 2\pi.$$

Then,

$$\mathrm{Re}\left(\mathrm{e}^{-i\theta} T_\phi\right) = \begin{bmatrix} 0 & z \\ \bar{z} & 0 \end{bmatrix}, \quad z = -i\sin\theta + \mathrm{e}^{-i(\phi+\theta)},$$



(a) The boundaries of $W(T_{\phi_k})$ and of W(T) (thick line) for $\phi_k = \dfrac{k\pi}{25}$, $k = 1, ..., 50$.



(b) The curves $\mathscr{C}_1$ (full curve) and $\mathscr{C}_2$ (dashed curve)

Figure 1: $W(T)$ for the operator in the example 5.1.

being the eigenvalues of $\mathrm{Re}\left(\mathrm{e}^{-i\theta}T_\phi\right)$

$$\lambda_{\pm}(\theta,\phi) = \pm\sqrt{\frac{3}{2} + \cos\phi - \frac{1}{2}\cos(2\theta) - \cos(\phi+2\theta)}.$$

By easy computations the extreme values of $\lambda_+(\theta,\phi)$ *with respect to* ϕ *are obtained*

$$f_{1,2}(\theta) = \sqrt{1+\sin^2\theta \pm 2\sin\theta} = 1 \pm \sin\theta.$$

Additional computations show that the searched boundary generating curves are the circles $|z-1| = 1$ *and* $|z+1| = 1$*. Considering the convex hull of these two circles we obtain*

$$\partial W(T) = \{\mathrm{e}^{i\psi}+1 : 0 \le \psi \le \pi\} \cup \{1+it : -1 \le t \le 1\} \cup \{\mathrm{e}^{i\psi}-1 : \pi \le \psi \le 2\pi\} \cup \{1-it : -1 \le t \le 1\}.$$

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