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Long distance expansion for the NJL model with $SU(3)$ and $U_A(1)$ breaking

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Abstract

This work is a follow up of recent investigations, where we study the implications of a generalized heat kernel expansion, constructed to incorporate non-perturbatively the effects of a non-commutative quark mass matrix in a fully covariant way at each order of the expansion. As underlying Lagrangian we use the Nambu–Jona-Lasinio model of QCD, with $SU_f(3)$ and $U_A(1)$ breaking, the latter generated by the ’t Hooft flavour determinant interaction. The associated bosonized Lagrangian is derived in leading stationary phase approximation (SPA) and up to second order in the generalized heat kernel expansion. Its symmetry breaking pattern is shown to have a complex structure, involving all powers of the mesonic fields allowed by symmetry. The considered Lagrangian yields a reliable playground for the study of the implications of symmetry and vacuum structure on the mesonic spectra, which we evaluate for the scalar and pseudoscalar meson nonets and compare with other approaches and experiment.

1. Introduction

The heat kernel expansion [1] is known as a useful and effective tool to study the properties of low-energy QCD [2–4]. Depending on the physical problem, it can be used either

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in the form of a derivative expansion [5], or as an inverse mass expansion [6]. Based on the powerful method of Schwinger–DeWitt [7], it allows for calculations of effective meson Lagrangians directly in coordinate space by integrating out the quadratic fluctuations of quark fields in presence of a background of classical mesonic fields. The result is cast as an asymptotic expansion of the effective action in powers of proper time with Seeley–DeWitt coefficients a_n, which accumulate the whole dependence on the background fields. The remarkable property of the method is that each order of the expansion is fully gauge and chiral covariant.

In the case of massive quantum fields with a degenerate mass matrix $M = \mathrm{diag}(m, m, \ldots)$, it is not difficult to derive from the proper time expansion an expansion in inverse powers of m^2, since the mass dependence is easily factorized and a subsequent integration over the proper time leads to the desired result. The resulting asymptotic coefficients remain unchanged.

If the mass matrix is however non-degenerate $M = \mathrm{diag}(m_1, m_2, \ldots)$ its total factorization is impossible because of the non-commutativity of the matrix M with the rest of the elliptic operator. It has been shown recently [8,9] that masses can be redistributed among the mass-dependent factors by performing resummations in the series. This leads to new covariant asymptotic coefficients. The algorithm for the resummations was derived and the generalized heat kernel coefficients b_n for the $SU_f(2)$ [8] and $SU_f(3)$ [9] flavour cases were obtained. In [10] the relation of the new coefficients with the standard ones has been clarified.

Given the success in the mathematical formulation of the problem, it is now a natural step to apply the new asymptotic expansion in the construction of effective chiral Lagrangians. This expansion provides a reasonable approximation to the physics of massive and heavy quantum fields with a non-degenerate mass matrix. This is the case, for instance, of low-energy QCD. Here a light current quark mass matrix which is non-degenerate is replaced by a non-degenerate mass matrix of heavy constituent quarks through the non-perturbative mechanism of spontaneous breakdown of chiral symmetry. This area of physics opens a window where our generalized heat kernel expansion can be applied.

Several different approaches based on the standard heat kernel series have already been used to study the above mentioned task [2–4]. The main difference between them is hidden in the definition of the vacuum state. The generalized heat kernel expansion also leads to its own prescription for the vacuum. It is clear that the closer one is to the physical vacuum, the more realistic the description of the spectrum of the mesonic exitations will be. From this point of view we hope that our method is a useful tool for the accurate description of the hadronic vacuum state at low energies.

In the present work we shall choose a well-known quark model [11] to describe the formation of the hadronic vacuum and its mesonic exitations. It is an effective microscopic low energy Lagrangian combining the $U_L(3) \times U_R(3)$ chiral four-quark Nambu–Jona-Lasinio (NJL) interactions together with the ’t Hooft determinantal six-quark flavour-mixing interaction, responsible for $U_A(1)$ breaking [12]. By including a mass term for the light u, d and strange s quarks one can explicitly break the remaining $SU_L(3) \times SU_R(3)$ chiral symmetry to the $SU_f(3)$ flavour group or its subgroups. This Lagrangian has been previously used in [13,14] to calculate the low lying meson mass spectrum at leading order.[1] In the recent works [16] we have analyzed the quasi-classical corrections stemming from the 't Hooft interaction, and presented a fully analytical solution for the bosonized Lagrangian and effective potential. We will use here these results. They represent a necessary step for the extension to a larger group of the earlier applications of the method in the $SU(2) \times SU(2)$ NJL model [17,18].

The paper is organized as follows. In Section 2 we introduce the model and present the main results of [16] needed for the present work. To summarize, these results are the following. Using path integral methods, the bosonization of the fermionic Lagrangian which involves the six-quark interaction requires the introduction of two sets of bosonic auxiliary fields, each of the scalar and pseudoscalar type, say (s, p) and (σ, ϕ). Then the integration over the fermionic fields can be cast in quadratic form, which can be done exactly. The remaining integrations are over one of the sets of auxiliary bosonic variables, (s, p) which are done in the stationary phase approximation. The solutions to the stationary path integral equations can be expressed as an infinite series in powers of the bosonic scalar and pseudoscalar fields (σ, ϕ), with coefficients that are known at all orders. In particular also the symmetry breaking piece of the bosonized Lagrangian contains an infinite number of terms involving powers of (σ, ϕ), and are a consequence of the flavour determinantal interaction. The piece of the bosonized Lagrangian which comes from the integration over the fermionic degrees of freedom will be dealt with our generalized heat kernel technique in Section 3. Here we also show how to deal with the gap equations combined with the requirement of covariance of the generalized Seeley–DeWitt coefficients and the symmetry breaking pattern of the original Lagrangian which must be not altered. We derive the expressions for the masses of the pseudoscalar and scalars in Section 4. In Section 5 we present numerical results and conclude with a summary and outlook in Section 6.

2. The model

To model low-energy QCD, we use the global $U_L(3) \times U_R(3)$ chiral symmetric four-quark interaction of the NJL-type model

$$\mathcal{L}_{\mathrm{NJL}} = \frac{G}{2}\left[(\bar{q}\lambda_a q)^2 + (\bar{q}i\gamma_5\lambda_a q)^2\right], \tag{1}$$

where $\lambda_a,\ a = 0, 1, \ldots, 8$, are the standard Gell-Mann matrices acting in flavour space and normalized by the condition $\mathrm{tr}(\lambda_a\lambda_b) = 2\delta_{ab}$, combined with the 't Hooft six-quark flavor determinantal interaction [12]

$$\mathcal{L}_{\mathrm{det}} = \kappa(\det\bar{q}P_L q + \det\bar{q}P_R q), \tag{2}$$

where the matrices $P_{L,R} = (1 \mp \gamma_5)/2$ are projectors on the left- and right-handed quarks. The Lagrangian $\mathcal{L}_{\mathrm{det}}$ lifts the unwanted $U_A(1)$ symmetry of $\mathcal{L}_{\mathrm{NJL}}$ for massless quarks,

[1] An early approach but without 't Hooft term can be found in [15].

as required by the $U_A(1)$ Adler–Bell–Jackiw anomaly of the $SU_f(3)$ singlet axial current $\bar{q}\gamma_\mu\gamma_5 q$ in QCD. The total fermionic Lagrangian reads

$$\mathcal{L} = \bar{q}\left(i\gamma^\mu\partial_\mu - \hat{m}\right)q + \mathcal{L}_{\mathrm{int}}, \tag{3}$$

with the interaction Lagrangian

$$\mathcal{L}_{\mathrm{int}} = \mathcal{L}_{\mathrm{NJL}} + \mathcal{L}_{\mathrm{det}}. \tag{4}$$

The quark fields have color ($N_c = 3$) and flavor ($N_f = 3$) indices which range over the set $i = 1, 2, 3$. The current quark mass, $\hat{m}$, is a diagonal matrix with elements $\mathrm{diag}(\hat{m}_u, \hat{m}_d, \hat{m}_s)$, which explicitly breaks the global chiral $SU_L(3) \times SU_R(3)$ symmetry of the Lagrangian.

This approach contains several commonly used simplifications which can be excluded in a more elaborate consideration. Let us comment first on the four-point interaction (1). The most general form of this vertex, based on phenomenological arguments, needs only to be compatible with the symmetry group of low-energy QCD and can be chosen to be invariant under the $SU(3)_c \times SU_L(3) \times SU_R(3) \times U_V(1) \times U_A(1)$ group. The six-point interaction (2) corresponds to the $N_c \to \infty$ limiting case and is modified by the tensor term at next to the leading $1/N_c$ order as it follows from the instanton dynamics [19]. We also assume that all interactions between quarks are taken in the long wavelength limit (low momenta) where they are effectively local. The explicit chiral symmetry breaking term, $\bar{q}\hat{m}q$, is standard for QCD. There are some doubts in the literature regarding this structure in the context of the NJL Lagrangian [20]. Further discussion of this point based on an instanton approach to the QCD vacuum can be found in [21].

In order to access the natural degrees of freedom of low-energy QCD in the mesonic sector, we proceed to bosonize the fermionic Lagrangian, by introducing in the vacuum persistence amplitude

$$Z = \int \mathcal{D}q\,\mathcal{D}\bar{q}\,\exp\left(i\int \mathrm{d}^4x\,\mathcal{L}\right) \tag{5}$$

the functional unity [14]

$$\begin{aligned} 1 &= \int \prod_a \mathcal{D}s_a\,\mathcal{D}p_a\,\delta(s_a - \bar{q}\lambda_a q)\delta(p_a - \bar{q}i\gamma_5\lambda_a q) \\ &= \int \prod_a \mathcal{D}s_a\,\mathcal{D}p_a\,\mathcal{D}\sigma_a\,\mathcal{D}\phi_a \\ &\quad\times \exp\left\{i\int \mathrm{d}^4x\left[\sigma_a(s_a - \bar{q}\lambda_a q) + \phi_a(p_a - \bar{q}i\gamma_5\lambda_a q)\right]\right\}, \end{aligned} \tag{6}$$

thus obtaining

$$\begin{aligned} Z &= \int \prod_a \mathcal{D}\sigma_a\,\mathcal{D}\phi_a\,\mathcal{D}q\,\mathcal{D}\bar{q}\,\exp\left(i\int \mathrm{d}^4x\,\mathcal{L}_q(\bar{q}, q, \sigma, \phi)\right) \\ &\quad\times \int \prod_a \mathcal{D}s_a\,\mathcal{D}p_a\,\exp\left(i\int \mathrm{d}^4x\,\mathcal{L}_r(\sigma, \phi, s, p)\right), \end{aligned} \tag{7}$$

where

$$\mathcal{L}_q = \bar{q}\big(i\gamma^\mu\partial_\mu - \sigma - i\gamma_5\phi\big)q, \tag{8}$$

$$\mathcal{L}_r = \frac{G}{2}\big[(s_a)^2 + (p_a)^2\big] + s_a(\sigma_a - \hat{m}_a) + p_a\phi_a + \frac{\kappa}{32}A_{abc}s_a(s_b s_c - 3p_b p_c), \tag{9}$$

and where the totally symmetric constants A_{abc} are related to the flavour determinant, and equal to

$$\begin{aligned} A_{abc} &= \frac{1}{3!}\epsilon_{ijk}\epsilon_{mnl}(\lambda_a)_{im}(\lambda_b)_{jn}(\lambda_c)_{kl} \\ &= \frac{2}{3}d_{abc} + \sqrt{\frac{2}{3}}(3\delta_{a0}\delta_{b0}\delta_{c0} - \delta_{a0}\delta_{bc} - \delta_{b0}\delta_{ac} - \delta_{c0}\delta_{ab}). \end{aligned} \tag{10}$$

We use the standard definitions for antisymmetric f_{abc} and symmetric d_{abc} structure constants of $U(3)$ flavour symmetry. One can find, for instance, the following useful relations

$$\begin{aligned} &f_{eac}A_{bfc} + f_{ebc}A_{fac} + f_{efc}A_{abc} = 0, \\ &d_{eac}A_{bfc} + d_{ebc}A_{fac} + d_{efc}A_{abc} = \sqrt{6}\delta_{e0}A_{abf}. \end{aligned} \tag{11}$$

Here and throughout the paper we use $\sigma = \sigma_a\lambda_a$, and so on for all auxiliary fields, ϕ, s, p, and use the following representation of the scalar and pseudoscalar fields

$$\frac{\lambda_a\sigma_a}{\sqrt{2}} = \begin{pmatrix} \frac{\sigma_u}{\sqrt{2}} & a_0^+ & K_0^{*+} \\ a_0^- & \frac{\sigma_d}{\sqrt{2}} & K_0^{*0} \\ K_0^{*-} & \bar{K}_0^{*0} & \frac{\sigma_s}{\sqrt{2}} \end{pmatrix}, \qquad \frac{\lambda_a\phi_a}{\sqrt{2}} = \begin{pmatrix} \frac{\phi_u}{\sqrt{2}} & \pi^+ & K^+ \\ \pi^- & \frac{\phi_d}{\sqrt{2}} & K^0 \\ K^- & \bar{K}^0 & \frac{\phi_s}{\sqrt{2}} \end{pmatrix} \tag{12}$$

with the following identifications $\phi_u = \eta_{\rm ns} + \pi^0$, $\phi_d = \eta_{\rm ns} - \pi^0$, $\phi_s = \sqrt{2}\eta_{\rm s}$, $\sigma_u = \epsilon_{\rm ns} + a_0^0$, $\sigma_d = \epsilon_{\rm ns} - a_0^0$, and $\sigma_s = \sqrt{2}\epsilon_{\rm s}$ for the correctly normalized states in the flavour basis (see Eq. (B.7) in Appendix B). Here the subscripts ns and s denote non-strange and strange, respectively.

For the set of auxiliary mesonic fields s, p the symmetry transformation properties are the same as the ones for σ, ϕ and follow from the chiral transformations of quark fields

$$\delta q = i(\alpha + \gamma_5\beta)q, \qquad \delta\bar{q} = -i\bar{q}(\alpha - \gamma_5\beta), \tag{13}$$

where the parameters of the infinitesimal global transformations α and β are Hermitian flavour matrices. One has, for example,

$$\delta s = i[\alpha, s] + \{\beta, p\}, \qquad \delta p = i[\alpha, p] - \{\beta, s\}. \tag{14}$$

The symmetry breaking piece of the Lagrangian is contained in $\mathcal{L}_r$, since

$$\delta\mathcal{L}_q = 0, \qquad \delta\mathcal{L}_r = \delta\mathcal{L}_{\rm SB} \neq 0, \tag{15}$$

where

$$\mathcal{L}_{\rm SB} = -\frac{1}{2}\,\mathrm{tr}(\hat{m}s) + \frac{\kappa}{64}\big(\det(s + ip) + \det(s - ip)\big). \tag{16}$$

We see that $\mathcal{L}_{\mathrm{SB}}$ is not invariant under a global chiral transformation due to explicit symmetry breaking, governed by the first term, and due to the 't Hooft interaction, given by the second term

$$\delta\mathcal{L}_{\mathrm{SB}} = \frac{1}{2}\,\mathrm{tr}\big(i\alpha[\hat{m}, s] - \beta\{\hat{m}, p\}\big) + i\beta_0\frac{\kappa\sqrt{6}}{32}\big(\det(s-ip) - \det(s+ip)\big). \tag{17}$$

In the following we shall consider the case with diagonal matrix $\hat{m}$ where $\hat{m}_u = \hat{m}_d \neq \hat{m}_s$, i.e., the chiral symmetry is explicitly broken down to the vectorial isotopic $SU_I(2) \times U(1)_Y$ symmetry. The non-vanishing term proportional to κ signals $U_A(1)$ breaking leading to the OZI-violating effects related to the Adler–Bell–Jackiw anomaly of the $SU(3)$ singlet axial current.

The Fermi fields in Eq. (8) enter the action bilinearly and the integration over them is exact. The result is given in the next section. It is necessary to shift the scalar fields in (7), $\sigma_a(x) \to \sigma_a(x) + m_a$. It is well known that in nature the global chiral symmetry $SU_L(3) \times SU_R(3)$ is spontaneously broken down to the Eightfold Way symmetry and the shift takes this into account. In the new vacuum state the vacuum expectation values of the shifted fields vanish $\langle 0|\sigma_a(x)|0\rangle = 0$. The new vacuum is determined by the tadpole mechanism demanding that all tadpole graphs must sum to zero. The constants m_a denoting the constituent quark masses will be fixed by the gap equations.

In [14] the lowest order stationary phase approximation (SPA) has been used to estimate the leading contribution from the 't Hooft determinant in Eq. (9) in the functional integrals over s_a and p_a

$$\mathcal{Z}[\sigma + m, \phi] \equiv \mathcal{N} \int_{-\infty}^{+\infty} \prod_a \mathcal{D}s_a\,\mathcal{D}p_a \exp\left(i \int \mathrm{d}^4x\,\mathcal{L}_r(\sigma + m, \phi, s, p)\right), \tag{18}$$

where $\mathcal{N}$ is chosen such that $\mathcal{Z}[m, 0] = 1$. In the SPA the functional integral is dominated by the stationary trajectories $r_{\mathrm{st}}^a = (s_{\mathrm{st}}^a,\ p_{\mathrm{st}}^a)$, leading to

$$\int \prod_a \mathcal{D}s_a\,\mathcal{D}p_a \exp\left(i \int \mathrm{d}^4x\,\mathcal{L}_r(\sigma + m, \phi, s, p)\right) \simeq \exp\left(i \int \mathrm{d}^4x\,\mathcal{L}_r(r_{\mathrm{st}})\right), \tag{19}$$

where $\hbar$ corrections are neglected. The stationary point, $r_{\mathrm{st}}^a(\sigma, \phi; m)$, is a solution of the equations $\mathcal{L}_r'(s, p) = 0$:

$$\begin{cases} Gs_a + (\sigma + \Delta)_a + \frac{3\kappa}{32} A_{abc}(s_b s_c - p_b p_c) = 0, \\ Gp_a + \phi_a - \frac{3\kappa}{16} A_{abc} s_b p_c = 0, \end{cases} \tag{20}$$

where $\Delta_a = m_a - \hat{m}_a$. This system is well known from [14]. Using expressions (9) and (20) we obtain

$$\mathcal{L}_r(r_{\mathrm{st}}) = \frac{G}{6}\left[\left(s_{\mathrm{st}}^a\right)^2 + \left(p_{\mathrm{st}}^a\right)^2\right] + \frac{2}{3}\big((\sigma + \Delta)_a s_{\mathrm{st}}^a + \phi_a p_{\mathrm{st}}^a\big). \tag{21}$$

One solves Eqs. (20) exactly, looking for solutions s_{st}^a and p_{st}^a in the form of increasing powers in fields σ_a, ϕ_a

$$\begin{aligned} s_{\mathrm{st}}^a &= h_a + h_{ab}^{(1)}\sigma_b + h_{abc}^{(1)}\sigma_b\sigma_c + h_{abc}^{(2)}\phi_b\phi_c + h_{abcd}^{(1)}\sigma_b\sigma_c\sigma_d + h_{abcd}^{(2)}\sigma_b\phi_c\phi_d + \cdots, \\ p_{\mathrm{st}}^a &= h_{ab}^{(2)}\phi_b + h_{abc}^{(3)}\phi_b\sigma_c + h_{abcd}^{(3)}\sigma_b\sigma_c\phi_d + h_{abcd}^{(4)}\phi_b\phi_c\phi_d + \cdots \end{aligned} \tag{22}$$

with coefficients depending on m_a and coupling constants. Putting these expansions in Eqs. (20) one obtains a series of self-consistent equations to determine coefficients h_a, $h^{(1)}_{ab}$, $h^{(2)}_{ab}$ and so on. The first three of them are

$$\begin{aligned}
&Gh_a + \Delta_a + \frac{3\kappa}{32} A_{abc} h_b h_c = 0, \\
&\left(G\delta_{ac} + \frac{3\kappa}{16} A_{acb} h_b \right) h^{(1)}_{ce} = -\delta_{ae}, \\
&\left(G\delta_{ac} - \frac{3\kappa}{16} A_{acb} h_b \right) h^{(2)}_{ce} = -\delta_{ae}.
\end{aligned} \tag{23}$$

All the other equations can be written in terms of the already known coefficients, for instance, we have [16]

$$\begin{aligned}
&h^{(1)}_{abc} = \frac{3\kappa}{32} h^{(1)}_{a\bar a} h^{(1)}_{b\bar b} h^{(1)}_{c\bar c} A_{\bar a\bar b\bar c}, \qquad h^{(2)}_{abc} = -\frac{3\kappa}{32} h^{(1)}_{a\bar a} h^{(2)}_{b\bar b} h^{(2)}_{c\bar c} A_{\bar a\bar b\bar c}, \\
&h^{(3)}_{abc} = -\frac{3\kappa}{16} h^{(2)}_{a\bar a} h^{(2)}_{b\bar b} h^{(1)}_{c\bar c} A_{\bar a\bar b\bar c}, \qquad h^{(1)}_{abcd} = \frac{3\kappa}{16} h^{(1)}_{a\bar a} h^{(1)}_{b\bar b} h^{(1)}_{\bar c cd} A_{\bar a\bar b\bar c}, \\
&h^{(2)}_{abcd} = \frac{3\kappa}{16} h^{(1)}_{a\bar a} \left(h^{(1)}_{b\bar b} h^{(2)}_{\bar c cd} - h^{(2)}_{c\bar b} h^{(3)}_{\bar c db} \right) A_{\bar a\bar b\bar c}, \quad \ldots .
\end{aligned} \tag{24}$$

One can see from these equations that the terms quadratic and higher order in mesonic fields in Eqs. (22) are generated by the 't Hooft interaction and will disappear if $\kappa = 0$. Let us also give the relations following from (23) which have been used to obtain (24)

$$h_b = (Gh_a + 2\Delta_a) h^{(1)}_{ab} = -(3Gh_a + 2\Delta_a) h^{(2)}_{ab}. \tag{25}$$

As a result the effective Lagrangian (21) can be expanded in powers of meson fields. Such an expansion, up to and including the terms which are cubic in σ_a, ϕ_a, looks like

$$\begin{aligned}
\mathcal{L}_r(r_{\rm st}) = {}& h_a\sigma_a + \frac{1}{2} h^{(1)}_{ab} \sigma_a\sigma_b + \frac{1}{2} h^{(2)}_{ab} \phi_a\phi_b \\
&+ \frac{1}{3}\sigma_a \left[h^{(1)}_{abc}\sigma_b\sigma_c + \left(h^{(2)}_{abc} + h^{(3)}_{bca} \right) \phi_b\phi_c \right] + \mathcal{O}\left(\text{field}^4\right).
\end{aligned} \tag{26}$$

The coefficients h_a are determined by couplings G, κ and the mean field Δ_a. This field has in general only three non-zero components with indices $a = 0$, 3, 8, according to the symmetry breaking pattern. The same is true for h_a because of the first equation in (23). It means that there is a system of only three equations to determine $h = h_a\lambda_a = \mathrm{diag}(h_u, h_d, h_s)$,

$$\Delta_i + Gh_i + \frac{\kappa}{32} \sum_{j,k} t_{ijk} h_j h_k = 0. \tag{27}$$

Here the totally symmetric coefficients t_{ijk} are zero except for the case with different values of indices $i \neq j \neq k$ when $t_{uds} = 1$. The Latin indices i, j, k mark the flavour states $i = u, d, s$ which are linear combinations of states with indices 0, 3 and 8. In Appendix A we collect the matrices which project one set to the other and write out exact solutions for Eq. (23). Let us note that Eqs. (27) must be solved self-consistently with the gap equations (see Eq. (43) below) to yield the constituent quark masses in leading SPA order.

3. Heat kernel expansion

Eq. (26) contains the piece of the bosonized effective Lagrangian, which has no kinetic terms and is obtained in the weak field limit. Now we turn to the evaluation of the fermionic functional integral in Eq. (7), which after the shift $\sigma_a(x) \to \sigma_a(x) + m_a$, reads

$$Z[Y] = \int \mathcal{D}q\,\mathcal{D}\bar{q}\,\exp\biggl(i\int \mathrm{d}^4x\,\bar{q}\bigl[i\gamma^\mu\partial_\mu - (m+\sigma+i\gamma_5\phi)\bigr]q\biggr), \tag{28}$$

where Y collects the background field dependence as indicated below. This fermion determinant accounts for the remaining part of the effective Lagrangian and leads, in general, to non-local mesonic vertices with unphysical cuts (the quark deconfinement problem). We have resorted here to the Schwinger–DeWitt representation for the real part of the corresponding effective action, $W[Y]$, to obtain in the end the asymptotics for $W[Y]$ in terms of local polynomials of background fields and their derivatives given by the heat kernel coefficients at coinciding arguments,

$$Z[Y] = \exp\bigl(W[Y]\bigr),$$
$$W[Y] = \ln|\det D| = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}t}{t}\rho\bigl(t\Lambda^2\bigr)\,\mathrm{Tr}\exp\bigl(-tD_\mathrm{E}^\dagger D_\mathrm{E}\bigr), \tag{29}$$

where Tr designates functional trace, the operator D_E stands for the Euclidean Dirac operator in presence of the background fields σ, ϕ and

$$D_\mathrm{E}^\dagger D_\mathrm{E} = m^2 - \partial^2 + Y, \tag{30}$$

with the definition

$$Y = i\gamma_\mu(\partial_\mu\sigma + i\gamma_5\partial_\mu\phi) + \sigma^2 + \{m,\sigma\} + \phi^2 + i\gamma_5[\sigma+m,\phi]. \tag{31}$$

For the regulator $\rho(t\Lambda^2)$, needed to keep the integral convergent at $t=0$, we use two Pauli–Villars subtractions[2]

$$\rho\bigl(t\Lambda^2\bigr) = 1 - \bigl(1+t\Lambda^2\bigr)\exp\bigl(-t\Lambda^2\bigr), \tag{32}$$

where the cut-off Λ is a free dimensionfull parameter. The regularization function $\rho(t\Lambda^2)$, being written in terms of a dimensionless variable $\tau = t\Lambda^2$, fulfills the necessary conditions: $\rho(\tau) \sim \tau^2/2$ at $\tau \to 0$ and $\rho(\tau) \to 1$ at $\tau \to \infty$. It is important to know to what extent the specific form of this function affects our results. It is obvious that the type of used regulator does not affect the chiral invariance of the heat kernel expansion, since the generalized heat kernel coefficients b_i [9], which carry the whole symmetry properties of the heat kernel expansion, do not depend on it

$$W[Y] = -\int \frac{\mathrm{d}^4x_\mathrm{E}}{32\pi^2}\sum_{i=0}^\infty I_{i-1}\,\mathrm{tr}(b_i). \tag{33}$$

[2] A regularization function ρ must be introduced to define the coincidence limit for the Schwinger representation. The regularization of the quark determinant in general should be done in accordance with certain requirements (see, for example, the review of R.D. Ball in [1]). Some of them are discussed also in [22].

Here the expressions for the first four b_i in the case of $SU(2)_I \times U(1)_Y$ flavour symmetry $m_u = m_d \neq m_s$ are

$$\begin{aligned}
b_0 &= 1, \\
b_1 &= -Y, \\
b_2 &= \frac{Y^2}{2} + \frac{\Delta_{us}}{\sqrt{3}}\lambda_8 Y, \\
b_3 &= -\frac{Y^3}{3!} + \frac{\Delta_{us}^2}{6\sqrt{3}}\lambda_8 Y - \frac{\Delta_{us}}{2\sqrt{3}}\lambda_8 Y^2 - \frac{1}{12}(\partial Y)^2,
\end{aligned} \tag{34}$$

where we used the definition $\Delta_{ij} \equiv m_i^2 - m_j^2$. In (33) the trace is to be taken over colour, flavour and Dirac 4-spinors indices and the regulator-dependent integrals I_i are the weighted sums [9]

$$I_i = \frac{1}{3}\left(2J_i\left(m_u^2\right) + J_i\left(m_s^2\right)\right) \tag{35}$$

with

$$J_i\left(m_j^2\right) = \int_0^\infty \frac{\mathrm{d}t}{t^{2-i}}\rho\left(t\Lambda^2\right)\exp\left(-tm_j^2\right). \tag{36}$$

For the chosen form of the cut-off function we obtain, for instance,

$$J_0\left(m^2\right) = \Lambda^2 - m^2\ln\left(1 + \frac{\Lambda^2}{m^2}\right), \tag{37}$$

$$J_1\left(m^2\right) = \ln\left(1 + \frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{\Lambda^2 + m^2}. \tag{38}$$

Both of them are divergent in the limiting case $\Lambda \to \infty$.

Thus, the effective Lagrangian depends on the integrals I_i. The more terms of the heat kernel series are taken into account, the more the final result depends on the form of the cut-off function $\rho(\tau)$ and, therefore, the more careful one should be choosing a regulator. In the following we restrict our study to the two non-trivial terms, b_1 and b_2, in the asymptotic expansion of $W[Y]$. In this case only two integrals, I_0 and I_1, are involved. If we introduced in $\rho(\tau)$ two independent parameters, instead of one, Λ, the outcome would not depend at all on the form of the regulator, because one can always fix these parameters by fixing independently couplings I_0 and I_1 from experimental data. Actually, we slightly simplified our calculations working with only one parameter Λ, paying for that the price of having some dependence on the regularization procedure, which is finally inherited by the constituent quark masses.

The heat kernel series (33) defines the asymptotics of the effective action for a physical system with the mass matrix m being large compared to the rest of the background fields and their derivatives. It corresponds exactly to the considered case of low-energy QCD, where the small meson exitations of the quark sea take place in the "superconducting" phase with heavy constituent quarks. It is interesting to stress that in comparison with the standard Seeley–DeWitt coefficients, which transform covariantly with respect to the action of the chiral group, our coefficients b_i possess more specific transformation properties. Indeed, in the broken vacuum state an arbitrary infinitesimal variation $\delta\,\mathrm{tr}(b_i)$, induced by global transformations of the background fields

$$\delta\sigma = i[\alpha, \sigma + m] + \{\beta, \phi\}, \qquad \delta\phi = i[\alpha, \phi] - \{\beta, \sigma + m\}, \tag{39}$$

depends on the variation δY which is equal to

$$\delta Y = i\big[\alpha + \gamma_5\beta, Y + m^2\big]. \tag{40}$$

One can see that already the first coefficient b_1 transforms non-covariantly, because m^2 does not commute with $\alpha + \gamma_5\beta$ in (40). Nevertheless, one can prove that $\delta\,\mathrm{tr}(b_i) = 0$ for all generalized coefficients b_i [9].

In the present calculations we truncate the heat kernel series at b_2. In this approximation the effective Lagrangian $\mathcal{L}$ is given by the sum of only two local terms $\mathcal{L} = \mathcal{L}(b_1) + \mathcal{L}(b_2) + \cdots$, where

$$\begin{aligned}
&\mathcal{L}(b_1) = \mathcal{L}_{\mathrm{tad}}(b_1) + \mathcal{L}_{\mathrm{mass}}(b_1),\\
&\mathcal{L}(b_2) = \mathcal{L}_{\mathrm{tad}}(b_2) + \mathcal{L}_{\mathrm{kin}}(b_2) + \mathcal{L}_{\mathrm{mass}}(b_2) + \mathcal{L}_{\mathrm{int}}(b_2).
\end{aligned} \tag{41}$$

Here we distinguish the tadpole terms, $\mathcal{L}_{\mathrm{tad}}$, from mass terms, $\mathcal{L}_{\mathrm{mass}}$, kinetic terms, $\mathcal{L}_{\mathrm{kin}}$, and interaction terms, $\mathcal{L}_{\mathrm{int}}$. We have, for instance,

$$\begin{aligned}
&\mathcal{L}_{\mathrm{tad}}(b_1) = \frac{N_c I_0}{4\pi^2}\big[m_u(\sigma_u + \sigma_d) + m_s\sigma_s\big],\\
&\mathcal{L}_{\mathrm{tad}}(b_2) = -\frac{N_c I_1}{12\pi^2}\Delta_{us}\big[m_u(\sigma_u + \sigma_d) - 2m_s\sigma_s\big].
\end{aligned} \tag{42}$$

Joined together with the tadpole contribution from Lagrangian (26), they lead to the gap equations

$$\begin{cases}
h_u + \frac{N_c}{6\pi^2} m_u(3I_0 - \Delta_{us}I_1) = 0,\\
h_s + \frac{N_c}{6\pi^2} m_s(3I_0 + 2\Delta_{us}I_1) = 0.
\end{cases} \tag{43}$$

The mass-part of the heat kernel effective Lagrangian contains two contributions and is given by

$$\begin{aligned}
\mathcal{L}_{\mathrm{mass}}^{(b_1+b_2)} = {}& \frac{N_c I_0}{4\pi^2}\big(\sigma_a^2 + \phi_a^2\big) - \frac{N_c I_1}{12\pi^2}\big\{\Delta_{us}\big[2\sqrt{2}(3\sigma_0\sigma_8 + \phi_0\phi_8) - \phi_8^2 + \phi_i^2\big]\\
&+ 2\big(2m_u^2 + m_s^2\big)\sigma_0^2 + \big(m_u^2 + 5m_s^2\big)\sigma_8^2 + \big(7m_u^2 - m_s^2\big)\sigma_i^2\\
&+ (m_u + m_s)(m_u + 2m_s)\sigma_f^2 + (m_s - m_u)(2m_s - m_u)\phi_f^2\big\},
\end{aligned} \tag{44}$$

where we assume that the indices i and f range over the subsets $i = 1, 2, 3$ and $f = 4, 5, 6, 7$ of the set $a = 0, 1, \ldots, 8$. Thus we have

$$\begin{aligned}
&\phi_i^2 = 2\pi^+\pi^- + \big(\pi^0\big)^2, \qquad \phi_f^2 = 2\big(K^+K^- + \bar{K}^0K^0\big),\\
&\sigma_i^2 = 2a_0^+a_0^- + \big(a_0^0\big)^2, \qquad \sigma_f^2 = 2\big(K_0^{*+}K_0^{*-} + \bar{K}_0^{*0}K_0^{*0}\big).
\end{aligned} \tag{45}$$

The kinetic term, $\mathcal{L}_{\text{kin}}(b_2)$, after continuation to Minkowski space, has a non-standard factor

$$\mathcal{L}_{\text{kin}}(b_2) = \frac{N_c I_1}{16\pi^2} \operatorname{tr}\left[(\partial_\mu \sigma)^2 + (\partial_\mu \phi)^2\right]. \tag{46}$$

It should be rescaled by the redefinition of mesonic fields

$$\sigma_a = g\sigma_a^{\text{R}}, \qquad \phi_a = g\phi_a^{\text{R}}, \qquad g^2 = \frac{4\pi^2}{N_c I_1}, \tag{47}$$

where the index R stands for the new renormalized fields.

By virtue of the PCAC hypothesis the coupling g is related to the weak decay constants of the pion, f_π, or the kaon, f_K,

$$f_\pi = \frac{m_u}{g}, \qquad f_K = \frac{m_s + m_u}{2g}. \tag{48}$$

To see this let us recall Eq. (6), where the quarks bilinears $\bar{q}\lambda_a q$ and $\bar{q}i\gamma_5\lambda_a q$ have been replaced by the auxiliary fields s_a and p_a. The SPA approximation used to estimate the path integral over these variables in (19) restricts them to the stationary trajectories s_{st}^a, p_{st}^a, given by Eq. (22). Thus, we have

$$i\bar{q}\gamma_5\lambda_a q = p_a^{\text{st}}, \qquad \bar{q}\lambda_a q = s_a^{\text{st}}. \tag{49}$$

The quark operators are finally represented by expansions in increasing powers of bosonic fields σ_a and ϕ_a. This is a convenient form to establish a connection to some current algebra results, such as the PCAC hypothesis or the Gell-Mann–Oakes–Renner (GOR) relation [23].

For instance, one easily finds from (49),

$$\langle \pi^- | \bar{d}\gamma_5 u | 0\rangle = \frac{ig\langle \pi^- | \phi_{\pi^+}^{\text{R}} | 0\rangle}{\sqrt{2}G(1+\omega_s)} = \frac{im_\pi^2}{\sqrt{2}\hat{m}_u}\left(\frac{m_u}{g}\right)\langle \pi^- | \phi_{\pi^+}^{\text{R}} | 0\rangle, \tag{50}$$

where result (56) has been used to obtain the last equality. In exactly the same way one derives with the help of Eq. (57)

$$\langle K^- | \bar{s}\gamma_5 u | 0\rangle = \frac{ig\langle K^- | \phi_{K^+}^{\text{R}} | 0\rangle}{\sqrt{2}G(1+\omega_u)} = \frac{i\sqrt{2}m_K^2}{(\hat{m}_u + \hat{m}_s)}\left(\frac{m_u + m_s}{2g}\right)\langle K^- | \phi_{K^+}^{\text{R}} | 0\rangle. \tag{51}$$

Let us assume that (48) holds, then these equations coincide with the well-known PCAC relations.

One can use the second equation in (49) to estimate the quark condensates in the vacuum. As far as the isotopic invariance is implemented here we have

$$\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle = \frac{h_u}{2}, \qquad \langle 0|\bar{s}s|0\rangle = \frac{h_s}{2}. \tag{52}$$

Combining these equations with Eqs. (56), (57) and (48) one finds the GOR relations (up to the last terms in the round brackets, which are proportional to the current quark masses and give some model corrections to the leading order result)

$$m_\pi^2 f_\pi^2 = -2\hat{m}_u \langle 0|\bar{u}u|0\rangle \left(1 + \frac{\hat{m}_u}{\Delta_u}\right), \tag{53}$$

$$m_K^2 f_K^2 = -\frac{1}{2}(\hat{m}_u + \hat{m}_s)\langle 0|\bar{u}u + \bar{s}s|0\rangle \left(1 + \frac{\hat{m}_u + \hat{m}_s}{\Delta_u + \Delta_s}\right). \tag{54}$$

4. Mass spectrum

We proceed now to extract the mass terms for the low-lying pseudoscalar and scalar nonets. We discuss first the pseudoscalar spectrum. The quadratic terms in the fields from Eq. (26) and Eq. (44) combine to yield for instance

$$\mathcal{L}_{\text{mass}}(\pi) = \phi_i^2 \left[\frac{N_c}{12\pi^2}(3I_0 - \Delta_{us} I_1) - \frac{1}{2G(1+\omega_s)}\right] = -\frac{\hat{m}_u \phi_i^2}{2Gm_u(1+\omega_s)}. \tag{55}$$

To get this result we used the gap equation (43) and the stationary phase conditions (27). Let us also remind that some of our notations and results are explained in Appendix A. Finally the pion mass is obtained by introducing physical fields (47)

$$m_\pi^2 = \frac{g^2 \hat{m}_u}{Gm_u(1+\omega_s)}. \tag{56}$$

In exactly the same way one can obtain the masses of the other members of the pseudoscalar nonet

$$m_K^2 = \frac{g^2(\hat{m}_u + \hat{m}_s)}{G(m_u + m_s)(1+\omega_u)}, \tag{57}$$

$$m_\eta^2 = \frac{g^2}{2}\left(A + B - \sqrt{(A-B)^2 + 4D^2}\right), \tag{58}$$

$$m_{\eta'}^2 = \frac{g^2}{2}\left(A + B + \sqrt{(A-B)^2 + 4D^2}\right). \tag{59}$$

We also have

$$\begin{aligned} A + B &= \frac{h_u}{m_u} + \frac{h_s}{m_s} + \frac{2-\omega_s}{G\mu_-}, \\ A - B &= \frac{1}{3}\left(\frac{h_u}{m_u} - \frac{h_s}{m_s} + \frac{8\omega_u + \omega_s}{G\mu_-}\right), \\ D &= \frac{\sqrt{2}}{3}\left(\frac{h_u}{m_u} - \frac{h_s}{m_s} + \frac{\omega_s - \omega_u}{G\mu_-}\right), \end{aligned} \tag{60}$$

where $\mu_\pm = (1 \pm \omega_s - 2\omega_u^2)$. The argument of the square root is

$$(A-B)^2 + 4D^2 = \left(\frac{h_u}{m_u} - \frac{h_s}{m_s} + \frac{\omega_s}{G\mu_-}\right)^2 + 8\left(\frac{\omega_u}{G\mu_-}\right)^2. \tag{61}$$

It is known that for $m_u = m_d \neq m_s$ there is mixing in the 0, 8 channels. This part of the Lagrangian has been diagonalized by introducing physical fields η and η' via an orthogonal transformation, as it is discussed in Appendix B, with the mixing angle $\theta_{\rm p}$ (in the singlet–octet basis) defined from the diagonalization requirement.

In the limit of vanishing 't Hooft interaction, $\kappa = 0$, the mixing angle $\theta_{\rm p}$ is equal to the ideal one with $\tan(2\theta_{\rm id}) = 2\sqrt{2}$ and one can conclude that $\eta \sim \eta_{\rm ns}$, $\eta' \sim -\eta_{\rm s}$. We find in this case

$$m_\pi^2 = m_{\eta_{\rm ns}}^2 = \frac{g^2 \hat{m}_u}{G m_u}, \qquad m_K^2 = \frac{g^2(\hat{m}_u + \hat{m}_s)}{G(m_u + m_s)}, \qquad m_{\eta_{\rm s}}^2 = \frac{g^2 \hat{m}_s}{G m_s}. \tag{62}$$

Using the gap equations one obtains the relations

$$\frac{m_K^2 - m_\pi^2}{2m_u(m_s - m_u)} = \frac{m_s}{m_u}, \qquad \frac{m_{\eta_{\rm s}}^2 - m_K^2}{2m_u(m_s - m_u)} = 1, \tag{63}$$

which show the mass splittings within the nonet.

In the $SU(3)$ limit $m_u = m_d = m_s$ for non-vanishing κ there is no ϕ_0–ϕ_8 mixing, since $D = 0$. One obtains immediately the masses

$$m_\pi^2 = m_K^2 = m_{88}^2 = \frac{g^2 \hat{m}_u}{G m_u(1 + \omega)}, \tag{64}$$

with the singlet–octet mass splitting

$$m_{00}^2 - m_{88}^2 = \frac{3g^2\omega}{G(1 + \omega)(1 - 2\omega)}, \tag{65}$$

where

$$\omega = \frac{\kappa h}{16G} = \frac{1}{2}\left(\sqrt{1 - \frac{\kappa \Delta_u}{4G^2}} - 1\right) \tag{66}$$

is a solution of the stationary phase equation (27) for the $SU(3)$ case. In the chiral limit, $\hat{m} = 0$, the singlet mass m_{00} takes a non-vanishing value. The would be $U(1)$ Goldstone boson receives a mass as a result of the 't Hooft interaction.

We turn now to the scalar sector. The masses of the scalar mesons are as follows. For the mesons usually referred to as a_0 $(I^G(J^{PC}) = 1^-(0^{++}))$ we obtain

$$m_{a_0}^2 = g^2\left(\frac{h_u}{m_u} + \frac{1}{G(1 - \omega_s)}\right) + 4m_u^2 = m_\pi^2 + 4m_u^2 + \frac{2g^2\omega_s}{G(1 - \omega_s^2)}, \tag{67}$$

and for the strange K_0^* $(I(J^P) = \frac{1}{2}(0^+))$ we have

$$m_{K_0^*}^2 = g^2\left(\frac{1}{G(1 - \omega_u)} + \frac{h_u + h_s}{m_u + m_s}\right) + 4m_s m_u = m_K^2 + 4m_s m_u + \frac{2g^2\omega_u}{G(1 - \omega_u^2)}. \tag{68}$$

In the 0, 8 channels one must diagonalize the states. Diagonalization proceeds as in the pseudoscalar case and the resulting scalar states are denoted by ϵ and ϵ', respectively, indicating a set of f_0 $(I^G(J^{PC}) = 0^+(0^{++}))$ mesons. The mixing angle $\theta_{\rm s}$ is defined in the $(0, 8)$ basis. As a result we obtain for the corresponding masses

$$m_{\epsilon}^2 = \frac{g^2}{2}\Big(\mathcal{A}+\mathcal{B}-\sqrt{(\mathcal{A}-\mathcal{B})^2+4\mathcal{D}^2}\Big),$$
$$m_{\epsilon'}^2 = \frac{g^2}{2}\Big(\mathcal{A}+\mathcal{B}+\sqrt{(\mathcal{A}-\mathcal{B})^2+4\mathcal{D}^2}\Big), \tag{69}$$

where

$$\mathcal{A}+\mathcal{B} = \frac{h_u}{m_u}+\frac{h_s}{m_s}+\frac{N_c I_1}{\pi^2}\big(m_s^2+m_u^2\big)+\frac{2+\omega_s}{G\mu_+},$$
$$\mathcal{A}-\mathcal{B} = \frac{h_u}{m_u}-\frac{h_s}{m_s}-\frac{8\omega_u+\omega_s}{3G\mu_+},$$
$$\mathcal{D} = \sqrt{2}\left(\frac{h_u}{m_u}-\frac{h_s}{m_s}-\frac{\omega_s-\omega_u}{3G\mu_+}\right), \tag{70}$$

and

$$(\mathcal{A}-\mathcal{B})^2+4\mathcal{D}^2 = \left[3\left(\frac{h_u}{m_u}-\frac{h_s}{m_s}\right)-\frac{\omega_s}{G\mu_+}\right]^2+8\left(\frac{\omega_u}{G\mu_+}\right)^2. \tag{71}$$

Supposing for a moment that $\kappa = 0$, we find the mixing angle θ_{s} to be equal θ_{id}, the ϵ-meson is a pure non-strange state, ϵ_{ns}, and the ϵ' is purely strange, $-\epsilon_{\mathrm{s}}$. The scalar masses become

$$m_{a_0}^2 = m_{\epsilon_{\mathrm{ns}}}^2 = m_\pi^2+4m_u^2,$$
$$m_{K_0^*}^2 = m_K^2+4m_u m_s,$$
$$m_{\epsilon_{\mathrm{s}}}^2 = m_{\eta_{\mathrm{s}}}^2+4m_s^2, \tag{72}$$

giving the following mass splittings within the nonet

$$m_{K_0^*}^2-m_{a_0}^2 = 2(m_s-m_u)(m_s+2m_u),$$
$$m_{\epsilon_{\mathrm{s}}}^2-m_{K_0^*}^2 = 2(m_s-m_u)(2m_s+m_u). \tag{73}$$

The latter is three times bigger than in the pseudoscalar case

$$m_{\epsilon_{\mathrm{s}}}^2-m_{\epsilon_{\mathrm{ns}}}^2 = 3\big(m_{\eta_{\mathrm{s}}}^2-m_{\eta_{\mathrm{ns}}}^2\big) = 6\big(m_s^2-m_u^2\big). \tag{74}$$

Let us consider now the $SU(3)$ limit $m_u = m_d = m_s$ for $\kappa \neq 0$. One has

$$m_{a_0}^2 = m_{K_0^*}^2 = M_{88}^2 = m_\pi^2+4m_u^2+\frac{2g^2\omega}{G(1-\omega^2)}. \tag{75}$$

There is no mixing here, since $\mathcal{D} = 0$, and the singlet state is splitted due to the 't Hooft interaction

$$M_{00}^2-M_{88}^2 = -\frac{3g^2\omega}{G(1-\omega)(1+2\omega)}. \tag{76}$$

Comparing the $SU(3)$ limit of singlet–octet mass splittings in the pseudoscalar, Eq. (65), and scalar, Eq. (76), channels, one observes that these expressions have opposite signs for the physically reasonable sets of parameters ($0 < \omega < 1/2$), where μ_- and μ_+ are positive.

The 't Hooft interaction pulls the singlet pseudoscalar state up and the singlet scalar state down with respect to the corresponding octet ones.

To summarize, the pseudoscalar and scalar masses are obtained by means of a specific asymptotic expansion[3] of the heat kernel in the framework of a simple model for low-energy QCD. It can be improved in different ways. We have already mentioned some of them in Section 2. Here we also would like to point out that in truncating the heat kernel series at second order we are neglecting finite size momentum dependent contributions to the one-loop fermion determinant that become more important for the heavier particles, so that the pole position for extraction of the masses can be modified in a sizable way. However it is well known that the lack of confinement in the NJL model introduces serious difficulties with the crossing of non-physical thresholds associated with the production of free quark–antiquark pairs, which one may encounter by formally continuing the full Euclidean action to Minkowski space. These are the main reasons why we decided in this simplified version of the model to truncate the series, taking into account only the divergent contributions. On one hand, in doing so, we admittedly deviate from the original NJL Lagrangian, however in a way which relies heavily on its symmetries and asymptotic dynamics, which are fully taken into account. On the other hand, this approach gives us, in principle, a chance to correct systematically the coefficients I_i of the heat kernel series by introducing new parameters in the regularization function $\rho(t, \Lambda_1, \Lambda_2, \ldots)$ and fixing them in accordance with phenomenological requirements. This procedure, hopefully, can be developed similarly to QCD sum rules, like it has been done in [24] and discussed, in particular, in relation with NJL-type models in [25].

5. Numerical results and discussion

The parameters of the model, $\hat{m}_u, \hat{m}_s, G, \kappa$ and Λ are shown in Table 1.

In Table 2 is the pseudoscalar spectrum, together with the weak decay constants f_π, f_K and mixing angle θ_{p}; the masses and mixing angle θ_{s} of the scalars are given in Table 3. Inputs are indicated by (*). The Latin letter labels on the left-hand side identify the sets in the tables.

The following empirical values are taken from [26]: $m_\pi^\pm = 139.57018 \pm 0.00035$ [MeV], $m_K^\pm = 493.677 \pm 0.016$ [MeV], $m_\eta = 547 \pm 0.12$ [MeV], $m_{\eta'} = 957.78 \pm 0.14$ [MeV] for the masses in the low lying pseudoscalar sector. The weak decay constants $F_\pi^{\mathrm{exp}} = 130.7 \pm 0.1 \pm 0.36$ [MeV], $F_K^{\mathrm{exp}} = 159.8 \pm 1.4 \pm 0.44$ [MeV] relate to ours through a $\sqrt{2}$ normalization factor, thus $f_\pi^{\mathrm{exp}} \simeq 92.4$ MeV and $f_K^{\mathrm{exp}} \simeq 113$ MeV.

The scalar masses up to $\simeq 2$ GeV are presently known to be: $a_0(980) = 984.7 \pm 1.2$ [MeV], $a_0(1450) = 1474 \pm 19$ [MeV], $f_0(600) = 400$–1200 [MeV], $f_0(980) = 980 \pm 10$ [MeV], $f_0(1370) = 1200$–1500 [MeV], $f_0(1500) = 1500 \pm 5$ [MeV], $f_0(1710) = 1713 \pm 6$ [MeV], $K_0^*(1430) = 1412 \pm 6$ [MeV], where the name of the particle is identified with its mass, in order not to clutter the notation. In [27] there is reported the possibil-

[3] A summation over all constant meson fields in this series leads to a derivative (long wavelength) expansion.

Table 1
The main parameters of the model given in the following units: $[m] = \text{MeV}$, $[G] = \text{GeV}^{-2}$, $[\kappa] = \text{GeV}^{-5}$, $[\Lambda] = \text{GeV}$

	$\hat{m}_u$	(m_u)	$\hat{m}_s$	(m_s)	G	$-\kappa$	Λ
a	4.9	(302)	167	(519)	9.3	0*	0.95
b	2.8	(211)	85	(356)	2.8	157	1.4
c	2.7	(214)	92	(397)	3.1	88	1.4
d	1.2	(171)	41	(310)	1.1	11	2.3
e	0.7	(155)	24	(296)	0.6	1.6	3.2
f	3.2	(227)	105	(405)	3.7	173	1.3
g	4.9	(296)	161	(493)	7.6	664	0.95
h	2.2	(199)	75	(375)	2.3	45	1.6
i	3.6	(242)	122	(437)	4.6	205	1.2
j	3.6	(235)	109	(382)	3.7	422	1.2
k	4.7	(286)	155	(485)	7.2	477	0.98
l	1.5	(179)	50	(317)	1.5	23.4	2.0

Table 2
The pseudoscalar nonet parameters in units of MeV (except for the angle θ_{p}, which is given in degrees)

	m_π	m_K	f_π	f_K	m_η	$m_{\eta'}$	θ_{p}
a	138*	494*	92*	125*	138	612	35
b	138*	494*	92*	124	547*	1504	2
c	138*	494*	92*	131	526	958*	−4
d	138*	494*	92*	129	547*	1078	2
e	138*	495*	92*	134	545*	958*	2
f	137*	496*	92*	128	532	1109	−2
g	137*	496*	92*	122*	507	1089	−7
h	138*	495*	92*	133	535	958*	−3*
i	138*	495*	92*	129	516	958*	−7*
j	138*	494*	92*	121*	547*	2187	2
k	138*	494*	92*	124*	497	958*	−10
l	138*	494*	92*	127*	547*	1156	2

ity of existence of a low lying strange scalar meson K_0^*. A broad resonance with mass $K_0^*(800) = 797 \pm 19 \pm 43$ [MeV] is observed in [28].

We start the discussion of the scalar and pseudoscalar sectors with the following special case shown in set (a). This pattern corresponds to $SU(3)$ breaking ($\hat{m}_u \neq \hat{m}_s$) without $U_A(1)$ breaking ($\kappa = 0$) and has been considered in detail in Section 4 (see Eq. (62) for the pseudoscalars and Eq. (72) for the scalars).

The overall description of mass spectra is reasonable, given the simplicity of the model. Particular trends are as follows. Fixing m_π, m_K, f_π, and m_η (set b) or $m_{\eta'}$ (set c) to their empirical values, results in reducing the parameter κ of the 't Hooft interaction by approximately a factor 2 in going from (b) to (c) (dropping slightly with increasing value of the cutoff). The masses for the scalars and $m_{\eta'}$ are highly sensitive to the choice of the η mass:

Table 3
The different fits for the masses of the scalar NJL nonet in units of MeV (except of the angle θ_S which is given in degrees), as compared with a putative nonet family $a_0(980)$, $K_0^*(800)$, $f_0(600)$ and $f_0(980)$. The symbols of resonances stand for their masses

	a_0 $\{a_0(980)\}$	K_0^* $\{K_0^*(800)\}$	ϵ $\{f_0(600)\}$	ϵ' $\{f_0(980)\}$	θ_S
a	620	933	620	1205	35
b	1215	1164	346	1199	14
c	888	976	423	1097	22
d	985*	968	249	1017	16
e	900	895	224	954	18
f	985*	1050	441	1153	20
g	985*	1150	601	1295	22
h	891	954	384	1063	22
i	889	1021	489	1252	23
j	1447	1346	399	1364	12
k	907	1087	339	1248	24
l	1036	1009	263	1053	15

only a 4% reduction of m_η value in (b) is needed to get the empirical $m_{\eta'}$ (c), corresponding however to a 35% drop of the latter with respect to its value in (b). Fixing η to its empirical mass in (b) not only yields a much too heavy η', but also too heavy scalars a_0, K_0^* and ϵ' (Table 3).

Although the order of magnitude for the scalar masses in set (c) is reasonable, e.g., the mass of a_0 is obtained within 10% of its experimental value and the K_0^* mass within 20%, the general trend for a large set of parameters is $m_{a_0} < m_{K_0^*} < m_{\epsilon'}$, as opposed to the present empirical evidence $m_{K_0^*} < m_{a_0} \simeq m_{f_0(980)}$. The latter ordering can be obtained for sufficiently low values of κ, see set (d), with $m_{a_0} \simeq m_{\epsilon'}$ within 2% of the empirical value, but at the expense of a very light ϵ and too low values of current and constituent quark masses. The mass of K_0^*, being almost degenerate with a_0, remains too large by 20%.

In set (e) we fix the 5 parameters of the model completely in the pseudoscalar sector, through $m_\pi, m_K, f_\pi, m_\eta, m_{\eta'}$. This constrains the κ and G parameters to comparatively very low values and yields also small quark masses; the a_0 and K_0^* masses are almost degenerate, the K_0^* mass being slightly smaller than the a_0 mass.

In sets (f) and (g) three model parameters are fixed through m_π, m_K, f_π, in the pseudoscalar sector and one in the scalar sector m_{a_0}, requiring that the average value of the η, η' masses be within 10% of the empirical value.

In sets (h), (i) we fix $m_\pi, m_K, f_\pi, m_{\eta'}$ and the mixing angle in the pseudoscalar channels. Results are also quite sensitive to the choice of f_K, see for instance sets (j) and (k), where the four input values of sets (b) and (c) have been kept respectively, fixing the remaining freedom by reducing slightly the values of f_K. In set (j) a reduction of f_K implies an increase in the magnitude of κ, increasing the splitting and turning therefore the η' significantly heavier (η remained fixed). The masses of the scalars increase by about 20%, as compared to their values in set (b), the lower f_0 a bit less, by 15%. In set (k) the reduction of f_K implies also an increase in κ and therefore in the splitting, this time reducing the value of m_η (since $m_{\eta'}$ was kept fixed). The splitting in the scalars is also enhanced, the ϵ' is pushed up and ϵ down. The masses of a_0 and K_0^* increase only slightly.

In set (l) the input parameters of (b) were kept, but f_K chosen larger. The parameter κ gets reduced and the conclusions are opposite to the ones of set (j).

The values of the mixing angles θ_{p} and θ_{s} shown in Tables 2 and 3 are consistent with results obtained in [29] in the framework of the linear σ model with broken $U(3) \times U(3)$ symmetry, where $\theta_{\mathrm{p}} = -5^\circ$ and $\theta_{\mathrm{s}} = 21.9^\circ$, and with the values $\theta_{\mathrm{p}} \approx 2^\circ$, $\phi_{\mathrm{s}} \approx -14^\circ$ reported in [30]. The last angle here describes the mixing in the flavour basis and corresponds to $\bar{\psi}_{\mathrm{s}}$ (see Appendix B) in our notations. This agreement is not accidental, since the bosonized NJL model is closely related to the linear sigma model [31,32].

6. Concluding remarks

We have analyzed the Nambu–Jona-Lasinio model of QCD in the light of a new generalized heat kernel expansion. The result is an effective Lagrangian of low-energy QCD, incorporating the complete original symmetry pattern, but eliminating all non-physical thresholds associated with quark–antiquark pair formation due to the lack of confinement of the original Lagrangian. We applied the so obtained Lagrangian in the extraction of the low lying spectra of pseudoscalars and scalars. The pseudoscalar spectrum turns out to be quite satisfactory and we used it partly to fix the main parameters of the model. As can be seen from Table 3 the predictions for scalar mesons are also not too far from the experimental masses of the lightest known scalars, which is remarkable in view of the simplicity of the model.

There is growing evidence that an isovector $a_0(980)$, an isospinor $K_0^*(800)$, as well as two isoscalars $f_0(600)$ and $f_0(980)$, are members of the same low-lying scalar nonet [30, 33–35]. There are however different opinions about their origin. In our calculation we considered the lightest scalar nonet as being $q\bar{q}$ states. It is in line with ideas presented in [29].

The outcome of the model is obtained in the leading order stationary phase approximation and can be implemented. There are different sources for corrections both at leading order and next to leading order. For instance the inclusion of vector and axial-vector mesons can be important for the physical picture, because they contribute already at leading order through the pseudoscalar–axial-vector and scalar–vector mixings. There are also several contributions at next to leading order, e.g., meson loop corrections [36] and semi-classical corrections to the ’t Hooft determinant [16]. As discussed in Section 4 of [16] there are two distinct regimes of chiral symmetry breaking, related to small/large six-quark fluctuations. For large fluctuations the quantum corrections may be numerically relevant. Our aim in the present work was to show that the considered new method for the asymptotic expansion of the heat kernel, which is in full agreement with all symmetry requirements, leads already in its minimal form to realistic results for mass spectra. A more detailed description of the scalar nonet in the framework of our method, including its decay properties, will be given elsewhere.

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Appendix A. Consequences of Eq. (23)

The first equation in (23) can be written in terms of quark-flavour components h_i (see Eq. (27)). In general the (u, d, s) basis can be transformed to the basis $(0, 3, 8)$ by the use of the following matrices ω_{ia} and e_{ai} defined as [16]

$$e_{ai} = \frac{1}{2\sqrt{3}}\begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{3} & -\sqrt{3} & 0 \\ 1 & 1 & -2 \end{pmatrix}, \qquad \omega_{ia} = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & -\sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \end{pmatrix}. \tag{A.1}$$

Here the index a runs $a = 0, 3, 8$ (for the other values of a the corresponding matrix elements are assumed to be zero). We have then for instance $h_a = e_{ai}h_i$, and $h_i = \omega_{ia}h_a$. Similar relations can be obtained for Δ_i and Δ_a. In accordance with this notation we use, for instance, that $h^{(1)}_{ci} = \omega_{ia}h^{(1)}_{ca}$. The following properties of matrices (A.1) are straightforward: $\omega_{ia}e_{aj} = \delta_{ij}$, $e_{ai}\omega_{ib} = \delta_{ab}$, $e_{ai}e_{aj} = \delta_{ij}/2$ and $\omega_{ia}\omega_{ie} = 2\delta_{ae}$. The coefficients t_{ijk} are related to the coefficients A_{abc} by the embedding formula $3\omega_{ia}A_{abc}e_{bj}e_{ck} = t_{ijk}$. The $SU(3)$ matrices λ_a with index i are defined in a slightly different way $2\lambda_i = \omega_{ia}\lambda_a$ and $\lambda_a = 2e_{ai}\lambda_i$. In this case it follows that, for instance, $\sigma = \sigma_a\lambda_a = \sigma_i\lambda_i = \mathrm{diag}(\sigma_u, \sigma_d, \sigma_s)$, but $2\sigma_a\Delta_a = \sigma_i\Delta_i$.

The solutions of Eq. (27) are given in [16]. One can express all other coefficients $h_{a\ldots}$ in terms of these basic variables. We quote further our result for h_{ab}, splitting the range of running indices a, b on three subsets: $r, s = 0, 8$, $n, m = 1, 2, 3$ and $f, g = 4, 5, 6, 7$,

$$h^{(1,2)}_{nm} = \frac{-\delta_{nm}}{G(1 \mp \omega_s)}, \qquad h^{(1,2)}_{fg} = \frac{-\delta_{fg}}{G(1 \mp \omega_u)}. \tag{A.2}$$

For the 2×2 matrix with indices 0, 8 we have

$$h^{(1,2)}_{rs} = \frac{-1}{3G\mu_\pm}\begin{pmatrix} 3 \mp (4\omega_u - \omega_s) & \pm\sqrt{2}(\omega_u - \omega_s) \\ \pm\sqrt{2}(\omega_u - \omega_s) & 3 \pm 2(2\omega_u + \omega_s) \end{pmatrix}_{rs}, \tag{A.3}$$

with $\mu_\pm = (1 \pm \omega_s - 2\omega_u^2)$ and

$$\omega_i = \frac{\kappa h_i}{16G}. \tag{A.4}$$

Quite often the stationary phase equations considered together with the gap-equations help us to simplify essentially the results. Here is an useful example that shows Eqs. (27) and (43) at work.

Example. Let us consider the expression for the mass of kaons following from our mesonic Lagrangian. It is not difficult to obtain that

$$m_K^2 = g^2\left[\frac{1}{G(1 + \omega_u)} + \frac{1}{2}\left(\frac{h_s}{m_s} + \frac{h_u}{m_u}\right)\right] + (m_s - m_u)^2. \tag{A.5}$$

One notices, by using

$$g^2\left(\frac{h_s}{m_s} - \frac{h_u}{m_u}\right) = 2\left(m_s^2 - m_u^2\right), \tag{A.6}$$

which is a direct consequence of the gap equations, that the following relation is fulfilled

$$\frac{g^2}{2}\left(\frac{h_s}{m_s}+\frac{h_u}{m_u}\right)+(m_s-m_u)^2=g^2\frac{h_u+h_s}{m_u+m_s}. \tag{A.7}$$

Therefore, we obtain

$$m_K^2=g^2\left(\frac{1}{G(1+\omega_u)}+\frac{h_u+h_s}{m_u+m_s}\right), \tag{A.8}$$

which can be further reduced to the final result indicated in the Eq. (57), by observing that

$$h_u+h_s=-\frac{\Delta_u+\Delta_s}{G(1+\omega_u)}. \tag{A.9}$$

This last expression follows immediately from Eq. (27).

Appendix B. Diagonalization of the mass matrix and physical states

To illustrate how the physical fields are chosen in the main part of the text we recall here some useful details of the diagonalization procedure and how to relate to several different conventions adopted in the literature. Our starting point is a quadratic form $\mathcal{Q}$ written in the singlet–octet basis (X_0, X_8)

$$\mathcal{Q}=(X_0,X_8)\begin{pmatrix} A & D\\ D & B\end{pmatrix}\begin{pmatrix} X_0\\ X_8\end{pmatrix} \tag{B.1}$$

which can be diagonalized by an orthogonal transformation to the physical states $(X, \bar{X})$

$$\begin{pmatrix} X\\ \bar{X}\end{pmatrix}=\begin{pmatrix} \cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix} X_0\\ X_8\end{pmatrix}. \tag{B.2}$$

The angle θ is extracted from the equation

$$\tan 2\theta=\frac{2D}{A-B}. \tag{B.3}$$

After some trigonometry the θ-dependence of the diagonalized matrix $\mathcal{Q}$ can be absorbed in just one term

$$\mathcal{Q}=\frac{1}{2}(X,\bar{X})\begin{pmatrix} A+B+\frac{A-B}{\cos 2\theta} & 0\\ 0 & A+B-\frac{A-B}{\cos 2\theta}\end{pmatrix}\begin{pmatrix} X\\ \bar{X}\end{pmatrix}. \tag{B.4}$$

It is easy to see that

$$\frac{A-B}{\cos 2\theta}=\operatorname{sgn}\left(\frac{A-B}{\cos 2\theta}\right)\sqrt{(A-B)^2+4D^2} \tag{B.5}$$

and therefore

$$\begin{aligned}
&\mathcal{Q}=m_X^2X^2+m_{\bar{X}}^2\bar{X}^2,\\
&m_X^2=\frac{1}{2}\left[A+B+\operatorname{sgn}\left(\frac{A-B}{\cos 2\theta}\right)\sqrt{(A-B)^2+4D^2}\right],\\
&m_{\bar{X}}^2=\frac{1}{2}\left[A+B-\operatorname{sgn}\left(\frac{A-B}{\cos 2\theta}\right)\sqrt{(A-B)^2+4D^2}\right].
\end{aligned} \tag{B.6}$$

Finally, to identify the fields $X, \bar{X}$ with the physical ones, one should proceed as follows. Firstly, find the angle θ from Eq. (B.3), choosing, for instance, the principal value of $\arctan 2\theta$, i.e., $-(\pi/4) \leqslant \theta \leqslant (\pi/4)$. Secondly, determine the sign of the ratio $(A-B)/\cos 2\theta$. Only after having established which value of $(m_X, m_{\bar{X}})$ is bigger, should one proceed with identification of the physical fields, writing down the corresponding rotation (B.2).

Alternatively, one can use the non-strange–strange basis $(X_{\rm ns}, X_{\rm s})$, where

$$\begin{pmatrix} X_{\rm ns} \\ X_{\rm s} \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} & 1 \\ 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} X_0 \\ X_8 \end{pmatrix}. \tag{B.7}$$

Our definition (B.2), taken together with Eq. (B.7), leads to the explicit representation

$$\begin{pmatrix} \bar{X} \\ X \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} X_{\rm ns} \\ X_{\rm s} \end{pmatrix}, \tag{B.8}$$

or

$$\begin{pmatrix} X \\ \bar{X} \end{pmatrix} = \begin{pmatrix} \cos\bar{\psi} & \sin\bar{\psi} \\ -\sin\bar{\psi} & \cos\bar{\psi} \end{pmatrix} \begin{pmatrix} X_{\rm ns} \\ -X_{\rm s} \end{pmatrix}. \tag{B.9}$$

The angle ψ here is equal to $\psi = \theta + \bar{\theta}_{\rm id}$, where $\bar{\theta}_{\rm id}$ $(\theta_{\rm id} + \bar{\theta}_{\rm id} = \pi/2)$ is determined by the equations $\sin\bar{\theta}_{\rm id} = \sqrt{2/3}$, $\cos\bar{\theta}_{\rm id} = 1/\sqrt{3}$, and therefore $\psi = \theta + \arctan\sqrt{2} \simeq \theta + 54.74°$. It means that ψ is restricted to the range $9.74° \lesssim \psi \lesssim 99.74°$. The angle $\bar{\psi} = \psi - (\pi/2) = \theta - \theta_{\rm id}$ and belongs to the interval $-80.26° \lesssim \bar{\psi} \lesssim 9.74°$. These two angles correspond to two alternative phase conventions for a strange $\bar{s}s$-component.

Here are examples that illustrate the physical interpretation of the given formulae, using the results of our calculations obtained in Section 4.

Example 1. In the case of pseudoscalars with broken $SU(3)$ symmetry but without $U_A(1)$ breaking $(\kappa = 0)$ the ϕ_0 and ϕ_8 components are mixed with the angle $\theta = \theta_{\rm id}$ and $(A - B) < 0$. Hence, one can conclude from Eq. (B.6) that the $\bar{X}$-state is a heavier one and corresponds to η'. We have from (B.8) $\eta' \equiv -\eta_{\rm s}$ and $\eta \equiv \eta_{\rm ns}$. However, if $U_A(1)$ symmetry is broken $(\kappa \neq 0)$, one has $(A - B) > 0$ (these are exactly the cases (b)–(l) shown in the Table 2) and we must identify the physical fields in opposite order $\eta' \equiv X$, $\eta \equiv \bar{X}$.

Example 2. In the case of scalar mesons there is no difference between the two patterns $\kappa = 0$ and $\kappa \neq 0$. In both cases we have $(\mathcal{A} - \mathcal{B}) < 0$, i.e., $\bar{X} \equiv \epsilon'$, $X \equiv \epsilon$. If $\kappa = 0$, they are pure flavour states $\bar{X} \equiv -\epsilon_{\rm s}$ and $X \equiv \epsilon_{\rm ns}$.

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