

REGULARITY AND ALGEBRAIC PROPERTIES OF CERTAIN LATTICE IDEALS

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Dedicated to Professor Aron Simis on the occasion of his 70th birthday

ABSTRACT. We study the regularity and the algebraic properties of certain lattice ideals. We establish a map $I \mapsto \tilde{I}$ between the family of graded lattice ideals in an \mathbb{N} -graded polynomial ring over a field K and the family of graded lattice ideals in a polynomial ring with the standard grading. This map is shown to preserve the complete intersection property and the regularity of I but not the degree. We relate the Hilbert series and the generators of I and \tilde{I} . If $\dim(I) = 1$, we relate the degrees of I and \tilde{I} . It is shown that the regularity of certain lattice ideals is additive in a certain sense. Then, we give some applications. For finite fields, we give a formula for the regularity of the vanishing ideal of a degenerate torus in terms of the Frobenius number of a semigroup. We construct vanishing ideals, over finite fields, with prescribed regularity and degree of a certain type. Let X be a subset of a projective space over a field K . It is shown that the vanishing ideal of X is a lattice ideal of dimension 1 if and only if X is a finite subgroup of a projective torus. For finite fields, it is shown that X is a subgroup of a projective torus if and only if X is parameterized by monomials. We express the regularity of the vanishing ideal over a bipartite graph in terms of the regularities of the vanishing ideals of the blocks of the graph.

1. INTRODUCTION

Let $S = K[t_1, \dots, t_s] = \bigoplus_{d=0}^{\infty} S_d$ and $\tilde{S} = K[t_1, \dots, t_s] = \bigoplus_{d=0}^{\infty} \tilde{S}_d$ be polynomial rings, over a field K , with the gradings induced by setting $\deg(t_i) = d_i$ for all i and $\deg(t_i) = 1$ for all i , respectively, where d_1, \dots, d_s are positive integers. Let $F = \{f_1, \dots, f_s\}$ be a set of algebraically independent homogeneous polynomials of \tilde{S} of degrees d_1, \dots, d_s and let

$$\phi: S \rightarrow K[F]$$

be the isomorphism of K -algebras given by $\phi(g) = g(f_1, \dots, f_s)$, where $K[F]$ is the K -subalgebra of \tilde{S} generated by f_1, \dots, f_s . For convenience we denote $\phi(g)$ by \tilde{g} . Given a graded ideal $I \subset S$ generated by g_1, \dots, g_m , we associate to I the graded ideal $\tilde{I} \subset \tilde{S}$ generated by $\tilde{g}_1, \dots, \tilde{g}_m$ and call \tilde{I} the *homogenization* of I with respect to f_1, \dots, f_s . The ideal \tilde{I} is independent of the generating set g_1, \dots, g_m and \tilde{I} is a graded ideal with respect to the standard grading of $K[t_1, \dots, t_s]$.

If $f_i = t_i^{d_i}$ for $i = 1, \dots, s$, the map $I \mapsto \tilde{I}$ induces a correspondence between the family of graded lattice ideals of S and the family of graded lattice ideals of \tilde{S} . The first aim of this paper is to study this correspondence and to relate the algebraic invariants (regularity and degree)

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and properties of I and \tilde{I} (especially the complete intersection property). For finite fields, the interest in this correspondence comes from the fact that any given vanishing ideal $I(X) \subset \tilde{S}$, over a degenerate projective torus X , arise as a toric ideal $I \subset S$ of a monomial curve, i.e., $\tilde{I} = I(X)$ for some graded toric ideal I of S of dimension 1 (see [22, Proposition 3.2]). In this paper, we extend the scope of [22] to include lattice ideals of arbitrary dimension. The second aim of this paper is to use our methods to study the regularity of graded vanishing ideals and to give classifications of this type of ideals. The algebraic invariants (degree, regularity) and the complete intersection property of vanishing ideals over finite fields, are of interest in algebraic coding theory [14, 28, 33] and commutative algebra [1, 2, 8, 11, 12, 24, 36]. The length, dimension and minimum distance of evaluation codes arising from complete intersections have been studied in [7, 13, 17, 19, 29].

The contents of this paper are as follows. In Section 2, we introduce the notions of degree and index of regularity via Hilbert functions. Lattices and lattice ideals are also introduced in this section. We present some of the results that will be needed throughout the paper. All the results of this section are well known.

In Section 3, we establish a map $I \mapsto \tilde{I}$ between the graded ideals of S and \tilde{S} . We relate the minimal graded resolutions, the Hilbert series, and the regularities of I and \tilde{I} . In general, the map $I \mapsto \tilde{I}$ does not preserve the height of I (Example 3.2). Let I be a graded ideal of S and assume that $K[F] \subset \tilde{S}$ is an integral extension. We show that $\dim(S/I) \geq \dim(\tilde{S}/\tilde{I})$ with equality if I is a monomial ideal (Lemma 3.3). Then, using the Buchsbaum-Eisenbud acyclicity criterion [5, Theorem 1.4.13], we show that if

$$0 \rightarrow \bigoplus_{j=1}^{b_g} S(-a_{gj}) \rightarrow \cdots \rightarrow \bigoplus_{j=1}^{b_1} S(-a_{1j}) \rightarrow S \rightarrow S/I \rightarrow 0$$

is the minimal graded free resolution of S/I , then

$$0 \rightarrow \bigoplus_{j=1}^{b_g} \tilde{S}(-a_{gj}) \rightarrow \cdots \rightarrow \bigoplus_{j=1}^{b_1} \tilde{S}(-a_{1j}) \rightarrow \tilde{S} \rightarrow \tilde{S}/\tilde{I} \rightarrow 0$$

is the minimal graded free resolution of \tilde{S}/\tilde{I} (Lemma 3.5). By the *regularity* of S/I , denoted by $\text{reg}(S/I)$, we mean the Castelnuovo-Mumford regularity. This notion is introduced in Section 3. We denote the Hilbert series of S/I by $F_I(t)$.

This close relationship between the graded resolutions of I and \tilde{I} allows us to relate the Hilbert series and the regularities of I and \tilde{I} .

Theorem 3.6 *Let I be a graded ideal of S , then $\text{reg}(S/I) = \text{reg}(\tilde{S}/\tilde{I})$ and*

$$F_{\tilde{I}}(t) = \lambda_1(t) \cdots \lambda_s(t) F_I(t), \text{ where } \lambda_i(t) = 1 + t + \cdots + t^{d_i-1}.$$

For the rest of the introduction we will assume that $f_i = t_i^{d_i}$ for $i = 1, \dots, s$. Accordingly, $\tilde{I} \subset \tilde{S}$ will denote the homogenization of a graded ideal $I \subset S$ with respect to $t_1^{d_1}, \dots, t_s^{d_s}$.

In Section 4, we examine the map $I \mapsto \tilde{I}$ between the family of graded lattice ideals of S and the family of graded lattice ideals of \tilde{S} . Let $\mathcal{L} \subset \mathbb{Z}^s$ be a *homogeneous lattice*, with respect to d_1, \dots, d_s , and let $I(\mathcal{L}) \subset S$ be its graded lattice ideal. It is well known that the height of $I(\mathcal{L})$ is the rank of \mathcal{L} (see for instance [23, Proposition 7.5]). If D is the non-singular matrix $\text{diag}(d_1, \dots, d_s)$, then $I(\tilde{\mathcal{L}})$ is the lattice ideal of $\tilde{\mathcal{L}} = D(\mathcal{L})$ and the height of $I(\mathcal{L})$ is equal to the height of $I(\tilde{\mathcal{L}})$.

We come to the main result of Section 4 that relates the generators of $I(\mathcal{L})$ and $I(\tilde{\mathcal{L}})$.

Theorem 4.2 *Let $\mathcal{B} = \{t^{a_i} - t^{b_i}\}_{i=1}^m$ be a set of binomials. If $\tilde{\mathcal{B}} = \{t^{D(a_i)} - t^{D(b_i)}\}_{i=1}^m$, then $I(\mathcal{L}) = (\mathcal{B})$ if and only if $I(\tilde{\mathcal{L}}) = (\tilde{\mathcal{B}})$.*

Then, using the fact that $\text{ht}(I(\mathcal{L})) = \text{ht}(I(\tilde{\mathcal{L}}))$, we show that $I(\tilde{\mathcal{L}})$ is a complete intersection if and only if $I(\mathcal{L})$ is a complete intersection (Corollary 4.4).

In Section 5, we study the restriction of the map $I \mapsto \tilde{I}$ to the family of graded lattice ideals of dimension 1. In this case, the degrees of I and \tilde{I} are nicely related.

Theorem 5.3 *If I is a graded lattice ideal of dimension 1, then*

$$\deg(\tilde{S}/\tilde{I}) = \frac{d_1 \cdots d_s}{\max\{d_1, \dots, d_s\}} \deg(S/I).$$

Let $r = \gcd(d_1, \dots, d_s)$ and let \mathcal{S} be the numerical semigroup $\mathbb{N}(d_1/r) + \cdots + \mathbb{N}(d_s/r)$. The *Frobenius number* of \mathcal{S} , denoted by $g(\mathcal{S})$, is the largest integer not in \mathcal{S} . The Frobenius number occurs in many branches of mathematics and is one of the most studied invariants in the theory of semigroups. A great deal of effort has been directed at the effective computation of this number, see the monograph of Ramírez-Alfonsín [27].

The next result gives an explicit formula for the regularity of \tilde{I} , in terms of the Frobenius number of \mathcal{S} , when I is the toric ideal of a monomial curve. This formula can be used to compute the regularity using some available algorithms to compute Frobenius numbers [27].

Theorem 5.5 *If I is the toric ideal of $K[y_1^{d_1}, \dots, y_1^{d_s}] \subset K[y_1]$, then*

$$\text{reg}(\tilde{S}/\tilde{I}) = r \cdot g(\mathcal{S}) + 1 + \sum_{i=1}^s (d_i - 1).$$

Let \succ be the reverse lexicographical order. If I is a graded lattice ideal and $\dim(S/I) = 1$, we show the following equalities

$$\text{reg}(S/I) = \text{reg}(\tilde{S}/\tilde{I}) = \text{reg}(\tilde{S}/\text{in}(\tilde{I})) = \text{reg}(S/\text{in}(I)),$$

where $\text{in}(I)$, $\text{in}(\tilde{I})$ are the initial ideals of I , \tilde{I} , with respect to \succ , respectively (Corollary 5.7).

We come to the last main result of Section 5 showing that the regularity of the saturation of certain one dimensional graded ideals is additive in a certain sense. This will be used in Section 7 to study the regularity of vanishing ideals over bipartite graphs.

Theorem 5.8 *Let V_1, \dots, V_c be a partition of $V = \{t_1, \dots, t_s\}$ and let ℓ be a positive integer. If I_k is a graded binomial ideal of $K[V_k]$ such that $t_i^\ell - t_j^\ell \in I_k$ for $t_i, t_j \in V_k$ and \mathcal{I} is the ideal of $K[V]$ generated by all binomials $t_i^\ell - t_j^\ell$ with $1 \leq i, j \leq s$, then*

$$\text{reg } K[V]/(I_1 + \cdots + I_c + \mathcal{I} : h^\infty) = \sum_{k=1}^c \text{reg } K[V_k]/(I_k : h_k^\infty) + (c-1)(\ell-1),$$

where $h = t_1 \cdots t_s$ and $h_k = \prod_{t_i \in V_k} t_i$ for $k = 1, \dots, c$. Here $K[V_k]$ and $K[V]$ are polynomial rings with the standard grading.

In Section 6, we study graded vanishing ideals over arbitrary fields and give some applications of the results of Section 5. Let K be a field and let \mathbb{P}^{s-1} be the projective space of dimension $s-1$ over K . Given a sequence $v = (v_1, \dots, v_s)$ of positive integers, the set

$$\{[(x_1^{v_1}, \dots, x_s^{v_s})] \mid x_i \in K^* \text{ for all } i\} \subset \mathbb{P}^{s-1}$$

is called a *degenerate projective torus* of type v , where $K^* = K \setminus \{0\}$. If $v_i = 1$ for all i , this set is called a *projective torus* in \mathbb{P}^{s-1} and it is denoted by \mathbb{T} . If X is a subset of \mathbb{P}^{s-1} , the *vanishing ideal* of X , denoted by $I(X)$, is the ideal of \tilde{S} generated by the homogeneous polynomials that vanish on all X .

For finite fields, we give formulae for the degree and regularity of graded vanishing ideals over degenerate tori. The next result was shown in [22] under the hypothesis that $I(X)$ is a complete intersection.

Corollary 6.2 *Let $K = \mathbb{F}_q$ be a finite field and let X be a degenerate projective torus of type $v = (v_1, \dots, v_s)$. If $d_i = (q - 1) / \gcd(v_i, q - 1)$ for $i = 1, \dots, s$ and $r = \gcd(d_1, \dots, d_s)$, then*

$$\operatorname{reg}(\tilde{S}/I(X)) = r \cdot g(\mathcal{S}) + 1 + \sum_{i=1}^s (d_i - 1) \quad \text{and} \quad \deg(\tilde{S}/I(X)) = d_1 \cdots d_s / r.$$

This result allows us to construct graded vanishing ideals over finite fields with prescribed regularity and degree of a certain type (Proposition 6.4).

We characterize when a graded lattice ideal of dimension 1 is a vanishing ideal in terms of the degree (Proposition 6.5). Then we classify the vanishing ideals that are lattice ideals of dimension 1. For finite fields, it is shown that X is a subgroup of a projective torus if and only if X is parameterized by monomials (Proposition 6.7). For infinite fields we show a formula for the vanishing ideal of an algebraic toric set parameterized by monomials (Theorem 6.9). For finite fields, a formula for the vanishing ideal was shown in [28, Theorem 2.1].

In Section 7, we study graded vanishing ideals over bipartite graphs. Let G be a simple graph with vertex set $V_G = \{y_1, \dots, y_n\}$ and edge set E_G . We refer to [4] for the general theory of graphs. Let $\{v_1, \dots, v_s\} \subset \mathbb{N}^n$ be the set of all characteristic vectors of the edges of the graph G . We may identify the edges of G with the variables t_1, \dots, t_s of a polynomial ring $K[t_1, \dots, t_s]$. The set $X \subset \mathbb{T}$ parameterized by the monomials y^{v_1}, \dots, y^{v_s} is called the *projective algebraic toric set parameterized by the edges of G* .

For bipartite graphs, if the field K is finite, we are able to express the regularity of the vanishing ideal of X in terms of the regularities of the vanishing ideals of the projective algebraic toric sets parameterized by the edges of the blocks of the graph. The blocks of G are essentially the maximal 2-connected subgraphs of G (see Section 7). If K is infinite, the vanishing ideal of X is the toric ideal of $K[t^{v_1}z, \dots, t^{v_s}z]$ (as is seen in Theorem 6.9), in this case $I(X)$ is Cohen-Macaulay [31, 32] (because G is bipartite) and formulae for the regularity and the a -invariant are given in [35, Proposition 4.2].

We come to the main result of Section 7.

Theorem 7.4 *Let G be a bipartite graph without isolated vertices and let G_1, \dots, G_c be the blocks of G . If K is a finite field with q elements and X (resp X_k) is the projective algebraic toric set parameterized by the edges of G (resp. G_k), then*

$$\operatorname{reg} K[E_G]/I(X) = \sum_{k=1}^c \operatorname{reg} K[E_{G_k}]/I(X_k) + (q - 2)(c - 1).$$

Let P be the toric ideal of $\mathbb{F}_q[y^{v_1}, \dots, y^{v_s}]$ and let I be the binomial ideal $I = P + \mathcal{I}$, where \mathcal{I} is the binomial ideal

$$\mathcal{I} = (\{t_i^{q-1} - t_j^{q-1} \mid t_i, t_j \in E_G\}).$$

We relate the regularity of $I(X)$ with the Hilbert function of S/I and the primary decompositions of I (Proposition 7.5). As a byproduct one obtains a method that can be used to compute the regularity (Proposition 7.5(d)). For an arbitrary bipartite graph, Theorem 7.4 and [37, Theorem 2.18] can be used to bound the regularity of $I(X)$.

For all unexplained terminology and additional information, we refer to [10, 23] (for the theory of binomial and lattice ideals), [6, 8, 9, 16, 34, 36] (for commutative algebra and the theory of Hilbert functions), and [4] (for the theory of graphs).

2. PRELIMINARIES

In this section, we introduce the notions of degree and index of regularity—via Hilbert functions—and the notion of a lattice ideal. We present some of the results that will be needed throughout the paper.

Let $S = K[t_1, \dots, t_s]$ be a polynomial ring over a field K and let I be an ideal of S . The vector space of polynomials of S (resp. I) of degree at most d is denoted by $S_{\leq d}$ (resp. $I_{\leq d}$). The functions

$$H_I^a(d) = \dim_K(S_{\leq d}/I_{\leq d}) \quad \text{and} \quad H_I(d) = H_I^a(d) - H_I^a(d-1)$$

are called the *affine Hilbert function* and the *Hilbert function* of S/I respectively. We denote the Krull dimension of S/I by $\dim(S/I)$. If $k = \dim(S/I)$, according to [16, Remark 5.3.16, p. 330], there are unique polynomials

$$h_I^a(t) = \sum_{i=0}^k a_i t^i \in \mathbb{Q}[t] \quad \text{and} \quad h_I(t) = \sum_{i=0}^{k-1} c_i t^i \in \mathbb{Q}[t]$$

of degrees k and $k-1$, respectively, such that $h_I^a(d) = H_I^a(d)$ and $h_I(d) = H_I(d)$ for $d \gg 0$. By convention the zero polynomial has degree -1 .

Definition 2.1. The integer $a_k(k!)$, denoted by $\deg(S/I)$, is called the *degree* of S/I .

Notice that $a_k(k!) = c_{k-1}((k-1)!)$ for $k \geq 1$. If $k = 0$, then $H_I^a(d) = \dim_K(S/I)$ for $d \gg 0$ and the degree of S/I is just $\dim_K(S/I)$. If $S = \bigoplus_{d=0}^{\infty} S_d$ has the standard grading and I is a graded ideal, then

$$H_I^a(d) = \sum_{i=0}^d \dim_K(S_d/I_d)$$

where $I_d = I \cap S_d$. Thus, one has $H_I(d) = \dim_K(S_d/I_d)$ for all d .

Definition 2.2. The *index of regularity* of S/I , denoted by $r(S/I)$, is the least integer $\ell \geq 0$ such that $h_I(d) = H_I(d)$ for $d \geq \ell$.

If S has the standard grading and I is a graded Cohen-Macaulay ideal of S of dimension 1, then $\text{reg}(S/I)$, the Castelnuovo Mumford regularity of S/I , is equal to the index of regularity of S/I (see Theorem 3.4).

Proposition 2.3. ([16, Lemma 5.3.11], [26]) *If I is an ideal of S and $I = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_m$ is a minimal primary decomposition, then*

$$\deg(S/I) = \sum_{\text{ht}(\mathfrak{q}_i) = \text{ht}(I)} \deg(S/\mathfrak{q}_i).$$

Definition 2.4. Let \mathbb{P}^{s-1} be a projective space over K and let $X \subset \mathbb{P}^{s-1}$. If S has the standard grading, the *vanishing ideal* of X , denoted by $I(X)$, is the ideal of S generated by the homogeneous polynomials of S that vanish on all X .

Corollary 2.5. [12] *If $X \subset \mathbb{P}^{s-1}$ is a finite set, then $\deg(S/I(X)) = |X|$.*

Recall that a binomial in S is a polynomial of the form $t^a - t^b$, where $a, b \in \mathbb{N}^s$ and where, if $a = (a_1, \dots, a_s) \in \mathbb{N}^s$, we set

$$t^a = t_1^{a_1} \dots t_s^{a_s} \in S.$$

A binomial of the form $t^a - t^b$ is usually referred to as a *pure binomial* [10], although here we are dropping the adjective “pure”. A *binomial ideal* is an ideal generated by binomials.

Given $c = (c_i) \in \mathbb{Z}^s$, the set $\text{supp}(c) = \{i \mid c_i \neq 0\}$ is called the *support* of c . The vector c can be uniquely written as $c = c^+ - c^-$, where c^+ and c^- are two nonnegative vectors with

disjoint support, the *positive* and the *negative* part of c respectively. If t^a is a monomial, with $a = (a_i) \in \mathbb{N}^s$, the set $\text{supp}(t^a) = \{t_i \mid a_i > 0\}$ is called the *support* of t^a .

Definition 2.6. A *lattice ideal* is an ideal of the form $I(\mathcal{L}) = (t^{a^+} - t^{a^-} \mid a \in \mathcal{L}) \subset S$ for some subgroup \mathcal{L} of \mathbb{Z}^s . A subgroup \mathcal{L} of \mathbb{Z}^s is called a *lattice*.

The class of lattice ideals has been studied in many places, see for instance [10, 23] and the references there. This concept is a natural generalization of a toric ideal.

The following is a well known description of lattice ideals that follows from [10, Corollary 2.5].

Theorem 2.7. [10] *If I is a binomial ideal of S , then I is a lattice ideal if and only if t_i is a non-zero divisor of S/I for all i .*

Given a subset $I \subset S$, its *variety*, denoted by $V(I)$, is the set of all $a \in \mathbb{A}_K^a$ such that $f(a) = 0$ for all $f \in I$, where \mathbb{A}_K^a is the affine space over K .

Lemma 2.8. [20] *Let $I \subset S$ be a graded binomial ideal such that $V(I, t_i) = \{0\}$ for all i . Then the following hold.*

- (a) *If I is Cohen-Macaulay, then I is a lattice ideal.*
- (b) *If \mathfrak{p} is a prime ideal containing (I, t_k) for some $1 \leq k \leq s$, then $\mathfrak{p} = (t_1, \dots, t_s)$.*

3. HILBERT SERIES AND ALGEBRAIC INVARIANTS

We continue to use the notation and definitions used in Section 1. In this section we establish a map $I \mapsto \tilde{I}$ between the graded ideals of S and \tilde{S} . We relate the minimal graded resolutions, the Hilbert series, and the regularities of I and \tilde{I} .

In what follows $S = K[t_1, \dots, t_s] = \bigoplus_{d=0}^{\infty} S_d$ and $\tilde{S} = K[t_1, \dots, t_s] = \bigoplus_{d=0}^{\infty} \tilde{S}_d$ are polynomial rings graded by the grading induced by setting $\deg(t_i) = d_i$ for all i and the standard grading induced by setting $\deg(t_i) = 1$ for all i , respectively, where d_1, \dots, d_s are positive integers. Let f_1, \dots, f_s be a set of algebraically independent homogeneous polynomials of \tilde{S} of degrees d_1, \dots, d_s and let

$$\phi: S \rightarrow K[f_1, \dots, f_s]$$

be the isomorphism of K -algebras given by $\phi(g) = g(f_1, \dots, f_s)$, where $K[f_1, \dots, f_s]$ is the K -subalgebra of \tilde{S} generated by f_1, \dots, f_s . For convenience we denote $\phi(g)$ by \tilde{g} .

Definition 3.1. Given a graded ideal $I \subset S$ generated by g_1, \dots, g_m , we associate to I the graded ideal $\tilde{I} \subset \tilde{S}$ generated by $\tilde{g}_1, \dots, \tilde{g}_m$. We call \tilde{I} the *homogenization* of I with respect to f_1, \dots, f_s .

The ideal \tilde{I} is independent of the generating set g_1, \dots, g_m and \tilde{I} is a graded ideal with respect to the standard grading of $K[t_1, \dots, t_s]$. In general the map $I \mapsto \tilde{I}$ does not preserve the height of I , as the following example shows.

Example 3.2. The polynomials $f_1 = t_1$, $f_2 = t_2$, $f_3 = t_1 t_2 - t_1 t_3$ are algebraically independent over \mathbb{Q} . The homogenization of $I = (t_1, t_2, t_3)$ with respect to f_1, f_2, f_3 is $\tilde{I} = (t_1, t_2, t_1 t_2 - t_1 t_3)$ which is equal to (t_1, t_2) . Thus, $\text{ht}(\tilde{I}) < \text{ht}(I)$.

Lemma 3.3. (a) *If $K[f_1, \dots, f_s] \subset \tilde{S}$ is an integral extension and I is a graded ideal of S , then $\dim(S/I) \geq \dim(\tilde{S}/\tilde{I})$.* (b) *If I is a monomial ideal, then $\dim(S/I) \leq \dim(\tilde{S}/\tilde{I})$.*

Proof. (a): By the Noether normalization lemma [40, Corollary 2.1.8], there is an integral extension

$$(*) \quad K[h_1, \dots, h_d] \xrightarrow{\mathcal{J}} S/I, \quad h_i \mapsto h_i + I$$

where h_1, \dots, h_d are in S and $d = \dim(S/I)$. In particular, h_1, \dots, h_d are algebraically independent over K . The map ϕ induces an isomorphism of K -algebras between $K[h_1, \dots, h_d]$ and $K[\tilde{h}_1, \dots, \tilde{h}_d]$. Thus, $\tilde{h}_1, \dots, \tilde{h}_d$ are also algebraically independent elements of $K[f_1, \dots, f_s]$. Hence, there are homomorphisms of K -algebras

$$K[\tilde{h}_1, \dots, \tilde{h}_d] \xrightarrow{\tilde{\mathcal{J}}} (K[f_1, \dots, f_s] + \tilde{I})/\tilde{I} \hookrightarrow \tilde{S}/\tilde{I}, \quad \tilde{h}_i \mapsto \tilde{h}_i + \tilde{I}, \quad g + \tilde{I} \mapsto g + \tilde{I}.$$

Notice that $1 \notin \tilde{I}$ because \tilde{I} is graded. From Eq. (*), $t_i + I$ is integral over $K[h_1, \dots, h_d]$. Hence, using ϕ , we get that $f_i + \tilde{I}$ is integral over $K[\tilde{h}_1, \dots, \tilde{h}_d]$. Thus, since $K[f_1, \dots, f_s] \subset \tilde{S}$ is integral, we get integral extensions

$$K[\tilde{h}_1, \dots, \tilde{h}_d]/\ker(\tilde{\mathcal{J}}) \hookrightarrow (K[f_1, \dots, f_s] + \tilde{I})/\tilde{I} \hookrightarrow \tilde{S}/\tilde{I}.$$

Altogether, using the fact that the Krull dimension is preserved under integral extensions of rings [18, Theorem 2.2.5], we get

$$\dim(S/I) = d = \text{ht}(\ker(\tilde{\mathcal{J}})) + \dim(\tilde{S}/\tilde{I}) \geq \dim(\tilde{S}/\tilde{I}).$$

(b): Pick a minimal prime \mathfrak{p} of I of height $g = \text{ht}(I)$. We may assume that $\mathfrak{p} = (t_1, \dots, t_g)$. It is not hard to see that $\tilde{I} \subset (f_1, \dots, f_g)$. Thus, by Krull principal ideal theorem, $\text{ht}(\tilde{I}) \leq g$. \square

Let I be a graded ideal of S and let $F_I(t)$ be the Hilbert series of S/I . The a -invariant of S/I , denoted by $a(S/I)$, is the degree of $F_I(t)$ as a rational function. Let

$$\mathbf{F} : \quad 0 \rightarrow \bigoplus_{j=1}^{b_g} S(-a_{g,j}) \rightarrow \cdots \rightarrow \bigoplus_{j=1}^{b_1} S(-a_{1,j}) \rightarrow S \rightarrow S/I \rightarrow 0$$

be the minimal graded free resolution of S/I as an S -module. The free modules in the resolution of S/I can be written as

$$F_i = \bigoplus_{j=1}^{b_i} S(-a_{i,j}) = \bigoplus_j S(-j)^{b_{i,j}}.$$

The numbers $b_{i,j} = \text{Tor}_i(K, S/I)_j$ are called the *graded Betti numbers* of S/I and $b_i = \sum_j b_{i,j}$ is called the i^{th} *Betti number* of S/I . The *Castelnuovo-Mumford regularity* or simply the *regularity* of S/I is defined as

$$\text{reg}(S/I) = \max\{j - i \mid b_{i,j} \neq 0\}.$$

Theorem 3.4. ([8, p. 521], [40, Proposition 4.2.3]) *If $I \subset S$ is a graded Cohen-Macaulay ideal, then*

$$a(S/I) = \text{reg}(S/I) - \text{depth}(S/I) - \sum_{i=1}^s (d_i - 1) = \text{reg}(S/I) + \text{ht}(I) - \sum_{i=1}^s d_i.$$

Lemma 3.5. *Let I be a graded ideal of S . If $K[f_1, \dots, f_s] \subset \tilde{S}$ is an integral extension and*

$$\mathbf{F} : \quad 0 \rightarrow \bigoplus_{j=1}^{b_g} S(-a_{g,j}) \rightarrow \cdots \rightarrow \bigoplus_{j=1}^{b_1} S(-a_{1,j}) \rightarrow S \rightarrow S/I \rightarrow 0$$

is the minimal graded free resolution of S/I , then

$$\tilde{\mathbf{F}} : \quad 0 \rightarrow \bigoplus_{j=1}^{b_g} \tilde{S}(-a_{g,j}) \rightarrow \cdots \rightarrow \bigoplus_{j=1}^{b_1} \tilde{S}(-a_{1,j}) \rightarrow \tilde{S} \rightarrow \tilde{S}/\tilde{I} \rightarrow 0$$

is the minimal graded free resolution of \tilde{S}/\tilde{I} .

Proof. For $1 \leq i \leq g$ consider the map $\varphi_i: \bigoplus_{j=1}^{b_i} S(-a_{i,j}) \rightarrow \bigoplus_{j=1}^{b_{i-1}} S(-a_{i-1,j})$ of the resolution of S/I . The entries of the matrix φ_i are homogeneous polynomials of S , and accordingly the ideal $L_i = I_{r_i}(\varphi_i)$ generated by the r_i -minors of φ_i is graded with respect to the grading of S , where r_i is the rank of φ_i (that is, the largest size of a nonvanishing minor).

Let $\tilde{\varphi}_i$ be the matrix obtained from φ_i by replacing each entry of φ_i by its image under the map $S \mapsto \tilde{S}$, $g \mapsto \tilde{g} = g(f_1, \dots, f_s)$. Since this map is an injective homomorphism of K -algebras, the rank of $\tilde{\varphi}_i$ is equal to r_i , $\tilde{\varphi}_{i-1}\tilde{\varphi}_i = 0$, and $I_{r_i}(\tilde{\varphi}_i) = \tilde{L}_i$ for all i . Therefore, one has a graded complex

$$0 \rightarrow \bigoplus_{j=1}^{b_g} \tilde{S}(-a_{g,j}) \xrightarrow{\tilde{\varphi}_g} \cdots \rightarrow \bigoplus_{j=1}^{b_i} \tilde{S}(-a_{i,j}) \xrightarrow{\tilde{\varphi}_i} \cdots \rightarrow \bigoplus_{j=1}^{b_1} \tilde{S}(-a_{1,j}) \xrightarrow{\tilde{\varphi}_1} \tilde{S} \rightarrow \tilde{S}/\tilde{I} \rightarrow 0.$$

To show that this complex is exact, by the Buchsbaum-Eisenbud acyclicity criterion [5, Theorem 1.4.13, p. 24], it suffices to verify that $\text{ht}(I_{r_i}(\tilde{\varphi}_i))$ is at least i for $i \geq 1$. As \mathbf{F} is an exact complex, using Lemma 3.3, we get

$$\text{ht}(I_{r_i}(\tilde{\varphi}_i)) = \text{ht}(\widetilde{I_{r_i}(\varphi_i)}) \geq \text{ht}(I_{r_i}(\varphi_i)) \geq i,$$

as required. \square

Theorem 3.6. *Let I be a graded ideal of S and let $F_I(t)$ be the Hilbert series of S/I . If $K[f_1, \dots, f_s] \subset \tilde{S}$ is an integral extension, then*

- (a) $F_{\tilde{I}}(t) = \lambda_1(t) \cdots \lambda_s(t) F_I(t)$, where $\lambda_i(t) = 1 + t + \cdots + t^{d_i-1}$.
- (b) $\text{reg}(S/I) = \text{reg}(\tilde{S}/\tilde{I})$.

Proof. (a): The Hilbert series of \tilde{S} and S , denoted by $F(\tilde{S}, t)$ and $F(S, t)$ respectively, are related by $F(\tilde{S}, t) = \lambda_1(t) \cdots \lambda_s(t) F(S, t)$. Hence, using Lemma 3.5 and the additivity of Hilbert series, we get the required equality.

(b): This follows at once from Lemma 3.5 \square

Lemma 3.7. *Let $I \subset S$ be a graded ideal and let $f(t)/\prod_{i=1}^s(1-t^{d_i})$ be the Hilbert series of S/I , where $f(t) \in \mathbb{Z}[t]$. If $J \subset S$ is the ideal generated by all $g(t_1^r, \dots, t_s^r)$ with $g \in I$, then*

- (a) $f(t^r)/\prod_{i=1}^s(1-t^{d_i r})$ is the Hilbert series of S/J .
- (b) If $d_i = 1$ for all i and I is a Cohen-Macaulay ideal such that $\text{ht}(I) = \text{ht}(J)$, then

$$\text{reg}(S/J) = \text{ht}(I)(r-1) + r \cdot \text{reg}(S/I).$$

Proof. (a): Clearly J is also a graded ideal of S . If

$$0 \rightarrow \bigoplus_{j=1}^{b_g} S(-a_{g,j}) \rightarrow \cdots \rightarrow \bigoplus_{j=1}^{b_1} S(-a_{1,j}) \rightarrow S \rightarrow S/I \rightarrow 0$$

is the minimal graded free resolution of S/I , then it is seen that

$$0 \rightarrow \bigoplus_{j=1}^{b_g} S(-ra_{g,j}) \rightarrow \cdots \rightarrow \bigoplus_{j=1}^{b_1} S(-ra_{1,j}) \rightarrow S \rightarrow S/J \rightarrow 0$$

is the minimal graded free resolution of S/J . This can be shown using the method of proof of Lemma 3.5. Hence, by the additivity of the Hilbert series, the result follows.

(b): This follows from part (a) and Theorem 3.4. \square

4. COMPLETE INTERSECTIONS AND ALGEBRAIC INVARIANTS

We continue to use the notation and definitions used in Section 1. In this section, we establish a map $I \mapsto \tilde{I}$ between the family of graded lattice ideals of a positively graded polynomial ring and the graded lattice ideals of a polynomial ring with the standard grading. We relate the lattices and generators of I and \tilde{I} , then we show that I is a complete intersection if and only if \tilde{I} is a complete intersection.

Let K be a field and let $S = K[t_1, \dots, t_s] = \bigoplus_{d=0}^{\infty} S_d$ and $\tilde{S} = K[t_1, \dots, t_s] = \bigoplus_{d=0}^{\infty} \tilde{S}_d$ be polynomial rings graded by the grading induced by setting $\deg(t_i) = d_i$ for all i and the standard grading induced by setting $\deg(t_i) = 1$ for all i , respectively, where d_1, \dots, d_s are positive integers.

Throughout this section, $\mathcal{L} \subset \mathbb{Z}^s$ will denote a *homogeneous lattice*, with respect to the positive vector $\mathbf{d} = (d_1, \dots, d_s)$, i.e., $\langle \mathbf{d}, a \rangle = 0$ for $a \in \mathcal{L}$, and $I(\mathcal{L}) \subset S$ will denote the graded lattice ideal of \mathcal{L} .

Let D be the non-singular matrix $D = \text{diag}(d_1, \dots, d_s)$. Consider the homomorphism of \mathbb{Z} -modules:

$$D: \mathbb{Z}^s \rightarrow \mathbb{Z}^s, \quad e_i \mapsto d_i e_i.$$

The lattice $\tilde{\mathcal{L}} = D(\mathcal{L})$ is called the *homogenization* of \mathcal{L} with respect to d_1, \dots, d_s . Notice that $\tilde{I}(\tilde{\mathcal{L}}) = I(\tilde{\mathcal{L}})$, i.e., $I(\tilde{\mathcal{L}})$ is the homogenization of $I(\mathcal{L})$ with respect to $t_1^{d_1}, \dots, t_s^{d_s}$. The lattices that define the lattice ideals $I(\mathcal{L})$ and $I(\tilde{\mathcal{L}})$ are homogeneous with respect to the vectors $\mathbf{d} = (d_1, \dots, d_s)$ and $\mathbf{1} = (1, \dots, 1)$, respectively.

Lemma 4.1. *The map $t^a - t^b \mapsto t^{D(a)} - t^{D(b)}$ induces a bijection between the binomials $t^a - t^b$ of $I(\mathcal{L})$ whose terms t^a, t^b have disjoint support and the binomials $t^{a'} - t^{b'}$ of $I(\tilde{\mathcal{L}})$ whose terms $t^{a'}, t^{b'}$ have disjoint support.*

Proof. If $f = t^a - t^b$ is a binomial of $I(\mathcal{L})$ whose terms have disjoint support, then $a - b \in \mathcal{L}$. Consequently, the terms of $\tilde{f} = t^{D(a)} - t^{D(b)}$ have disjoint support because

$$\text{supp}(t^a) = \text{supp}(t^{D(a)}) \quad \text{and} \quad \text{supp}(t^b) = \text{supp}(t^{D(b)}),$$

and \tilde{f} is in $I(\tilde{\mathcal{L}})$ because $D(a) - D(b) \in \tilde{\mathcal{L}}$. Thus, the map is well defined. The map is clearly injective. To show that the map is onto, take a binomial $f' = t^{a'} - t^{b'}$ in $I(\tilde{\mathcal{L}})$ such that $t^{a'}$ and $t^{b'}$ have disjoint support. Hence, $a' - b' \in \tilde{\mathcal{L}}$ and there is $c = c^+ - c^- \in \mathcal{L}$ such that $a' - b' = D(c^+) - D(c^-)$. As a' and b' have disjoint support, we get $a' = D(c^+)$ and $b' = D(c^-)$. Thus, the binomial $f = t^{c^+} - t^{c^-}$ is in $I(\mathcal{L})$ and maps to $t^{a'} - t^{b'}$. \square

Theorem 4.2. *Let $\mathcal{B} = \{t^{a_i} - t^{b_i}\}_{i=1}^m$ be a set of binomials. If $\tilde{\mathcal{B}} = \{t^{D(a_i)} - t^{D(b_i)}\}_{i=1}^m$, then $I(\mathcal{L}) = (\mathcal{B})$ if and only if $I(\tilde{\mathcal{L}}) = (\tilde{\mathcal{B}})$.*

Proof. We set $g_i = t^{a_i} - t^{b_i}$ and $h_i = t^{D(a_i)} - t^{D(b_i)}$ for $i = 1, \dots, m$. Notice that h_i is equal to $\tilde{g}_i = g_i(t_1^{d_1}, \dots, t_s^{d_s})$, the evaluation of g_i at $(t_1^{d_1}, \dots, t_s^{d_s})$.

\Rightarrow) By Lemma 4.1, one has the inclusion $(\tilde{\mathcal{B}}) \subset I(\tilde{\mathcal{L}})$. To show the reverse inclusion take a binomial $0 \neq f \in I(\tilde{\mathcal{L}})$. We may assume that $f = t^{a^+} - t^{a^-}$. Then, by Lemma 4.1, there is $g = t^{c^+} - t^{c^-}$ in $I(\mathcal{L})$ such that $f = t^{D(c^+)} - t^{D(c^-)}$. By hypothesis we can write $g = \sum_{i=1}^m f_i g_i$

for some f_1, \dots, f_m in S . Then, evaluating both sides of this equality at $(t_1^{d_1}, \dots, t_s^{d_s})$, we get

$$f = t^{D(c^+)} - t^{D(c^-)} = g(t_1^{d_1}, \dots, t_s^{d_s}) = \sum_{i=1}^m f_i(t_1^{d_1}, \dots, t_s^{d_s}) g_i(t_1^{d_1}, \dots, t_s^{d_s}) = \sum_{i=1}^m \tilde{f}_i h_i,$$

where $\tilde{f}_i = f_i(t_1^{d_1}, \dots, t_s^{d_s})$ for all i . Then, $f \in (\tilde{\mathcal{B}})$.

\Leftarrow) We may assume that h_1, \dots, h_r is a minimal set of generators of $I(\mathcal{L})$ for some $r \leq m$. Consider the K -vector spaces $\tilde{V} = I(\tilde{\mathcal{L}})/\mathfrak{m}I(\tilde{\mathcal{L}})$ and $V = I(\mathcal{L})/\mathfrak{m}I(\mathcal{L})$, where $\mathfrak{m} = (t_1, \dots, t_s)$. Since the images, in \tilde{V} , of h_1, \dots, h_r form a K -basis for \tilde{V} , it follows that the images, in V , of g_1, \dots, g_r are linearly independent. On the other hand, by Lemma 3.5, the minimum number of generators of $I(\mathcal{L})$ is also r and $r = \dim_K(V)$. Thus, the images, in V , of g_1, \dots, g_r form a K -basis for V . Consequently \mathcal{B} generates $I(\mathcal{L})$ by Nakayama's lemma (see [40, Proposition 2.5.1 and Corollary 2.5.2]). \square

Definition 4.3. An ideal $I \subset S$ is called a *complete intersection* if there exists g_1, \dots, g_m such that $I = (g_1, \dots, g_m)$, where m is the height of I .

Recall that a graded binomial ideal $I \subset S$ is a complete intersection if and only if I is generated by a homogeneous regular sequence, consisting of binomials, with $\text{ht}(I)$ elements (see for instance [40, Proposition 1.3.17, Lemma 1.3.18]).

Corollary 4.4. $I(\tilde{\mathcal{L}})$ is a complete intersection (resp. Cohen-Macaulay, Gorenstein) if and only if $I(\mathcal{L})$ is a complete intersection (resp. Cohen-Macaulay, Gorenstein).

Proof. The rank of \mathcal{L} is equal to the height of $I(\mathcal{L})$ [23, Proposition 7.5]. Since D is non-singular, \mathcal{L} and $\tilde{\mathcal{L}} = D(\mathcal{L})$ have the same rank. Thus, $I(\mathcal{L})$ and $I(\tilde{\mathcal{L}})$ have the same height. Therefore, the result follows from Theorem 4.2 and Lemma 3.5. \square

5. LATTICE IDEALS OF DIMENSION 1

We continue to use the notation and definitions used in Section 4. In this section, we study the map $I \mapsto \tilde{I}$, when I is a graded lattice ideals of dimension 1. In this case, we relate the degrees of I and \tilde{I} and show a formula for the regularity and the degree of \tilde{I} when I is the toric ideal of a monomial curve. We show that the regularity of the saturation of certain one dimensional graded ideals is additive in a certain sense.

We begin by identifying some elements in the torsion subgroup of $\mathbb{Z}^s/D(\mathcal{L})$. The proof of the following lemma is straightforward.

Lemma 5.1. *If \mathcal{L} is a homogeneous lattice, with respect to d_1, \dots, d_s , of rank $s - 1$, then for each i, j there is a positive integer $\eta_{i,j}$ such that $\eta_{i,j}(d_j e_i - d_i e_j)$ is in \mathcal{L} and $\eta_{i,j}(d_i d_j e_i - d_i d_j e_j)$ is in $D(\mathcal{L})$. In particular, $e_i - e_j$ is in $T(\mathbb{Z}^s/D(\mathcal{L}))$ for any i, j .*

Lemma 5.2. *Let $\mathcal{L} \subset \mathbb{Z}^s$ be a homogeneous lattice of rank $s - 1$. If $\gcd(d_1, \dots, d_s) = 1$, then*

$$|T(\mathbb{Z}^s/D(\mathcal{L}))| = |\mathbb{Z}^s/D(\mathbb{Z}^s)| |T(\mathbb{Z}^s/\mathcal{L})| = \det(D) |T(\mathbb{Z}^s/\mathcal{L})|.$$

Proof. The second equality follows from the equality $|\mathbb{Z}^s/D(\mathbb{Z}^s)| = \det(D)$. Next, we show the first equality. Consider the following sequence of \mathbb{Z} -modules:

$$0 \rightarrow T(\mathbb{Z}^s/\mathcal{L}) \xrightarrow{\sigma} T(\mathbb{Z}^s/D(\mathcal{L})) \xrightarrow{\rho} \mathbb{Z}^s/D(\mathbb{Z}^s) \rightarrow 0,$$

where $a + \mathcal{L} \xrightarrow{\sigma} D(a) + D(\mathcal{L})$ and $a + D(\mathcal{L}) \xrightarrow{\rho} a + D(\mathbb{Z}^s)$. It suffices to show that this sequence is exact. It is not hard to see that σ is injective and that $\text{im}(\sigma) = \ker(\rho)$. Next, we show that ρ

is onto. We need only show that $e_k + D(\mathbb{Z}^s)$ is in the image of ρ for $k = 1, \dots, s$. For simplicity of notation we assume that $k = 1$. There are integers $\lambda_1, \dots, \lambda_s$ such that $1 = \sum_i \lambda_i d_i$. Then, using that $D(\mathbb{Z}^s)$ is generated by $d_1 e_1, \dots, d_s e_s$, we obtain

$$\begin{aligned} e_1 + D(\mathbb{Z}^s) &= \lambda_1 d_1 e_1 + \lambda_2 d_2 e_1 + \dots + \lambda_s d_s e_1 + D(\mathbb{Z}^s) \\ &= \lambda_1 d_1 e_1 + \lambda_2 d_2 (e_1 - e_2) + \dots + \lambda_s d_s (e_1 - e_s) + D(\mathbb{Z}^s) \\ &= \lambda_2 d_2 (e_1 - e_2) + \dots + \lambda_s d_s (e_1 - e_s) + D(\mathbb{Z}^s). \end{aligned}$$

Hence, by Lemma 5.1, the element $\lambda_2 d_2 (e_1 - e_2) + \dots + \lambda_s d_s (e_1 - e_s) + D(\mathcal{L})$ is a torsion element of $\mathbb{Z}^s / D(\mathcal{L})$ that maps to $e_1 + D(\mathbb{Z}^s)$ under the map ρ . \square

Theorem 5.3. *If $I = I(\mathcal{L})$ is a graded lattice ideal of dimension 1, then*

$$\deg(\tilde{S}/\tilde{I}) = \frac{d_1 \cdots d_s}{\max\{d_1, \dots, d_s\}} \deg(S/I).$$

Proof. We set $r = \gcd(d_1, \dots, d_s)$ and $D' = \text{diag}(d_1/r, \dots, d_s/r)$. As I and \tilde{I} are a graded lattice ideals of dimension 1, according to some results of [26] and [21], one has

$$\deg(S/I) = \frac{\max\{d_1, \dots, d_s\}}{r} |T(\mathbb{Z}^s/\mathcal{L})| \quad \text{and} \quad \deg(\tilde{S}/\tilde{I}) = |T(\mathbb{Z}^s/\tilde{\mathcal{L}})|,$$

where $\tilde{\mathcal{L}} = D(\mathcal{L})$. Hence, by Lemma 5.2, we get

$$\begin{aligned} \deg(\tilde{S}/\tilde{I}) &= |T(\mathbb{Z}^s/D(\mathcal{L}))| = r^{s-1} |T(\mathbb{Z}^s/D'(\mathcal{L}))| \\ &= r^{s-1} |\mathbb{Z}^s/D'(\mathbb{Z}^s)| |T(\mathbb{Z}^s/\mathcal{L})| = r^{s-1} \det(D') |T(\mathbb{Z}^s/\mathcal{L})| \\ &= \frac{d_1 \cdots d_s}{r} |T(\mathbb{Z}^s/\mathcal{L})| = \frac{d_1 \cdots d_s}{\max\{d_1, \dots, d_s\}} \deg(S/I). \end{aligned}$$

The second equality can be shown using that the order of $T(\mathbb{Z}^s/D(\mathcal{L}))$ is the gcd of all $s - 1$ minors of a presentation matrix of $\mathbb{Z}^s/D(\mathcal{L})$. \square

Definition 5.4. If \mathcal{S} is a numerical semigroup of \mathbb{N} , the *Frobenius number* of \mathcal{S} , denoted by $g(\mathcal{S})$, is the largest integer not in \mathcal{S} .

The next result gives an explicit formula for the regularity in terms of Frobenius numbers, that can be used to compute the regularity using some available algorithms (see the monograph [27]). Using *Macaulay2* [15], we can use this formula to compute the Frobenius number of the corresponding semigroup.

Theorem 5.5. *If I is the toric ideal of $K[y_1^{d_1}, \dots, y_1^{d_s}] \subset K[y_1]$ and $r = \gcd(d_1, \dots, d_s)$, then*

- (a) $\text{reg}(\tilde{S}/\tilde{I}) = r \cdot g(\mathcal{S}) + 1 + \sum_{i=1}^s (d_i - 1)$, where $\mathcal{S} = \mathbb{N}(d_1/r) + \dots + \mathbb{N}(d_s/r)$.
- (b) $\text{deg}(\tilde{S}/\tilde{I}) = d_1 \cdots d_s / r$.

Proof. (a): We set $d'_i = d_i/r$ for $i = 1, \dots, s$. Let L be the toric ideal of $K[y_1^{d'_1}, \dots, y_1^{d'_s}]$ and let \tilde{L} be the homogenization of L with respect to $t_1^{d'_1}, \dots, t_s^{d'_s}$. It is not hard to see that the toric ideals I and L are equal. Let $F_L(t)$ be the Hilbert series of S/L , where S has the grading induced by setting $\text{deg}(t_i) = d'_i$ for all i . As $\gcd(d'_1, \dots, d'_s) = 1$, \mathcal{S} is a numerical semigroup, and by [3, Remark 4.5, p. 200] we can write $F_L(t) = f(t)/(1-t)$, where $f(t)$ is a polynomial in $\mathbb{Z}[t]$ of degree $g(\mathcal{S}) + 1$. Then, by Theorem 3.4, we get

$$\text{reg}(S/L) = \text{deg}(F_L(t)) - \text{ht}(L) + \sum_{i=1}^s d_i/r = g(\mathcal{S}) - (s-1) + \sum_{i=1}^s d_i/r.$$

Notice that \tilde{I} and \tilde{L} are Cohen-Macaulay lattice ideals of height $s - 1$. Since \tilde{I} is the homogenization of \tilde{L} with respect to t_1^r, \dots, t_s^r , by Lemma 3.7(b) and Theorem 3.6, we get

$$\begin{aligned} \operatorname{reg}(\tilde{S}/\tilde{I}) &= (s-1)(r-1) + r \cdot \operatorname{reg}(\tilde{S}/\tilde{L}) = (s-1)(r-1) + r \cdot \operatorname{reg}(S/L) \\ &= (s-1)(r-1) + r(g(\mathcal{S}) - (s-1) + \sum_{i=1}^s d_i/r) \\ &= r \cdot g(\mathcal{S}) + 1 + \sum_{i=1}^s (d_i - 1). \end{aligned}$$

(b): By a result of [26], $\operatorname{deg}(S/I) = \max\{d_1, \dots, d_s\}/r$. Hence, the formula follows from Theorem 5.3. \square

Definition 5.6. The *graded reverse lexicographical order* (GRevLex for short) is defined as $t^b \succ t^a$ if and only if $\operatorname{deg}(t^b) > \operatorname{deg}(t^a)$ or $\operatorname{deg}(t^b) = \operatorname{deg}(t^a)$ and the last nonzero entry of $b - a$ is negative. The *reverse lexicographical order* (RevLex for short) is defined as $t^b \succ t^a$ if and only if the last nonzero entry of $b - a$ is negative.

Corollary 5.7. *Let \succ be the RevLex order. If I is a graded lattice ideal and $\dim(S/I) = 1$, then*

$$\operatorname{reg}(S/I) = \operatorname{reg}(\tilde{S}/\tilde{I}) = \operatorname{reg}(\tilde{S}/\operatorname{in}(\tilde{I})) = \operatorname{reg}(S/\operatorname{in}(I)),$$

where $\operatorname{in}(I)$, $\operatorname{in}(\tilde{I})$ are the initial ideals of I , \tilde{I} , with respect to \succ , respectively.

Proof. The quotient rings \tilde{S}/\tilde{I} and $\tilde{S}/\operatorname{in}(\tilde{I})$ are Cohen-Macaulay standard algebras of dimension 1 because t_s is a regular element of both rings. Hence, these two rings have the same Hilbert function and the same index of regularity. Therefore $\operatorname{reg}(\tilde{S}/\tilde{I})$ is equal to $\operatorname{reg}(\tilde{S}/\operatorname{in}(\tilde{I}))$. As $\operatorname{in}(\tilde{I}) = \widetilde{\operatorname{in}(I)}$, by Theorem 3.6, we get

$$\operatorname{reg}(S/I) = \operatorname{reg}(\tilde{S}/\tilde{I}) = \operatorname{reg}(\tilde{S}/\operatorname{in}(\tilde{I})) = \operatorname{reg}(\tilde{S}/\widetilde{\operatorname{in}(I)}) = \operatorname{reg}(S/\operatorname{in}(I)),$$

as required. \square

Theorem 5.8. *Let V_1, \dots, V_c be a partition of $V = \{t_1, \dots, t_s\}$ and let ℓ be a positive integer. Suppose that $K[V_k]$ and $K[V]$ are polynomial rings with the standard grading for $k = 1, \dots, c$. If I_k is a graded binomial ideal of $K[V_k]$ such that $t_i^\ell - t_j^\ell \in I_k$ for $t_i, t_j \in V_k$ and \mathcal{I} is the ideal of $K[V]$ generated by all binomials $t_i^\ell - t_j^\ell$ with $1 \leq i, j \leq s$, then*

$$(i) \quad (I_1 + \dots + I_c + \mathcal{I} : h^\infty) = (I_1 : h_1^\infty) + \dots + (I_c : h_c^\infty) + \mathcal{I}, \text{ and}$$

$$(ii) \quad \operatorname{reg} K[V]/(I_1 + \dots + I_c + \mathcal{I} : h^\infty) = \sum_{k=1}^c \operatorname{reg} K[V_k]/(I_k : h_k^\infty) + (c-1)(\ell-1),$$

where $h = t_1 \cdots t_s$ and $h_k = \prod_{t_i \in V_k} t_i$ for $k = 1, \dots, c$.

Proof. The proofs of (i) and (ii) are by induction on c . If $c = 1$ the asserted equalities hold because in this case $\mathcal{I} \subset I_1$. We set

$$J = (I_1 + \dots + I_c + \mathcal{I} : h^\infty), \quad J_k = (I_k : h_k^\infty),$$

$$L = I_1 + \dots + I_{c-1} + \mathcal{I}', \text{ where } \mathcal{I}' = (\{t_i^\ell - t_j^\ell \mid t_i, t_j \in V'\}) \text{ and } V' = V_1 \cup \dots \cup V_{c-1},$$

and $g = \prod_{t_i \in V'} t_i$. First, we show the case $c = 2$. (i): We set $J' = J_1 + J_2 + \mathcal{I}$. Clearly one has the inclusion $J' \subset J$. To show the reverse inclusion it suffices to show that J' is a lattice ideal. Since this ideal is graded of dimension 1 and $V(J', t_i) = 0$ for $i = 1, \dots, s$, we need only show that J' is Cohen-Macaulay (see Lemma 2.8). As J_1, J_2 are lattice ideals of dimension 1, they are Cohen-Macaulay. Hence, $J_1 + J_2$ is Cohen-Macaulay of dimension 2 (see [38, Lemma 4.1]). Pick $t_i \in V_1$ and $t_j \in V_2$. The binomial $f = t_i^\ell - t_j^\ell$ is regular modulo $J_1 + J_2$. Indeed, if f is

in some associated prime \mathfrak{p} of $J_1 + J_2$, then $\mathcal{I} \subset \mathfrak{p}$ and consequently the height of \mathfrak{p} is at least $s - 1$, a contradiction because all associated primes of $J_1 + J_2$ have height $s - 2$. Hence, since $J_1 + J_2 + \mathcal{I}$ is equal to $J_1 + J_2 + (f)$, the ideal J' is Cohen-Macaulay. (ii): There is an exact sequence

$$0 \longrightarrow (K[V]/(J_1 + J_2))[-\ell] \xrightarrow{f} K[V]/(J_1 + J_2) \longrightarrow K[V]/J \longrightarrow 0.$$

Hence, by the additivity of Hilbert series, we get

$$H_J(t) = H_{(J_1+J_2)}(t)(1-t^\ell) = H_{J_1}(t)H_{J_2}(t)(1-t^\ell),$$

where $H_J(t)$ is the Hilbert series of $K[V]/J$ (cf. arguments below). The ideals J, J_1, J_2 are graded lattice ideals of dimension 1, hence they are Cohen-Macaulay. Therefore (cf. arguments below) we obtain

$$\text{reg } K[V]/J = \text{reg } K[V_1]/J_1 + \text{reg } K[V_2]/J_2 + (\ell - 1),$$

which gives the formula for the regularity.

Next, we show the general case. (i): Applying the case $c = 2$, by induction, we get

$$\begin{aligned} J &= (L + I_c + \mathcal{I} : h^\infty) = (L : g^\infty) + (I_c : h_c^\infty) + \mathcal{I} \\ &= [(I_1 : h_1^\infty) + \cdots + (I_{c-1} : h_{c-1}^\infty) + \mathcal{I}'] + (I_c : h_c^\infty) + \mathcal{I}. \end{aligned}$$

Thus, $J = (I_1 : h_1^\infty) + \cdots + (I_c : h_c^\infty) + \mathcal{I}$, as required. (ii): We set $Q_1 = (L : g^\infty)$ and $Q = Q_1 + J_c$. From the isomorphism

$$K[V]/Q \simeq (K[V']/Q_1) \otimes_K (K[V_c]/J_c),$$

we get that $H_Q(t) = H_1(t)H_2(t)$, where $H_Q(t)$ is the Hilbert series of $K[V]/Q$ and $H_1(t), H_2(t)$ are the Hilbert series of $K[V']/Q_1$ and $K[V_c]/J_c$, respectively. Since Q_1 and J_c are graded lattice ideals of dimension 1, they are Cohen-Macaulay and their Hilbert series can be written as $H_i(t) = f_i(t)/(1-t)$, with $f_i(t) \in \mathbb{Z}[t]$ for $i = 1, 2$ and such that $\deg(f_1) = \text{reg } K[V']/Q_1$ and $\deg(f_2) = \text{reg } K[V_c]/J_c$. Fix $t_i \in V'$ and $t_j \in V_c$. If $f = t_i^\ell - t_j^\ell$, then by the case $c = 2$ there is an exact sequence

$$0 \longrightarrow (K[V]/Q)[- \ell] \xrightarrow{f} K[V]/Q \longrightarrow K[V]/J \longrightarrow 0.$$

Hence, by the additivity of Hilbert series, we get

$$H_J(t) = H_Q(t)(1-t^\ell) = H_1(t)H_2(t)(1-t^\ell) = \frac{f_1(t)f_2(t)(1+t+\cdots+t^{\ell-1})}{(1-t)}.$$

Therefore, as J is a graded lattice ideal of dimension 1, by induction we get

$$\begin{aligned} \text{reg } K[V]/J &= \text{reg } K[V']/Q_1 + \text{reg } K[V_c]/J_c + (\ell - 1) \\ &= \left[\sum_{k=1}^{c-1} \text{reg } K[V_k]/(I_k : g_k^\infty) + (c-2)(\ell-1) \right] + \text{reg } K[V_c]/J_c + (\ell - 1), \end{aligned}$$

which gives the formula for the regularity. \square

6. VANISHING IDEALS

In this section we study graded vanishing ideals over arbitrary fields and give some applications of the results of Section 5. For finite fields, we give formulae for the degree and regularity of graded vanishing ideals over degenerate tori. Given a sequence of positive integers, we construct vanishing ideals, over finite fields, with prescribed regularity and degree of a certain type. We characterize when a graded lattice ideal of dimension 1 is a vanishing ideal in terms of the degree. We show that the vanishing ideal of X is a lattice ideal of dimension 1 if and only if X is a finite

subgroup of a projective torus. For finite fields, it is shown that X is a subgroup of a projective torus if and only if X is parameterized by monomials.

Let $K \neq \mathbb{F}_2$ be a field and let \mathbb{P}^{s-1} be a projective space of dimension $s - 1$ over the field K . If X is a subset of \mathbb{P}^{s-1} , the *vanishing ideal* of X , denoted by $I(X)$, is the ideal of \tilde{S} generated by all the homogeneous polynomials that vanish on X .

Definition 6.1. Given a sequence $v = (v_1, \dots, v_s)$ of positive integers, the set

$$\{[(x_1^{v_1}, \dots, x_s^{v_s})] \mid x_i \in K^* \text{ for all } i\} \subset \mathbb{P}^{s-1}$$

is called a *degenerate projective torus* of type v , where $K^* = K \setminus \{0\}$. If $v_i = 1$ for all i , this set is called a *projective torus* in \mathbb{P}^{s-1} and it is denoted by \mathbb{T} .

The next result was shown in [22] under the hypothesis that $I(X)$ is a complete intersection.

Corollary 6.2. *Let $K = \mathbb{F}_q$ be a finite field and let X be a degenerate projective torus of type $v = (v_1, \dots, v_s)$. If $d_i = (q - 1) / \gcd(v_i, q - 1)$ for $i = 1, \dots, s$ and $r = \gcd(d_1, \dots, d_s)$, then*

$$\text{reg}(\tilde{S}/I(X)) = r \cdot g(\mathcal{S}) + 1 + \sum_{i=1}^s (d_i - 1) \quad \text{and} \quad \text{deg}(\tilde{S}/I(X)) = d_1 \cdots d_s / r,$$

where $\mathcal{S} = \mathbb{N}(d_1/r) + \cdots + \mathbb{N}(d_s/r)$ is the semigroup generated by $d_1/r, \dots, d_s/r$.

Proof. Let I be the toric ideal of $K[y_1^{d_1}, \dots, y_s^{d_s}]$ and let \tilde{I} be the homogenization of I with respect to $t_1^{d_1}, \dots, t_s^{d_s}$. According to [22, Lemma 3.1], \tilde{I} is equal to $I(X)$. Hence, the result follows from Theorem 5.5. \square

Lemma 6.3. *Given positive integers d_1, \dots, d_s , there is a prime number p such that d_i divides $p - 1$ for all i .*

Proof. We set $m = \text{lcm}(d_1, \dots, d_s)$ and $a = 1$. As a and m are relatively prime positive integers, by a classical theorem of Dirichlet [30, p. 25, p. 61], there exist infinitely many primes p such that $p \equiv a \pmod{m}$. Thus, we can write $p - 1 = km$ for some integer k . This proves that d_i divides $p - 1$ for all i . \square

The next result allows us to construct vanishing ideals over finite fields with prescribed regularity and degree of a certain type.

Proposition 6.4. *Given a sequence d_1, \dots, d_s of positive integers, there is a finite field $K = \mathbb{F}_q$ and a degenerate projective torus X such that*

$$\text{reg}(\tilde{S}/I(X)) = r \cdot g(\mathcal{S}) + 1 + \sum_{i=1}^s (d_i - 1) \quad \text{and} \quad \text{deg}(\tilde{S}/I(X)) = d_1 \cdots d_s / r,$$

where $r = \gcd(d_1, \dots, d_s)$ and $\mathcal{S} = \mathbb{N}(d_1/r) + \cdots + \mathbb{N}(d_s/r)$.

Proof. By Lemma 6.3, there is a prime number q such that d_i divides $q - 1$ for $i = 1, \dots, s$. We set $K = \mathbb{F}_q$ and $v_i = (q - 1) / d_i$ for all i . If X is a degenerate torus of type $v = (v_1, \dots, v_s)$ then, by Corollary 6.2, the result follows. \square

Proposition 6.5. *Let L be a graded lattice ideal of \tilde{S} of dimension 1 over an arbitrary field K and let $L = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$ be a minimal primary decomposition of L . Then, $\text{deg}(\tilde{S}/L) \geq m$ with equality if and only if $L = I(X)$ for some finite set X of a projective torus \mathbb{T} of \mathbb{P}^{s-1} .*

Proof. The inequality $\text{deg}(\tilde{S}/L) \geq m$ follows at once from Proposition 2.3. Assume that $\text{deg}(\tilde{S}/L) = m$. Let $\mathfrak{q} = \mathfrak{q}_i$ be any primary component of L . Then, $\text{deg}(\tilde{S}/\mathfrak{q}) = 1$. Consider the reduced Gröbner basis $\mathcal{G} = \{g_1, \dots, g_p\}$ of \mathfrak{q} relative to the graded reverse lexicographical

order of \tilde{S} . As usual, we denote the initial term of g_i by $\text{in}(g_i)$. As the degree and the Krull dimension of \tilde{S}/\mathfrak{q} are equal to 1, $H_{\mathfrak{q}}(d) = 1$ for $d \gg 0$, i.e., $\dim_K(\tilde{S}/\mathfrak{q})_d = 1$ for $d \gg 0$. Using that t_i is not a zero divisor of \tilde{S}/\mathfrak{q} for $i = 1, \dots, s$, we get that t_s does not divide $\text{in}(g_i)$ for any i . Then, $t_s^d + \mathfrak{q}$ generates $(\tilde{S}/\mathfrak{q})_d$ as a K -vector space for $d \gg 0$. Hence, for $i = 1, \dots, s-1$, there is $\mu_i \in K^*$ such that $t_i t_s^{d-1} - \mu_i t_s^d \in \mathfrak{q}$. Thus, $t_i t_s^{d-1}$ is in the initial ideal $\text{in}(\mathfrak{q})$ of \mathfrak{q} which is generated by $\text{in}(g_1), \dots, \text{in}(g_p)$. In particular, $t_i \in \text{in}(L)$ for $i = 1, \dots, s-1$ and $p = s-1$ because \mathcal{G} is reduced. Therefore for $i = 1, \dots, s-1$, using that \mathcal{G} is a reduced Gröbner basis, we may assume that $g_i = t_i - \lambda_i t_s$ for some $\lambda_i \in K^*$. If $Q = (\lambda_1, \dots, \lambda_{s-1}, 1)$, it is seen that \mathfrak{q} is the vanishing ideal of $[Q]$. Let $X = \{[Q_1], \dots, [Q_m]\}$ be the set of points in the projective torus $\mathbb{T} \subset \mathbb{P}^{s-1}$ such that \mathfrak{q}_i is the vanishing ideal of $[Q_i]$, then

$$I(X) = \bigcap_{i=1}^m I_{[Q_i]} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_m = L,$$

where $I_{[Q_i]}$ is the vanishing ideal of $[Q_i]$.

To show the converse, assume that L is the vanishing ideal of a finite set of points X in a projective torus \mathbb{T} . Let $[Q] = [(\alpha_i)]$ be a point in X and let $I_{[Q]}$ be the vanishing ideal of $[Q]$. It is not hard to see that the ideal $I_{[Q]}$ is given by

$$I_{[Q]} = (\alpha_1 t_2 - \alpha_2 t_1, \alpha_1 t_3 - \alpha_3 t_1, \dots, \alpha_1 t_s - \alpha_s t_1).$$

The primary decomposition of $L = I(X)$ is $I(X) = \bigcap_{[Q] \in X} I_{[Q]}$ because $I_{[Q]}$ is a prime ideal of \tilde{S} for any $[Q] \in X$. To complete the proof notice that $\deg(\tilde{S}/I_{[Q]}) = 1$ for any $[Q] \in X$ and $\deg(\tilde{S}/I(X)) = |X|$. \square

A similar statement holds for non-graded lattice ideals of dimension 0.

Proposition 6.6. *Let L be a lattice ideal of S of dimension 0 and let $L = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_m$ be a minimal primary decomposition of L . Then, $\deg(S/L) \geq m$ with equality if and only if $L = I(X^*)$ for some finite set X^* contained in an affine torus \mathbb{T}^* of K^s .*

Let K be a field with $K \neq \mathbb{F}_2$ and let y^{v_1}, \dots, y^{v_s} be a finite set of monomials. As usual if $v_i = (v_{i1}, \dots, v_{in}) \in \mathbb{N}^n$, then we set

$$y^{v_i} = y_1^{v_{i1}} \dots y_n^{v_{in}}, \quad i = 1, \dots, s,$$

where y_1, \dots, y_n are the indeterminates of a ring of polynomials with coefficients in K . The projective algebraic toric set parameterized by y^{v_1}, \dots, y^{v_s} is the set:

$$\{[(x_1^{v_{11}} \dots x_n^{v_{1n}}, \dots, x_1^{v_{s1}} \dots x_n^{v_{sn}})] \mid x_i \in K^* \text{ for all } i\} \subset \mathbb{P}^{s-1}.$$

A set of this form is said to be *parameterized by monomials*.

Proposition 6.7. *Let $K \neq \mathbb{F}_2$ be an arbitrary field and let $X \subset \mathbb{P}^{s-1}$. Then the following hold.*

- (a) *$I(X)$ is a lattice ideal of dimension 1 if and only if X is a finite subgroup of \mathbb{T} .*
- (b) *If K is finite, then X is a subgroup of \mathbb{T} if and only if X is parameterized by monomials.*

Proof. (a): (\Rightarrow) The set X is finite because $\dim \tilde{S}/I(X) = 1$ (see [6, Proposition 6, p. 441]). Let $[\alpha] = [(\alpha_i)]$ be a point of X and let $I_{[\alpha]}$ be its vanishing ideal. We may assume that $\alpha_k = 1$ for some k . Since the ideal

$$(*) \quad I_{[\alpha]} = (t_1 - \alpha_1 t_k, \dots, t_{k-1} - \alpha_{k-1} t_k, t_{k+1} - \alpha_{k+1} t_k, \dots, t_s - \alpha_s t_k)$$

is a minimal prime of $I(X)$, $\alpha_i \neq 0$ for all i because t_i is not a zero divisor of $\tilde{S}/I(X)$. Thus, $[\alpha] \in \mathbb{T}$. This proves that $X \subset \mathbb{T}$. Next, we show that X is a subgroup of \mathbb{T} . Let g_1, \dots, g_r be a

generating set of $I(X)$ consisting of binomials and let $[\alpha] = [(\alpha_i)]$, $[\beta] = [(\beta_i)]$ be two elements of X . We set $\gamma = \alpha \cdot \beta = (\alpha_i \beta_i)$. Since the entries of γ are all non-zero, we may assume that $\gamma_s = 1$. Since $g_i(\alpha) = 0$ and $g_i(\beta) = 0$ for all i , we get that $g_i(\gamma) = 0$ for all i . Hence, $I(X) \subset I_{[\gamma]}$ and consequently $I_{[\gamma]}$ is a minimal prime of $I(X)$. Hence there is $[\gamma'] \in X$, with $\gamma'_s = 1$ such that $I_{[\gamma]} = I_{[\gamma']}$. It follows that $\gamma = \gamma'$. Thus, $[\gamma] \in X$. By a similar argument it follows that $[\alpha]^{-1} = [(\alpha_i^{-1})]$ is in X .

\Leftarrow) The ideal $I(X)$ is generated by binomials, this follows from [10, Proposition 2.3(a)] and its proof. Since $I(X)$ is equal to $\bigcap_{[\alpha] \in X} I_{[\alpha]}$, using Eq. (*), we get that t_i is not a zero divisor of $S/I(X)$ for all i . Hence, by Theorem 2.7, $I(X)$ is a lattice ideal.

(b): (\Rightarrow) By the fundamental theorem of finitely generated abelian groups, X is a direct product of cyclic groups. Hence, there are $[\alpha_1], \dots, [\alpha_n]$ in X such that

$$X = \{ [\alpha_1]^{i_1} \cdots [\alpha_n]^{i_n} \mid i_1, \dots, i_n \in \mathbb{Z} \}.$$

If β is a generator of (K^*, \cdot) , we can write

$$\alpha_1 = (\beta^{v_{11}}, \dots, \beta^{v_{s1}}), \dots, \alpha_n = (\beta^{v_{1n}}, \dots, \beta^{v_{sn}})$$

for some v_{ij} 's in \mathbb{N} . Then, $[\gamma]$ is in X if and only if we can write

$$[\gamma] = [((\beta^{i_1})^{v_{11}} \cdots (\beta^{i_n})^{v_{1n}}), \dots, (\beta^{i_1})^{v_{s1}} \cdots (\beta^{i_n})^{v_{sn}})]$$

for some $i_1, \dots, i_n \in \mathbb{Z}$. Therefore, X is parameterized by the monomials y^{v_1}, \dots, y^{v_s} , where $v_i = (v_{i1}, \dots, v_{is})$ for $i = 1, \dots, s$.

(b): (\Leftarrow) If $X \subset \mathbb{P}^{s-1}$ is a projective algebraic toric set parameterized by y^{v_1}, \dots, y^{v_s} , then by the exponent laws it is not hard to show that X is a multiplicative group under componentwise multiplication. \square

The next structure theorem allows us—with the help of *Macaulay2* [15]—to compute the vanishing ideal of an algebraic toric set parameterized by monomials over a finite field.

Theorem 6.8. [28, Theorem 2.1] *Let $B = K[t_1, \dots, t_s, y_1, \dots, y_n, z]$ be a polynomial ring over the finite field $K = \mathbb{F}_q$ and let X be the algebraic toric set parameterized by y^{v_1}, \dots, y^{v_s} . Then*

$$I(X) = (\{t_i - y^{v_i} z\}_{i=1}^s \cup \{y_i^{q-1} - 1\}_{i=1}^n) \cap \tilde{S}$$

and $I(X)$ is a Cohen-Macaulay radical lattice ideal of dimension 1.

The following theorem takes care of the infinite field case.

Theorem 6.9. *Let $B = K[t_1, \dots, t_s, y_1, \dots, y_n, z]$ be a polynomial ring over an infinite field K . If X is an algebraic toric set parameterized by monomials y^{v_1}, \dots, y^{v_s} , then*

$$I(X) = (\{t_i - y^{v_i} z\}_{i=1}^s) \cap \tilde{S}$$

and $I(X)$ is the toric ideal of $K[x^{v_1} z, \dots, x^{v_s} z]$.

Proof. We set $I' = (t_1 - y^{v_1} z, \dots, t_s - y^{v_s} z) \subset B$. First we show the inclusion $I(X) \subset I' \cap \tilde{S}$. Take a homogeneous polynomial $F = F(t_1, \dots, t_s)$ of degree d that vanishes on X . We can write

$$(6.1) \quad F = \lambda_1 t^{m_1} + \cdots + \lambda_r t^{m_r} \quad (\lambda_i \in K^*; m_i \in \mathbb{N}^s),$$

where $\deg(t^{m_i}) = d$ for all i . Write $m_i = (m_{i1}, \dots, m_{is})$ for $1 \leq i \leq r$. Applying the binomial theorem to expand the right hand side of the equality

$$t_j^{m_{ij}} = [(t_j - y^{v_j} z) + y^{v_j} z]^{m_{ij}}, \quad 1 \leq i \leq r, 1 \leq j \leq s,$$

and then substituting all the $t_j^{m_{ij}}$ in Eq. (6.1), we obtain that F can be written as:

$$(6.2) \quad F = \sum_{i=1}^s g_i(t_i - y^{v_i} z) + z^d F(y^{v_1}, \dots, y^{v_s}) = \sum_{i=1}^s g_i(t_i - y^{v_i} z) + z^d G(y_1, \dots, y_n)$$

for some g_1, \dots, g_s in B . Thus to show that $F \in I' \cap S$ we need only show that $G = 0$. We claim that G vanishes on $(K^*)^n$. Take an arbitrary sequence x_1, \dots, x_n of elements of K^* . Making $t_i = x^{v_i}$ for all i in Eq. (6.2) and using that F vanishes on X , we obtain

$$(6.3) \quad 0 = F(x^{v_1}, \dots, x^{v_s}) = \sum_{i=1}^s g'_i(x^{v_i} - y^{v_i} z) + z^d G(y_1, \dots, y_n).$$

We can make $y_i = x_i$ for all i and $z = 1$ in Eq. (6.3) to get that G vanishes on (x_1, \dots, x_n) . This completes the proof of the claim. Therefore G vanishes on $(K^*)^n$ and since the field K is infinite it follows that $G = 0$.

Next we show the inclusion $I(X) \supset I' \cap \tilde{S}$. Let \mathcal{G} be a Gröbner basis of I' with respect to the lexicographical order $y_1 \succ \dots \succ y_n \succ z \succ t_1 \succ \dots \succ t_s$. By Buchberger's algorithm [6, Theorem 2, p. 89] the set \mathcal{G} consists of binomials and by elimination theory [6, Theorem 2, p. 114] the set $\mathcal{G} \cap \tilde{S}$ is a Gröbner basis of $I' \cap \tilde{S}$. Hence $I' \cap \tilde{S}$ is a binomial ideal. Thus to show the inclusion $I(X) \supset I' \cap \tilde{S}$ it suffices to show that any binomial in $I' \cap \tilde{S}$ is homogeneous and vanishes on X . Take a binomial $f = t^a - t^b$ in $I' \cap \tilde{S}$, where $a = (a_i)$ and $b = (b_i)$ are in \mathbb{N}^s . Then we can write

$$(6.4) \quad f = \sum_{i=1}^s g_i(t_i - y^{v_i} z)$$

for some polynomials g_1, \dots, g_s in B . Making $y_i = 1$ for $i = 1, \dots, n$ and $t_i = y^{v_i} z$ for $i = 1, \dots, s$, we get

$$z^{a_1} \dots z^{a_s} - z^{b_1} \dots z^{b_s} = 0 \implies a_1 + \dots + a_s = b_1 + \dots + b_s.$$

Hence f is homogeneous. Take a point $[P]$ in X with $P = (x^{v_1}, \dots, x^{v_s})$. Making $t_i = x^{v_i}$ in Eq. (6.4), we get

$$f(x^{v_1}, \dots, x^{v_s}) = \sum_{i=1}^s g'_i(x^{v_i} - y^{v_i} z).$$

Hence making $y_i = x_i$ for all i and $z = 1$, we get that $f(P) = 0$. Thus f vanishes on X . Thus, we have shown the equality $I(X) = I' \cap \tilde{S}$.

By [40, Proposition 7.1.9] $I(X)$ is the toric ideal of $K[x^{v_1} z, \dots, x^{v_s} z]$. □

7. VANISHING IDEALS OVER GRAPHS

In this section, we study graded vanishing ideals over bipartite graphs. For a projective algebraic toric set parameterized by the edges of a bipartite graph, we are able to express the regularity of the vanishing ideal in terms of the corresponding regularities for the blocks of the graph. For bipartite graphs, we introduce a method that can be used to compute the regularity.

Let $K = \mathbb{F}_q$ be a finite field with q elements and let G be a simple graph with vertex set $V_G = \{y_1, \dots, y_n\}$ and edge set E_G . We refer to [4] for the general theory of graphs.

Definition 7.1. Let $e = \{y_i, y_j\}$ be an edge of G . The *characteristic vector* of e is the vector $v = e_i + e_j$, where e_i is the i^{th} unit vector in \mathbb{R}^n .

In what follows $\mathcal{A} = \{v_1, \dots, v_s\}$ will denote the set of all characteristic vectors of the edges of the graph G . We may identify the edges of G with the variables t_1, \dots, t_s of a polynomial ring $K[t_1, \dots, t_s]$ and refer to t_1, \dots, t_s as the edges of G .

Definition 7.2. If X is the projective algebraic toric set parameterized by y^{v_1}, \dots, y^{v_s} , we call X the *projective algebraic toric set parameterized by the edges of G* .

Definition 7.3. A graph G is called *bipartite* if its vertex set can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G has one end in V_1 and one end in V_2 . The pair (V_1, V_2) is called a *bipartition* of G .

Let G be a graph. A vertex v (resp. an edge e) of G is called a *cutvertex* (resp. *bridge*) if the number of connected components of $G \setminus \{v\}$ (resp. $G \setminus \{e\}$) is larger than that of G . A maximal connected subgraph of G without cutvertices is called a *block*. A graph G is *2-connected* if $|V_G| > 2$ and G has no cutvertices. Thus a block of G is either a maximal 2-connected subgraph, a bridge or an isolated vertex. By their maximality, different blocks of G intersect in at most one vertex, which is then a cutvertex of G . Therefore every edge of G lies in a unique block, and G is the union of its blocks (see [4, Chapter III] for details).

We come to the main result of this section.

Theorem 7.4. *Let G be a bipartite graph without isolated vertices and let G_1, \dots, G_c be the blocks of G . If K is a finite field with q elements and X (resp X_k) is the projective algebraic toric set parameterized by the edges of G (resp. G_k), then*

$$\operatorname{reg} K[EG]/I(X) = \sum_{k=1}^c \operatorname{reg} K[EG_k]/I(X_k) + (q-2)(c-1).$$

Proof. We denote the set of all characteristic vectors of the edges of G by $\mathcal{A} = \{v_1, \dots, v_s\}$. Let P be the toric ideal of $K[\{y^v | v \in \mathcal{A}\}]$, let \mathcal{A}_k be the set of characteristic vectors of the edges of G_k and let P_k be the toric ideal of $K[\{y^v | v \in \mathcal{A}_k\}]$. The toric ideal P is the kernel of the epimorphism of K -algebras

$$\varphi: S = K[t_1, \dots, t_s] \longrightarrow K[\{y^v | v \in \mathcal{A}\}], \quad t_i \longmapsto y^{v_i}.$$

Permitting an abuse of notation, we may denote the edges of G by t_1, \dots, t_s . As G is a bipartite graph and $E_{G_i} \cap E_{G_j} = \emptyset$ for $i \neq j$, from [39, Proposition 3.1], it follows that $P = P_1 + \dots + P_c$. Setting

$$\mathcal{I} = (\{t_i^{q-1} - t_j^{q-1} | t_i, t_j \in E_G\}) \quad \text{and} \quad \mathcal{I}_k = (\{t_i^{q-1} - t_j^{q-1} | t_i, t_j \in E_{G_k}\}),$$

by [28, Corollary 2.11], we get

$$((P + \mathcal{I}): \prod_{t_i \in E_G} t_i) = I(X) \quad \text{and} \quad ((P_k + \mathcal{I}_k): \prod_{t_i \in E_{G_k}} t_i) = I(X_k).$$

Therefore the formula for the regularity follows from Theorem 5.8. \square

This result is interesting because it reduces the computation of the regularity to the case of 2-connected bipartite graphs. Next, we compare the ideals $I(X)$ and $I = P + \mathcal{I}$, where \mathcal{I} is the ideal $(\{t_i^{q-1} - t_j^{q-1} | t_i, t_j \in E_G\})$, and relate the regularity of $I(X)$ with the Hilbert function of S/I and the primary decompositions of I .

Proposition 7.5. *Let G be a bipartite graph which is not a forest and let P be the toric ideal of $K[y^{v_1}, \dots, y^{v_s}]$. If $I = P + \mathcal{I}$ and X is the projective algebraic toric set parameterized by the edges of G , then the following hold:*

- (a) $I \subsetneq I(X)$ and I is not unmixed.

- (b) *There is a minimal primary decomposition $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m \cap \mathfrak{q}'$, where $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ are prime ideals such that $I(X) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m$ and \mathfrak{q}' is an \mathfrak{m} -primary ideal. Here \mathfrak{m} denotes the maximal ideal (t_1, \dots, t_s) .*
- (c) *If $t^a \in \mathfrak{q}'$, then $\mathfrak{q} := I + (t^a)$ is \mathfrak{m} -primary, $(I : t^a) = I(X)$ and $I = I(X) \cap \mathfrak{q}$.*
- (d) *If i_0 is the least integer $i \geq |a|$ such that $H_I(i) - H_{\mathfrak{q}}(i) = |X|$, then $\text{reg } S/I(X) = i_0 - |a|$.*

Proof. (a): Since G has at least one even cycle of length at least 4, using [25, Theorem 5.9.] it follows that $I \subsetneq I(X)$. To show that I is not unmixed, we proceed by contradiction. Assume that I is unmixed, i.e., all associated primes of I have height $s - 1$. Then, by Lemma 2.8, I is a lattice ideal, i.e., I is equal to $(I : (t_1 \cdots t_s)^\infty)$, a contradiction because G is bipartite and according to [28, Corollary 2.11] one has $(I : (t_1 \cdots t_s)^\infty) = I(X)$.

(b): As I is graded, by (a), there is a minimal primary decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m \cap \mathfrak{q}'$, where \mathfrak{q}_i is \mathfrak{p}_i -primary of height $s - 1$ for all i and \mathfrak{q}' is \mathfrak{m} -primary. By Lemma 2.8, \mathfrak{q}_i is a lattice ideal for all i . Hence

$$I(X) = (I : (t_1 \cdots t_s)^\infty) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m.$$

As $I(X)$ is a radical ideal, so is \mathfrak{q}_i for $i = 1, \dots, m$, i.e., $\mathfrak{q}_i = \mathfrak{p}_i$ for all i .

(c): Let $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m \cap \mathfrak{q}'$ be a minimal primary decomposition as in (b). Pick any monomial t^a in \mathfrak{q}' . Then, by Lemma 2.8, $\mathfrak{q} = I + (t^a)$ is \mathfrak{m} -primary and

$$(I : t^a) = (\mathfrak{p}_1 : t^a) \cap \cdots \cap (\mathfrak{p}_m : t^a) \cap (\mathfrak{q}' : t^a) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m = I(X).$$

From the equality $(I : t^a) = I(X)$, it follows readily that $I = I(X) \cap \mathfrak{q}$.

(d): Let t^a be any monomial of \mathfrak{q}' and let $\ell = \deg(t^a)$. If $\mathfrak{q} = I + (t^a)$, by (c), there is an exact sequence

$$0 \longrightarrow S/I(X)[- \ell] \xrightarrow{t^a} S/I \longrightarrow S/\mathfrak{q} \longrightarrow 0.$$

Hence, by the additivity of Hilbert functions, $H_X(i - \ell) = H_I(i) - H_{\mathfrak{q}}(i)$ for $i \geq 0$. Since $I(X)$ is Cohen-Macaulay of dimension 1, $\text{reg } S/I(X)$ is equal to the index of regularity of $S/I(X)$. Thus, $\text{reg}(S/I(X))$, is the least integer $r \geq 0$ such that $H_X(d) = |X|$ for $d \geq r$. Thus, $r = i_0 - |a|$. \square

Theorem 7.6. ([25], [37]) *Let G be a connected bipartite graph with bipartition (V_1, V_2) and let X be the projective algebraic toric set parameterized by the edges of G . If $|V_2| \leq |V_1|$, then*

$$(|V_1| - 1)(q - 2) \leq \text{reg } S/I(X) \leq (|V_1| + |V_2| - 2)(q - 2).$$

Furthermore, equality on the left occurs if G is a complete bipartite graph or if G is a Hamiltonian graph and equality on the right occurs if G is a tree.

For an arbitrary bipartite graph, Theorems 7.4 and 7.6 can be used to bound the regularity of $I(X)$.

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