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Multi-quark interactions with a globally stable vacuum

A.A. Osipov [*,1], B. Hiller, J. da Providência

Centro de Física Teórica, Departamento de Física da Universidade de Coimbra, 3004-516 Coimbra, Portugal

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Abstract

It is shown that $U(3)_L \times U(3)_R$ eight-quark interactions stabilize the asymmetric ground state of the well-known model with four-quark Nambu–Jona-Lasinio and six-quark ’t Hooft interactions. The result remains when the reduced $SU(3)$ flavour symmetry is explicitly broken by the general current quark mass term with $\hat{m}_u \neq \hat{m}_d \neq \hat{m}_s$.

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1. Introduction

Phenomenological parametrizations based on some simple ansatz with solid symmetry grounds are frequently used in low-energy QCD. One of the most common and important outputs of this approach is to get a clue of how high-energy QCD may influence low-energy observables. Unfortunately, in spite of all remarkable successes of the QCD sum rules method [1], or chiral perturbation theory [2], this picture is still far away from being completed.

Some features of the large distance hadron dynamics can be understood in the framework of effective chiral Lagrangians written in terms of quark degrees of freedom [3]. They are efficient for the description of spontaneous chiral symmetry breaking, or for the study of the quark structure of light mesons. The parameters of such Lagrangians can be related to the characteristics of the QCD vacuum given in form of the vacuum expectation values of the relevant quark bilinears or gluons (if they are included). In many respects this approach corresponds to a Landau–Ginzburg-like description of the flavour dynamics.

There are a number of instructive models that assume the existence of underlying multi-quark interactions and their importance for physics of hadrons. Well-known examples include the Nambu and Jona-Lasinio (NJL) model [4], where the four-fermion interactions have been used to study dynamical breaking of chiral symmetry[2]; the instanton inspired models [8], where $2N_f$-quark interactions (N_f is the number of quark flavours) offer a possible framework to discuss the $U_{\mathrm{A}}(1)$ problem [9]; the potential-type quark models which are successfully applied to the evaluation of hadronic parameters [10].

In this Letter, we propose to extend the phenomenologically interesting three-flavour quark model which combines the chiral $U(3)_L \times U(3)_R$ NJL-type Lagrangian with the ’t Hooft six-quark determinant (NJLH), by supplying it with flavour mixing eight-quark interactions. The original NJLH Lagrangian gives a good description of the pseudoscalar nonet, especially the η and η' masses and mixing [11], and in this form the model has been widely and successfully explored at the mean-field level [12–14].

This approximation was refined by works of Reinhardt and Alkofer [15], who used the functional integral method to bosonize the model. This approach hinges decisively on the stationary phase asymptotics of the generating functional and

* Corresponding author.
E-mail addresses: osipov@nusun.jinr.ru, alexguest@malaposta.fis.uc.pt (A.A. Osipov), brigitte@teor.fis.uc.pt (B. Hiller), providencia@teor.fis.uc.pt (J. da Providência).
[1] On leave from Joint Institute for Nuclear Research, Laboratory of Nuclear Problems, 141980 Dubna, Moscow Region, Russia.

[2] Later on a modified form of this interaction has been used to derive the QCD effective action at long distances [5–7].

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allows to calculate the contribution of the classical path already at lowest order. This lowest order result sums all tree diagrams in the perturbative series in powers of the coupling constant of the 't Hooft interaction [16].

The functional treatment of the model reveals one essential problem: the model has actually several classical trajectories which belong to the interval of the functional integration, and therefore contribute to the integral [16]. If one takes them into account, the effective potential of the theory gets unbounded from below, i.e., the system does not have a ground state.

We argue here that this drawback of the NJLH model can be removed. The eight-quark interactions added to the original Lagrangian reduce (under given conditions) the number of stationary phase trajectories to one and, as a result, the theory has a stable global minimum, attributed to a spontaneous symmetry breakdown. It should be remarked that the stationary phase equations which appear in this approach are of cubic order and have, in general, more than one admissible solution. We obtain inequalities for coupling constants to distinguish those solutions further and show that these constraints can be finally understood as the stability criteria of the whole system. We consider the most general eight-quark spin-zero interactions invariant under $U(3)_L \times U(3)_R$ chiral symmetry and assume that current quarks have realistic masses: $\hat{m}_u \neq \hat{m}_d \neq \hat{m}_s$. It is shown that our result is independent both of the specific form of eight-quark interactions and values of current quark masses.

Let us note that only one type of the eight-quark interactions considered are flavour mixing (the first part of the eight-quark Lagrangian studied here, i.e., $\mathcal{L}_1$ in Eq. (4)) and have been used previously [17] in a different context, namely, to introduce OZI-violating effects [18] in a NJL-type model with the $U_A(1)$ anomaly term inspired by the works of Di Vecchia and Veneziano [19], and independently by Rosenzweig, Schechter and Trahern [20]. Recently, by describing the properties of nuclear matter with two-flavour NJL models, eight-fermion interactions of the $\mathcal{L}_1$-type have been also analyzed in [21].

2. The model

The dynamics of the model considered is determined by the Lagrangian density

$$\mathcal{L}_{\text{eff}} = \bar{q}\big(i\gamma^\mu\partial_\mu - \hat{m}\big)q + \mathcal{L}_{\text{NJL}} + \mathcal{L}_{\text{H}} + \mathcal{L}_{8q}, \tag{1}$$

where it is assumed that quark fields have colour ($N_c = 3$) and flavour ($N_f = 3$) indices. The current quark mass, $\hat{m}$, is a diagonal matrix with elements $\operatorname{diag}(\hat{m}_u, \hat{m}_d, \hat{m}_s)$, which explicitly breaks the global chiral $SU_L(3) \times SU_R(3)$ symmetry of the Lagrangian. The flavour symmetry of the model becomes $SU(3)$, if $\hat{m}_u = \hat{m}_d = \hat{m}_s$; and one gets the reduced symmetries of isospin and hypercharge conservation, if $\hat{m}_u = \hat{m}_d \neq \hat{m}_s$. Putting $\hat{m}_u \neq \hat{m}_d \neq \hat{m}_s$, one obtains the most general pattern of the explicit symmetry breakdown in the model.

We suppose that quark vertices are effectively local, this being a frequently used approximation. Even in this essentially simplified form the Lagrangian has all basic ingredients to describe the dynamical symmetry breaking of the hadronic vacuum and find its stability condition.

The interaction Lagrangian of the NJLH model in the scalar and pseudoscalar channels is given by two terms

$$\mathcal{L}_{\text{NJL}} = \frac{G}{2}\big[(\bar{q}\lambda_a q)^2 + (\bar{q}i\gamma_5\lambda_a q)^2\big], \tag{2}$$

$$\mathcal{L}_{\text{H}} = \kappa(\det\bar{q}P_L q + \det\bar{q}P_R q). \tag{3}$$

The first one is the $U_L(3) \times U_R(3)$ chiral symmetric interaction specifying the local part of the effective four-quark Lagrangian in channels with quantum numbers $J^P = 0^+, 0^-$. The Gell-Mann flavour matrices λ_a, $a = 0, 1, \ldots, 8$, are normalized such that $\operatorname{tr}(\lambda_a\lambda_b) = 2\delta_{ab}$. The second term represents the 't Hooft determinantal interactions [9]. The matrices $P_{L,R} = (1 \mp \gamma_5)/2$ are projectors and the determinant is over flavour indices. The determinantal interaction breaks explicitly the axial $U_A(1)$ symmetry [22] and Zweig's rule.

The new feature of the model is the inclusion of $U(3)_L \times U(3)_R$ symmetric eight-quark forces, which we add to the standard NJLH Lagrangian to obtain the stable ground state. They are described by the term $\mathcal{L}_{8q} = \mathcal{L}_1 + \mathcal{L}_2$, where

$$\mathcal{L}_1 = 8g_1\big[(\bar{q}_i P_R q_m)(\bar{q}_m P_L q_i)\big]^2, \tag{4}$$

$$\mathcal{L}_2 = 16g_2(\bar{q}_i P_R q_m)(\bar{q}_m P_L q_j)(\bar{q}_j P_R q_k)(\bar{q}_k P_L q_i). \tag{5}$$

The flavour indices $i, j, \ldots = 1, 2, 3 = u, d, s$, and g_1, g_2 stand for the various symmetric eight-quark coupling strengths. The first term $\mathcal{L}_1$ coincides with the OZI-violating eight-quark interactions considered in [17]. The second term $\mathcal{L}_2$ represents interactions without violation of Zweig's rule. $\mathcal{L}_{8q}$ is the most general Lagrangian which describes the spin zero eight-quark interactions without derivatives. It is the lowest order term in number of quark fields which is relevant to the case. We restrict our consideration to these forces, because in the long wavelength limit the higher-dimensional operators are suppressed.

Large N_c arguments can be also used to justify this step if the dimensionful coupling constants $[G] = M^{-2}$, $[\kappa] = M^{-5}$, $[g_1] = [g_2] = M^{-8}$ count at large N_c as $G \sim 1/N_c$, $\kappa \sim 1/N_c^{N_f}$, $g_1, g_2 \sim 1/N_c^4$. In this case the NJL interactions (2) dominate over $\mathcal{L}_{\text{H}}$ and $\mathcal{L}_{8q}$ at large N_c, as it should be, because Zweig's rule is exact at $N_c = \infty$. On the other hand, with these counting rules the Lagrangians $\mathcal{L}_{\text{H}}$ and $\mathcal{L}_{8q}$ contribute at the same N_c order, thus the effects coming from them are comparable and must be considered together.[3]

It is clear that our considerations are also relevant if the multi-quark interactions create a hierarchy [23] similar to the hierarchy found within the gluon field correlators [24]. In this case the lowest four-quark interaction forms a stable vacuum corresponding to spontaneously broken chiral symmetry. The higher multi-quark interactions in the hierarchy must not destroy this state, otherwise they would be as important as the lowest order terms. Since, however, the 't Hooft interaction, which is the next term in the hierarchy, destroys the ground state [16], one cannot truncate the tower of multi-quark interactions at this level. The next natural candidate is the eight-quark

[3] Let us note that our counting for g_1 differs from the prescription of paper [17] where $g_1 \sim 1/N_c^3$.

term $\mathcal{L}_{8q}$. We show that its inclusion is sufficient to stabilize the ground state.

3. The eight-quark term at work

The many-fermion vertices of Lagrangian $\mathcal{L}_{\rm eff}$ can be presented in the bilinear form by introducing the functional unity [15] in the vacuum-to-vacuum amplitude of the theory. The specific details of this bosonization procedure are given in our recent work [16]. The new interaction term $\mathcal{L}_{8q}$, which we add now to the effective quark Lagrangian, does not create additional problems, and the method can be simply extended to the present case. This is why we take as a starting point the corresponding functional integral already in its bosonized form

$$Z=\int \mathcal{D}q\,\mathcal{D}\bar q\prod_a \mathcal{D}\sigma_a\prod_a\mathcal{D}\phi_a\exp\left(i\int \mathrm{d}^4x\,\mathcal{L}_q(\bar q,q,\sigma,\phi)\right)$$
$$\times\int_{-\infty}^{+\infty}\prod_a\mathcal{D}s_a\prod_a\mathcal{D}p_a\exp\left(i\int \mathrm{d}^4x\,\mathcal{L}_r(\sigma,\phi,\Delta;s,p)\right),\tag{6}$$

where

$$\mathcal{L}_q=\bar q\left(i\gamma^\mu\partial_\mu-m-\sigma-i\gamma_5\phi\right)q,\tag{7}$$

$$\begin{aligned}\mathcal{L}_r={}&s_a(\sigma_a+\Delta_a)+p_a\phi_a+\frac{G}{2}\left(s_a^2+p_a^2\right)\\&+\frac{\kappa}{32}A_{abc}s_a(s_bs_c-3p_bp_c)+\frac{g_1}{8}\left(s_a^2+p_a^2\right)^2\\&+\frac{g_2}{8}\big[d_{abe}d_{cde}(s_as_bs_cs_d+2s_as_bp_cp_d+p_ap_bp_cp_d)\\&+4f_{ace}f_{bde}s_as_bp_cp_d\big].\end{aligned}\tag{8}$$

It is worth to observe that we did not use any approximations to obtain this result.

Let us explain our notations. The bosonic fields σ_a and ϕ_a are the composite scalar and pseudoscalar nonets which will be identified later with the corresponding physical states. The auxiliary fields s_a and p_a must be integrated out from the effective mesonic Lagrangian $\mathcal{L}_r$. We assume that $\sigma=\sigma_a\lambda_a$, and so on for all bosonic fields σ,ϕ,s,p. The quarks obtain their constituent masses $m=m_a\lambda_a=\mathrm{diag}(m_u,m_d,m_s)$ due to dynamical chiral symmetry breaking in the physical vacuum state, $\Delta_a=m_a-\hat m_a$. The totally symmetric constants A_{abc} are related to the flavour determinant, and equal to

$$A_{abc}=\frac{1}{3!}\epsilon_{ijk}\epsilon_{mnl}(\lambda_a)_{im}(\lambda_b)_{jn}(\lambda_c)_{kl}.\tag{9}$$

The eight-quark interactions change drastically the semi-classical asymptotics of the functional integral over s_a,p_a in (6), as compared to the case, when $g_1,g_2=0$. To see this one should first find all real stationary phase trajectories $s_a^{\rm st}=s_a(\sigma,\phi)$, $p_a^{\rm st}=p_a(\sigma,\phi)$ given by the equations

$$\frac{\partial\mathcal{L}_r}{\partial s_a}=0,\qquad\frac{\partial\mathcal{L}_r}{\partial p_a}=0.\tag{10}$$

We seek these solutions in form of expansions in the external mesonic fields, σ_a,ϕ_a,

$$s_a^{\rm st}=h_a+h_{ab}^{(1)}\sigma_b+h_{abc}^{(1)}\sigma_b\sigma_c+h_{abc}^{(2)}\phi_b\phi_c+\cdots,$$

$$p_a^{\rm st}=h_{ab}^{(2)}\phi_b+h_{abc}^{(3)}\phi_b\sigma_c+\cdots.\tag{11}$$

The coefficients $h_{a\ldots}^{(i)}$ depend on the coupling constants G, κ, g_1, g_2 and quark masses Δ_a. The higher index coefficients $h_{a\ldots}^{(i)}$ are recurrently expressed in terms of the lower ones. The one-index coefficients h_a are the solutions of the following system of cubic equations

$$\begin{aligned}&\Delta_a+Gh_a+\frac{3\kappa}{32}A_{abc}h_bh_c\\&\quad+\frac{g_1}{2}h_ah_b^2+\frac{g_2}{2}d_{abe}d_{cde}h_bh_ch_d=0.\end{aligned}\tag{12}$$

The trivial solution $h_a=0$, corresponds to the perturbative vacuum $\Delta_a=0$. There are also non-trivial ones. In accordance with the pattern of explicit symmetry breaking the mean field Δ_a can have only three non-zero components at most with indices $a=0,3,8$. It means that in general we have a system of only three equations to determine $h_a\lambda_a=\mathrm{diag}(h_u,h_d,h_s)$

$$\begin{cases}Gh_u+\Delta_u+\frac{\kappa}{16}h_dh_s+\frac{g_1}{4}h_u\left(h_u^2+h_d^2+h_s^2\right)+\frac{g_2}{2}h_u^3=0,\\Gh_d+\Delta_d+\frac{\kappa}{16}h_uh_s+\frac{g_1}{4}h_d\left(h_u^2+h_d^2+h_s^2\right)+\frac{g_2}{2}h_d^3=0,\\Gh_s+\Delta_s+\frac{\kappa}{16}h_uh_d+\frac{g_1}{4}h_s\left(h_u^2+h_d^2+h_s^2\right)+\frac{g_2}{2}h_s^3=0.\end{cases}\tag{13}$$

Our aim now is to show that parameters can be fixed in such a way that this system will have only one real solution. We start by summing the first two equations, which leads to the cubic equation

$$\begin{aligned}&x^3+tx=b,\\&t=\frac{1}{g_1+g_2}\left(8G+\frac{\kappa}{2}h_s+y^2(g_1+3g_2)+2g_1h_s^2\right),\\&b=-\frac{8(\Delta_u+\Delta_d)}{g_1+g_2},\end{aligned}\tag{14}$$

where $x=h_u+h_d$, $y=h_u-h_d$.

Note that deviations of the variable y from zero are a measure of isospin breaking effects due to electromagnetic forces, as the difference h_u-h_d does not vanish for $\hat m_u\neq\hat m_d$. The function $t(y,h_s)$ has a minimum (if $g_1>0$ and $g_1+3g_2>0$) at $y=0$ and $h_s=-\kappa/(8g_1)$, thus the inequality $t>0$ always holds for coupling constants fixed by

$$G>\frac{1}{g_1}\left(\frac{\kappa}{16}\right)^2.\tag{15}$$

In this case the cubic equation has for any given value of b just one real root

$$\begin{aligned}&x_{(1)}=\left(\frac{b}{2}+\sqrt{D}\right)^{1/3}+\left(\frac{b}{2}-\sqrt{D}\right)^{1/3},\\&D=\left(\frac{t}{3}\right)^3+\left(\frac{b}{2}\right)^2.\end{aligned}\tag{16}$$

Since $b<0$ (provided that $\Delta_u+\Delta_d>0$), this function is negative. Its minimum is located at the point $y=0$, $h_s=-\kappa/(8g_1)$, and the surface $x=0$ is an asymptotic one to $x_{(1)}$.

Subtracting the second equation from the first one we obtain a quadratic equation with respect to x. Its solutions are given by

$$x_{(2)} = \pm\left[\frac{-1}{g_1+3g_2}\left(8G - \frac{\kappa}{2}h_s + 2g_1h_s^2 + y^2(g_1+g_2) + \frac{8(\Delta_u-\Delta_d)}{y}\right)\right]^{1/2}. \tag{17}$$

Since we only allow real solutions, the following inequality must hold

$$8G - \frac{\kappa}{2}h_s + 2g_1h_s^2 + y^2(g_1+g_2) < \frac{8}{y}(\Delta_d-\Delta_u). \tag{18}$$

For definiteness, we suppose that $\Delta_d - \Delta_u > 0$. This assumption represents one of two possible alternatives. Our final mathematical conclusions do not depend on the choice made. However, it is not obvious which of them should be required physically. Next, the function $f(h_s) = 8G - \kappa h_s/2 + 2g_1h_s^2 > 0$, since the minimum value $f_{\min} = 8G - \kappa^2/(32g_1) > 0$ in the parameter region of (15). Therefore, y ranges over the half-open interval $0 < y \leqslant y_{\max}(h_s)$. The lower bound $y = 0$ is an asymptotic surface for the function $x_{(2)}$. The upper bound $y_{\max}(h_s)$ is the unique real solution of the equation $y^3(g_1+g_2) + yf(h_s) + 8(\Delta_u - \Delta_d) = 0$. It follows that $y_{\max} \propto (\Delta_d - \Delta_u)$, i.e., the electromagnetic forces which are responsible for the isospin symmetry breaking determine the length of the segment $[0, y_{\max}]$, which is relatively small as compared with intervals determined by the strong interaction. As a consequence a negative branch of the function $x_{(2)}$ grows rapidly with y from $-\infty$ at $y = 0$ up to 0 at $y = y_{\max}$. On the contrary, functions $x_{(1)}(y)$ and $x_{(3)}(y)$ (see Eq. (19) below) remain almost unchanged in the interval $0 < y < y_{\max}$, because here the strong driving forces totally cover electromagnetic effects.

Let us consider now the third equation which yields

$$x_{(3)} = \pm 4\sqrt{\frac{v(h_s,y)}{-(\kappa+8g_1h_s)}}, \tag{19}$$

where we have introduced the notation

$$v(h_s,y) = (g_1+2g_2)h_s^3 + \frac{h_s}{2}\left(8G + g_1y^2\right) - \frac{\kappa}{16}y^2 + 4\Delta_s. \tag{20}$$

The expression under the square root is positive, if conditions

$$v(h_s,y) > 0, \qquad \kappa + 8g_1h_s < 0, \tag{21}$$

are fulfilled. The alternative case does not have solutions, since we assume that $\Delta_s > 0$ and $\kappa < 0$ (phenomenological requirements). Inequalities (21) hold with h_s belonging to the half-open interval $h_s^{\min} \leqslant h_s < h_s^{\max}$. Here $h_s^{\max} = -\kappa/(8g_1) > 0$, and $h_s^{\min} < 0$. The lower bound is a solution of the equation $v(h_s,y) = 0$. This cubic equation has only one real root which is negative for

$$g_1 + 2g_2 > 0, \qquad 8G + g_1y^2 > 0,$$
$$4\Delta_s - \frac{\kappa}{16}y^2 > 0. \tag{22}$$

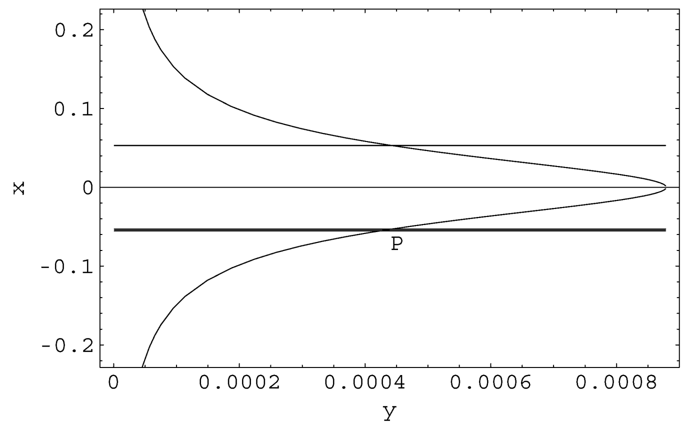

Fig. 1. The label x stands for the curves $x_{(i)}(h_s, y)$ plotted as function of y at fixed $h_s = -0.03$ GeV^3. The bell-shaped curve corresponds to the two branches of $x_{(2)}$, its peak is located at $y_{\max}(h_s) = 8.8 \times 10^{-4}$ GeV^3. The thick line indicates the negative branch of curve $x_{(3)}$ and also the curve $x_{(1)}$, which are degenerate at this scale, the upper line is the positive branch of $x_{(3)}$. The two branches of $x_{(3)}$ meet at $y \simeq 0.14$ GeV^3, outside the indicated range in the plot. The real solution of the system (13) is indicated by the point P. The corresponding parameters are $G = 6.1$ GeV^{-2}, $g_1 = g_2 = 4 \times 10^3$ GeV^{-8}, $\kappa = -705$ GeV^{-5}, $\Delta_u = 313$ MeV, $\Delta_d = 318$ MeV, $\Delta_s = 345$ MeV.

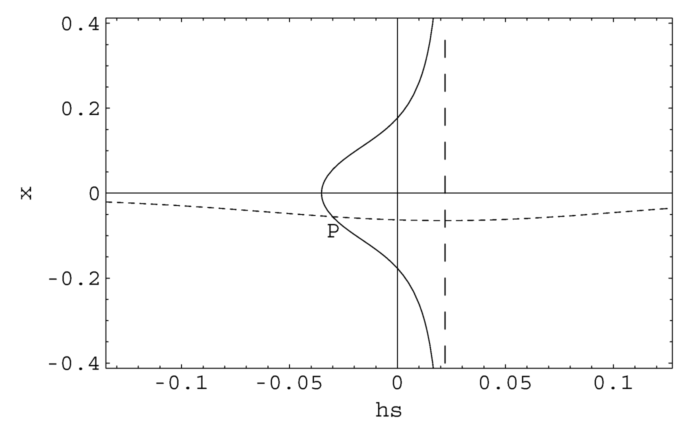

Fig. 2. The curves $x_{(1)}$ of Eq. (16) (small dashed line) and $x_{(3)}$ of Eq. (19) (solid line) are shown as functions of h_s for the parameter set of Fig. 1, at fixed $y = 4.2 \times 10^{-4}$ GeV^3. The solution P has $h_u = -0.02747$, $h_d = -0.02789$, $h_s = -0.03$ in units GeV^3. The vertical dashed line corresponds to $h_s = -\kappa/(8g_1)$.

Under the assumptions made above these inequalities are obviously fulfilled.

We illustrate the case in two figures. The y-dependence is shown in Fig. 1. Since $x_{(2)}$ is a monotonic function of y in the region $x < 0$, $0 < y < y_{\max}$ at any fixed value of h_s, the question whether the system (13) has one or more solutions is now reduced to a careful check of the number of intersections for curves $x_{(1)}$ and $x_{(3)}$ as functions of h_s at a fixed value of y. Actually, for this purpose one can choose any value of y from the interval $0 < y < y_{\max}$, because functions $x_{(1)}$ and $x_{(3)}$ are almost insensitive to this value.

In Fig. 2 we show $x_{(1)}(h_s, y)$ and $x_{(3)}(h_s, y)$ as functions of h_s, at fixed y given by the solution P of Fig. 1. It is quite easy to verify that the line $h_s = h_s^{\max}$, being the asymptote for the curve given by Eq. (19), crosses the other curve (16) in its minimum, dividing it in two monotonic parts. Thus, both functions decrease monotonically with increasing h_s in the third quadrant of the Cartesian coordinates system formed by the line $h_s = h_s^{\max}$ and the axis of abscissas. The curves have only one intersection, which corresponds to an unique solution of the system (13).

The fact that the cubic equations (13) have only one set of real roots over a certain range of values of parameters G, κ, g_1, g_2, Δ_i is crucial for the ground state of the theory: it makes the vacuum globally stable.[4]

Unfortunately, we merely can find the solution (h_u, h_d, h_s) numerically, apart from the simplest case with the octet flavour symmetry, where current quarks have equal masses $\hat{m}_u = \hat{m}_d = \hat{m}_s$, and the system (13) reduces to a cubic equation for only one variable $h_u = h_d = h_s$

$$h_u^3 + \frac{\kappa}{12\lambda} h_u^2 + \frac{4G}{3\lambda} h_u + \frac{4\Delta}{3\lambda} = 0, \tag{23}$$

with $\lambda = g_1 + (2/3)g_2$. Making the replacement $h_u = \bar{h}_u - \kappa/(36\lambda)$, one obtains from (23)

$$\bar{h}_u^3 + t' \bar{h}_u = b', \tag{24}$$

where

$$t' = \frac{4}{3}\left[\frac{G}{\lambda} - \left(\frac{\kappa}{24\lambda}\right)^2\right],$$
$$b' = \frac{4}{3}\left\{\frac{\kappa}{36\lambda}\left[\frac{G}{\lambda} - \frac{2}{3}\left(\frac{\kappa}{24\lambda}\right)^2\right] - \frac{\Delta}{\lambda}\right\}. \tag{25}$$

It is clear now that this cubic equation has one real root, if $t' > 0$, i.e.,

$$\frac{G}{\lambda} > \left(\frac{\kappa}{24\lambda}\right)^2. \tag{26}$$

In this particular case the proof of existence and uniqueness of the solution is straightforward. Let us also note that the solution found above for the general case deviates not much from the case with octet symmetry, i.e., we have approximately $h_u \simeq h_d \simeq h_s$.

4. Effective potential

Since the system of equations (10) can be solved, we are able to obtain the semi-classical asymptotics of the integral over s_a, p_a in (6). One has the following result which is valid at lowest order of the stationary phase approximation:

$$\mathcal{Z}[\sigma, \phi, \Delta] = \int_{-\infty}^{+\infty} \prod_a \mathcal{D}s_a \prod_a \mathcal{D}p_a \exp\left(i \int d^4x\, \mathcal{L}_r(\sigma, \phi, \Delta; s, p)\right)$$
$$\sim \mathcal{N} \sum_{j=1}^{n} \exp\left(i \int d^4x\, \mathcal{L}_r\left(\sigma, \phi, \Delta; s_{\rm st}^{(j)}, p_{\rm st}^{(j)}\right)\right) \quad (\hbar \to 0), \tag{27}$$

where n is the number of real solutions $(s_a^{\rm st}, p_a^{\rm st})^{(j)}$ of Eq. (10).

The information about the vacuum state is contained in the effective potential of the theory. To obtain it let us consider the linear term in the σ_a field. The resulting contribution, as it follows from Eq. (27), is

$$\mathcal{Z} \sim \exp\left(i \int d^4x \sum_{j=1}^{n} h_a^{(j)} \sigma_a + \cdots\right). \tag{28}$$

This part of the Lagrangian is responsible for the dynamical symmetry breaking in the multi-quark system and taken together with the corresponding part from the Gaussian integration over quark fields in Eq. (6) leads us to the gap equations (for each of quark's flavours $i = u, d, s$),

$$\sum_{j=1}^{n} h_i^{(j)} + \frac{N_c}{2\pi^2} m_i J_0\left(m_i^2\right) = 0, \tag{29}$$

where $J_0(m_i^2)$ is the tadpole quark loop contribution with a high-momentum cutoff Λ

$$J_0\left(m_i^2\right) = \Lambda^2 - m_i^2 \ln\left(1 + \frac{\Lambda^2}{m_i^2}\right). \tag{30}$$

Using standard techniques [25], we obtain from the gap-equations the effective potential $U(m_i)$ as a function of the constituent quark masses m_i which corresponds, in general, to the case with n real roots. Here it is more convenient to use (h_u, h_d, h_s) as independent variables, with masses m_i being determined by Eqs. (13). In particular, if the parameters of the model are fixed in such a way that Eqs. (13) have only one real solution, the effective potential (up to an unessential constant, which is omitted here) is

$$U(h_u, h_d, h_s) = \frac{1}{16}\left(4G h_i^2 + \kappa h_u h_d h_s + \frac{3g_1}{2}\left(h_i^2\right)^2 + 3g_2 h_i^4\right) - \frac{1}{2}\left(v\left(m_u^2\right) + v\left(m_d^2\right) + v\left(m_s^2\right)\right), \tag{31}$$

where $h_i^2 = h_u^2 + h_d^2 + h_s^2$, $h_i^4 = h_u^4 + h_d^4 + h_s^4$, and

$$v\left(m_i^2\right) = \frac{N_c}{8\pi^2}\left[m_i^2 J_0\left(m_i^2\right) + \Lambda^4 \ln\left(1 + \frac{m_i^2}{\Lambda^2}\right)\right]. \tag{32}$$

In the specific and limited case where one deals with the octet $SU(3)$ symmetric model and $\hat{m}_i = 0$ the effective potential $U(m)$ is an even function of m for $\kappa = 0$ and its plot has the standard form of the double well ("Mexican hat") with two symmetric minima, at $m = \pm m_{\min}$, and one local maximum, at $m = 0$. The 't Hooft interaction ($\kappa \neq 0$) makes this curve asymmetric: if $\kappa < 0$, the minimum located at positive values of m gets deeper as compared with the other minimum at negative m, becoming therefore the global minimum for the whole effective potential. It corresponds to the stable ground state of the system with spontaneously broken chiral symmetry.

[4] Let us recall that putting $g_1 = g_2 = 0$, one obtains from (13) the system of quadratic equations to find h_u, h_d, h_s. It has been shown in [16,25] that such equations have two real solutions (for a physical set of parameters) in the $SU(3)$ case and three real solutions in the $SU(2) \times U(1)$ case. This is exactly the underlying reason for the vacuum instability.

To appreciate the correlation found between the number of critical points and stability let us consider the same $SU(3)$ symmetric model in the range with three real roots. In this case

$$\sum_{j=1}^{3} h_u^{(j)} = -\frac{\kappa}{12\lambda}, \tag{33}$$

and we find

$$V(m) = \frac{\kappa}{8\lambda} m - \frac{3N_c}{16\pi^2}\left[m^2 J_0(m^2) + \Lambda^4 \ln\left(1 + \frac{m^2}{\Lambda^2}\right)\right]. \tag{34}$$

As opposed to $U(m)$ the potential with three real roots, described by the function $V(m)$ has at most a metastable vacuum, for $\kappa/\lambda > 0$. If $\kappa/\lambda < 0$, the effective potential does not have extrema in the region $m > 0$. In both cases the theory related with $V(m)$ is unbounded from below and is physically nonsensical.

Several special properties of the eight-quark interactions are expressed in these results.

Firstly, the couplings of the considered model can always be chosen to fulfill the inequalities required, i.e.,

$$g_1 > 0, \qquad g_1 + 3g_2 > 0, \qquad Gg_1 > (\kappa/16)^2, \tag{35}$$

and therefore the model can be simply driven to its stable regime where only one critical point determines the asymptotical dynamics of the system. The NJLH model does not have a proper mechanism for that.

Secondly, the eight-quark terms dominate at large values of h_i, making $U(h_i)$ positive in all directions $h_i \to \pm\infty$. As a result, the function $U(h_i)$ is bounded from below, and exhibits a global ground state.

Thirdly, the effective potential (31) coincides at $g_1, g_2 = 0$ with the potential obtained in the framework of the NJLH model by the mean-field method [13]. This is probably at the heart of the success of NJHL: although the limit $g_1, g_2 \to 0$ in $U(h_i)$ is formally not allowed as soon as inequality (15) does not hold (instead the system is then described by the unstable potential $V(h_i)$), the eight-quark terms in Eq. (31) are not likely to destroy the results of the mean field approach. Nevertheless, one should expect some new noticeable effects from it.

We must conclude that the eight-quark interactions play a fundamental role in the formation of the stable ground state for the unstable system described by the NJLH Lagrangian. We consider this finding as the main result of our study.

5. Summary and discussion

Let us summarize what we have found.

(1) An eight-quark extension of the conventional three-flavour NJL model with the explicit $U_A(1)$ breaking by the 't Hooft determinant has been suggested. We have taken the eight-quark interactions in its most general form for spin zero states. Eight-quark interactions prove to be essential in stabilizing the vacuum of the theory: the quark model considered follows the general trend of spontaneous breakdown of chiral symmetry, and possesses a globally stable ground state, when relevant inequalities in terms of the coupling constants hold.

(2) The so ensured stability of the ground state is crucial for applications of the model to the study of cases in which corrections (radiative, temperature, density, and so on effects) may qualitatively change the structure of the theory, e.g., by turning minima in the effective potential into maxima. Presently the $U(3)_L \times U(3)_R$ chiral symmetric NJL model with the six-quark 't Hooft interactions is frequently used for that. The eight-quark extension of the model considered here is needed for well-founded calculations in this field.

(3) The eight-quark interactions are an additional (to the 't Hooft determinant) source of OZI-violating effects. They are of the same order, for $g_1 \sim 1/N_c^4$. It is important to take them into account from the phenomenological point of view: the details of OZI-violation are still a puzzle of nonperturbative QCD [26].

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