ON IDENTITIES OF A TERNARY QUATERNION ALGEBRA

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This article studies a simple 4-dimensional ternary algebra $\mathcal{A}$ which appears analogously to the quaternions from the Lie algebra $\mathfrak{sl}(2)$. We describe the heights 1 and 2 identities, and the derivations of $\mathcal{A}$. Based on $\mathcal{A}$, some ternary enveloping algebras for ternary Filippov algebras are constructed.

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INTRODUCTION

A few years ago, the theory of n-Lie algebras attracted a lot of attention because of its close relation with the Nambu Mechanics proposed in Nambu [11]. This connection was revealed in Takhtajan [15], where those algebras appear under the name of Nambu–Lie algebras. Specifically, the notion of n-Lie algebra is the implicit algebraic concept underlying the Nambu Mechanics—the proposal of Nambu to obtain generalized Hamiltonian equations of the movement. The definition of n-Lie algebra ($n \geq 2$) was introduced in Filippov [6] as a natural generalization of the Lie algebra concept. Following Grabowski and Marmo [7] and Pojidaev [12], in memory of Filippov's remarkable work, we use the term Filippov algebra instead of n-Lie algebra.

Let us mention that Faulkner [5] discussed a class of triple systems that were called alternating triple systems. He encountered them while studying which trilinear identities are satisfied by a triple system such that its left multiplication operators are derivations and its derivation algebra acts irreducibly on it. By definition, a triple system is alternating if its multiplication is alternating and its left multiplication operators are derivations, properties that also characterize Lie algebras. So, one can require the same for an n-linear multiplication on a vector space to naturally generalize the concept of Lie algebra. This was done by Filippov, who achieved this generalization of the Jacobi identity.

In this article, we consider the ternary Filippov algebra A_1 equipped with a bilinear, symmetric, and nondegenerate form. We define, in Section 1, a new multiplication on the underlying vector space of A_1. The obtained algebra $\mathcal{A}$ is said to be a ternary quaternion algebra because it appears analogously to the construction of the quaternions from the Lie algebra $\mathfrak{sl}(2)$. Moreover, in Pojidaev [12], where a question on the existence of n-ary systems playing a role of enveloping algebras for n-Lie algebras was investigated, it was proved that $\mathcal{A}$ is an enveloping algebra for A_1. We notice that $\mathcal{A}$ may be defined on quaternions by $\{x, y, z\} \doteq x\bar{y}z$ (equality up to a coefficient). The ternary algebras with such product were considered by many authors, for example, Brown and Gray [3], Elduque [4], Kamiya and Okubo [8], Pojidaev [13], Shaw [14]. As against the cited articles, we investigate the identities of small level or height (where height refers to the number of operations in each term) in these ternary algebras.

Speaking finite dimensionally, it is possible to determine by hand identities of some heights valid in an algebra that has a small dimension. But when we deal with an algebra with a considerable dimension or we are looking for identities of high height, it is imperative to substitute the hand calculations by computational algebra. The articles of Bremner, Hentzel, and Peresi illustrate the applications of computational linear algebra to the study of identities for nonassociative algebras using the expansion matrix and the representation theory of the symmetric group $\mathcal{S}_n$. See, for instance, Bremner and Hentzel [1] and Bremner and Peresi [2].

Concerning the identities of height 1 valid in $\mathcal{A}$, we first calculate them by hand in Section 2. The problem arose while investigating the height 2 identities of $\mathcal{A}$ which imply the work with the permutations of $\mathcal{S}_5$. At this point, we decided to use a computational algebra software and Bremner's method, the expansion matrix. Concretely, as in Bremner and Hentzel [1], the information about the structure of the space of identities is given by the nullspace of that matrix, and this linear-algebraic data can be translated back into the identities we seek.

In the two following sections, where the previously obtained identities have a relevant function, we continue studying the properties of $\mathcal{A}$. In Section 3 we consider, namely, its simplicity and derivations. As far as Section 4, applying the deduced identities of $\mathcal{A}$, we construct some ternary enveloping algebras for ternary Filippov algebras.

The final section contains some open problems related to the algebra $\mathcal{A}$. On one hand, we make some remarks on higher level identities of $\mathcal{A}$ than the ones that we consider in this article. On the other hand, a few questions concerning possible generalizations of the presented work are raised.

In what follows, the symbol $:=$ denotes an equality by definition, Φ is a ground field, $\mathrm{ch}(\Phi)$ is the characteristic of Φ ($\mathrm{ch}(\Phi) \neq 2$) and $\langle \Upsilon \rangle$ is the linear span of the set Υ over Φ.

1. PRELIMINARIES

Given a vector space U over Φ, U is an *Ω-algebra* over Φ if Ω is a system of multilinear algebraic operations defined on U:

$$\Omega = \{w_i : |w_i| = n_i \in \mathbb{N}, i \in I\},$$

where $|w_i|$ denotes the arity of w_i. In particular, we say that U is a *triple system* (or a *ternary algebra*) over Φ if U is equipped with a trilinear map $p : U^3 \to U$, i.e., $\Omega = \{w\}$ with $|w| = 3$. We omit the arity whenever it is clear from the context.

Given a ternary algebra U with multiplication $(\cdot,\cdot,\cdot)$, $U^{(-)}$ denotes the *commutator algebra of U*, that is, the ternary algebra with multiplication $(\cdot,\cdot,\cdot)_c$ given by

$$(x_1, x_2, x_3)_c := \sum_{\sigma\in\mathcal{S}_3} sgn(\sigma)(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}),$$

the ternary version of the generalized commutator.

Let L be an Ω-algebra over Φ equipped with a single n-ary operation $[\cdot,\ldots,\cdot]$; L is a *Filippov n-algebra* (or *n-ary Filippov algebra* or *n-Lie algebra*, $n \geq 2$) over Φ if, for all $x_1,\ldots,x_n, y_2,\ldots,y_n \in L$ and $\sigma \in \mathcal{S}_n$:

$$[x_1,\ldots,x_n] = sgn(\sigma)[x_{\sigma(1)},\ldots,x_{\sigma(n)}], \tag{1}$$

$$[[x_1,\ldots,x_n], y_2,\ldots,y_n] = \sum_{i=1}^{n}[x_1,\ldots,[x_i, y_2,\ldots,y_n],\ldots,x_n]. \tag{2}$$

If (1) holds, then $[\cdot,\ldots,\cdot]$ is said to be *anticommutative*; (2) is called the *generalized Jacobi identity*.

The following example of an $(n+1)$-dimensional n-Lie algebra, an analogue of the 3-dimensional Lie algebra with the cross product as multiplication, appears among the first ones given by Filippov [6].

Example 1.1. Let L be an $(n+1)$-dimensional Euclidean vector space over $\mathbb{R}$ equipped with the multiplication $[\cdot,\ldots,\cdot]$, which is the vector product of n elements in L, $n \geq 2$. If $x_1,\ldots,x_n \in L$ and $\mathcal{E}_{n+1} = \{e_1,\ldots,e_{n+1}\}$ is an orthonormal basis of L, then we have:

$$[x_1,\ldots,x_n] = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} & e_1 \\ x_{21} & x_{22} & \cdots & x_{2n} & e_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{(n+1)1} & x_{(n+1)2} & \cdots & x_{(n+1)n} & e_{n+1} \end{vmatrix},$$

where $x_{1i},\ldots,x_{(n+1)i}$ are the coordinates of x_i. The vector product is completely determined by the rule

$$[e_1,\ldots,e_{i-1},\widehat{e_i},e_{i+1},\ldots,e_{n+1}] = (-1)^{n+1+i}e_i, \tag{3}$$

$i \in \{1, \ldots, n+1\}$, where the symbol $\widehat{e_i}$ means that e_i is omitted. The remaining products of the basis vectors are either zero or obtained from (3) and (1). The basis $\mathscr{E}_{n+1}$ for which the product is written in the form (3) is said to be canonical. Filippov [6], showed that this Ω-algebra is an n-Lie algebra. Following Pojidaev [12], we shall denote it by A_1, clarifying its dimension whenever it may not be clear.

Let L be a Filippov algebra. A subspace I of L is an *ideal* of L provided that $[I, L, \ldots, L] \subseteq I$. If $[L, \ldots, L] \neq \{0\}$ and L lacks ideals other than $\{0\}$ and L, then we say that L is *simple*.

In Ling [9], it was proved that A_1 is the only $(n+1)$-dimensional simple Filippov n-algebra over an algebraically closed field of characteristic zero, up to isomorphism.

Consider the 4-dimensional ternary Filippov algebra A_1 over Φ equipped with a bilinear, symmetric, and nondegenerate form $(\cdot, \cdot)$, and the canonical basis $\mathscr{E} = \{e_1, e_2, e_3, e_4\}$. We define a new multiplication on the underlying vector space of A_1 in the following way:

$$\{x, y, z\} := -(y, z)x + (x, z)y - (x, y)z + [x, y, z]. \tag{4}$$

We denote the obtained algebra by $\mathscr{A}$, differing the multiplication (4) from that which appears in Pojidaev [12] by a scalar. Note that $\mathbb{R} \cdot 1 \oplus \mathfrak{sl}(2)$ with the product

$$(\alpha + x)(\beta + y) = \alpha\beta - (x, y) + \alpha y + \beta x + [x, y]$$

is isomorphic to the quaternions. This is the main reason for investigating $\mathscr{A}$ as a ternary associative algebra, which is an enveloping algebra for A_1.

Let us emphasize that, in (4), $[\cdot, \cdot, \cdot]$ is a ternary vector cross product on the underlying vector space of A_1 equipped with $(\cdot, \cdot)$. According to Brown and Gray [3], recall that this means that $[\cdot, \cdot, \cdot]$ is a trilinear map from $\mathscr{A}^3$ to $\mathscr{A}$ satisfying

$$([a_1, a_2, a_3], a_i) = 0, \tag{5}$$

$$([a_1, a_2, a_3], [a_1, a_2, a_3]) = \det[(a_i, a_j)], \tag{6}$$

for all $a_1, a_2, a_3 \in \mathscr{A}$ and $i, j \in \{1, 2, 3\}$. Notice that the skewsymmetry of $[\cdot, \cdot, \cdot]$ is an immediate consequence of (6); $([\cdot, \cdot, \cdot], \cdot)$ is skewsymmetric in its arguments by the skewsymmetry of $[\cdot, \cdot, \cdot]$ and by (5).

Let C be a 4-dimensional composition algebra over a field F of characteristic $\neq 2$. Then C is a simple Filippov algebra with respect to the product

$$[x, y, z] = x\bar{y}z - (y, z)x + (x, z)y - (x, y)z,$$

Pojidaev [13]. Applying $x\bar{y} + y\bar{x} = 2(x, y) = \bar{x}y + \bar{y}x$, we see that $\{x, y, z\} \doteq x\bar{y}z$. Thus, we may use the last equality as an equivalent definition of $\{x, y, z\}$. We use both definitions in this work.

We start the study of $\mathscr{A}$ deducing its identities of heights 1 and 2 (1-identities and 2-identities, respectively). These identities will be mainly found using the method of the expansion matrix. More details on this process can be found in Bremner and Hentzel [1] and Bremner and Peresi [2].

2. IDENTITIES OF $\mathcal{A}$

Level 1. The first step is the seeking of the 1-identities valid in $\mathcal{A}$, i.e., we search for the identities of type:

$$\sum_{\sigma\in\mathcal{S}_3} \alpha_\sigma\{x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\} = 0, \qquad \alpha_\sigma \in \Phi. \tag{7}$$

Theorem 2.1. *All 1-identities for $\mathcal{A}$ follow from*

$$\{b, a, a\} = \{a, a, b\}. \tag{8}$$

Proof. In this demonstration, we use a manual approach. Considering some valuations for x_i in $\mathcal{E}$, from (7), we obtain a system of linear equations on α_σ. The resolution of this system gives that the 1-identities of $\mathcal{A}$ are of the shape

$$\alpha\{a, b, c\} + (\alpha - \beta)\{a, c, b\} + \beta\{b, a, c\} + (\beta - \alpha)\{b, c, a\} - \beta\{c, a, b\} - \alpha\{c, b, a\} = 0,$$

where, for the sake of simplicity, the scalars were denoted by α and β. Linearizing $\{b, a, a\} = \{a, a, b\}$, we see that all 1-identities of $\mathcal{A}$ are implied by (8). □

Observe that, despite (8), the ternary algebra $\mathcal{A}$ is not a *quadratic triple system* (as defined in Kamiya and Okubo [8]), since $\{a, a, b\} = (a, a)b$ does not hold in $\mathcal{A}$. By the same reason, $(\mathcal{A}, (\cdot,\cdot), \{\cdot,\cdot,\cdot\})$ is not a *ternary composition algebra* (see the definition in Elduque [4] and Shaw [14]).

Corollary 2.2. *The following identity holds in $\mathcal{A}$:*

$$\{a, b, c\} + \{a, c, b\} - \{b, c, a\} - \{c, b, a\} = 0. \tag{9}$$

Proof. This is just the linearization of (8). □

In order to confirm the conclusions obtained for the 1-identities of $\mathcal{A}$, we now use, but with $\mathrm{ch}(\Phi) = 0$, the expansion matrix method. On the other hand, it is an opportunity to recall the mentioned process, applying it to a smaller problem than the one that will be considered for height 2.

Second Proof of Theorem 2.1. The $\{\mathcal{S}_3\}$-*monomials* (the terms arising from the action of the six permutations in $\mathcal{S}_3$ over the arguments of $\{a, b, c\}$) can be seen as elements of the $\mathcal{S}_3$-module generated by the *basic monomials* $(b, c)a$, $(a, c)b$, $(a, b)c$ and $[a, b, c]$. Note that, for all $\sigma \in \mathcal{S}_3$, we have $[\sigma(a), \sigma(b), \sigma(c)] = sgn(\sigma)[a, b, c]$ and $(\sigma(a), \sigma(b))\sigma(c) \in \langle (b, c)a, (a, c)b, (a, b)c\rangle$.

Denote by $\mathcal{X}$ the expansion matrix of $\{\cdot,\cdot,\cdot\}$ in height 1. Its columns are given by the expansion of the $\{\mathcal{S}_3\}$-monomials as a linear combination of the basic monomials. So, $\mathcal{X}$ is a 4×6 matrix whose entry x_{ij} is the coefficient of the ith basic monomial in the expansion of the jth $\{\mathcal{S}_3\}$-monomial. According to Bremner and Hentzel [1], the 1-identities are given by the nullspace of this matrix. Then, if we choose the lexicographical order for the permutations of the arguments of $\{a, b, c\}$, we have

$$\mathcal{X} = \begin{bmatrix} -1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

A basis for the nullspace of $\mathcal{X}$ is $\{(1, 1, 0, -1, 0, -1), (0, -1, 1, 1, -1, 0)\}$. The basis vectors represent the following 1-identities of $\mathcal{A}$

$$\{a, b, c\} + \{a, c, b\} - \{b, c, a\} - \{c, b, a\} = 0,$$
$$\{a, c, b\} - \{b, a, c\} - \{b, c, a\} + \{c, a, b\} = 0.$$

It is clear that the second identity can be obtained by the action of the transposition $(a\ b) \in \mathcal{S}_3$ over the first one. So, the first identity generates the whole space of 1-identities of $\mathcal{A}$ under the action of $\mathcal{S}_3$. □

Level 2. We now seek the 2-identities of $\mathcal{A}$

$$\sum_{\sigma\in\mathcal{S}_5} \big(\alpha_\sigma\{\{x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\}, x_{\sigma(4)}, x_{\sigma(5)}\} + \beta_\sigma\{x_{\sigma(1)}, \{x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}\}, x_{\sigma(5)}\}$$
$$+\gamma_\sigma\{x_{\sigma(1)}, x_{\sigma(2)}, \{x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)}\}\}\big) = 0, \qquad \alpha_\sigma, \beta_\sigma, \gamma_\sigma \in \Phi. \tag{10}$$

It is always possible to obtain identities of a certain height d ($d > 1$) from identities of height $d - 1$, procedure which appears in Bremner and Peresi [2] under the name of *lift*. The two possible ways to do it, obtaining *liftings*, are the following: replacing one variable by a triple, or embedding the identity in a triple. But the interesting part related to the problem of searching 2-identities of $\mathcal{A}$ is to find the new ones, that is, those that cannot be obtained, in any manner, from the ones of the previous height. Next results give two of them, as justified in Remark 2.5.

Proposition 2.3 (Pojidaev [12]). *In $\mathcal{A}$ the following identity holds:*

$$\{\{a, b, c\}, d, e\} = \{a, b, \{c, d, e\}\}. \tag{11}$$

Proof. As we proved before, $\{x, y, z\} \doteq x\bar{y}z$. Now, (11) follows immediately. □

According to the terminology in Kamiya and Okubo [8], and since (11) holds in $\mathcal{A}$, we may conclude that this ternary algebra is an *associative triple system*. We presented an easier way to prove this fact, via quaternions, than the one indicated in Pojidaev [12].

Proposition 2.4. *The following identity holds in $\mathcal{A}$:*

$$\{\{a, b, c\}, d, e\} = \{a, \{d, c, b\}, e\}. \tag{12}$$

Proof. We have $\{\{a, b, c\}, d, e\} = a\bar{b}c\bar{d}e = \{a, \{d, c, b\}, e\}$. □

Notice that, by Loos [10], (11) and (12) are the defining identities for an *associative triple system of the second kind.*

Remark 2.5. We constructed a 600×360 matrix in GAP to store all height 2 liftings of (9) under the action of $\mathcal{S}_5$. As (11) and (12) do not belong to the row space of this matrix, they won't belong to the $\mathcal{S}_5$-module generated by the liftings of (9). Therefore, the mentioned identities are not consequences of (8). Moreover, we can conclude that (11) is not a consequence of (12) through a similar reasoning.

Returning to our purpose, with $\mathrm{ch}(\Phi) = 0$, all identities of height 2 can be found using the expansion matrix method. The calculations, in the context of linear algebra with large matrices, were made using GAP. See The GAP Team [16] for more details on this system for computational algebra.

Theorem 2.6. *The identities* (8), (11), *and* (12) *imply all* 2*-identities of* $\mathcal{A}$.

Proof. As considered in (10), for the operation $\{\cdot, \cdot, \cdot\}$, in height two, we have three association types represented by the $\mathcal{S}_5$-monomials: $\{\{a, b, c\}, d, e\}$, $\{a, \{b, c, d\}, e\}$, $\{a, b, \{c, d, e\}\}$. By Propositions 2.3 and 2.4, we only have to consider the first type for the construction of the expansion matrix.

We have four types of basic monomials that we describe and count below taking into account the symmetry of $(\cdot, \cdot)$, the skewsymmetry of $[\cdot, \cdot, \cdot]$, the generalized Jacobi identity and the skewsymmetry of $([\cdot, \cdot, \cdot], \cdot)$.

(a) The monomials of type $(a, b)(c, d)e$; we have 15 monomials of this type.
(b) The monomials of type $([a, b, c], d)e$; we have 5 of them.
(c) The monomials of type $(a, b)[c, d, e]$; there are 10 of these monomials.
(d) Finally, the monomials of type $[[a, b, c], d, e]$; we have 6 of them.

Thus, we have 36 basic monomials and the expansion matrix $\mathcal{X}$, in height 2, has size 36×120. The 120 columns of $\mathcal{X}$ are labeled by the $\{\mathcal{S}_5\}$-monomials of the first association type. This matrix has rank 35, so its nullspace $\mathcal{N}$ has dimension 85.

Now, we obtain a basis of $\mathcal{N}$ with GAP. One of the basis vectors represents the following 2-identity of $\mathcal{A}$:

$$\begin{aligned}&\{\{b, d, c\}, a, e\} + \{\{a, b, e\}, d, c\} - \{\{e, b, a\}, d, c\}\\ &\quad - \{\{b, e, a\}, d, c\} - \{\{e, a, b\}, d, c\} + \{\{b, d, c\}, e, a\} = 0. \end{aligned} \tag{13}$$

This identity generates, under the action of $\mathcal{S}_5$, a subspace of dimension 85, that is, the whole space $\mathcal{N}$. So, (11), (12), and (13) imply all 2-identities of $\mathcal{A}$.

From (9), we have

$$\{\{a, b, e\}, d, c\} + \{\{a, e, b\}, d, c\} - \{\{e, b, a\}, d, c\} - \{\{b, e, a\}, d, c\} = 0, \tag{14}$$

$$\{\{b, d, c\}, a, e\} + \{\{b, d, c\}, e, a\} - \{e, a, \{b, d, c\}\} - \{a, e, \{b, d, c\}\} = 0. \tag{15}$$

Applying (11) in (15), we obtain

$$\{\{b, d, c\}, a, e\} + \{\{b, d, c\}, e, a\} - \{\{e, a, b\}, d, c\} - \{\{a, e, b\}, d, c\} = 0. \qquad (16)$$

Thus, adding member to member (14) and (16), we arrive at (13). Therefore, we conclude that (13) is a consequence of (8) and (11). □

3. SOME PROPERTIES OF $\mathcal{A}$

Simplicity. Fix $a_1, a_2 \in \mathcal{A}$. For the operation $\{\cdot, \dots, \cdot\}$, the right, left and outer multiplication operators R_{a_1,a_2}, L_{a_1,a_2}, and M_{a_1,a_2}, respectively, are the linear mappings from $\mathcal{A}$ to $\mathcal{A}$ defined in the following way:

$$R_{a_1,a_2} : x \mapsto \{x, a_1, a_2\}, \qquad L_{a_1,a_2} : x \mapsto \{a_1, a_2, x\}, \qquad M_{a_1,a_2} : x \mapsto \{a_1, x, a_2\}.$$

Recall that the associative algebra $\mathcal{A}^*$, called the *multiplication algebra of* $\mathcal{A}$, is generated by the previous operators. $\mathcal{A}^*$ is a subalgebra of the associative algebra of linear endomorphisms of the underlying vector space of the ternary algebra $\mathcal{A}$. A subalgebra I is an *ideal* of $\mathcal{A}$ provided that $\{I, \mathcal{A}, \mathcal{A}\}, \{\mathcal{A}, I, \mathcal{A}\}, \{\mathcal{A}, \mathcal{A}, I\} \subseteq I$; $\mathcal{A}$ is *simple* if and only if $\{\mathcal{A}, \mathcal{A}, \mathcal{A}\} \neq \{0\}$, and the only ideals of $\mathcal{A}$ are $\{0\}$ and $\mathcal{A}$.

We can give the definition of ideal for $\mathcal{A}$ in terms of the multiplication algebra of $\mathcal{A}$. Concretely, I is an *ideal* of $\mathcal{A}$ if I is a subspace of $\mathcal{A}$ that is invariant under the multiplication algebra $\mathcal{A}^*$. Note that $\mathcal{A}$ is *simple* if and only if $\mathcal{A}^*$ is an irreducible algebra of linear transformations.

Denote the right, left, and outer multiplication algebras generated by the right, left, and outer operators by $\mathcal{A}^*_r$, $\mathcal{A}^*_l$ and $\mathcal{A}^*_m$, respectively; the number of inversions in the 3-tuple (i, j, k) by $inv(i, j, k)$.

Lemma 3.1. *Let $F \in \{L, R\}$. If i, j, k, l are different elements in $\{1, 2, 3, 4\}$ then we have*

$$Id = -F_{e_i,e_i}; \qquad F_{e_k,e_l} = (-1)^{inv(i,j,k)+l+\gamma} F_{e_i,e_j}, \qquad \gamma = \begin{cases} 0 & \text{if } F = L \\ 1 & \text{if } F = R \end{cases}$$

$$F_{e_i,e_j} = -F_{e_j,e_i}; \quad F_{e_i,e_j} F_{e_i,e_k} = (-1)^{i+j+k+inv(i,j,k)} F_{e_i,e_l}.$$

Proof. For example, let us prove that $R_{e_i,e_j} R_{e_i,e_k} = (-1)^{i+j+k+inv(i,j,k)} R_{e_i,e_l}$. By (11) and (4), we have

$$\{\{x, e_i, e_k\}, e_i, e_j\} = \{x, e_i, \{e_k, e_i, e_j\}\} = (-1)^{i+j+k+inv(i,j,k)} \{x, e_i, e_l\}.$$

Other equalities may be proved analogously. □

By direct computation, $\mathcal{A}^* \cong M_4(\Phi)$; therefore, we have the following theorem.

Theorem 3.2. *The algebra $\mathcal{A}$ is simple.*

Denote the Lie algebra generated by the multiplication operators (right, left, and outer) by $\mathcal{A}_L$.

Theorem 3.3. $\mathcal{A}_L \cong \mathfrak{sl}(4)$.

Proof. We can see that for $i, j \in \{1, 2, 3, 4\}$ and $i \neq j$, $e_{ij} \in [\mathcal{A}_m^*, \mathcal{A}_m^*]$ and, for $i = 1, 2, 3$, $e_{ii} - e_{i+1,i+1} \in [\mathcal{A}_r^*, \mathcal{A}_m^*]$. □

Derivations. Recall that a linear mapping $D : \mathcal{A} \to \mathcal{A}$ such that, for every $a, b, c \in \mathcal{A}$,

$$D(\{a, b, c\}) = \{D(a), b, c\} + \{a, D(b), c\} + \{a, b, D(c)\} \tag{17}$$

is called a *derivation* of $\mathcal{A}$. We denote by $Der(\mathcal{A})$ the derivation Lie algebra of $\mathcal{A}$. In this subsection, we use the obtained identities of $\mathcal{A}$ to describe $Der(\mathcal{A})$.

Lemma 3.4. *In $\mathcal{A}$ the following identity holds:*

$$\{c, \{a, b, d\}, e\} - \{\{c, a, b\}, d, e\} = \{c, \{d, a, b\}, e\} - \{\{c, d, a\}, b, e\}. \tag{18}$$

Proof. By (12), we have

$$\{c, \{d, b, a\}, e\} - \{\{c, a, b\}, d, e\} = \{c, \{b, a, d\}, e\} - \{\{c, d, a\}, b, e\}.$$

To finish the proof, using (9), we only have to notice that

$$\{c, \{d, b, a\}, e\} - \{c, \{b, a, d\}, e\} = \{c, \{a, b, d\}, e\} - \{c, \{d, a, b\}, e\}.$$

□

Let $x, y \in \mathcal{A}$ fixed, and

$$D_{x,y} := R_{x,y} - L_{x,y}. \tag{19}$$

Proposition 3.5. *For every fixed $x, y \in \mathcal{A}$, $D_{x,y} \in Der(\mathcal{A})$.*

Proof. Take $u, v, w \in \mathcal{A}$. We have

$$D_{x,y}(\{u, v, w\}) = \{\{u, v, w\}, x, y\} - \{x, y, \{u, v, w\}\}.$$

On the other hand,

$$\begin{aligned}\{D_{x,y}(u), v, w\} &+ \{u, D_{x,y}(v), w\} + \{u, v, D_{x,y}(w)\}\\ &= \{\{u, x, y\}, v, w\} - \{\{x, y, u\}, v, w\} + \{u, \{v, x, y\}, w\}\\ &\quad - \{u, \{x, y, v\}, w\} + \{u, v, \{w, x, y\}\} - \{u, v, \{x, y, w\}\}.\end{aligned}$$

Therefore, by (11) and (18), we obtain

$$\begin{aligned}&D_{x,y}(\{u, v, w\}) - \{D_{x,y}(u), v, w\} - \{u, D_{x,y}(v), w\} - \{u, v, D_{x,y}(w)\}\\ &\quad = -\{\{u, x, y\}, v, w\} - \{u, \{v, x, y\}, w\} + \{u, \{x, y, v\}, w\} + \{u, v, \{x, y, w\}\} = 0.\end{aligned}$$

□

Theorem 3.6. *$Der(\mathcal{A}) \cong \mathfrak{o}(4)$.*

Proof. $D \in Der(\mathcal{A}) \Leftrightarrow$ the matrix $[D]_{\mathscr{E}}$ is skewsymmetric. $\square$

Previously, we pointed out that $(\mathcal{A}, (\cdot,\cdot), \{\cdot,\cdot,\cdot\})$ is not a ternary composition algebra. Even so, the above result coincides with the one obtained in Elduque [4], for the derivations of a ternary composition algebra. In the mentioned article, from any composition algebra of dimension 4 or 8, the properties of the associated triple products are deduced from properties of the composition algebras. Our approach for the algebra $\mathcal{A}$ is different, and we reach many of the properties applying the determined heights 1 and 2 identities.

Corollary 3.7. $D \in Der(\mathcal{A}) \Leftrightarrow D \in \langle D_{x,y} : x, y \in \mathcal{A}\rangle$.

Proof. We only have to notice that $R_{e_i,e_j} - L_{e_i,e_j} = 2(e_{ij} - e_{ji})$. $\square$

Recall that a derivation is called *inner* if it belongs to the Lie algebra $\mathcal{A}_L$.

Corollary 3.8. *All derivations of $\mathcal{A}$ are inner.*

4. ENVELOPES FOR TERNARY FILIPPOV ALGEBRAS

We see that the ternary algebra $\mathcal{A}$ satisfies a lot of interesting identities which are similar to those of associative algebras, namely, (11). Also, notice that (8) looks like a weak commutativity of $\mathcal{A}$.

In what follows, we assume that $ch(\Phi) \neq 2, 3$. Let $\mathscr{V}A$ be the variety of ternary algebras over Φ given by the following identities (of $\mathcal{A}$):

$$(a, b, b) = (b, b, a), \tag{20}$$

$$((a, b, c), d, e) = (a, b, (c, d, e)), \tag{21}$$

$$((a, b, c), d, e) = (a, (d, c, b), e). \tag{22}$$

Theorem 4.1. *Given $\mathscr{B} \in \mathscr{V}A$, $\mathscr{B}^{(-)}$ is a ternary Filippov algebra.*

Proof. Equip $\mathscr{B}$ with the ternary commutator

$$(x_1, x_2, x_3)_c = \sum_{\sigma \in \mathscr{S}_3} sgn(\sigma)(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}).$$

By construction, $(\cdot,\cdot,\cdot)_c$ is skewsymmetric. Applying the linearized form of (20), we have

$$(a, b, c)_c = 3((a, b, c) - (c, b, a)).$$

Using (21), the generalized Jacobi identity holds in $\mathscr{B}^{(-)}$ if and only if

$$((a, d, e), b, c) + (c, b, (e, d, a)) + (a, (b, d, e), c) - (c, (b, d, e), a) \\ - (a, (e, d, b), c) + (c, (e, d, b), a) - ((c, d, e), b, a) - (a, b, (e, d, c)) = 0.$$

From the linearized form of (20) and by (21), we can write

$$((a, d, e), b, c) + ((c, b, e), d, a) - (a, (b, e, d), c) + (a, (d, e, b), c)$$
$$+ (c, (b, e, d), a) - (c, (d, e, b), a) - ((c, d, e), b, a) - ((a, b, e), d, c) = 0.$$

Thus, invoking (22), we conclude that the generalized Jacobi identity holds in $\mathcal{B}^{(-)}$. Therefore, $\mathcal{B}^{(-)}$ is a ternary Filippov algebra. □

5. FINAL REMARKS

In this article, we studied a ternary analogue $\mathcal{A}$ of the quaternion algebra. Namely, we have considered the 1-identities and the 2-identities valid in $\mathcal{A}$.

Observe that one could also consider the trilinear operations defined by

$$a(y, z)x + b(x, z)y + c(x, y)z + [x, y, z], \tag{23}$$

where a, b, c are some bilinear symmetric forms. These algebras are enveloping algebras for A_1 as well. One of the open problems is to study the polynomial identities satisfied by the family of operations defined in (23). Specifically, to know how those identities depend on a, b, c. Our research focused on the particular case where $a = c = -1$ and $b = 1$ due to already pointed reasons.

A very interesting problem arises from the fact that the 2-identities of $\mathcal{A}$ are just the familiar identities for associative triple systems. Concretely, a natural question is to know whether or not there are any new identities of height 3. Since the height 2 identities of $\mathcal{A}$ imply ternary associativity, we just have to consider one association type in height 3, for instance,

$$\{\{\{,,\},,\},,\}.$$

Therefore, the size of the expansion matrix in the mentioned height may be reduced in order to make the computation feasible. Depending on the chosen programme to implement the method, three problems might appear: time and space requirements of exact rational arithmetic imposing limits on the height of the identities that can be studied; arbitrarily large integers may not be available; in the last case, all calculations should be done in modular arithmetic with respect to a suitable prime p, holding the results only in characteristic p.

As we pointed out, the algebra $\mathcal{A}$ appears analogously to the quaternions from the Lie algebra $\mathfrak{sl}(2)$. This raises a relevant question: Is there some generalization of the ternary operation defined in (4) that would produce an n-ary associative algebra from the simple Filippov n-algebra of dimension $n+1$?

It is very interesting to describe the polynomial identities of height ≤ 3 of the ternary algebra defined on the octonions by $(x\bar{y})z$. Note that the operation

$$[x, y, z] := (x\bar{y})z - (y, z)x + (x, z)y - (x, y)z$$

equips the octonions with the structure of a ternary Malcev algebra (see Pojidaev [13]). Thus, one may expect that the study of the identities for the ternary product $(x\bar{y})z$ on the octonions leads to an interesting class of ternary alternative algebras.

Finally, we highlight the obtained envelopes for ternary Filippov algebras since, until now, no good class of algebras has been found playing a role of the universal enveloping algebras for Filippov algebras.

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