REDUCED n-LIE ALGEBRAS

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INTRODUCTION

One of the interesting properties of n-Lie and n-ary Malcev algebras [5] is the possibility of obtaining other algebras of the same class, but being the arity of the new operations reduced in one unity (which suggests the name of *reduced algebras* for the latter ones). More generally, if L is a given Ω-algebra over a field F - that is, L is a vector space over F with a system of multilinear algebraic operations

$$\Omega = \{\omega_i : |\omega_i| = n_i \in N,\ i \in I\},$$

where $|\omega_i|$ denotes the arity of ω_i - and $a \in L$ is fixed, L can be equipped with a system of new multilinear algebraic operations

$$\Omega^a = \{\omega^a : \omega \in \Omega\},$$

where

$$\omega^a(x_1, \ldots, x_{n-1}) = \omega(a, x_1, \ldots, x_{n-1})$$

and $|\omega^a| = |\omega| - 1 = n - 1$. Then, L turns into an Ω^a-algebra, denoted by L_a and called a *reduced algebra of L*. If an Ω-algebra L is anticommutative (*i.e.*,

ω is anticommutative for all $\omega \in \Omega$), then its center is nonzero. In this case, we will call reduced algebra just to the quotient algebra $\overline{L_a} = L_a/Z(L_a)$ and not to L_a.

In the present article, we study the reduced algebras of n-Lie algebras. Recall that an *n-Lie algebra* L is an Ω-algebra with one n-linear operation $[\cdot, \dots, \cdot]$ satisfying the identities:

$$[x_1, \dots, x_n] = \operatorname{sgn}(\sigma)[x_{\sigma(1)}, \dots, x_{\sigma(n)}]$$

$$[[x_1, \dots, x_n], y_2, \dots, y_n] = \sum_{i=1}^{n} [x_1, \dots, [x_i, y_2, \dots, y_n], \dots, x_n],$$

where σ is a permutation of the symmetric group S_n and $\operatorname{sgn}(\sigma)$ stands for the sign of σ. If L is an n-Lie algebra ($n \geq 3$) then, by [1], L_a is an $(n-1)$-Lie algebra for any $a \in L$, with multiplication defined by

$$[x_1, \dots, x_{n-1}]_a = [a, x_1, \dots, x_{n-1}]. \tag{1}$$

To simplify notations, each reduced $(n-1)$-Lie algebra $\overline{L_a}$ will be identified with L_a.

It is important to relate the structure of the original algebra with the one which corresponds to its reduced algebras. In the first section we recall the definitions of k-solvable and k-nilpotent ideal of an n-Lie algebra L, and show that if L is k-solvable (k-nilpotent), with $k < n$, then for any element a of an arbitrary basis of L, its reduced algebra L_a is k-solvable (k-nilpotent) too. The special case $k = n$ is also analyzed in both situations. Further, we establish the relation between the k-radical (2-nilradical) of L and the k-radical (2-nilradical) of each L_a.

It is known that any 3-Lie algebra can be considered as a vector space equipped with a system of binary operations such that it becomes a Lie algebra with respect to each operation, and these are related to each other. In the second section we consider the reciprocal problem, presenting a procedure of obtaining a 3-Lie algebra starting from a family of Lie algebras and illustrating with a concrete example how this can be done. Another interesting result, included in this section, states that the reduced algebras of an n-Lie algebra of type A_1 (as defined in [1]) which are obtained by fixing the elements of its canonical basis, are isomorphic to an $(n-1)$-Lie algebra of the same type, under the condition of the ground field being algebraically closed with characteristic zero. However, this doesn't hold for any reduced algebra of A_1.

Studying the reduced algebras can be a source of new examples of simple and semisimple algebras. In the last two sections we present the main results of this article, concerning two classes of simple n-Lie algebras introduced in [4]: $A(n, t)$ and $E(n, t)$. We study its reduced algebras which arise from fixing the elements of the respective canonical basis. In the case of a characteristic zero field, we prove that those reduced algebras of $A(n, t)$ and $E(n, t)$ are new examples of semisimple n-Lie algebras. In the third section we compute the radical of any of those reduced algebras of $A(n, t)$ in the case of a modular field and prove that the quotient of this algebra by its radical is isomorphic to $A(n-1, t')$, for some t'. In the last section we prove that, when $n \equiv 0 \pmod p$ any of those reduced algebras of $E(n, t)$ is simple, while when $n \not\equiv 0 \pmod p$ its square is simple. In both cases, the obtained reduced simple algebras are $(p^{k(n-1)} - 2)$-dimensional, where $p = \operatorname{char} F$ and $p^k = |F|$. It remains an open question to say if these are isomorphic to any known simple n-Lie algebra (namely, the possibility of being isomorphic to $A(n-1, t_1)$, for some t_1).

1 k-SOLVABILITY AND k-NILPOTENCE OF REDUCED $(n-1)$-LIE ALGEBRAS

The purpose of this section is to study the relations between n-Lie algebras and its reduced $(n-1)$-Lie algebras in which concerns k-solvability and k-nilpotence. These notions were introduced in [2] and we recall them below.

In what follows, L denotes an n-Lie algebra over a field F of arbitrary characteristic and, for a fixed element a of a given basis of L, L_a is a reduced $(n-1)$-Lie algebra of L. Further, if I is an ideal of L, it is clear that it is also an ideal of L_a. Indeed, we have

$$[I, L_a, \ldots, L_a]_a = [I, L, \ldots, L]_a = [a, I, L, \ldots, L] \subseteq [I, L, \ldots, L] \subseteq I. \quad (2)$$

If we want to precise that I is being considered as an ideal of L_a we denote it by I_a.

Given an ideal I of L and a fixed $k \in \{2, \ldots, n\}$, let $I^{(s,k)}$ denote the descending sequence of ideals of L, defined by:

$$\begin{cases} I^{(0,k)} = I \\ I^{(s+1,k)} = [\underbrace{I^{(s,k)}, \ldots, I^{(s,k)}}_{k}, L, \ldots, L], \; s \geq 0 \end{cases}$$

We say that I is a *k-solvable ideal* of L if $I^{(s,k)} = 0$ for some $s \in N_0$. L is said to be k-solvable if $L^{(s,k)} = 0$ for some $s \in N_0$. The ideal $I^{(1,n)} = [I, \ldots, I]$ is called the *square* of I.

Lemma 1.1. *For $k \in \{2, \ldots, n-1\}$, if I is a k-solvable ideal of L, then I_a is a k-solvable ideal of L_a. Furthermore, if I is an n-solvable ideal of L with index of n-solvability equal to r and $a \in I^{(r-1,n)}$, then I_a is an $(n-1)$-solvable ideal of L_a.*

Proof. Fixing $k \in \{2, \ldots, n-1\}$, we will show by induction that $I_a^{(s,k)} \subseteq I^{(s,k)}$ for all $s \in N_0$, which is sufficient to prove the first assertion. Clearly, we have $I_a^{(0,k)} = I = I^{(0,k)}$. By definition, we now have

$$I_a^{(s,k)} = [\underbrace{I_a^{(s-1,k)}, \ldots, I_a^{(s-1,k)}}_{k}, L_a, \ldots, L_a]_a.$$

Thus, using the induction hypothesis,

$$\begin{aligned}\Big[\underbrace{I_a^{(s-1,k)}, \ldots, I_a^{(s-1,k)}}_{k}, L_a, \ldots, L_a\Big]_a &\subseteq [\underbrace{I^{(s-1,k)}, \ldots, I^{(s-1,k)}}_{k}, L, \ldots, L]_a \\ &= [a, \underbrace{I^{(s-1,k)}, \ldots, I^{(s-1,k)}}_{k}, L, \ldots, L] \\ &\subseteq [I^{(s-1,k)}, \ldots, I^{(s-1,k)}, L, \ldots, L] = I^{(s,k)}.\end{aligned}$$

Suppose now that I is n-solvable, being $I^{(r,n)} = 0$ and $I^{(r-1,n)} \neq 0$. Admitting that $a \in I^{(r-1,n)}$, we will prove by induction that $I_a^{(s,n-1)} \subseteq I^{(s,n)}$ for all $s \in \{0, \ldots, r\}$. This is clearly true when $s = 0$. Further, suppose that $I_a^{(s-1,n-1)} \subseteq I^{(s-1,n)}$ with $s \leq r$. Then

$$I_a^{(s,n-1)} \subseteq [\underbrace{I^{(s-1,n)}, \ldots, I^{(s-1,n)}}_{n-1}]_a = [a, \underbrace{I^{(s-1,n)}, \ldots, I^{(s-1,n)}}_{n-1}].$$

Since $a \in I^{(r-1,n)} \subseteq I^{(s-1,n)}$ for all $s \leq r$, this last ideal is included in $I^{(s,n)}$. The second assertion follows then from the inclusion $I_a^{(r,n-1)} \subseteq I^{(r,n)}$, and the lemma is proved. □

Remark 1.1. Of course, if $I = L$, the additional condition appearing in the second assertion of the above lemma is redundant.

Next we give two properties that are left without proof (which can be found [2]).

Lemma 1.2. *Let $k \in \{2, \ldots, n\}$ be fixed.*

1) *If L is k-solvable, then all subalgebras of L are k-solvable and all ideals of L are k-solvable ideals of L;*

2) *If I and J are k-solvable ideals of L, then $I+J$ is a k-solvable ideal of L.*

The second property justifies the existence of a maximal k-solvable ideal of L for each $k \in \{2,\ldots,n\}$, which is called the k-radical of L, denoted by $\mathrm{Rad}_k(L)$. Further, L is said to be k-semisimple if $\mathrm{Rad}_k(L) = 0$.

Theorem 1.3. *Let k be a fixed integer in $\{2,\ldots,n-1\}$. If L_a is k-semisimple, then L is k-semisimple.*

Proof. By Lemma 1.1, we can write

$$\mathrm{Rad}_k(L_a) \supseteq \mathrm{Rad}_k(L), \tag{3}$$

for all $k \in \{2,\ldots,n-1\}$. The assertion is then a consequence of (3). □

Remark 1.2. The inclusion reciprocal to (3) is not valid in general. This will be a consequence of Theorem 3.5.

Given an ideal I of L and a fixed $k \in \{2,\ldots,n\}$, we can consider another descending sequence of ideals of L, denoted by $I^{s,k}$, defined as follows:

$$\begin{cases} I^{1,k} = I \\ I^{s+1,k} = [I^{s,k}, \underbrace{I,\ldots,I}_{k-1}, L,\ldots,L], s \geq 1\cdot \end{cases}$$

We say that I is a *k-nilpotent ideal* of L if $I^{s,k} = 0$ for some $s \in N$. Of course, if $L^{s,k} = 0$ for some $s \in N$, L is said to be k-nilpotent.

Lemma 1.4. *Let $k \in \{2,\ldots,n-1\}$ be fixed. If I is a k-nilpotent ideal of L, then I_a is a k-nilpotent ideal of L_a. Further, if $a \in I$ and I is an n-nilpotent ideal of L, then I_a is an $(n-1)$-nilpotent ideal of L_a.*

Proof. Fix $k \in \{2,\ldots,n-1\}$ and prove by induction that $I_a^{s,k} \subseteq I^{s,k}$ for all $s \geq 1$. Clearly, $I_a^{1,k} = I = I^{1,k}$. Assuming that $I_a^{s-1,k} \subseteq I^{s-1,k}$, we have

$$\begin{aligned} I_a^{s,k} &= [I_a^{s-1,k}, \underbrace{I_a,\ldots,I_a}_{k-1}, L_a,\ldots,L_a]_a \subseteq [I^{s-1,k}, \underbrace{I,\ldots,I}_{k-1}, L,\ldots,L]_a \\ &= [a, I^{s-1,k}, \underbrace{I,\ldots,I}_{k-1}, L,\ldots,L] \subseteq [I^{s-1,k}, \underbrace{I,\ldots,I}_{k-1}, L,\ldots,L] = I^{s,k}. \end{aligned}$$

The first assertion follows from this inclusion.

Now, for $a \in I$ we prove that $I_a^{s,n-1} \subseteq I^{s,n}$ for all $s \geq 1$. The inclusion is trivial when $s = 1$. Assume that $I_a^{s-1,n-1} \subseteq I^{s-1,n}$. Then

$$\begin{aligned} I_a^{s,n-1} &= [I_a^{s-1,n-1}, \underbrace{I_a, \ldots, I_a}_{n-2}]_a \subseteq [I^{s-1,n}, \underbrace{I, \ldots, I}_{n-2}]_a \\ &= [a, I^{s-1,n}, \underbrace{I, \ldots, I}_{n-2}] \subseteq [I^{s-1,n}, \underbrace{I, \ldots, I}_{n-1}] = I^{s,n}. \end{aligned}$$

Thus, from this inclusion we conclude that the second assertion is valid too. The lemma is proved. □

In general, when $k > 2$ the sum of two k-nilpotent ideals of L is not a k-nilpotent ideal of L. But since the opposite holds when $k = 2$, then there exists a maximal 2-nilpotent ideal of L, called the *2-nilradical* of L and denoted by $N(L)$.

Theorem 1.5. *If $n > 2$, then $N(L) \subseteq N(L_a)$.*

Proof. By the previous lemma, this inclusion is immediate. □

For simplicity, in the subsequent sections we use the words solvability, semisimplicity and radical instead of n-solvability, n-semisimplicity and n-radical, respectively.

2 ON THE FAMILY OF LIE ALGEBRAS DEFINING A 3-LIE ALGEBRA

Let L be a 3-Lie algebra, $\{e_1, \ldots, e_m\}$ a basis of L and consider the reduced Lie algebras which arise from L by fixing e_i, $i = 1, \ldots, m$. To simplify, hereafter we denote by L_i the reduced Lie algebra L_{e_i} and the corresponding multiplication by $[\cdot,\cdot]_i$. It is clear that

$$[x, y]_a = \sum_{j=1}^{m} \alpha_j [x, y]_j, \quad \text{for every } a = \sum_{j=1}^{m} \alpha_j e_j \in L,$$

so every reduced Lie algebra of L can be defined by means of the reduced Lie algebras L_i.

Observe now that, as a consequence of the definition of 3-Lie algebra, every three reduced multiplications $[\cdot,\cdot]_i$, $[\cdot,\cdot]_j$, $[\cdot,\cdot]_k$ on L are related by the following equalities:

$$\begin{cases} [x, e_j]_i = [e_i, x]_j; \\ [[e_k, y]_i, z]_j = [y, [z, e_j]_i]_k + [[z, e_j]_k, y]_i + [e_k, [y, z]_j]_i. \end{cases} \quad (4)$$

Reciprocally, let $\{L_1, \ldots, L_m\}$ be a family of Lie algebras all with dimension m, and V an m-dimensional vector space over a field F. Suppose that, for $i = 1, \ldots, m$, $\dim Z(L_i) \geq 1$, and $\varphi_i : L_i \to V$ are linear injective mappings such that:

1) It is possible to choose a basis $\{e_1, \ldots, e_m\}$ with $e_i \in \varphi_i(Z(L_i))$, $i = 1, \ldots, m$;
2) If V is equipped with m multiplications $\circ_i$ defined by

$$x \circ_i y = \varphi_i([\varphi_i^{-1}(x), \varphi_i^{-1}(y)]_i), \quad \text{for } i = 1, \ldots, m,$$

then (4) holds. Under these assumptions, V can be equipped with a ternary multiplication defined by

$$[z, x, y] = \sum_{i=1}^{m} \alpha_i (x \circ_i y), \quad \text{for every } z = \sum_{i=1}^{m} \alpha_i e_i \in V. \quad (5)$$

It is easy to prove that $(V, [\,,\,,\,])$ is a 3-Lie algebra.

When F is an algebraically closed field of characteristic zero, the following result establishes an important relation between a semisimple n -Lie algebra and its reduced $(n-1)$-Lie algebras relative to the canonical basis.

Theorem 2.1. *Let L be an m-dimensional semisimple n-Lie algebra over an algebraically closed field F of characteristic zero. Then, the reduced algebras $L_1, \ldots, L_m$ of L which arise by fixing the elements of its canonical basis, are simple, isomorphic to A_1.*

Proof. Under the above assumptions, we know [6, Thms. 2.7 and 3.9] that L is a direct sum of simple ideals $L = \oplus_{i=1}^{s} W_i$, where each W_i is isomorphic to the $(n+1)$-dimensional simple n-Lie algebra A_1 defined by V.T. Filippov in [1]. Being $\dim L = m = (n+1)s$, we can rearrange the elements of the canonical basis of L,

$$\{e_1^1, \ldots, e_{n+1}^1, \ldots, e_1^s, \ldots, e_{n+1}^s\},$$

in such a way that $\{e_1^j, \ldots, e_{n+1}^j\}$ is a basis of W_j, for $j = 1, \ldots, s$. Thus, it is possible to define a family of $(n-1)$-Lie algebras of L,

$$\{L_i^j : i = 1, \ldots, n+1; j = 1, \ldots, s\}$$

by fixing e_i^j. It is easy to see that the center of each algebra is equal to

$$Z(L_i^j) = \langle e_1^1, \ldots, e_{n+1}^1, \ldots, \widehat{e_1^j}, \ldots, e_i^j, \ldots, \widehat{e_{n+1}^j}, \ldots, e_1^s, \ldots, e_{n+1}^s \rangle_F,$$

and thus, $\dim Z(L_i^j) = (n+1)s - n = m - n$. Hence, we have

$$L_i^j = Z(L_i^j) \oplus A,$$

where A is a simple n-dimensional $(n-1)$-Lie algebra isomorphic to A_1. Recalling the identification of $L_i^j / Z(L_i^j)$ with L_i^j, we may now conclude that the reduced algebras L_i^j of L are pairwise isomorphic, of type A_1, due to the above decomposition and since the centers are pairwise isomorphic. The theorem is proved. □

Remark 2.1. We emphasize that the conclusions of the above result are restricted to the reduced algebras obtained by fixing the elements of the canonical basis. Indeed, it is easy to observe that if a is an arbitrary nonzero element of L, then the reduced $(n-1)$-Lie algebra L_a may be solvable (thus, not isomorphic to A_1).

Using ideas from the previous theorem and the preceding considerations we illustrate with a concrete example how a 3-Lie algebra can be built from a family of Lie algebras.

Example 2.2. Consider a 4-dimensional Lie algebra $(L, [\cdot,\cdot])$ over F generated by $\{e, f, g, z\}$ and such that

$$L = sl(2) \oplus Z$$

with $Z = \langle z \rangle_F$. Let now $V = \langle e_1, \ldots, e_4 \rangle_F$ be a 4-dimensional vector space over F and consider the following injective linear maps $\phi_i : L \mapsto V$ defined on the basis of L by the rules:

$$\begin{array}{llll} \phi_1(e) = e_2 & \phi_2(e) = e_1 & \phi_3(e) = e_1 & \phi_4(e) = e_1 \\ \phi_1(f) = e_3 & \phi_2(f) = -e_4 & \phi_3(f) = e_2 & \phi_4(f) = -e_2 \\ \phi_1(g) = -e_4 & \phi_2(g) = e_3 & \phi_3(g) = -e_4 & \phi_4(g) = -e_3 \\ \phi_1(z) = e_1 & \phi_2(z) = e_2 & \phi_3(z) = e_3 & \phi_4(z) = e_4 \end{array}$$

It is clear that $e_i \in \phi_i(Z)$. We can define 4 multiplications $\circ_i$ on V by means of

$$x \circ_i y = \phi_i([\phi_i^{-1}(x), \phi_i^{-1}(y)]),$$

under which each $(V, \circ_i)$ is a Lie algebra isomorphic to $(L, [\cdot,\cdot])$. It can be verified by direct computations that the multiplications $\circ_i$ satisfy (4). So, for every $x, y \in V$ and $w = \sum_{i=1}^{4} \alpha_i e_i \in V$ the operation

$$[w, x, y] = \sum_{i=1}^{4} \alpha_i (x \circ_i y)$$

defines a ternary multiplication, under which V becomes a 3-Lie algebra.

3 REDUCED $(n-1)$-LIE ALGEBRAS OF $A(n, t)$

In this section we investigate the structure of the reduced $(n-1)$-Lie algebras of $A(n, t)$ obtained by fixing the vectors of its canonical basis, when $\operatorname{char} F = 0$ and when $\operatorname{char} F = p$, but $F \neq Z_2$. First, we recall the definition of the algebra $A(n, t)$.

Let A be a vector space over a field F generated by the elements e_a such that $a \in F^n$, that is,

$$A = \langle e_a : a \in F^n \rangle_F.$$

For each fixed $t \in F^n$, define an n-ary anticommutative multiplication $[., \ldots, .]$ on A by the rule:

$$[e_{a_1}, \ldots, e_{a_n}] = |a_1, \ldots, a_n| e_{a_1 + \cdots + a_n + t}, \tag{6}$$

where $a_i = (a_{i1}, \ldots, a_{in}) \in F^n$ and $|a_1, \ldots, a_n| = \det(a_{ij})$, and extend it by n-linearity to all vectors in A. Equipped with any such multiplication, A is an Ω-algebra denoted by $\bar{A}(n, t)$ (or $\bar{A}(n)$ if $t = 0$). Further, it has already been proved [3] that $\bar{A}(n, t)$ is an n-Lie algebra.

Consider now the quotient n-Lie algebra $\tilde{A}(n, t) = \bar{A}(n, t)/Fe_0$, where $Fe_0 = \langle e_0 \rangle_F$ and define the n-Lie algebra $A(n, t)$ as follows:

$$A(n, t) = \begin{cases} \tilde{A}(n, t), & \text{if } t = 0 \\ \tilde{A}^{(1\, n)}(n, t), & \text{if } t \neq 0 \end{cases}.$$

Then

$$A(n, t) = \langle \overline{e_a} = e_a + Fe_0 : a \in F^n \backslash \{0, t\} \rangle_F. \tag{7}$$

It is easy to see that $A(n, t)$ is infinite dimensional if char $F = 0$, while if char $F = p$ and $|F| = p^k$, then

$$\dim A(n, t) = \begin{cases} p^{nk} - 1, & \text{if } t = 0 \\ p^{nk} - 2, & \text{if } t \neq 0 \end{cases}.$$

By [4], $A(n, t)$ is a simple n-Lie algebra.

Let us fix an element e_v of the canonical basis of $A(n, t)$ (simplifying notations, we identify hereinafter $\overline{e_v} = e_v + Fe_0$ with e_v) and put $L = A(n, t)_{e_v}$.

Lemma 3.1. *$Z = \langle e_a : a \in \langle v \rangle \backslash \{0, t\} \rangle_F$ is the center of L.*

Proof. If $a \in \langle v \rangle$ $(a \neq 0, t)$ then it is obvious that $e_a \in Z(L)$. Reciprocally, suppose that e_a is a nonzero vector of $Z(L)$ but $a \notin \langle v \rangle$. Then it is possible to take $a_3, \ldots, a_n \in F^n$ such that

$$\dim\langle v, a, a_3, \ldots, a_n \rangle = n \quad \text{and} \quad v + a + a_3 + \cdots + a_n \neq -t.$$

Then, $e_a \notin Z(L)$, contradicting the assumption. The lemma is proved. □

We are going to describe the ideals of the reduced algebra $\bar{L} = L/Z(L)$. In order to simplify notations, each $e_a + Z$ will be identified with e_a and $F^n \setminus (\langle v \rangle_F \cup \{t\})$ with Φ.

Lemma 3.2. *If I is a proper ideal of $\bar{L}$, then I belongs to the family of ideals*

$$\Xi = \{I(k; \alpha_1, \ldots, \alpha_k; \gamma_1, \ldots, \gamma_k),\ k \in N, \alpha_i, \gamma_i \in F\},$$

where

$$I(k; \alpha_1, \ldots, \alpha_k; \gamma_1, \ldots, \gamma_k) = \left\langle e_a + \sum_{i=1}^{k} \alpha_i e_{a+\gamma_i v} \ :\ a \in \Phi \right\rangle_F. \tag{8}$$

Proof. Let I be a proper ideal of $\bar{L}$ and consider $u = \sum_{i=1}^{k+1} \alpha_i e_{a_i} \in I$ an element with minimal length. Without loss of generality, we may assume that $\alpha_1 = 1$. Suppose that $k \geq 1$. By definition of ideal, $[u, e_{b_3}, \ldots, e_{b_n}]_{e_v} \in I$ for all $e_{b_i} \in \bar{L}$. Observe now that

$$[u, e_{b_3}, \ldots, e_{b_n}]_{e_v} = [e_v, u, e_{b_3}, \ldots, e_{b_n}] = \sum_{i=1}^{k+1} \alpha_i \beta_i e_{v+a_i+\bar{b}}, \tag{9}$$

where $\beta_i = |v, a_i, b_3, \ldots, b_n| \in F$, $i = 1, \ldots, k+1$ and $\bar{b} = b_3 + \cdots + b_n + t$. If there exists some $j \neq 1$ such that $a_j \notin \langle v, a_1 \rangle$, since $b_3, \ldots, b_n$ are arbitrary, we can take $b_3 = a_1$ and, if $n > 3$, choose $b_4, \ldots, b_n$ satisfying $\dim\langle v, a_j, a_1, b_4, \ldots, b_n \rangle = n$. This implies that $\beta_1 = 0$ and, since at least $\beta_j \neq 0$, we may conclude that the length of $[u, e_{b_3}, \ldots, e_{b_n}]_{e_v} \neq 0$ is less then $k+1$, contradicting the hypothesis. So, all $a_j \in \langle v, a_1 \rangle$.

We now investigate under what conditions we have $v + a_i + \bar{b} \in \langle v, v + a_1 + \bar{b} \rangle$. It follows from $v + a_i + \bar{b} \in \langle v, v + a_1 + \bar{b} \rangle$ that there exist $\delta_i, \gamma_i \in F$ such that

$$v + a_i + \bar{b} = \gamma_i v + \delta_i (v + a_1 + \bar{b}), \tag{10}$$

whence

$$(1 - \delta_i)\bar{b} = (\gamma_i + \delta_i - 1)v + \delta_i a_1 - a_i. \tag{11}$$

Suppose that $\delta_i \neq 1$ for some i. Then, since $a_i \in \langle v, a_1 \rangle$, it follows from (11) that $\bar{b} \in \langle v, a_1 \rangle$. But, being $b_3, \ldots, b_n$ arbitrary, it is possible to choose them such that $\bar{b} \notin \langle v, a_1 \rangle$. Thus, $\delta_i = 1$, $i = 1, \ldots, k+1$. From this and from (10) we obtain $a_i = a_1 + \gamma_i v$, and since $\alpha_1 = 1$, we conclude that every $u \in I$ can be written in the form

$$u = e_{a_1} + \sum_{i=2}^{k+1} \alpha_i e_{a_1 + \gamma_i v}. \tag{12}$$

To complete the proof, we now show that if $u_y = e_y + \sum_{i=2}^{k+1} \alpha_i e_{y + \gamma_i v}$, then $u_y \in I$ for all $y \in \Phi$. Assume first that $y - t \notin \langle v, a_1 \rangle$. If $n > 3$, we can choose $b_3, \ldots, b_{n-1}$ such that $\dim\langle v, a_1, b_3, \ldots, b_{n-1}, y - t \rangle = n$. By setting $b_n = y - t - v - a_1 - b_3 - \cdots - b_{n-1}$ it follows from (12) that

$$\begin{aligned} [u, e_{b_3}, \ldots, e_{b_n}]_{e_v} &= [e_v, u, e_{b_3}, \ldots, e_{b_n}] \\ &= |v, a_1, b_3, \ldots, b_{n-1}, y - t| \left(e_y + \sum_{i=2}^{k+1} \alpha_i e_{y + \gamma_i v} \right) \in I \end{aligned}$$

and $u_y \in I$. If $n = 3$, we just have to put $b_3 = y - t - v - a_1$ arriving to the same conclusion. Suppose now that $y - t \in \langle v, a_1 \rangle$ and consider $z \in \Phi$ satisfying $z \notin \langle v, a_1 \rangle$. Then $y - t \notin \langle v, z \rangle$ and, analogously, we have $u_y \in I$.

If the length of the element u is equal to 1 then, similarly to what was done with the element y we would conclude that $I = \bar{L}$ which contradicts the assumption that I is a proper ideal. The lemma is finally proved. □

We now want to investigate the structure of some specific ideals in Ξ. Namely, we are interested in the ideals

$$I_\alpha = \langle e_a - e_{a+\alpha v} : a \in \Phi \rangle_F,$$

where α is fixed in F^*. To do so, we need an auxiliary number theory result, suggested by A.P. Pozhidaev.

Lemma 3.3. ([1]) *Let $n \in N$ and $p \leq n$ be a prime number. Then for any $i = 0, 1, \ldots, p-1$*

$$\sum_{\substack{k=0 \\ k \equiv i (\mathrm{mod}\, p)}}^{n} (-1)^k C_n^k \equiv 0 \,(\mathrm{mod}\, p).$$

Proof. Define a mapping ϕ from $Z[x]$ into the ring $Z^p[x]$ of truncated polynomials given by

$$\phi(mx^k) = m_p x^{k_p},$$

where $k \equiv k_p \,(\mathrm{mod}\, p)$, $m \equiv m_p \,(\mathrm{mod}\, p)$, $0 \leq m_p, k_p < p$, $x^0 = 1$, and extended to $Z[x]$ by linearity. It is easy to see that ϕ is an homomorphism. Then we have

$$\phi((1-x)^n) = \phi\left(\sum_{k=0}^{n} (-1)^k C_n^k x^k\right) = \sum_{i=0}^{p-1} \gamma_i x^i,$$

where

$$\gamma_i \equiv \left(\sum_{\substack{k=0 \\ k \equiv i (\mathrm{mod}\, p)}}^{n} (-1)^k C_n^k\right) (\mathrm{mod}\, p).$$

On the other hand, $\phi((1-x)^n) = 0$ since $\phi((1-x)^p) = 0$ and ϕ is an homomorphism. This means that $\gamma_i \equiv 0 (\mathrm{mod}\, p)$ for $i = 0, \ldots, p-1$, and the lemma is proved. □

[1]This lemma was suggested by A.P. Pozhidaev.

Lemma 3.4. *For a fixed* $\alpha \in F^*$, I_α *is a proper ideal of* $\bar{L}$, *which is*

—*non-solvable, if* $\operatorname{char} F = 0$;
—*solvable, if* $\operatorname{char} F = p$.

Proof. Without loss of generality, we may assume that $\alpha = 1$, since otherwise v can be replaced by $v' = \alpha v$. Put $u_a^1 = e_a - e_{a+v}$ and $I = \langle e_a - e_{a+v} : a \in \Phi \rangle_F$. Note that developing the product

$$[u_{a_1}^1, \dots, u_{a_{n-1}}^1]_{e_v} \tag{13}$$

according to (6), all the determinants are equal and for each k the coefficient of $e_{\bar{a}+kv}$ (where $\bar{a} = a_1 + \cdots + a_{n-1} + v + t$), is equal to the coefficient of x^k on the binomial development of $(1-x)^{n-1}$. To simplify notations, let us denote by $u_{\bar{a}}^2$ the element obtained from (13) after dividing by that determinant. Repeating the previous calculation with $u_{\bar{a}}^2$ instead of u_a^1, a similar situation occurs for the coefficients, which are now those of the binomial development of $(1-x)^{(n-1)^2}$. And similarly, for the ith iteration, the coefficients are those of $(1-x)^{(n-1)^i}$. Thus, if $\operatorname{char} F = 0$ it is obvious that I is a non-solvable ideal of $\bar{L}$. In the case of $\operatorname{char} F = p$, taking into account Lemma 3.3 we conclude that I is a solvable ideal, ending the proof of this lemma. □

Theorem 3.5. *If* $\operatorname{char} F = 0$ *then* $\bar{L}$ *is a semisimple* $(n-1)$*-Lie algebra. If* $\operatorname{char} F = p$ *then* $R = \sum_{\alpha \in F^*} I_\alpha + Z(L)$ *is the radical of* L*; further,* L/R *is a semisimple algebra isomorphic to the simple* $(n-1)$*-Lie algebra* $A(n-1, t')$ *for some* $t' \in \Phi$ *and in the case* $(n-2) \not\equiv 0 \pmod{p}$ *there exists a subalgebra* S *of* L *such that* $L = R \oplus S$ *and* $S \cong L/R$.

Proof. If $\operatorname{char} F = 0$ then, using Lemma 3.4, it is easy to see that there are no solvable ideals in $\bar{L}$. Therefore, $\bar{L}$ is semisimple.

Let F be a field of characteristic p. By Lemma 3.4, it is obvious that $\operatorname{Rad}(L) \subseteq R$. Note that

$$\bar{e}_a = e_a + R = e_{a+\beta v} + R = \bar{e}_{a+\beta v}, \quad \text{for any } \beta \in F.$$

On the other hand, if $a - b \notin \langle v \rangle$, $a, b \notin \langle v \rangle$ then it is easy to see that $\bar{e}_a$ and $\bar{e}_b$ are linearly independent. Let $v, \epsilon_1, \dots, \epsilon_{n-1}$ be a basis of F^n. Denote the vector space $\langle \epsilon_1, \dots, \epsilon_{n-1} \rangle_F$ by F_ϵ^{n-1}. Then $L/R = \langle \bar{e}_a : a \in F_\epsilon^{n-1} \rangle_F$. By Corollary 2.1 of [3] there are linear mappings $h_1, \dots, h_{n-1}$ from F_ϵ^{n-1} into F such that $|v, \cdot, \dots, \cdot| = h_1 \wedge \cdots \wedge h_{n-1}$ and for some $t' \in \Phi$ we have

$$L/R \cong A(n-1, t')$$

by [3]. In the case $(n-2) \not\equiv 0 \,(\mathrm{mod}\, p)$ we can choose a subalgebra S as follows

$$S = \langle e_{a-\frac{1}{n+2}v} : a \in F_{\epsilon}^{n-1} \rangle_F.$$

It is easy to see that $L = R \oplus S$ and $S \cong L/R$. The theorem is proved. □

4 REDUCED $(n-1)$-LIE ALGEBRAS OF $E(n,t)$

Analogously to the previous section, our aim is to investigate the structure of the reduced $(n-1)$-Lie algebras of $E(n,t)$ obtained by fixing the elements of the respective canonical basis. This family of algebras was also introduced in [4] and we recall below its definition.

Consider a vector space $\langle e_a : a \in F^{n-1} \rangle_F$, and for t fixed in F^{n-1} define a multilinear anticommutative multiplication by

$$[e_{a_1}, \ldots, e_{a_n}] = \begin{vmatrix} a_{11} & \cdots & a_{1\,n-1} & 1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{n\,n-1} & 1 \end{vmatrix} e_{a_1+\cdots+a_n+t}, \tag{14}$$

where, for each $i = 1, \ldots, n$, $a_i = (a_{i1}, \ldots, a_{i\,n-1}) \in F^{n-1}$. To simplify, we shall denote the above determinant by $|a_1, \ldots, a_n|_1$. Clearly, the resulting n-Lie algebra, denoted by $\bar{E}(n,t)$, is isomorphic to an n-Lie subalgebra of $\bar{A}(n,t)$.

Let us define an n-Lie subalgebra of $\bar{E}(n,t)$ in the following way:

$$E(n,t) = \begin{cases} \bar{E}(n,t), & \text{if } n \not\equiv 0 \ (\mathrm{mod}\, p) \text{ or char } F = 0 \\ \bar{E}^{(1,n)}(n,t), & \text{if } n \equiv 0 \ (\mathrm{mod}\, p) \end{cases}$$

It is easy to see that if $\mathrm{char}\, F = p$ and $|F| = p^k$, then

$$\dim E(n,t) = \begin{cases} p^{k(n-1)}, & \text{if } n \not\equiv 0 \ (\mathrm{mod}\, p) \\ p^{k(n-1)} - 1, & \text{if } n \equiv 0 \ (\mathrm{mod}\, p) \end{cases},$$

while in the case $\mathrm{char}\, F = 0$, $E(n,t)$ is obviously infinite dimensional. By [4], $E(n,t)$ is a simple n-Lie algebra.

Let us fix an element e_v on the canonical basis of $E(n,t)$ ($v \neq t$ if $n \equiv 0 \pmod p$) and put $L = E(n,t)_{e_v}$, with $n \geq 3$.

Lemma 4.1. *$Z = \langle e_v \rangle$ is the center of L.*

Proof. We have $e_a \in Z(L)$ if and only if $[e_a, e_{a_3}, \ldots, e_{a_n}]_{e_v} = 0$ for all basis elements $e_{a_3}, \ldots, e_{a_n} \in L$, which is equivalent to $|v, a, a_3, \ldots, a_n|_1 e_{v+a+a_3+\cdots+a_n+t} = 0$. If $a = v$, it is clear that $e_a \in Z(L)$. Suppose that $a \neq v$. Then $e_a \in Z(L)$ if and only if $|v, a, a_3, \ldots, a_n|_1 = 0$, that is, if and only if $a - v, a_3 - v, \ldots, a_n - v$ are linearly dependent for all $a_3, \ldots, a_n \in F^{n-1}$. Observe that we can choose $a_3, \ldots, a_n \in F^{n-1}$ such that $\dim\langle a - v, a_3 - v, \ldots, a_n - v\rangle = n - 1$. Therefore, $e_a \notin Z(L)$ if $a \neq v$, and the lemma is proved. □

We now consider the reduced algebra $\bar{L} = L/Z$ and, for simplicity, make the identification $\bar{e}_a = e_a$, where $\bar{e}_a = e_a + Z$.

Theorem 4.2. *For $n \geq 3$ ($F \neq Z_2$ if $n = 3$), the algebra $\bar{L}$ is a simple $(n-1)$-Lie algebra if $n \equiv 0 \pmod p$, and its square is a simple $(n-1)$-Lie algebra of codimension 1 in $\bar{L}$ if $n \not\equiv 0 \pmod p$ or $\operatorname{char} F = 0$.*

Proof. Computing the square of $\bar{L}$, it is possible to observe that $e_{t+nv} \notin \bar{L}^{(1,n-1)}$ when $n \not\equiv 0 \pmod p$ or $\operatorname{char} F = 0$. Thus, to simplify notations, we are going to put

$$\bar{W} = \begin{cases} \bar{L}, & \text{if } n \equiv 0 \pmod p \\ \bar{L}^{(1,n-1)}, & \text{if } n \not\equiv 0 \pmod p \text{ or } \operatorname{char} F = 0 \end{cases}$$

This way, $\bar{W} = \langle e_a : a \in \Phi \rangle$, where $\Phi = F^{n-1} \setminus \{v, t + nv\}$ ([2]).

Let I be a proper ideal of $\bar{W}$ and $u = \sum_{i=1}^{k+1} \alpha_i e_{a_i} \in I$ be an element of minimal length, $k+1$, with $\alpha_1 = 1$. Suppose that $k \geq 1$. For all basis elements $e_{b_i} \in \bar{W}$ we have $[u, e_{b_3}, \ldots, e_{b_n}]_{e_v} \in I$. Observe now that

$$[u, e_{b_3}, \ldots, e_{b_n}]_{e_v} = [e_v, u, e_{b_3}, \ldots, e_{b_n}] = \sum_{i=1}^{k+1} \alpha_i \beta_i e_{v+a_i+\bar{b}}, \tag{15}$$

where $\beta_i = |v, a_i, b_3, \ldots, b_n|_1 = |a_i - v, b_3 - v, \ldots, b_n - v| \in F$, $i = 1, \ldots, k+1$ and $\bar{b} = b_3 + \cdots + b_n + t$. If there exists some $j \neq 1$ such that

[2](observe that if $n \equiv 0 \pmod p$, then $t + nv \equiv t \pmod p$ and $\bar{L}$ can also be represented this way).

$a_j - v \notin \langle a_1 - v\rangle$, since $b_3, \ldots, b_n$ are arbitrary, we can take $b_3 = a_1$ and, if $n > 3$, choose $b_4, \ldots, b_n$ satisfying

$$\dim\langle a_j - v, a_1 - v, b_4 - v, \ldots, b_n - v\rangle = n - 1.$$

This implies that $\beta_1 = 0$ and, at least, $\beta_j \neq 0$. Whence, the length of the nonzero vector $[u, e_{b_3}, \ldots, e_{b_n}]_{e_v}$ is less then $k+1$, contradicting the hypothesis. So,

$$a_i - v \in \langle a_1 - v\rangle, \quad i = 1, \ldots, k+1, \tag{16}$$

that is,

$$a_i = (1 - \gamma_i)v + \gamma_i a_1 \quad \text{for some } \gamma_i \in F. \tag{17}$$

If we take $b_3, \ldots, b_n \in \Phi$ such that $|v, a_i, b_3, \ldots, b_n|_1 \neq 0$, then $e_{v+a_i+\bar{b}} \in I$ and by (16) we must have

$$v + a_i + \bar{b} - v \in \langle v + a_1 + \bar{b} - v\rangle,$$

that is, for each $i = 1, \ldots, k+1$ there exists $\delta_i \in F$ such that

$$a_i = \delta_i a_1 + (\delta_i - 1)\bar{b}, \tag{18}$$

where $\delta_1 = 1$. By (17) and (18),

$$(\gamma_i - \delta_i)a_1 + (1 - \gamma_i)v + (1 - \delta_i)\bar{b} = 0. \tag{19}$$

for some $\delta_i, \gamma_i \in F$. Consider $n > 3$. Supposing that $\delta_j \neq 1$ for some $j \neq 1$, (19) implies that $\bar{b} \in \langle v, a_1\rangle$. But this contradicts the arbitrariness of $\bar{b}$, since we can choose $b_3, \ldots, b_n \in \Phi$ such that $\bar{b} \notin \langle v, a_1\rangle$. Thus, $\delta_i = 1$, for $i = 1, \ldots, k+1$ and from (18) it follows $a_i = a_1$, $i = 1, \ldots, k+1$. Let $n = 3$. If $\operatorname{char} F = 0$, or if $\operatorname{char} F = p$ and $v \in \langle a_1\rangle$ the previous reasoning is still valid. Consider then the modular case with $v \notin \langle a_1\rangle$. If we admit that $\gamma_j \neq 1$ for some $j \neq 1$, from (19) we get that $v \in \langle \bar{b}, a_1\rangle$, for all $\bar{b}$. So, in particular, for a choice of $b_3 \in \Phi$ such that $\bar{b} \in \langle a_1\rangle$, we get $v \in \langle a_1\rangle$, which is impossible. Thus, $\gamma_i = 1$, for $i = 1, \ldots, k+1$ and from (17) it follows $a_i = a_1$, $i = 1, \ldots, k+1$. So, for $n \geq 3$, we must conclude that $e_a \in I$ for some $a \in \Phi$.

Consider then $e_a \in I$ for some $a \in \Phi$, where I is a nonzero ideal of $\bar{W}$. We now proceed to show that $e_y \in I$ for all $y \in \Phi$. Given a fixed basis element $e_y \in \bar{W}$, if $n > 3$, one can always take arbitrary $b_3, \ldots, b_{n-1}$ and put $b_n = y - (v + a + b_3 + \cdots + b_{n-1} + t)$. By doing so,

$$\begin{aligned} I \ni & [e_a, e_{b_3}, \ldots, e_{b_n}]_{e_v} \\ &= |a - v, b_3 - v, \ldots, b_{n-1} - v, y - 2v - a - b_3 - \ldots - b_{n-1} - t|e_y \\ &= |a - v, b_3 - v, \ldots, b_{n-1} - v, y - (t + nv)|e_y. \end{aligned} \tag{20}$$

Whence, $e_y \in I$ if and only if

$$y - (t + nv) \notin \langle a - v \rangle_F. \tag{21}$$

If $n = 3$, we just have to take $b_3 = y - (v + a + t)$ to conclude that

$$[e_a, e_{b_3}]_{e_v} = |a - v, y - (t + 3v)|e_y,$$

obtaining the same conclusion as in (21), with $n = 3$. In both cases, we can also choose $b_3, \ldots, b_n \in \Phi$ such that

$$\begin{aligned} & a - v \notin \langle a + b_3 + \cdots + b_n + t \rangle_F \\ & \text{and} \quad |a - v, b_3 - v, \ldots, b_n - v| \neq 0. \end{aligned} \tag{22}$$

Putting $z = v + a + b_3 + \cdots + b_n + t$, we conclude by (20) that $e_z \in I$, and using (21), $e_y \in I$ if and only if

$$y - (t + nv) \notin \langle z - v \rangle_F. \tag{23}$$

It is easy to see that for every $y \in \Phi$ we have (21) or (23). The simplicity of $\bar{W}$ follows from here and the theorem is proved.

Remark 4.3. When $n = 3$, if $F = Z_2$ some of the arguments in the above proof are not valid. However, it can be easily proved that $\bar{W}$ is simple only if $t = (0, 0)$.

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