Modules over linear spaces admitting a multiplicative basis

Antonio J. Calderón Martín*, Francisco J. Navarro Izquierdo, and José M. Sánchez Delgado

Department of Mathematics
Faculty of Sciences, University of Cádiz
Campus de Puerto Real, 11510, Puerto Real, Cádiz, Spain.
ajesus.calderon@uca.es; javi.navarroiz@uca.es; txema.sanchez@uca.es

Abstract. We study the structure of certain modules V over linear spaces W with restrictions neither on the dimensions nor on the base field $\mathbb{F}$. A basis $\mathfrak{B} = \{v_i\}_{i\in I}$ of V is called multiplicative respect to the basis $\mathfrak{B}' = \{w_j\}_{j\in J}$ of W if for any $i \in I, j \in J$ we have either $v_i w_j = 0$ or $0 \neq v_i w_j \in \mathbb{F}v_k$ for some $k \in I$. We show that if V admits a multiplicative basis then it decomposes as the direct sum $V = \bigoplus_k V_k$ of well-described submodules admitting each one a multiplicative basis. Also the minimality of V is characterized in terms of the multiplicative basis and it is shown that the above direct sum is by means of the family of its minimal submodules, admitting each one a multiplicative basis.

Keywords: Multiplicative basis, infinite dimensional linear space, module over an algebra, representation theory, structure theory.

1 Introduction and previous definitions

We begin by noting that throughout this paper linear spaces V and W are considered of arbitrary dimensions and over an arbitrary base field $\mathbb{F}$, and also the increasing interest in the study of modules over different classes of algebras, and so over linear spaces, specially motivated by their relation with mathematical physics (see [6], [7], [8], [9], [11], [12], [13]).

Definition 1. Let V be a vector space over an arbitrary base field $\mathbb{F}$. It is said that V *is moduled by a linear space* W (over same base field $\mathbb{F}$), or just that V is a *module* over W if it is endowed with a bilinear map $V \times W \to V, \quad (v, w) \mapsto vw$.

Any kind of algebra is an example of a module over itself. Since the even part L^0 of the standard embedding of a Lie triple system T is a Lie algebra, the natural action of L^0 over T makes of T a (Lie) module over L^0. Hence the present paper extend the results in [5].

* The first and the third authors are supported by the PCI of the UCA 'Teoría de Lie y Teoría de Espacios de Banach', by the PAI with project numbers FQM298, FQM7156 and by the project of the Spanish Ministerio de Educación y Ciencia MTM2010-15223. Third author acknowledges the University of Cadiz for the contract research.

Definition 2. Let V be a module over the linear space W. Given a basis $\mathfrak{B}' = \{w_j\}_{j\in J}$ of W we say that a basis $\mathfrak{B} = \{v_i\}_{i\in I}$ of V is *multiplicative* respect $\mathfrak{B}'$ if for any $i \in I$ and $j \in J$ we have either $v_i w_j = 0$ or $0 \neq v_i w_j \in \mathbb{F}v_k$ for some (unique) $k \in I$.

To construct examples of modules over linear spaces admitting a multiplicative basis, we just have to fix two non-empty sets I, J and two arbitrary mappings $\alpha : I \times J \to I$ and $\beta : I \times J \to \mathbb{F}$. Then the $\mathbb{F}$-linear space V with basis $\mathfrak{B} = \{v_i\}_{i\in I}$ is a module respect to the $\mathbb{F}$-linear space W with basis $\mathfrak{B}' = \{w_j\}_{j\in J}$, under the action induced by $v_i w_j := \beta(i,j)v_{\alpha(i,j)}$, admitting $\mathfrak{B}$ as multiplicative basis respect to $\mathfrak{B}'$.

Remark 1. Definition 2 agree with the one for arbitrary algebras given in [5], and it is a little bit more general than the usual one in the literature ([1–4, 10]).

2 Connections in the set of indexes. Decompositions

From now on and throughout the paper, V denotes a module over the linear space W, both endowed with respective basis $\mathfrak{B} = \{v_i\}_{i\in I}, \mathfrak{B}' = \{w_j\}_{j\in J}$, and being $\mathfrak{B}$ multiplicative respect to $\mathfrak{B}'$. We denote by $\mathcal{P}(I)$ the power set of I.

We begin this section by developing connection techniques among the elements in the set of indexes I as the main tool in our study. For each $j \in J$, a new variable $\overline{j} \notin J$ is introduced and we denote by $\overline{J} := \{\overline{j} : j \in J\}$ the set of all these new symbols. We will also write $\overline{(\overline{j})} := j \in J$.

We consider the operation $\star : I \times (J \,\dot\cup\, \overline{J}) \to \mathcal{P}(I)$ given by:

- If $i \in I$ and $j \in J$,
$$i \star j := \begin{cases} \emptyset & \text{if } 0 = v_i w_j \\ \{k\} & \text{if } 0 \neq v_i w_j \in \mathbb{F}v_k \end{cases}$$
- If $i \in I$ and $\overline{j} \in \overline{J}$,
$$i \star \overline{j} := \{k : 0 \neq v_k w_j \in \mathbb{F}v_i\}$$

Now, we also consider the mapping $\phi : \mathcal{P}(I) \times (J \,\dot\cup\, \overline{J}) \to \mathcal{P}(I)$ defined as $\phi(U,j) := \bigcup_{i\in U}(i \star j)$.

Lemma 1. *Let $a, b \in I$ be. Given $j \in J \,\dot\cup\, \overline{J}$ we have that $a \in b \star j$ if and only if $b \in a \star \overline{j}$.*

Proof. Let us suppose that $a \in b \star j$. If $j \in J$ then $v_b w_j \in \mathbb{F}v_a$, and if $j \in \overline{I}$ we have $v_a w_{\overline{j}} \in \mathbb{F}v_b$. In any case $b \in a \star \overline{j}$. To prove the converse we can argue in a similar way.

Lemma 2. *Given $j \in J \,\dot\cup\, \overline{J}$ and $U \subset \mathcal{P}(I)$ then $i \in \phi(U,j)$ if and only if $\phi(\{i\}, \overline{j}) \cap U \neq \emptyset$.*

Proof. Let us suppose that $i \in \phi(U,j)$. Then there exists $k \in U$ such that $i \in k \star j$. By Lemma 1 we have $k \in i \star \overline{j} = \phi(\{i\}, \overline{j})$. So $k \in \phi(\{i\}, \overline{j}) \cap U \neq \emptyset$. By arguing in a similar way the converse can be proven.

Definition 3. Let $i, k \in I$ be with $i \neq k$. We say that i is *connected* to k if there exists a subset $\{j_1, \dots, j_n\} \subset J \,\dot{\cup}\, \overline{J}$, such that the following conditions hold:

1. $\phi(\{i\}, j_1) \neq \emptyset$, $\phi(\phi(\{i\}, j_1), j_2) \neq \emptyset, \dots, \phi(\phi(\dots\phi(\{i\}, j_2)\dots), j_{n-1}) \neq \emptyset$.
2. $k \in \phi(\phi(\dots\phi(\{i\}, j_1)\dots), j_n)$.

We say that $\{j_1, \dots, j_n\}$ is a *connection* from i to k and we accept i is connected to itself.

Lemma 3. *Let $\{j_1, j_2, \dots, j_{n-1}, j_n\}$ be any connection from some i to some k where $i, k \in I$ with $i \neq k$. Then the set $\{\overline{j}_n, \overline{j}_{n-1}, \dots, \overline{j}_2, \overline{j}_1\}$ is a connection from k to i.*

Proof. Let us prove it by induction on n. For $n = 1$ we have that $k \in \phi(\{i\}, j_1)$. It means that $k \in i \star j_1$ and so, by Lemma 1, $i \in k \star \overline{j}_1 = \phi(\{k\}, \overline{j}_1)$. Hence $\{\overline{j}_1\}$ is a connection from k to i.

Let us suppose that the assertion holds for any connection with $n \geq 1$, elements and let us show this assertion also holds for any connection $\{j_1, j_2, \dots, j_n, j_{n+1}\}$.

By denoting the set $U := \phi(\phi(\dots\phi(\{i\}, j_1)\dots), j_n)$ and taking into the account the second condition of Definition 3 we have that $k \in \phi(U, j_{n+1})$. Then, by Lemma 2, $\phi(\{k\}, \overline{j}_{n+1}) \cap U \neq \emptyset$ and so we can take $h \in U$ such that

$$h \in \phi(\{k\}, \overline{j}_{n+1}). \tag{1}$$

Since $h \in U$ we have that $\{j_1, j_2, \dots, j_{n-1}, j_n\}$ is a connection from i to h. Hence $\{\overline{j}_n, \overline{j}_{n-1}, \dots, \overline{j}_2, \overline{j}_1\}$ connects h with i. From here and by Equation (1) we obtain $i \in \phi(\phi(\dots\phi(\phi(\{k\}, \overline{j}_{n+1}), \overline{j}_n)\dots), \overline{j}_1)$. So $\{\overline{j}_{n+1}, \dots, \overline{j}_2, \overline{j}_1\}$ connects k with i.

Proposition 1. *The relation $\sim$ in I, defined by $i \sim k$ if and only if i is connected to k, is an equivalence relation.*

Proof. The reflexive and symmetric character is given by Definition 3 an Lemma 3.

If we consider the connections $\{a_1, \dots, a_m\}$ and $\{b_1, \dots, b_n\}$ from a to b and from b to c respectively, then is easy to prove that $\{a_1, \dots, a_m, b_1, b_2, \dots, b_n\}$ is a connection from a to c. So $\sim$ is transitive and consequently an equivalence relation.

By the above Proposition we can introduce the quotient set $I/\sim := \{[i] : i \in I\}$, becoming $[i]$ the set of elements in I which are connected to i.

Recall that a *submodule* Y of a module V (respect to the linear space W) is a linear subspace of V such that $YW \subset Y$. Our next aim is to associate an (adequate) submodule to each $[i] \in I/\sim$. We define the linear subspace $V_{[i]} := \bigoplus_{j \in [i]} \mathbb{F} v_j$.

Proposition 2. *For any $i \in I/\sim$ we have that $V_{[i]}$ is a submodule of V.*

Proof. We need to check $V_{[i]} W \subset V_{[i]}$. Suppose there exist $i_1 \in [i], j_1 \in [j]$ such that $0 \neq v_{i_1} w_{j_1} \in v_n$, for some $n \in I$. Therefore $n \in \phi(\{i_1\}, j_1)$. Considering the connection $\{j_1\}$ we get $i_1 \sim n$, and by transitivity $n \in [i]$.

Definition 4. We say that a submodule $Y \subset V$ admits a multiplicative basis $\mathfrak{B}_Y$ *inherited* from $\mathfrak{B}$ if $\mathfrak{B}_Y \subset \mathfrak{B}$.

Observe that any submodule $V_{[i]} \subset V$ admits an inherited basis $\mathfrak{B}_{[i]} := \{v_j : j \in [i]\}$. So we can assert

Theorem 1. *Let V be a module admitting a multiplicative basis $\mathfrak{B}$ respect to a fixed basis of W. Then*

$$V = \bigoplus_{[i] \in I/\sim} V_{[i]},$$

being any $V_{[i]} \subset V$ a submodule admitting a multiplicative basis $\mathfrak{B}_{[i]}$ inherited from $\mathfrak{B}$.

We recall that a module V is *simple* if its only submodules are $\{0\}$ and V.

Corollary 1. *If V is simple then any couple of elements of I are connected.*

Proof. The simplicity of V applies to get that $V_{[i]} = V$ for some $[i] \in I/\sim$. Hence $[i] = I$ and so any couple of elements in I are connected.

3 The minimal components

In this section we show that, under mild conditions, the decomposition of V of Theorem 1 can be given by means of the family of its minimal submodules. We begin by introducing a concept of minimality for V that agree with the one for algebras in [5].

Definition 5. A module V, (over a linear space W), admitting a multiplicative basis $\mathfrak{B}$ respect to fixed basis of W, is said to be *minimal* if its unique nonzero submodule admitting a multiplicative basis inherited from $\mathfrak{B}$ is V.

Let us also introduce the concept of $\star$-multiplicativity in the framework of modules over linear spaces in a similar way to the analogous one for arbitrary algebras (see [5] for these notions and examples).

Definition 6. We say that a module V respect W admits a $\star$-*multiplicative basis* $\mathfrak{B} = \{v_i\}_{i \in I}$ respect to a fixed basis $\mathfrak{B}' = \{l_j\}_{j \in J}$ of W, if it is multiplicative and given $a, b \in I$ such that $b \in a \star j$ for some $j \in J \dot{\cup} \overline{J}$ then $v_b \in v_a W$.

Theorem 2. *Let V be a module respect W admitting a $\star$-multiplicative basis $\mathfrak{B} = \{v_i\}_{i \in I}$ respect to the basis $\mathfrak{B}' = \{w_j\}_{j \in J}$ of W. Then V is minimal if and only if the set of indexes I has all of its elements connected.*

Proof. The first implication is similar to Corollary 1. To prove the converse, consider a nonzero submodule $Y \subset V$ admitting a multiplicative basis inherited from $\mathfrak{B}$. Then, for a certain $\emptyset \neq I_Y \subset I$, we can write $Y = \bigoplus_{i \in I_Y} \mathbb{F} v_i$. Fix some $i_0 \in I_Y$ being then

$$0 \neq v_{i_0} \in Y. \tag{2}$$

Let us show by induction on n that if $\{j_1, j_2, \ldots, j_n\}$ is any connection from i_0 to some $k \in I$ then for any $h \in \phi(\phi(\cdots\phi(\{i_0\}, j_1)\ldots), j_n)$ we have that $0 \neq v_h \in Y$.

In case $n = 1$, we get $h \in \phi(\{i_0\}, j_1)$. Hence $h \in i_0 \star j_1$, then, taking into account that Y is a submodule of V, by $\star$-multiplicativity of $\mathfrak{B}$ and Equation (2) we obtain $v_h \in v_{i_0}W \subset Y$.

Suppose now the assertion holds for any connection $\{j_1, j_2, \ldots, j_n\}$ from i_0 to any $r \in I$ and consider some arbitrary connection $\{j_1, j_2, \ldots, j_n, j_{n+1}\}$ from i_0 to any $k \in I$. We know that for $x \in U$, where $U := \phi(\phi(\cdots\phi(\{i_0\}, j_1)\cdots), j_n)$, the element

$$0 \neq v_x \in Y. \tag{3}$$

Taking into account that the fact $h \in \phi(\phi(\cdots\phi(\{i_0\}, j_1)\ldots), j_{n+1})$ means $h \in \phi(U, j_{n+1})$, we have that $h \in x\star j_{n+1}$ for some $x \in U$. From here, the $\star$-multiplicativity of $\mathfrak{B}$ and Equation (3) allow us to get $v_h \in v_xW \subset Y$ as desired.

Since given any $k \in I$ we know that i_0 is connected to k, we can assert by the above observation that $\mathbb{F}v_k \subset Y$. We have shown $V = \bigoplus_{k\in I} \mathbb{F}v_k \subset Y$ and so $Y = V$.

Theorem 3. *Let V be a module, over the linear space W, admitting a $\star$-multiplicative basis $\mathfrak{B}$ respect to a fixed basis of W. Then $V = \bigoplus_k V_k$ is the direct sum of the family of its minimal submodules, each one admitting a $\star$-multiplicative basis inherited from $\mathfrak{B}$.*

Proof. By Theorem 1 we have $V = \bigoplus\limits_{[i]\in I/\sim} V_{[i]}$ is the direct sum of the submodules $V_{[i]}$. Now for any $V_{[i]}$ we have that $\mathfrak{B}_{[i]}$ is a $\star$-multiplicative basis where all of the elements of $[i]$ are connected. Applying Theorem 2 to any $V_{[i]}$ we have that the decomposition $V = \bigoplus\limits_{[i]\in I/\sim} V_{[i]}$ satisfies the assertions of the theorem.

References

1. Bautista, R., Gabriel, P., Roiter, A.V. and Salmeron, L.: Representation-finite algebras and multiplicative basis. Invent. math. 81 (1985), 217–285.
2. Bovdi, V.: On a filtered multiplicative bases of group algebras. II. Algebr. Represent. Theory 6 (2003), no. 3, 353–368.
3. Bovdi, V., Grishkov, A. and Siciliano, S.: Filtered multiplicative bases of restricted enveloping algebras. Algebr. Represent. Theory 14 (2011), no. 4, 601–608.
4. Bovdi, V., Grishkov, A. and Siciliano, S.: On filtered multiplicative bases of some associative algebras. Algebr. Represent. Theory. DOI 10.1007/s10468-014-9494-7.
5. Calderón, A.J. and Navarro, F.J.: Arbitrary algebras with a multiplicative basis. Linear Algebra and its Applications. In press.
6. Chu, Y.J., Huang, F., Zheng, Z.J.: A commutant of $\beta\gamma$-system associated to the highest weight module V_4 of $sl(2, \mathbb{C})$. J. Math. Phys. 51(9), 092301, 32 pp, (2010).
7. Dimitrov, I., Futorny, V., Penkov, I.: A reduction theorem for highest weight modules over toroidal Lie algebras. Comm. Math. Phys. 250(1), 47–63, (2004).
8. Grantcharov, D., Jung, J.H., Kang, S.J., Kim, M.: Highest weight modules over quantum queer superalgebra $U_q(\mathfrak{q}(n))$. Comm. Math. Phys. 296(3), 827-860, (2010).

9. Iohara, K.: Unitarizable highest weight modules of the $N = 2$ super Virasoro algebras: untwisted sectors. Lett. Math. Phys. 91(3), 289-305, (2010).
10. Kupisch, H. and Waschbusch, J.: On multiplicative basis in quasi-Frobenius algebras. Math. Z. 186, (1984), 401-405.
11. Liu, D., Gao, S., Zhu, L.: Classification of irreducible weight modules over W-algebra $W(2,2)$. J. Math. Phys. 49(1), 113503, 6 pp, (2008).
12. Takemura, K.: The decomposition of level-1 irreducible highest-weight modules with respect to the level-0 actions of the quantum affine algebra. J. Phys. A 31, n. 5., 1467-1485, (1998).
13. Zapletal, A.: Difference equations and highest-weight modules of $U_q[\mathrm{sl}(n)]$. J. Phys. A 31, n. 47, 9593-9600, (1998).