ON THE LARGEST SIZE OF AN ANTICHAIN IN THE BRUHAT ORDER FOR $\mathscr{A}(2k,k)$

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ABSTRACT. We discuss a problem proposed by Brualdi and Deaett on the largest size of an antichain in the Bruhat order for the interesting combinatorial class of binary matrices of $\mathscr{A}(2k,k)$.

1. INTRODUCTION

Let m and n be two positive integers and let $R = (r_1, \dots, r_m)$ and $S = (s_1, \dots, s_n)$ be vectors of non–negative integers with $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j$. The set of all $m \times n$ matrices over $\{0,1\}$ with ith row sum equal to r_i, for $1 \leqslant i \leqslant m$, and jth column sum equal to s_j, for $1 \leqslant j \leqslant n$, is commonly denoted by $\mathscr{A}(R,S)$.

Since 1957, the combinatorial properties of $\mathscr{A}(R,S)$ have been a prolific source of several interesting and still open problems (cf. e.g. [2, 3, 4, 5, 7, 8, 9, 16] and references therein). The Gale–Ryser Theorem, originally proved independently in [10] and [17], describing when $(0,1)$–matrices with given row and column sum vectors exist, lies at the heart of the classical combinatorial mathematics. In 1963, Herbert J. Ryser wrote in the preface of his fascinating book [16, p.x]:

> Combinatorial mathematics is tremendously alive at this moment, and we believe that its greatest truths are still to be revealed.

The interesting case in which the nonemptiness is guarantee emerges when $m = n$, k is a positive integer such that $0 \leqslant k \leqslant n$, and $R = S = (k, \dots, k)$ is the constant vector having each component equal to k. In this case we simply write $\mathscr{A}(n,k)$ for $\mathscr{A}(R,S)$.

Motivated by a characterization of the Bruhat order on S_n, the symmetric group of n elements, in [5] Brualdi and Hwang defined a Bruhat partial order $\preccurlyeq$ on a nonempty class $\mathscr{A}(R,S)$. Specifically, for an $m \times n$

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matrix $A = (a_{ij})$, let $\Sigma_A = (\sigma_{ij}(A))$ be the $m \times n$ matrix defined by

$$\sigma_{ij}(A) = \sum_{k=1}^{i} \sum_{\ell=1}^{j} a_{k\ell}\,, \qquad \text{for} \qquad 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\,.$$

If $A_1, A_2 \in \mathscr{A}(R,S)$, then $A_1 \preccurlyeq A_2$ if and only if $\Sigma_{A_1} \geqslant \Sigma_{A_2}$ in the entrywise order, i.e., $\sigma_{ij}(A_1) \geqslant \sigma_{ij}(A_2)$, for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.

Later on, Brualdi and Deaett [4, Theorem 5.1] characterized all families of the class $\mathscr{A}(n,k)$ for which there is a unique minimal element, which are when $k \in \{0, 1, n-1, n\}$ or $n = 2k$.

Since $\mathscr{A}(n,k) \simeq \mathscr{A}(n,n-k)$, $|\mathscr{A}(n,0)| = 1$, and $\mathscr{A}(n,1) \simeq S_n$, the most interesting case is in fact $\mathscr{A}(2k,k)$, for which the minimal matrix is

$$P_k = J_k \oplus J_k = \begin{pmatrix} J_k & O_k \\ O_k & J_k \end{pmatrix},$$

where J_k is the matrix of all 1's and O_k is the zero matrix, both of order k. As an immediate consequence, the unique maximal element is

$$Q_k = \begin{pmatrix} O_k & J_k \\ J_k & O_k \end{pmatrix}.$$

We point out that the sequence of $|\mathscr{A}(2k,k)|$ ($k \in \mathbb{N}$) is coined as A058527, cf. [18], in the *The On–Line Encyclopedia of Integer Sequences*. We observe also that computing a closed manageable formula for such sequence is a still open problem which looks quite hard (cf., e.g., [1, 6, 11, 12, 13, 14, 15, 19, 20] and the references therein for some partial results).

In [4, Section 6] an example is provided to show that Bruhat order $\preccurlyeq$ is not graded, and it is asked what the largest size of an antichain in the Bruhat order in the class $\mathscr{A}(2k,k)$ is. Recall that an antichain in $\mathscr{A}(2k,k)$ is a set of pairwise incomparable elements in that class. In this brief note, carrying on the investigation started in [7], we provide the first estimates which prove that the answer is $O(k^8)$. We remark that this value is asymptotically much greater than the size of the largest chain, which is k^4, as it was shown in [7].

2. The main result

We start this section with our main result.

Theorem 2.1. *For any integer $k \geqslant 2$, let $\vartheta(k)$ be the largest size of an antichain in the Bruhat order in $\mathscr{A}(2k,k)$. Then*

$$\left(\left\lfloor \frac{k}{2} \right\rfloor^4 + 1 \right)^2 \leqslant \vartheta(k) \leqslant \left\lfloor \frac{k^8}{4} \right\rfloor + 1\,,$$

where $\lfloor x \rfloor$ *stands for the largest integer not greater than* x.

Proof. We start proving the upper bound for $\vartheta(k)$.

As an immediate consequence of the definition of antichain, we have

$$\vartheta(k) \leqslant 1 + \max_{A \in \mathscr{A}(2k,k)} \Gamma(A)\,,$$

where

$$\Gamma(A) = |\{M \in \mathscr{A}(2k,k) \text{ such that } M \text{ is incomparable with } A\}|\ .$$

By definition of Bruhat order, it is evident that A and M in $\mathscr{A}(2k,k)$ are incomparable if and only if there exist (u,v) and (w,z) with $1 \leqslant u,v,w,z \leqslant 2k$ such that $\sigma_{uv}(A) > \sigma_{uv}(M)$ and $\sigma_{wz}(A) < \sigma_{wz}(M)$.

Moreover, since $\mathscr{A}(2k,k)$ admits a minimum P_k and a maximum Q_k, obviously

$$\sigma_{ij}(P_k) \leqslant \sigma_{ij}(A) \leqslant \sigma_{ij}(Q_k)\,,$$

for all $1 \leqslant i,j \leqslant 2k$.

For any fixed $A \in \mathscr{A}(2k,k)$, we split Σ_M, for any $M \in \mathscr{A}(2k,k)$, as the disjoint union of

$$\begin{aligned}
\Sigma^{<} &= \{\sigma_{ij}(M),\ \text{with } 1 \leqslant i,j \leqslant 2k,\ \text{such that } \sigma_{ij}(M) < \sigma_{ij}(A)\}\,,\\
\Sigma^{=} &= \{\sigma_{ij}(M),\ \text{with } 1 \leqslant i,j \leqslant 2k,\ \text{such that } \sigma_{ij}(M) = \sigma_{ij}(A)\}\,,\\
\Sigma^{>} &= \{\sigma_{ij}(M),\ \text{with } 1 \leqslant i,j \leqslant 2k,\ \text{such that } \sigma_{ij}(M) > \sigma_{ij}(A)\}\,,
\end{aligned}$$

and clearly an upper bound for $\Gamma(A)$ is given by η_1, the number of all possible choices for $\Sigma^{<}$, times η_2, the number of all possible choices for $\Sigma^{>}$.

In [7] it is shown that

$$\varphi(P_k,Q_k) := \sum_{i=1}^{m}\sum_{j=1}^{n} [\sigma_{ij}(P_k) - \sigma_{ij}(Q_k)] = k^4\,,$$

and an algorithm is presented showing that for any integer value $0 \leqslant c \leqslant k^4$ there exists at least a matrix N such that $\varphi(P_k,N) = c$ and $\varphi(N,Q_k) = k^4 - c$.

Hence we get $\eta_1 + \eta_2 \leqslant k^4$.

We restrict now to the case $k \equiv 0 \pmod 2$. Since the real variables function f defined by $f(x,y) = xy$ in the domain $x > 0$, $y > 0$, and $x+y \leqslant k^4$, admits only a maximum when $x = y = \frac{k^4}{2}$, we may conclude that $\max_{A \in \mathscr{A}(2k,k)} \Gamma(A)$ is achieved when A is such that

$$\varphi(P_k,A) = \varphi(A,Q_k) = \frac{k^4}{2}, \tag{1}$$

and both η_1 and η_2 admit as an upper bound $\frac{k^4}{2}$, and therefore

$$\max_{A\in\mathscr{A}(2k,k)} \Gamma(A) \leqslant \frac{k^8}{4}\,.$$

If $k \equiv 1 \pmod 2$, analogously we get

$$\max_{A\in\mathscr{A}(2k,k)} \Gamma(A) \leqslant \left(\frac{k^4-1}{2}\right)\left(\frac{k^4+1}{2}\right) = \frac{k^8-1}{4} \leqslant \left\lfloor \frac{k^8}{4} \right\rfloor\,.$$

Next, we present a lower bound for $\vartheta(k)$ when $k \equiv 0 \pmod 2$.

Let us consider the matrix

$$A = \left(\begin{array}{cc|cc} J_{\frac{k}{2}} & J_{\frac{k}{2}} & O_{\frac{k}{2}} & O_{\frac{k}{2}} \\ O_{\frac{k}{2}} & O_{\frac{k}{2}} & J_{\frac{k}{2}} & J_{\frac{k}{2}} \\ \hline O_{\frac{k}{2}} & O_{\frac{k}{2}} & J_{\frac{k}{2}} & J_{\frac{k}{2}} \\ J_{\frac{k}{2}} & J_{\frac{k}{2}} & O_{\frac{k}{2}} & O_{\frac{k}{2}} \end{array}\right) = \left(\begin{array}{c|c|c} J_{\frac{k}{2}} & & O_{\frac{k}{2}} \\ O_{\frac{k}{2}} & P_{\frac{k}{2}} & J_{\frac{k}{2}} \\ \hline O_{\frac{k}{2}} & & J_{\frac{k}{2}} \\ J_{\frac{k}{2}} & Q_{\frac{k}{2}} & O_{\frac{k}{2}} \end{array}\right)$$

$$= \left(\begin{array}{c|c|c} J^*_{\frac{k}{2}} & & O^*_{\frac{k}{2}} \\ O^*_{\frac{k}{2}} & P^\bullet_{\frac{k}{2}} & J^*_{\frac{k}{2}} \\ \hline O^\dagger_{\frac{k}{2}} & & J^\dagger_{\frac{k}{2}} \\ J^\dagger_{\frac{k}{2}} & Q^\odot_{\frac{k}{2}} & O^\dagger_{\frac{k}{2}} \end{array}\right)$$

which satisfies (1) (and actually it is the matrix generated at step $\frac{k^4}{2}$ by the algorithm in [7]). We use symbols $^\bullet$, $^\odot$, *, and † just to mark and indicate the corresponding submatrices of A. Note that $^\bullet \simeq {}^* \simeq P_{\frac{k}{2}}$ and $^\odot \simeq {}^\dagger \simeq Q_{\frac{k}{2}}$.

The *Chain* algorithm of [7] generates a chain of maximal length n^4 between P_n and Q_n, for any integer $n \geqslant 2$, and it is straightforward to see that it can be reverted, viz. we can consider the *Rev–Chain* algorithm which generates the same chain backwards from Q_n and P_n.

Clearly applying simultaneously Chain and Rev–Chain algorithms to $^\bullet$ and $^\odot$, and denoting this operation as *central–antichain* algorithm, we get $(\frac{k}{2})^4+1$ elements incomparable, and the same is true considering submatrices * and †. This last operation is denoted by *lateral–antichain* algorithm .

In fact, it is possible to apply independently both central–antichain and lateral–antichain algorithms to A and still getting an antichain, viz. $Z = \{A^{ij} \,|\, 0 \leqslant i,j \leqslant (\frac{k}{2})^4\}$ is an antichain, where A^{ij} is the matrix obtained from A applying i–times the central–antichain algorithm and j–times the lateral–antichain algorithm, so we get an instance of an antichain having size

$$\left(\left(\frac{k}{2}\right)^4+1\right)^2\,.$$

It is easy to see that Z is an antichain because the upper half of the matrix A is the disjoint union of two submatrices $P_{\frac{k}{2}}$, whereas the lower half is the disjoint union of two submatrices $Q_{\frac{k}{2}}$, hence for any transformation we apply, the upper half goes up in the Bruhat order, and the lower half goes down, and therefore the resulting elements are incomparable.

For any integer $k \geqslant 3$, not necessary even, we obviously have $\vartheta(k-1) \leqslant \vartheta(k)$, and the desired result follows. $\square$

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