NOTES ON THE PRODUCT OF LOCALES

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Dedicated to Richard N. Ball on the occasion of his retirement

Abstract. Products of locales (generalized spaces) are coproducts of frames. Because of the algebraic nature of the latter they are often viewed as algebraic objects without much topological connotation. In this paper we first analyze the frame construction emphasizing its tensor product carrier. Then we show how it can be viewed topologically, that is, in the sum-of-the-open-rectangles perspective. The main aim is to present the product from different points of view, as an algebraic and a geometric object, and persuade the reader that both of them are fairly transparent.

Introduction

Pointfree spaces (locales) can be, roughly (but not very roughly) speaking, viewed as generalized topological spaces. Therefore, one wishes and expects to be able to carry out with these objects as with spaces, and hopes for a transparent relationship between the corresponding concepts.

One of such constructions is the product, an operation without which one would not make headway with any serious development of the theory. Now there are (at least) two circumstances indicating that a product of locales may not be very intuitive, and that its connection with the product of spaces may not be very transparent:

— the category of locales is substantially bigger than that of topological spaces (more precisely, sober topological spaces but this does not really influence the situation); thus, the product (the *categorical* product) confronts the spaces entering the operation with a much

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bigger class of objects than they have to cope with, in contrast to the classical case of spaces,

— and, from the technical point of view, the category of locales, although it can be given a covariant space-like interpretation, is first of all the dual of the category of frames, a category of algebraic objects that may require a specific algebraic construction; this is something one does not have in classical spaces and hence the geometric intuition is in fact missing.

Nevertheless, the localic product can be made transparent, and this is the aim of the present paper.

In Section 2 we concentrate on the algebraic aspects of the construction. First, we point out the striking similarity of the construction with the construction of the tensor product of abelian groups. Then we show that this similarity has some flaws, and finally explain what the exact parallels are: frames enrich a certain structure in a distributive manner in full analogy with that of the multiplication of commutative rings enriching the abelian group structure (and both are part of a much more general phenomenon) and there is indeed a tensor product involved. Instead of one imperfect analogy we obtain two perfect fittings.

In Section 3 we explain what happens in a product of spaces respectively locales from the geometric (topological) point of view. We show that as for spaces, where the open sets are unions of rectangles $U \times V$ (U, V open), we have here again "sums of rectangles" $a \oplus b$ and that the difference between $X \times Y$ in the category of spaces and in the category of locales is in that the sums may be somewhat looser, and we explain when they are exactly the same.

In the last section we mention a few cases showing that this difference can in fact be beneficent rather than an unpleasant consequence of the generalization.

It is not the intention of this paper to present new results but to enable the reader to see the locale product from a number of different perspectives based on well-known constructions in various categories. In the process we hope that we have thereby emphasised that it is something natural and, in particular, geometrically transparent.

1. Preliminaries

1.1. Recall that a *frame* L is a complete lattice satisfying the distribution law

$$(\bigvee A) \wedge b = \bigvee \{a \wedge b \mid a \in A\},$$

and that a *frame homomorphism* preserves all joins (including the bottom element 0) and all finite meets (including the top element 1). The resulting *category of frames* will be denoted by **Frm**.

A typical frame is the lattice $\Omega(X)$ of all open sets of a topological space X, and if $f\colon X \to Y$ is a continuous map we have a frame homomorphism $\Omega(f) = (U \mapsto f^{-1}[U])\colon \Omega(Y) \to \Omega(X)$. Thus one obtains a contravariant functor $\Omega\colon \mathbf{Top} \to \mathbf{Frm}$. The dual to **Frm** is called the *category of locales* and will be denoted by **Loc**, and we now have a covariant functor

$$\Omega\colon \mathbf{Top} \to \mathbf{Loc}. \tag{covΩ}$$

Not every locale can be represented as an $\Omega(X)$. Those that can are said to be *spatial*.

1.2. A filter F in L is said to be *completely prime* if we have for arbitrary joins

$$\bigvee A \in F \quad \Rightarrow \quad \exists a \in A,\ a \in F.$$

A typical completely prime filter in $\Omega(X)$ is

$$\mathcal{F}(x) = \{U \in \Omega(X) \mid x \in U\},$$

the system of all open neighbourhoods of a point $x \in X$. A space X is *sober* if there are no other completely prime filters in $\Omega(X)$ (that is, "each completely prime filter has a center"; thus, sobriety is a sort of very weak completeness requirement). For instance, every Hausdorff space is sober.

The functor Ω restricted to the subcategory of sober spaces $\mathbf{Sob} \subseteq \mathbf{Top}$ is a full embedding. Thus, we can think of **Loc** as an extension of **Sob** to a more general type of spaces (in fact, one usually views locales as generalized topological spaces, sober or not).

1.3. Spectrum. One can think of a point x in a topological space as represented by the $\mathcal{F}(x)$; this is what one does when realistically contemplating points in a space: not as an entity without extent, but as a system of spots with diminishing extent concentrating to it.

This naturally leads to the concept of a *point in a frame* L defined as a completely prime filter $F \subseteq L$. The set ΣL of all points of L endowed with the topology

$$\{\Sigma_a \mid a \in L\} \ \text{ where } \ \Sigma_a = \{F \in \Sigma L \mid a \in F\}$$

is called the *spectrum* of L. Note that:

(1) $\Sigma_0 = \emptyset$, $\Sigma_1 = \Sigma L$, $\Sigma_{a \wedge b} = \Sigma_a \cap \Sigma_b$ and $\Sigma_{\bigvee a_i} = \bigcup \Sigma_{a_i}$.
(2) If X is a sober space then $\Sigma\Omega(X)$ is naturally isomorphic to X by the map $x \mapsto \mathcal{F}(x)$.

For a frame homomorphism $h\colon L \to M$ we have a continuous map $\Sigma h\colon \Sigma M \to \Sigma L$ sending F to $h^{-1}[F]$ (note that $\Omega(f)^{-1}[\mathcal{F}(x)] = \{U \in \Omega(Y) \mid f^{-1}[U] \ni x\} = \{U \in \Omega(Y) \mid f(x) \in U\} = \mathcal{F}(f(x))$.

Thus we obtain the *spectrum functor*

$$\Sigma\colon \mathbf{Loc} \to \mathbf{Top}.$$

When interpreting Ω and Σ geometrically – that is, as something happening with spaces – we use the localic notation and interpretation, however computing is usually done in frames. No confusion should arise.

We have the adjunction

$$\Omega \dashv \Sigma \quad (\Omega \text{ to the left, } \Sigma \text{ to the right})$$

connecting the category of topological spaces with the category of more general spaces, that is, locales, sending $h\colon \Omega(X) \to L$ in **Loc** to $\overline{h} = (x \mapsto h^{-1}[\mathcal{F}(x)])\colon X \to \Sigma L$, and $f\colon X \to \Sigma L$ to $\widetilde{f} = (a \mapsto f^{-1}[\Sigma_a])\colon L \to \Omega(X)$.

1.4. We will also consider the category **SLat** of (bounded) meet-semilattices (that is, meet-semilattices with 0 and 1). Note that in this category the cartesian product carries the biproduct

$$L_i \xrightarrow{\iota_i} L_1 \times L_2 \xrightarrow{\pi_j} L_j \quad (i, j = 1, 2)$$

with $\iota_1(x) = (x, 1)$, $\iota_2(x) = (1, x)$ and $\pi_i(x_1, x_2)$, characterized by the equalities

$$\pi_i \iota_j = \begin{cases} \mathrm{id}_{L_i} & \text{if } i = j, \\ 1 & \text{if } i \neq j \end{cases} \qquad \text{and} \qquad \iota_1\pi_1 \wedge \iota_2\pi_2 = \mathrm{id}_{L_1 \times L_2}.$$

1.5. In a poset we write, as usual,

$$\downarrow M = \{x \mid x \leq m \text{ for some } m \in M\},$$

and abbreviate $\downarrow\{m\}$ to $\downarrow m$. For a (bounded) meet-semilattice L we set

$$\mathfrak{D}(L) = \{U \subseteq L \mid \emptyset \neq U = \downarrow U\}.$$

$\mathfrak{D}(L)$ is a frame (with the intersection for meet and the union for join) and we have the following

Fact. *Define $\alpha_L\colon L \to \mathfrak{D}(L)$ by $\alpha_L(x) = \downarrow x$. Let M be a frame. Then for every semilattice homomorphism $f\colon L \to M$ there is precisely one frame homomorphism $h\colon \mathfrak{D}(L) \to M$ such that $h \cdot \alpha_L = f$.*

(Namely the mapping defined by $h(U) = \bigvee\{f(x) \mid x \in U\}$.)

Thus, the system $\alpha_L\colon L \to \mathfrak{D}(L)$ constitutes a free extension of semilattices to frames. A similar construction will be discussed in 3.1 below.

For more about frames the reader can consult, e.g., [10, 18] or [15].

2. A striking similarity and its flaws

2.1. Recall the construction of the coproduct of two objects L_1, L_2 in the category of frames. See the following diagram:

$$\begin{tikzcd}
 & & L_1 \oplus L_2 = \mathfrak{D}(L_1 \times L_2)/R \arrow[ddr, "h"] & \\
 & \mathfrak{D}(L_1 \times L_2) \arrow[ur, dashed, "\kappa"] \arrow[drr, "\overline{f}"] & & \\
L_1 \times L_2 \arrow[ur, dashed, "\alpha"] \arrow[rrr, "f"] & & & M \\
L_j \arrow[u, dashed, "i_j"] \arrow[urrr, "h_j"'] \arrow[uuurr, bend left=60, "\iota_j"] & & &
\end{tikzcd}$$

Take, first, the product (carried by the cartesian product) $L_1 \times L_2$. Then, take the downset frame $\mathfrak{D}(L_1 \times L_2)$ and the embedding

$$\alpha = (a \mapsto {\downarrow} a)\colon L_1 \times L_2 \to \mathfrak{D}(L_1 \times L_2).$$

Finally, take the quotient

$$\kappa\colon \mathfrak{D}(L_1 \times L_2) \to \mathfrak{D}(L_1 \times L_2)/R$$

by (the congruence generated by) the relation R consisting of all the pairs

$$\Big({\downarrow}\big(\bigvee a_i, b\big), \bigcup {\downarrow}(a_i, b)\Big) \quad \text{and} \quad \Big({\downarrow}\big(a, \bigvee b_i\big), \bigcup {\downarrow}(a, b_i)\Big). \tag{2.1.1}$$

If we consider the mappings $i_j\colon L_j \to L_1 \times L_2$ defined by $i_1(a) = (a, 1)$, $i_2(a) = (1, a)$ we easily see that the resulting $\iota_j = \kappa\alpha i_j$ are frame homomorphisms. Now let $h_j\colon L_j \to M$ be arbitrary frame homomorphisms. Realizing that the cartesian product $L_1 \times L_2$ is a biproduct in the category of bounded semilattices we obtain a semilattice homomorphism $f\colon L_1 \times L_2 \to M$ such that $f \cdot i_j = h_j$. It is not a frame homomorphism; this is partly mended by lifting to a frame homomorphism $\overline{f}$ such that $\overline{f} \cdot \alpha = f$. This is still not good enough: the αi_j are not yet frame homomorphisms. But the factorization κ finishes the job: we have the already mentioned frame homomorphisms $\iota_j = \kappa\alpha i_j$, and $\overline{f}$ is easily seen to factorize through κ to the desired h such that $h \cdot \iota_j = h_j$.

2.2. Concentrating on the path from the L_j to the $\mathfrak{D}(L_1 \times L_2)/R$ we see a striking similarity between this construction and the construction of a tensor product of two abelian groups A_1 and A_2: there one starts with taking the cartesian product, forgetting the structure (we have forgotten at this stage a part of the structure of $L_1 \times L_2$ as well), then takes the free group over the product, and, finally, one adjusts it to fit the additive structure. See the following table in which we have omitted, for easier comparison, the $\downarrow$'s.

coproduct in **Frm**	tensor product in **Ab**
$L_1 \times L_2$ cartesian product	$A_1 \times A_2$ cartesian product
$\mathfrak{D}(L_1 \times L_2)$ free construction	$\mathfrak{F}(A_1 \times A_2)$ free construction
adjustment for $\bigvee$: $\big(\bigvee a_i, b\big) \sim \bigvee (a_i, b)$ $\big(a, \bigvee b_i\big) \sim \bigvee (a, b_i)$	adjustment for $+$: $(a_1 + a_2, b) \sim (a_1, b) + (a_2, b)$ $(a, b_1 + b_2) \sim (a, b_1) + (a, b_2)$

2.3. At a closer scrutiny, however, we see that we have neglected important differences as well. See the following table, where the similarity somewhat fades.

coproduct in **Frm**	tensor product in **Ab**
$L_1 \times L_2$ biproduct in **SLat** STRUCTURED	$A_1 \times A_2$ cartesian product UNSTRUCTURED
$\mathfrak{D}(L_1 \times L_2)$ free construction **SLat**$\to$**Frm** (the semilattice structure is important)	$\mathfrak{F}(A_1 \times A_2)$ free group OVER A SET
adjustment for $\bigvee$: $\big(\bigvee a_i, b\big) \sim \bigvee (a_i, b)$ $\big(a, \bigvee b_i\big) \sim \bigvee (a, b_i)$ TO MAKE INJECTIONS HOMOMORPHIC	adjustment for $+$: $(a_1 + a_2, b) \sim (a_1, b) + (a_2, b)$ $(a, b_1 + b_2) \sim (a, b_1) + (a, b_2)$ TO MAKE A BIMORPHISM INTO A MORPHISM

First, on the left hand side it is important that the $L_1 \times L_2$ is a biproduct in **SLat**, something structurally very specific; at the same stage on the right hand side we have simply a cartesian product. In the next line, on the left hand side there is a free construction linking a weaker (but fundamentally engaged) structure with a stronger one, while on the other side one has a standard free algebra formed on an unstructured set. The last line seems to be really quite the same thing: adjusting the additive structure. But the purpose is different: on the left hand side we make the injections into homomorphisms, on the right hand side we make *bi*homomorphisms into homomorphisms.

2.4. On the other hand, look at the coproduct in the category of commutative rings with unit (see for instance [4]). It can be constructed via the underlying abelian additive structure. One has the free construction

$$\mathfrak{F}' \colon \mathbf{CSgr} \to \mathbf{CRing}$$

from commutative semigroups to rings carried by the free group $\mathfrak{F}'(S)$ which can be thought of as the set of formal linear combinations $\sum_{x\in S} k_x x$ with k_x integers and all but finitely many of them 0, endowed with the multiplication

$$\Big(\sum_{x\in S} k_x x\Big) \cdot \Big(\sum_{y\in S} l_y y\Big) = \sum_{x,y\in S} (k_x l_y)(xy).$$

Proving that the coproduct is obtained goes along the same lines as the procedure used for frames. Now we have a perfect fit for coproducts as depicted in the following table.

coproduct in **Frm**	coproduct in **CRing**
$L_1 \times L_2$ structure partially forgotten biproduct in **SLat**	$A_1 \times A_2$ structure partially forgotten biproduct in **CSgr**
$\mathfrak{D}(L_1 \times L_2)$ free functor **SLat**→**Frm**	$\mathfrak{F}'(A_1 \times A_2)$ free functor **CSgr**→**CRing**
adjustment for $\bigvee$: $\big(\bigvee a_i, b\big) \sim \bigvee(a_i, b)$ $\big(a, \bigvee b_i\big) \sim \bigvee(a, b_i)$ to make the injections homomorphic	adjustment for $+$: $(a_1 + a_2, b) \sim (a_1, b) + (a_2, b)$ $(a, b_1 + b_2) \sim (a, b_1) + (a, b_2)$ to make the injections homomorphic

In the original observation in 2.2, however, we had a striking similarity between the construction of a *coproduct* in one context and a *tensor product* in another one. The tensor product, which has now vanished from the scene, is an important construction providing a universal bimorphism. The universality of coproducts is of a different nature and we have here different categorical phenomena. The question arises whether, besides the coproduct vs. coproduct fit as above, there also is a tensor vs. tensor one. In the next section we will show that there is a tensor product directly connected with the coproduct of frames, fitting well into the left hand side of the picture.

3. Tensor products in **SupLat**

3.1. Recall from [11] the category

$$\mathbf{SupLat}$$

of complete lattices with $\bigvee$-preserving maps; in this context one usually speaks of *sup-lattices* and *sup-homomorphisms*.

The downset construction from 1.5 can be modified to a functor

$$\mathfrak{D}'\colon \mathbf{Pos} \to \mathbf{SupLat}, \quad \mathfrak{D}'(X, \le) = \Big(\{M \subseteq X \mid \ {\downarrow} M = M\}, \subseteq\Big)$$

($M = \emptyset$ is now allowed) and we have again a free extension.

3.1.1. Proposition. *Set $\alpha(x) = {\downarrow} x$. Then $\alpha\colon X \to \mathfrak{D}'(X)$ is monotone, and for every monotone $f\colon X \to Y$ with X a poset and Y a complete lattice there is precisely one sup-homomorphism $\overline{f}\colon \mathfrak{D}'(X) \to Y$ such that $\overline{f} \cdot \alpha = f$.*

Proof. Since $U \in \mathfrak{D}'(X)$ is the join $\bigcup\{{\downarrow} u \mid u \in U\}$ there is at most one such $\overline{f}$ and we have $\overline{f}(U) = \bigvee_{u\in U} \overline{f}({\downarrow} u) = \bigvee_{u\in U} f(u)$, which, on the other hand, is a formula defining a sup-homomorphism. □

3.2. Recall computing a quotient of a frame from e.g. [18, 15] (basically, already from [10]). It can be done as follows. For a binary relation $R \subseteq L \times L$ call an element $s \in L$ *R-saturated* if

$$\forall a, b, c \ \ aRb \ \Rightarrow \ (a \wedge c \le s \text{ iff } b \wedge c \le s). \qquad \text{(f-sat)}$$

Then one has the following:

- Any meet of R-saturated elements is R-saturated and hence set $\mu(x) = \bigwedge\{s \text{ saturated} \mid x \le s\}$.
- μ is monotone, $x \le \mu(x)$, $\mu\mu(x) = \mu(x)$ and $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$.
- If we set
$$L/R = \{x \in L \mid x = \mu(x)\}$$
we obtain a frame (with the supremum given by the formula $\bigsqcup a_i = \mu(\bigvee a_i)$) and an onto frame homomorphism $\mu\colon L \to L/R$.
- For all $(a, b) \in R$ we have $\mu(a) = \mu(b)$ and for every frame homomorphism $h\colon L \to M$ such that $(a, b) \in R$ implies $h(a) = h(b)$ there is precisely one homomorphism $\overline{h}\colon L/R \to M$ such that $\overline{h} \cdot \mu = h$; furthermore, $\overline{h} = h|_{L/R}$.

3.3. Computing quotients of sup-lattices is similar. For an $R \subseteq L \times L$ call an element $s \in L$ R-$\bigvee$*saturated* if

$$\forall a, b \;\; aRb \;\Rightarrow\; (a \leq s \text{ iff } b \leq s). \qquad (\textstyle\bigvee\text{-sat})$$

Obviously, again, any meet of R-$\bigvee$saturated sets is R-$\bigvee$saturated; consequently, we have the R-$\bigvee$saturated

$$\nu(a) = \bigwedge\{s \in L \mid s \text{ is } R\text{-}\bigvee \text{ saturated and } a \leq s\}$$

and we easily see that one has

$$x \leq \nu(x), \quad x \leq y \;\Rightarrow\; \nu(x) \leq \nu(y), \quad \text{and} \;\; \nu\nu(x) = \nu(x)$$

and if we set

$$L/R = \nu[L] = \{x \in L \mid \nu(x) = x\}$$

we have a factorization theorem quite like in frames.

3.3.1. Theorem. *L/R is a sup-lattice with suprema $\bigsqcup x_i = \nu(\bigvee x_i)$ and the corestriction $\nu_R \colon L \to L/R$ of ν to $L \to L/R$ is a(n onto) sup-homomorphism such that for aRb we have $\nu_R(a) = \nu_R(b)$.*

For every sup-homomorphism $h \colon L \to M$ such that

$$aRb \;\Rightarrow\; h(a) = h(b)$$

there is a sup-homomorphism $\overline{h} \colon L/R \to M$ such that $\overline{h} \cdot \nu_R = h$. Moreover, $\overline{h}(a) = h(a)$ for all $a \in L/R$.

Proof. If $x \in L/R$ is such that $x \geq x_j$ for all j then $x \geq \bigvee x_i$ and $x = \nu(x) \geq \nu(\bigvee x_i) = \bigsqcup x_i$. Further, we have $\nu(\bigvee x_i) \leq \nu(\bigvee \nu(x_i)) = \bigsqcup \nu(x_i) \leq \nu(\bigvee x_i)$, the last inequality being trivial.

Now if aRb then $b \leq \nu(a)$ since $a \leq \nu(a)$ and $\nu(a)$ is R-$\bigvee$saturated. Hence $\nu(b) \leq \nu(a)$ and by symmetry $\nu(b) = \nu(a)$.

Finally if $h \colon L \to M$ is a sup-homomorphism such that aRb implies $h(a) = h(b)$ set $\sigma(x) = \bigvee\{y \mid h(y) \leq h(x)\}$. Obviously $x \leq \sigma(x)$ and $h\sigma(x) = h(x)$. If aRb and $a \leq \sigma(x)$ then $h(b) = h(a) \leq h\sigma(x) = h(x)$ and hence $b \leq \sigma(x)$. Thus, σ is R-$\bigvee$saturated. and we see that $x \leq \nu(x) \leq \sigma(x)$ and hence $h(x) \leq h\nu(x) \leq h\sigma(x) = h(x)$ so that $h\nu(x) = h(x)$ and the statement follows. $\square$

3.4. An important observation. If L is a frame and if $R \subseteq L \times L$ respects the meet (that is, if aRb implies $(a \wedge c)R(b \wedge c)$ for all c) then the formula (f-sat) can be replaced by ($\bigvee$-sat). Thus we have

3.4.1. Corollary. *If L is a frame and if $R \subseteq L \times L$ respects the meet then L/R taken as a quotient frame coincides with the L/R taken as a quotient sup-lattice. In particular this holds for the relation R given by* (2.1.1). $\square$

3.5. Tensor product in SupLat. Let X_1, X_2 and Y be sup-lattices. A mapping $\phi\colon X_1 \times X_2 \to Y$ is said to be a *bimorphism* if each $\phi(x_1, -)\colon X_2 \to Y$ and each $\phi(-, x_2)\colon X_1 \to Y$ is a morphism.

For X_1, X_2 in **SupLat** construct $X_1 \otimes X_2$ as $\mathfrak{D}'(X_1 \times X_2)/R$ with the R from 2.1 (extended for sup-lattices). We will speak of $X_1 \otimes X_2$ as the *tensor product* of X_1 and X_2.

3.5.1. Theorem. *The mapping $\nu = \kappa\alpha\colon X_1 \times X_2 \to X_1 \otimes X_2$ is a bimorphism and for every bimorphism $\phi\colon X_1 \times X_2 \to Y$ there is precisely one morphism $\widetilde{\phi}\colon X_1 \otimes X_2 \to Y$ such that $\widetilde{\phi}\nu = \phi$. This property determines $X_1 \otimes X_2$ up to isomorphism.*

If X_1, X_2 are frames then the tensor product $X_1 \otimes X_2$ coincides with the coproduct $X_1 \oplus X_2$.

Proof. Consider the following diagram (note that it is almost identical to the diagram in 2.1, but the interpretation slightly differs).

$$\begin{tikzcd}
 & & X_1 \otimes X_2 = \mathfrak{D}'(X_1 \times X_2)/R \arrow[ddr, "\widetilde{\phi}"] & \\
 & \mathfrak{D}'(X_1 \times X_2) \arrow[ur, "\kappa"] \arrow[drr, "\overline{\phi}"] & & \\
X_1 \times X_2 \arrow[ur, "\alpha"] \arrow[rrr, dashed, "\phi"] & & & Y
\end{tikzcd}$$

Since ϕ is obviously monotone, we have by 3.1.1 a sup-homomorphism $\overline{\phi}$ such that $\overline{\phi}\alpha = \phi$. We have

$$\overline{\phi}\big(\bigcup \downarrow(a_i, b)\big) = \bigvee \overline{\phi}(\downarrow(a_i, b)) = \bigvee \phi(a_i, b) = \phi\big(\bigvee a_i, b\big) = \overline{\phi}\big(\downarrow(\bigvee a_i, b)\big)$$

and similarly $\overline{\phi}\big(\bigcup \downarrow (a, b_i)\big) = \overline{\phi}\big(\downarrow (a, \bigvee b_i)\big)$ and hence there is a sup-homomorphism $\widetilde{\phi}$ such that $\widetilde{\phi}\kappa = \overline{\phi}$ and finally $\widetilde{\phi}\nu = \widetilde{\phi}\kappa\alpha = \phi$.

The uniqueness is obvious. $\square$

3.5.2. Note. This tensor product can be extended to a closed monoidal structure in **SupLat**. Namely, for sup-lattices Y, Z consider

$$\mathsf{Hom}(Y, Z) = \{h\colon Y \to Z \mid h \in \mathbf{SupLat}\}$$

naturally ordered by $h_1 \le h_2$ if for all $y \in Y$, $h_1(y) \le h_2(y)$. Then we have the supremum $\bigvee_{j\in J} h_j$ defined by $(\bigvee_{j\in J} h_j)(y) = \bigvee_{j\in J} h_j(y)$. Setting for a sup-preserving $f\colon X \otimes Y \to Z$, $f'(x)(y) = (f\nu)(x, y)$ we easily check that we have defined a mapping $f'\colon X \to \mathsf{Hom}(Y, Z)$; on the other hand, for a $g\colon X \to \mathsf{Hom}(Y, Z)$ we have obviously a bimorphism $h\colon X \times Y \to Z$ defined by $h(x, y) = g(x)(y)$ and we can define $g^\circ\colon X \otimes Y \to Z$ by $g^\circ\nu = h$. Then

$(g^\circ)'(x)(y) = (g^\circ\nu)(x,y) = g(x)(y)$ and $(f')^\circ\nu(x,y) = f'(x)(y) = f\nu(x,y)$ making also $(f')^\circ = f$; checking that the correspondences $(f \mapsto f')$ and $(g \mapsto g^\circ)$ are natural is straightforward.

3.5.3. Now we have the situation from 2.3 mended to

tensor product in **SupLat**	tensor product in **Ab**
$L_1 \times L_2$ product of posets	$A_1 \times A_2$ product of sets
$\mathfrak{D}(L_1 \times L_2)$ free construction **Pos**$\to$**SupLat**	$\mathfrak{F}(A_1 \times A_2)$ free construction **Set**$\to$**Ab**
adjustment for $\bigvee$: $\big(\bigvee a_i, b\big) \sim \bigvee(a_i, b)$ $\big(a, \bigvee b_i\big) \sim \bigvee(a, bi)$ to make a bimorphism into a morphism	adjustment for $+$: $(a_1 + a_2, b) \sim (a_1, b) + (a_2, b)$ $(a, b_1 + b_2) \sim (a, b_1) + (a, b_2)$ to make a bimorphism into a morphism

There is still a difference, but not a very important one: instead of sets we have partially ordered sets. But the products on the both sides play the same role, there is no biproduct property involved.

3.6. The coproduct as an almost tensor product in Frm. Contemplating 3.4.1 we can try to interpret the diagram from the previous subsection in the category **Frm**. Now if we define a bimorphism $\phi\colon L_1 \times L_2 \to M$ as a mapping such that all the $\phi(a,-)\colon L_2 \to M$ and all the $\phi(-,b)\colon L_1 \to M$ are frame homomorphisms, the $\overline{\phi}$ (not to speak of the induced map $\widetilde{\phi}$) is in general not a frame homomorphism. The trouble is with the meet. Luckily enough, the mapping ν has a special property which can be used to mend the definition. Namely we have $\nu(x_1, x_2) = \nu(x_1, 1) \wedge \nu(1, x_2)$ and if we define a *frame bimorphism in* **Frm** as a mapping

$$\phi\colon L_1 \times L_2 \to M$$

such that all the $\phi(a,-)\colon L_2 \to M$ and all the $\phi(-,b)\colon L_1 \to M$ are frame homomorphisms and moreover

$$\forall x_i \in L_i, \quad \phi(x_1, x_2) = \phi(x_1, 1) \wedge \phi(1, x_2) \tag{3.6.1}$$

we obtain a (very restricted) tensor behavior.

3.6.1. Proposition. *Let L_1, L_2 be frames. Then (replacing in 3.5.1 X_i by L_i) we have a frame bimorphism $\nu\colon L_1 \times L_2 \to L_1 \otimes L_2$ universal in the sense that for every frame bimorphism $\phi\colon L_1 \times L_2 \to M$ there is precisely one frame homomorphism $\widetilde{\phi}$ such that $\widetilde{\phi}\nu = \phi$.*

Proof. We will prove that $\overline{\phi}$ is a frame homomorphism (for which it suffices to check it preserves meets). Then it follows that $\widetilde{\phi}$ is a frame homomorphism (and the uniqueness is obvious). We have

$$\begin{aligned}
\overline{\phi}(U) \wedge \overline{\phi}(V) &= \bigvee_{(u_1,u_2)\in U} \phi(u_1,u_2) \wedge \bigvee_{(v_1,v_2)\in V} \phi(v_1,v_2) \\
&= \bigvee\{\phi(u_1,u_2) \wedge \phi(v_1,v_2) \mid (u_1,u_2)\in U, (v_1,v_2)\in V\} \\
&= \bigvee\{\phi(u_1,1) \wedge \phi(1,u_2) \wedge \phi(v_1,1) \wedge \phi(1,v_2) \mid \cdots\} \\
&= \bigvee\{\phi(u_1\wedge v_1,1) \wedge \phi(1,u_2\wedge v_2) \mid \cdots\} \\
&= \bigvee\{\phi(u_1\wedge v_1,u_2\wedge v_2) \mid (u_1,u_2)\in U, (v_1,v_2)\in V\} \\
&\le \bigvee\{\phi(x_1,x_2) \mid (x_1,x_2)\in U\cap V\} \\
&= \overline{\phi}(U\cap V) \le \overline{\phi}(U) \wedge \overline{\phi}(V). \quad \square
\end{aligned}$$

3.6.2. Notes. (1) Recall the construction from 2.1. The coproduct $L_1 \oplus L_2$ is obviously join-generated by the $\kappa(\downarrow(x_1,x_2))$. These elements will be denoted by $x_1 \oplus x_2$. The condition (3.6.1) is necessary to have $\widetilde{\phi}$ a frame homomorphism. We have $\nu(x_1,x_2) = \kappa(\downarrow(x_1,x_2)) = x_1 \oplus x_2$ and hence in particular

$$\begin{aligned}
\phi(x_1,x_2) = \widetilde{\phi}\nu(x_1,x_2) = \widetilde{\phi}(x_1\oplus x_2) = \widetilde{\phi}((x_1\oplus 1)\wedge(1\oplus x_2)) = \\
= \widetilde{\phi}(x_1\oplus 1)\wedge\widetilde{\phi}(1\oplus x_2) = \phi(x_1,1)\wedge\phi(1,x_2).
\end{aligned}$$

(2) One cannot extend this tensor product to a closed tensor structure in **Frm** as in 3.5.2. Indeed, if L, M are frames, $\mathsf{Hom}(L,M)$ is generally not a frame and even if it were the $f\colon K\oplus L \to M$ associated with a frame homomorphism $g\colon K \to \mathsf{Hom}(L,M)$ would violate (3.6.1): we would have $f\nu(x,1)\wedge f\nu(1,y) = g(x)(1)\wedge g(1)(y) = 1$, which is very seldom equal to $f\nu(x,y)$.

3.7. Note. The construction of the coproduct we have outlined in 2.1 can be obviously generalized. What one needed was the biproduct behaviour of the semilattices, and a transparent quotient κ, as one has in commutative quantales with unit. A more general treatment of the coproducts of enriched objects obtained from tensor products of underlying objects enriched by an additional operation can be found in [1]. Even more generally one can produce in this vein all colimits in a category in which the objects are endowed by an operation distributing over a basic structure (see [3]).

4. The geometry of the product of locales

4.1. The category of locales is much larger than the category of topological spaces (more exactly, than the category of sober spaces, but the distinction is not important for our analysis of the phenomena here). Thus, we cannot expect the product in **Loc** to be an immediate extension of the product in **Top**. Yet its geometric nature is in fact more transparent than what one might expect. This will be seen when analysing the phenomena in view of the spectrum adjunction connecting **Loc** and **Top** (recall 1.3).

4.2. The functor Ω is a left adjoint and hence it should not be expected to preserve products, and indeed it does not. But the situation is not as bad as one might fear, and we easily gain insight into the situation.

Consider a product of topological spaces $p_i\colon X_1 \times X_2 \to X_i$ $(i = 1, 2)$ and the diagram

$$\begin{tikzcd} \Omega(X_1) \oplus \Omega(X_2) \arrow[rr, "\pi"] & & \Omega(X_1 \times X_2) \\ & \Omega(X_i) \arrow[ul, "\iota_i"] \arrow[ur, "\Omega(p_i)"'] & \end{tikzcd}$$

where π is the unique homomorphism satisfying $\pi\iota_i = \Omega(p_i)$, $i = 1, 2$.

4.2.1. Recall the $a \oplus b = \kappa({\downarrow}(a, b))$ from 3.6.2. It is a standard and often used fact that $a \oplus 0 = 0 \oplus b = 0$, $a \oplus b = 0$ only if $a = 0$ or $b = 0$, and if $a_1, a_2 \neq 0$ and $a_1 \oplus a_2 \leq b_1 \oplus b_2$ then $a_i \leq b_i$ for both $i = 1, 2$.

The basic elements $U_1 \oplus U_2$ in $\Omega(X_1) \oplus \Omega(X_2)$ are closely associated with the basic open rectangles in the space $X_1 \times X_2$. We have

Observation. *For open $U_i \subseteq X_i$, $\pi(U_1 \oplus U_2) = U_1 \times U_2$.*

(Indeed, $\pi(U_1 \oplus U_2) = \pi((U_1 \oplus 1) \wedge (1 \oplus U_2)) = \pi(\iota_1(U_1) \wedge \iota_2(U_2)) = p_1^{-1}[U_1] \wedge p_2^{-1}[U_2] = U_1 \times U_2$.)

4.2.2. Proposition. *π is an onto dense homomorphism. Thus, in the localic language, π embeds $\Omega(X_1 \times X_2)$ into $\Omega(X_1) \oplus \Omega(X_2)$ as a dense sublocale.*

Proof. π is onto since the $U_1 \times U_2$ generate the topology of $X_1 \times X_2$ by unions. Now if $\pi(x) = \emptyset$ then

$$\emptyset = \pi\Big(\bigvee\{U_1 \oplus U_2 \mid U_1 \oplus U_2 \leq x\}\Big) = \bigcup\{U_1 \times U_2 \mid U_1 \oplus U_2 \leq x\}$$

so that if $U_1 \oplus U_2 \leq x$ then either U_1 or U_2 is $\emptyset$, and hence $x = 0$. $\square$

4.3. Thus, the localic product $\Omega(X_1) \times \Omega(X_2)$ (that is, frame coproduct $\Omega(X_1) \oplus \Omega(X_2)$) is not quite so far from $\Omega(X_1 \times X_2)$, the product of spaces embedded into **Loc**. This can be made more intuitive as follows.

The topology of the product $X \times Y$ of spaces consists of the unions of rectangles $\bigcup_{i\in J}(U_i \times V_i)$ with U_i open in X and V_i open in Y. This can be also viewed as taking "free joins" $\bigvee_{i\in I}(U_i \times V_i)$ and factorizing the set of such free joins by the equivalence

$$\bigvee_{i\in J}(U_i \times V_i) \sim \bigvee_{i\in J'}(U_i' \times V_i') \quad \text{iff} \quad \bigcup_{i\in J}(U_i \times V_i) = \bigcup_{i\in J'}(U_i' \times V_i'). \qquad (*)$$

The condition on the right hand side is of course heavily point dependent. In some very special particular cases such an equality of unions can be freed of this dependence, though. Namely, this is the case of the unions where all the V_i and V_i' coincide with the common value V, or the unions where all the U_i and U_i' coincide with the common value U, that is, of the unions

$$\bigcup_{i\in J}(U_i \times V) = \bigcup_{i\in J'}(U_i' \times V) \quad \text{and} \quad \bigcup_{i\in J}(U \times V_i) = \bigcup_{i\in J'}(U \times V_i'). \qquad (**)$$

These special cases produce an equivalence $\approx$ generated by

$$\bigvee_{i\in J}(U_i \times V) \approx \bigvee_{i\in J'}(U_i' \times V) \quad \text{and} \quad \bigvee_{i\in J}(U \times V_i) \approx \bigvee_{i\in J'}(U \times V_i').$$

The equivalence $\approx$ is in general weaker than $\sim$, causing the $\Omega(X_1)\oplus\Omega(X_1)$ to have in general more elements than $\Omega(X_1 \times X_2)$. But in some important cases the equivalence $\approx$ does after all coincide with $\sim$.

4.4. A necessary and sufficient condition. The full embedding $\Omega\colon$ **Sob** $\to$ **Top** from 1.1 produces an isomorphic copy **Sob**$'$ of **Sob** in **Loc**. Thus, whenever we have that for X, Y sober, $\Omega(X) \oplus \Omega(Y)$ is spatial, then it is a copy of $X \times Y$ in **Sob**$'$. Hence we have

Proposition. *For sober spaces X, Y the equivalence $\approx$ coincides with $\sim$ (and hence the product of X and Y as locales is the same as their product as spaces) iff $\Omega(X) \oplus \Omega(X)$ is spatial.* $\square$

Here are two important special cases:

4.4.1. Proposition. *Let either X and Y be* (1) *sober locally compact or* (2) *admit complete metrics. Then their product as locales coincide with their product as spaces.*

Proof. (1): Proposition 2.13 in Chapter II of [10]; see also [8] and [6].
(2): It is well known that a space X admits a (complete) metric iff it admits a (complete) countable generated uniformity. Then the coproduct $\Omega(X) \oplus \Omega(Y)$ admits a countably generated uniformity and by [19] it admits a complete one; by [2] (see also Theorem X.2.2 of [15]) this makes $\Omega(X) \oplus \Omega(Y)$ spatial. □

4.5. The product seen from the perspective of the spectrum. A locale (frame) L is generally a richer space than the picture obtained by exploring it by means of spectral points, that is than ΣL. Still, it is useful to realize that in this perspective the $L_1 \oplus L_2$ always appears as the classical product.

To be more precise, since Σ is a right adjoint, the ψ in the diagram

$$\begin{tikzcd} \Sigma(L_1 \oplus L_2) \arrow[rr, "\text{isomorphism } \psi"] \arrow[dr, "\Sigma\iota_i"'] & & \Sigma L_1 \times \Sigma L_2 \arrow[dl, "p_i"] \\ & \Sigma(L_i) & \end{tikzcd}$$

satisfying $p_i\psi = \Sigma\iota_i$ (where p_i are the cartesian projections, $i = 1, 2$) is an isomorphism (that is, a homeomorphism of spaces). If we write (F_1, F_2) for $\psi(F)$ we have $F_i = \Sigma\iota_i(F) = \iota_i^{-1}[F]$. Now consider the open set $\Sigma_{a_1 \oplus a_2}$. We have

$$a_1 \in F_1 \text{ iff } a_1 \oplus 1 \in F \quad \text{and} \quad a_2 \in F_1 \text{ iff } 1 \oplus a_2 \in F$$

yielding that

$$\begin{aligned} (F_1, F_2) \in & \Sigma_{a_1} \times \Sigma_{a_2} \quad \text{iff} \quad a_i \in F_i \quad \text{iff} \quad a_1 \oplus 1 \in F \text{ and } 1 \oplus a_2 \in F \\ & \text{iff} \quad a_1 \oplus a_2 = (a_1 \oplus 1) \wedge (1 \oplus a_2) \in F \quad \text{iff} \quad F \in \Sigma_{a_1 \oplus a_2} \end{aligned}$$

so that the homeomorphism ψ translates $\Sigma_{a_1 \oplus a_2}$ into $\Sigma_{a_1} \times \Sigma_{a_2}$ and consequently, by taking unions, a general open set in $\Sigma(L_1 \oplus L_2)$ into a general open set in $\Sigma L_1 \times \Sigma L_2$.

5. The discrepancy helps

In the previous section we have seen that the difference in the product of spaces as spaces and their product as locales is limited (the former is a dense sublocale of the latter, recall 4.2.2). Still, it can be substantial, and the question naturally arises whether this tribute to the generalizing of the concept of space is not an unpleasant complication of the theory. In fact, it is often rather beneficent as we will illustrate on a few examples.

5.1. Paracompact locales. The class of paracompact spaces is an important generalization of the metric ones, often appearing in applications. These spaces, however, behave very badly in constructions: even a product of a paracompact space with a metric one is not necessarily paracompact. In contrast with this, the category of paracompact locales is very well behaved: it is reflective in the category of all locales. This is due to the product that always exists (although it is not necessarily spatial for spatial factors). The satisfactory behaviour of this category is also connected with Isbell's beautiful characterization of paracompactness [7]:

a locale is paracompact iff it admits a complete uniformity,

a fact that has no counterpart in classical spaces.

5.2. Uniformities in the point-free context. As in spaces, a uniformity on a frame (locale) L can be introduced as a special system of covers of L, or as a suitable system of neighbourhoods of the (co)diagonal in $L \oplus L$. While the former is a straightforward extension of the space concept, the latter is not, since $L \oplus L$ (in the case of a space X, $\Omega(X) \oplus \Omega(X)$) does not exactly corresponds to the product $X \times X$. However, somewhat surprisingly, the two approaches can be shown to be equivalent also here [14, 16]. This equivalence is now a deeper fact, and sometimes a mightier tool in proofs.

5.3. Localic groups. A topological group is not always a localic one because the operation $X \times X \to X$ results just in a localic morphism $\Omega(X \times X) \to \Omega(X)$, not in an operation $\Omega(X) \oplus \Omega(X) \to \Omega(X)$, and neither can be necessarily lifted (over the π from 4.2) to one. It turns out that the nice topological groups (roughly speaking, those that are complete in the natural uniformity) are localic. In particular we have the Closed Subgroup Theorem ([9, 16], Chapter XV of [15]):

each localic subgroup of a localic group is closed,

again a fact without a spatial counterpart.

5.4. Connectedness. The intuition of connectedness is expressed by the connected locally connected spaces better than by the plainly connected one. This is seen in the behaviour of locales where the connected locally connected ones behave as expected while the plainly connected do not ([12, 13], Chapter XIII of [15]). In particular, the product of two connected locally connected locales is connected locally connected, but the product of two plainly connected locales is not necessarily connected.

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