ON THE REFLECTIVENESS OF SPECIAL HOMOGENEOUS SURJECTIONS OF MONOIDS

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Dedicated to Manuela Sobral on the occasion of her seventieth birthday

Abstract. We prove that the so-called *special homogeneous surjections* are reflective amongst surjective homomorphisms of monoids. To do so, we use a general result in categorical Galois theory, and the recent result that the special homogeneous surjections are the normal (= central) extensions with respect to the admissible Galois structure Γ_{Mon} determined by the *Grothendieck group* adjunction together with the classes of surjective homomorphisms.

1. Introduction

In the recent paper [13] we showed that the special homogeneous surjections of monoids (in the sense of [3, 4]) are the *central* extensions, for the admissible Galois structure [6, 7] obtained via the Grothendieck group construction. Moreover, for this Galois structure, the central extensions coincide with the normal extensions. In categorical Galois theory, the central extensions are also called *covering morphisms* [9].

The aim of our present work is to answer the following question: *Is the category of* special homogeneous surjections *of monoids a reflective subcategory of the category of surjective monoid homomorphisms?* The positive answer to this question is a consequence of a general Galois-theoretical result, namely Theorem 4.2 in [9].

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Before specialising to monoids, we first focus on the above mentioned reflectiveness result in a general Galois-theoretic setting (Section 2). Then, in Section 3, we recall the definitions and main results concerning special homogenous surjections from [3, 4] as well as the main results from [13], giving the link between special homogeneous surjections and central extensions, which are needed throughout the subsequent section. In Section 4 we explain how the technique of Section 2 is applicable in order to obtain Theorem 4.1—thus answering our question.

2. Reflectiveness of normal extensions

2.1. **Galois structures.** We start by recalling the definition of *(admissible) Galois structure* as well as the concepts of *trivial*, *normal* and *central extension* arising from it [6, 7, 8]. We consider the context of Barr-exact categories [1] which is general enough for our purposes and allows us to avoid some technical difficulties.

Definition 2.1. *A* ***Galois structure*** $\Gamma = (\mathscr{C}, \mathscr{X}, I, H, \eta, \epsilon, \mathscr{E}, \mathscr{F})$ *consists of an adjunction*

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "I"] \arrow[r, phantom, "\perp"] & \mathscr{X} \arrow[l, shift left, "H"] \end{tikzcd}$$

with unit $\eta\colon 1_{\mathscr{C}} \Rightarrow HI$ *and counit* $\epsilon\colon IH \Rightarrow 1_{\mathscr{X}}$ *between Barr-exact categories* $\mathscr{C}$ *and* $\mathscr{X}$*, as well as classes of morphisms* $\mathscr{E}$ *in* $\mathscr{C}$ *and* $\mathscr{F}$ *in* $\mathscr{X}$ *such that:*

- (G1) $\mathscr{E}$ *and* $\mathscr{F}$ *contain all isomorphisms;*
- (G2) $\mathscr{E}$ *and* $\mathscr{F}$ *are pullback-stable;*
- (G3) $\mathscr{E}$ *and* $\mathscr{F}$ *are closed under composition;*
- (G4) $H(\mathscr{F}) \subseteq \mathscr{E}$*;*
- (G5) $I(\mathscr{E}) \subseteq \mathscr{F}$.

We call the morphisms in $\mathscr{E}$ *and* $\mathscr{F}$ ***fibrations*** [7]*. The following definitions are given with respect to a Galois structure* Γ.

Definition 2.2. *A* ***trivial extension*** *is a fibration* $f\colon A \to B$ *in* $\mathscr{C}$ *such that the square*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & HI(A) \arrow[d, "HI(f)"] \\ B \arrow[r, "\eta_B"'] & HI(B) \end{tikzcd}$$

is a pullback. A ***central extension*** *is a fibration* f *whose pullback* $p^*(f)$ *along* some *fibration* p *is a trivial extension. A* ***normal extension*** *is a fibration such that its kernel pair projections are trivial extensions.*

It is well known and easy to see that trivial extensions are always central extensions and that any normal extension is automatically central.

For any object B in $\mathscr{C}$, there is an induced adjunction

$$
\begin{tikzcd}
(\mathscr{E} \downarrow B) \arrow[r, shift left, "I^B"] \arrow[r, phantom, "\perp"] & (\mathscr{F} \downarrow I(B)), \arrow[l, shift left, "H^B"]
\end{tikzcd}
$$

where we write $(\mathscr{E} \downarrow B)$ for the full subcategory of the slice category $(\mathscr{C} \downarrow B)$ determined by morphisms in $\mathscr{E}$; similarly for $(\mathscr{F} \downarrow I(B))$. The functor I^B is the restriction of I, and H^B sends a fibration $g\colon X \to I(B)$ to the pullback

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "H^B(g)"'] \arrow[dr, phantom, "\lrcorner", very near start] & H(X) \arrow[d, "H(g)"] \\
B \arrow[r, "\eta_B"'] & HI(B)
\end{tikzcd}
$$

of $H(g)$ along η_B.

Definition 2.3. *An object $B \in \mathscr{C}$ is said to be* ***admissible*** *when the functor H^B is full and faithful. A Galois structure $\Gamma = (\mathscr{C}, \mathscr{X}, I, H, \eta, \epsilon, \mathscr{E}, \mathscr{F})$ is* ***admissible*** *when every object B in $\mathscr{C}$ is admissible.*

The admissibility condition amounts to reflectiveness of trivial extensions amongst fibrations. More precisely, we have that:

(1) by Proposition 2.1 below, the replete image of the functor H^B is precisely the category $\mathsf{Triv}(B)$ of trivial extensions over B;
(2) $\mathsf{Triv}(B)$ is a reflective subcategory of $(\mathscr{E} \downarrow B)$;
(3) $H^B I^B\colon (\mathscr{E} \downarrow B) \to \mathsf{Triv}(B)$ is its reflector.

By Proposition 5.8 in [5], we obtain a left adjoint, which we will call the **trivialisation functor**

$$\mathrm{Triv}\colon \mathsf{Fib}(\mathscr{C}) \to \mathsf{TExt}(\mathscr{C}),$$

to the inclusion of the category $\mathsf{TExt}(\mathscr{C})$ of trivial extensions in $\mathscr{C}$ into the full subcategory $\mathsf{Fib}(\mathscr{C})$ of the category of arrows in $\mathscr{C}$ determined by the fibrations.

Proposition 2.1. [9, Proposition 2.4] *If Γ is an admissible Galois structure, then $I\colon \mathscr{C} \to \mathscr{X}$ preserves pullbacks along trivial extensions. Hence a fibration is a trivial extension if and only if it is a pullback of some fibration in $H(\mathscr{X})$. In particular, the trivial extensions are pullback-stable, so that every trivial extension is a normal extension.* $\square$

Although the reflectiveness of trivial extensions amongst fibrations follows just from admissibility, for the reflectiveness of central extensions, one needs more specific conditions on the Galois structure Γ (Theorem 2.2).

2.2. **A result on reflectiveness of central extensions.** Consider an admissible Galois structure Γ as in Definition 2.3. Given an object B in $\mathscr{C}$, we let $\mathsf{Centr}(B)$ denote the full subcategory of $(\mathscr{E} \downarrow B)$ determined by the central extensions over B. When it exists, the left adjoint to the inclusion functor $\mathsf{Centr}(B) \hookrightarrow (\mathscr{E} \downarrow B)$ will be written as $\mathrm{Centr}\colon (\mathscr{E} \downarrow B) \to \mathsf{Centr}(B)$ and called the **centralisation functor**. Moreover, given a fibration $p\colon E \to B$, we denote by $\mathsf{Spl}(E,p)$ the full subcategory of $(\mathscr{E} \downarrow B)$ whose objects are the fibrations whose pullback along p is a trivial extension. We recall from [9] the following result:

Theorem 2.2. [9, Theorem 4.2] *For a Galois structure Γ as in Definition 2.3, let $\mathscr{C}$ admit pullbacks, pushouts and filtered colimits, and for each $B \in \mathscr{C}$ let $(\mathscr{E} \downarrow B)$ be closed in $(\mathscr{C} \downarrow B)$ under these colimits; the latter is certainly the case if $\mathscr{E}$ is closed under these colimits in $\mathscr{C}^2$. Let $p\colon E \to B$ be a fibration, with E admissible. Then $\mathsf{Spl}(E,p)$ is reflective in $(\mathscr{E} \downarrow B)$ if, for some regular cardinal α, each pullback functor $f^*\colon (\mathscr{C} \downarrow A) \to (\mathscr{C} \downarrow C)$ (where $f\colon C \to A$ is any morphism in $\mathscr{C}$) preserves α-filtered colimits; this is a fortiori the case if pullbacks commute with α-filtered colimits in $\mathscr{C}$.* □

In Section 4 we shall explain, following the same ideas as in Subsection 8.4 of [9], how this result may be used to prove that special homogeneous surjections of monoids are reflective amongst surjective monoid homomorphisms.

3. Revision of some known results for monoids

In this section we recall the main results from [13], where it is shown that the group completion of monoids determines an admissible Galois structure with respect to surjective homomorphisms. Moreover, the corresponding central (= normal) extensions are precisely the special homogeneous surjections.

3.1. **The Grothendieck group of a monoid.** The **Grothendieck group** (or **group completion**) of a monoid $(M,\cdot,1)$ is given by a group $\mathrm{Gp}(M)$ and a monoid homomorphism $M \to \mathrm{Gp}(M)$ which is universal with respect to monoid homomorphisms from M to a group [10, 11, 12]. Explicitly, we can define $\mathrm{Gp}(M) = \mathrm{GpF}(M)/\mathrm{N}(M)$, where $\mathrm{GpF}(M)$ denotes the free group on M and $\mathrm{N}(M)$ is the normal subgroup generated by elements of the form $[m_1][m_2][m_1 \cdot m_2]^{-1}$. We shall simply write m_1m_2 instead of $m_1 \cdot m_2$ from now on. This gives us an equivalence relation $\equiv$ on $\mathrm{GpF}(M)$ generated by $[m_1][m_2] \equiv [m_1m_2]$ with equivalence classes $\overline{[m_1][m_2]} = \overline{[m_1m_2]}$. An arbitrary element in $\mathrm{Gp}(M)$ is an equivalence class of words, which may be represented by a word of the form

$$[m_1][m_2]^{-1}[m_3][m_4]^{-1}\cdots[m_n]^{\iota(n)} \quad \text{or} \quad [m_1]^{-1}[m_2][m_3]^{-1}[m_4]\cdots[m_n]^{\iota(n)},$$

where $\iota(n) = \pm 1$, $n \in \mathbb{N}$, $m_1, \ldots, m_n \in M$ and no further cancellation is possible.

We write Mon for the category of monoids and Gp for the category of groups. The Grothendieck group construction determines an adjunction

$$\begin{tikzcd} \mathsf{Mon} \arrow[r, shift left, "\mathrm{Gp}"] \arrow[r, phantom, "\perp"] & \mathsf{Gp}, \arrow[l, shift left, "\mathrm{Mon}"] \end{tikzcd} \tag{3.1}$$

where Mon is the forgetful functor. To simplify notation, we write $\mathrm{Gp}(M)$ instead of $\mathrm{MonGp}(M)$ when referring to the monoid structure of $\mathrm{Gp}(M)$. The counit is $\epsilon = 1_{\mathsf{Gp}}$ and the unit is defined, for any monoid M, by

$$\eta_M \colon M \to \mathrm{Gp}(M) \colon m \mapsto \overline{[m]}.$$

By choosing the classes of morphisms $\mathscr{E}$ and $\mathscr{F}$ to be the surjections in Mon and Gp, respectively, we obtain a Galois structure

$$\Gamma_{\mathrm{Mon}} = (\mathsf{Mon}, \mathsf{Gp}, \mathrm{Gp}, \mathrm{Mon}, \eta, \epsilon, \mathscr{E}, \mathscr{F}).$$

This Galois structure was studied in the article [13], with as main result its Theorem 2.2:

Theorem 3.1. *The Galois structure Γ_{Mon} is admissible.* □

3.2. **Special homogeneous surjections.** We recall the definition and some results concerning special homogenous surjections from [3, 4] which are needed in the sequel.

Definition 3.1. *Let f be a split epimorphism of monoids, with a chosen splitting s, and N its (canonical) kernel*

$$\begin{tikzcd} N \arrow[r, tail, "k"'] & X \arrow[r, two heads, shift left, "f"] & Y. \arrow[l, shift left, "s"] \end{tikzcd} \tag{3.2}$$

The split epimorphism (f, s) is said to be ***right homogeneous*** *when, for every element $y \in Y$, the function $\mu_y \colon N \to f^{-1}(y)$ defined through multiplication on the right by $s(y)$, so $\mu_y(n) = n\, s(y)$, is bijective. Similarly, we can define a* ***left homogeneous*** *split epimorphism: the function $N \to f^{-1}(y) \colon n \mapsto s(y)\, n$ is a bijection for all $y \in Y$. A split epimorphism is said to be* ***homogeneous*** *when it is both right and left homogeneous.*

Definition 3.2. *Given a surjective homomorphism g of monoids and its kernel pair*

$$\begin{tikzcd} \mathrm{Eq}(g) \arrow[r, shift left=2, "\pi_1"] \arrow[r, shift right=2, "\pi_2"'] & X \arrow[l, "\Delta"' description] \arrow[r, two heads, "g"] & Y, \end{tikzcd}$$

the morphism g is called a ***special homogeneous surjection*** *when (π_1, Δ) (or, equivalently, (π_2, Δ)) is a homogeneous split epimorphism.*

The next two results illustrate the connection between special homogeneous surjections and the notions of trivial, central and normal extensions arising from the Galois structure Γ_{Mon}. The admissibility of Γ_{Mon} is essential to the coincidence of central and normal extensions in this context.

Proposition 3.2. [13, Proposition 4.2] *For a split epimorphism f of monoids, the following statements are equivalent:*

(i) *f is a trivial extension;*
(ii) *f is a special homogeneous surjection.* □

Theorem 3.3. [13, Theorem 4.4] *For a surjective homomorphism g of monoids, the following statements are equivalent:*

(i) *g is a central extension;*
(ii) *g is a normal extension;*
(iii) *g is a special homogeneous surjection.* □

4. Reflectiveness of special homogeneous surjections

We finish by showing that special homogeneous surjections are reflective amongst surjective homomorphisms of monoids. In order to do this, we can apply Theorem 2.2, thanks to the following considerations (which are analogous to the ones of Subsection 8.4 of [9]):

- the category Mon of monoids is complete and cocomplete;
- $(\mathscr{E} \downarrow B)$ is closed in $(\mathscr{C} \downarrow B)$ under colimits because the class $\mathscr{E}$ of surjections is closed under colimits in $\mathscr{C}^2$;
- the Galois structure Γ_{Mon} is admissible;
- the functor H preserves filtered colimits since these are formed both in Mon and in Gp as they are in the category of sets;
- for each morphism $f\colon C \to A$ in Mon, the pullback functor
$$f^*\colon (\mathscr{C} \downarrow A) \to (\mathscr{C} \downarrow C)$$
preserves filtered colimits because pullbacks commute with filtered colimits in Mon since they do in the category of sets.

Hence all the hypotheses of Theorem 2.2 are satisfied. As an immediate consequence, we get that, for any monoid B, the full subcategory $\mathsf{Centr}(B)$ of central extensions over B, with respect to the Galois structure Γ_{Mon}, is reflective in the category $(\mathscr{E} \downarrow B)$ of surjective monoid homomorphisms over B. Thanks to Theorem 3.3, we get our main result.

Theorem 4.1. *Special homogeneous surjections are reflective amongst surjective monoid homomorphisms.* □

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