Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 57 (2007) 977–990
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Reduction of Lagrangian mechanics on Lie algebroids

José F. Cariñena[a], Joana M. Nunes da Costa[b], Patrícia Santos[c,*]

[a] Departamento de Física Teórica, Universidad de Zaragoza, 50009 Zaragoza, Spain
[b] Departamento de Matemática, Universidade de Coimbra, 3001-454 Coimbra, Portugal
[c] Departamento de Física e Matemática, Instituto Superior de Engenharia de Coimbra, 3030-199 Coimbra, Portugal

Received 23 November 2005; received in revised form 21 July 2006; accepted 1 August 2006
Available online 1 September 2006

Abstract

We prove that if a surjective submersion which is a homomorphism of Lie algebroids is given, then there exists another homomorphism between the corresponding prolonged Lie algebroids and a relation between the dynamics on these Lie algebroid prolongations is established. We also propose a geometric reduction method for dynamics on Lie algebroids defined by a Lagrangian and the method is applied to regular Lagrangian systems with nonholonomic constraints.

MSC: 37J15; 37J60; 70H33; 53D

Keywords: Reduction; Lie algebroids; Lagrangian mechanics

1. Introduction

The Lie algebroids were introduced by Pradines [20] as infinitesimal objects for differential groupoids, and since then they are receiving an increasing interest from mathematics and theoretical physicists. Recall that a *Lie algebroid* is a vector bundle $p : A \to M$ over a manifold M together with a vector bundle morphism $\rho : A \to TM$ over the identity map on M (called the anchor) and a Lie bracket $[\cdot, \cdot]_A$ on the $C^\infty(M)$-module $\Gamma(A)$ of sections for p satisfying

$$[v, fw]_A = f[v, w]_A + (\rho(v)f)w$$

for every pair of sections v and w of A and any smooth function f on M. A Lie algebroid can be seen as a generalization of both a Lie algebra and a tangent bundle, these being the simplest (nontrivial) examples of Lie algebroids. Another relevant example of Lie algebroid with equal importance to mathematics and physics is the gauge algebroid TP/G associated to a principal bundle $P(M, G)$, wherein the classical field theory M is the space–time manifold and G is the gauge group. For the basic properties and literature on the subject we refer to the book by Cannas [1] and the survey paper and book by Mackenzie [12,13].

* Corresponding author. Tel. +351 965281797.
E-mail addresses: jfc@unizar.es (J.F. Cariñena), jmcosta@mat.uc.pt (J.M. Nunes da Costa), patricia@isec.pt (P. Santos).

0393-0440/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2006.08.001

The aim of this paper is to study the reduction of the dynamics on Lie algebroids, defined by a Lagrangian function, which can be carried out by using the prolongation of a Lie algebroid over a map, introduced by Higgins and Mackenzie [9]. The study of Lagrangian mechanics on Lie algebroids was first proposed by Weinstein [23] (see also [11]), and then Martínez [15], adapting the definition of prolongation of a Lie algebroid over a map, developed a formalism for Lagrangian mechanics on Lie algebroids using the generalization of the fundamental ingredients of geometric Lagrangian mechanics (the vertical endomorphism, the Liouville vector field and the Cartan forms). Afterwards, several papers on related subjects were developed, see e.g. [10] and references therein. It is fair to mention that for studying Lagrangian mechanics on Lie algebroids other approaches are possible, as the one developed by Grabowska et al. [8] that generalize the work developed by Tulczyjew [21], although in this paper we use the formalism developed by Martínez.

The theory of reduction has many applications and has been shown to be extremely useful for a deep understanding of many physical theories including, among others, systems with symmetry, Poisson structures, stability theory and integrable systems. The reduction of the dynamics has been previously considered in many papers (see [14] and the references therein) but is not a well known subject in Lie algebroids dynamics. This happens because the meaning of Lie algebroid reduction has not been clearly stated; this issue was clarified in a previous paper [4] where the concept of Lie algebroid reduction is defined by means of a surjective morphism. In [4], given a surjective morphism of vector bundles $(\Pi, \pi) : (A, p, M) \to (\widehat{A}, \widehat{p}, \widehat{M})$ between two Lie algebroids $(A, \rho, [\cdot,\cdot]_A)$ and $(\widehat{A}, \widehat{\rho}, [\cdot,\cdot]_{\widehat{A}})$, respectively, $\widehat{A}$ is said to be a *reduced Lie algebroid* of A if Π is a homomorphism of Lie algebroids (see [9,22]). With the study of reduction of the dynamics on Lie algebroids defined by a Lagrangian we generalize a previous work by Rodríguez-Olmos [19], where the author reduced the dynamics of Lie algebroids with symmetry, that is, a Lie algebroid where a Lie group acts and whose action is defined by a Lie algebroid representation of the Lie group.

The paper is organized as follows. In Section 2, we prove that given a surjective submersion $\Pi : A \to \widehat{A}$ that is a homomorphism of Lie algebroids, there exists a surjective map between their prolonged Lie algebroids $\overline{\Pi}|_{\mathcal{T}A} : \mathcal{T}A \to \mathcal{T}\widehat{A}$ that is a homomorphism of Lie algebroids too, i.e. we can define a Lie algebroid reduction between the corresponding prolonged Lie algebroids. The particular case of Lie algebroids with symmetry is analyzed with a special attention to the gauge algebroid. In Section 3 we show how the dynamics can be reduced, establishing a relation between the dynamics in the Lie algebroid prolongation $\mathcal{T}A$ and the dynamics in the reduced Lie algebroid prolongation $\mathcal{T}\widehat{A}$. Finally, in the last section the Chetaev formulation for nonholonomic systems on Lie algebroids is given and a reduction procedure for Lagrangian systems on Lie algebroids with nonholonomic constraints is explained. We show that the dynamics of a system with nonholonomic constraints can be reduced if the system has a regular and Π-invariant Lagrangian $L = l \circ \Pi$ with $l \in C^\infty(\widehat{A})$.

2. Prolongation of a reduced Lie algebroid

In this section, we will show that, given a Lie algebroid A and a reduced Lie algebroid $\widehat{A}$ of A, there exists a homomorphism of Lie algebroids between the prolongation of A and the prolongation of $\widehat{A}$, in such a way that the prolongation of $\widehat{A}$ is a reduced Lie algebroid of the prolongation of A. This construction is a generalization of the work developed in [19].

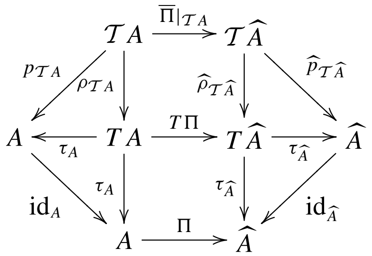
Recall that the *prolongation of the Lie algebroid* $p : A \to M$ (see [9,10,15]) is a vector bundle $\mathcal{T}A$ over A, where $\mathcal{T}A$ is the total space of the pullback of the vector bundle $Tp : TA \to TM$ by the anchor map $\rho : A \to TM$. The projection $p_{\mathcal{T}A} : \mathcal{T}A \to A$ is defined by $p_{\mathcal{T}A}(b, v) = p_{TA}(v) = a \in A$, with $p_{TA} : TA \to A$ the canonical projection of the tangent bundle TA over the base A. An element (b, v) of $\mathcal{T}A$ will be denoted by (a, b, v), where $a \in A$ is the point where v is tangent to A. With this notation, $\mathcal{T}A = \{(a, b, v) \in A \times A \times TA \mid p(a) = p(b), \rho(b) = T_a p(v), \text{ with } v \in T_a A\}$. The vector bundle $p_{\mathcal{T}A} : \mathcal{T}A \to A$ can be endowed with a Lie algebroid structure, where the anchor is the map $\rho_{\mathcal{T}A} : \mathcal{T}A \to TA$, given by $\rho_{\mathcal{T}A}(a, b, v) = v$, and the Lie bracket on the space of sections is defined by setting [15,10]:

$$[V_1, V_2]_{\mathcal{T}A}(a) = (a, [\sigma_1, \sigma_2]_A(p(a)), [X_1, X_2](a))$$

for all $a \in A$ and all projectable sections $V_1, V_2 \in \Gamma(A)$, i.e. sections of the form $V_i(a) = (a, \sigma_i(p(a)), X_i(a))$ where $\sigma_i \in \Gamma(A)$ and $X_i \in \mathfrak{X}(A)$ are such that $Tp \circ X_i = \rho(\sigma_i) \circ p$, with $i = 1, 2$. For example, if A is the tangent bundle to a manifold Q, $A = TQ$, endowed with its usual Lie algebroid structure, the prolongation of the Lie algebroid A is the tangent bundle $T(TQ)$ to TQ endowed with its usual structure of Lie algebroid over TQ (see [15]).

Let A and $\widehat{A}$ be vector bundles endowed with the Lie algebroid structures $(\rho, [\cdot,\cdot]_A)$ and $(\widehat{\rho}, [\cdot,\cdot]_{\widehat{A}})$, respectively. Suppose that $(\Pi, \pi) : (A, p, M) \to (\widehat{A}, \widehat{p}, \widehat{M})$ is a surjective submersion of vector bundles that is a homomorphism of Lie algebroids, i.e. $\widehat{A}$ is a reduced Lie algebroid of A.

Let us consider the surjective morphism of vector bundles $\overline{\Pi} = (\Pi, \Pi, T\Pi)$ over Π. We will show that the restriction of $\overline{\Pi}$ to $\mathcal{T}A$ is a surjective map with values in $\mathcal{T}\widehat{A}$ and a homomorphism of Lie algebroids too. First, we will show that $\overline{\Pi}|_{\mathcal{T}A}(\mathcal{T}A)$ is contained in $\mathcal{T}\widehat{A}$. Let $(a, b, v) \in \mathcal{T}A$, then $\overline{\Pi}(a, b, v) = (\Pi(a), \Pi(b), T\Pi(v))$. Since $p(a) = p(b)$, we have that $\pi(p(a)) = \pi(p(b))$ and so $\widehat{p}(\Pi(a)) = \widehat{p}(\Pi(b))$. On the other hand, $\widehat{\rho}(\Pi(b)) = T\pi(\rho(b)) = T\pi(Tp(v)) = T\widehat{p}(T\Pi(v))$; therefore, $\overline{\Pi}(a, b, v) \in \mathcal{T}\widehat{A}$. Before proving that $\overline{\Pi}|_{\mathcal{T}A} : \mathcal{T}A \to \mathcal{T}\widehat{A}$ is a homomorphism of Lie algebroids over $\Pi : A \to \widehat{A}$, we need two auxiliary results.



Lemma 2.1. *Every section V of $p_{\mathcal{T}A} : \mathcal{T}A \to A$ is written in the form $V = \sum_{i\in I} f_i V_i + Z$, where $Z \in \Gamma(\mathrm{Ker}\,\overline{\Pi})$ and, for each $i \in I$, $f_i \in C^\infty(A)$ and $V_i(a) = (a, \sigma_i(p(a)), X_i(a))$ is a projectable section of $\mathcal{T}A$, with $\Pi \circ \sigma_i = \sigma_i' \circ \pi$ for $\sigma_i' \in \Gamma(\widehat{A})$ and $T\Pi(X_i) = X_i' \circ \Pi$ for $X_i' \in \mathfrak{X}(\widehat{A})$.*

Proof. The $\overline{\Pi}$-projection of a section V of $\mathcal{T}A$ is given by

$$\overline{\Pi} \circ V = \sum_i f_i (V_i' \circ \Pi),$$

with $f_i \in C^\infty(A)$ and V_i' a projectable section of $\mathcal{T}\widehat{A}$. Each section V_i' is given by $V_i'(a') = (a', \sigma_i'(\widehat{p}(a')), X_i'(a'))$ for all $a' \in \widehat{A}$. Then,

$$\overline{\Pi} \circ V(a) = \sum_i f_i(a)(\Pi(a), \sigma_i' \circ \widehat{p} \circ \Pi(a), X_i' \circ \Pi(a)) = \sum_i f_i(a)(\Pi(a), \sigma_i' \circ \pi \circ p(a), X_i' \circ \Pi(a))$$

because $\widehat{p}\circ\Pi = \pi \circ p$. Once Π and $T\Pi$ are surjective, there exist $\sigma_i \in \Gamma(A)$ and $X_i \in \mathfrak{X}(A)$, such that, $\Pi\circ\sigma = \sigma_i'\circ\pi$ and $T\Pi \circ X = X_i' \circ \Pi$. Thus,

$$\overline{\Pi} \circ V(a) = \sum_i f_i(a)(\Pi(a), \Pi\circ\sigma_i \circ p(a), T\Pi \circ X_i(a)) = \sum_i f_i(a)(\overline{\Pi} \circ V_i)(a) = \overline{\Pi} \circ \left(\sum_i f_i V_i\right)(a)$$

with $V_i(a) = (a, \sigma_i(p(a)), X_i(a))$, where $\Pi\circ\sigma_i = \sigma_i'\circ\pi$ and $T\Pi\circ X_i = X_i' \circ \Pi$. Therefore, $V = \sum_i f_i V_i + Z$ for $Z \in \mathrm{Ker}\,\overline{\Pi}$. □

Lemma 2.2. *Let Z be a section of* $\mathrm{Ker}\,\overline{\Pi}$ *and V a projectable section of $\mathcal{T}A$ fulfilling the conditions of the above lemma. Then, the following conditions are satisfied:*

(1) $\rho_{\mathcal{T}A}(Z) \in \mathrm{Ker}\,T\Pi$;
(2) $[V, Z]_{\mathcal{T}A} \in \mathrm{Ker}\,\overline{\Pi}$;
(3) $\Gamma(\mathrm{Ker}\,\overline{\Pi})$ *is a Lie subalgebra of* $\Gamma(\mathcal{T}A)$.

Proof. If $Z \in \Gamma(\mathrm{Ker}\,\overline{\Pi})$ then Z is of the form $Z(a) = (a, \sigma(p(a)), X(a))$ for all $a \in A$, where $X \in \mathfrak{X}^V(A)$ is a Π-vertical vector field on A and $\Pi\circ\sigma = 0\circ\pi$. Since $\rho_{\mathcal{T}A}(Z)(a) = \rho_{\mathcal{T}A}(a, \sigma(p(a)), X(a)) = X(a)$, then (1) holds. The bracket between the sections Z and V, with $V(a) = (a, \eta(p(a)), Y(a))$, for all $a \in A$, is given by

$$[V, Z]_{\mathcal{T}A}(a) = (a, [\eta, \sigma]_A(p(a)), [Y, X](a)).$$

Since

$$T\Pi \circ [Y, X] = [Y', 0] \circ \Pi = 0 \circ \Pi$$

and

$$\Pi \circ [\eta, \sigma]_A = [\eta', 0]_{\widehat{A}} \circ \pi = 0 \circ \pi,$$

we have that $\overline{\Pi} \circ [V, Z]_{\mathcal{T}A} = (\Pi(a), 0_{\pi(p(a))}, 0_{\Pi(a)})$, that is, condition (2) holds. Now, we suppose that $Z_1, Z_2 \in \Gamma(\operatorname{Ker} \overline{\Pi})$. Then, $Z_1(a) = (a, \sigma_1(p(a)), X_1(a))$ and $Z_2(a) = (a, \sigma_2(p(a)), X_2(a))$, with $X_1, X_2 \in \mathfrak{X}^V(A)$, $\Pi \circ \sigma_1 = 0 \circ \pi$ and $\Pi \circ \sigma_2 = 0 \circ \pi$. Thus,

$$[Z_1, Z_2]_{\mathcal{T}A}(a) = (a, [\sigma_1, \sigma_2]_A(p(a)), [X_1, X_2](a)).$$

Since

$$T\Pi \circ [X_1, X_2] = 0 \circ \Pi \qquad \text{and} \qquad \Pi \circ [\sigma_1, \sigma_2]_A = 0 \circ \pi,$$

we can conclude that $\overline{\Pi} \circ [Z_1, Z_2]_{\mathcal{T}A} = (\Pi(a), 0_{\pi(p(a))}, 0_{\Pi(a)})$. □

Now, we prove the main result of this section: "reduction commutes with prolongation", i.e. the prolongation of a reduced Lie algebroid is a reduced Lie algebroid of the Lie algebroid prolongation.

Theorem 2.3. *The map $\overline{\Pi}|_{\mathcal{T}A} : \mathcal{T}A \to \mathcal{T}\widehat{A}$ is a homomorphism of Lie algebroids over $\Pi : A \to \widehat{A}$. Therefore, $\mathcal{T}\widehat{A}$ is a reduced Lie algebroid of $\mathcal{T}A$.*

Proof. We first remark that $T\Pi \circ \rho_{\mathcal{T}A} = \rho_{\mathcal{T}\widehat{A}} \circ \overline{\Pi}$. In fact, if $(a, b, v) \in \mathcal{T}A$, then

$$T\Pi \circ \rho_{\mathcal{T}A}(a, b, v) = T\Pi(v) = \rho_{\mathcal{T}\widehat{A}}(\Pi(a), \Pi(b), T\Pi(v)) = \rho_{\mathcal{T}\widehat{A}} \circ \overline{\Pi}(a, b, v).$$

Now, let us prove the condition (3) stated by Higgins and Mackenzie in [9]. By the above lemmas and the Leibniz condition of the Lie bracket $[\cdot, \cdot]_{\mathcal{T}A}$, we just have to prove the condition on projectable sections of $\mathcal{T}A$ of the form $V_i(a) = (a, \sigma_i(p(a)), X_i(a))$, with $\Pi \circ \sigma_i = \sigma_i' \circ \pi$ for $\sigma_i' \in \Gamma(\widehat{A})$ and $T\Pi(X_i) = X_i' \circ \Pi$ for $X_i' \in \mathfrak{X}(\widehat{A})$, with $i = 1, 2$. Thus,

$$[V_1, V_2]_{\mathcal{T}A}(a) = (a, [\sigma_1, \sigma_2]_A(p(a)), [X_1, X_2](a)),$$

for all $a \in A$. Then, by the definition of $\mathcal{T}\Pi$, we have

$$\mathcal{T}\Pi \circ [V_1, V_2]_{\mathcal{T}A}(a) = (\Pi(a), \Pi \circ [\sigma_1, \sigma_2]_A(p(a)), T\Pi \circ [X_1, X_2](a)),$$

for all $a \in A$. Once $\Pi : A \to \widehat{A}$ and $T\Pi : TA \to T\widehat{A}$ are Lie algebroid homomorphisms, we have

$$\mathcal{T}\Pi \circ [V_1, V_2]_{\mathcal{T}A}(a) = (\Pi(a), [\sigma_1', \sigma_2']_{\widehat{A}}(\pi(p(a))), [X_1', X_2'](\Pi(a))),$$

that is,

$$\mathcal{T}\Pi \circ [V_1, V_2]_{\mathcal{T}A}(a) = [V_1', V_2'] \circ \Pi(a),$$

for all $a \in A$, where $V_i'(a') = (a', \sigma_i'(\widehat{p}(a')), X_i'(a'))$, for all $a' \in \widehat{A}$, with $i = 1, 2$. □

This proposition is a particular case of a more general result obtained in [16]. The statement of the Theorem 2.3 means that $d_{\mathcal{T}A} \circ (\overline{\Pi}|_{\mathcal{T}A})^* = (\overline{\Pi}|_{\mathcal{T}A})^* \circ d_{\mathcal{T}\widehat{A}}$, where $d_{\mathcal{T}A}$ and $d_{\mathcal{T}\widehat{A}}$ are the exterior derivatives of the Lie algebroids $\mathcal{T}A$ and $\mathcal{T}\widehat{A}$, respectively. Moreover, one can easily prove that the Lie algebroid structure on $\mathcal{T}\widehat{A}$ is the unique structure for which $\overline{\Pi}|_{\mathcal{T}A} : \mathcal{T}A \to \mathcal{T}\widehat{A}$ is a homomorphism of Lie algebroids over $\Pi : A \to \widehat{A}$.

By hypothesis, Π is a homomorphism of Lie algebroids, i.e. $d_A \circ \Pi^* = \Pi^* \circ d_{\widehat{A}}$. We also have that $p_2 : \mathcal{T}A \to A$ is a homomorphism of Lie algebroids (see [10]) so, $d_{\mathcal{T}A} \circ p_2^* = p_2^* \circ d_A$. Then, $d_{\mathcal{T}A} \circ (\Pi \circ p_2)^* = (\Pi \circ p_2)^* \circ d_{\widehat{A}}$, that is, $\widehat{A}$ is a reduced Lie algebroid of $\mathcal{T}A$.

2.1. Lie algebroids with symmetry

Let Φ be a representation of the Lie group G on the Lie algebroid $(A, \rho, [\cdot, \cdot]_A)$ in the sense of [4]. Suppose that Φ and its contragradient representation define free and proper actions of G on the fibre bundles (A, p, M) and (A^*, τ, M), respectively. In these conditions, $\overline{\Phi} = (\Phi, \Phi, T\Phi)$ defines a Lie algebroid representation of G in the Lie algebroid prolongation $\mathcal{T}A$ of A, that is:

Proposition 2.4. *The morphism $\overline{\Phi} : G \to Aut(\mathcal{T}A)$ is a Lie algebroid representation of the Lie group G on the Lie algebroid $\mathcal{T}A$.*

Proof. First of all we need to prove that $\overline{\Phi}_g := \overline{\Phi}(g)$ preserves $\mathcal{T}A$, i.e. $\overline{\Phi}_g(\mathcal{T}A) \subset \mathcal{T}A$. Let $(a, b, v) \in \mathcal{T}_a A$, then $\overline{\Phi}_g(a, b, v) = (\Phi_g(a), \Phi_g(b), T\Phi_g(v))$, with $\Phi_g := \Phi(g)$. Since $p(a) = p(b)$, $\phi_g(p(a)) = \phi_g(p(b))$, where ϕ_g is the base map of Φ_g; then $p \circ \Phi_g(a) = p \circ \Phi_g(b)$. We have $Tp(T\Phi_g(v)) = T(p \circ \Phi_g)(v) = T\phi_g \circ Tp(v) = T\phi_g \circ \rho(b)$. Since Φ_g is a Lie algebroid representation, then $Tp(T\Phi_g(v)) = \rho(\Phi_g(b))$. Therefore $\overline{\Phi}_g(a, b, v) \in \mathcal{T}_{\Phi_g(a)}A$.

Now, in order to show that $\overline{\Phi}_g$ is an automorphism of Lie algebroids, we need to prove that $T\Phi_g \circ \rho_{\mathcal{T}A} = \rho_{\mathcal{T}A} \circ \overline{\Phi}_g$ and $\overline{\mathcal{R}}_g([V_1, V_2]_{\mathcal{T}A}) = [\overline{\mathcal{R}}_g(V_1), \overline{\mathcal{R}}_g(V_2)]_{\mathcal{T}A}$, for all $V_1, V_2 \in \Gamma(\mathcal{T}A)$, where $\overline{\mathcal{R}}_g(V) = \overline{\Phi}_g \circ V \circ \Phi_{g^{-1}}$, for all $V \in \Gamma(\mathcal{T}A)$. For the first condition, we have, for all $(a, b, v) \in \mathcal{T}_a A$,

$$\rho_{\mathcal{T}A} \circ \overline{\Phi}_g(a, b, v) = \rho_{\mathcal{T}A}(\Phi_g(a), \Phi_g(b), T\Phi_g(v)) = T\Phi_g(v) = T\Phi_g \circ \rho_{\mathcal{T}A}(a, b, v).$$

Suppose now that $V_1, V_2 \in \Gamma(\mathcal{T}A)$ are projectable sections, that is, $V_i(a) = (a, \sigma_i(p(a)), X_i(a))$, with $i = 1, 2$. Then $\overline{\mathcal{R}}_g(V_i)(a) = (a, \mathcal{R}(\sigma)(p(a)), X_i'(a))$, where $\mathcal{R}(\sigma) = \Phi_g \circ \sigma \circ \phi_{g^{-1}}$ and $X_i' = T\Phi_g \circ X_i \circ \Phi_{g^{-1}}$, with $i = 1, 2$. Since

$$\overline{\mathcal{R}}_g([V_1, V_2]_{\mathcal{T}A})(a) = (a, \Phi_g([\sigma_1, \sigma_2]_A)(\phi_{g^{-1}}(p(a))), T\Phi_g([X_1, X_2])(\Phi_{g^{-1}}(a))), \quad a \in A,$$

and Φ_g and $T\Phi_g$ are homomorphisms of Lie algebroids, we conclude

$$\overline{\mathcal{R}}_g([V_1, V_2]_{\mathcal{T}A})(a) = (a, [\mathcal{R}_g(\sigma_1), \mathcal{R}_g(\sigma_2)]_A(p(a)), [X_1', X_2'](a)) = [\overline{\mathcal{R}}_g(V_1), \overline{\mathcal{R}}_g(V_2)]_{\mathcal{T}A}(a). \quad \square$$

Let Π be the canonical projection of A onto A/G. We know (see [4]) that A/G is endowed with a vector bundle structure in such a way that $(\Pi, \pi) : (A, p, M) \to (A/G, \widehat{p}, M/G)$ is a surjective submersion of vector bundles. Moreover, A/G is endowed with a Lie algebroid structure such that Π is a homomorphism of Lie algebroids. Thus, by Theorem 2.3, $\overline{\Pi}|_{\mathcal{T}A}$ is a homomorphism of Lie algebroids over Π and $\mathcal{T}(A/G)$ is a reduced Lie algebroid of $\mathcal{T}A$. One can easily prove that $(\mathcal{T}A)/G$ is endowed with a unique structure of Lie algebroid in such a way that the projection $\underline{\Pi} : \mathcal{T}A \to (\mathcal{T}A)/G$ defined by $\overline{\Phi} = (\Phi, \Phi, T\Phi)$, is a Lie algebroid homomorphism over $\Pi : A \to A/G$.

Next, we will prove the main theorem of this section that deals with reduction and prolongation of Lie algebroids, in this case reduction of Lie algebroids is defined by a Lie group of symmetry.

Lemma 2.5. *For each $a \in A$, the map $\overline{\Pi}_a : \mathcal{T}_a A \to \mathcal{T}_{[a]}(A/G)$ is an isomorphism.*

Proof. Let $(a, b, v), (a, b', v') \in \mathcal{T}_a A$ such that $\overline{\Pi}_a(a, b, v) = \overline{\Pi}_a(a, b', v')$, that is, $(\Pi(a), \Pi(b), T\Pi(v)) = (\Pi(a), \Pi(b'), T\Pi(v'))$. So $\Pi(b) = \Pi(b')$ and $T\Pi(v) = T\Pi(v')$. Since Π is the canonical projection defined by the Lie algebroid representation Φ of the Lie group G on A, there exists $g \in G$ such that $b = \Phi_g(b')$. Then, $p(b) = p \circ \Phi_g(b')$, that is, $p(a) = \phi_g \circ p(b') = \phi_g(p(a))$, where ϕ_g is the base map of Φ_g that defines a free and proper action of the Lie group G on the vector bundle $p : A \to M$. Therefore, $g = e$ and so $b = b'$.

Now, we prove that $v = v'$. We have $v - v' \in \operatorname{Ker} T_a\Pi$. In these conditions, $\operatorname{Ker} T_a\Pi$ is generated by the fundamental vector fields in $a \in A$, defined by the free and proper action associated to the representation Φ of G on A. So, $v = v' + \sum_i \lambda_i X_A^i(a)$, where X^i represents the elements of a basis of the Lie algebra $\mathfrak{g}$ of G. Since $b = b'$, we know that $T_a p(v) = T_a p(v')$, then

$$T_a p(v') + \sum_i \lambda_i T_a p(X_A^i(a)) = T_a p(v'),$$

that is,

$$\sum_i \lambda_i T_a p(X_A^i(a)) = 0.$$

But $\sum_i \lambda_i T_a p(X_A^i(a)) = 0$ is equivalent to $\sum_i \lambda_i X_M^i(m) = 0$, where $p(a) = m$ and X_M^i is the fundamental vector field on M associated to the element X^i of $\mathfrak{g}$. Since the fundamental vector fields associated to elements of a basis of $\mathfrak{g}$ are independent, then, all λ_i are null. Therefore, $v = v'$. $\square$

Theorem 2.6. *The Lie algebroids $\mathcal{T}(A/G)$ and $(\mathcal{T}A)/G$ are isomorphic.*

Proof. The canonical projection $\underline{\Pi} : \mathcal{T}A \to \mathcal{T}A/G$ is, in each point of A, an isomorphism. So, $\mathcal{T}_aA \simeq (\mathcal{T}A/G)_{[a]}$ and therefore, by the above lemma, $\mathcal{T}_{[a]}(A/G) \simeq (\mathcal{T}A/G)_{[a]}$, for each $a \in A$. We can easily verify that $\mathcal{I} : \mathcal{T}(A/G) \to (\mathcal{T}A)/G$, defined by $\mathcal{I}(\overline{\Pi}(a, b, v)) = \underline{\Pi}(a, b, v)$, is an isomorphism of Lie algebroids over the identity map on A/G. □

For example, consider a principal fibre bundle $P(M, G)$ and the gauge algebroid associated $(TP/G, p, M)$ (see [12]). The canonical projection $\Pi : TP \to TP/G$ determines a homomorphism of Lie algebroids $\overline{\Pi}|_{\mathcal{T}(TP)} : \mathcal{T}(TP) \to \mathcal{T}(TP/G)$ over Π. Let ϕ be the (right) action of the group G on P. Then, $\Phi(g) := T\phi_g$ defines a Lie algebroid representation of the group G in TP. Thus, $\overline{\Phi} = (\Phi, \Phi, T\Phi)$ is a Lie algebroid representation of the group G in $\mathcal{T}(TP) \equiv T(TP)$. By the above proposition $\mathcal{T}(TP)/G \cong \mathcal{T}(TP/G)$, that is, $T(TP)/G \cong \mathcal{T}(TP/G)$. In the recent paper of de Léon et al. [10], the above results for gauge algebroids are proven by a different approach.

3. Reduction of a Lagrangian dynamics

In this section, given a dynamical system on a Lie algebroid A defined by a regular and "invariant" Lagrangian function, we will show how the dynamics is reduced.

Let (A, p, M) and $(\widehat{A}, \widehat{p}, \widehat{M})$ be two vector bundles endowed with the Lie algebroid structures $(\rho, [\cdot, \cdot]_A)$ and $(\widehat{\rho}, [\cdot, \cdot]_{\widehat{A}})$, respectively, and suppose that $(\Pi, \pi) : (A, p, M) \to (\widehat{A}, \widehat{p}, \widehat{M})$ is a surjective submersion of vector bundles that is a homomorphism of Lie algebroids.

Definition 3.1. A function $F \in C^\infty(A)$ is said to be Π-invariant if there exists a function $f \in C^\infty(\widehat{A})$ such that $F = f \circ \Pi$.

The prolongation of the Lie algebroid $p : A \to M$ plays a relevant role in the definition of Lagrangian mechanics on Lie algebroids. See, for example [15], for the definitions of vertical endomorphism S on the Lie algebroid prolongation $\mathcal{T}A$ and Liouville section Δ of $\mathcal{T}A$.

Lemma 3.2. *The morphism $\overline{\Pi}$ intertwines the vertical endomorphism S on $\mathcal{T}A$ with the vertical endomorphism S' on $\mathcal{T}\widehat{A}$, and the Liouville section Δ of $\mathcal{T}A$ is $\overline{\Pi}$-related with the Liouville section Δ' of $\mathcal{T}\widehat{A}$, i.e. $\overline{\Pi} \circ S = S' \circ \overline{\Pi}$ and $\overline{\Pi} \circ \Delta = \Delta' \circ \Pi$.*

Proof. Let $a \in A$, then

$$\overline{\Pi}(b^V(a)) = (\Pi(a), 0, T\Pi(b_a^V)).$$

Since

$$T\Pi(b_a^V)f = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \Pi)(a + tb)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} f(\Pi(a) + t\Pi(b))|_{t=0}$$

for all $f \in C^\infty(A)$, we have

$$T\Pi(b_a^V) = (\Pi(b))^V_{\Pi(a)}.$$

Therefore,

$$\overline{\Pi}(b^V(a)) = (\Pi(b))^V(\Pi(a)). \tag{3.1}$$

From the above equality, we show that $\overline{\Pi} \circ S = S' \circ \overline{\Pi}$. In fact, for each $(a, b, v) \in \mathcal{T}_aA$, we have that

$$S' \circ \overline{\Pi}(a, b, v) = S'(\Pi(a), \Pi(b), T_a\Pi(v)) = (\Pi(b))^V(\Pi(a)).$$

By (3.1) we conclude that $S' \circ \overline{\Pi}(a, b, v) = \overline{\Pi}(b^V(a)) = \overline{\Pi} \circ S(a, b, v)$. Now, let us prove that $\overline{\Pi} \circ \Delta = \Delta' \circ \Pi$. We have $\overline{\Pi} \circ \Delta(a) = \overline{\Pi}(a^V(a))$, for all $a \in A$. From (3.1), we may write

$$\overline{\Pi} \circ \Delta(a) = (\Pi(a))^V(\Pi(a))$$

and therefore $\overline{\Pi} \circ \Delta(a) = \Delta'(\Pi(a)) = \Delta' \circ \Pi(a)$, for all $a \in A$. □

Next, we will prove the main result of this section, that shows how the dynamics is reduced. This result generalizes the one obtained in [19] for the reduction of dynamics on Lie algebroids with symmetry.

Theorem 3.3. *Let us suppose that the Lagrangian $L \in C^\infty(A)$ of a dynamical system on the Lie algebroid A is Π-invariant, that is, there exists $l \in C^\infty(\widehat{A})$ such that $L = l \circ \Pi$. Then, the following conditions are satisfied:*

(i) *if E_L and E'_l are the energies of the dynamics on the Lie algebroids A and $\widehat{A}$, respectively, then $E'_l \circ \Pi = E_L$;*
(ii) *if θ_L and θ'_l are the Cartan 1-forms defined by L and l on the Lie algebroids A and $\widehat{A}$, respectively, then $\overline{\Pi}^*\theta'_l = \theta_L$. As a consequence, we have the following relation between the Cartan 2-forms $\overline{\Pi}^*\omega'_l = \omega_L$.*
Moreover, in the case where L is regular we can state:
(iii) *the induced Lagrangian l is regular;*
(iv) *if V_L and V'_l are the solutions of the dynamics on the Lie algebroids A and on $\widehat{A}$, respectively, then $\overline{\Pi} \circ V_L = V'_l \circ \Pi$.*

Therefore, the dynamics on A induced by a regular and Π-invariant Lagrangian $L = l \circ \Pi$ reduces to the Lagrangian dynamics on $\widehat{A}$ given by l.

Proof. (i) We have $E'_l := \rho_{\mathcal{T}\widehat{A}}(\Delta')l - l$. Then,

$$E'_l \circ \Pi(a) = \rho_{\mathcal{T}\widehat{A}}(\Delta')l \circ \Pi(a) - l \circ \Pi(a) = \rho_{\mathcal{T}\widehat{A}}(\Delta'(\Pi(a)))l - L(a)$$

for all $a \in A$. By Lemma 3.2 we have

$$E'_l \circ \Pi(a) = \rho_{\mathcal{T}\widehat{A}}(\overline{\Pi}(\Delta(a)))l - L(a)$$

and by $\rho_{\mathcal{T}\widehat{A}} \circ \overline{\Pi} = T\Pi \circ \rho_{\mathcal{T}A}$ we obtain

$$E'_l \circ \Pi(a) = T\Pi \circ \rho_{\mathcal{T}A}(\Delta(a))l - L(a) = \rho_{\mathcal{T}A}(\Delta(a))(l \circ \Pi) - L(a)$$

for all $a \in A$. Therefore, $E'_l \circ \Pi = E_L$.
(ii) Let us prove that $\overline{\Pi}^*\theta'_l = \theta_L$. Recall that $\theta'_l := d_{\mathcal{T}\widehat{A}}l \circ S'$. So, if $V \in \mathcal{T}_a A$, we have

$$\overline{\Pi}^*\theta'_l(V) = \theta'_l(\overline{\Pi}(V)) = d_{\mathcal{T}\widehat{A}}l \circ S'(\overline{\Pi}(V))$$

and using the results of the Lemma 3.2 we may write

$$\overline{\Pi}^*\theta'_l(V) = d_{\mathcal{T}A}l \circ \overline{\Pi} \circ S(V) = \overline{\Pi}^* \circ d_{\mathcal{T}A}l \circ S(V).$$

But $\overline{\Pi}^* \circ d_{\mathcal{T}A} = d_{\mathcal{T}\widehat{A}} \circ \overline{\Pi}^*$, and then

$$\overline{\Pi}^*\theta'_l(V) = d_{\mathcal{T}\widehat{A}} \circ \overline{\Pi}^* l \circ S(V) = d_{\mathcal{T}\widehat{A}}L \circ S(V) = \theta_L(V),$$

that is, $\overline{\Pi}^*\theta'_l = \theta_L$. Thus, by definition of the Cartan 2-form ω'_l, we deduce $\overline{\Pi}^*\omega'_l = -\overline{\Pi}^*(d_{\mathcal{T}\widehat{A}}\theta'_l)$. The exterior derivative commutes with the morphism $\overline{\Pi}^*$, so

$$\overline{\Pi}^*\omega'_l = -d_{\mathcal{T}A}(\overline{\Pi}^*\theta'_l) = -d_{\mathcal{T}A} \circ \theta_L =: \omega_L.$$

(iii) If $L \in C^\infty(A)$ is regular then ω_L is symplectic. By the above condition we have $\overline{\Pi}^*\omega'_l = \omega_L$, then, ω'_l is also symplectic because $\overline{\Pi}$ is a surjective morphism. So the reduced Lagrangian $l \in C^\infty(\widehat{A})$ is regular.
(iv) We have

$$\overline{\Pi}^*\omega'_l(V_L, X) = \omega'_l(\overline{\Pi}(V_L), \overline{\Pi}(X)) = i(\overline{\Pi}(V_L))\omega'_l(\overline{\Pi}(X)), \tag{3.2}$$

for all $X \in \Gamma(\mathcal{T}A)$. On the other hand,

$$\overline{\Pi}^*\omega'_l(V_L, X) = \omega_L(V_L, X) = i(V_L)\omega_L(X) = d_{\mathcal{T}A}E_L(X). \tag{3.3}$$

Since $E_L = E'_l \circ \overline{\Pi} = \overline{\Pi}^* E'_l$, then by (3.2) and (3.3) we have

$$i(\overline{\Pi}(V_L))\omega'_l(\overline{\Pi}(X)) = (d_{\mathcal{T}A} \circ \overline{\Pi}^*)E'_l(X) = (\overline{\Pi}^* \circ d_{\mathcal{T}\widehat{A}})E'_l(X) = d_{\mathcal{T}\widehat{A}}E'_l(\overline{\Pi}(X)),$$

for all $X \in \Gamma(\mathcal{T}A)$. So $\overline{\Pi}(V_L)$ is a global solution of the dynamics in $\mathcal{T}\widehat{A}$. Once L is regular so is l, then the Cartan 2-form ω'_l is symplectic and so the dynamical equation has just one solution. Therefore, $\overline{\Pi}(V_L) = V'_l$, that is, $\overline{\Pi} \circ V_L = V'_l \circ \Pi$. □

In general, the regularity of l does not imply the regularity of L. However, in the case of Lie algebroids with symmetry, since $\overline{\Pi}$ is an isomorphism in each fibre, we have that a projectable Lagrangian $L = l \circ \Pi$ on A is regular iff the reduced Lagrangian l on A/G is regular. Moreover, if $\mathcal{F}l : \widehat{A} \to \widehat{A}^*$ denotes the Legendre transformation associated with the Lagrangian $l \in C^\infty(\widehat{A})$ we have

$$\mathcal{F}L(a)(b) = \frac{\mathrm{d}}{\mathrm{d}t} L(a + tb)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} l(\Pi(a) + t\Pi(b))|_{t=0} = \mathcal{F}l(\Pi(a))(\Pi(b))$$

for all $a, b \in A$, then $\mathcal{F}L = \Pi^* \circ \mathcal{F}l \circ \Pi$.

We can weaken the conditions of the Theorem 3.3, by considering a Π-invariant (possibly degenerated) Lagrangian $L \in C^\infty(A)$ that admits a global dynamics, i.e. there exists a globally defined section V of $\mathcal{T}A$ satisfying the equation $i(V)\omega_L = d_{\mathcal{T}A} E_L$. With these hypotheses we have

$$i(\overline{\Pi}(V))\omega'_l = d_{\mathcal{T}\widehat{A}} E'_l,$$

that is, the reduced dynamics admits a global solution given by $\overline{\Pi}(V)$. If the solution of the initial dynamics V is $\overline{\Pi}$-projectable and a *second order differential equation* (SODE), i.e. $S(V) = \Delta$, then the solution of the reduced dynamics is a SODE too, because

$$S'(\overline{\Pi}(V)) = \overline{\Pi}(S(V)) = \overline{\Pi}(\Delta) = \Delta'.$$

Of course, if the Cartan 2-form ω'_l is symplectic then $\overline{\Pi}(V)$ is always a SODE.

3.1. Examples of dynamical reduction

1. **Reduction of degenerated Lagrangian systems.** In standard classical dynamics, let us consider a Lagrangian $L \in C^\infty(TQ)$ satisfying the following conditions (see [2]):
   (A1) the Cartan 2-form ω_L is presymplectic, i.e. it is a constant rank closed form;
   (A2) the Lagrangian L admits a global dynamics;
   (A3) the foliation defined by ω_L is regular, i.e. the quotient space $TQ/\mathrm{Ker}\,\widehat{\omega}_L$ has a differentiable manifold structure and the projection $\Pi : TQ \to TQ/\mathrm{Ker}\,\widehat{\omega}_L$ is a surjective submersion, where $\widehat{\omega}_L(X) = i(X)\omega_L = \omega_L(X, \cdot)$ for all $X \in \mathfrak{X}(TQ)$.
   Under these conditions, let $p : TQ \to Q$ be the canonical projection of TQ and let us suppose that:
   (A4) the distribution $D = Tp(\mathrm{Ker}\,\widehat{\omega}_L)$ defines a regular foliation of Q, i.e. the space of the leaves $\widehat{Q} = Q/D$ admits a structure of differentiable manifold for which the canonical projection $\pi : Q \to \widehat{Q}$ is a surjective submersion.
   We can prove that there exists a unique vector bundle structure in the quotient manifold $\widehat{TQ} = TQ/\mathrm{Ker}\,\widehat{\omega}_L$ such that $\widehat{p} \circ \Pi = \pi \circ p$, where $\widehat{p} : \widehat{TQ} \to \widehat{Q}$ is given by $\widehat{p}([X]) = \pi(p(X))$, for each $X \in TQ$, such that $\Pi(X) = [X]$.
   We know that the tangent bundle TQ is a Lie algebroid over Q whose anchor is the identity map on TQ and whose Lie algebra structure on the set of sections is given by the usual bracket of vector fields on Q. If the surjective submersion of vector bundles $(\Pi, \pi) : (TQ, p, Q) \to (\widehat{TQ}, \widehat{p}, \widehat{Q})$ satisfies the conditions of the reduction theorem stated in [4], then the bundle $\widehat{TQ}$ is endowed with a (reduced) Lie algebroid structure, such that (Π, π) is a homomorphism of Lie algebroids. From what we have proved so far, if L is Π-invariant, the dynamics in $T(TQ) = \mathcal{T}(TQ)$ reduces to the dynamics in $\mathcal{T}\widehat{TQ}$. In other words, the dynamics solution in $T(TQ)$, given by a vector field V on TQ, projects into a section of $\mathcal{T}\widehat{TQ}$ that satisfies the dynamics equation in $\mathcal{T}\widehat{TQ}$. In these conditions, we can conclude that there exists a unique symplectic form $\widetilde{\omega}$ on $\mathcal{T}\widehat{TQ}$ such that $\omega_L = \overline{\Pi}^* \widetilde{\omega}$ and, therefore, $\omega'_l = \widetilde{\omega}$ is a symplectic form, where $L = l \circ \Pi$. So the solution of the reduced dynamics is a SODE, i.e. $S'(\overline{\Pi}(V)) = \Delta$.
2. **Reduction of a principal fibre bundle.** Let $P(M, G)$ be a principal fibre bundle. We saw in [4] that the gauge algebroid TP/G is a reduced Lie algebroid of the tangent bundle TP endowed with its usual Lie algebroid structure, where the canonical projection (surjective submersion) $\Pi : TP \to TP/G$ is a homomorphism of Lie algebroids. Given a Π-invariant Lagrangian $L = l \circ \Pi \in C^\infty(TP)$ with a global dynamics (solution) V on TP, we have that $\overline{\Pi}(V)$ is a global dynamics (solution) on TP/G, i.e. $i(\overline{\Pi}(V))\omega'_l = d_{\mathcal{T}(TP/G)} E'_l$. If $V' = \overline{\Pi}(V)$ is a SODE, then the reduced dynamics equation is equivalent to $\pounds_{V'} \theta'_l = d_{\mathcal{T}(TP/G)} l$, with $\theta'_l = d_{\mathcal{T}(TP/G)} l \circ S'$. Recall that, in this case we have $T(TP)/G \cong \mathcal{T}(TP/G)$.

4. Reduction of nonholonomic systems on Lie algebroids

In this section, we will prove that dynamical systems on Lie algebroids with constraints can be reduced, if we impose on the Lagrangian similar conditions to the Theorem 3.3.

4.1. Nonholonomic systems on a Lie algebroid

The first time nonholonomic systems in the framework of Lie algebroids is dealt with, was in [5].

Let us consider a system on the Lie algebroid $p : A \to M$ with nonholonomic constraints given by a vector subbundle B of A, where the submanifold B is defined by the vanishing of a set of independent linear functions $\{\phi_a = \phi_{a\beta}\mathbf{v}^\beta \mid a = 1, \dots, k\}$. Parallel to the usual formalism in classical mechanics on the tangent bundle (see e.g. [3]), the constrained system equations of motion can be written in a global form

$$\begin{cases} (i(V)\omega_L - d_{\mathcal{T}A}E_L)|_B \in \Gamma(S^*((\mathcal{T}B)^0)) \\ V|_B \in \Gamma(\mathcal{T}B), \end{cases} \tag{4.1}$$

where $\mathcal{T}B = \{(b, c, v) \in B \times B \times TB \mid p(b) = p(c),\ \rho(c) = Tp(v),\ \text{with } v \in T_bB\}$[1] is a subbundle of $p_{\mathcal{T}A}|_B : \mathcal{T}_BA \to B$ and $(\mathcal{T}B)^0 \subset (\mathcal{T}A)^*$ denotes the annihilator of $\mathcal{T}B$. The above formulation of the nonholonomic system in the Lie algebroid A is called *Chetaev formulation*.

Note that V is a SODE of $\mathcal{T}A$ since sections in $S^*((\mathcal{T}B)^0)$ are semibasic, that is, vanishes on vertical sections of $\mathcal{T}A$. The semibasic sections $S^*(d_{\mathcal{T}A}\phi_a) = \partial\phi_a/\partial\mathbf{v}^\beta\, \mathcal{X}^\beta$ are called *reaction forces* on the Lie algebroid A.

Definition 4.1. A nonholonomic constraint ϕ on a Lie algebroid is called ideal if the Liouville section of $\mathcal{T}_BA$ is a section of $\mathcal{T}B$, where B is the subbundle of A defined by $\phi = 0$, that is, $\pounds_\Delta\, \phi = d_{\mathcal{T}A}\phi(\Delta) = 0$ in B.

With this definition of ideal constraint, we can show:

Proposition 4.2. *When the constrains ϕ_α are ideal and V is a solution of (4.1), the energy of the system is conserved, that is, $\pounds_V E_L = 0$ on B.*

Proof. We have, on B,

$$\pounds_V E_L = \rho_{\mathcal{T}A}(V)E_L = \lambda^a S^*(d_{\mathcal{T}A}\phi_a)(V) = \lambda^a d_{\mathcal{T}A}\phi_a(S(V))$$

and, since V is a SODE, then $S(V) = \Delta$ and

$$\pounds_V E_L = \lambda^a d_{\mathcal{T}A}\phi_\alpha(\Delta) = \lambda^a \pounds_\Delta\, \phi_a.$$

But as the constraints were assumed to be ideal, the right hand term is zero. □

The solution of the nonholonomic system (4.1) is given by

$$V = V_L + \lambda^a Z_a,$$

where V_L is a solution of the initial dynamics (without constraints), Z_a is a vertical section of $\mathcal{T}A$ given by $i(Z_a)\omega_L = -S^*(d_{\mathcal{T}A}\phi_a)$ and λ^a is a function on A determined in such a way that $V|_B \in \Gamma(\mathcal{T}B)$, i.e. $\pounds_V\, \phi_a = 0$, for all $a = 1, \dots, k$.

If $(q^1, \dots, q^n, \mathbf{v}^1, \dots, \mathbf{v}^s)$ is a system of local coordinates of $p : A \to M$ associated with the choice of a basis of local sections $\{e_\alpha \mid \alpha = 1, \dots, s\}$, the Euler–Lagrangian equations of the constrained system (4.1) are:

$$\begin{cases} \dot{q}^i = \rho^i{}_\alpha \mathbf{v}^\alpha \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \mathbf{v}^\alpha}\right) = \rho^i{}_\alpha \dfrac{\partial L}{\partial q^i} + \mathbf{v}^\beta c_{\beta\alpha}{}^\gamma \dfrac{\partial L}{\partial \mathbf{v}^\gamma} + \lambda^a \dfrac{\partial \phi_a}{\partial \mathbf{v}^\alpha} \end{cases}$$

with $\lambda^a \in C^\infty(A)$, for all $a = 1, \dots, k$, where $\rho^i{}_\alpha$ and $c_{\beta\alpha}{}^\gamma$ are the *structure functions of the Lie algebroid A* relative to $\{e_\alpha\}$ (see [1]). We can use a new set of local coordinates adapted to the constraints, that is, let us consider

[1] In order to simplify the text, we write p (resp. ρ) instead of $p|_B$ (resp. $\rho|_B$).

a new set of local coordinates on the Lie algebroid A, $\{(q^i, \mathbf{w}^\alpha) \mid i = 1, \ldots, n,\ \alpha = 1, \ldots, s\}$, associated with the basis of sections $\{f_\alpha \mid \alpha = 1, \ldots, s\}$ of A that satisfy:

$$\mathbf{w}^\alpha = \widehat{\Phi_\alpha}(q, \mathbf{v}) = \Phi_{\alpha\beta}(q)\mathbf{v}^\beta, \qquad \mathbf{v}^\alpha = \widehat{\Psi_\alpha}(q, \mathbf{w}) = \Psi_{\alpha\beta}(q)\mathbf{w}^\beta, \tag{4.2}$$

for all $\alpha = 1, \ldots, s$, where $\widehat{\Phi_\alpha}$ and $\widehat{\Psi_\alpha}$ are linear functions on A associated to the A-1-forms Φ_α and Ψ_α, respectively, that satisfies $\Psi_{\alpha\beta}\Phi_{\beta\gamma} = \delta_{\alpha\gamma}$ and are defined by

$$\Phi = \begin{pmatrix} I_{s-k} & 0_{(s-k)\times k} \\ A_{21} & A_{22} \end{pmatrix}, \qquad \Psi = \begin{pmatrix} I_{s-k} & 0_{(s-k)\times k} \\ B_{21} & B_{22} \end{pmatrix},$$

where the matrix $A = (A_{21}\ A_{22})$ is given by $A_{a\beta} = \phi_{a\beta}$ for all $a = 1, \ldots, k$ and $\beta = 1, \ldots, s$, with A_{22} invertible, and the matrix $B = (B_{21}\ B_{22})$ is given by $B_{21} = -A_{22}^{-1}A_{21}$ and $B_{22} = A_{22}^{-1}$. We have the following relations between the local sections:

$$f_\alpha = \Psi_{\beta\alpha}e_\beta, \qquad e_\alpha = \Phi_{\beta\alpha}f_\beta.$$

In these new coordinates, the Euler–Lagrangian equations of the nonholonomic system on the Lie algebroid A are given by:

$$\begin{cases} \dot{q}^i = \rho^i{}_\beta \Psi_{\beta\bar{a}}\mathbf{w}^{\bar{a}} \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \mathbf{w}^{\bar{a}}}\right) = \rho^i{}_\beta \Psi_{\beta\bar{a}}\dfrac{\partial L}{\partial q^i} + \mathbf{w}^{\bar{b}}\gamma^\beta_{\bar{b}\bar{a}}\dfrac{\partial L}{\partial \mathbf{w}^\beta} \end{cases}$$

for all $\bar{a} = 1, \ldots, s-k$, where $[f_{\bar{b}}, f_{\bar{a}}]_A = \gamma^\beta_{\bar{b}\bar{a}}f_\beta$. These are, precisely, the equations (5) obtained by Mestdag et al. in [17], because:

$$\begin{aligned} \mathbf{w}^{\bar{b}}\gamma^\beta_{\bar{b}\bar{a}}\frac{\partial L}{\partial \mathbf{w}^\beta} &= \mathbf{w}^{\bar{b}}\gamma^\beta_{\bar{b}\bar{a}}\Psi_{\eta\beta}\frac{\partial L}{\partial \mathbf{v}^\eta} \\ &= \mathbf{w}^{\bar{b}}\left\langle e^\eta, [f_{\bar{b}}, f_{\bar{a}}]_A\right\rangle\frac{\partial L}{\partial \mathbf{v}^\eta} \\ &= \mathbf{w}^{\bar{b}}\left(\Psi_{\beta\bar{b}}\Psi_{\alpha\bar{a}}c_{\beta\alpha}{}^\eta + \rho^i{}_\beta\Psi_{\beta\bar{b}}\frac{\partial \Psi_{\eta\bar{a}}}{\partial q^i} - \rho^i{}_\alpha\Psi_{\alpha\bar{a}}\frac{\partial \Psi_{\eta\bar{b}}}{\partial q^i}\right)\frac{\partial L}{\partial \mathbf{v}^\eta}. \end{aligned}$$

Now, suppose that the subbundle B of A is a Lie subalgebroid of A, that is, there exists an injective morphism $\iota : B \to A$ such that ι is a homomorphism of Lie algebroids. In these conditions we have that $\bar{\iota} = (\iota, \iota, T\iota)|_{\mathcal{T}B} : \mathcal{T}B \to \mathcal{T}A$ is a homomorphism of Lie algebroids. Note that, $\bar{\iota} \circ V|_B = V \circ \iota$, for all $V \in \Gamma(\mathcal{T}A)$ such that $V|_B \in \Gamma(\mathcal{T}B)$. Then, we can prove the following result:

Lemma 4.3. $\bar{\iota}^*(S^*(d_{\mathcal{T}A}\phi_a)) = 0$.

Proof. Indeed

$$\bar{\iota}^*(S^*(d_{\mathcal{T}A}\phi_a)) = S_B^*(\iota^* d_{\mathcal{T}A}\phi_a),$$

where S_B is the vertical endomorphism on $\mathcal{T}B$. Then, because ι is a homomorphism of Lie algebroids, we have

$$\bar{\iota}^*(S^*(d_{\mathcal{T}A}\phi_a)) = S_B^*(d_{\mathcal{T}B}(\bar{\iota}^*\phi_a))$$

and, since $\bar{\iota}^*\phi_a = 0$, we get $\bar{\iota}^*(S^*(d_{\mathcal{T}A}\phi_a)) = 0$. □

Consider the set $\mathcal{T}^A B = \{(b, a, v) \in B \times A \times TB \mid p(b) = p(a), \rho(a) = Tp(v), \text{ with } v \in T_bB\}$. The vector bundle $\tau : \mathcal{T}^A B \to B$, with $\tau(b, a, v) = b$, is endowed with a Lie algebroid structure (see [10,17]), and is called the prolongation of the bundle B with respect to A (or prolongation of A over the map $p : B \to M$ in the terminology of [10]). Note that

$$\iota = I^A \circ I \tag{4.3}$$

where $I : \mathcal{T}B \to \mathcal{T}^A B$ is defined by $I(b, c, v) = (b, \iota(c), v)$ and $I^A : \mathcal{T}^A B \to \mathcal{T}A$ is defined by $I^A(b, c, v) = (\iota(b), c, T\iota(v))$; both I^A and I are Lie algebroids homomorphisms. In these conditions, we can give a geometric proof of the relation (14) obtained by Mestdag et al. [17].

Proposition 4.4. *The condition $i(V|_B)\delta\widetilde{\theta}_L = -\delta\widetilde{E}_L$ holds, where $\widetilde{\theta}_L = (I^A)^*\theta_L$, $\widetilde{E}_L = (I^A)^*E_L$, $\delta = I^* \circ d_{\mathcal{T}^A B}$ and V is the SODE solution to the constrained system (4.1).*

Proof. From (4.1) and Lemma 4.3, we have

$$\bar{\iota}^*(\, i(V)\omega_L - d_{\mathcal{T}A}E_L\,) = 0,$$

that is,

$$\bar{\iota}^*(i(V)\omega_L) = \bar{\iota}^*(d_{\mathcal{T}A}E_L) = \delta\widetilde{E}_L,$$

which is equivalent to

$$i(V|_B)(\bar{\iota}^*\omega_L) = \delta\widetilde{E}_L.$$

Since $\bar{\iota}^*\omega_L = -\bar{\iota}^*(d_{\mathcal{T}A}\theta_L) \overset{(4.3)}{=} -I^* \circ d_{\mathcal{T}^A B}((I^A)^*\theta_L) = -\delta\widetilde{\theta}_L$, we obtain $i(V|_B)\delta\widetilde{\theta}_L = -\delta\widetilde{E}_L$. □

Since $\delta = I^* \circ d_{\mathcal{T}^A B} = d_{\mathcal{T}B} \circ I^*$, from Proposition 4.4 we can write

$$i(V|_B)d_{\mathcal{T}B}(\bar{\iota}^*\theta_L) = -d_{\mathcal{T}B}(\bar{\iota}^*E_L),$$

which is equivalent to

$$i(V|_B)d_{\mathcal{T}B}\theta_{\overline{L}} = -d_{\mathcal{T}B}E_{\overline{L}},$$

with $\theta_{\overline{L}} = \bar{\iota}^*\theta_L$ and $E_{\overline{L}} = \bar{\iota}^*E_L$, where $\overline{L} = L \circ \iota : B \to \mathbb{R}$ is a differentiable function on B. Therefore,

$$i(V|_B)\omega_{\overline{L}} = d_{\mathcal{T}B}E_{\overline{L}}, \tag{4.4}$$

with $\omega_{\overline{L}} = -d_{\mathcal{T}B}\theta_{\overline{L}}$; in general this 2-form is degenerate.

4.2. Reduction of nonholonomic systems

Suppose that the surjective submersion of vector bundles $(\Pi, \pi) : (A, p, M) \to (\widehat{A}, \widehat{p}, \widehat{M})$ is a homomorphism of the Lie algebroids $(A, \rho, [\cdot,\cdot]_A)$ and $(\widehat{A}, \widehat{\rho}, [\cdot,\cdot]_{\widehat{A}})$. Let $\widehat{B}$ be a subbundle of $\widehat{A}$ given by $\operatorname{Im}\Pi|_B$, $\Pi|_B(B) = \widehat{B}$. Next, we will prove that the constrained dynamics on A reduces into a dynamics on $\widehat{A}$ whose solution is a section of $\mathcal{T}\widehat{B}$.

First of all, we will show, for all $b \in B$, that $\overline{\Pi}(V(b))$ belongs to the subbundle $\mathcal{T}\widehat{B}$ of $\mathcal{T}_{\widehat{B}}\widehat{A}$, with total space

$$\mathcal{T}\widehat{B} = \left\{(b', c', v') \in \widehat{B} \times \widehat{B} \times T\widehat{B} \mid \widehat{p}(b') = \widehat{p}(c'),\ \widehat{\rho}(c') = T\widehat{p}(v'), \text{with } v \in T_{b'}\widehat{B}\right\},$$

where V is the solution of the constrained dynamics on A that satisfies the system (4.1). Let $V(b) = (b, c, v) \in \mathcal{T}_b B$, then $\overline{\Pi}(V(b)) = (\Pi(b), \Pi(c), T\Pi(v))$. Thus, $\widehat{p}(\Pi(b)) = \pi(p(b)) = \pi(p(c)) = \widehat{p}(\Pi(c))$. On the other hand, we have

$$\begin{aligned} T\widehat{p}(T\Pi(v)) &= T(\widehat{p} \circ \Pi)(v) = T(\pi \circ p)(v) \\ &= T\pi(Tp(v)) = T\pi(\rho(b)) \\ &= (\widehat{\rho} \circ \Pi)(b) = \widehat{\rho}(\Pi(b)). \end{aligned}$$

Therefore, $\overline{\Pi}(V(b)) \in \mathcal{T}\widehat{B}$, for all $b \in B$. Moreover, if we suppose that V is $\overline{\Pi}$-projectable, then $V' = \overline{\Pi}(V)$ is a SODE since

$$S'(V') = S'(\overline{\Pi}(V)) = \overline{\Pi}(S(V)) = \overline{\Pi}(\Delta) = \Delta'.$$

In the next lemma we prove that $i(V')\omega'_l - d_{\mathcal{T}\widehat{A}}E'_l$ is equal to a external force of the reduced system on $\widehat{A}$.

Lemma 4.5. *With the same notation as before, we have $i(V')\omega'_l - d_{\mathcal{T}\widehat{A}}E'_l = S'^*(\Phi)$ on $\widehat{B}$, where $\overline{\Pi}^*(\Phi) \subset (\mathcal{T}B)^0 + \operatorname{Ker} S^*$.*

Proof. On B, we know that

$$i(V)\omega_L - d_{\mathcal{T}A}E_L \in S^*((\mathcal{T}B)^0).$$

So,

$$i(V)(\overline{\Pi}^*\omega'_l) - d_{\mathcal{T}A}(\overline{\Pi}^*E'_l) \in S^*((\mathcal{T}B)^0),$$

that is,

$$\overline{\Pi}^*[i(V')\omega'_l - d_{\mathcal{T}\widehat{A}}E'_l] \in S^*((\mathcal{T}B)^0).$$

But V' is a SODE, then $i(V')\omega'_l - d_{\mathcal{T}\widehat{A}}E'_l$ is a semibasic 1-form. Therefore, there exists $\Phi \in (\mathcal{T}\widehat{A})^*$ such that

$$i(V')\omega'_l - d_{\mathcal{T}\widehat{A}}E'_l = S'^*(\Phi).$$

Since $\overline{\Pi}^*(S'^*(\Phi)) \in S^*((\mathcal{T}B)^0)$ and $\overline{\Pi} \circ S = S' \circ \overline{\Pi}$, then, $S^*(\overline{\Pi}^*(\Phi)) \in S^*((\mathcal{T}B)^0)$. Therefore, $\overline{\Pi}^*(\Phi) \in (\mathcal{T}B)^0 + \operatorname{Ker} S^*$. □

Thus, we have proved:

Theorem 4.6. *The constrained dynamics on A reduces into a dynamics on $\widehat{A}$ whose solution satisfies*

$$\begin{cases} (i(V')\omega'_l - d_{\mathcal{T}\widehat{A}}E'_l)|_{\widehat{B}} \in \Gamma(S'^*(\mathcal{C})) \\ V'|_{\widehat{B}} \in \Gamma(\mathcal{T}\widehat{B}), \end{cases}$$

where $\overline{\Pi}^*(\mathcal{C}) \subset (\mathcal{T}\widehat{B})^0 + \operatorname{Ker} S^*$.

Since the solution of the nonholonomic system (4.1) is given by $V = V_L + \lambda^\alpha Z_\alpha$, then $V' = \overline{\Pi}(V) = \overline{\Pi}(V_L) + \overline{\Pi}(\lambda^\alpha Z_\alpha)$, where $V'_l = \overline{\Pi}(V_L)$ is the SODE solution of the reduced dynamics without constraints and $\overline{\Pi}(\lambda^\alpha Z_\alpha)$ is a vertical section of $\mathcal{T}\widehat{A}$ such that $i(\overline{\Pi}(\lambda^\alpha Z_\alpha))\omega'_l = S'^*(\Phi)$ is a semibasic section.

Proposition 4.7. *We have the following relation, $\pounds_V E_L = \pounds_{V'} E'_l \circ \Pi$.*

Proof. From the relation $\overline{\Pi}^* E'_l = E_L$, we have

$$\pounds_V E_L = d_{\mathcal{T}A}E_L(V) = d_{\mathcal{T}A}(\overline{\Pi}^*E'_l)(V).$$

The map $\overline{\Pi}$ is a homomorphism of Lie algebroids, then

$$\pounds_V E_L = \overline{\Pi}^*(d_{\mathcal{T}\widehat{A}}E'_l)(V) = d_{\mathcal{T}\widehat{A}}E'_l(V') \circ \Pi,$$

that is, $\pounds_V E_L = \pounds_{V'} E'_l \circ \Pi$. □

As an immediate consequence we have that:

Corollary 4.8. *The energy of the constrained system on A is conserved iff the energy of the reduced system on $\widehat{A}$ is conserved.*

4.3. Example: Non-Abelian Čaplygin systems

A non-Abelian Čaplygin system is a constrained system whose configuration space is a principal fibre bundle $\pi : P \to M = P/G$ endowed with a connection given by the constraint distribution H such that $TP = H \oplus V$, where V is the vertical bundle; therefore, the constraints ϕ_α are linear in the velocities and the energy of the system is conserved (see [3] and references therein). The Lagrangian $L \in C^\infty(TP)$ of the system is supposed to be regular and invariant for the lifted action of the Lie group G on P, i.e. $L = l \circ \Pi$ where $\Pi : TP \to TP/G$ is the canonical projection defined by the lifted action. The constrained system on TP can be formulated as follows:

$$\begin{cases} (i(V)\omega_L - d_{T(TP)}E_L)|_H \in \Gamma(S^*((\mathcal{T}H)^0)) \\ V|_H \in \Gamma(\mathcal{T}H). \end{cases} \tag{4.5}$$

The solution of the system is of the form $V = V_L + \lambda^a Z_a$, where V_L is the solution of the system without constraints, Z_a is a vertical section of $T(TP)$ such that $i(Z_a)\omega_L = -S^*(d_{T(TP)}\phi_a)$ and $£_V\, \phi_a = 0$.

The canonical projection $\Pi : TP \to TP/G$ maps the subbundle H of $A = TP$ onto the subbundle $H' = H/G \cong TM$ of $\widehat{A} = TP/G$, and it is a homomorphism of Lie algebroids, [4]. The inclusion $\iota' : TM \to TP/G$ of TM in TP/G is given by the horizontal lift of a vector field on M into a vector field on P. Thus, by Theorem 4.6, the reduced constrained system on TP/G has a solution that satisfies

$$i(V'|_{TM})\delta\widetilde{\theta}'_l = -\delta\widetilde{E}'_l + \overline{\iota'}^*(\Phi'),$$

where $\Phi' = i(\Pi(\lambda^a Z_a))\omega'_l$ is a semibasic section, $\overline{\iota'} = I^{TP/G} \circ I' : \mathcal{T}H' = T(TM) \to \mathcal{T}(TP/G)$, with $I' : T(TM) \to \mathcal{T}^{TP/G}(TM)$ defined by $I'(a, b, v) = (a, \iota'(b), v)$ and $I^{TP/G} : \mathcal{T}^{TP/G}(TM) \to \mathcal{T}(TP/G)$ defined by $I^{TP/G}(a, b, v) = (\iota'(a), b, T\iota'(v))$, and $\delta = I'^* \circ d_{\mathcal{T}^{TP/G}(TM)}$. Let $d_{T(TM)}$ be a differential operator given by $d_{T(TM)} \circ I'^* = \delta = I'^* \circ d_{\mathcal{T}^{TP/G}(TM)}$. So,

$$i(V'|_{TM})d_{T(TM)}(\overline{\iota'}^*\theta'_l) = -d_{T(TM)}(\overline{\iota'}^* E'_l) + \overline{\iota'}^*(\Phi').$$

Note that, $\overline{\iota'}^*\theta'_l = \theta_{\bar{l}}$ and $\overline{\iota'}^* E'_l = E_{\bar{l}}$, with $\bar{l} : TM \to M$ defined by $\bar{l}(Y_{\bar{q}}) = L(Y^H_q)$ for all $\bar{q} = \pi(q)$, $Y_{\bar{q}} \in T_{\bar{q}}M$, where Y^H denotes the horizontal lift to P of a vector field Y on M. Therefore,

$$i(V'|_{TM})\omega_{\bar{l}} = d_{T(TM)}(\overline{\iota'}^* E'_l) - \overline{\iota'}^*(\Phi'),$$

with $\omega_{\bar{l}} = -d_{T(TM)}(\overline{\iota'}^*\theta'_l)$. As in [3] we have $i(V'|_{TM})\overline{\iota'}^*(\Phi') = 0$, because

$$i(V'|_{TM})\overline{\iota'}^*(\Phi') = \Phi'(\overline{\iota'} \circ V'|_{TM}) = \Phi'(V') = \omega'_l(\overline{\Pi}(\lambda^a Z_a), \overline{\Pi}(V)) = \omega_L(\lambda^a Z_a, V) = 0.$$

Once this work had been finished we have realized that some similar results had been announced in [6], and are given in [7] and [18].

Acknowledgments

The authors acknowledge the financial support from PRODEP/5.3/2003, POCI/MAT/58452/2004, CMUC/FCT and the project BFM-2003-02532. We thank the authors Cortés et al. for sending us the preprint [7] with their results.

References

[1] A. Cannas da Silva, A. Weinstein, Lectures on Geometrical Models for Noncommutative Algebra, University of California, Berkeley, 1998.
[2] F. Cantrijn, J.F. Cariñena, M. Crampin, Reduction of degenerate Lagrangians systems, J. Geom. Phys. 3 (1986) 353–400.
[3] F. Cantrijn, M. de León, J.C. Marrero, D.M. de Diego, Reduction of nonholonomic mechanical systems with symmetries, Rep. Math. Phys. 42 (1998) 25–45.
[4] J.F. Cariñena, J.M. Nunes da Costa, P. Santos, Reduction of Lie algebroids structures, Int. J. Geom. Methods Mod. Phys. 2 (2005) 965–991.
[5] J. Cortés, E. Martínez, Mechanical control systems on Lie algebroids, IMA J. Math. Control Inform. 21 (2004) 457–492.
[6] J. Cortés, M. de Léon, J.C. Marrero, D. Martín de Diego, E. Martínez, A survey of Lagrangian mechanics and control on Lie algebroids and groupoids, Int. J. Geom. Methods Mod. Phys. 3 (3) (2006) 509–558. arXiv:math-ph/0511009.
[7] J. Cortés, M. de Léon, J.C. Marrero, E. Martínez, Nonholonomic Lagrangian systems on Lie algebroids, arXiv:math-ph/0512003.
[8] K. Grabowska, J. Grabwoski, P. Urbański, Geometrical mechanics on algebroids, Int. J. Geom. Methods Mod. Phys. 3 (2006) 559–575.
[9] P.J. Higgins, K. Mackenzie, Algebraic constructions in the category of Lie algebroids, J. Algebra 129 (1990) 194–230.
[10] M. de León, J.C. Marrero, E. Martínez, Lagrangian submanifolds and dynamics on Lie algebroids, J. Phys. A: Math. Gen. 38 (2005) R241–R308.
[11] P. Libermann, Lie algebroids and mechanics, Arch. Math. (Brno) 32 (1996) 147–162.
[12] K. Mackenzie, Lie Groupoids and Lie Algebroids in Differential Geometry, in: London Math. Soc. Lect. Note Series, vol. 124, Cambridge University Press, 1987.
[13] K. Mackenzie, Lie algebroids and Lie pseudoalgebras, Bull. London Mat. Soc. 27 (1995) 97–147.
[14] J.E. Marsden, T. Ratiu, Reduction of Poisson manifolds, Lett. Math. Phys. 11 (1986) 161–169.
[15] E. Martínez, Lagrangian mechanics on Lie algebroids, Acta Appl. Math. 67 (2001) 295–320.
[16] E. Martínez, Classical field theory on Lie algebroids: Multisymplectic formalism, math.DG/0411352 (preprint).
[17] T. Mestdag, B. Langerock, A Lie algebroid framework for non-holonomic systems, J. Phys. A: Math. Gen. 38 (2005) 1097–1111.
[18] T. Mestdag, Lagrangian reduction by stages for non-holonomic systems in a Lie algebroid framework, J. Phys. A: Math. Gen. 38 (2005) 10157–10179.

[19] M. Rodríguez-Olmos, Formalismo Lagrangiano en algebroides de Lie: reducción Lagrangiana, Technical Report.
[20] J. Pradines, Théorie de Lie pour les groupoïdes différentiables. Calcul différentiel dans la catégorie des groupoïdes infinitésimaux, C.R. Acad. Sci. Paris Sér. A 264 (1967) 245–248.
[21] W. Tulczyjew, Hamiltonian systems, Lagrangian systems, and the Legendre transformation, Symposia Math. 14 (1974) 101–114.
[22] A. Vaintrob, Lie Algebroids and homological vector fields, Russian Math. Surveys 52 (1997) 428–429.
[23] A. Weinstein, Lagrangian mechanics and groupoids, in: W.F. Shadwick, P.S. Krishnaprasad, T.S. Ratiu (Eds.), Mechanics Day, Amer. Math. Soc., Fields Inst. Commun. 7 (1996), 207–231.