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HEAT CONDUCTION IN THE HOLLOW SPHERE WITH A POWER-LAW VARIATION OF THE EXTERNAL HEAT TRANSFER COEFFICIENT

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ABSTRACT

The conduction phenomenon in an insulated sphere is re-worked through a dimensionless approach, where the heat transfer coefficient dependence on the external radius and on the surface temperature, as in the case of forced and free convection, is taken into account. Assuming a power law variation of the convection coefficient [1, 2], and using the results of Sparrow [3], equations and graphs for the most important dimensionless parameters are presented. The developed equations show, for example, that as the insulation thickness increases the heat transfer rate tends to a positive value, independent of the considered case: constant convection coefficient, forced or free convection.

Introduction

The study of radial heat conduction in the hollow sphere, or in the insulation of spherical bodies, may be improved with the use of a dimensionless approach, allowing the graphical representation of the phenomenon, through the most important governing parameters. Natural convection, assuming a power-law variation of the external convection coefficient, is analyzed, whereas forced convection and the constant convection coefficient cases (eg. Incropera and DeWitt [4]) are treated as particular situations.

Sparrow [3] used this kind of variation of the convection coefficient

$$h_o \propto r_o^{-p}\left|T_o - T_\infty\right|^n \quad (1)$$

in the study of the critical insulation radius, with $p \geq 0$ and $n \geq 0$, to obtain the following implicit equation

$$r_{o,crit} = \frac{2-p}{1+n}\frac{k}{h_{o,crit}} \quad (2)$$

Balmer [5] used the correlation developed by Yuge [6] for free convection around spheres, in the development of an equation for the critical insulation radius; similarly to equation (2) it is an implicit equation, since it uses the unknown insulation surface temperature. For the case of constant heat transfer coefficient, Russo [7] presented an equation for the minimum amount of insulation necessary to minimize the heat loss.

The following development extends these results, under the assumption of a power law variation of the external convection coefficient. Explicit solutions for the critical insulation radius, but also for the heat transfer rate and the temperature distribution are presented.

Convection Over Spheres

When far enough of the limiting case of pure conduction heat transfer, forced and free convection around spheres can be modeled using a power law. In the range of Reynolds number from 17 to 70000, McAdams [1] recommends the following equation

$$Nu_D = 0.37(Re_D)^{0.6} \quad (3)$$

In the case of free convection, an equation indicated by Schlichting [2] is

$$Nu_D = 0.429(Gr_D)^{0.25} \quad (4)$$

where Nu_D is the Nusselt number, Re_D the Reynolds number and Gr_D the Grashof number, all based in the outer diameter, D.

Typical general expressions [4, 8] for forced and natural convection around spheres are of the form

$$Nu_D = 2 + \alpha(N)^{\beta} Pr^{\gamma} \quad (5)$$

where $N = Re_D$ for forced convection and $N = Gr_D$ for free convection. This kind of equation combines the cases of pure conduction heat transfer ($N = 0$, $Nu_D = 2$) with a power law, characteristic of greater Reynolds or Grashof numbers. As examples, for the case of forced convection, we have the correlation of Ranz and Marshall [9]

$$Nu_D = 2 + 0.6\, Re_D^{\,1/2} Pr^{1/3} \quad (6)$$

and in the case of free convection the equations of Yuge [6] ($Pr \approx 1$, $1 < Ra_D < 10^5$)

$$Nu_D = 2 + 0.43 Ra_D^{1/4} \tag{7}$$

and Churchill [10] ($Pr \geq 0.7$, $Ra_D \leq 10^{11}$)

$$Nu_D = 2 + 0.589\Big/\left(1 + (0.469/\mathrm{Pr})^{9/16}\right)^{4/9} Ra_D^{1/4} \tag{8}$$

All the referred equations, when far enough of the pure conduction limit, can be written as power law equations. In the case of forced convection

$$Nu_D = B(Re_D)^m Pr^{1/3} \tag{9}$$

and in the free convection case

$$Nu_D = C(Ra_D)^n \tag{10}$$

where Ra_D is the Rayleigh number. Constants B, m, C and n can be found in the literature.

Heat Conduction Under Free External Convection

A sphere of external radius r_o, covered with an external insulation layer of thickness $e = r_o - r_i$ and thermal conductivity k, is losing heat to a surrounding fluid, in free convection regime. The analysis presented in this work is based in the following choice of dimensionless parameters

$$Bi = \frac{h_i r_i}{k}, \quad r^* = \frac{r}{r_i}, \quad r_o^* = \frac{r_o}{r_i}, \quad T^* = \frac{T - T_\infty}{T_i - T_\infty}, \quad q^* = \frac{q}{q_i} \quad \text{and} \quad h^* = \frac{h}{h_i} \tag{11}$$

The subscript 'i' stands for the inner insulation surface, 'o' for the outer insulation surface and '∞' for the surrounding fluid. Bi is the characteristic Biot number, based on h_i, the convection coefficient in the absence of insulation. The dimensionless radial coordinate is r^*, T^* is the dimensionless temperature difference, q^* is the dimensionless heat transfer rate and the dimensionless convection coefficient is h^*.

In the case of natural convection, equation (10) may be written in a dimensionless form, similar to equation (1), as

$$h_o^* = (r_o^*)^{m-1}(T_o^*)^n \tag{12}$$

Forced convection and the constant heat transfer coefficient case may be treated as specific examples of this more general one. In the free convection case $m = 3n$ and under forced convection $n = 0$. In the case of constant convection coefficient, $m = 1$ and $n = 0$. In the following development, m, n and fluid properties, calculated at an average film temperature, are considered constants.

Temperature Distribution

For steady-state conditions, no internal heat sources, and constant properties for the insulating material, the conduction equation in spherical coordinates reduces to

$$\frac{d^2}{dr^{*2}}(r^*T^*) = 0 \tag{13}$$

The boundary conditions – known internal temperature and prescribed external heat transfer coefficient – can be written as

$$T^*(r_i^*) = 1 \quad \text{and} \quad \left.\frac{dT^*}{dr^*}\right|_{r=r_o} = -Bi\,h_o^*T_o^* \tag{14}$$

Integration of equation (13) under these conditions, originates the following temperature distribution across the insulation

$$T^* = 1 - BiT_o^*h_o^*(r_o^*)^2\left(1-\frac{1}{r^*}\right) = 1-(1-T_o^*)\frac{1-1/r^*}{1-1/r_o^*} \tag{15}$$

The external surface temperature can be obtained introducing equation (12) into equation (15)

$$T_o^* = 1 - Bi(r_o^*)^{1+m}(T_o^*)^{1+n}\left(1-\frac{1}{r_o^*}\right) \tag{16}$$

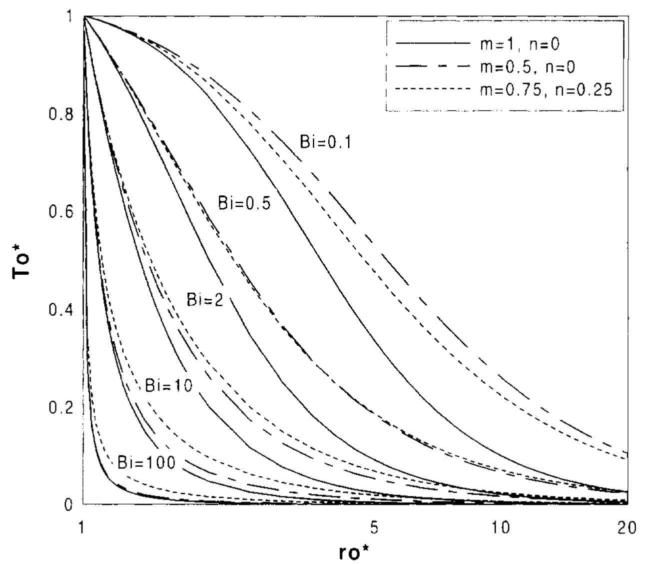

FIG. 1
Variation of the dimensionless surface temperature of the insulation layer, as a function of the Biot number, for for m=1, n=0, m=0.5, n=0 and m=0.75, n=0.25.

This equation can be solved numerically, and T_o substituted into equation (15) to obtain the temperature distribution in the radial direction. Figure 1 represents the variation of the insulation surface temperature as the insulation thickness increases, for different Biot numbers.

Heat Transfer Rate Through the Insulation

The dimensionless heat transfer rate under the prescribed variation of the heat transfer coefficient is

$$q^* = (r_o^*)^{1+m}(T_o^*)^{1+n} \tag{17}$$

This equation and equation (16) can be combined to yield

$$q^* = (r_o^*)^{1+m}\left(1 - Bi\left(1 - \frac{1}{r_o^*}\right)q^*\right)^{1+n} \tag{18}$$

showing that the insulation efficiency depends on its thickness and on the Biot number.

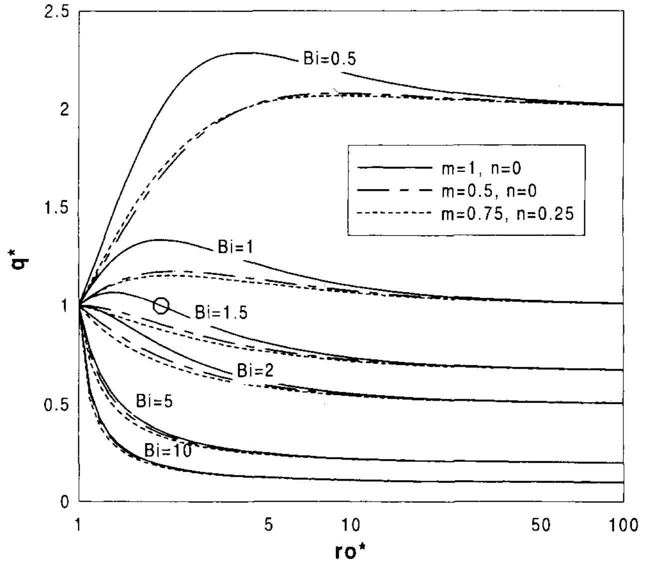

FIG. 2
Variation of the dimensionless heat flow rate as a function of the external radius of the insulation and the Biot number, for m=1, n=0, m=0.5, n=0 and m=0.75, n=0.25.

Equation (18) is graphically represented in Fig. 2. It can be seen that Bi controls the shape of the different curves and that, with low Bi, losses may increase. Equation (18) and Fig. 2 also reveal that, for an infinite insulation thickness, the dissipated heat tends asymptotically to a positive value

$$\lim_{r_o^* \to \infty} q^* = \frac{1}{Bi} \tag{19}$$

This means that, in a spherical container, the heat losses could only be eliminated using a perfect insulating material. It can be also seen that, when $Bi < 1$, the heat loss is always greater than the heat loss for the case of the non-insulated sphere. For a cylindrical geometry, similar equations show that, even with a poor insulating material, an adiabatic system can be simulated using a sufficiently large insulation thickness.

Critical Radius and Minimum Insulation Radius

The external radius of insulation that maximizes heat losses can be obtained by differentiating equation (18) with respect to r_o^*, and equating the result to zero, leading to

$$q_{max}^* = \frac{r_{o,crit}^*}{Bi((1+n)/(1+m)+(r_{o,crit}^*-1))} \tag{20}$$

$$(r_{o,crit}^*)^m = \frac{1+m}{1+n}\frac{1}{Bi}\left(1+\frac{1+m}{1+n}(r_{o,crit}^*-1)\right)^n \tag{21}$$

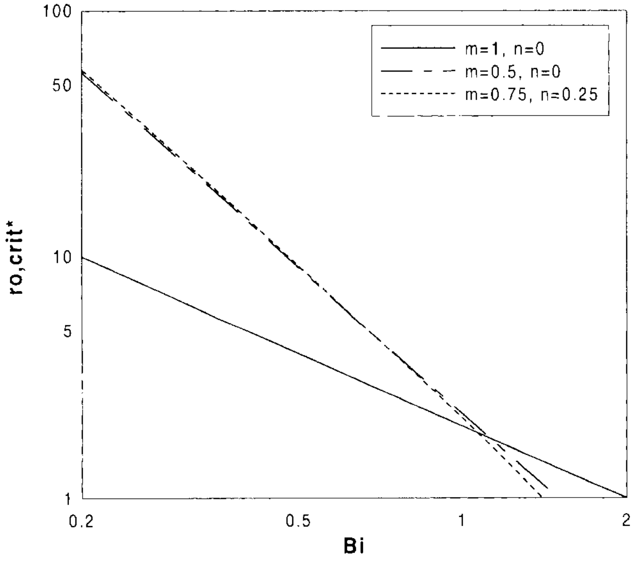

FIG. 3
Variation of the critical radius of insulation as a function of the Biot number, for for m=1, n=0, m=0.5, n=0 and m=0.75, n=0.25.

We can conclude that, with a power law variation of the external convection coefficient, a necessary condition for the existence of a critical radius is

$$Bi < \frac{1+m}{1+n} \tag{22}$$

When the Biot number satisfies condition (22) and the limiting value of the dimensionless heat transfer rate – equation. (19) – is smaller than one

$$1 < Bi < \frac{1+m}{1+n} \tag{23}$$

the insulation is only effective, Chapman [11], above the minimum insulation radius, $r_{o,min}$, indicated by points with "+" marks in Fig. 2. The minimum insulation radius can be obtained finding the non trivial solution of

$$(r^*_{o,min})^{m-n}(r^*_{o,min} - Bi\,(r^*_{o,min} - 1))^{1+n} = 1 \tag{24}$$

Outside the interval defined in equation (23) a minimum insulation radius does not exist.

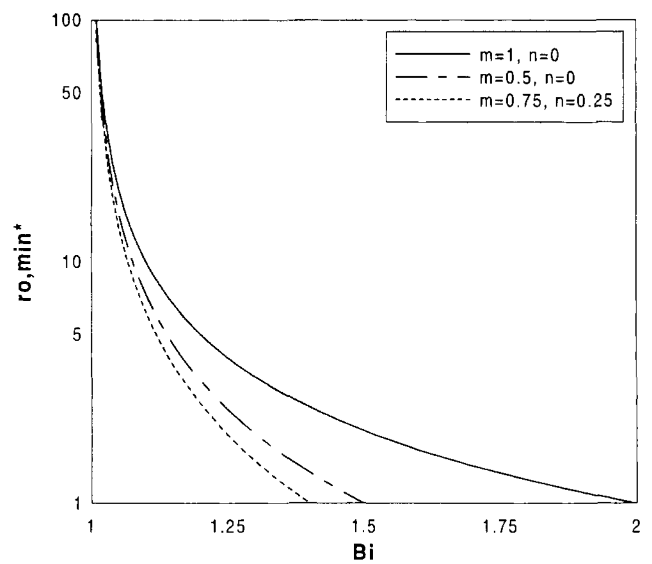

FIG. 4
Variation of the minimum radius of insulation as a function of the Biot number, for for m=1, n=0, m=0.5, n=0 and m=0.75, n=0.25.

The critical and the minimum insulation radius are represented in Figs. 3 and 4. Practical applications of this phenomenon can be found in the field of electricity and electronics (Russo and StCyr [12]).

Forced Convection and Constant Convection Coefficient

In the cases of forced convection and of constant heat transfer coefficient, equations (15-24) can be re-written setting $n = 0$, or $m = 1$ and $n = 0$, respectively. Also the case of pure conduction heat transfer in the surrounding fluid (Nu_D=2) can be represented setting $m = 0$ and $n = 0$. In this late case, in the absence of a convection phenomenon, a critical insulation radius does not exist, the defined Biot number being the ratio between the thermal conductivity of the fluid and of the insulating material.

The obtained results are presented in Table 1. Figures 1–4, besides the previously referred case of free convection, also illustrate the variation of the insulation external surface temperature, dimensionless heat flux, critical radius, minimum insulation radius for the situations of h= *const.* and forced convection (m= 0.5, n= 0).

TABLE 1
Heat Conduction in an Insulated Sphere: Forced Convection, Constant Heat Transfer Coefficient and Pure Conduction Heat Transfer

	Forced convection (n = 0, 0< m<1)	h = *const.* (n = 0, m = 1)	Nu_D = 2 (n = 0, m = 0)
T^*	$1-\dfrac{Bi(r_o^*)^{1+m}(1-1/r^*)}{1+Bi(r_o^*)^{1+m}(1-1/r_o^*)}$	$1-\dfrac{Bi(r_o^*)^2(1-1/r^*)}{1+Bi(r_o^*)^2(1-1/r_o^*)}$	$1-\dfrac{Bi\,r_o^*(1-1/r^*)}{1+Bi\,r_o^*(1-1/r_o^*)}$
q^*	$\dfrac{(r_o^*)^{1+m}}{1+Bi(r_o^*)^{1+m}(1-1/r_o^*)}$	$\dfrac{(r_o^*)^2}{1+Bi(r_o^*)^2(1-1/r_o^*)}$	$\dfrac{r_o^*}{1+Bi(r_o^*-1)}$
$r_{o,crit}^*$	$\left(\dfrac{1+m}{Bi}\right)^{1/m}$	$\dfrac{2}{Bi}$	–
$r_{o,min}^*$	$(r_{o,min}^*)^m(r_{o,min}^*-Bi(r_{o,min}^*-1))=1$	$\dfrac{1}{Bi-1}$	–
q_{max}^*	$\dfrac{(1+m)^{(m+1)/m}}{Bi((1+m)^{(m+1)/m}-m\,Bi^{1/m})}$	$\dfrac{4}{Bi(4-Bi)}$	–

Conclusion

With a more realistic representation of the convection coefficient, equations (9) and (10), than the usually assumed (h_o= *const.*), Figs. 1 to 4 show that:

As expected, the outer surface temperature of the insulation is higher, Fig. 1, while the heat transfer rate is lower, Fig. 2, than the observed with a rougher approach. For Bi close to unity, the critical insulation radius assumes a smaller value; but for lower values of Bi there is a bias inversion. Through equation (2) it can be seen that, although the corrective factor $(1-p)/(1+n)$ is smaller than unity, the dependence of h_o with the insulation radius ($h_o < h_i$) has a superimposing and opposite effect. The minimum insulation radius decreases, as can be seen in Fig. 4. These conclusions are in agreement with the expected trends, since the external convection coefficient, given, for example, by equations (3) and (4), decreases as the insulation thickness increases.

As the insulation thickness increases the dimensionless heat transfer rate converges to $1/Bi$, and as the defined Biot number increases, the error made assuming constant convection coefficient – except in the evaluation of the critical or minimum insulation radius – diminishes.

Acknowledgements

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Nomenclature

B, C	proportionality constants in equations (9) and (10), dimensionless
Bi	Biot number, $h_i r_i / k$, dimensionless
D	external diameter, m
e	thickness of the insulation layer, m
Gr_D	Grashof number based on diameter, dimensionless
h, h^*	convection coefficient (W/m^{2}K) and dimensionless convection coefficient
k	thermal conductivity of the insulation, W/mK
m, n, p	exponents in equations (1), (9) and (10), dimensionless
N	characteristic dimensionless number, in equation (5), dimensionless
Nu_D	Nusselt number based on diameter, dimensionless
Pr	Prandtl number, dimensionless
q, q^*	heat transfer rate across the insulation (W) and dimensionless heat transfer rate (q/q_i)
r, r^*	radial coordinate (m) and dimensionless radial coordinate (r/r_i)
Ra_D	Rayleigh number based on diameter, dimensionless
Re_D	Reynolds number based on diameter, dimensionless
T, T^*	temperature along the insulation (K) and dimensionless temperature difference
α	constant in equation (5), dimensionless
β, γ	exponents in equation (5), dimensionless

Subscripts and Superscripts

crit	critical radius

i	inner surface of insulation
min	minimum insulation radius
o	outer surface of insulation
*	dimensionless variable
∞	surrounding fluid

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