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Fundamental solutions for transient heat transfer by conduction and convection in an unbounded, half-space, slab and layered media in the frequency domain

Nuno Simões, António Tadeu*

Department of Civil Engineering, University of Coimbra, Pinhal de Marrocos, 3030-290 Coimbra, Portugal

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Abstract

Analytical Green's functions in the frequency domain are presented for the three-dimensional diffusion equation in an unbounded, half-space, slab and layered media. These proposed expressions take into account the conduction and convection phenomena, assuming that the system is subjected to spatially sinusoidal harmonic heat line sources and do not require any type of discretization of the space domain. The application of time and spatial Fourier transforms along the two horizontal directions allows the solution of the three-dimensional time convection-diffusion equation for a heat point source to be obtained as a summation of one-dimensional responses. The problem is recast in the time domain by means of inverse Fourier transforms using complex frequencies in order to avoid aliasing phenomenon. Further, no restriction is placed on the source time dependence, since the static response is obtained by limiting the frequency to zero and the high frequency contribution to the response is small.

The proposed functions have been verified against analytical time domain solutions, known for the case of an unbounded medium, and the Boundary Element Method solutions for the case of the half-space, slab and layered media.

Keywords: Transient heat transfer; Conduction; Convection; 2.5D Green's functions; Layered media

1. Introduction

The transient heat diffusion is a fundamental phenomenon observed in several applications, such as building physics and thermal engineering. One of the most important reference works for transient heat transfer is by Carslaw and Jaeger [1]. This book contains a set of analytical solutions and Green's functions for the diffusion equation, which gives the response, i.e. temperature field or/and heat fluxes, of the diffusion equation in the presence of a transient heat process.

Most of the known techniques to solve transient convection-diffusion heat problems have been formulated in the time domain or using Laplace transforms. In the 'time marching' approach, the solution is assessed step by step at consecutive time intervals after an initially specified state has been assumed. Using the Laplace transform, a numerical transform inversion is required to calculate the physical variables in the real space, after the solution has been obtained for a sequence of values of the transformed parameter.

In the 'time marching' approach, the result at each time step is computed directly in the time domain. Chang et al. [2] and Shaw [3] used a time-dependent fundamental solution for studying transient heat processes. Later, Wrobel and Brebbia [4] implemented a Boundary Element Method BEM formulation for axisymmetric diffusion problems. Dargush and Banerjee [5] proposed a BEM approach in the time domain, where planar, three-dimensional and axisymmetric analyses are all addressed with a time-domain convolution. Lesnic et al. [6] studied the unsteady diffusion equation in both one and two dimensions by a time marching BEM model, taking into account the treatment of singularities. Davies [7] used a time-domain analysis to compute the heat flow across a multi-layer wall, considering surface films to model the radiant and convective exchange.

* Corresponding author. Tel.: +351 239 797201; fax: +351 239 797190.
E-mail address: tadeu@dec.uc.pt (A. Tadeu).

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An alternative to the 'time marching' approach is to remove the time dependent derivative, using instead a transformed variable. The Laplace transform technique has been extensively reported in the literature, to shift the solution from the time domain to a transformed domain, for solving diffusion problems. However, an inverse transform is then required to reconstitute the solution in the time domain, which is associated with a loss of accuracy and may lead to the amplification of small truncation errors. Several researchers have tried to overcome this major drawback: algorithms for Laplace inversion have been proposed by Stehfest [8] and Papoulis [9]. Rizzo and Shippy [10] used a numerical approach that incorporates a Laplace transform to create a time-independent boundary integration in a transform domain. Other authors have proposed different solutions for analyzing the diffusion-type problem by means of Laplace transforms, such as those presented by Cheng et al. [11], Zhu and Satravaha ([12,13]).

The search for Green's functions has been extensively researched, given their interest as benchmark solutions and in the development of numerical methods, such as the BEM (e.g. Ochiai [14]) and the Method of the Fundamental Solutions (MFS) (e.g. Šarler [15]). Feng [16] used a method based on modified Green's functions to compute the unsteady heat transfer of a homogeneous or a composite solid body. Haji-Sheikh et al. [17] present different types of Green's functions that are solutions of the heat conduction diffusion equation in multi-dimensional, multi-layer bodies for different boundary conditions, calculating eigenvalues.

This work presents Green's functions for calculating the transient heat transfer wave field in the presence of an unbounded, half-space, slab and multi-layer formations, with the occurrence of conduction and convection phenomena. The problem is formulated in the frequency domain using time Fourier transforms. The proposed technique allows the use of any type of heat source, and deals with the static response.

The problem of multi-layer heat transfer has been broadly studied. The Özisik's book [18] includes a review of one-dimensional composite media, referring to orthogonal expansions, Green's functions and Laplace transform techniques. Monte [19] analysed the transient heat conduction of multi-layer composite slabs, applying the method of separation of variables to the heat conduction partial differential equation. A frequency-domain regression method was developed by Wang and Chen [20] to compute the heat flow for a one-dimensional multi-layer model.

The present work defines the transient heat transfer in a multi-layer system subject to a point, a linear or a plane source in the presence of both convection and conduction. This work extends previous work carried out by the authors to define the response of layered solid media subjected to a spatially sinusoidal harmonic heat conduction line source, e.g. Tadeu et al. [21], where only the conduction phenomenon was addressed. The proposed fundamental solutions relate the heat field variables (fluxes or temperatures) at some position in the domain caused by a heat source placed elsewhere in the media, in the presence of both conduction and convection phenomena.

As in the previous work, the technique requires the knowledge of the Green's function for the unbounded media, which are developed by first applying a time Fourier transform to the time diffusion equation for a heat point source and then a spatial Fourier transform to the resulting Helmholtz equation, along the z direction, in the frequency domain. So these functions are written first as a superposition of cylindrical heat waves along one horizontal direction (z) and then as a superposition of heat plane sources.

The Green's functions for a layered formation are formulated as the sum of the heat source terms equal to those in the full-space and the surface terms required to satisfy the boundary conditions at the interfaces, i.e. continuity of temperatures and normal fluxes between layers, and null normal fluxes or null temperatures at the outer surface. The total heat field is achieved by adding the heat source terms, equal to those in the unbounded space, to the sets of surface terms arising within each layer and at each interface.

The scope of this paper is to present first the three-dimensional formulation, explaining the mathematical manipulation to obtain the Green's functions for a heat line source as a continuous superposition of heat plane sources in the frequency domain. The procedure to retrieve the time domain solutions is also given. This methodology is verified by comparing the results obtained with the exact time solutions for one, two and three-dimensional point heat sources placed in an unbounded medium.

This paper then goes on to describe the formulation of a sinusoidal line heat load applied to a half-space, a slab, a slab over a half-space medium and a layered formation. The continuity of temperature and heat fluxes need to be achieved between two neighbouring layers, while null temperatures or null heat fluxes may be prescribed along an external interface of the boundary. The full set of expressions is corroborated by comparing its solutions with those provided by the Boundary Element Method, which requires the discretization of all layer interfaces.

2. Three-dimensional problem formulation and Green's functions in an unbounded medium

The transient heat transfer by conduction and convection in the domain with constant velocities along the x, y and z directions is expressed by the equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)T -\frac{1}{K}\left(V_x\frac{\partial}{\partial x}+V_y\frac{\partial}{\partial y}+V_z\frac{\partial}{\partial z}\right)T =\frac{1}{K}\frac{\partial T}{\partial t}, \quad (1)$$

in which V_x, V_y and V_x are the velocity components in the direction x, y and z, respectively, t is time, $T(t,x,y,z)$ is temperature, $K=k/(\rho c)$ is the thermal diffusivity, k is the thermal conductivity, ρ is the density and c is the specific heat. The application of a Fourier transformation in the time domain to the Eq. (1) gives the equation

$$\left(\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)-\frac{1}{K}\left(V_x\frac{\partial}{\partial x}+V_y\frac{\partial}{\partial y}+V_z\frac{\partial}{\partial z}\right)+\left(\sqrt{\frac{-\mathrm{i}\omega}{K}}\right)^2\right)\hat{T}(\omega,x,y,z)=0, \quad (2)$$

where $\mathrm{i}=\sqrt{-1}$ and ω is the frequency. Eq. (2) differs from the Helmholtz equation by the insertion of a convective term. For a heat point source, applied at (0,0,0) in an unbounded medium, of the form $p(\omega, x, y, z, t)=\delta(x)\ \delta(y)\ \delta(z)^{i(\omega t)}$, where $\delta(x)$, $\delta(y)$ and $\delta(z)$ are Dirac-delta functions, the fundamental solution of Eq. (2) can be expressed as

$$\hat{T}_f(\omega,x,y,z) =\frac{e^{(V_x x+V_y y+V_z z)/2K}}{2k\sqrt{x^2+y^2+z^2}}e^{-i\sqrt{-(V_x^2+V_y^2+V_z^2)/4K^2-\mathrm{i}\omega/K}\sqrt{x^2+y^2+z^2}}. \quad (3)$$

As the geometry of the problem remains constant along the z direction, the full three-dimensional problem can be expressed as a summation of simpler two-dimensional solutions. This requires the application of a Fourier transformation along that direction, writing this as a summation of two-dimensional solutions with different spatial wavenumbers k_z (Tadeu and Kausel [22]). The application of a spatial Fourier transformation to

$$\frac{e^{-i\sqrt{-(V_x^2+V_y^2+V_z^2)/4K^2-\mathrm{i}\omega/K}\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}, \quad (4)$$

along the z direction, leads to this fundamental solution

$$\tilde{T}_\mathrm{f}(\omega,x,y,k_z)=\frac{-\mathrm{i}e^{(V_x x+V_y y+V_z z)/2K}}{4k}\times H_0\left(\sqrt{-\frac{V_x^2+V_y^2+V_z^2}{4K^2}-\frac{\mathrm{i}\omega}{K}-(k_z)^2}r_0\right), \quad (5)$$

where $H_0()$ are Hankel functions of the second kind and order 0, and $r_0=\sqrt{x^2+y^2}$.

The full three-dimensional solution is then achieved by applying an inverse Fourier transform along the k_x domain to the expression $(-\mathrm{i}/2)H_0 > \left(\sqrt{-(V_x^2+V_y^2+V_z^2)/4K^2 -(\mathrm{i}\omega/K)-(k_z)^2}r_0\right)$. This inverse Fourier transformation can be expressed as a discrete summation if we assume the existence of virtual sources, equally spaced at L_z, along z, which enables the solution to be obtained by solving a limited number of two-dimensional problems,

$$\hat{T}(\omega,x,y,z)=\frac{2\pi}{L}\frac{\mathrm{e}^{(V_x x+V_y y+V_z z)/2K}}{2k\sqrt{x^2+y^2+z^2}}\sum_{m=-M}^{M}H_0 \times\left(\sqrt{-\frac{V_x^2+V_y^2+V_z^2}{4K^2}-\frac{\mathrm{i}\omega}{K}-(k_{zm})^2}r_0\right)\mathrm{e}^{-ik_{zm}z}, \quad (6)$$

with k_{zm} being the axial wavenumber given by $K_{zm}=(2\pi/L_x)m$. The distance L_z is chosen so as to prevent spatial contamination from the virtual sources, i.e. it must be sufficiently large (Bouchon and Aki [23]). This technique is an adaptation and extension of other mathematical and numerical formulations used to solve problems such as wave propagation (Tadeu et al. [24] and Godinho et al. [25]).

Note that Eq. (5) with $V_z=0$ becomes the fundamental solution of the differential equation obtained from Eq. (2) after the application of a spatial Fourier transformation along the z direction, namely

$$\left(\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-\frac{1}{K}\left(V_x\frac{\partial}{\partial x}+V_y\frac{\partial}{\partial y}\right)+\left(\sqrt{\frac{-\mathrm{i}\omega}{K}-(k_z)^2}\right)^2\right)\tilde{T}(\omega,x,y,k_z)=0, \quad (7)$$

when $V_z=0$.

Eq. (5), which results when a spatially sinusoidal harmonic heat line source is applied at the point (0,0) along the z direction, subject to convection velocities

V_x, V_y and V_z, can be further manipulated and written as a continuous superposition of heat plane phenomena,

$$\tilde{T}_\mathrm{f}(\omega,x,y,k_z)=\frac{-\mathrm{i}e^{(V_x x+V_y y+V_z z)/2K}}{4\pi k}\times\int_{-\infty}^{+\infty}\left(\frac{\mathrm{e}^{-i\nu|y|}}{\nu}\right)\mathrm{e}^{-ik_x(x-x_0)}\mathrm{d}k_x' \quad (8)$$

where

$$\nu=\sqrt{-\frac{V_x^2+V_y^2+V_z^2}{2K}-\frac{\mathrm{i}\omega}{K}-(k_z)^2-k_x^2}$$

and $\mathrm{Im}(\nu)\le 0$, and the integration is performed with respect to the horizontal wavenumber (k_x) along the x direction.

Assuming the existence of an infinite number of virtual sources, we can discretize these continuous integrals. The integral in the above equation can be transformed into a summation if an infinite number of such sources are distributed along the x direction, spaced at equal intervals L_x. The above equation can then be written as

$$\tilde{T}_{\mathrm{f}}(\omega, x, y, k_z) = \frac{-\mathrm{i}\, \mathrm{e}^{(V_x x + V_y y + V_z z)/2K}}{4k} E_0 \sum_{n=-\infty}^{n=+\infty} \left(\frac{E}{\nu_n}\right) E_{\mathrm{d}}, \quad (9)$$

where $E_0 = -\mathrm{i}/2kL_x$, $E = \mathrm{e}^{-i\nu_n|y|}$, $E_{\mathrm{d}} = \mathrm{e}^{-ik_{xn}(x)}$, $\nu_{\mathrm{n}} = \sqrt{-(V_x^2 + V_y^2 + V_z^2)/4K^2 - (\mathrm{i}\omega/K) - (k_z)^2 - k_{xn}^2}$ and $\mathrm{Im}(\nu_n) \leq 0$, and $k_{xn} = (2\pi/L_x)n$,which can in turn be approximated by a finite sum of equations (N). Notice that $k_z = 0$ corresponds to the two-dimensional case.

2.1. Responses in the time domain

The heat in the spatial-temporal domain is calculated by applying a numerical inverse fast Fourier transform in k_z, k_x and in the frequency domain. The computations are performed using complex frequencies with a small imaginary part of the form $\omega_{\mathrm{c}} = \omega\text{-}i\eta$ (with $\eta = 0.7\Delta\omega$, and $\Delta\omega$ being the frequency step) to prevent interference from aliasing phenomena. In the time domain, this effect is removed by rescaling the response with an exponential window of the form $\mathrm{e}^{\eta t}$. The time variation of the source can be arbitrary. The time Fourier transformation of the source heat field defines the frequency domain to be computed. The response may need to be computed from 0.0 Hz up to very high frequencies. However, as the heat responses decay very rapidly with increasing frequency, we may limit the upper frequency for which the solution is required. The frequency 0.0 Hz corresponds to the static response that can be computed when the frequency is zero. The use of complex frequencies allows the solution to be obtained because, when $\omega_{\mathrm{c}} = \omega\text{-}i\eta$ (for 0.0 Hz), the arguments of the Hankel function of the equations are not zero.

The technique proposed in this paper uses Fourier transformations, which can be written as discrete summations over wavenumbers and frequencies. The mathematical formulation entails the use of sources equally spaced in the z-axis and x-axis by spatial separations $L_z = 2\pi/\Delta k_z$ and $L_x = 2\pi/\Delta k_x$, and also by temporal intervals $T = 2\pi/\Delta\omega$, with Δk_z and Δk_x being the wavenumber steps. Note that the use of complex frequencies diminishes the contribution from the periodic (fictitious) sources to the response at the time window T.

2.2. Verification of the solution

The formulation described above was implemented and used to compute the heat field in an unbounded medium. In order to verify this formulation, the solution is compared with the analytical response in the time domain.

The exact solution of the convective diffusion, expressed by Eq. (1), in an unbounded medium subjected to a unit heat source is well known and it allows the computation of the heat field given by both conduction and convection phenomena in the presence of three, two or one-dimensional problems. When the heat source is applied at the point (0,0,0) at time $t = t_0$, the temperature at (x,y,z) is given by the expression

$$T(t, x, y, z) = \frac{\mathrm{e}^{(-(-\tau V_x + x)^2 - (-\tau V_y + y)^2 - (-\tau V_z + z)^2)/4K\tau}}{\rho c (4\pi K \tau)^{d/2}}, \quad (10)$$

if $t > t_0$,

where $\tau = t\text{-}t_0$, $d = 3$, $d = 2$ and $d = 1$ when in the presence of a three, two and one-dimensional problems, respectively (Carslaw and Jaeger [1] and Hagentoft [26]).

In the verification procedure, a homogeneous unbounded medium, with thermal properties that allowed $k = 1.4\ \mathrm{W\,m^{-1}\ {}^{\circ}C^{-1}}$, $c = 880.0\ \mathrm{J\,Kg^{-1}\,{}^{\circ}C^{-1}}$ and $\rho = 2300\ \mathrm{Kg\,m^{-3}}$, was excited at $t = 277.8$ h by a unit heat source placed at $x = 0.0$ m, $y = 0.0$ m, $z = 0.0$ m. The convection velocities applied in the x, y and z direction were equal to $1 \times 10^{-6}\ \mathrm{m\,s^{-1}}$.

The responses were calculated along a line of 40 receivers placed from ($x = -1.5$,$y = 0.35$,$z = 0.0$) to ($x = 1.5$,$y = 0.35$, $z = 0.0$), for a plane ($d = 1$), cylindrical ($d = 2$) and spherical ($d = 3$) unit heat source.

The calculations were first performed in the frequency range [0, 1024×10^{-7}]Hz with an increment of $\Delta\omega = 10^{-7}$ Hz, which defines a time window of $T = 2777.8$ h. The solution for the two-dimensional case (cylindrical unit heat source) was found with Eq. (5), while the results for a plane unit heat source propagating along the y axis was obtained ascribing $k_z = 0$ and $k_{xn} = 0$ to Eq. (9), multiplied by L_x. Complex frequencies of the form $\omega_c = \omega - i0.7\Delta\omega$ have been used to avoid the aliasing phenomenon. The spatial period has been set as $L_x = L_z = 2\sqrt{k/(\rho c \Delta f)}$.

In Fig. 1, the solid line represents the exact time solution given by Eq. (10) while the marks show the response obtained using the proposed Green's functions. There is good agreement between these two solutions. Notice that, as we have assigned a convection velocity in the x direction, the temperature response along the line of receivers is not symmetrical. In addition, lower temperatures were registered at the three-dimensional case, since the energy emitted by the heat source is dissipated in the three directions.

3. Green's functions in a half-space

In this section, a semi-infinite medium bounded by a surface with null heat fluxes or null temperatures is considered. The required Green's functions for a half-space can be expressed as the sum of the surface terms and the source terms. The surface terms need to satisfy the boundary condition of the surface (null heat fluxes or null temperatures), while the source terms are equal to those presented for the infinite unbounded medium. The surface terms for a heat source located at (x_0,y_0)can be expressed by

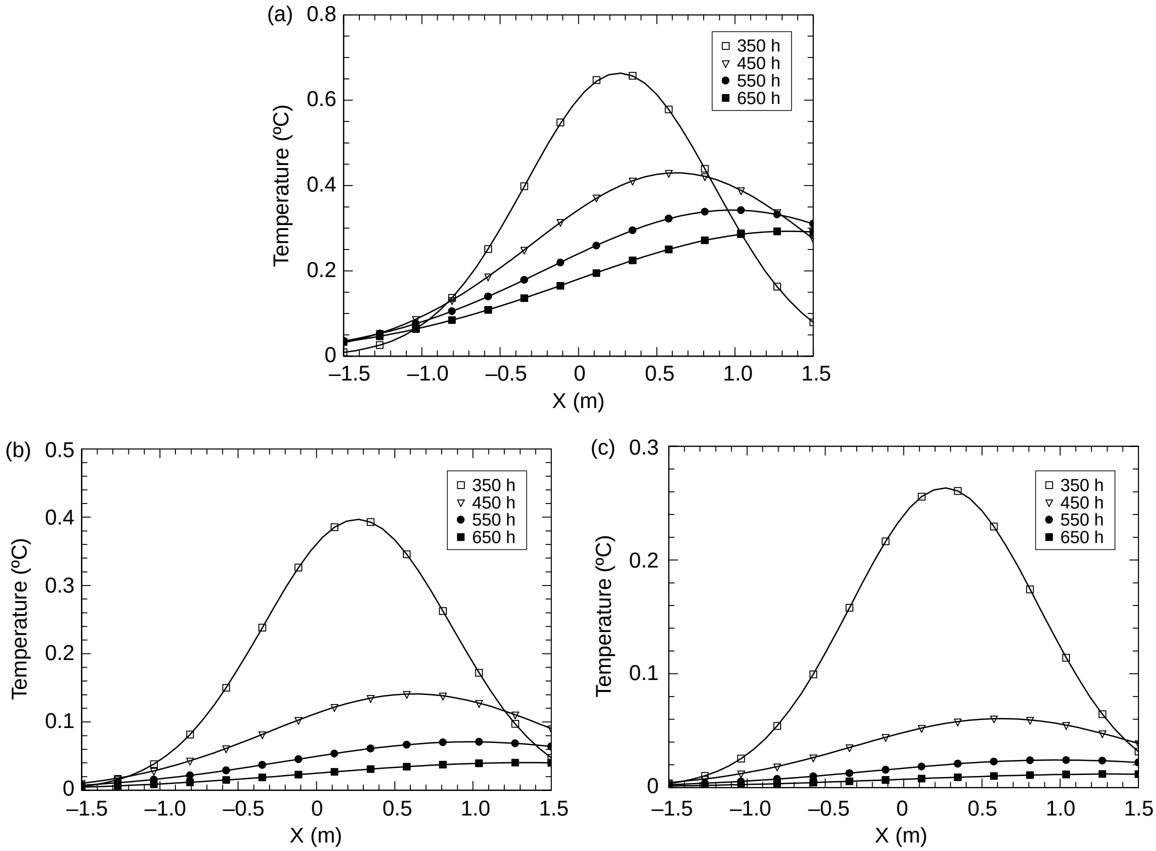

Fig. 1. Temperature along a line of 40 receivers, at times 350, 450, 550 and 650 h: (a) for a plane ($d=1$) unit heat source; (b) for a cylindrical ($d=2$) unit heat source and (c) for a spherical ($d=3$) unit heat source.

$$\tilde{T}_1(\omega,x,y,k_z) = E_0 \mathrm{e}^{((V_x(x-x_0)+V_y(y-y_0))/2K)} \sum_{n=-\infty}^{n=+\infty} \left(\frac{E_\mathrm{a}}{\nu_\mathrm{n}} A_\mathrm{n}\right) E_\mathrm{d}, \quad (11)$$

where $E_\mathrm{a} = \mathrm{e}^{-i\nu_n y}$. A_n is the unknown coefficient to be computed, so that the heat field produced simultaneously by the source and surface terms should produce $\tilde{T}_1(\omega,x,y,k_z)=0$ or $(\partial\tilde{T}_1(\omega,x,y,k_z)/\partial y)=0$ at $y=0$.

The computation of the unknown coefficient is obtained for each value of n. These coefficients are given below for the two cases of null heat fluxes and null temperatures at the surface $y=0$.

Null normal flux at $y=0$,

$$A_n = \mathrm{e}^{-i\nu_n y_0};$$

Null temperature at $y=0$,

$$A_n = -\mathrm{e}^{-i\nu_n y_0}. \quad (12)$$

Replacing these coefficients in Eq. (11), we may compute the heat terms associated with the surface.

Null normal flux at $y=0$,

$$\tilde{T}_1(\omega,x,y,k_z) = E_0 \mathrm{e}^{((V_x(x-x_0)+V_y(y-y_0))/2K)} \sum_{n=-\infty}^{n=+\infty} \left(\frac{E_\mathrm{af}}{\nu_n}\right) E_\mathrm{d};$$

Null temperature at $y=0$,

$$\tilde{T}_1(\omega,x,y,k_z) = E_0 \mathrm{e}^{((V_x(x-x_0)+V_y(y-y_0))/2K)} \sum_{n=-\infty}^{n=+\infty} \left(\frac{E_\mathrm{at}}{\nu_\mathrm{n}}\right) E_\mathrm{d} \quad (13)$$

where $E_\mathrm{af} = \mathrm{e}^{-i\nu_n(y+y_0)}$ and $E_\mathrm{at} = -\mathrm{e}^{-i\nu_n(y+y_0)}$.

The final fundamental solutions for a half-space are given by adding both terms: the source and the surface terms, which leads to

Null normal flux at $y=0$,

$$\tilde{T}(\omega,x,y,k_z) = E_0 \mathrm{e}^{((V_x(x-x_0)+V_y(y-y_0))/2K)} \sum_{n=-\infty}^{n=+\infty} \left(\frac{E+E_\mathrm{af}}{\nu_n}\right) E_\mathrm{d}; \quad (14)$$

Null temperature at $y=0$,

$$\tilde{T}(\omega,x,y,k_z) = E_0 \mathrm{e}^{((V_x(x-x_0)+V_y(y-y_0))/2K)} \sum_{n=-\infty}^{n=+\infty} \left(\frac{E-E_\mathrm{at}}{\nu_n}\right) E_\mathrm{d}. \quad (15)$$

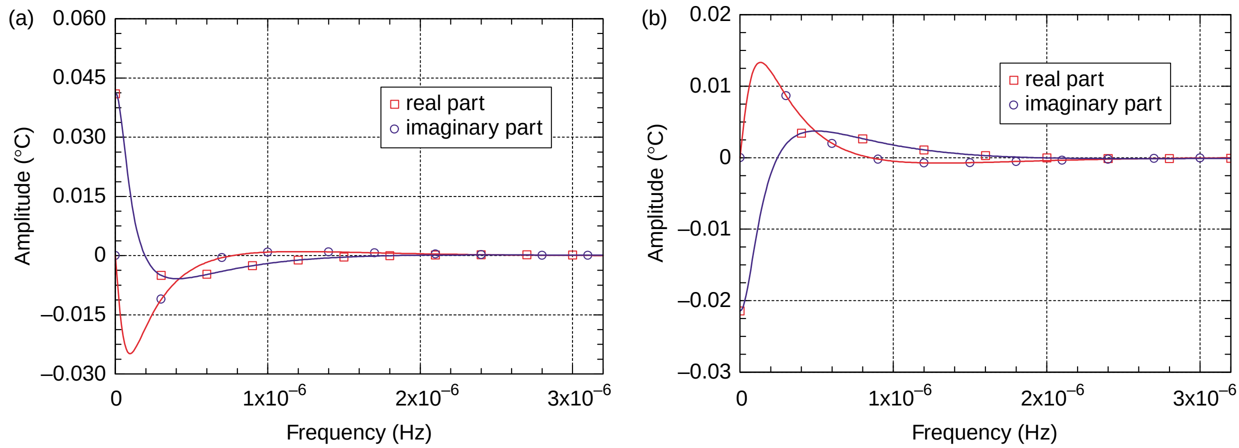

Fig. 2. Real and imaginary parts of the response for a half-space formation ($K_z=0.4$ rad m^{-1}): (a) Null normal flux at $y=0$; and (b) Null temperature at $y=0$.

3.1. Verification of the solution

A BEM model was used to compute the temperature field when a heat source is placed in a semi-infinite medium bounded by a surface with null temperatures or null normal heat fluxes. This technique entails high computational costs, since it needs the discretization of the boundary. In order to simulate the half-space, the boundary is modelled through a large number of elements distributed along as much of the surface as necessary. The limited discretization of the interfaces was achieved by introducing an imaginary part to the frequencies, $\omega_c=\omega-i\eta$ (with $\eta=0.7\Delta\omega$), which introduces damping.

The verification of the solution is illustrated for a homogeneous half-space medium with thermal material properties that allow $k=1.4$ W m^{-1} °C^{-1}, $c=880.0$ J Kg^{-1} °C^{-1} and $\rho=2300$ Kg m^{-3}. The convection velocity, in the y direction, applied to the half-space medium is 5×10^{-7} m s^{-1}. It is assumed that the origin of convection coincides with the source position. This structure is excited at ($x=0.0$ m, $y=1.0$ m) by a line heat source with spatial sinusoidal variation ($k_z=0.4$ rad m^{-1}). Fig. 2 gives the results obtained at the receiver ($x=0.2$ m, $y=0.5$ m) in the frequency range $[0, 032\times10^{-7}]$ Hz with a frequency increment of 1×10^{-7} Hz. In this plot, the solid lines represent the results provided by the proposed solutions while the markers correspond to the solution computed using the Boundary Element technique. These results show that the responses are similar.

4. Green's functions in a slab formation

For a slab structure with thickness h, the Green's functions can be achieved taking into account the boundary conditions prescribed at each surface, i.e. null heat fluxes or null temperatures. They can be expressed by adding the surface and source terms, which are equal to those in the full-space.

Three scenarios can be considered: the prescription of null heat fluxes at the top and bottom interfaces (Case I); or null temperatures in both surface boundaries (Case II); or even the consideration of different conditions at each surface (Case III). At the top and bottom interfaces, surface terms can be generated and expressed in a form similar to that of the source term.

Top surface medium

$$\tilde{T}_1(\omega,x,y,k_z)=E_0 e^{((V_x(x-x_0)+V_y(y-y_0))/2K)}\sum_{n=-\infty}^{n=+\infty}\left(\frac{E_a}{\nu_n}A_n^t\right)E_d;$$

Bottom surface medium

$$\tilde{T}_2(\omega,x,y,k_z)$$

$$=E_0 e^{((V_x(x-x_0)+V_y(y-y_0))/2K)}\sum_{n=-\infty}^{n=+\infty}\left(\frac{E_b}{\nu_n}A_n^b\right)E_d, \quad (16)$$

where $E_b=e^{-i\nu_n|y-h|}$. A_n^t and A_n^b are as yet unknown coefficients to be determined by imposing the appropriate boundary conditions, so that the field originated simultaneously by the source and the surface terms guarantees null heat fluxes or null temperatures at $y=0$ and at $y=h$.

This problem formulation leads to a system of two equations in the two unknown constants for each value of n.

Case I. -null heat fluxes at the top and bottom surfaces.

$$\begin{bmatrix}\left(\frac{V_y}{2K}-i\nu_n\right) & \left(\frac{V_y}{2K}+i\nu_n\right)e^{-i\nu_n h}\\ \left(\frac{V_y}{2K}-i\nu_n\right)e^{-i\nu_n h} & \left(\frac{V_y}{2K}+i\nu_n\right)\end{bmatrix}\begin{bmatrix}A_n^t\\ A_n^b\end{bmatrix}$$

$$=\begin{bmatrix}-\left(\frac{V_y}{2K}+i\nu_n\right)e^{-i\nu_n y_0}\\ -\left(\frac{V_y}{2K}-i\nu_n\right)e^{-i\nu_n|h-y_0|}\end{bmatrix}. \quad (17)$$

Case II. -null temperatures at the top and bottom surfaces.

$$\begin{bmatrix} 1 & -e^{-i\nu_n h} \\ e^{-i\nu_n h} & 1 \end{bmatrix}\begin{bmatrix} A_n^t \\ A_n^b \end{bmatrix} = \begin{bmatrix} -e^{-i\nu_n y_0} \\ -e^{-i\nu_n|h-y_0|} \end{bmatrix} \tag{18}$$

Case III. -null heat fluxes at the top surface and null temperatures at the bottom surface.

$$\begin{bmatrix} \left(\dfrac{V_y}{2K} - i\nu_n\right) & \left(\dfrac{V_y}{2K} + i\nu_n\right) e^{-i\nu_n h} \\ e^{-i\nu_n h} & 1 \end{bmatrix}\begin{bmatrix} A_n^t \\ A_n^b \end{bmatrix} = \begin{bmatrix} -\left(\dfrac{V_y}{2K} + i\nu_n\right) e^{-i\nu_n y_0} \\ -e^{-i\nu_n|h-y_0|} \end{bmatrix}. \tag{19}$$

Once this system of equations has been solved, the amplitude of the surface terms has been fully defined, and the heat in the slab can thus be found. The final expressions for the Green's functions are then derived from the sum of the source terms and the surface terms originated in the two slab surfaces, which leads to the following expressions,

$$\tilde{T}(\omega, x, y, k_z) = \frac{-i}{4k} e^{((V_x(x-x_0)+V_y(y-y_0))/2K)} H_0(K_t r_0) + E_0 e^{((V_x(x-x_0)+V_y(y-y_0))/2K)} \sum_{n=-\infty}^{n=+\infty} \left(\frac{E_a}{\nu_n} A_n^t + \frac{E_b}{\nu_n} A_n^b\right) E_d, \tag{20}$$

where

$$K_t = \sqrt{-\frac{V_x^2 + V_y^2 + V_z^2}{2K}\,\frac{-i\omega}{K} - (k_z)^2}.$$

4.1. Verification of the solution

The formulation described above related to the slab formation was used to compute the responses at a receiver placed in a slab 3.0 m thick, subjected to a spatially harmonic varying line load in the z direction. The results for the three different cases of boundary conditions were then compared with those achieved by using a Boundary Element Method.

In this verification procedure, the medium properties and convection velocity remain the same as those assumed for the half-space ($k = 1.4$ W m^{-1} °C^{-1}, $c = 880.0$ J Kg^{-1} °C^{-1} and $\rho = 2300$ Kg m^{-3}). The slab is heated by a harmonic point heat source applied at ($x = 0.0$ m, $y = 1.0$ m).

The response is performed in the frequency range $[0, 32 \times 10^{-7}]$ Hz with a frequency increment of $\Delta\omega = 10^{-7}$ Hz. The imaginary part of the frequency has been set to $\eta = 0.7\Delta\omega$. To validate the results, the response is computed for a single value of k_z ($k_z = 0.4$ rad m^{-1}). Fig. 3 shows the real and imaginary parts of the responses at the receiver ($x = 0.2$ m, $y = 0.5$ m). The solid lines represent the discrete analytical responses, while the marked points correspond to the Boundary Element Method. The results confirm that the solutions to the three cases are in very close agreement.

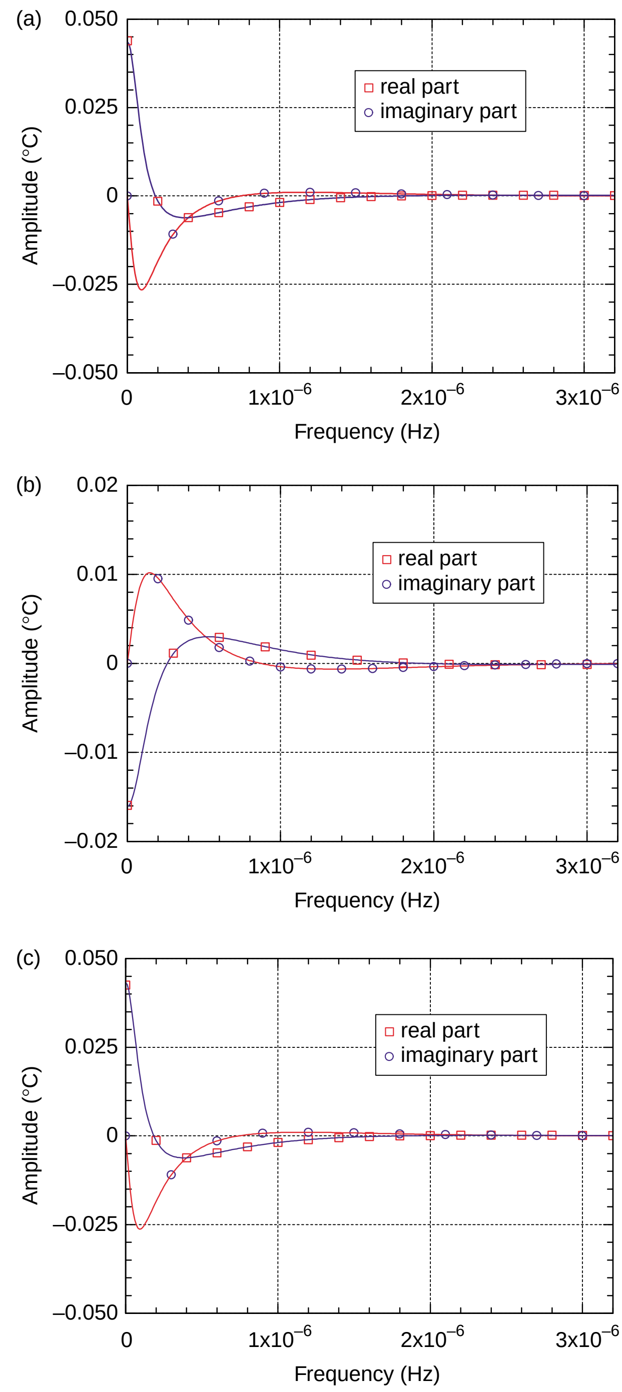

Fig. 3. Real and imaginary parts of the responses for a slab formation, when a heat source is applied at the point ($x = 0.0$ m, $y = 0.0$ m): (a) Case I (null heat fluxes at the top and bottom surfaces); (b) Case II (null temperatures at the top and bottom surfaces); and (c) Case III (null temperatures at the top surface and null heat fluxes at the bottom surface).

5. Green's functions in a layered formation

The solutions for more complex structures, such as a layer over a half-space, a layer bounded by two semi-infinite media and a multi-layer can be established imposing the required boundary conditions at the interfaces and at the free surface.

5.1. Layer over a half-space

Assuming the presence of a layer, h_1 thick, over a half-space, we may prescribe null temperature or null heat fluxes at the free surface (top), while at the interface we need to satisfy the continuity of temperature and normal heat fluxes. The solution is again expressed as the sum of the source terms (the incident field) equal to those in the full-space and the surface terms. At the interfaces 1 and 2, surface terms are generated, which can be expressed in a form analogous to that of the source term.

Layer interface 1

$$\tilde{T}_{11}(\omega,x,y,k_z) = E_{01}\mathrm{e}^{(V_{y1}(y-y_0)/2K_1)}\sum_{n=-\infty}^{n=+\infty}\left(\frac{E_{11}}{\nu_{n1}}A_{n1}^{t}\right)E_{\mathrm{d}}. \tag{21}$$

Layer interface 2

$$\tilde{T}_{12}(\omega,x,y,k_z) = E_{01}\mathrm{e}^{(V_{y1}(y-y_0)/2K_1)}\sum_{n=-\infty}^{n=+\infty}\left(\frac{E_{12}}{\nu_{n1}}A_{n1}^{b}\right)E_{\mathrm{d}}. \tag{22}$$

Half-space (interface 2)

$$\tilde{T}_{21}(\omega,x,y,k_z) = E_{02}\mathrm{e}^{(V_{y2}(y-y_0)/2K_2)}\sum_{n=-\infty}^{n=+\infty}\left(\frac{E_{21}}{\nu_{n2}}A_{n2}^{t}\right)E_{\mathrm{d}}. \tag{23}$$

where $E_{0j} = (-\mathrm{i}/2k_jL_x)$, $E_{11} = \mathrm{e}^{-i\nu_{n1}y}$ $E_{12} = \mathrm{e}^{-i\nu_{n1}|y-h_1|}$, $E_{21} = \mathrm{e}^{-i\nu_{n2}|y-h_1|}$ $\nu_{nj} = \sqrt{-(V_{yj}/2K_j)^2-(i\omega/K_j)-k_z^2-k_{xn}^2}$ with $\mathrm{Im}(\nu_{nj})\leq 0$ and h_1 is the layer thickness ($j=1$ stands for the layer (medium 1) while $j=2$ indicates the half-space (medium 2)). Meanwhile, $K_j=k_j/\rho_jc_j$) is the thermal diffusivity in the medium j (k_j, ρ_j and c_j are the thermal conductivity, the density and the specific heat of the material in the medium j, respectively).

The coefficients A_{n1}^{t}, A_{n1}^{b} and A_{n2}^{t} are as yet unknown. They are defined in order to ensure the appropriate boundary conditions: the field produced simultaneously by the source and surface terms allows the continuity of heat fluxes and temperatures at $y=h_1$, and null heat fluxes (Case I) or null temperatures (Case II) at $y=0$.

Imposing the three stated boundary conditions for each value of n, a system of three equations in the three unknown coefficients is defined.

Case I. -null heat fluxes at $y=0$.

$$\begin{bmatrix} \left(\frac{V_{y1}}{2K_1}-\mathrm{i}\nu_{n1}\right) & \left(\frac{V_{y1}}{2K_1}+\mathrm{i}\nu_{n1}\right)\mathrm{e}^{-i\nu_{n1}h_1} & 0 \\ \left(\frac{V_{y1}}{2K_1}-\mathrm{i}\nu_{n1}\right)\mathrm{e}^{-i\nu_{n1}h_1} & \left(\frac{V_{y1}}{2K_1}+\mathrm{i}\nu_{n1}\right) & -\frac{\nu_{n1}}{\nu_{n2}}\left(\frac{V_{y2}}{2K_2}-i\nu_{n2}\right)\frac{\mathrm{e}^{(V_{y2}(h_1-y_0)/2K_2)}}{\mathrm{e}^{(V_{y1}(h_1-y_0)/2K_1)}} \\ \mathrm{e}^{-i\nu_{n1}h_1} & 1 & -\frac{k_1\nu_{n1}}{k_2\nu_{n2}}\frac{\mathrm{e}^{(V_{y2}(h_1-y_0)/2K_2)}}{\mathrm{e}^{(V_{y1}(h_1-y_0)/2K_1)}} \end{bmatrix}\begin{bmatrix} A_{n1}^{t} \\ A_{n1}^{b} \\ A_{n2}^{t}\end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3\end{bmatrix}, \tag{24}$$

where

$$b_1 = -\left[\frac{\mathrm{V}_{y1}}{2K_1}+\mathrm{i}\nu_{n1}\right]\mathrm{e}^{-i\nu_{n1}y_0},$$

$$b_2 = -\left[\frac{\mathrm{V}_{y1}}{2K_1}-\mathrm{i}\nu_{n1}\right]\mathrm{e}^{-i\nu_{n1}|h_1-y_0|}, \qquad b_3 = -\mathrm{e}^{-i\nu_{n1}|h_1-y_0|}$$

when the source is in the layer ($y_0 < h_1$),

while

$$b_1 = 0,$$

$$b_2 = \left[\frac{V_{y2}}{2K_2}+\mathrm{i}\nu_{n2}\right]\frac{\nu_{n1}}{\nu_{n2}}\frac{e^{(V_{y2}(h-y_0)/2K_2)}}{e^{(V_{y1}(h-y_0)/2K_1)}}e^{-\mathrm{i}\nu_{n2}|h_1-y_0|},$$

$$b_3 = \frac{k_1\nu_{n1}}{k_2\nu_{n2}}\frac{e^{(\mathrm{V}_{y2}(h-y_0)/2K_2)}}{e^{(\mathrm{V}_{y1}(h-y_0)/2K_1)}}e^{-\mathrm{i}\nu_{n2}|h_1-y_0|},$$

when the source is in the half-space ($y_0 > h_1$).

Case II. -null temperatures at $y=0$.

$$\begin{bmatrix} 1 & \mathrm{e}^{-i\nu_{n1}h_1} & 0 \\ \left(\frac{V_{y1}}{2K_1}-\mathrm{i}\nu_{n1}\right)\mathrm{e}^{-i\nu_{n1}h_1} & \left(\frac{V_{y1}}{2K_1}+\mathrm{i}\nu_{n1}\right) & -\frac{\nu_{n1}}{\nu_{n2}}\left(\frac{V_{y2}}{2K_2}-\mathrm{i}\nu_{n2}\right)\frac{\mathrm{e}^{(V_{y2}(h_1-y_0)/2K_2)}}{\mathrm{e}^{(V_{y1}(h_1-y_0)/2K_1)}} \\ \mathrm{e}^{-i\nu_{n1}h_1} & 1 & -\frac{k_1\nu_{n1}}{k_2\nu_{n2}}\frac{\mathrm{e}^{(V_{y2}(h_1-y_0)/2K_2)}}{\mathrm{e}^{(V_{y1}(h_1-y_0)/2K_1)}} \end{bmatrix}\begin{bmatrix} A_{n1}^{t} \\ A_{n1}^{b} \\ A_{n2}^{t}\end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3\end{bmatrix}. \tag{25}$$

where

$$b_1 = -\mathrm{e}^{-i\nu_{n1} y_0}, \qquad b_2 = -\left[\frac{\mathrm{V}_{y1}}{2K_1} - \mathrm{i}\nu_{n1}\right]\mathrm{e}^{-i\nu_{n1}|h_1 - y_0|},$$

$$b_3 = -\mathrm{e}^{-i\nu_{n1}|h_1 - y_0|},$$

when the source is in the layer ($y_0 < h_1$),

while

$$b_1 = 0,$$

$$b_2 = \left[\frac{\mathrm{V}_{y2}}{2K_2} + \mathrm{i}\nu_{n2}\right]\frac{\nu_{n1}}{\nu_{n2}}\frac{\mathrm{e}^{(V_{y2}(h-y_0)/2K_2)}}{\mathrm{e}^{(V_{y1}(h-y_0)/2K_1)}}\mathrm{e}^{-i\nu_{n2}|h_1 - y_0|},$$

$$b_3 = \frac{k_1\nu_{n1}}{k_2\nu_{n2}}\frac{\mathrm{e}^{(V_{y2}(h-y_0)/2K_2)}}{\mathrm{e}^{(V_{y1}(h-y_0)/2K_1)}}\mathrm{e}^{-i\nu_{n2}|h_1 - y_0|},$$

when the source is in the half–space ($y_0 > h_1$).

The heat field produced within the two media results from the contribution of both the surface terms generated at the various interfaces and the source term.

$y_0 < h_1$ (source in the medium 1)

$$\tilde{T}(\omega,x,y,k_z) = \frac{-\mathrm{i}}{4k_1}\mathrm{e}^{(V_{y1}(y-y_0)/2K_1)}H_0(K_{t1}r_0) + E_{01}\mathrm{e}^{(V_{y1}(y-y_0)/2K_1)}\sum_{n=-\infty}^{n=+\infty}\left(\frac{E_{11}}{\nu_{n1}}A_{n1}^t + \frac{E_{12}}{\nu_{n1}}A_{n1}^b\right)E_d,$$
$$\text{if } y < h_1; \quad \tilde{T}(\omega,x,y,k_z) = E_{02}\mathrm{e}^{(\mathrm{V}_{y2}(y-y_0)/2K_2)} \times \sum_{n=-\infty}^{n=+\infty}\left(\frac{E_{21}}{\nu_{n2}}A_{n2}^t\right)E_\mathrm{d}, \quad \text{if } y < h_1. \tag{26}$$

with

$$K_{tj} = \sqrt{-\left(\frac{\mathrm{V}_{yj}}{2K_j}\right)^2 - \frac{\mathrm{i}\omega}{K_j} - (k_z)^2}\,(j = 1,2).$$

5.2. Layer bounded by two semi-infinite media

For the case of the layer placed between two semi-infinite media, the solution needs additionally to account for the continuity of temperature and heat fluxes at the interface 1, since the heat propagation also occurs through the top semi-infinite space (medium 0), which can be expressed by

$$\tilde{T}_{02}(\omega,x,y,k_z) = E_{00}\mathrm{e}^{(V_{y0}(y-y_0)/2K_0)}\sum_{n=-\infty}^{n=+\infty}\left(\frac{E_{01}}{\nu_{n0}}A_{n0}^b\right)E_\mathrm{d}, \tag{28}$$

where $E_{0j} = (-\mathrm{i}/2k_jL_x)$, $E_{00} = \mathrm{e}^{-i\nu_{n0}y}$, and $\nu_{n0} = \sqrt{-(V_{y0}/2K_0)^2 - (\mathrm{i}\omega/K_0) - k_z^2 - k_{xn}^2}$.

The surface terms produced in the interfaces 1 and 2, in the media 1 and 2 (bottom semi-infinite medium) are expressed in Eqs. (21)–(23).

The coefficients A_{n0}^b, A_{n1}^t, A_{n1}^b and A_{n2}^t are defined by respecting the continuity of heat fluxes and temperatures at $y = h_1$ and $y = 0$. These surface conditions lead to the following system of four equations when the heat source is placed within the layer.

$$\begin{bmatrix} \left(\frac{V_{y0}}{2K_0} + \mathrm{i}\nu_{n0}\right)\frac{c_{30}}{c_{41}} & -\left(\frac{V_{y1}}{2K_1} - \mathrm{i}\nu_{n1}\right) & -\left(\frac{V_{y1}}{2K_1} + \mathrm{i}\nu_{n1}\right)\mathrm{e}^{-i\nu_{n1}h_1} & 0 \\ \frac{k_1c_{30}}{k_0c_{41}} & -1 & -\mathrm{e}^{-i\nu_{n1}h_1} & 0 \\ 0 & \left(\frac{V_{y1}}{2K_1} - \mathrm{i}\nu_{n1}\right)\mathrm{e}^{-i\nu_{n1}h_1} & \left(\frac{V_{y1}}{2K_1} + \mathrm{i}\nu_{n1}\right) & -\left[\frac{V_{y2}}{2K_2} - \mathrm{i}\nu_{n2}\right]\frac{c_{42}}{c_{31}} \\ 0 & \mathrm{e}^{-i\nu_{n1}h_1} & 1 & -\frac{k_1c_{42}}{k_2c_{31}} \end{bmatrix}\begin{bmatrix} A_{n0}^b \\ A_{n1}^t \\ A_{n1}^b \\ A_{n2}^t \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \tag{29}$$

$y_0 > h_1$ (source in the medium 2)

$$\tilde{T}(\omega,x,y,k_z) = E_{01}\mathrm{e}^{(\mathrm{V}_{y1}(y-y_0)/2K_1)}\sum_{n=-\infty}^{n=+\infty}\left(\frac{E_{11}}{\nu_{n1}}A_{n1}^t + \frac{E_{12}}{\nu_{n1}}A_{n1}^b\right)E_d,$$
$$\text{if } y < h_1; \tilde{T}(\omega,x,y,k_z) = \frac{-i}{4k_2}\mathrm{e}^{(\mathrm{V}_{y2}(y-y_0)/2K_2)}H_0(K_{t2}r_0) + E_{02}\mathrm{e}^{(\mathrm{V}_{y2}(y-y_0)/2K_2)}\sum_{n=-\infty}^{n=+\infty}\left(\frac{E_{21}}{\nu_{n2}}A_{n2}^t\right)E_\mathrm{d}, \text{if } y > h_1. \tag{27}$$

where

$$c_{3j} = \frac{\mathrm{e}^{\left(V_{yj}\left(\sum_{l=1}^{j} h_l - y_0\right)/2K_j\right)}}{\nu_{nj}},$$

$$c_{4j} = \frac{\mathrm{e}^{\left(V_{yj}\left(\sum_{l=1}^{j-1} h_l - y_0\right)/2K_j\right)}}{\nu_{nj}},$$

$$b_1 = \left[\frac{V_{y1}}{2K_1} + i\nu_{n1}\right] e^{-i\nu_{n1} y_0}, \qquad b_2 = e^{-i\nu_{n1} y_0},$$

$$b_3 = -\left[\frac{V_{y1}}{2K_1} - i\nu_{n1}\right] e^{-i\nu_{n1}|h_1 - y_0|}, \qquad b_4 = -e^{-i\nu_{n1}|h_1 - y_0|},$$

when the source is in the layer ($0 < y_0 < h_1$),

while

$$b_1 = -\left[\frac{V_{y0}}{2K_0} - i\nu_{n0}\right] \frac{c_{30}}{c_{41}} e^{-i\nu_{n0}|y_0|}, \qquad b_2 = -\frac{k_1 c_{30}}{k_0 c_{41}} e^{-i\nu_{n0}|y_0|},$$

$$b_3 = 0, \qquad b_4 = 0$$

when the source is in the half–space ($y_0 < h_1$).

and

$$b_1 = 0, \qquad b_2 = 0, \qquad b_3 = \left[\frac{V_{y2}}{2K_2} + i\nu_{n2}\right] \frac{c_{42}}{c_{31}} e^{-i\nu_{n2}|h_1 - y_0|},$$

$$b_4 = \frac{k_1 c_{42}}{k_2 c_{31}} e^{-i\nu_{n2}|h_1 - y_0|},$$

when the source is in the half–space ($y_0 > h_1$).

The temperatures for the three media are then computed by adding the contribution of the source terms to those associated with the surface terms originated at the various interfaces. This procedure produces the following expressions for the temperatures in the three media.

$$\tilde{T}(\omega, x, y, k_z) = E_{00} e^{(V_{y0}(y-y_0)/2K_0)} \sum_{n=-\infty}^{n=+\infty} \left(\frac{E_{01}}{\nu_{n0}} A^b_{n0}\right) E_d, \text{if } y < 0;$$

$$\tilde{T}(\omega, x, y, k_z) = \frac{-i}{4k_1} e^{(V_{y1}(y-y_0)/2K_1)} H_0(K_{t1} r_0) + E_{01} e^{(V_{y1}(y-y_0)/2K_1)} \times \sum_{n=-\infty}^{n=+\infty} \left(\frac{E_{11}}{\nu_{n1}} A^t_{n1} + \frac{E_{12}}{\nu_{n1}} A^b_{n1}\right) E_d, \text{if } 0 < y < h_1;$$

$$\tilde{T}(\omega, x, y, k_z) = E_{02} e^{(V_{y2}(y-y_0)/2K_2)} \sum_{n=-\infty}^{n=+\infty} \left(\frac{E_{21}}{\nu_{n2}} A^t_{n2}\right) E_d, \text{if } y > h_1. \tag{30}$$

The derivation presented assumed that the spatially sinusoidal harmonic heat source is placed within the layer. However, the equations can be easily manipulated to accommodate another position of the source.

5.3. Multi-layer

The Green's functions for a multi-layer are established using the required boundary conditions at all interfaces.

Consider a system built from a set of m plane layers of infinite extent bounded by two flat, semi-infinite media, as shown in Fig. 4. The top semi-infinite medium is called medium 0, while the bottom semi-infinite medium is assumed to be the medium $m+1$. The thermal material properties and thickness of the various layers may differ.

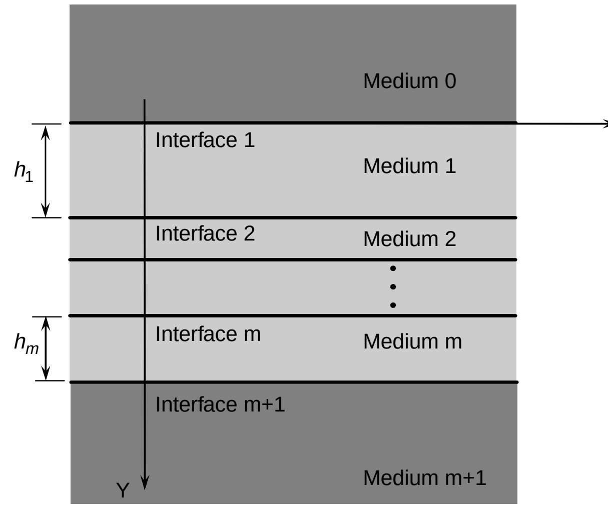

Fig. 4. Geometry of the problem for a multi-layer bounded by two semi-infinite media.

Different vertical convection velocities can be ascribed at each layer. The convection is computed assuming that the origin of convection coincides with the conduction source. The system of equations is achieved considering that the multi-layer is excited by a spatially sinusoidal heat source located in the first layer (medium 1). The heat field at some position in the domain is computed, taking into account both the surface heat terms generated at each interface and the contribution of the heat source term.

For the layer j, the heat surface terms on the upper and lower interfaces can be expressed as

$$\tilde{T}_{j1}(\omega, x, y, k_z) = E_{0j} e^{(V_{yj}(y-y_0)/2K_j)} \sum_{n=-\infty}^{n=+\infty} \left(\frac{E_{j1}}{\nu_{nj}} A^t_{nj}\right) E_d,$$

$$\tilde{T}_{j2}(\omega, x, y, k_z) = E_{0j} e^{V_{yj}(y-y_0)/2K_j} \sum_{n=-\infty}^{n=+\infty} \left(\frac{E_{j2}}{\nu_{nj}} A^b_{nj}\right) E_d, \tag{31}$$

where $E_{0j} = -i/2k_j L_x$, $E_{j1} = e^{-i\nu_{nj}\left|y - \sum_{l=1}^{j-1} h_l\right|}$, $E_{j2} = e^{-i\nu_{nj}\left|y - \sum_{l=1}^{j} h_l\right|}$, $\nu_{nj} = \sqrt{-(V_y/2K_j)^2 + (-i\omega/K_j) - kJ_z^2 - k_{xn}^2}$, with $\mathrm{Im}(\nu_{xj}) \le 0$ and h_l is the thickness of the layer l. The heat surface terms produced at interfaces 1 and $m+1$, which govern the heat that propagates through the top and bottom semi-infinite media, are ,respectively, expressed by

$$\tilde{T}_{02}(\omega, x, y, k_z) = E_{00} e^{(V_{y0}(y-y_0)/2K_0)} \sum_{n=-\infty}^{n=+\infty} \left(\frac{E_{01}}{\nu_{n0}} A^b_{n0}\right) E_d,$$

$$\tilde{T}_{(m+1)2}(\omega, x, y, k_z) = E_{0(m+1)} e^{(V_{y(m+1)}(y-y_0)/2K_{(m+1)})} \sum_{n=-\infty}^{n=+\infty} \times \left(\frac{E_{(m+1)2}}{\nu_{n(m+1)}} A^t_{n(m+1)}\right) E_d. \tag{32}$$

A system of $2(m+1)$ equations is derived, ensuring the continuity of temperatures and heat fluxes along the $m+1$ interfaces between layers. Each equation takes into account the contribution of the surface terms and the involvement of the incident field. All the terms are organized according to the form $\underline{\underline{F}}\,\underline{a}=\underline{b}$

$$\begin{bmatrix} c_{10}c_{30} & -c_{21}c_{41} & -c_{11}c_{41}c_{51} & \cdots & 0 & 0 & 0 \\ \dfrac{c_{30}}{k_0} & -\dfrac{c_{41}}{k_1} & -\dfrac{c_{41}c_{51}}{k_1} & \cdots & 0 & 0 & 0 \\ 0 & c_{21}c_{31}c_{51} & c_{11}c_{31} & \cdots & 0 & 0 & 0 \\ 0 & \dfrac{c_{31}c_{51}}{k_1} & \dfrac{c_{31}}{k_1} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -c_{2m}c_{4m} & -c_{1m}c_{4m}c_{5m} & 0 \\ 0 & 0 & 0 & \cdots & -\dfrac{c_{4m}}{k_m} & -\dfrac{c_{41}c_{5m}}{k_m} & 0 \\ 0 & 0 & 0 & \cdots & c_{2m}c_{3m}c_{5m} & c_{1m}c_{3m} & -c_{2(m+1)}c_{4(m+1)} \\ 0 & 0 & 0 & \cdots & \dfrac{c_{3m}c_{5m}}{k_m} & \dfrac{c_{3m}}{k_m} & -\dfrac{c_{4(m+1)}}{k_{(m+1)}} \end{bmatrix} \begin{bmatrix} A^b_{n0} \\ A^t_{n1} \\ A^b_{n1} \\ \cdots \\ A^t_{nm} \\ A^b_{nm} \\ A^t_{n(m+1)} \end{bmatrix} = \begin{bmatrix} c_{11}c_{41}\mathrm{e}^{-i\nu_{n1}y_0} \\ \dfrac{c_{41}}{k_1}\mathrm{e}^{-i\nu_{n1}y_0} \\ -c_{21}c_{31}\mathrm{e}^{-i\nu_{n1}|h_1-y_0|} \\ -\dfrac{c_{31}}{k_1}\mathrm{e}^{-i\nu_{n1}|h_1-y_0|} \\ \cdots \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{33}$$

where

$$c_{1j}=\left[\frac{\mathrm{V}_{yj}}{2K_j}+\mathrm{i}\nu_{nj}\right],\qquad c_{2j}=\left[\frac{\mathrm{V}_{yj}}{2K_j}-\mathrm{i}\nu_{nj}\right],$$

$$c_{3j}=\frac{\mathrm{e}^{V_{yj}\left(\sum_{l=1}^{j}h_l-y_0\right)/2K_j}}{\nu_{nj}},\qquad c_{4j}=\frac{\mathrm{e}^{V_{yj}\left(\sum_{l=1}^{j-1}h_l-y_0\right)/2K_j}}{\nu_{nj}}\quad\text{and}$$

$$c_{5j}=\mathrm{e}^{-i\nu_{nj}h_j}.$$

The resolution of the system gives the amplitude of the surface terms in each interface. The temperature field for each layer formation is obtained by adding these surface terms to the contribution of the incident field, leading to the following equations.

Top semi-infinite medium (medium 0)

$$\tilde{T}(\omega,x,y,k_z)=E_{00}\mathrm{e}^{(\mathrm{V}_{y0}(y-y_0)/2K_0)}\sum_{n=-\infty}^{n=+\infty}\left(\frac{E_{01}}{\nu_{n0}}A^b_{n0}\right)E_{\mathrm{d}},$$

if $y<0$,

Layer 1 (source position)

$$\tilde{T}(\omega,x,y,k_z)=\frac{-\mathrm{i}}{4k_1}\mathrm{e}^{(\mathrm{V}_{y1}(y-y_0)/2K_1)}H_0(K_1r_0)+E_{01}\mathrm{e}^{(\mathrm{V}_{y1}(y-y_0)/2K_1)}$$
$$\times\sum_{n=-\infty}^{n=+\infty}\left(\frac{E_{11}}{\nu_{n1}}A^t_{n1}+\frac{E_{12}}{\nu_{n1}}A^b_{n1}\right)E_{\mathrm{d}},\ \text{if } 0<y<h_1;$$

Layer j $(j\neq 1)$

$$\tilde{T}(\omega,x,y,k_z)=E_{0j}\mathrm{e}^{(\mathrm{V}_{yj}(y-y_0)/2K_j)}\sum_{n=-\infty}^{n=+\infty}\left(\frac{E_{j1}}{\nu_{nj}}A^t_{nj}+\frac{E_{j2}}{\nu_{nj}}A^b_{nj}\right)E_{\mathrm{d}},$$

if $\sum_{l=1}^{j-1}h_l<y<\sum_{l=1}^{j}h_l$;

Bottom semi-infinite medium (medium $m+1$)

$$\tilde{T}_{(m+1)2}(\omega,x,y,k_z)$$
$$=E_{0(m+1)}\mathrm{e}^{(\mathrm{V}_{y(m+1)}(y-y_0)/2K_{m+1})}\sum_{n=-\infty}^{n=+\infty}\left(\frac{E_{(m+1)2}}{\nu_{n(m+1)}}A^t_{n(m+1)}\right)E_{\mathrm{d}}. \tag{34}$$

Notice that when the position of the heat source is changed, the matrix $\underline{\underline{F}}$ remains the same, while the independent terms of $\underline{\underline{b}}$ are different. However, as the equations can be easily manipulated to consider another position for the source, they are not included here.

5.4. Verification of the solution

Next, the results are found for the three scenarios. First, a flat layer, 3.0 m thick, is assumed to be bounded by one half-space. Null heat fluxes or null temperatures are prescribed at the top surface. Then, a flat layer, also 3.0 m thick, bounded by two semi-infinite media, is used to evaluate the accuracy of the proposed formulation. The convection velocities applied to the three media were 5×10^{-7}, 8×10^{-7} and 1×10^{-6} m s^{-1} for the top medium, intermediate layer and bottom medium, respectively. The thermal material properties used in the intermediate layer were $k=1.4$ W m^{-1} °C^{-1}, $c=880.0$ J Kg^{-1} °C^{-1} and $\rho=2300.0$ Kg m^{-3}, while at the top and bottom media were $k=63.9$ W m^{-1} °C^{-1}, $c=434.0$ J Kg^{-1} °C^{-1} and $\rho=7832.0$ Kg m^{-3}.

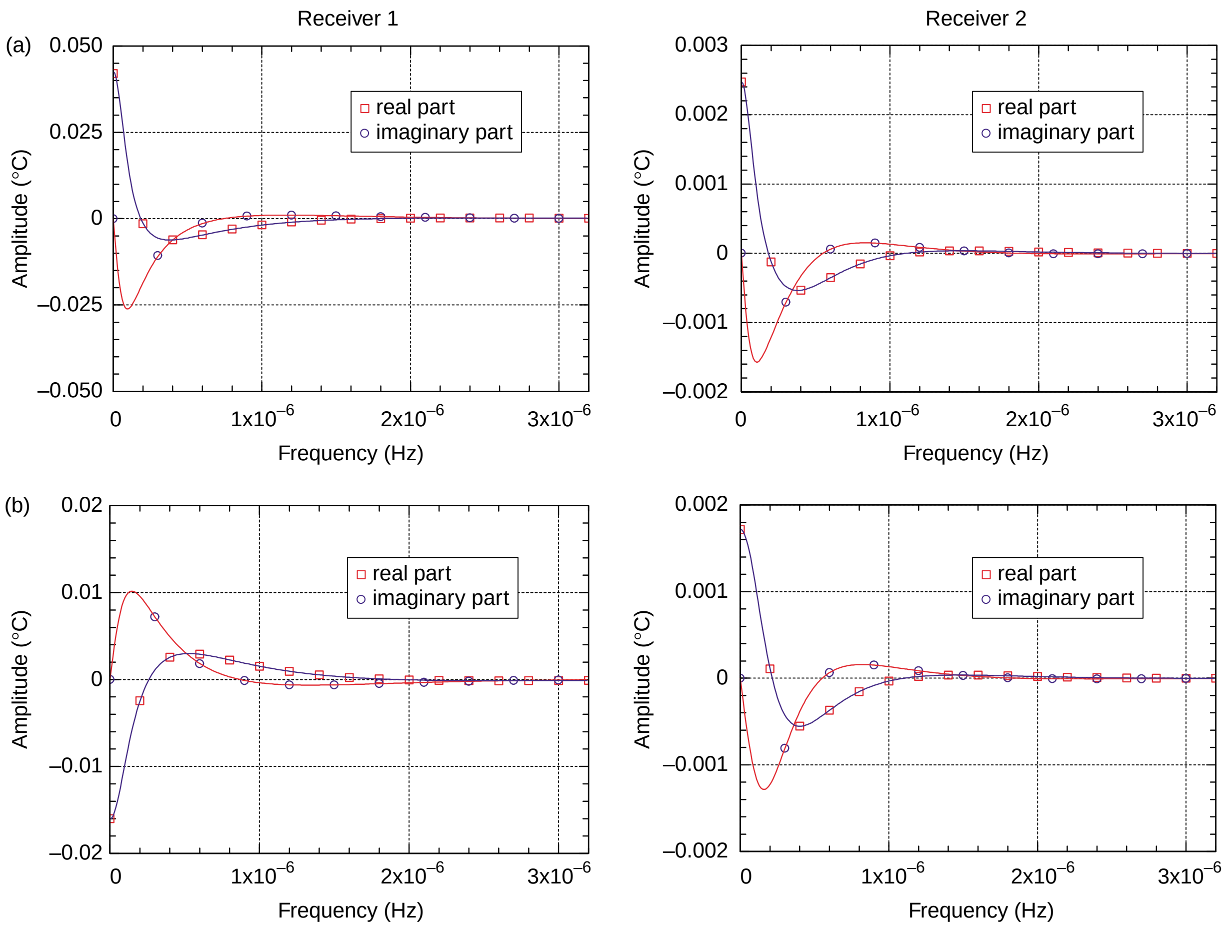

Fig. 5. Real and imaginary parts of the responses for a layer over a half-space: (a) Case I-null heat fluxes at $y=0$, responses at receivers 1 and 2; and (b) Case II-null temperatures at $y=0$, responses at receivers 1 and 2.

The calculations have been performed in the frequency domain from 0 to 32×10^{-7} Hz, with a frequency increment of $\Delta\omega=10^{-7}$ Hz and considering a single value of k_z equal to 0.4 rad m^{-1}. The amplitude of the response for two receivers placed in two different media was computed for a heat point source applied at ($x=0.0$ m, $y=1.0$ m). The real and imaginary parts of the response at the receiver 1 ($x=0.2$ m, $y=3.5$ m) and the receiver 2 ($x=0.2$ m, $y=3.5$ m) are displayed in Figs. 5 and 6, with the imaginary part of the frequency set to $\eta=0.7\Delta\omega$. The solid lines represent the analytical responses, while the marked points correspond to the BEM solution.

As can be seen, these two solutions seem to be in very close agreement, and equally good results were obtained

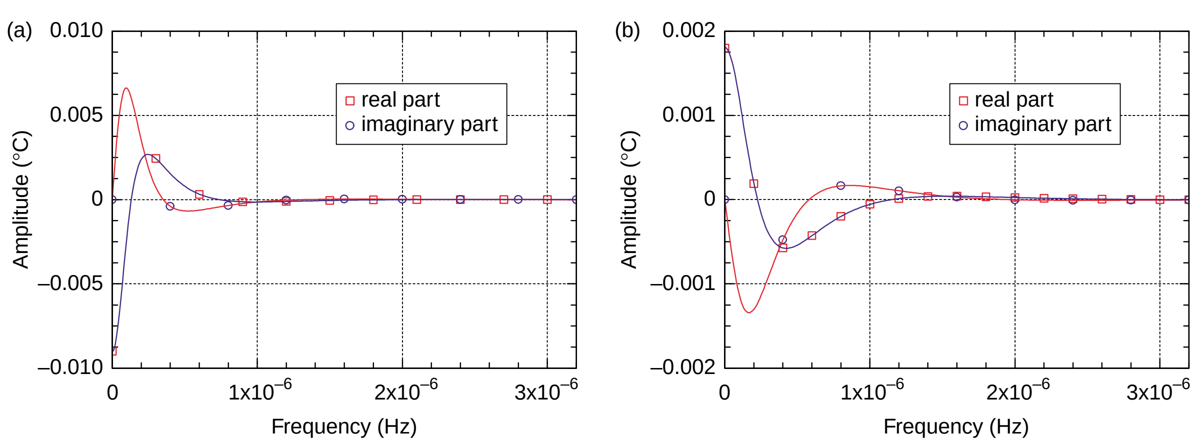

Fig. 6. Real and imaginary parts of the responses for a layer bounded by two semi-infinite media: (a) Receiver 1; and (b) Receiver 2.

from tests in which heat sources and receivers were situated at different points.

6. Conclusions

This paper has presented the 2.5 Green's functions for computing the transient heat transfer by conduction and convection in an unbounded medium, half-space, slab and layered media. In this approach the calculations are first performed in the frequency domain. The results for a layered formation are obtained adding the heat source term and the surface terms, required to satisfy the interface boundary conditions (temperature and heat fluxes continuity). Notice that null convection velocities can be prescribed for the different layers, allowing solid layers to be modelled.

The unbounded medium formulation was corroborated by comparing its time responses and the exact time solutions. In turn, the analytical solutions used in the half-space, slab systems and layered media formulation were verified using a BEM algorithm. Very good agreement was found between the solutions.

The 2.5 Green's functions for a layered medium can be useful for solving problems such as the heat performance of a multi-layer construction element. Using these fundamental solutions together with the BEM or MFS algorithms can be helpful in resolving engineering problems, such as the case of layered formations with buried inclusions.

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