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DESCENT THEORY OF (T,V)-CATEGORIES: GLOBAL-DESCENT AND ÉTALE-DESCENT

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Abstract

The Thesis investigates Descent Theory in categories of lax algebras, in particular with respect to two different classes of morphisms: the class given by all morphisms, when one usually speaks of *global-descent* (or simply omits the prefix), and the class of étale morphisms, speaking of *étale-descent*. Having this goal in mind, we start by giving an overview of the general problem of descent, with respect to an arbitrary category with pullbacks. Different descriptions of the problem are given, namely in terms of monads, with respect to a fibration and with respect to an indexed category. Particularly interesting for us is the framework of descent in **Top** (i.e., Topological Descent Theory), and its passage from the finite case to the infinite case. We retain also useful to give a perspective of the world of $(\mathbb{T},\mathbf{V})$-categories, in particular with respect to a **Set**-monad $\mathbb{T}=(T,\mu,\eta)$ and a quantale **V**. Our study of the problem of descent in categories of lax algebras, with respect to the class given by all morphisms, starts with the generalization to the non-flat case of the already known results due to M.M. Clementino and D. Hofmann, investigating also the role played by the triquotient maps (for which we introduce a suitable definition in the context of $(\mathbb{T},\mathbf{2})$-categories based on the already known characterization in **Top** in terms of ultrafilter convergence). The relation between the effective descent morphisms in $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat**, where $\overline{\mathbb{T}}$ is the Barr extension to **Rel** of $\mathbb{T}$, and in $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat**, where $\widetilde{\mathbb{T}}$ is (what we call) the uniform extension to **V**-**Rel** of $\mathbb{T}$, is investigated. In particular, with respect to the M-ordered monad $\mathbb{M}$ and the free-monoid monad $\mathbb{W}$, we give a complete characterization of the effective descent morphisms in $(\mathbb{M},\mathbf{V},\widetilde{\mathbb{M}})$-**Cat** and $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-**Cat**. The latter case, under suitable hypotheses on **V**, gives also a characterization of the effective descent morphisms in $(\mathbb{W},\mathbf{V},\mathbb{W}^{\wedge})$-**Cat**. We use informations about the morphisms which are effective for descent in $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat** to get results for the problem of descent in $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat**. An useful method is also given, although it represents only a sufficient condition. Based on the work of M.M. Clementino and D. Hofmann, versions of the Van Kampen Theorem in categories of lax algebras are also given, in particular with respect to the free-monoid monad $\mathbb{W}$ and the powerset monad $\mathbb{P}$. Considering the problem of descent with respect to the class of étale morphisms, we recall our first contribution given by the characterization of the effective étale-descent morphisms in M-**Ord**, the category of M-ordered sets and monotone maps, based on the already known characterization in **Cat** of the effective descent morphisms with respect to the class of discrete (co)fibrations. Two different sufficient conditions for the effective étale-descent morphisms are given in **V**-**Cat**, the category of **V**-categories and **V**-relations, one based on the method used to study the passage from $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat** to $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat** in case of global-descent, and one on direct arguments using techniques from the characterization in **Ord**.

Resumo

A Tese investiga a Teoria da Descida nas categorias de álgebras lassas, em particular no que diz respeito a duas classes diferentes de morfismos: a classe que inclui todos os morfismos, onde se fala de *descida global* (ou simplesmente de *descida*), e a classe dos morfismos *étale*, ou homeomorfismos locais, que se refere à *descida étale*. Tendo este objectivo em mente, começamos com uma panorâmica do problema geral da descida, em relação a uma categoria arbitrária com produtos fibrados. São dadas descrições diferentes do problema, nomeadamente em termos de mónadas, em relação a uma fibração e a uma categoria indexada. De particular interesse para nós é a estrutura de descida em **Top** (ou seja, Teoria Topológica da Descida), e a sua passagem do caso finito ao caso infinito. Parece-nos igualmente útil apresentar uma perspectiva do mundo das $(\mathbb{T},\mathbf{V})$-categorias, em particular no que respeita a mónada $\mathbb{T}=(T,\mu,\eta)$ na categoria dos conjuntos e o quantale **V**. O nosso estudo do problema da descida nas categorias de lax álgebras, em relação à classe de todos os morfismos, começa com a generalização ao caso não plano dos resultados já conhecidos devido ao trabalho de M.M. Clementino e D. Hofmann, relativos também ao papel desempenhado pelos triquocientes (para os quais apresentamos uma adequada definição no contexto das $(\mathbb{T},\mathbf{2})$-categorias baseada na já conhecida caracterização em **Top** em termos de convergência de ultrafiltros). A relação entre os morfismos de descida efectiva em $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat**, onde $\overline{\mathbb{T}}$ é a extensão de Barr à categoria **Rel** das relações de $\mathbb{T}$, e em $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat**, onde $\widetilde{\mathbb{T}}$ é (aquela que chamamos) a extensão uniforme à categoria **V**-**Rel** das relações com valor em **V** de $\mathbb{T}$, é investigada. Em particular, em relação à mónada M-ordenada $\mathbb{M}$ e à mónada do monóide livre $\mathbb{W}$, é apresentada a completa caracterização dos morfismos de descida efectiva em $(\mathbb{M},\mathbf{V},\widetilde{\mathbb{M}})$-**Cat** e $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-**Cat**. O último caso, com hipóteses adequadas sobre **V**, fornece também a caracterização dos morfismos de descida efectiva em $(\mathbb{W},\mathbf{V},\mathbb{W}^{\wedge})$-**Cat**. Utilizamos informações sobre os morfismos de descida efectiva em $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat**, transpondo os resultados para o problema da descida em $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat**. Apresentamos também um método útil, ainda que represente somente uma condição suficiente. Sã também presentadas algumas versões do Teorema de Van Kampen nas categorias de álgebras lassas, baseadas no trabalho de M.M. Clementino e D. Hofmann, em particular no que diz respeito à mónada do monóide livre $\mathbb{W}$ e à mónada das partes de um conjunto. Tendo em consideração o problema da descida em relação à classe dos morfismos *étale*, recordamos a nossa primeira contribução relativa à caracterização dos morfismos de descida étale efectiva em M-**Ord**, a categoria dos conjuntos M-ordenados e funções monótonas, baseada na já conhecida caracterização dos morfismos de descida efectiva em relação à classe das (co)fibrações discretas em **Cat**. São também dadas duas condições suficientes para os morfismos de descida efectiva em **V**-**Cat**, uma baseada no método utilizado no estudo da passagem de $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat** a $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat** no caso de descida global, e outra assente na utilização de técnicas a partir da caracterização em **Ord**.

Contents

Introduction

Descent Theory was developed by A. Grothendieck in [26] and [27] in the abstract context of fibred categories (see also M. Demazure [23] and J. Giraud [25]). In [33] G. Janelidze and W. Tholen show its strict connection with Sheaf Theory, in particular investigating how Descent Theory gives an immediate access to the fundamental idea of passing from global data to local data. They base their work on the categorical *theory of monads*, which was not yet in place when A. Grothendieck developed descent. In this context, for any morphism $p : E \to B$ in any category $\mathbf{C}$ with pullbacks, Descent Theory asks whether objects of $\mathbf{C} \downarrow B$ can be given in terms of algebraic objects of $\mathbf{C} \downarrow E$, i.e., whether the *pullback functor*

$$p^* : \mathbf{C} \downarrow B \to \mathbf{C} \downarrow E$$

is monadic, so that $\mathbf{C} \downarrow B$ is (up to isomorphism) the category of Eilenberg-Moore algebras over $\mathbf{C} \downarrow E$. More precisely, the pullback functor p^* admits a left-adjoint $p_!$, given by the composition with p from left, for which the category of Eilenberg-Moore algebras over $\mathbf{C} \downarrow E$, defined by the monad induced by the adjunction $p_! \dashv p^*$, turns out to be equivalent to the category $\mathrm{Des}(p)$ of *descent data (relative to p)*, so that one can state that p is an *(effective) descent morphism* if and only if p^* is pre(monadic). If one restricts to a subclass $\mathbb{E}$ of $\mathbf{C}$, assumed to be pullback stable and closed under composition with isomorphisms, the problem of descent can be still stated but it is not always the case that the category $\mathrm{Des}_{\mathbb{E}}(p)$ is equivalent to a category of Eilenberg-Moore algebras. The monadic description covers descent also in the abstract context of *bifibred* categories satisfying the *Beck-Chevalley condition*, as observed by J. Bénabou and J. Roubaud (see [3]) and by J.M. Beck. In [34] G. Janelidze and W. Tholen present also the problem of descent with respect to an *indexed category*, where methods of internal category theory are applied to show that the split epimorphisms in a category $\mathbf{C}$ are precisely the morphisms which are effective for descent with respect to any fibration over $\mathbf{C}$ (or to any $\mathbf{C}$-indexed category).
The key role of convergence in the papers [47] and [30] where, respectively, J. Reiterman and W. Tholen characterize the effective descent maps of topological spaces and G. Janelidze and M. Sobral characterize special classes of maps between finite spaces (including the effective descent maps), and the consequent description of effective descent maps given in [9] by M.M. Clementino and D. Hofmann, were the starting point of the study of Descent Theory in the context of *lax algebras* (or $(\mathbb{T},\mathbf{V})$*-categories*). Moreover, the result given by M. Barr in [1], where **Top** is proved to be (up to isomorphism) a category of *relational algebras*, represents itself one of the principal roots of the area *Monoidal Topology* whose theory provides a unifying framework for fundamental ordered, metric and topological structures. The main reference for this area can be now considered the book [29], although the theory of $(\mathbb{T},\mathbf{V})$-categories (as presented in the book itself) started with [10] (M.M. Clementino

and D. Hofmann) and [21] (M.M. Clementino and W. Tholen). Hence, for these reasons, along the Thesis, **Top** will be our *leading* category, in the sense that examples, remarks and results will be particularly stressed in this category. Considering the case of finite topological spaces, in [31] G. Janelidze and M. Sobral give also a complete characterization in **Ord** (an example of a category of $(\mathbb{T}, \mathbf{V})$-categories) of the effective descent maps with respect to the class of étale morphisms, or local homeomorphisms, which is still an open problem in **Top**.
In the effective general context of $(\mathbb{T}, \mathbf{V})$-categories, first results on Descent Theory are given in [11] by M.M. Clementino and D. Hofmann, based on the key-stone characterization in [47], with respect to *flat* lax extensions of a **Set**-monad $\mathbb{T} = (T, \mu, \eta)$ to the category **V**-**Rel** of **V**-*relations*. In particular the authors show how suitable hypotheses on the functor T and on the quantale **V** can lead to an unifying treatment of the arguments. Further developments can be found in [20] (in particular referring to the *Barr extension* to **Rel**), in [14] (where it is given a characterization of the descent morphisms in categories of lax algebras), and in [18] (where effective descent morphisms in different categories of relational algebras are fully characterized). More recents developments can be found in [15], where several results on Descent Theory in the category **V**-**Cat** of **V**-*categories* and **V**-*functors* are given, in particular concerning the characterization of effective descent morphisms. Our first contribution to this topic is given in [2], where we give a complete characterization of the effective étale-descent morphisms in M-**Ord**, the category of *M-ordered sets* and *monotone maps* (an example of a category of $(\mathbb{T}, \mathbf{V})$-categories involving the *M-ordered monad*).
The Thesis presents new results about Descent Theory in $(\mathbb{T}, \mathbf{V})$-categories, in particular with respect to the class of all $(\mathbb{T}, \mathbf{V})$-functors (i.e., *global-descent*) and with respect to the class of étale morphisms (i.e., *étale-descent*). At the same time the Thesis wants to serve as a "journey" along Descent Theory in categories of lax algebras. In fact most of the known results are here collected in order to present to the Reader a very global picture of the topic. More specifically: Chapter 1 is devoted to Descent Theory and it is mostly based on the papers [33] and [34] by G. Janelidze and W. Tholen. In particular, descriptions of Descent Theory in terms of monads, with respect to a fibration and with respect to an indexed category are given, respectively, in Section 1.1, 1.2 and 1.3. The aim is to give a global overview of the problem of descent, in particular focusing on the arguments that are going to be used in the contex of $(\mathbb{T}, \mathbf{V})$-categories. Section 1.4 is about Topological Descent Theory, collecting most of the known results on descent in **Top**, in particular with respect to the class of all continuous maps and with respect to the class of local homeomorphisms. The finite case, that is when the topological spaces are finite, is particularly stressed. In [30] and [31] several developments are given by G. Janelidze and M. Sobral, not only characterizing the (effective) descent maps and effective étale-descent maps but also suggesting, concretely, how the finite case can help looking for counter-examples and how it can also help for hints to the infinite case. The last section of Chapter 1 is about the Van Kampen Theorem since, as shown in [5] by R. Brown and G. Janelidze, its categorical formulation is strictly connected to the problem of descent.
In Chapter 2, having as main reference the book [29], we introduce the notions of *lax extensions* and lax algebras (or $(\mathbb{T}, \mathbf{V})$-categories), firstly recalling some basic concepts about quantales and **V**-relations, and then giving the proper definitions. The Barr extension to **Rel**, given by M. Barr in [1], and the uniform construction of lax extensions to **V**-**Rel** (which we call *uniform extension*), given by M.M. Clementino and D. Hofmann in [12], are of our main interest.

In Chapter 3 and Chapter 4 our contribution is given, in particular for what concerns, respectively, the study of the effective global-descent morphisms and of the effective étale-descent morphisms in categories of lax algebras. Chapter 3 starts with a generalization to the *non-flat* case of the results given in [11]. In Section 3.2 the notion of *triquotient map* is introduced in the more general context of $(\mathbb{T},\mathbf{2})$-categories, using the same technique given in [9] by M.M. Clementino and D. Hofmann, where a characterization of triquotient maps in **Top** in terms of ultrafilter convergence is given. Moreover, the role in Descent Theory of the triquotient maps is studied. As we mentioned previously for what concerns Chapter 2, the Barr extension $\overline{\mathbb{T}}$ (to **Rel**) and the uniform extension $\widetilde{\mathbb{T}}$ to (**V**-**Rel**) of a **Set**-monad $\mathbb{T} = (T,\mu,\eta)$ are of our main interest, and the reason is the following: the uniform extension, under suitable hypotheses, is an extension itself of the Barr extension and, in particular, it can be described in terms of it. Therefore we study whether is the case that informations about the effective descent morphisms in $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat** (where there are several examples of complete characterizations of effective descent morphisms) can give results for the study of the effective descent morphisms in $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat**. The cases where $\mathbb{T}$ is the M-ordered monad $\mathbb{M}$ and the free-monoid monad $\mathbb{W}$ are analyzed. These examples suggest that more general arguments can be given and this is what we do in Section 3.3.3. In Section 3.3.4, having in mind the same goal, we present a method, called the *relational method*, to study effective descent morphisms in $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat**, even if we need to restrict to the case where the quantale **V** is a frame. We end up the chapter with a direct path to Chapter 1: we give versions of the Van Kampen Theorem in categories of lax algebras, mostly based on the results given in [14] by M.M. Clementino and D. Hofmann.

In Chapter 4 we focus on Descent Theory in categories of lax algebras with respect to the class of étale morphisms. After having analyzed the notion of étale morphism in **Top** (with its characterization in terms of ultrafilter convergence given in [16] by M.M. Clementino, D. Hofmann and G. Janelidze) and its formulation in the more general context of $(\mathbb{T},\mathbf{V})$-categories, we recall the complete characterization of the effective étale-descent morphisms in M-**Ord** (see [2]), based on the results given in [50] by M. Sobral, concerning the characterization of the effective descent morphisms in **Cat**, with respect to the class of discrete (co)fibrations. Moreover, two different sufficient conditions for the effective étale-descent morphisms in **V**-**Cat** are given, respectively, in Section 4.3 and in Section 4.3.3. Both of them, in a different way, use informations from the characterization in **Ord**. We end up the chapter, and so the Thesis, comparing (effective) global-descent and (effective) étale-descent morphisms in the general context of $(\mathbb{T},\mathbf{V})$-categories.

Chapter 1

Descent Theory

We follow the theory developed in [33] and [34] by G. Janelidze and W. Tholen to introduce the problem of descent. We decide to start by describing Descent Theory in terms of monads, although in [33] it represents a first step of generalization. The topological approach to Grothendieck's idea of descent will be given only after having presented the description of Descent Theory also with respect to a fibration and with respect to and indexed category. This choice is motivated by the fact that we collect most the know results about Descent Theory in **Top** and, to do that, more general settings of the problem of descent (such as with respect to a fibration and with respect to and indexed category) are needed.

1.1 Monadic Descent Theory

Let $\mathbf{C}$ be a category with pullbacks and let $\mathbb{E}$ be a class of morphisms in $\mathbf{C}$ closed under composition with isomorphisms. For an object E in $\mathbf{C}$, consider the full subcategory $\mathbb{E}(E)$ of the comma category $\mathbf{C} \downarrow E$ with objects in $\mathbb{E}$; that is, objects in $\mathbb{E}(E)$ are pairs (C, γ), where C is an object in $\mathbf{C}$ and $\gamma : C \to E$ is a morphism in $\mathbf{C}$ belonging to $\mathbb{E}$, and a morphism $f : (C, \gamma) \to (C', \gamma')$ in $\mathbb{E}(E)$ is given by a morphism $f : C \to C'$ in $\mathbf{C}$ such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\;f\;} & C' \\ & {\scriptstyle\gamma}\searrow \quad \swarrow{\scriptstyle\gamma'} & \\ & E & \end{array}$$

commutes.
Let $p : E \to B$ be a morphisms in $\mathbf{C}$ such that $\mathbb{E}$ is stable under pullback along p. Let

$$\begin{array}{ccc} E \times_B C & \xrightarrow{\;\pi_2\;} & C \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle p\cdot\gamma} \\ E & \xrightarrow[\;p\;]{} & B \end{array}$$

be the pullback in $\mathbf{C}$ of $p \cdot \gamma$ along p. The category

$$\mathrm{Des}_{\mathbb{E}}(p)$$

of *descent data* (*relative to* p) is given by triples (C, γ, ξ), where (C, γ) is an object in $\mathbb{E}(E)$ and

$$\xi : E \times_B C \to C \tag{1.1}$$

is a morphism in $\mathbf{C}$ such that the diagrams

$$\begin{array}{ccc} C & \xrightarrow{<\gamma, 1_C>} & E \times_B C \\ {\scriptstyle 1_C}\downarrow & \swarrow{\scriptstyle \xi} & \downarrow{\scriptstyle \pi_1} \\ C & \xrightarrow[\gamma]{} & E \end{array}$$

$$\begin{array}{ccc} E \times_B (E \times_B C) & \xrightarrow{1_E \times_B \xi} & E \times_B C \\ {\scriptstyle 1_E \times_B \pi_2}\downarrow & & \downarrow{\scriptstyle \xi} \\ E \times_B C & \xrightarrow[\xi]{} & C \end{array}$$

commute.
A morphism $h : (C, \gamma, \xi) \to (C', \gamma', \xi')$ in $\mathrm{Des}_{\mathbb{E}}(p)$ is a morphism $h : (C, \gamma) \to (C', \gamma')$ in $\mathbb{E}(E)$ compatible with the descent data, that is, such that the diagram

$$\begin{array}{ccc} E \times_B C & \xrightarrow{1_E \times_B h} & E \times_B C' \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \xi'} \\ C & \xrightarrow[h]{} & C' \end{array}$$

commutes.
If we start with an object (A, α) in $\mathbb{E}(B)$, and we pull it back along p

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\mathrm{pr}_2} & A \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B, \end{array} \tag{1.2}$$

we get an object $(E \times_B A, \mathrm{pr}_1)$ in $\mathbb{E}(E)$, since the class $\mathbb{E}$ is assumed to be stable under pullback along p. This defines a functor

$$p^* : \mathbb{E}(B) \to \mathbb{E}(E), \quad (A, \alpha) \mapsto (E \times_B A, \mathrm{pr}_1)$$

which is usually called *pullback functor* or *change-of-base functor*. We will sometime stress the class $\mathbb{E}$ if necessary, that is refer to p^* by $p^*_{\mathbb{E}}$.

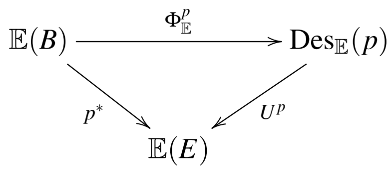
The object $p^*(A,\alpha) = (E \times_B A, \mathrm{pr}_1)$ comes equipped with a *canonical descent data*

$$1_E \times_B \mathrm{pr}_2 : E \times_B (E \times_B A) \to E \times_B A$$

which allows to define a functor

$$\Phi^p_{\mathbb{E}} : \mathbb{E}(B) \to \mathrm{Des}_{\mathbb{E}}(p), \quad (A,\alpha) \mapsto (E \times_B A, \mathrm{pr}_1, 1_E \times_B \mathrm{pr}_2) \tag{1.3}$$

such that the diagram



commutes, where U^p is the obvious forgetful functor.
The functor $\Phi^p_{\mathbb{E}}$ is usually called *comparison functor* for a reason that we will explain soon.

Definition 1.1.1 Let **C** be a category with pullbacks and let $p : E \to B$ be a morphism in **C**. Let $\mathbb{E}$ be a class of morphisms in **C** closed under composition with isomorphisms and stable under pullback along p. The morphism p is said to be $\mathbb{E}$-*descent* if $\Phi^p_{\mathbb{E}}$ is full and faithful, and it is an *effective* $\mathbb{E}$-*descent morphism* if $\Phi^p_{\mathbb{E}}$ is an equivalence of categories.

The definition says that for a morphism $p : E \to B$ to be $\mathbb{E}$-descent means that morphisms $f : (A,\alpha) \to (A',\alpha')$ in $\mathbb{E}(B)$ are completely described by morphisms

$$h : (E \times_B A, \mathrm{pr}_1, 1_E \times_B \mathrm{pr}_2) \to (E \times_B A', \mathrm{pr}'_1, 1_E \times_B \mathrm{pr}'_2),$$

so that h is of the form $h = 1_E \times_B f$. For a morphism $p : E \to B$, to be effective $\mathbb{E}$-descent means that, in addition, objects (C,γ,ξ) in $\mathrm{Des}_{\mathbb{E}}(p)$ are of the form $(E \times_B A, \mathrm{pr}_1, 1_E \times_B \mathrm{pr}_2)$.

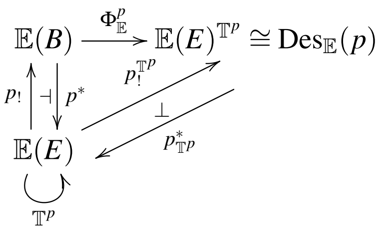
To study necessary and sufficient conditions for a morphism p to be (effective) for $\mathbb{E}$-descent one can assume that, in addition to the conditions given in Definition 1.1.1, the class $\mathbb{E}$ is stable under composition with p from the left, that is,

$$p \cdot \gamma \in \mathbb{E} \quad \text{if} \quad \gamma \in \mathbb{E}.$$

In this case, the functor $p^* : \mathbb{E}(B) \to \mathbb{E}(E)$ has a left adjoint $p_!$ given by the composition with p from the left

$$p_! : \mathbb{E}(E) \to \mathbb{E}(B), \quad (C,\gamma) \mapsto (C, p \cdot \gamma).$$

This pair of adjoint functors induces a monad $\mathbb{T}^p$ on $\mathbb{E}(E)$ and the category of Eilenberg-Moore algebras $\mathbb{E}(E)^{\mathbb{T}^p}$, defined by the monad $\mathbb{T}^p$, is exactly the category $\mathrm{Des}_{\mathbb{E}}(p)$ of descent data. These general arguments of monads theory can be summarized by the following diagram



where

$$\mathbb{E}(E) \underset{p^*_{\mathbb{T}^p}}{\overset{p_!^{\mathbb{T}^p}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbb{E}(E)^{\mathbb{T}^p} \cong \mathrm{Des}_{\mathbb{E}}(p)$$

is the adjunction induced by $\mathbb{T}^p$.

Proposition 1.1.2 [33, Proposition 2.2] *If $\mathbb{E}$ is closed under composition with p from the left, then* $\mathrm{Des}_{\mathbb{E}}(p)$ *is exactly the Eilenberg-Moore category of the monad induced by the adjunction $p_! \dashv p^*$, and p is an (effective) $\mathbb{E}$-descent morphism if and only if p^* is premonadic (monadic).*

This is the reason why the functor $\Phi^p_{\mathbb{E}}$ in (1.3) is usually called comparison functor. Therefore, one can explore the Beck's monadicity criterion to study necessary and sufficient conditions for a morphism p to be (effective) for $\mathbb{E}$-descent.

Definition 1.1.3 We say that $p : E \to B$ is an $\mathbb{E}$-*universal regular epimorphism* if the class of morphisms which are pullbacks of p along a morphism in $\mathbb{E}$ is contained in the class of regular epimorphisms.

We denote by $\mathbb{E}^*(p)$ the class of morphisms which are pullbacks of p along a morphism in $\mathbb{E}$.

Theorem 1.1.4 [33, Theorem 2.3] *Let $\mathbb{E}$ be a class of morphisms in* **C** *stable under pullback along p and under composition with p from the left. Assume that* **C** *has coequalizers of parallel pairs of morphisms in $\mathbb{E}^*(p)$.The morphism p is an $\mathbb{E}$-descent morphism of* **C** *if and only if p is an $\mathbb{E}$-universal regular epimorphism of* **C**. *The $\mathbb{E}$-descent morphism p is effective if $\mathbb{E}$ is right cancellable with respect to those regular epimorphisms of* **C** *which are coequalizers of $\mathbb{E}^*(p)$-morphisms over B and if these coequalizers are stable under pullback along p.*

The technique used to prove this theorem is given by a direct construction of the left adjoint

$$\Psi^p_{\mathbb{E}} : \mathrm{Des}_{\mathbb{E}}(p) \to \mathbb{E}(B)$$

of the comparison functor $\Phi^p_{\mathbb{E}}$. For each object (C, γ, ξ) in $\mathrm{Des}_{\mathbb{E}}(p)$, since both π_2 and ξ are in $\mathbb{E}^*(p)$, we can construct the coequalizer (Q, π)

$$E \times_B C \underset{\xi}{\overset{\pi_2}{\rightrightarrows}} C \xrightarrow{\pi} Q \tag{1.4}$$

of the parallel pair (π_2, ξ). By the universal property of the coequalizer, there is a unique morphism $\delta : Q \to B$

$$\begin{array}{ccccc} E \times_B C & \overset{\pi_2}{\underset{\xi}{\rightrightarrows}} & C & \xrightarrow{\pi} & Q \\ & & \downarrow{\scriptstyle p\cdot\gamma} & \swarrow{\scriptstyle \delta} & \\ & & B & & \end{array} \tag{1.5}$$

such that $\delta \cdot \pi = p \cdot \gamma$. Diagram (1.5) usually refers to a *descent situation describing Q*. By the right cancellability of $\mathbb{E}$, δ is in $\mathbb{E}(B)$. Therefore just define

$$\Psi^p_{\mathbb{E}}(C, \gamma, \xi) := (Q, \delta)$$

to get the left adjoint. The Beck's monadicity criterion given in [36] says that the comparison functor $\Phi^p_{\mathbb{E}}$ is full and faithful if and only if the components of the counit of the adjunction $p_! \dashv p^*$ are regular epimorphisms, and, moreover, the unit of the adjunction $\Psi^p_{\mathbb{E}} \dashv \Phi^p_{\mathbb{E}}$ is an isomorphism if and only if the functor p^* preserves the coequalizer diagram (1.4).

It is important to stress that if in Proposition 1.1.2 we remove the condition on $\mathbb{E}$ to be closed under composition with p from the left, the result might fail, in the sense there are classes $\mathbb{E}$ of morphisms in $\mathbf{C}$ for which p is not effective for $\mathbb{E}$-descent, although the functor p^* is monadic. An example is given in [33, Section 3.10] for $\mathbf{C} = \mathbf{Top}$, the category of topological spaces and continuous maps. Consider $\mathbb{E}_c$ be the class of closed-subspace embeddings in **Top** and let $p : E \to B$ be the identity map where E is the space given by a set X (with at least two points) equipped with the discrete topology on X, and B is given by the same set X equipped with the indiscrete topology, i.e., only X and the empty-set $\emptyset$ are open. An inspection reveals that the category $\mathrm{Des}_{\mathbb{E}_c}(p)$ is equivalent to the category $\mathbb{E}_c(E)$, which is itself equivalent to the (partially ordered) powerset of X, hence non-equivalent to $\mathbb{E}_c(B)$ (which is the 2-element chain), although $p^*_{\mathbb{E}_c}$ is monadic. A more general sufficient condition for which the category of descent data is isomorphic to a category of Eilenberg-Moore algebras will be given in Section 1.2.

In Definition 1.1.1, if $\mathbb{E}$ is the class of all morphisms in $\mathbf{C}$, we speak of *(effective) global-descent*. We remark that in literature one may also simply speak of *(effective) descent*, omitting the prefix. In this case, since this class is trivially closed under composition with p from the left, from Theorem 1.1.4, one obtains the following corollary.

Corollary 1.1.5 [33, Corollary 2.4] *A morphism p is a global-descent morphism if and only if p is a universal regular epimorphism. A global-descent morphism $p : E \to B$ is an effective global-descent morphism if the coequalizer of every parallel pair of universal regular epimorphisms over B exists and is stable under pullback along p.*

Therefore, if the category $\mathbf{C}$ (with pullbacks and coequalizers) is locally cartesian closed, Descent Theory gets easier. In fact, the effective descent morphisms are exactly the regular epimorphisms, which are necessarily universal. One of the techniques to study (effective) descent morphisms in an arbitrary category is to fully embed it (if possible) into a locally cartesian closed category, and then interpret the result in terms of the original category.

Let $\mathbb{E}_0$ and $\mathbb{E}_1$ be two classes of morphisms of a category with pullbacks, both stable under pullback along $p : E \to B$ and under composition with isomorphisms. Assume that

$$\mathbb{E}_0 \subseteq \mathbb{E}_1.$$

Proposition 1.1.6 [33, Proposition 2.6] *$\mathbb{E}_1$-descent for p implies $\mathbb{E}_0$-descent for p. The effective $\mathbb{E}_1$-descent morphism p is an effective $\mathbb{E}_0$-descent morphism if and only if the following condition holds: for every pullback diagram*

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B, \end{array}$$

$\pi_1 \in \mathbb{E}_0$ and $\alpha \in \mathbb{E}_1$ implies $\alpha \in \mathbb{E}_0$.

Corollary 1.1.7 [33, Corollary 2.7]

1. *For* **C** *with pullbacks and $\mathbb{E}$ stable under pullback along the effective global-descent morphism p of* **C***, p is an effective $\mathbb{E}$-descent morphism if and only if in every pullback square*

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B, \end{array}$$

   $\pi_1 \in \mathbb{E}$ implies $\alpha \in \mathbb{E}$.

2. *For* **D** *with pullbacks and* **C** *a full subcategory closed under pullback in* **D***, a morphism p of* **C** *which is an effective global-descent morphism in* **D** *is also an effective global-descent morphism in* **C** *if and only if in every pullback square*

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B \end{array}$$

   of **D***, $E \times_B A \in$* **C** *implies $A \in$* **C**.

Statement 2. of Corollary 1.1.7 is precisely the technique we described before. If we can fully embed a category **C** into a locally cartesian closed category **D**, where effective global-descent morphisms are known to be exactly the regular epimorphisms, the condition above gives us a criterion for studying effective global-descent morphisms in **C**.

Statement 1. of Corollary 1.1.7 turns out to be useful when, knowing the effective global-descent morphisms in a category **C**, we are interested in studying effective descent morphisms with respect to some subclass $\mathbb{E}$ of morphisms.

1.2 Descent Theory with respect to fibrations

Let $F : \mathbf{D} \to \mathbf{C}$ be an arbitrary functor and $p : E \to B$ a morphism in $\mathbf{C}$. The *fibre* $\mathbf{D}(B) := F^{-1}(B)$ *of F at B* is the (non-full) subcategory of $\mathbf{D}$ whose morphisms are mapped by F to the identity morphism, i.e., morphisms $f : A \to A'$ such that $F(f) = 1_B$. For an object $A \in \mathbf{D}(B)$, a pair (C, c), where C is an object in $\mathbf{D}(E)$ and $c : C \to A$ is a morphism in $\mathbf{D}$ with $F(c) = p$, is called an *F-lifting of p at A*. A morphism $c : C \to A$ in $\mathbf{D}$ is called *F-cartesian* if it is a terminal F-lifting of $F(c)$ at A, i.e., for any morphism $d : D \to A$ in $\mathbf{D}$ and any morphism $q : F(D) \to F(C)$ with $F(c) \cdot q = F(d)$, there is a unique morphism $g : D \to C$ with $c \cdot g = d$ and $F(g) = q$,

$$\begin{array}{ccc} D & \overset{\exists! g}{\dashrightarrow} & C \\ & \searrow^{d} & \downarrow c \\ & & A \end{array} \qquad \mapsto \qquad \begin{array}{ccc} F(D) & \xrightarrow{q} & F(C) \\ & \searrow^{F(d)} & \downarrow F(c) \\ & & F(A). \end{array}$$

Definition 1.2.1 A functor $F : \mathbf{D} \to \mathbf{C}$ is a *(cloven) fibration* if every morphism $p : E \to B$ admits a (specified) cartesian lifting at every object A in $\mathbf{D}(B)$.

In the presence of a fibration $F : \mathbf{D} \to \mathbf{C}$ or, more generally, if every object A in $\mathbf{D}(B)$ admits a cartesian lifting $(p^*A, \theta_p A)$ of p at A

$$\begin{array}{ccc} p^*A & \overset{\theta_p A}{\dashrightarrow} & A \\ & \downarrow & \\ E & \xrightarrow[p]{} & B, \end{array}$$

then one can obtain a functor

$$p^* : \mathbf{D}(B) \to \mathbf{D}(E),$$

called *inverse-image functor*, and a *cleavage*

$$\theta_p : J_E \cdot p^* \to J_B,$$

where $J_E : \mathbf{D}(E) \to \mathbf{D}$ and $J_B : \mathbf{D}(B) \to \mathbf{D}$ are the inclusion functors, with $F\theta_p$ the constant natural transformation. Therefore $(-)^* : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$

$$\begin{array}{ccc} B & \longmapsto & \mathbf{D}(B) \\ p \uparrow & & \downarrow p^* \\ E & \longmapsto & \mathbf{D}(E) \end{array}$$

defines a pseudo-functor. Note that the cleavages of a fibration $F : \mathbf{D} \to \mathbf{C}$ define a right-adjoint right-inverse of every functor $F_A : \mathbf{D} \downarrow A \to \mathbf{C} \downarrow F(A) (A \in \mathbf{D})$ induced by F. Conversely, with the right-adjoint right-inverses of these functors, F becomes a (cloven) fibration.

Example 1.2.2

(1) Let **C** be a category and let $\mathbb{E}$ be a class of morphisms in **C**. The comma categories $\mathbb{E}(C)(C \in \mathbf{C})$, considered in Section 1.1, can be interpreted as the fibres of the codomain functor

$$F_{\mathbb{E}} : \mathbb{E}^2 \to \mathbf{C},$$

where $\mathbb{E}^2$ is the category whose objects are all morphisms in $\mathbb{E}$ and whose morphisms $(p', p) : \alpha' \to \alpha$ (with $\alpha, \alpha' \in \mathbb{E}$) are commutative diagrams in **C** of the form

$$\begin{array}{ccc} \cdot & \xrightarrow{p'} & \cdot \\ {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \cdot & \xrightarrow[p]{} & \cdot \end{array}$$

If the diagram above is a pullback then it represents an $F_{\mathbb{E}}$-cartesian lifting morphism of $\mathbb{E}^2$. On the other hand, if the class $\mathbb{E}$ contains all isomorphisms of **C**, then every $F_{\mathbb{E}}$-cartesian morphism of **C** is given by a pullback diagram. Moreover, if **C** has pullbacks and $\mathbb{E}$ is stable under pullback, then $F_{\mathbb{E}}$ is a (cloven) fibration. In fact, for a morphism $p : E \to B$ in **C**, we have a cartesian $F_{\mathbb{E}}$-lifting of p at every $(A, \alpha) \in \mathbb{E}(B)(= \mathbb{E}^2(B))$ given by the pullback diagram (1.2). The inverse-image functor p^* and the cleavage θ_p are obtained from a choice of the pullback (1.2).

(2) Let ΠX be the fundamental groupoid of a topological space X. Every continuous map $p : E \to B$ gives a groupoid homomorphism $\Pi p : \Pi E \to \Pi B$ so that $\Pi : \mathbf{Top} \to \mathbf{Grpd}$ is a functor from **Top** to **Grpd**, the category of groupoids and groupoid homomorphisms. It can be seen also as a 2-functor $\Pi : \mathbf{Top} \to \mathbf{Cat}$, with **Top** considered as a groupoid-enriched category. If p is a *Hurewicz fibration* (see [52]) then Πp is a (cloven) fibration.

Let $F : \mathbf{D} \to \mathbf{C}$ be a fibration, where **C** is a category with pullbacks. For a morphism $p : E \to B$ in **C** consider (p_1, p_2) the kernel pair of p

$$\begin{array}{ccc} E \times_B E & \xrightarrow{p_2} & E \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\ E & \xrightarrow[p]{} & B. \end{array} \qquad (1.6)$$

A *descent data relative to* p is given by a pair $(C, \hat{\xi})$, where C is an object in $\mathbf{D}(E)$ and

$$\hat{\xi} : p_1^*(C) \to p_2^*(C)$$

is a morphism in $\mathbf{D}(E \times_B E)$ from $p_1^* = C \times_E (E \times_B E)$ to $p_2^* = (E \times_B E) \times_E C$ such that the following diagrams commute

$$\begin{array}{ccc} p_1^*(C) & \xrightarrow{\hat{\xi}} & p_2^*(C) \\ {\scriptstyle \delta_1}\nwarrow & & \swarrow{\scriptstyle \theta_{p_2}C} \\ & C & \end{array}$$

$$\begin{array}{ccccc} & \pi_1^* p_2^* C & \xrightarrow{j} & \pi_2^* p_1^* C & \\ {}^{\pi_1^*\hat{\xi}}\nearrow & & & & \searrow^{\pi_2^*\hat{\xi}} \\ \pi_1^* p_1^* C & & & & \pi_2^* p_2^* C \\ {}_{j_1^{-1}}\searrow & & & & \nearrow_{j_2} \\ & \pi^* p_1^* C & \xrightarrow[\pi^*\hat{\xi}]{} & \pi^* p_2^* C. & \end{array}$$

The morphism $\delta_1 : C \to p_1^* C$ is the unique morphism induced by the cartesian lifting $(p_1^* C, \theta_{p_1} C)$ with respect to the morphism $\delta : E \to E \times_B E$ for which $p_1 \cdot \delta = p_2 \cdot \delta = 1_E$, i.e., δ_1 is the unique morphism such that $F(\delta_1) = \delta$ and $\theta_{p_1} C \cdot \delta_1 = 1_C$. The morphism $\pi : (E \times_B E) \times_E (E \times_B E) \to E \times_B E$ is the one induced by the pair $(p_1 \cdot \pi_1, p_2 \cdot \pi_2)$, where π_1 and π_2 are given by the pullback diagram

$$\begin{array}{ccc} (E \times_B E) \times_E (E \times_B E) & \xrightarrow{\pi_2} & E \times_B E \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ E \times_B E & \xrightarrow[p_2]{} & E. \end{array}$$

The morphisms j, j_1 and j_2 are the canonical isomorphisms arising from the identities $p_1 \cdot \pi_2 = p_2 \cdot \pi_1$, $p_1 \pi = p_1 \cdot \pi_1$ and $p_2 \cdot \pi = p_2 \cdot \pi_2$. In [44] an explicit proof of the fact that $\hat{\xi}$ must be an isomorphism is given. Descent data relative to p form the object-part of the category

$$\mathrm{Des}_{\mathbf{D}}(p)$$

whose morphisms $h : (C, \hat{\xi}) \to (C', \hat{\xi}')$ are morphisms $h : C \to C'$ in $\mathbf{D}(E)$ such that the diagram

$$\begin{array}{ccc} p_1^* C & \xrightarrow{p_1^* h} & p_1^* C' \\ {\scriptstyle \hat{\xi}}\downarrow & & \downarrow{\scriptstyle \hat{\xi}'} \\ p_2^* C & \xrightarrow[p_2^* h]{} & p_2^* C' \end{array}$$

commutes. For every object A in $\mathbf{D}(B)$, $p^* A$ comes equipped with a *canonical descent data*

$$\hat{\varphi} = (j_{p,p_2}^{-1} A)(j_{p,p_1} A) : p_1^* p^* A \to p_2^* p^* A,$$

where $j_{p,p_1} : p_1^* p^* \to (p \cdot p_1)^*$ and $j_{p,p_2} : p_2^* p^* \to (p \cdot p_2)^*$ are natural equivalences. Therefore one can define a functor $\Phi^p : \mathbf{D}(B) \to \mathrm{Des}_{\mathbf{D}}(p)$

$$A \mapsto (p^* A, \hat{\varphi}),$$

which makes the diagram

$$\begin{array}{ccc} \mathbf{D}(B) & \xrightarrow{\Phi^p} & \mathrm{Des}_{\mathbf{D}}(p) \\ & {}_{p^*}\searrow \quad \swarrow_{U^p} & \\ & \mathbf{D}(E) & \end{array}$$

commutative. The functor U^p is the obvious forgetful functor.

Definition 1.2.3 Let $F : \mathbf{D} \to \mathbf{C}$ be a fibration with $\mathbf{C}$ a category with pullbacks. For a morphism $p : E \to B$ in $\mathbf{C}$ one says that p is an *(effective)* $\mathbf{D}$*-descent morphism* if Φ^p is full and faithful (an equivalence of categories).

For $F = F_{\mathbb{E}}$, there is a bijective correspondence between the descent data $\hat{\xi}$ and the one given in Section 1.1 so that the categories $\mathrm{Des}_{\mathbf{D}}(p)$, with $\mathbf{D} = \mathbb{E}^2$, and the category $\mathrm{Des}_{\mathbb{E}}(p)$, defined in Section 1.1, are isomorphic. In Proposition 1.1.2 we saw under which conditions the category of descent data is exactly a category of (Eilenberg-Moore) algebras. This fact can be viewed also in a more general context concerning fibrations. In particular, assume that the functor $F : \mathbf{D} \to \mathbf{C}$ is a (cloven) *bifibration*, so that both F and $F^{\mathrm{op}} : \mathbf{D}^{\mathrm{op}} \to \mathbf{C}^{\mathrm{op}}$ are (cloven) fibrations. Hence, for a morphism $p : E \to B$ in $\mathbf{C}$, one has, dually with respect the inverse-image functor p^* and the cleavage θ_p, the *direct-image functor*

$$p_! : \mathbf{D}(E) \to \mathbf{D}(B)$$

and the *co-cleavage*

$$\vartheta_p : J_E \to J_B p_!,$$

giving rise to the adjunction

$$\mathbf{D}(B) \underset{p^*}{\overset{p_!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{D}(E).$$

The unit and counit of the adjoint situation above are given, respectively, by the natural transformations

$$\eta_p : 1_{\mathbf{D}(E)} \to p^* \cdot p_!, \qquad \varepsilon_p : p_! \cdot p^* \to 1_{\mathbf{D}(B)},$$

with $(\theta_p p_!)(J_E \eta_p) = \vartheta_p$ and $(J_B \varepsilon_p)(\vartheta_p p^*) = \theta_p$. Moreover, considering the kernel pair (p_1, p_2) of p (see diagram (1.6)), one has the so-called *Beck-transformation*

$$\beta_p : (p_2)_! p_1^* \to p^* p_!, \quad \text{with} \quad (J_E \beta_p)(\vartheta_{p_2} p_1^*) = (J_E \eta_p)\theta_{p_1}.$$

The bifibration F *satisfies the Beck-Chevalley condition for* p if β_p is a natural equivalence. This condition, as proved by J. Bénabou and J. Roubaud in [3], and by J.M. Beck, turns out to play a key role for the Monadic Descent Theory.

Theorem 1.2.4 *Let $F : \mathbf{D} \to \mathbf{C}$ be a bifibration, with $\mathbf{C}$ a category with pullbacks. For a morphism $p : E \to B$ in $\mathbf{C}$ such that the Beck-Chevalley condition is satisfied for p, the category $\mathrm{Des}_{\mathbf{D}}(p)$ of descent data is isomorphic to the category of Eilenberg-Moore algebras of the monad induced by the adjunction $p_! \dashv p^*$. Therefore p is an (effective) $\mathbf{D}$-descent morphism if and only if p^* is premonadic (monadic).*

The theorem above reveals how the monadic description given in Section 1.1 covers the problem of descent in the abstract context of bifibred category satisfying the Beck-Chevalley condition.

1.3 Descent Theory with respect to indexed categories

Since fibrations (over **C**) are essentially equivalent to pseudo-functors $\mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$, the descent problem can be treated equivalently in the context of indexed categories, as developed in [34] by G. Janelidze and W. Tholen. The authors found convenient to work with indexed categories (instead of fibrations) to study, in any category **C** with pullbacks, which morphisms are effective for descent with respect to *every* fibration $\mathbf{D} \to \mathbf{C}$. In this general context, they show that they are precisely the split epimorphisms of **C** (see Theorem 1.3.4).
Let **C** be a category with pullbacks. A **C**-*indexed category* $\mathbb{A}$, in the sense of [37], is a pseudo-functor

$$\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT};$$

that is, it consists of the following data:

- for every object D in **C**, a category $\mathbb{A}^D$;
- for every pair of objects E, D in **C**, a functor

$$\mathbf{C}(E,D) \to \mathbf{CAT}(\mathbb{A}^D, \mathbb{A}^E), \quad f \mapsto f^*;$$

- for every $E \xrightarrow{f} D \xrightarrow{g} C$ in **C**, a natural isomorphism

$$j^{f,g} : f^* \cdot g^* \to (g \cdot f)^*; \tag{1.7}$$

- for every object D in **C**, a natural isomorphism

$$i^D : 1_{\mathbb{A}^D} \to (1_D)^*. \tag{1.8}$$

The natural isomorphisms (1.7) and (1.8) are required to satisfy the following coherence axioms:

(*i*) for every triple of morphisms $E \xrightarrow{f} D \xrightarrow{g} C \xrightarrow{h} B$ in **C**, the diagram

$$\begin{array}{ccc} f^* \cdot g^* \cdot h^* & \xrightarrow{f^* j^{g,h}} & f^* \cdot (h \cdot g)^* \\ {\scriptstyle j^{f,g} h^*}\downarrow & & \downarrow{\scriptstyle j^{f,h \cdot g}} \\ (g \cdot f)^* \cdot h^* & \xrightarrow[j^{g \cdot f,h}]{} & (h \cdot g \cdot f)^* \end{array} \tag{1.9}$$

commutes;

(*ii*) for every morphism $f : E \to D$ in **C**, the diagram

$$\begin{array}{ccc} f^* & \xrightarrow{f^* i^D} & f^* \cdot (1_D)^* \\ {\scriptstyle i^E f^*}\downarrow & \searrow^{1_{f^*}} & \downarrow{\scriptstyle j^{f,1_D}} \\ (1_E)^* \cdot f^* & \xrightarrow[j^{1_E,f}]{} & f^* \end{array} \tag{1.10}$$

commutes.

Example 1.3.1

(1) An example of a **C**-indexed category is given by the comma categories $\mathbf{C} \downarrow D$, for an object D in **C**. In fact, to each object D in **C**, we assign $\mathbb{A}^D = \mathbf{C} \downarrow D$ and to each morphism $f : E \to D$ the pullback functor $f^* : \mathbf{C} \downarrow D \to \mathbf{C} \downarrow E$, given by pulling back along f. This is precisely the situation studied in Section 1.1 and it is easy to check that this defines a pseudo-functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$;

(2) The second example of a **C**-indexed category is given by a (cloven) fibration. More precisely, as already remarked in Section 1.2, in the presence of a fibration $F : \mathbf{D} \to \mathbf{C}$ one can define a pseudo-functor $()^* : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$ by considering F-fibres of objects and F-cartesian liftings.

Let $\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$ be a **C**-indexed category. In order to define the category $\mathrm{Des}_{\mathbb{A}}(p)$ of *descent data*, for a morphism $p : E \to B$ in **C**, one needs to define the category $\mathbb{A}^D$, for D an internal category in **C**. Let us first recall what an internal category, of a category **C** with pullbacks, is. Let **C** be a category with pullbacks. An *internal category* D of **C** is given by a diagram

$$D_2 \; \overset{\pi_2}{\underset{\pi_1}{\overset{m}{\rightrightarrows\!\!\!\to}}} \; D_1 \; \overset{d}{\underset{c}{\overset{e}{\rightleftarrows\!\!\!\to}}} \; D_0 \tag{1.11}$$

of objects and morphisms in **C**, with D_2, π_1, π_2 given by the pullback diagram

$$\begin{array}{ccc} D_2 & \xrightarrow{\pi_2} & D_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle e} \\ D_1 & \xrightarrow[d]{} & D_0 \end{array}$$

in **C**, such that the following conditions are satisfied:

(1) $d \cdot e = 1_{D_0} = c \cdot e$;

(2) $d \cdot m = d \cdot \pi_2$, $c \cdot m = c \cdot \pi_1$;

(3) $m \cdot (1_{D_1} \times m) = m \cdot (m \times 1_{D_1})$;

(4) $m \cdot < 1_{D_1}, e \cdot d >= 1_{D_1} = m \cdot < e \cdot c, 1_{D_1} >$.

If $\mathbf{C} = \mathbf{Set}$, it is precisely the definition of a (small) category. An *internal functor* $f : D \to D'$ of internal categories D, D' of a category **C** is given by two morphisms

$$f_0 : D_0 \to D_0', \quad f_1 : D_1 \to D_1'$$

in **C** such that

(5) $f_0 \cdot d = d' \cdot f_1$, $f_0 \cdot c = c' \cdot f_1$;

(6) $f_1 \cdot e = e' \cdot f_0$, $f_1 \cdot m = m' \cdot f_2$,

where $f_2 = f_1 \times f_1 : D_1 \times_{D_0} D_1 \to D'_1 \times_{D'_0} D'_1$.

Defining the composition of internal functors as in $\mathbf{C}$, we obtain the category $\mathbf{cat}(\mathbf{C})$ of internal categories of $\mathbf{C}$. It is actually a 2-category; in fact one can define an *internal natural transformation* $\alpha : f \to g$ between internal functors $f, g : D \to D'$ as a morphism

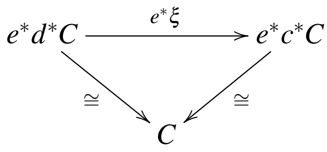
$$\alpha : D_0 \to D'_1$$

in $\mathbf{C}$ such that

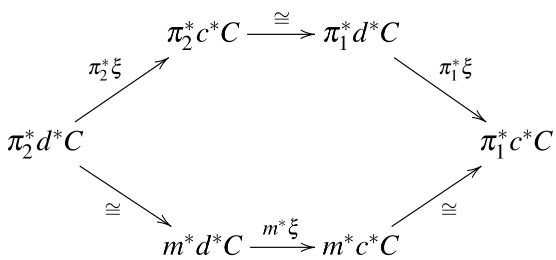
(7) $d' \cdot \alpha = f_0$, $c' \cdot \alpha = g_0$;

(8) $m' \cdot < \alpha \cdot c, f_1 >= m' \cdot < g_1, \alpha \cdot d >$.

The category $\mathbb{A}^D$, for $\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$ a pseudo-functor and D an internal category in $\mathbf{C}$, is then defined as follows: objects are given by pairs (C, ξ), where C is an object of $\mathbb{A}^{D_0}$ and $\xi : d^*C \to c^*C$ is a morphism in $\mathbb{A}^{D_1}$ such that the following diagrams

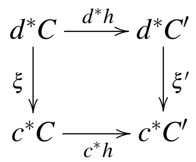


(1.12)



(1.13)

commute in $\mathbb{A}^{D_0}$ and $\mathbb{A}^{D_1}$, respectively. All isomorphisms come from the conditions (1) and (2) in the definition of the internal category D and from the natural isomorphisms in diagrams (1.9) and (1.10). A morphism $h : (C, \xi) \to (C', \xi')$ in $\mathbb{A}^D$ is defined to be a morphism $h : C \to C'$ of $\mathbb{A}^{D_0}$ such that the diagram



commutes in $\mathbb{A}^{D_1}$. Composition and identity morphisms are defined as in $\mathbb{A}^{D_0}$.
Every object D_0 in $\mathbf{C}$ can be seen as a *discrete* internal category D in $\mathbf{C}$. In fact, in diagram (1.11), just take $D_2 = D_1 = D_0$ and all morphisms to be identities. By diagram (1.12), for every $\mathbf{C}$-indexed category $\mathbb{A}$, each object C in $\mathbb{A}^{D_0}$ has only one structure ξ, so that $\mathbb{A}^D \cong \mathbb{A}^{D_0}$. This allows to define a full embedding $\mathbf{C} \hookrightarrow \mathbf{cat}(\mathbf{C})$.

Theorem 1.3.2 [34, Theorem 2.5] *For every* $\mathbf{C}$*-indexed category* $\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$*, the assignment* $D \mapsto \mathbb{A}^D$ *is part of a pseudo-functor* $\mathbb{A} : \mathbf{cat}(\mathbf{C})^{\mathrm{op}} \to \mathbf{CAT}$ *of 2-categories.*

Let $\mathbf{C}$ be a category with pullbacks and let $p : E \to B$ be a morphism in $\mathbf{C}$. The kernel pair of p, given by the pullback diagram

$$\begin{array}{ccc} E\times_B E & \xrightarrow{\pi_2} & E \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ E & \xrightarrow[p]{} & B, \end{array}$$

induces an equivalence relation (π_1, π_2). This gives rise to an internal category, denoted by $\mathrm{Eq}(p)$ and defined by the diagram

$$(E\times_B E)\times_E (E\times_B E) \cong E\times_B E\times_B E \;\begin{array}{c}\xrightarrow{\pi_{2,3}}\\ \xrightarrow{\pi_{1,3}}\\ \xrightarrow[\pi_{1,2}]{}\end{array}\; E\times_B E \;\begin{array}{c}\xrightarrow{\pi_2}\\ \xleftarrow{e}\\ \xrightarrow[\pi_1]{}\end{array}\; E$$

where $e =< 1_E, 1_E >$.

Let $\mathbb{A}$ be a $\mathbf{C}$-indexed category. For a morphism $p : E \to B$ in $\mathbf{C}$, the *category of* $\mathbb{A}$*-descent data (relative to p)* is defined to be

$$\mathrm{Des}_{\mathbb{A}}(p) := \mathbb{A}^{\mathrm{Eq}(p)}.$$

The internal functor $\bar{p} : \mathrm{Eq}(p) \to B$, given by $\bar{p}_0 = p$ and $\bar{p}_1 = p\cdot\pi_1 = p\cdot\pi_2$, where B is considered as a discrete internal category, gives rise to a factorization of the discrete internal functor $p : E \to B$

$$\begin{array}{ccccc} E & & \xrightarrow{\quad p \quad} & & B \\ & {\scriptstyle \delta}\searrow & & \nearrow{\scriptstyle \bar{p}} & \\ & & \mathrm{Eq}(p) & & \end{array}$$

where δ is given by $\delta_0 = 1_E$ and $\delta_1 = e$. The diagram above induces the diagram

$$\begin{array}{ccccc} \mathbb{A}^B & & \xrightarrow{\quad \Phi^p_{\mathbb{A}}=\bar{p}^* \quad} & & \mathrm{Des}_{\mathbb{A}}(p) \\ & {\scriptstyle p^*}\searrow & & \swarrow{\scriptstyle \delta^*} & \\ & & \mathbb{A}^E & & \end{array}$$

in $\mathbf{CAT}$ which commutes up to natural isomorphism.

Definition 1.3.3 The morphism p is called an *(effective)* $\mathbb{A}$*-descent morphism* if the *comparison functor* $\Phi^p_{\mathbb{A}} = \bar{p}^*$ is full and faithful (an equivalence of categories). The morphism p is an *absolutely effective descent morphism* of $\mathbf{C}$ if it is an effective $\mathbb{A}$-descent morphism for *every* $\mathbf{C}$-indexed category $\mathbb{A}$.

If $\mathbb{A}$ is given by a (cloven) fibration $F : \mathbf{D} \to \mathbf{C}$, (see Example 1.3.1) the category $\mathrm{Des}_{\mathbb{A}}(p)$ is isomorphic to the category $\mathrm{Des}_{\mathbf{D}}(p)$ and the notion of (effective) $\mathbb{A}$-descent coincide with the notion of (effective)

D-descent.

One of the main results of [34] is the following theorem.

Theorem 1.3.4 [34, Theorem 3.5] *A morphism of a category with pullbacks is an absolutely effective descent morphism if and only if it a split epimorphism.*

We end up the section observing how the problem of descent can be *moved* from a **C**-indexed category $\mathbb{A}$ to another **C**-indexed category $\mathbb{B}$.

Remark 1.3.5 Let $\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$ and $\mathbb{B} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$ be two **C**-indexed categories, where **C** is a category with pullbacks. An indexed functor $F : \mathbb{A} \to \mathbb{B}$, in the sense of [37], is given by the following data:

- for each object D in **C**, a functor $F_D : \mathbb{A}^D \to \mathbb{B}^D$,
- for each morphism $f : E \to D$ in **C**, a natural isomorphism $\tau_f : f^*_{\mathbb{B}} \circ F_D \to F_E \circ f^*_{\mathbb{A}}$, such that, for each pair of composable morphisms $C \xrightarrow{g} E \xrightarrow{f} D$, the diagram

$$\begin{array}{ccccc} g^*_{\mathbb{B}} f^*_{\mathbb{B}} F_D & \xrightarrow{g^*_{\mathbb{B}}\tau_f} & g^*_{\mathbb{B}} F_E f^*_{\mathbb{A}} & \xrightarrow{\tau_g f^*_{\mathbb{A}}} & F_C g^*_{\mathbb{A}} f^*_{\mathbb{A}} \\ {\scriptstyle j^{g,f}_{\mathbb{B}} F_D}\downarrow & & & & \downarrow{\scriptstyle F_C j^{g,f}_{\mathbb{A}}} \\ (fg)^*_{\mathbb{B}} F_D & & \xrightarrow[\tau_{fg}]{} & & F_C (fg)^*_{\mathbb{A}} \end{array}$$

  commutes.

An indexed functor $F : \mathbb{A} \to \mathbb{B}$ is an *equivalence of* **C***-indexed categories* if each functor $F_D : \mathbb{A}^D \to \mathbb{B}^D$, for $D \in \mathrm{Ob}(\mathbf{C})$ is an equivalence of categories. If $\mathbb{A}$ and $\mathbb{B}$ are equivalent as **C**-indexed categories, for a morphism $p : E \to B$ in **C**, one has that the category $\mathrm{Des}_{\mathbb{A}}(p)$ of descent data with respect to $\mathbb{A}$ is equivalent to the category $\mathrm{Des}_{\mathbb{B}}(p)$ of descent data with respect to $\mathbb{B}$. Moreover the diagram

$$\begin{array}{ccc} \mathbb{A}^B & \xrightarrow{\simeq} & \mathbb{B}^B \\ {\scriptstyle \Phi^p_{\mathbb{A}}}\downarrow & & \downarrow{\scriptstyle \Phi^p_{\mathbb{B}}} \\ \mathrm{Des}_{\mathbb{A}}(p) & \xrightarrow[\simeq]{} & \mathrm{Des}_{\mathbb{B}}(p) \end{array}$$

is commutative (up to equivalence). Therefore one can state that, if $\mathbb{A}$ and $\mathbb{B}$ are equivalent as **C**-indexed categories, then a morphism $p : E \to B$ in **C** is an (effective) $\mathbb{A}$-descent morphism if and only if it is an (effective) $\mathbb{B}$-descent morphism.

In the presence of an equivalence of **C**-indexed categories, in order to study effective descent morphisms in the category **C**, one can then move from an environment $\mathbb{A}$ to a (possibly friendlier) environment $\mathbb{B}$. This represents another technique to study Descent Theory for which, along the Thesis, we will see some application.

1.4 Topological Descent Theory

In this section we present an account of Descent Theory in the category **Top** of topological spaces and continuous maps, based on the work in [33] by G. Janelidze and W. Tholen. Let $\mathbb{E}$ be a class of continuous maps of topological spaces, closed under composition with isomorphisms. Let $p : E \to B$ be a continuous map and let (C,γ) be an object of the comma category $\mathbb{E}(E)$ (usually called $\mathbb{E}$-*bundle* over E). The pullback

$$\begin{array}{ccc} E\times_B C & \xrightarrow{\pi_2} & C \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle p\cdot\gamma} \\ E & \xrightarrow[p]{} & B \end{array}$$

is given by

$$E\times_B C = \{(x,z) \in E\times C : p(x) = p(\gamma(z))\},$$

equipped with the subspace topology of the topological product $E\times C$. For each pair of points x,x' in E such that $p(x) = p(x')$, we have canonical embeddings

$$i_{x,x'} : \gamma^{-1}(x') \to E\times_B C, \quad z \mapsto (x,z),$$

for which we can consider $E\times_B C$ as the join of the subspaces $i_{x,x'}(\gamma^{-1}(x'))$.
A *descent data for* (C,γ) *(relative to p)* is given by a family of continuous maps

$$\xi_{x,x'} : \gamma^{-1}(x) \to \gamma^{-1}(x'),$$

indexed by points $x,x' \in E$ with $p(x) = p(x')$, and such that

(i) $\xi_{x,x} = 1_{\gamma^{-1}(x)}$, for each $x \in E$;

(ii) $\xi_{x,x''} = \xi_{x',x''}\cdot\xi_{x,x'}$, for each $x,x',x'' \in E$ with $p(x) = p(x') = p(x'')$;

(iii) the unique map $\bar{\xi} : E\times_B C \to E\times_B C$, which makes all diagrams

$$\begin{array}{ccc} \gamma^{-1}(x) & \xrightarrow{\xi_{x,x'}} & \gamma^{-1}(x') \\ {\scriptstyle i_{x',x}}\downarrow & & \downarrow{\scriptstyle i_{x,x'}} \\ E\times_B C & \xrightarrow[\bar{\xi}]{} & E\times_B C \end{array}$$

commute, is continuous.

Conditions (i) and (ii) represent *functorial* properties for which we conclude that each $\xi_{x,x'}$ is an isomorphism (since $\xi_{x',x}\cdot\xi_{x,x'} = 1_{\gamma^{-1}(x)}$). Therefore, since explicitly $\bar{\xi}$ is defined as

$$\bar{\xi}(x',z) = (x,\xi_{x,x'}(z)) \quad \text{with} \quad x = \gamma(z),$$

also $\bar{\xi}$ is an isomorphism, where $\bar{\xi}^{-1} = \bar{\xi}$. Condition (iii) represents a *gluing* property. One can usually speak of

$$\bar{\xi} : E\times_B C \to E\times_B C \tag{1.14}$$

as a descent data for an $\mathbb{E}$-bundle (C,γ) over E.
Triples $(C,\gamma,\bar{\xi})$, where (C,γ) is an $\mathbb{E}$-bundle over E and $\bar{\xi}$ is a descent data (relative to p) for the $\mathbb{E}$-bundle (C,γ), form the objects of the category

$$\mathrm{Des}_{\mathbb{E}}(p)$$

of descent data. A morphism $h:(C,\gamma,\bar{\xi})\to(C',\gamma',\bar{\xi}')$ in $\mathrm{Des}_{\mathbb{E}}(p)$ is a morphism $h:(C,\gamma)\to(C',\gamma')$ in $\mathbb{E}(E)$ such that

$$h(\xi_{x,x'}(z))=\xi'_{x,x'}(h(z)), \tag{1.15}$$

for each $x,x'\in E$ with $p(x)=p(x')$ and $z\in\gamma^{-1}(x)$. Condition (1.15) can be equivalently expressed in terms of $\bar{\xi}$, that is

$$(1_E\times_B h)\cdot\bar{\xi}=\bar{\xi}'\cdot(1_E\times_B h).$$

As in Section 1.1, if the class $\mathbb{E}$ is assumed to be stable under pullback along $p:E\to B$, each $\mathbb{E}$-bundle (A,α) over B gives rise to an $\mathbb{E}$-bundle $(E\times_B A,\mathrm{pr}_1)$ over E, simply pulling back along p:

$$\begin{array}{ccc} E\times_B A & \xrightarrow{\mathrm{pr}_2} & A \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B. \end{array}$$

This defines the pullback functor

$$p^*:\mathbb{E}(B)\to\mathbb{E}(E),\quad (A,\alpha)\mapsto(E\times_B A,\mathrm{pr}_1).$$

Also in this case, if necessary, we will stress the class $\mathbb{E}$ denoting p^* by $p^*_{\mathbb{E}}$. The $\mathbb{E}$-bundle $p^*(A,\alpha)=(E\times_B A,\mathrm{pr}_1)$ comes equipped with a canonical descent data

$$\phi_{x,x'}:\mathrm{pr}_1^{-1}(x)\to\mathrm{pr}_1^{-1}(x'),\quad (x,w)\mapsto(x',w),$$

so that $\bar{\phi}:E\times_B(E\times_B A)\to E\times_B(E\times_B A)$ is the involution

$$(x',(x,w))\mapsto(x,(x',w)).$$

We then have the (comparison) functor

$$\Phi^p_{\mathbb{E}}:\mathbb{E}(B)\to\mathrm{Des}_{\mathbb{E}}(p),\quad (A,\alpha)\mapsto(E\times_B A,\mathrm{pr}_1,\bar{\phi})$$

which makes the following diagram commutative

$$\begin{array}{ccc} \mathbb{E}(B) & \xrightarrow{\Phi^p_{\mathbb{E}}} & \mathrm{Des}_{\mathbb{E}}(p) \\ & {\scriptstyle p^*}\searrow \quad \swarrow{\scriptstyle U^p} & \\ & \mathbb{E}(E), & \end{array} \tag{1.16}$$

where U^p is the obvious forgetful functor.

Definition 1.4.1 The continuous map p is an $\mathbb{E}$-*descent map* if $\Phi^p_{\mathbb{E}}$ is full and faithful, and it is an *effective* $\mathbb{E}$-*descent map* if $\Phi^p_{\mathbb{E}}$ is an equivalence of categories.

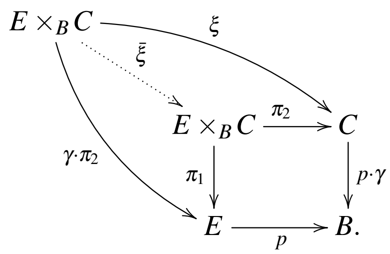
We remark that in [33] the authors presents Monadic Descent Theory (see Section 1.1) only as a first step of generalization of the Topological Descent Theory. As we mentioned in the introduction of the chapter, we present the topological framework only after the description of the problem of descent in terms of monads but, as shown in [33, Section 2.2], one can see explicitly the bijective correspondence between the descent data ξ given in (1.1) and the descent data $\bar{\xi}$ in (1.14). If we start with a descent data $\bar{\xi} : E \times_B C \to E \times_B C$ as in (1.14), just define

$$\xi := \pi_2 \cdot \bar{\xi}$$

to obtain a descent data in terms of algebra structure as in (1.1). On the other hand, starting with a descent data $\xi : E \times_B C \to C$ as in (1.1), define

$$\bar{\xi} :=< \gamma \cdot \pi_2, \xi >,$$

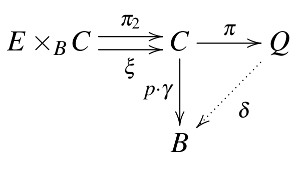
the morphism induced by the pair $(\gamma \cdot \pi_2, \xi)$ in the diagram below



This establishes a bijective correspondence between the descent data ξ and $\bar{\xi}$. Therefore the criteria given in Section 1.1 (such as for instance those involving the Beck's monadicity criterion, under further hypotheses on $\mathbb{E}$) can be applied to study descent in **Top**. In particular, in order to find criteria for a continuous map $p : E \to B$ to be effective for $\mathbb{E}$-descent, one can construct, as in Section 1.1, the left adjoint

$$\Psi^p_{\mathbb{E}} : \mathrm{Des}_{\mathbb{E}}(p) \to \mathbb{E}(B)$$

of the (comparison) functor $\Phi^p_{\mathbb{E}}$. As in (1.5), for an object $(C, \gamma, \bar{\xi}) \in \mathrm{Des}_{\mathbb{E}}(p)$, consider the diagram



(1.17)

where Q is the quotient space obtained by C under the equivalence relation $\sim_\xi$ defined by

$$z \sim z' \Leftrightarrow p(\gamma(z)) = p(\gamma(z')) \quad \text{and} \quad z' = \xi_{\gamma(z),\gamma(z')}(z),$$

for $z, z' \in C$. The map $\pi : C \to Q$ is the quotient map defined accordingly and $\delta : Q \to B$ is the map induced by the universal property of the quotient space. If $\mathbb{E}$ is stable under pullback along p and δ belongs to $\mathbb{E}$, for each $(C, \gamma, \bar{\xi}) \in \mathrm{Des}_{\mathbb{E}}(p)$, then the class $\mathbb{E}$ is called *descent stable with respect to p*. In this case the assignment

$$(C, \gamma, \bar{\xi}) \mapsto (Q, \delta)$$

defines the functor $\Psi^p_{\mathbb{E}}$ on objects.

Theorem 1.4.2 [33, Theorem 1.10] *Let $\mathbb{E}$ be descent stable with respect to $p : E \to B$. Then there is a pair of adjoint functors*

$$\Psi^p_{\mathbb{E}} \dashv \Phi^p_{\mathbb{E}} : \mathbb{E}(B) \to \mathrm{Des}_{\mathbb{E}}(p) \tag{1.18}$$

whose units are continuous bijections; this is also true for the counits, provided that p is a surjective map. Furthermore, in this case

(1) *p is an $\mathbb{E}$-descent map if and only if for every object in $\mathbb{E}(B)$ the counit is open;*

(2) *if p is an $\mathbb{E}$-descent map, then the category $\mathbb{E}(B)$ is equivalent to the full subcategory of those $(C, \gamma, \bar{\xi}) \in \mathrm{Des}_{\mathbb{E}}(p)$ for which the map*

$$1_E \times_B \pi : E \times_B C \to E \times_B Q, \quad (x, z) \mapsto (x, [z]),$$

with π as in diagram (1.17)*, is a quotient map; consequently,*

(3) *p is an effective $\mathbb{E}$-descent map if and only if p is an $\mathbb{E}$-descent map and the quotient condition of* (2) *holds for all objects* $\mathrm{Des}_{\mathbb{E}}(p)$.

Recall that in **Top** the class of regular epimorphisms coincides with the class of quotient maps so that $\mathbb{E}$-universal regular epimorphisms are also called $\mathbb{E}$-*universal quotient maps*. For what concerns $\mathbb{E}$-descent morphisms, the following results holds.

Proposition 1.4.3 [33, Proposition 1.6] *Let $\mathbb{E}$ be stable under pullback along $p : E \to B$. Then for the statements*

(i) *p is an $\mathbb{E}$-universal quotient map,*

(ii) *p is an $\mathbb{E}$-descent map,*

(iii) *p^* reflects isomorphisms,*

one has the implications $(i) \Rightarrow (ii) \Rightarrow (iii)$, and they are all equivalent if $\mathbb{E}$ is transferable along p.

One says that $\mathbb{E}$ is *transferable* along p if, for every pullback diagram

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B \end{array}$$

with $\pi_1 \in \mathbb{E}$ and π_2 a quotient map, one has $\alpha \in \mathbb{E}$. Of course, if $\mathbb{E}$ is the class of all continuous maps, the transferability property is trivially satisfied. This will lead, as we are going to see in the next section, to a complete characterization of the descent maps. But, in general, the equivalence of conditions (i), (ii) and (iii) is not true. As pointed out in [33, Section 4.4], if one considers $\mathbb{E}$ the class of local homeomorphisms, a bijective map $p : E \to B$, with E a 2-points discrete space and B a 2-points indiscrete space fails to be $\mathbb{E}$-descent, as can be seen by a direct inspection, although $p^*_{\mathbb{E}}$ reflects isomorphisms.

1.4.1 (Effective) global-descent maps

In this section we study when $\mathbb{E}$ is the class of all continuous maps. We recall that, in this case, one can speak of (effective) global-descent or simply of (effective) descent, omitting the prefix. We saw how one can explore monadicity criterion to study the problem of descent. In [32] G. Janelidze and W. Tholen, although they do not speak about descent, show that monadicity of the pullback functor $p^* : \mathbf{Top} \downarrow B \to \mathbf{Top} \downarrow E$, relative to a continuous map $p : E \to B$, turns out to be a local property. In particular, the main result of the paper states that p^* is monadic for every *locally sectionable map p*. Recall that a continuous map $p : E \to B$ in **Top** is *locally sectionable* or a *local-split epimorphism* if for every $y \in B$ there is a neighbourhood U in B such that the restriction $p_U : p^{-1}(U) \to U$ is a split epimorphism.

Remark 1.4.4 The fact that monadicity for p^*, and so to be effective for descent for p, is a local property has been investigated also in the more general context of indexed categories. In [34] G. Janelidze and W. Tholen generalize the notion of *locally-split epimorphism* to the context of an arbitrary indexed category $\mathbb{A}$, proving that, using Theorem 1.3.4, it still implies effective $\mathbb{A}$-descent.

The fact that continuous maps $p : E \to B$ for which $p^* : \mathbf{Top} \downarrow B \to \mathbf{Top} \downarrow E$ reflects isomorphisms coincide with the universal quotient maps can be immediately deduced by Proposition 1.4.3 but it was first proved in [32]. Universal quotient maps have been characterized by B.J. Day and G.M. Kelly in [22] as those continuous maps $p : E \to B$ satisfying the following condition:

> for every point $y \in B$ and for every family $(U_i)_{i \in I}$ of open sets in E which covers the fibre $p^{-1}(y)$, there are finitely many $i_1, \ldots, i_n \in I$ with y belonging to the interior of $p(U_{i_1} \cup \cdots \cup U_{i_n})$.

In particular, the following theorem holds.

Theorem 1.4.5 [32, Theorem 1.1] *The following conditions are equivalent for a map $p : E \to B$ in* **Top**:

(1) *p^* reflects isomorphisms;*

(2) *p is an universal quotient map;*

(3) *p^* reflects quotient maps;*

(4) *for every point $y \in B$ and for every family $(U_i)_{i \in I}$ of open sets in E which covers the fibre $p^{-1}(y)$, there are finitely many $i_1, \ldots, i_n \in I$ with y belonging to the interior of $p(U_{i_1} \cup \cdots \cup U_{i_n})$.*

Therefore, by Corollary 1.1.5, descent morphisms in **Top** are characterized as those continuous maps satisfying condition (4) of the theorem above. For what concerns the effective descent morphisms, a complete characterization is given in [47] by J. Reiterman and W. Tholen.

Theorem 1.4.6 [47, Theorem 1.5] *A surjective map $p : E \to B$ in* **Top** *is an effective descent morphism of* **Top** *if and only if the following condition condition holds:*

> *for every family of ultrafilters $\mathfrak{y}_i$ converging to $y_i \in B$, $i \in I$, if the y_i converge to y with respect to an ultrafilter $\mathfrak{i}$ on I, then there is an ultrafilter $\mathfrak{x}$ on E converging to a point $x \in p^{-1}(y)$ such that*
>
> $$\bigcup_{i \in U} A_i \in \mathfrak{x}$$
>
> *for all $U \in \mathfrak{i}$, with $A_i = p^{-1}(y_i) \cap \mathrm{adh}(p^{-1}(\mathfrak{y}_i))$ for $i \in I$, where $\mathrm{adh}(p^{-1}(\mathfrak{y}_i))$ is the set of adherence points of the filter base $p^{-1}(\mathfrak{y}_i)$.*

The technique they used to get such a characterization is precisely the one given in Corollary 1.1.7 2: embed **Top** into the locally cartesian closed category **PsTop** of *pseudo-topological* spaces, where effective descent morphisms are simply quotient maps, and reinterpret the characterization in topological terms using filter theory. A surjective map $p : E \to B$ in **PsTop** satisfying the condition of Theorem 1.4.6 is called a **-quotient map*. The following classes of maps are properly contained in the class of effective global-descent morphisms:

- open surjections (see [42] and [51]);
- proper surjections (see [53] and [47]);
- locally sectionable maps (see [32] and [34]).

Maps such as surjective local homeomorphisms and covering maps are locally sectionable and, therefore, effective for descent. In Chapter 3 we will see another description, always involving convergence of ultrafilters, of the effective descent maps in **Top**, given by M.M. Clementino and D. Hofmann in [9]. In [47] is also given an example of an universal quotient map in **Top** (therefore a descent morphism) which is not effective for descent. Anyway, for these kind of examples, it turned out to be useful working with finite topological spaces, as we will see soon in Section 1.4.3.

1.4.2 (Effective) étale-descent maps

When $\mathbb{E}$ is the class of étale maps (i.e., local homeomorphisms) one usually speaks of *(effective) étale-descent*. In [33] G. Janelidze and W. Tholen present criteria concerning the investigation of (effective) étale-descent morphisms. Let $p : E \to B$ be a continuous map of topological spaces and let $p_O : O(B) \to O(E)$ be its corresponding monotone map of (complete) lattices $O(B)$ and $O(E)$ given by the open sets of B and E, respectively. The map p can be factored as

$$E \xrightarrow{\bar{p}} p(E) \hookrightarrow B,$$

where $\bar{p}$ is the restriction of p to its image and $p(E) \hookrightarrow B$ is the subspace embedding.

Proposition 1.4.7 [33, Proposition 4.3] *A map $p: E \to B$ is an (effective) étale-descent map if and only if*

(i) *p is a descent map with respect to the class of open-subspace embeddings;*

(ii) *$\bar{p}$ is an (effective) étale-descent map.*

Moreover, if $p_O : O(B) \to O(E)$ is bijective, then p is an effective étale-descent map.

Condition (i) is equivalent to the fact that $p_O : O(B) \to O(E)$ is injective, as proved in [33, Proposition 4.2] where the effective descent maps with respect to the class of open embeddings (i.e., *effective open-descent maps*) are proved to be exactly the quotient maps. Therefore one can restrict the investigation to the case when p is surjective. A characterization of étale-descent maps is given in [33] in terms of étale systems.

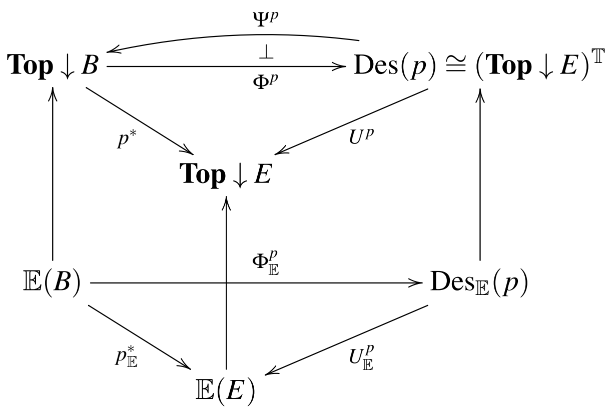
Definition 1.4.8 An *étale system* for an open subset U of a space B is a family $X = (X_{u,v})_{(u,v)\in U\times U}$ of open subsets of U such that, for all $u,v,w \in U$,

1. $u \in X_{u,u}$,
2. $X_{u,v} = X_{v,u}$,
3. $X_{u,v} \cap X_{v,w} \subseteq X_{u,w}$.

A subset $V \subseteq U$ is *X-admissible* if $V = \{v \in Y : v \in X_{u,v}\}$ for some open subset $Y \subseteq X_{u,u}$ with $u \in U$.

Theorem 1.4.9 [33, Theorem 4.6] *A surjective map $p: E \to B$ in* **Top** *is an étale-descent map if and only if for every open subspace U of B and every étale system X for U, all X-admissible subsets are open in B whenever the inverse image under p of all these sets are open in E.*

In [49] M. Sobral presents developments concerning the study of the effective étale-descent maps in **Top**. Let $p: E \to B$ be a continuous map of topological spaces. Consider the following commutative diagram



where the vertical arrows are full embeddings, the top-side and the botton-side triangles are diagrams (1.16) with respect to the class of all continuous maps and local homeomorphisms, respectively, $\mathbb{T}$ is the monad induced by the adjunction

$$\mathbf{Top}\downarrow B \underset{p^*}{\overset{p_!}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Top}\downarrow E,$$

where $p_!$ is the left adjoint of p^* given by the composition with p from the left, and Ψ^p is the left adjoint of Φ^p as in (1.18).

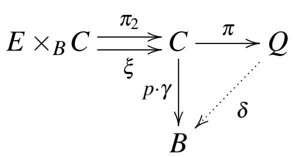
Proposition 1.4.10 [49, Proposition 3.4] *The surjective map p is effective for étale-descent if and only if the adjunction $\Psi^p \dashv \Phi^p : \mathbf{Top} \downarrow B \to \mathrm{Des}(p)$ restricts to an equivalence between $\mathbb{E}(B)$ and $\mathrm{Des}_{\mathbb{E}}(p)$.*

As a consequence we have the following result.

Theorem 1.4.11 [49, Theorem 3.5] *A surjective morphism p is effective étale-descent if and only if*

(i) p is an étale-universal regular epimorphism;

(ii) for each descent situation defining Q



δ is a local homeomorphism if γ is a local homeomorphism.

A complete characterization in **Top** of the effective étale-descent maps is still an open problem, at least for continuous maps between arbitrary topological spaces. But, if one restricts to the finite case, a complete characterization is given and this is what we are going to see in the next section.

1.4.3 The finite case

The restriction to the case where topological spaces are finite, investigated in [30] and [31] by G. Janelidze and M. Sobral, show how finite instances, expressed in the language of finite (pre)orders, motivate the results of Topological Descent Theory. Thanks to the isomorphism

$$\mathbf{FinTop} \cong \mathbf{FinOrd} \tag{1.19}$$

between the category **FinTop**, of finite topological spaces and continuous maps, and the category **FinOrd**, of finite (pre)orders and monotone maps, several kinds of continuous maps can be re-written in terms of relations of (pre)ordered sets. For every subset A of a finite topological space X, there is a smallest open set $\downarrow A$ containing A. Moreover,

$$\downarrow A = \bigcup_{x \in A} \downarrow x,$$

where $\downarrow x = \downarrow \{x\}$. Writing

$$y \to x \Leftrightarrow y \in \downarrow x$$

one obtains a reflexive and transitive relation, which establishes then the isomorphism (1.19). As we mentioned before, this allows for a description in terms of *convergence* of maps such as open maps

$$\begin{array}{c} X \\ {\scriptstyle f}\downarrow \\ Y \end{array} \qquad\qquad \begin{array}{ccc} \exists x_1 & \dashrightarrow & x_0 \\ \downarrow & & \downarrow \\ y_1 & \longrightarrow & f(x_0) \end{array} \tag{1.20}$$

(for every $x_0 \in X$ and $y_1 \to f(x_0)$ in Y, there exists $x_1 \in X$ with $x_1 \to x_0$ and $f(x_1) = y_1$), proper maps

$$\begin{array}{c} X \\ {\scriptstyle f}\downarrow \\ Y \end{array} \qquad\qquad \begin{array}{ccc} x_1 & \dashrightarrow & \exists x_0 \\ \downarrow & & \downarrow \\ f(x_1) & \longrightarrow & y_0 \end{array} \tag{1.21}$$

(for every $x_1 \in X$ and $f(x_1) \to y_0$ in Y, there exists $x_0 \in X$ with $x_1 \to x_0$ and $f(x_0) = y_0$), étale maps

$$\begin{array}{c} X \\ {\scriptstyle f}\downarrow \\ Y \end{array} \qquad\qquad \begin{array}{ccc} \exists! x_1 & \dashrightarrow & x_0 \\ \downarrow & & \downarrow \\ y_1 & \longrightarrow & f(x_0) \end{array} \tag{1.22}$$

(for every $x_0 \in X$ and $y_1 \to f(x_0)$ in Y, there exists a unique $x_1 \in X$ with $x_1 \to x_0$ and $f(x_1) = y_1$), and regular epimorphisms (i.e., quotient maps)

$$\begin{array}{cccccccc} x'_n & & & & & & & y_1 \\ \downarrow & & & & & & & \big\downarrow \\ x_{n-1} & \longrightarrow & x'_{n-1} & & & & & \big\downarrow \\ & & \downarrow & & & & & \big\downarrow \\ & & x_{n-2} & \longrightarrow & x'_{n-2} & & & \big\downarrow \\ & & & & \downarrow & & & \big\downarrow \\ & & & & \cdot & \dashrightarrow & \cdot & \big\downarrow \\ & & & & & & \vdots & \big\downarrow \\ & & & & & & \cdot \dashrightarrow x'_1 & \big\downarrow \\ & & & & & & \downarrow & \big\downarrow \\ & & & & & & x_0 & y_0 \end{array} \tag{1.23}$$

(if for each $y_1 \to y_0$ in B there exists a (finite) sequence in X as above with $f(x'_n) = y_1$, $f(x_0) = y_0$, $f(x_i) = f(x'_i)$, for $i = 1, \ldots, n-1$, and $x'_i \to x_{i-1}$, for $i = 1, \ldots, n$, in X). Also descent morphisms and effective descent morphisms, whose characterizations in **Top** are given in Theorem 1.4.5 and Theorem 1.4.6, respectively, have a description in terms of relations between points.

Proposition 1.4.12 [30, Proposition 2.5] *For a morphism $p : E \to B$ in* **FinTop***, the following conditions are equivalent:*

(i) *p is a descent map;*

(ii) *p is a pullback stable regular epimorphism;*

(iii) *for every $y_1 \to y_0$ in B there exists $x_1 \to x_0$ in E with $p(x_i) = y_i$, for $i = 0,1$,*

$$\begin{array}{ccc} E & \qquad & x_1 \dashrightarrow x_0 \\ {\scriptstyle p}\downarrow & & \downarrow \qquad \downarrow \\ B & & y_1 \longrightarrow y_0. \end{array}$$

Proposition 1.4.13 [30, Proposition 3.4] *For a morphism $p : E \to B$ in* **FinTop***, the following conditions are equivalent:*

(i) *p is an effective descent morphism;*

(ii) *for every $y_2 \to y_1 \to y_0$ in B there exists $x_2 \to x_1 \to x_0$ in E with $p(x_i) = y_i$, for $i = 0,1,2$,*

$$\begin{array}{ccc} E & \qquad & x_2 \dashrightarrow x_1 \dashrightarrow x_0 \\ {\scriptstyle p}\downarrow & & \downarrow \qquad \downarrow \qquad \downarrow \\ B & & y_2 \longrightarrow y_1 \longrightarrow y_0. \end{array}$$

For what concerns the étale case, as we mentioned, a complete characterization of the effective étale-descent maps in **Top** is not given. But, working with (pre)orders, it has been possible to get such a characterization, holding then for finite topological spaces. Let $p : E \to B$ a monotone map of (pre)ordered sets and denote by $\mathbb{E}$ the class of étale morphisms in **Ord**. The characterization is given in two steps, each one represented by changing environment

$$\begin{array}{ccc} \mathbb{E}(B) \xrightarrow{p^*} \mathbb{E}(E) & & \mathbf{Set}^{B^{\mathrm{op}}} \xrightarrow{\mathbf{Set}^{p^{\mathrm{op}}}} \mathbf{Set}^{E^{\mathrm{op}}} & & \mathbf{Set}^{B^{\mathrm{op}}} \xrightarrow{\mathbf{Set}^{p^{\mathrm{op}}}} \mathbf{Set}^{E^{\mathrm{op}}} \\ {\scriptstyle \Phi^p_{\mathbb{E}}}\downarrow \quad \nearrow {\scriptstyle U^p} & \simeq^{(1)} & {\scriptstyle k^p}\downarrow \quad \nearrow {\scriptstyle u^p} & \simeq^{(2)} & {\scriptstyle \mathbf{Set}^{\varphi^{\mathrm{op}}}}\downarrow \quad \nearrow {\scriptstyle \mathbf{Set}^{\psi^{\mathrm{op}}}} \\ \mathrm{Des}_{\mathbb{E}}(p) & & \mathbf{X} & & \mathbf{Set}^{Z(Eq(p))^{\mathrm{op}}} \end{array} \tag{1.24}$$

in an equivalent way. The first one has been done in [30] by G. Janelidze and M. Sobral. The passage from the standard diagram in Descent Theory, where the comparison functor $\Phi^p_{\mathbb{E}}$ and the category of descent data $\mathrm{Des}_{\mathbb{E}}(p)$ are involved, to the second diagram is given by the equivalence of **Ord**-indexed categories $\mathbb{E}(E)$, $\mathbb{E}(B)$ and $\mathbf{Set}^{E^{\mathrm{op}}}$, $\mathbf{Set}^{B^{\mathrm{op}}}$, respectively. Accordingly, by Remark 1.3.5, the pullback functor $p^* : \mathbb{E}(B) \to \mathbb{E}(E)$ can be identified (up to equivalence) with the functor $\mathbf{Set}^{p^{\mathrm{op}}} : \mathbf{Set}^{B^{\mathrm{op}}} \to \mathbf{Set}^{E^{\mathrm{op}}}$, which sends a functor $B^{\mathrm{op}} \to \mathbf{Set}$ to its composite $E^{\mathrm{op}} \xrightarrow{p^{\mathrm{op}}} B^{\mathrm{op}} \to \mathbf{Set}$ with p^{op}. The category $\mathrm{Des}_{\mathbb{E}}(p)$ is then identified with the category $\mathbf{X}$ of pairs (X, ξ) where $X : E^{\mathrm{op}} \to \mathbf{Set}$ is a functor and ξ is given by a family of maps $\xi_{x,x'} : X(x) \to X(x')$, defined for $x, x' \in E$ such that $p(x) = p(x')$, and satisfying the following conditions:

- $\xi_{x,x} = 1_{X(x)}$, for each $x \in E$;

- $\xi_{x',x''} \cdot \xi_{x,x'} = \xi_{x,x''}$, for each $x, x', x'' \in E$ such that $p(x) = p(x') = p(x'')$;

- $\xi_{x_1,x_1'} \cdot X(x_1, x_0) = X(x_1', x_0') \cdot \xi_{x_0,x_0'}$, for each $x_0, x_0', x_1, x_1' \in E$ such that $p(x_0) = p(x_0')$, $p(x_1) = p(x_1')$ and $x_1 \to x_0$, $x_1' \to x_0'$ in E.

The forgetful functor U^p is again a functor u^p forgetting the structure while k^p is defined by $k^p(F) = (F \cdot p^{\mathrm{op}}, 1)$, where 1 is the family of identity morphisms $1_{x,x'}$ of $F(p(x)) = F(p(x'))$ for all $x, x' \in E$ with $p(x) = p(x')$. Therefore, by step $\simeq^{(1)}$, one can say that the monotone map p is an effective étale-descent morphism if and only if the functor k^p is an equivalence of categories.
Step $\simeq^{(2)}$ is given in [31], where G. Janelidze and M. Sobral complete the characterization. It consists of the construction of the so-called category of *zigzags* $Z(\mathrm{Eq}(p))$, where $\mathrm{Eq}(p)$ is the internal category in **Ord** given by the kernel pair of p and $Z : \mathbf{DoubleCat} \to \mathbf{Cat}$ is the left adjoint of the functor $S : \mathbf{Cat} \to \mathbf{DoubleCat}$ which sends each category C to the double category of commutative squares in C. The category $Z(\mathrm{Eq}(p))$ is constructed as follows. Let:

- $\mathrm{Eq}(p)_0$ be the discrete category with objects as in $\mathrm{Eq}(p)$;

- $\mathrm{Eq}(p)_h$ and $\mathrm{Eq}(p)_v$ be categories with the same objects and the morphisms to be, respectively, the horizontal and the vertical arrows of $\mathrm{Eq}(p)$;

- $\mathrm{Eq}(p)_+$ the pushout in **Cat** of the embeddings $\mathrm{Eq}(p)_0 \to \mathrm{Eq}(p)_h$ and $\mathrm{Eq}(p)_0 \to \mathrm{Eq}(p)_v$;

then, for every square in $\mathrm{Eq}(p)$

$$\begin{array}{ccc} x_1 & \longrightarrow & x_1' \\ \downarrow & & \downarrow \\ x_0 & \longrightarrow & x_0' \end{array}$$

the pairs

$$\begin{array}{ccc} x_1 & \longrightarrow & x_1' \\ & & \downarrow \\ & & x_0' \end{array} \qquad \begin{array}{ccc} x_1 & & \\ \downarrow & & \\ x_0 & \longrightarrow & x_0' \end{array}$$

become morphisms in $\mathrm{Eq}(p)_+$ from x_1 to x_0', and one constructs $Z(\mathrm{Eq}(p))$ as the quotient category $\mathrm{Eq}(p)_+/\sim$ under the smallest equivalence relation $\sim$ for which

$$\begin{array}{ccc} x_1 & \longrightarrow & x_1' \\ & & \downarrow \\ & & x_0' \end{array} \quad \sim \quad \begin{array}{ccc} x_1 & & \\ \downarrow & & \\ x_0 & \longrightarrow & x_0', \end{array}$$

for all such pairs. Observe that a morphism in $Z(\mathrm{Eq}(p))$ from a point x_n to a point x'_0 is then given by an equivalent class of a morphism in $\mathrm{Eq}(p)_+$, say an n-zigzag z

$$\begin{array}{ccccccccc}
x'_n &&&&&&&& \\
\downarrow &&&&&&&& \\
x_{n-1} & \longrightarrow & x'_{n-1} &&&&&& \\
&& \downarrow &&&&&& \\
&& x_{n-2} & \longrightarrow & x'_{n-2} &&&& \\
&&&& \downarrow &&&& \\
&&&& \cdot & \dashrightarrow & \cdot && \\
&&&&&& \vdots && \\
&&&&&& \cdot & \dashrightarrow & x'_1 \\
&&&&&&&& \downarrow \\
&&&&&&&& x_0
\end{array}$$

where $p(x_i) = p(x'_i)$ for $i = 1,\ldots,n-1$, and $x'_i \to x_{i-1}$ in E for $i = 1,\ldots,n$. The notation for such an n-zigzag as above will be $z = (x_0,x'_0)\cdots(x_1,x'_1)$. The equivariant map $p : E \to B$ can be then factorized in **Cat** through the category $Z(\mathrm{Eq}(p))$

$$\begin{array}{ccc}
E & \xrightarrow{\quad p \quad} & B \\
& {\scriptstyle\psi}\searrow \quad \nearrow{\scriptstyle\varphi} & \\
& Z(\mathrm{Eq}(p)) &
\end{array} \tag{1.25}$$

where ψ is defined as the identity on objects and $\psi(x_1 \to x_0) = [x_1 \to x_0]$ on morphisms, while φ on objects acts as p and the image of an equivalent class of an n-zigzag $z = (x_0,x'_0)\cdots(x_n,x'_n)$ via φ is $\varphi([z]) = p(x_n) \to p(x'_{n-1}) \to \cdots \to p(x'_0) = p(x_n) \to p(x'_0)$. Thanks to the adjoint situation

$$\mathbf{Cat} \underset{S}{\overset{Z}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{DoubleCat} \tag{1.26}$$

an object (X,ξ) of $\mathrm{Des}_{\mathbb{E}}(p)$, i.e., a double functor from $\mathrm{Eq}(p)$ to $S(\mathbf{Set})$, is the same as a functor from $Z(\mathrm{Eq}(p))$ to **Set**, so that the equivalence $\simeq^{(2)}$ holds.

Theorem 1.4.14 [31, Theorem 1.2] *The monotone map $p : E \to B$ is an effective étale-descent morphism if and only if the functor $\varphi : Z(\mathrm{Eq}(p)) \to B$ is an equivalence of categories.*

Corollary 1.4.15 [31, Corollary 1.3] *The monotone map $p : E \to B$ is an effective étale-descent morphisms if and only if the following conditions hold:*

(i) *the map $p : E \to p(E)$ induced by p is a quotient map;*

(ii) *$Z(\mathrm{Eq}(p))$ is a (pre)ordered set;*

(*iii*) *$p : E \to B$ is essentially surjective on objects, i.e., for every $y \in B$ there exists an element $x \in E$ such that $p(x) \to y$ and $y \to p(x)$.*

Remarks 1.4.16

- [31, Proposition 2.1] For every two elements x_1 and x_0 in E, every two 1-zigzags $x_1 \to x_0$ are equivalent;

- [31, Corollary 2.2] If every 2-zigzag is equivalent to a 1-zigzag, then $Z(\mathrm{Eq}(p))$ is a (pre)ordered set.

As observed in [30], the (pre)order relation on a finite topological space represents convergence and this is the key ingredient to use while dealing with the infinite case. In fact, writing $\mathfrak{x} \to x$ when a filter $\mathfrak{x}$ converges to a point x, in the finite case one has that $\mathfrak{x} \to x$ if and only if $y \to x$ for every y belonging to the intersection of the elements of $\mathfrak{x}$. As we will see in the next chapter, the passage from topology in terms of open sets to topology in terms of (ultra)filters convergence determines an isomorphism of categories which extends the one given in the finite case by (1.19). Moreover, in Chapters 2 and 3, we are going to see how this translation of the pointwise convergence on a (pre)order set to the one in terms of (ultra)filters on a (infinite) topological space is reflected in the description of several maps, also for what concerns (effective) descent morphisms.

1.4.4 (Effective) global-descent versus (effective) étale-descent

What is the relations between the (effective) descent morphisms and the (effective) étale-descent morphisms in **Top**?

Theorem 1.4.17 [33, Theorem 4.7] *Every effective global-descent map is an effective étale-descent map.*

Of course this includes the case where topological spaces are finite. Anyway, thanks to characterizations of Proposition 1.4.13 and Corollary 1.4.15, a simple direct proof in the finite case can be given.

Proposition 1.4.18 [31, Proposition 2.3] *If p is an effective descent map, then every 2-zigzag is equivalent to a 1-zigzag.*

The proposition above, with Remark 1.4.16, gives immediately that every effective descent map between finite topological spaces is effective for étale-descent. The converse of Theorem 1.4.17 is not true, since there are non-surjective effective étale-descent maps as it can be deduced by Proposition 1.4.7. A concrete counter-example, even in the surjective case, has been given by M. Sobral in [49], involving precisely finite topological spaces. It actually represented the starting point of the study of Descent Theory in **Top** restricted to the finite case.

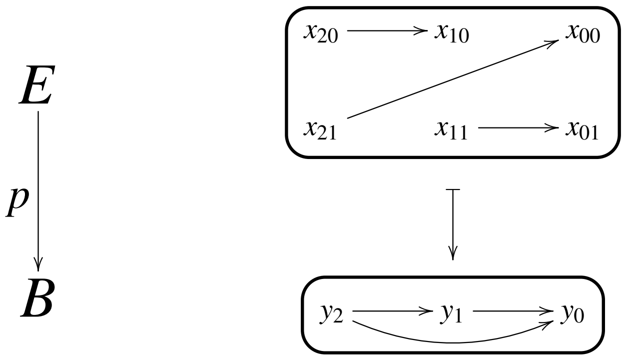
Proposition 1.4.19 [49, Proposition 3.6] *The class of effective global-descent maps is strictly contained in the class of surjective effective étale-descent maps.*

Take

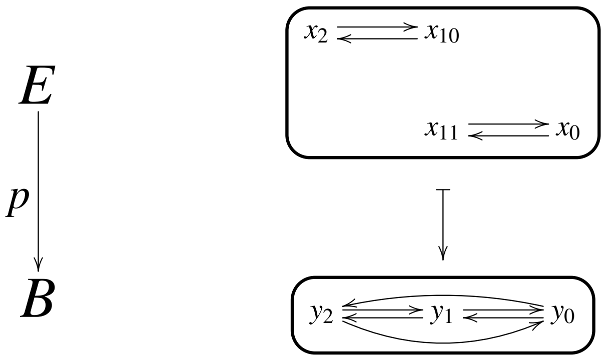
$$E = (\{x, x', x_1, x_2\}, \{\emptyset, E, \{x, x_1\}, \{x', x_2\}\}), \quad B = (\{y, y_1, y_2\}, \{\emptyset, B\}),$$

and $p : E \to B$ be the continuous map defined by $p(x) = p(x') = y$ and $p(x_i) = y_i$, for $i = 1, 2$. To prove that p is an effective étale-descent map one uses Theorem 1.4.11. The map p is an étale-regular epimorphism, since it is, in particular, an universal quotient map. Moreover, the transferability property given by condition (ii) in Theorem 1.4.11 is satisfied. Anyway, working directly with relations between points, a list of interesting examples are given in [30].

Example 1.4.20 [30, Example 8.2] A simple inspection using Proposition 1.4.12, Proposition 1.4.13 and Corollary 1.4.15 reveals that the following map is a descent morphism but neither effective for descent nor effective for étale-descent. This is precisely the finite version of the original counter-example given in [47].



Example 1.4.21 [30, Example 8.7] We already know by Proposition 1.4.19 that there are effective étale-descent maps not effective for descent, even in the surjective case. The following example gives a morphism effective for étale-descent but not even a descent map, as it can be deduced by a quick inspection using Proposition 1.4.12 and Corollary 1.4.15.



Summing up all the material studied so far, in **Top** the following picture of implications (taken from

[33]) holds.

$$\begin{array}{ccccc}
\text{effective global-descent} & & \Rightarrow & & \text{effective étale-descent} \\
\Downarrow & & & & \\
\text{global-descent} & \Leftrightarrow & \text{universal quotient} & & \Downarrow \\
 & & \Downarrow & & \\
 & & \text{quotient} & \Rightarrow & \text{étale-descent}
\end{array} \tag{1.27}$$

The fact quotient maps are étale-descent has been proved in [33] as a corollary of [33, Proposition 4.5] where a criterion for a surjective map in **Top** to be $\mathbb{E}$-descent is given for

$$\{\text{open-subspace embeddings}\} \subseteq \mathbb{E} \subseteq \{\text{local homeomorphisms}\}.$$

The one-direction implications are strict, as mostly suggested by Example 1.4.20 and Example 1.4.21. In fact in Example 1.4.20 a descent map not effective for descent is given. Moreover, being an universal quotient map, it is also étale-descent but it is not an effective étale-descent morphism. The map in Example 1.4.21 is an effective étale-descent map not effective for descent. Also it represents a quotient map that is not an universal quotient map. It remains to exhibit an example of an étale-descent map not quotient. Of course in the non-surjective case is easy since there are non-surjective effective étale-descent maps. But, as suggested in [33], also in the surjective case we have an example. Take the identity map of a 2-elements set where the domain is equipped with the Sierpiński topology while the codomain with the indiscrete topology. It is étale-descent but it fails to be a quotient map.

1.5 The categorical Van Kampen Theorem

We end up the first chapter with a section dedicated to the Van Kampen Theorem since its categorical version, given in [5] by R. Brown and G. Janelidze, shows a strict connection with Descent Theory. The classical Van Kampen Theorem is a tool to compute the fundamental group πX of a given topological space X. Let X be a topological space and let X_1 and X_2 two open and path-connected subspaces of X such that $X_1 \cup X_2 = X$ and $X_1 \cap X_2$ is non-empty and path-connected itself. Let $x \in X_1 \cap X_2$ be a fixed base point. The following square of inclusions

$$\begin{array}{ccc}
(X_1 \cap X_2, x) & \xrightarrow{i_1} & (X_1, x) \\
{\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\
(X_2, x) & \xrightarrow[j_2]{} & (X, x)
\end{array} \tag{1.28}$$

is a pushout in the category $\mathbf{Top}_*$ of pointed topological spaces and continuous maps preserving the base point. The classical Van Kampen Theorem asserts that the canonical morphism

$$k : \pi(X_1, x) +_{\pi(X_1 \cap X_2, x)} \pi(X_2, x) \to \pi(X, x) \tag{1.29}$$

induced by the universal property of the pushout

$$\begin{array}{ccccc} \pi(X_1 \cap X_2, x) & \xrightarrow{\pi i_1} & \pi(X_1, x) & & \\ \downarrow{\scriptstyle \pi i_2} & & \downarrow{\scriptstyle \tau_1} & \searrow{\scriptstyle \pi j_1} & \\ \pi(X_2, x) & \xrightarrow[\tau_2]{} & \pi(X_1,x) +_{\pi(X_1 \cap X_2, x)} \pi(X_2,x) & \overset{k}{\dashrightarrow} & \pi(X,x) \\ & \searrow_{\pi j_2} & & & \end{array}$$

is an isomorphism, i.e., the fundamental group of X is the pushout in category **Grp** of groups and group homomorphisms of the fundamental groups of X_1 and X_2 along $X_1 \cap X_2$. In the diagram above $\pi i_1, \pi i_2, \pi j_1$ and πj_2 are the morphisms induced by the inclusions i_1, i_2, j_1 and j_2, respectively, and τ_1 and τ_2 are the canonical injections of the pushout. In other words, the fundamental group functor $\pi : \mathbf{Top}_* \to \mathbf{Grp}$ preserves the pushout (1.28). A first generalization of it has been given in [4] by R. Brown where the fact of its computation in terms of a fixed base point, for which one usually restricts to path-connected spaces, has been replaced by considering, more generally, the fundamental groupoid ΠX of the space X, so that the hypotheses about path-connectedness can be avoided. In particular, let X be a topological space and let X_1 and X_2 be two subspaces of X such the join of their interiors covers X. The following square of inclusions

$$\begin{array}{ccc} X_1 \cap X_2 & \xrightarrow{i_1} & X_1 \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ X_2 & \xrightarrow[j_2]{} & X \end{array} \tag{1.30}$$

is a pushout in **Top**. A set A is called *representative* in X if A meets each path-component of X.

Theorem 1.5.1 [4, 6.7.2] *If A is representative in X_1, X_2 and $X_1 \cap X_2$, then the square*

$$\begin{array}{ccc} \Pi(X_1 \cap X_2, A) & \xrightarrow{\Pi i_1} & \Pi(X_1, A) \\ \downarrow{\scriptstyle \Pi i_2} & & \downarrow{\scriptstyle \Pi j_1} \\ \Pi(X_2, A) & \xrightarrow[\Pi j_2]{} & \Pi(X, A) \end{array}$$

induced by (1.30) *is a pushout in the category* **Grpd** *of groupoids and groupoid homomorphisms.*

If $A = X$ it simply says that the fundamental groupoid functor $\Pi : \mathbf{Top} \to \mathbf{Grpd}$ preserves the pushout (1.30).

1.5.1 General settings

Let $\mathbf{C}$ be a category with pullbacks and let $\mathbb{E}$ be a class of morphisms in $\mathbf{C}$ which contains all isomorphisms and is pullback stable. For a commutative diagram in $\mathbf{C}$ of the form

$$\begin{array}{ccc} X_0 & \xrightarrow{f_1} & X_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ X_2 & \xrightarrow[g_2]{} & X \end{array} \tag{1.31}$$

one defines

$$\mathbb{E}(X_1) \times_{\mathbb{E}(X_0)} \mathbb{E}(X_2)$$

to be the category of triples $((A_1,\alpha_1),(A_2,\alpha_2),\phi)$ where (A_1,α_1) is an object in $\mathbb{E}(X_1)$, (A_2,α_2) an object in $\mathbb{E}(X_2)$ and $\phi : f_1^*(A_1,\alpha_1) \to f_2^*(A_2,\alpha_2)$ is an isomorphism. Since the diagram (1.31) is commutative the pair (g_1,g_2) induces a functor

$$K_{g_1,g_2} : \mathbb{E}(X) \to \mathbb{E}(X_1) \times_{\mathbb{E}(X_0)} \mathbb{E}(X_2)$$

described in the following way: for an object $h : Y \to X$ in $\mathbb{E}(X)$ one considers the pullbacks of h along g_1 and g_2, respectively

$$\begin{array}{ccc} X_1 \times_X Y & \xrightarrow{\mathrm{pr}_2^{X_1}} & X \\ {\scriptstyle \mathrm{pr}_1^{X_1}}\downarrow & & \downarrow{\scriptstyle h} \\ X_1 & \xrightarrow[g_1]{} & X \end{array} \qquad\qquad \begin{array}{ccc} X_2 \times_X Y & \xrightarrow{\mathrm{pr}_2^{X_2}} & Y \\ {\scriptstyle \mathrm{pr}_1^{X_2}}\downarrow & & \downarrow{\scriptstyle h} \\ X_2 & \xrightarrow[g_2]{} & X \end{array}$$

so that $K_{g_1,g_2}(Y,h) = ((X_1 \times_X Y, \mathrm{pr}_1^{X_1}),(X_2 \times_X Y, \mathrm{pr}_1^{X_2}),\phi)$, where

$$\phi : f_1^*(X_1 \times_X Y, \mathrm{pr}_1^{X_1}) \to f_2^*(X_2 \times_X Y, \mathrm{pr}_1^{X_2})$$

is the isomorphism induced by the universal property of pullbacks.

Definition 1.5.2 One says that the class $\mathbb{E}$ satisfies the Van Kampen Theorem for a given commutative diagram (1.31) if the functor K_{g_1,g_2} is an equivalence of categories.

The classical formulation of the Van Kampen Theorem is covered by the definition above, in fact, as M. Brown proved in [4], if the space X is Hausdorff, locally connected and semi-locally 1-connected, then there is an equivalence of categories

$$\mathrm{Cov}(X) \simeq \mathbf{Set}^{\Pi X}$$

where $\mathrm{Cov}(X)$ is the comma category whose objects are coverings over X and $\mathbf{Set}^{\Pi X}$ is the presheaf category over the fundamental groupoid ΠX. Therefore (1.29) can be generalized and formulated in terms of coverings, that is,

$$\mathrm{Cov}(X) \simeq \mathrm{Cov}(X_1) \times_{\mathrm{Cov}(X_1 \cap X_2)} \mathrm{Cov}(X_2).$$

Hence one can state that the class of coverings satisfies the Van Kampen Theorem.

1.5.2 The general Van Kampen Theorem

To give a general formulation of the Van Kampen Theorem, and therefore explore its strict connection with Descent Theory, one needs to restrict the attention to lextensive categories. A category $\mathbf{C}$ with finite coproducts is called *extensive* if, for each pair of objects X_1, X_2 in $\mathbf{C}$, the canonical functor

$$+ : \mathbf{C} \downarrow X_1 \times \mathbf{C} \downarrow X_2 \to \mathbf{C} \downarrow (X_1 + X_2)$$

is an equivalence of categories. An equivalent formulation of extensive categories can be found in [6].

Proposition 1.5.3 [6, Proposition 2.14] *A category with finite coproducts and pullbacks along their injections is extensive if and only if the coproducts are universal and disjoint.*

Recall that, in a category with finite coproducts and pullbacks along their injections, a coproduct diagram

$$X_1 \longrightarrow X_1 + X_2 \longleftarrow X_2$$

is said to be *universal* if pulling it back along any morphism into $X_1 + X_2$ gives a coproduct diagram, while it is said to be *disjoint* if the pullback of the injections of a binary coproduct is the initial object, and all injections are monic. Roughly speaking, an extensive category is a category where coproducts exist and are well-behaved. An extensive category with all finite limits is called *lextensive*.

Lemma 1.5.4 [5, Lemma 3.1] *Let*

$$\begin{array}{ccc} X_0 & \xrightarrow{f_1} & X_1 \\ {\scriptstyle f_2}\big\downarrow & & \big\downarrow{\scriptstyle g_1} \\ X_2 & \xrightarrow[g_2]{} & X \end{array}$$

be a pullback diagram in a lextensive category $\mathbf{C}$*, in which* g_1 *and* g_2 *(and so also* f_1 *and* f_2*) are monomorphisms, and let* $\mathbb{E}$ *be a class of morphisms in* $\mathbf{C}$ *which is pullback stable and contains all isomorphisms. Let* $p : X_1 + X_2 \to X$ *denote the morphism induced by* g_1 *and* g_2*. Then there exists an equivalence of categories between* $\mathbb{E}(X_1) \times_{\mathbb{E}(X_0)} \mathbb{E}(X_2)$ *and* $\mathrm{Des}_{\mathbb{E}}(p)$ *such that the diagram*

$$\begin{array}{ccc} \mathbb{E}(X_1) \times_{\mathbb{E}(X_0)} \mathbb{E}(X_2) & \xrightarrow{\simeq} & \mathrm{Des}_{\mathbb{E}}(p) \\ & {\scriptstyle K_{g_1,g_2}}\nwarrow \quad \nearrow{\scriptstyle \Phi^p_{\mathbb{E}}} & \\ & \mathbb{E}(X) & \end{array}$$

commutes (up to isomorphism).

Theorem 1.5.5 [5, Proposition 3.2] *Let*

$$\begin{array}{ccc} X_0 & \xrightarrow{f_1} & X_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ X_2 & \xrightarrow[g_2]{} & X \end{array}$$

be a pullback diagram in a lextensive category **C** *in which g_1 and g_2 are monomorphisms. Let $\mathbb{E}$ be a class of morphisms in* **C** *which is pullback stable and contains all isomorphisms. Then the following are equivalent:*

(i) *$\mathbb{E}$ satisfies the Van Kampen Theorem for the pullback diagram above;*

(ii) *the morphism $X_1 + X_2 \to X$ induced by g_1 and g_2 is an effective $\mathbb{E}$-descent morphism.*

Note that if $\mathbb{E}$ is the class of all morphisms, then the pullback above is also a pushout. The theorem shows that the solution for a Van Kampen Theorem can be pursued studying Descent Theory. Moreover, one can now study the Van Kampen Theorem for different class of morphisms, as it has been done for example in [8] by M.M. Clementino, where a Van Kampen Theorem in **Top** with respect to the class of all continuous maps is given.

Theorem 1.5.6 [8, Corollary 2] *Given a pullback diagram in* **Top**

$$\begin{array}{ccc} X_0 & \xrightarrow{f_1} & X_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ X_2 & \xrightarrow[g_2]{} & X \end{array}$$

where g_1 and g_2 are embeddings, then the class $\mathbb{E}$ of all continuous maps satisfies the Van Kampen Theorem if and only if $\overline{X - X_2} \subseteq X_1$ and $\overline{X - X_1} \subseteq X_2$.

Chapter 2

Lax algebras

As proved by E. Manes in [39], the category **CHaus** of compact and Hausdorff spaces is equivalent to the category of Eilenberg-Moore algebras over **Set**, i.e., the forgetful functor **CHaus** $\to$ **Set** is monadic. Therefore compact Hausdorff spaces reveal an algebraic nature described in terms of ultrafilter convergence, i.e., a compact Hausdorf space X can be identified (up to isomorphism) with the (Eilenberg-Moore algebra) (X,a), where X is its underlying set and

$$a : UX \to X \tag{2.1}$$

is the map assigning to each ultrafilter $\mathfrak{x}$ on X ($\mathfrak{x} \in UX$) its limit point (recall that a topological space X is compact Hausdorff if and only if each ultrafilter on X converges to a unique point). The reflexive and transitive (algebras) properties can be expressed by the following commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\eta_X} & UX & \xleftarrow{Ua} & U^2X \\ & \searrow^{1_X} & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \mu_X} \\ & & X & \xleftarrow[a]{} & UX \end{array}$$

where $\eta_X : X \to UX$ and $\mu_X : U^2X \to UX$ are the maps defined, respectively, by

$$x \mapsto \{A \subseteq X : x \in A\}, \quad \mathfrak{X} \mapsto \bigcup_{\mathscr{A} \in \mathfrak{X}} \bigcap_{\mathfrak{x} \in \mathscr{A}} \mathfrak{x},$$

so that $A \subseteq X$ belongs to $\mu_X(\mathfrak{X})$ if and only if $A^\sharp = \{\mathfrak{x} \in UX | A \in \mathfrak{x}\} \in \mathfrak{X}$. They represent the components of the natural transformations $\eta : 1_{\mathbf{Set}} \to U$ and $\mu : U^2 \to U$ of the ultrafilter monad $\mathbb{U} = (U, \mu, \eta)$. For a general topological space X convergence of ultrafilters defines no longer a map (2.1) but a relation $a : UX \rightarrow\!\!\!\!\!\!\mapsto X$, since an ultrafilter on an arbitrary topological space might not be convergent and, even if it is, the set of its limit points might have more the one element. As M. Barr showed in [1], relaxing the axioms of reflexivity and transitivity

$$\begin{array}{ccccc} X & \xrightarrow{\eta_X} & UX & \overset{\overline{U}a}{\longleftarrow\!\!\!\!+} & U^2X \\ & \searrow^{\leq}_{1_X} & \downarrow a & \leq & \downarrow \mu_X \\ & & X & \underset{a}{\longleftarrow\!\!\!\!+} & UX, \end{array} \tag{2.2}$$

and considering a suitable *extension* $\overline{U}$ of the ultrafilter functor U : **Set** $\to$ **Set** to the category **Rel** of sets and relations, one is able to describe any topological space as a set X equipped with a (lax) algebraic structure $a : UX \to\!\!\!\!\!+ X$. The concept of *lax algebras* comes from this idea of relaxing the axioms of Eilenberg-Moore algebras. This presentation of topological spaces as lax Eilenberg-Moore algebras of the ultrafilter monad, together with description of metric spaces as small categories enriched over the complete (non-negative) half-real line, given by F.W. Lawvere in [35], represent the two principal roots of the theory developed in [10] by M.M. Clementino and D. Hofmann, where **Set** and **Rel** are replaced by different categories, and in [21] by M.M. Clementino and W. Tholen, where relations are replaced by more general **V**-*relations*, for **V** a monoidal closed category. Furthermore, in [48] G. Seal considers also a context with a different notion of extension. All these abstract constructions allow for a description of many other objects in mathematics such as, for instance, (pre)metric spaces, closure spaces and approach spaces. They are the subject of the area so-called *Monoidal Topology*, whose main reference can be now considered the book [29]. Along this chapter we follow its settings, in particular for what concerns the definition of *lax extension* and the choice of **V**.

2.1 Basic concepts

Definition 2.1.1 A *quantale* **V** is a complete lattice V which carries a monoid structure with neutral element k and where the binary operation, denoted as a *tensor* $\otimes$, distributes over suprema:

$$u \otimes \bigvee_{i\in I} v_i = \bigvee_{i\in I}(u\otimes v_i), \quad \bigvee_{i\in I} u_i \otimes v = \bigvee_{i\in I}(u_i \otimes v),$$

for all $u, v \in V$ and for all families $(u_i)_{i\in I}, (v_i)_{i\in I}$ of elements in V. A quantale **V** is denoted by a triple $\mathbf{V} = (V, \otimes, k)$.
A *lax homomorphism of quantales* $f : \mathbf{V} \to \mathbf{V}'$ is a monotone map $f : V \to V'$ of complete lattices such that

$$f(u) \otimes_{\mathbf{V}'} f(v) \leq f(u \otimes_{\mathbf{V}} v), \quad k_{\mathbf{V}'} \leq f(k_{\mathbf{V}}),$$

for all $u, v \in \mathbf{V}$.

Monotonicity of f means, equivalently, lax preservation of joins, i.e.,

$$\bigvee f(A) \leq f(\bigvee A)$$

for all $A \subseteq \mathbf{V}$.

Example 2.1.2

(1) $\mathbf{2} = (\{\bot, \top\}, \wedge, \top)$. It is the *two-chain* given by values *true* and *false*, where $k = \top$ is the top element and the tensor $\otimes$ is given by the meet $\wedge$.

(2) $\mathbf{P} = (P(M), \otimes, k)$. Given a monoid $(M, \cdot, 1_M)$, the complete lattice $(P(M), \subseteq)$ with tensor product $\otimes = \times$ defined by

$$A \times B = \{x \cdot y : x \in A, y \in B\}$$

for $A, B \subseteq M$, and unit $k = \{1_M\}$, defines a quantale.

(3) $\mathbf{R}_+ = ([0,\infty]^{\mathrm{op}}, +, 0)$. The complete half-real line $[0,\infty]$ is a complete lattice with its natural order $\leq$. We reverse it so that the top element $\top$ is 0 and the bottom $\bot$ is ∞. The tensor $\otimes$ is given by the addition extended via

$$u + \infty = \infty = \infty + u,$$

for all $u \in [0,\infty]$.

(4) $\mathbf{R}_* = ([0,\infty]^{\mathrm{op}}, *, 1)$. In (3) addition can be replaced by multiplication extended via

$$u * \infty = \infty = \infty * u,$$

for all $u \in [0,\infty]$.

(5) $\mathbf{R}_{\max} = ([0,\infty]^{\mathrm{op}}, \max, 0)$. In this case the complete half-real line $[0,\infty]$, with the reverse order $\geq$, is considered equipped with the tensor given by its meet operation which is the max with respect to the natural order $\leq$ of $[0,\infty]$.

(6) $\mathbf{I}_* = ([0,1], *, 1)$. The unit interval $[0,1]$ is a complete lattice with the usual order $\leq$. It is isomorphic to $[0,\infty]$ via the map

$$[0,1] \to [0,\infty], \quad u \mapsto -\ln(u),$$

where $-\ln(0) = \infty$. Under this isomorphism, the addition $+$ on $[0,\infty]$ corresponds to the multiplication $*$ on $[0,1]$.

(7) $\mathbf{I}_{\inf} = ([0,1], \inf, 1)$. The unit interval $[0,1]$ can be equipped with the tensor $\otimes = \wedge$ which is given by the infimum.

(8) $\mathbf{I}_{\oplus} = ([0,1], \oplus, 1)$. In this case the tensor is given by the *Łukasiewicz tensor* $\oplus$ defined by

$$u \oplus v = \max(0, u + v - 1),$$

for $u, v \in [0,1]$.

Quantale operations on $[0,1]$ are usually called *t-norms*. It is shown in [24] and [43] that every *continuous* t-norm $\otimes : [0,1] \times [0,1] \to [0,1]$ with neutral element 1 is a combination of the three operations on $[0,1]$ described above in (6), (7) and (8).

A quantale $\mathbf{V} = (V, \otimes, k)$ is said to be:

- *trivial*, if $|V| = 1$ or, equivalently, $\bot = k$;
- *commutative*, if it is commutative as a monoid i.e., $u \otimes v = v \otimes u$, for all $u, v \in \mathbf{V}$;
- *integral*, if the top element $\top$ coincides with the neutral element k;
- *totally ordered*, if the complete lattice V is totally ordered, i.e., $u \leq v$ or $v \leq u$, for all $u, v \in V$;
- *idempotent*, if each element is idempotent, i.e., $u \otimes u = u$, for all $u \in \mathbf{V}$;

A *frame* V is a complete lattice which satisfies the infinite distributive law

$$u \wedge \bigvee_{i \in I} v_i = \bigvee_{i \in I} (u \wedge v_i),$$

for all $u \in V$ and all families $(v_i)_{i \in I}$ of elements in V. Every frame becomes a commutative quantale when we put $\otimes = \wedge$ and $k = \top$. In fact, since for a commutative, integral and idempotent quantale $\mathbf{V} = (V, \otimes, k)$ one has $\otimes = \wedge$ (see [29, Exercise II.1.L]), one can identify frames as those commutative quantales which are integral and idempotent. In Example 2.1.2, (1), (5) and (7) are totally ordered frames, while (2) does not have any property mentioned above. It is commutative if the monoid M is commutative and it is integral when $M = 1$, the trivial monoid. The quantales $\mathbf{R}_+$, $\mathbf{I}_*$ and $\mathbf{I}_\oplus$ are commutative, integral and totally ordered but they are not idempotent. The quantale $\mathbf{R}_*$ in (4) is commutative and totally ordered but it is neither integral nor idempotent.

2.1.1 Completely distributive quantales

Let V be an ordered set, that is, a set equipped with a reflexive and transitive relation. For an element u in V,

$$\downarrow u = \{v \in V : v \leq u\}$$

is called the *down-set* of u in V. For a subset A of V, $\downarrow A = \bigcup_{u \in A} \downarrow u$ is the *down-closure* of A in V. We say that $A \subseteq V$ is *down-closed* if $\downarrow A = A$. Denoting by DnV the set

$$\mathrm{Dn} = \{A \subseteq V : \downarrow A = A\}$$

of down-closed sets of V, there is a full and faithful morphism

$$\downarrow : V \to \mathrm{Dn}V, \quad u \mapsto \downarrow u \tag{2.3}$$

where DnV is ordered by the inclusion. Another way to say that the ordered set V is complete is that the morphism (2.3) is right adjoint; equivalently, there is a morphism

$$\bigvee : \mathrm{Dn}V \to V \tag{2.4}$$

such that

$$\forall u \in V \quad (\bigvee S \leq u \iff S \subseteq \downarrow u),$$

for every $S \in \mathrm{Dn}V$.

Definition 2.1.3 A complete lattice V is called *completely distributive (cd)* if for each $\mathscr{A} \subseteq P(V)$

$$\bigwedge\{\bigvee A : A \in \mathscr{A}\} \cong \bigvee\{\bigwedge B : B \in \mathscr{A}^{\sharp}\}$$

where $\mathscr{A}^{\sharp}$ is the set of all subsets of the complete lattice V that have non-empty intersection with all members of $\mathscr{A}$.

Definition 2.1.4 A complete lattice V is called *constructively completely distributive (ccd)* if the left adjoint (2.4) has itself a left adjoint, i.e., if there is a morphism

$$\Downarrow : V \to \mathrm{Dn}V$$

such that

$$\Downarrow u \subseteq S \iff u \leq \bigvee S,$$

for all $u \in V$, $S \in \mathrm{Dn}V$.

As it has been shown in [45], the two notions of completely distributive and constructively completely distributive coincide if the Axiom of Choice (AC) is assumed.

Proposition 2.1.5 [45, Theorem 6.2]

$$(\mathrm{AC}) \iff ((cd) \Leftrightarrow (ccd)).$$

For $v \in \Downarrow u$, we write $v \ll u$ and we read v is *totally below* u. We have

$$v \ll u \iff \forall S \in \mathrm{Dn}V \quad (u \leq \bigvee S \Rightarrow v \in S)$$

or, equivalently,

$$v \ll u \iff \forall A \subseteq V \quad (u \leq \bigvee A \Rightarrow \exists w \in A : v \leq w).$$

The following properties follow:

1. $v \ll u \Rightarrow v \leq u$;
2. $v \leq v' \ll u' \leq u \Rightarrow v \ll u$;
3. $v \ll \bigvee A \Rightarrow \exists a \in A : v \ll a$;
4. $u = \bigvee\{v \in V : v \ll u\}$;
5. $u \ll \bigvee S \Rightarrow u \in S$, for all $S \in \mathrm{Dn}V$.

A quantale $\mathbf{V} = (V, \otimes, k)$ is said to be *(constructively) completely distributive ((c)cd)* if the complete lattice V is.

Proposition 2.1.6 [29, Proposition II.1.11.1] *If the complete lattice V allows for some relations $\sqsubset$ satisfying*

(1) $u \sqsubset v \leq w \Rightarrow u \sqsubset w$, *for all* u, v *and* w *in* V*;*

(2) $u \leq \bigvee\{v \in V : v \sqsubset u, v \sqsubset\text{-atomic}\}$,

then V *is ccd.*

Recall that an element $v \in V$ is $\sqsubset$*-atomic* if for all $A \subseteq V$, $v \sqsubset \bigvee A \Rightarrow \exists w \in A$ with $v \leq w$. All the quantales given in Example 2.1.2 are ccd. For the two-chain **2**, the totally below relation is given by $(v \ll u \Leftrightarrow u = \top)$. For quantales with complete lattices $[0,\infty]^{\mathrm{op}}$ and $[0,1]$, $(v \ll u \Leftrightarrow v > u)$ and $(v \ll u \Leftrightarrow v < u)$, respectively. For the lattice $P(M)$ one has $L \ll N \Leftrightarrow L = \{x\}$ for some $x \in N$, for $L, N \subseteq M$.

2.1.2 V-relations

In the category **Rel** a morphism is a relation $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ from a set X to a set Y. This can be seen as a map $r : X \times Y \to \mathbf{2}$, where $\mathbf{2} = (\{\bot, \top\}, \wedge, \top)$ is the two-chain given in Example 2.1.2. In fact, to each pair of elements $(x, y) \in X \times Y$, we give the value $\top$ (true) if x is in relation with y via r, or $\bot$ (false) otherwise. Therefore, to describe situations where quantitative informations are needed, one can ask for relations to take values in any quantale $\mathbf{V} = (V, \otimes, k)$ rather than just in **2**. We define then a **V**-*relation* $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ from the set X to the set Y to be a map

$$r : X \times Y \to \mathbf{V}.$$

Given two **V**-relations $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ and $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$, we can define the composition $s \cdot r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ by

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z), \tag{2.5}$$

for all $x \in X$ and $z \in Z$. One can easily verify that the composition defined above is associative. The **V**-relation $1_X : X \relbar\joinrel\mapstochar\joinrel\rightarrow X$, given by

$$(x, x') \mapsto \begin{cases} k, & \text{if } x = x', \\ \bot, & \text{otherwise,} \end{cases}$$

for $x, x' \in X$, acts as the identity morphism on X. Thus, sets and **V**-relations form a category denoted by **V**-**Rel**. We remark that, in the literature, this category is also denoted by **Mat**(**V**). The reason is that formula (2.7) can be interpreted as a "matrix multiplication". As one can expect, if $\mathbf{V} = \mathbf{2}$ then **2**-**Rel** $\cong$ **Rel**. The order of the quantale **V** induces a (pointwise) order on the hom-sets **V**-**Rel**(X, Y), for each pair of sets X and Y: given $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ and $r' : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, define

$$r \leq r' \iff \forall (x, y) \in X \times Y \quad (r(x, y) \leq r'(x, y)).$$

The order defined above inherits properties from the order on **V**: it is complete and allows the **V**-relational composition (2.7) to preserve suprema in each variable, i.e.,

$$s \cdot \bigvee_{i \in I} r_i = \bigvee_{i \in I} (s \cdot r_i), \quad \bigvee_{i \in I} r_i \cdot t = \bigvee_{i \in I} (r_i \cdot t),$$

for **V**-relations $(r_i : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y)_{i\in I}$, $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ and $t : W \relbar\joinrel\mapstochar\joinrel\rightarrow X$. Thus the category **V**-**Rel** is a 2-category or, more precisely, an ordered category.
For every **V**-relation $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, we can define the *opposite* (or *dual*) **V**-relation $r^{\mathrm{op}} : Y \relbar\joinrel\mapstochar\joinrel\rightarrow X$ by

$$r^{\mathrm{op}}(y,x) = r(x,y), \tag{2.6}$$

for all $y \in Y$ and $x \in X$. This is possible thanks to the isomorphism $\mathbf{V}\text{-}\mathbf{Rel}(X,Y) \cong \mathbf{V}\text{-}\mathbf{Rel}(Y,X)$, induced by the bijection between $X \times Y$ and $Y \times X$ for all sets X and Y. The opposite operation (2.6) has the following properties:

- $r \le r' \Rightarrow r^{\mathrm{op}} \le (r')^{\mathrm{op}}$, for all **V**-relations $r, r' : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$;
- $1_X^{\mathrm{op}} = 1_X$;
- $(r^{\mathrm{op}})^{\mathrm{op}} = r$;
- if **V** is commutative, $(s \cdot r)^{\mathrm{op}} = r^{\mathrm{op}} \cdot s^{\mathrm{op}}$, for all **V**-relations $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ and $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$.

2.1.3 From Set to V-Rel

A map $f : X \to Y$ can be interpreted as a **V**-relation $f : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ by

$$f(x,y) = \begin{cases} k, & \text{if } f(x) = y, \\ \bot, & \text{otherwise.} \end{cases}$$

We have then a functor

$$\mathbf{Set} \to \mathbf{V}\text{-}\mathbf{Rel} \tag{2.7}$$

which is faithful if and only if $\bot < k$ in **V**, i.e., **V** is not trivial. Therefore, from now on, we assume the quantale **V** to be non-trivial. The composition of **V**-relations, given by the formula (2.5), becomes much easier when maps, interpreted as **V**-relations, are involved:

1. $s \cdot f(x,z) = s(f(x),z)$,
2. $g \cdot r(x,z) = \bigvee_{y \in g^{-1}(z)} r(x,y)$,
3. $h^{\mathrm{op}} \cdot s(y,w) = s(y,h(w))$,
4. $t \cdot f^{\mathrm{op}}(y,z) = \bigvee_{x \in f^{-1}(y)} t(x,z)$,

for all maps $f : X \to Y$, $g : Y \to Z$, $h : W \to Z$, **V**-relations $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$, $t : X \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ and elements $x \in X$, $y \in Y$, $z \in Z$, $w \in W$. Moreover, composition of **V**-relations with maps is also compatible with the opposite operation (2.6), i.e.,

$$(s \cdot f)^{\mathrm{op}} = f^{\mathrm{op}} \cdot s^{\mathrm{op}} \quad \text{and} \quad (g \cdot r)^{\mathrm{op}} = r^{\mathrm{op}} \cdot g^{\mathrm{op}},$$

for all maps $f : X \to Y$, $g : Y \to Z$ and **V**-relations $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$. All the formulas above, where maps are involved, give the following inequalities in **V**-**Rel**, for every **Set**-morphism $f : X \to Y$:

$$1_X \le f^{\mathrm{op}} \cdot f \quad \text{and} \quad f \cdot f^{\mathrm{op}} \le 1_Y.$$

Moreover, given **Set**-morphisms $f : X \to Y$ and $g : Y \to Z$, we have the so called *adjunction rules*:

$$g \cdot r \leq t \iff r \leq g^{\mathrm{op}} \cdot t \quad \text{and} \quad t \cdot f^{\mathrm{op}} \leq s \iff t \leq s \cdot f, \tag{2.8}$$

for every **V**-relation $r : X {\longrightarrow\!\!\!\!\!\!\!\!\mapsto}\, Y$, $s : Y {\longrightarrow\!\!\!\!\!\!\!\!\mapsto}\, Z$ and $t : X {\longrightarrow\!\!\!\!\!\!\!\!\mapsto}\, Z$.

2.2 Lax extensions

In this section we are going to define *lax extensions* for a monad. In order to do that, we start by defining lax extensions for a functor. In particular, for our purpose, we restrict to **Set**-functors and **Set**-monads but we remark that, as in [10], this could be pursued in a more general setting.

Definition 2.2.1 Let **V** be a quantale and $T : \mathbf{Set} \to \mathbf{Set}$ be a functor. A *lax extension* of T to **V-Rel** is given by a lax functor

$$\widehat{T} : \mathbf{V}\text{-}\mathbf{Rel} \to \mathbf{V}\text{-}\mathbf{Rel}$$

which extends, laxly, the given **Set**-functor, i.e.,

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{T} & \mathbf{Set} \\ \downarrow & \geq & \downarrow \\ \mathbf{V}\text{-}\mathbf{Rel} & \xrightarrow[\widehat{T}]{} & \mathbf{V}\text{-}\mathbf{Rel} \end{array} \qquad \begin{array}{ccc} \mathbf{Set}^{\mathrm{op}} & \xrightarrow{T^{\mathrm{op}}} & \mathbf{Set}^{\mathrm{op}} \\ \downarrow & \geq & \downarrow \\ \mathbf{V}\text{-}\mathbf{Rel} & \xrightarrow[\widehat{T}]{} & \mathbf{V}\text{-}\mathbf{Rel} \end{array} \tag{2.9}$$

where the functor $\mathbf{Set} \to \mathbf{V}\text{-}\mathbf{Rel}$ is the embedding (2.7) and the functor $\mathbf{Set}^{\mathrm{op}} \to \mathbf{V}\text{-}\mathbf{Rel}$ is given by the opposite **V**-relation (2.6). The conditions above mean that

$$Tf \leq \widehat{T}f \quad \text{and} \quad (Tf)^{\mathrm{op}} \leq \widehat{T}(f^{\mathrm{op}}),$$

for all maps $f : X \to Y$.

Proposition 2.2.2 [29, Proposition III.1.4.3] *The following conditions are equivalent:*

(*i*) $Tf \leq \widehat{T}f$ *and* $(Tf)^{\mathrm{op}} \leq \widehat{T}(f^{\mathrm{op}})$ *for all* $f : X \to Y$*;*

(*ii*) $Tf \leq \widehat{T}f$ *and* $\widehat{T}(s \cdot f) = \widehat{T}s \cdot Tf$ *for all* $f : X \to Y$ *and* $s : Y {\longrightarrow\!\!\!\!\!\!\!\!\mapsto}\, Z$*;*

(*iii*) $(Tf)^{\mathrm{op}} \leq \widehat{T}(f^{\mathrm{op}})$ *and* $\widehat{T}(f^{\mathrm{op}} \cdot s) = (Tf)^{\mathrm{op}} \cdot \widehat{T}r$ *for all* $f : X \to Y$ *and* $r : Z {\longrightarrow\!\!\!\!\!\!\!\!\mapsto}\, Y$.

Corollary 2.2.3 [29, Corollary III.1.4.4] *For a lax extension* $\widehat{T} : \mathbf{V}\text{-}\mathbf{Rel} \to \mathbf{V}\text{-}\mathbf{Rel}$ *of a* **Set***-functor* T *one has*

$$\widehat{T}(s \cdot f) = \widehat{T}s \cdot \widehat{T}f = \widehat{T}s \cdot Tf, \quad \widehat{T}(f^{\mathrm{op}} \cdot r) = \widehat{T}(f^{\mathrm{op}}) \cdot \widehat{T}r = (Tf)^{\mathrm{op}} \cdot \widehat{T}r \tag{2.10}$$

for all maps $f : X \to Y$ *and* **V***-relations* $r : Z {\longrightarrow\!\!\!\!\!\!\!\!\mapsto}\, Y$, $s : Y {\longrightarrow\!\!\!\!\!\!\!\!\mapsto}\, Z$.

A lax extension $\widehat{T}$ of T is said to be *flat* if

$$\widehat{T}1_X = T1_X = 1_{TX},$$

i.e., if both diagrams (2.9) commute

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{T} & \mathbf{Set} \\ \downarrow & & \downarrow \\ \mathbf{V}\text{-}\mathbf{Rel} & \xrightarrow{\widehat{T}} & \mathbf{V}\text{-}\mathbf{Rel} \end{array} \qquad \begin{array}{ccc} \mathbf{Set}^{\mathrm{op}} & \xrightarrow{T^{\mathrm{op}}} & \mathbf{Set}^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \mathbf{V}\text{-}\mathbf{Rel} & \xrightarrow{\widehat{T}} & \mathbf{V}\text{-}\mathbf{Rel}. \end{array}$$

In fact, by Proposition 2.2.2, if $\widehat{T}$ is flat, we get

$$\widehat{T}f = \widehat{T}1_Y \cdot Tf = Tf \quad \text{and} \quad \widehat{T}(f^{\mathrm{op}}) = (Tf)^{\mathrm{op}} \cdot \widehat{T}1_X = (Tf)^{\mathrm{op}}$$

for all maps $f : X \to Y$.

Definition 2.2.4 A lax extension $\widehat{T}$ to **V-Rel** of a **Set**-functor T is said to be *left-whiskering* if

$$\widehat{T}(h \cdot r) = Th \cdot \widehat{T}r$$

for all $r : Y \to\!\!\!\!\!+ Z$, $h : Z \to W$. Similarly one says that $\widehat{T}$ is *right-whiskering* if

$$\widehat{T}(s \cdot f^{\mathrm{op}}) = \widehat{T}s \cdot (Tf)^{\mathrm{op}}$$

for all $f : X \to Y$, $s : X \to\!\!\!\!\!+ Z$.

Definition 2.2.5 Let $\mathbb{T} = (T, \mu, \eta)$ be a monad on **Set**. A *lax extension* $\widehat{\mathbb{T}}$ *of the monad* $\mathbb{T}$ is given by a lax extension $\widehat{T}$ of T, in the sense of Definition 2.2.1, which makes both $\mu : \widehat{T}^2 \to \widehat{T}$ and $\eta : 1_{\mathbf{V}\text{-}\mathbf{Rel}} \to \widehat{T}$ op-lax, i.e.,

(4) $\mu_Y \cdot \widehat{T}^2 r \leq \widehat{T}r \cdot \mu_X$,

(5) $\eta_Y \cdot r \leq \widehat{T}r \cdot \eta_X$,

for all **V**-relations $r : X \to\!\!\!\!\!+ Y$.

Conditions (4) and (5) of Definition 2.2.5, by the adjunction rules (2.8), can be equivalently expressed as:

(4′) $\widehat{T}^2 r \leq \mu_Y^{\mathrm{op}} \cdot \widehat{T}r \cdot \mu_X$,

(5′) $r \leq \eta_Y^{\mathrm{op}} \cdot \widehat{T}r \cdot \eta_X$,

for all **V**-relations $r : X \to\!\!\!\!\!+ Y$. Pointwise:

($4^\bullet$) $\widehat{T}^2 r(\mathfrak{X}, \mathfrak{Y}) \leq \widehat{T}r(\mu_X(\mathfrak{X}), \mu_Y(\mathfrak{Y}))$,

($5^\bullet$) $r(x,y) \leq \widehat{T}r(\eta_X(x), \eta_Y(y))$,

for all $x \in X$, $y \in Y$, $\mathfrak{X} \in T^2X$, $\mathfrak{Y} \in T^2Y$ and **V**-relations $r : X \to\!\!\!\!\!+ Y$.

A lax extension of a monad $\mathbb{T} = (T, \mu, \eta)$ will be denoted by $\widehat{\mathbb{T}} = (\widehat{T}, \mu, \eta)$. We say that a lax extension $\widehat{\mathbb{T}}$ of a monad $\mathbb{T}$ is *flat* if the lax extension $\widehat{T}$ of the functor T is flat. The same holds for *right-whiskering* and *left-whiskering* when referred to a lax extension. We point out that in literature, such as for example in [11], [14], [10] and [21], the authors may include flatness in the definition of lax extensions.

Example 2.2.6

(1) The identity monad $\mathbb{I} = (I,1,1)$ on **Set** has a flat lax extension given by the identity monad $\widehat{\mathbb{I}} = (I,1,1)$ on **V**-**Rel**.

(2) Each **Set**-monad $\mathbb{T}$ admits a largest lax extension to **V**-**Rel**, denoted by $\mathbb{T}^{\top}$, where the lax functor $T^{\top}$ is defined by

$$T^{\top}r : TX \times TY \to \mathbf{V}, \quad (\mathfrak{x},\mathfrak{y}) \mapsto \top$$

for all **V**-relations $r : X \rightarrow\!\!\!\!\!\!+\,\, Y$. This lax extension is not flat.

(3) If $\mathbf{V} = \mathbf{2}$, for the powerset monad $\mathbb{P} = (P,\mu,\eta)$, where $P : \mathbf{Set} \to \mathbf{Set}$ is the covariant powerset functor and $\eta : 1_{\mathbf{Set}} \to P$ and $\mu : P^2 \to P$ are natural transformations defined, componentwise, by

$$\eta_X : X \to PX, \quad x \mapsto \{x\},$$
$$\mu_X : P^2X \to PX, \quad \mathscr{A} \mapsto \bigcup \mathscr{A},$$

one can consider the lax extensions $\hat{\mathbb{P}}$, $\check{\mathbb{P}}$ with corresponding lax functor $\hat{P}, \check{P} : \mathbf{Rel} \to \mathbf{Rel}$ defined, respectively, by

$$\hat{P}r(A,B) = \begin{cases} \top, & \text{if} \quad \forall y \in B \quad \exists x \in A : r(x,y) = \top, \\ \bot, & \text{otherwise,} \end{cases}$$

$$\check{P}r(A,B) = \begin{cases} \top, & \text{if} \quad \forall x \in A \quad \exists y \in B : r(x,y) = \top, \\ \bot, & \text{otherwise,} \end{cases}$$

for every relation $r : X \rightarrow\!\!\!\!\!\!+\,\, Y$, and all $A \subseteq X$, $B \subseteq Y$. Both $\hat{\mathbb{P}}$ and $\check{\mathbb{P}}$ are not flat. On the other hand, the lax extension $\hat{\mathbb{P}}$ is left-whiskering but not right-whiskering, while the lax extension $\check{\mathbb{P}}$ behaves conversely.

2.2.1 The Barr extension

Given a relation $r : X \rightarrow\!\!\!\!\!\!+\,\, Y$, we denote by Γ_r its graph, i.e.,

$$\Gamma_r = \{(x,y) \in X \times Y : r(x,y) = \top\}.$$

Therefore we can represent r as a *span*

$$\begin{array}{ccc} & \Gamma_r & \\ {\scriptstyle \pi_X}\swarrow & & \searrow{\scriptstyle \pi_Y} \\ X & & Y \end{array}$$

where $\pi_X : \Gamma_r \to X$ and $\pi_Y : \Gamma_r \to Y$ are the restrictions to Γ_r of the product projections. We can write r as

$$r = \pi_Y \cdot \pi_X^{\mathrm{op}}$$

in **Rel**. Given a functor $T : \mathbf{Set} \to \mathbf{Set}$, the Barr extension $\overline{T} : \mathbf{Rel} \to \mathbf{Rel}$ of T, introduced by M. Barr in [1], is defined by

$$\overline{T}r := T\pi_Y \cdot (T\pi_X)^{\mathrm{op}}.$$

Pointwise, for elements $\mathfrak{x} \in TX$ and $\mathfrak{y} \in TY$, the Barr extension is given by

$$\overline{T}r(\mathfrak{x},\mathfrak{y}) = \begin{cases} \top, & \text{if } \exists \mathfrak{g} \in \Gamma_r : T\pi_X(\mathfrak{g}) = \mathfrak{x} \quad \& \quad T\pi_Y(\mathfrak{g}) = \mathfrak{y}, \\ \bot, & \text{otherwise.} \end{cases}$$

The following properties follow:

(1) $\overline{T}$ preserves the order on hom-sets, i.e., $r \leq r' \Rightarrow \overline{T}r \leq \overline{T}r'$, for all relations $r, r' : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$;

(2) $\overline{T}$ is an op-lax functor from **Rel** to **Rel**, i.e., $\overline{T}(s \cdot r) \leq \overline{T}s \cdot \overline{T}r$, for all relations $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ and $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$;

(3) $\overline{T}$ preserves the opposite operation, i.e., $\overline{T}(r^{\mathrm{op}}) = (\overline{T}r)^{\mathrm{op}}$, for all relations $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$;

(4) $\overline{T}$ extends the given functor T, i.e., $\overline{T}f = Tf$, for all maps $f : X \to Y$;

(5) $\overline{T}(g \cdot r) = Tg \cdot \overline{T}r$ and $\overline{T}(r \cdot f^{\mathrm{op}}) = \overline{T}r \cdot (Tf)^{\mathrm{op}}$, for all relations $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ and maps $f : A \to X$, $g : Y \to B$.

We have still to check whether the Barr extension is a lax extension, in the sense of Definition 2.2.1. In order to do that, we need the so called *Beck-Chevalley condition*. A commutative diagram in **Set**

$$\begin{array}{ccc} W & \xrightarrow{\;q\;} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[\;f\;]{} & Z \end{array} \tag{2.11}$$

is a *Beck-Chevalley square*, or simply a *BC-square*, if $q \cdot p^{\mathrm{op}} = g^{\mathrm{op}} \cdot f$, or equivalently, $p \cdot q^{\mathrm{op}} = f^{\mathrm{op}} \cdot g$.

Definition 2.2.7

(1) A **Set**-functor T satisfies the *Beck-Chevalley condition*, or simply *BC*, if it sends BC-squares to BC-squares, i.e.,

$$q \cdot p^{\mathrm{op}} = g^{\mathrm{op}} \cdot f \Rightarrow Tq \cdot (Tp)^{\mathrm{op}} = (Tg)^{\mathrm{op}} \cdot Tf$$

for all maps f, g, p, q as in diagram (2.11).

(2) A natural transformation $\alpha : S \to T$, between **Set**-functors S and T, satisfies BC if every naturality diagram

$$\begin{array}{ccc} SX & \xrightarrow{\alpha_X} & TX \\ {\scriptstyle Sf}\downarrow & & \downarrow{\scriptstyle Tf} \\ SY & \xrightarrow[\alpha_Y]{} & TY \end{array}$$

is a BC-square for all maps $f : X \to Y$.

Proposition 2.2.8 [29, Proposition III.1.11.3] *The following statements are equivalent for a* **Set**-*functor T:*

(i) *T satisfies BC;*

(ii) *T preserves weak pullback diagrams;*

(iii) *T transforms pullbacks into weak pullbacks and preserves the surjectivity of maps.*

The following result shows the key-role played by the Beck-Chevalley condition.

Theorem 2.2.9 [29, Theorem III.1.11.5] *For a functor T :* **Set** $\to$ **Set**, *the following assertions are equivalent:*

(i) *the functor T satisfies BC;*

(ii) *the Barr extension $\overline{T}$ is a flat lax extension of T to* **Rel** *and a functor $\overline{T}$:* **Rel** $\to$ **Rel***;*

(iii) *there is some functor $\widehat{T}$:* **Rel** $\to$ **Rel** *which is a lax extension of T to* **Rel**.

Moreover, any functor $\widehat{T}$: **Rel** $\to$ **Rel** *as in (iii) is uniquely determined, i.e., $\widehat{T} = \overline{T}$.*

Under the same assumptions, one can show that the Barr extension yields a lax extension of a **Set**-monad $\mathbb{T} = (T, \mu, \eta)$.

Theorem 2.2.10 [29, Theorem III.1.12.1] *For a monad $\mathbb{T} = (T, \mu, \eta)$ on* **Set**, *the following assertions are equivalent:*

(i) *the functor T satisfies BC;*

(ii) *the Barr extension yields a flat lax extension $\overline{\mathbb{T}} = (\overline{T}, \mu, \eta)$ of $\mathbb{T}$ to* **Rel**.

Example 2.2.11

(1) *Identity monad* - The Barr extension $\overline{1}_{\mathbf{Set}}$ of the identity functor $1_{\mathbf{Set}}$ on **Set** is the identity functor $1_{\mathbf{Rel}}$ on **Rel**. Of course the identity functor $1_{\mathbf{Set}}$ satisfies BC so that the Barr extension $1_{\mathbf{Rel}}$ is a flat lax extension of the identity monad $\mathbb{I} = (I, 1, 1)$.

(2) *Ultrafilter monad* - The ultrafilter functor $U : \mathbf{Set} \to \mathbf{Set}$ is defined by

$$\begin{array}{ccc} X & \longmapsto & UX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Uf} \\ Y & \longmapsto & UY \end{array}$$

where UX and UY are the sets of ultrafilters on the sets X and Y, respectively, and Uf is the map sending each ultrafilter $\mathfrak{x} \in UX$ to the ultrafilter $Uf(\mathfrak{x})$ generated by $\{f(A) : A \in \mathfrak{x}\}$. It satisfies BC so that its Barr extension $\overline{U}$ yields a flat lax extension to **Rel** of the ultrafilter monad $\mathbb{U} = (U, \mu, \eta)$. The functor U is the ultrafilter functor described above, while the natural transformations η and μ are defined, componentwise, by

$$\eta_X : X \to UX, \quad x \mapsto \{A \subseteq X : x \in A\}$$

$$\mu_X : U^2X \to UX, \quad \mathfrak{X} \mapsto \bigcup_{\mathscr{A} \in \mathfrak{X}} \bigcap_{\mathfrak{x} \in \mathscr{A}} \mathfrak{x}.$$

The functor $\overline{U} : \mathbf{Rel} \to \mathbf{Rel}$ is given by

$$\overline{U}r(\mathfrak{x}, \mathfrak{y}) = \begin{cases} \top, & \text{if } \forall B \in \mathfrak{y} \quad \{x \in X \quad | \quad \exists y \in B : r(x,y) = \top\} \in \mathfrak{x}, \\ \bot, & \text{otherwise,} \end{cases}$$

for each relation $r : X \relbar\joinrel\mapsto Y$ and $\mathfrak{x} \in UX, \mathfrak{y} \in UY$. We remark that the natural transformation μ satisfies BC and remains strict (in the sense that $\mu_Y \cdot \overline{U}^2 r = \overline{U}r \cdot \mu_X$ for any relation $r : X \relbar\joinrel\mapsto Y$), while η does not satisfy BC. However, the naturality diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & UX \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle \overline{U}r} \\ Y & \xrightarrow[\eta_Y]{} & UY \end{array}$$

is a BC-square, i.e., $(\overline{U}r)^{\mathrm{op}} \cdot \eta_Y = \eta_X \cdot r^{\mathrm{op}}$, provided that the relation r has *finite fibres* in the sense that, for every $y \in Y$, the set $\{x \in X : r(x,y) > \bot\}$ is finite.

(3) *M-ordered monad* - Let $M = (M, \cdot, 1_M)$ be a monoid. Consider the functor $M^\times = M \times - : \mathbf{Set} \to \mathbf{Set}$ given by

$$\begin{array}{ccc} X & \longmapsto & M \times X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle M \times f} \\ Y & \longmapsto & M \times Y \end{array}$$

where $(M \times f)(m,x) = (m, f(x))$, for each $m \in M$ and $x \in X$. This functor satisfies the Beck-Chevalley condition so that its Barr extension yields a flat lax extension to **Rel** of the M-ordered monad $\mathbb{M} = (M^\times, \mu, \eta)$. The natural transformations η and μ are defined by

$$\eta_X : X \to M \times X, \quad x \mapsto (1_M, x),$$

$$\mu_X : M \times M \times X \to M \times X, \quad (m,n,x) \mapsto (n \cdot m, x),$$

and the functor $\overline{M}^{\times} : \mathbf{Rel} \to \mathbf{Rel}$ is given by

$$\overline{M}^{\times} r((m,x),(n,y)) = \begin{cases} \top, & \text{if } m = n \quad \& \quad r(x,y) = \top, \\ \bot, & \text{otherwise,} \end{cases}$$

for each relation $r : X \to\!\!\!\!\mapsto Y$ and $m,n \in M, x \in X, y \in Y$.

(4) *Free-monoid monad* - The free-monoid monad $\mathbb{W} = (W, \mu, \eta)$ is given by the functor $W : \mathbf{Set} \to \mathbf{Set}$, which assigns to each set X the set WX of all finite words $(x_1, \ldots, x_n)(n \in \mathbb{N})$ of elements of X and to each map $f : X \to Y$ the **Set**-morphism $Wf : WX \to WY$ defined by

$$(x_1, \ldots, x_n) \mapsto (f(x_1), \ldots, f(x_n)).$$

The natural transformation $\eta : 1_{\mathbf{Set}} \to W$ is defined, elementwise, by

$$\eta_X : X \to WX, \quad x \mapsto (x),$$

while the components $\mu_X : W^2X \to WX$ of μ are given by removing the inner brackets. The functor W is a *cartesian functor*, that is, it sends pullback diagrams to pullback diagrams. Moreover, the naturality squares of η and μ are pullback squares making then the free-monoid monad $\mathbb{W}$ a *cartesian monad*. By Proposition 2.2.8, the BC-property is satisfied and the Barr extension $\overline{\mathbb{W}} = (\overline{W}, \eta, \mu)$, where $\overline{W} : \mathbf{Rel} \to \mathbf{Rel}$ is given by

$$\overline{W} r((x_1, \ldots, x_n), (y_1, \ldots, y_m)) = \begin{cases} \top, & \text{if } m = n \quad \& \quad r(x_i, y_i) = \top \text{ for } i = 1, \ldots, n, \\ \bot, & \text{otherwise,} \end{cases}$$

for each relation $r : X \to\!\!\!\!\mapsto Y, (x_1, \ldots, x_n) \in WX, (y_1, \ldots, y_m) \in WY$, yields a flat lax extension of $\mathbb{W}$ to **Rel**.

(5) *Powerset monad* - In Example 2.2.6 we already introduced two different (non-flat) lax extensions of the powerset monad $\mathbb{P} = (P, \mu, \eta)$. But, since the functor $P : \mathbf{Set} \to \mathbf{Set}$ has BC, also the Barr extension yields a flat lax extension $\overline{\mathbb{P}} = (\overline{P}, \mu, \eta)$ of $\mathbb{P}$ to **Rel**. The lax functor $\overline{P} : \mathbf{Rel} \to \mathbf{Rel}$ is defined by

$$\overline{P} r(A,B) = \begin{cases} \top, & \text{if } \forall x \in A \quad \exists y \in B : r(x,y) = \top \ \& \ \forall y \in B \quad \exists x \in A : r(x,y) = \top, \\ \bot, & \text{otherwise,} \end{cases}$$

for each relation $r : X \to\!\!\!\!\mapsto Y$ and $A \subseteq X$, $B \subseteq Y$.

2.2.2 Uniform construction of lax extensions to V-Rel

In [12] M.M. Clementino and D. Hofmann presente a uniform construction of an extension of a **Set**-monad, satisfying BC, into a lax monad of the 2-category **V**-**Rel**. By a *lax monad* $\mathbb{T} = (T, \mu, \eta)$ on **V**-**Rel** they mean a lax functor $T : \mathbf{V}\text{-}\mathbf{Rel} \to \mathbf{V}\text{-}\mathbf{Rel}$ for which the op-lax natural transformations

$\eta : 1_{\mathbf{V}\text{-}\mathbf{Rel}} \to T$ and $\mu : T^2 \to T$ satisfy, for every set X,

$$\begin{array}{ccc} T^3X & \xrightarrow{\mu_{TX}} & T^2X \\ {\scriptstyle T\mu_X}\downarrow & \leq & \downarrow{\scriptstyle \mu_X} \\ T^2X & \xrightarrow[\mu_X]{} & TX \end{array} \qquad \begin{array}{ccccc} TX & \xrightarrow{T\eta_X} & T^2X & \xleftarrow{\eta_{TX}} & TX \\ & {\scriptstyle 1_{TX}}\searrow \;\leq & \downarrow{\scriptstyle \mu_X} & \leq\; \swarrow{\scriptstyle T1_X} & \\ & & TX. & & \end{array}$$

The construction is given in three steps (1), (2) and (3) which we can sum up with the following diagram

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{T} & \mathbf{Set} \\ \downarrow & (1) & \downarrow \\ \mathbf{Rel} & \xrightarrow[\overline{T}]{} & \mathbf{Rel} \\ {\scriptstyle E}\downarrow & (2) & \downarrow{\scriptstyle E} \\ \mathbf{2}^{\mathbf{V}^{\mathrm{op}}}\text{-}\mathbf{Rel} & \xrightarrow{\widehat{\overline{T}}} & \mathbf{2}^{\mathbf{V}^{\mathrm{op}}}\text{-}\mathbf{Rel} \\ {\scriptstyle L}\downarrow & (3) & \downarrow{\scriptstyle L} \\ \mathbf{V}\text{-}\mathbf{Rel} & \xrightarrow[\widehat{\widehat{\overline{T}}}]{} & \mathbf{V}\text{-}\mathbf{Rel}. \end{array}$$

Starting with a **Set**-monad $\mathbb{T} = (T, \mu, \eta)$, the first step (1) consists of the Barr extension $\overline{T}$ of T to **Rel**. We know, by Theorem 2.2.10, that if T satisfies BC then $\overline{T}$ yields a flat lax extensions $\overline{\mathbb{T}}$ of $\mathbb{T}$ to **Rel**. The intermediate step (2) is given by the construction of an extension $\widehat{\overline{T}} : \mathbf{2}^{\mathbf{V}^{\mathrm{op}}}\text{-}\mathbf{Rel} \to \mathbf{2}^{\mathbf{V}^{\mathrm{op}}}\text{-}\mathbf{Rel}$

$$X \mapsto \widehat{\overline{T}}X := TX, \quad r \mapsto \widehat{\overline{T}}a(\mathfrak{x}, \mathfrak{y})(v) := \overline{T}(r_v)(\mathfrak{x}, \mathfrak{y})$$

for every set X, every $\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}$-relation $r : X \longrightarrow\!\!\!\!\!\!\!\mapsto Y$, $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$ and $v \in \mathbf{V}$, where $r_v : X \longrightarrow\!\!\!\!\!\!\!\mapsto Y$ is given by

$$r_v(x, y) = r(x, y)(v)$$

for every $(x, y) \in X \times Y$. The functor $E : \mathbf{Rel} \to \mathbf{2}^{\mathbf{V}^{\mathrm{op}}}\text{-}\mathbf{Rel}$ is induced by the embedding

$$E : \mathbf{2} \to \mathbf{2}^{\mathbf{V}^{\mathrm{op}}}$$

defined by $u \mapsto E(u) : \mathbf{V}^{\mathrm{op}} \to \mathbf{2}$ where

$$v \mapsto E(u)(v) = \begin{cases} u, & \text{if } v \leq k_{\mathbf{V}}, \\ \bot, & \text{otherwise.} \end{cases}$$

In general the lax functor $\widehat{\overline{T}}$ extents only laxly the Barr extension $\overline{T}$, i.e.,

$$\begin{array}{ccc} \mathbf{Rel} & \xrightarrow{\overline{T}} & \mathbf{Rel} \\ {\scriptstyle E}\downarrow & \geq & \downarrow{\scriptstyle E} \\ \mathbf{2}^{\mathbf{V}^{\mathrm{op}}}\text{-}\mathbf{Rel} & \xrightarrow[\widehat{\overline{T}}]{} & \mathbf{2}^{\mathbf{V}^{\mathrm{op}}}\text{-}\mathbf{Rel} \end{array} \qquad (2.12)$$

but if $\mathbf{V}$ is integral or $\overline{T}$ preserves the $\bot$-relation, as it is proved in [12, Theorem 4.1], $\widehat{\overline{T}}$ strictly extends $\overline{T}$, i.e., the diagram (2.12) is commutative.

Step (3) is given by the construction of the lax functor $\widetilde{\widehat{\overline{T}}} : \mathbf{V}\text{-}\mathbf{Rel} \to \mathbf{V}\text{-}\mathbf{Rel}$. This is defined by the following composition

$$\mathbf{V}\text{-}\mathbf{Rel} \xrightarrow{\widetilde{\widehat{\overline{T}}}} \mathbf{V}\text{-}\mathbf{Rel} := \mathbf{V}\text{-}\mathbf{Rel} \xrightarrow{Y} \mathbf{2}^{\mathbf{V}^{\mathrm{op}}}\text{-}\mathbf{Rel} \xrightarrow{\widehat{\overline{T}}} \mathbf{2}^{\mathbf{V}^{\mathrm{op}}}\text{-}\mathbf{Rel} \xrightarrow{L} \mathbf{V}\text{-}\mathbf{Rel}$$

where Y is the functor induced by the Yoneda embedding $Y : \mathbf{V} \to \mathbf{2}^{\mathbf{V}^{\mathrm{op}}}$

$$v \mapsto Y(v) : \mathbf{V}^{\mathrm{op}} \to \mathbf{2}, \quad Y(v)(u) = \top \iff u \leq v,$$

and L is the functor induced by the left adjoint $L : \mathbf{2}^{\mathbf{V}^{\mathrm{op}}} \to \mathbf{V}$ of Y

$$f \mapsto \bigvee\{v \in \mathbf{V} : f(v) = \top\}.$$

Also in this case it happens that the extension $\widetilde{\widehat{\overline{T}}}$ is only lax with respect to $\widehat{\overline{T}}$, i.e.,

$$\begin{array}{ccc} \mathbf{2}^{\mathbf{V}^{\mathrm{op}}}\text{-}\mathbf{Rel} & \xrightarrow{\widehat{\overline{T}}} & \mathbf{2}^{\mathbf{V}^{\mathrm{op}}}\text{-}\mathbf{Rel} \\ {\scriptstyle L}\downarrow & \geq & \downarrow{\scriptstyle L} \\ \mathbf{V}\text{-}\mathbf{Rel} & \xrightarrow[\widetilde{\widehat{\overline{T}}}]{} & \mathbf{V}\text{-}\mathbf{Rel} \end{array} \tag{2.13}$$

but, if $\mathbf{V}$ is ccd, the extension becomes strict. Summing up we have the following result:

Theorem 2.2.12 [12, Corollary 5.3] *Let (T,μ,η) be a monad in* **Set**. *If T satisfies BC,* $\mathbf{V}$ *is ccd and $k_{\mathbf{V}} = \top_{\mathbf{V}}$ or $\overline{T}$ preserves the $\bot$-relation, then $(\widetilde{\widehat{\overline{T}}},\widetilde{\widehat{\overline{\mu}}},\widetilde{\widehat{\overline{\eta}}})$ is a lax monad in* $\mathbf{V}$-**Rel** *that extends the given one.*

Hence, under the hypotheses of Theorem 2.2.12, a **Set**-monad $\mathbb{T}$ admits a flat lax extension $\widetilde{\mathbb{T}} = \widetilde{\widehat{\overline{\mathbb{T}}}}$ to $\mathbf{V}$-**Rel**, in the sense of Definition 2.2.1. The extension $\widetilde{T} : \mathbf{V}\text{-}\mathbf{Rel} \to \mathbf{V}\text{-}\mathbf{Rel}$ can described by the following formula: for $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$ and every $\mathbf{V}$-relation $r : X \nrightarrow Y$,

$$\widetilde{T}r(\mathfrak{x},\mathfrak{y}) = \bigvee\{v \in \mathbf{V} : \overline{T}r_v(\mathfrak{x},\mathfrak{y}) = \top\}, \tag{2.14}$$

where $r_v : X \nrightarrow Y$ is the relation defined by

$$r_v(x,y) = \begin{cases} \top, & \text{if } v \leq r(x,y) \\ \bot, & \text{otherwise.} \end{cases}$$

The extension $\widetilde{\mathbb{T}}$ of $\mathbb{T}$ to $\mathbf{V}$-**Rel**, given by (2.14), will be called *uniform extension*.

Example 2.2.13

(1) *Identity monad* - The uniform extension $\widetilde{\mathbb{I}}$ to $\mathbf{V}$-**Rel** of the identity monad $\mathbb{I} = (I,1,1)$ is itself the identity monad on $\mathbf{V}$-**Rel**.

(2) *Ultrafilter monad* - The uniform extension $\widetilde{\mathbb{U}} = (\widetilde{U}, \mu, \eta)$ on **V**-**Rel** of the ultrafilter monad $\mathbb{U}$ is given by

$$\widetilde{U}(\mathfrak{x}, \mathfrak{y}) = \bigwedge_{A \in \mathfrak{x}, B \in \mathfrak{y}} \bigvee_{x \in A, y \in B} r(x,y),$$

for $\mathfrak{x} \in UX$, $\mathfrak{y} \in UY$ and every **V**-relation $r : X \to\!\!\!\!\!\!\mapsto Y$.

(3) *M-ordered monad* - Let $\mathbb{M} = (M^\times, \mu, \eta)$ be the M-ordered monad. Its uniform extension $\widetilde{\mathbb{M}}$ to **V**-**Rel** is described by the lax functor $\widetilde{M}^\times$: **V**-**Rel** $\to$ **V**-**Rel** given by

$$\widetilde{M}^\times r((m,x),(n,y)) = \begin{cases} r(x,y), & \text{if } m = n, \\ \bot_{\mathbf{V}}, & \text{otherwise,} \end{cases}$$

for $m, n \in M$, $x \in X$, $y \in Y$ and every **V**-relation $r : X \to\!\!\!\!\!\!\mapsto Y$.

(4) *Free-monoid monad* - The lax functor $\widetilde{W}$: **V**-**Rel** $\to$ **V**-**Rel** of the uniform extension $\widetilde{\mathbb{W}} = (\widetilde{W}, \mu, \eta)$ to **V**-**Rel** of the free-monoid monad $\mathbb{W}$ is given by

$$\widetilde{W} r((x_1, \ldots, x_n),(y_1, \ldots, y_m)) = \begin{cases} \bigwedge_{i=1}^n r(x_i, y_i), & \text{if } m = n \\ \bot_{\mathbf{V}}, & \text{otherwise,} \end{cases} \tag{2.15}$$

for $(x_1, \ldots, x_n) \in WX$, $(y_1, \ldots, y_m) \in WY$ and every **V**-relation $r : X \to\!\!\!\!\!\!\mapsto Y$.

(5) *Powerset monad* - For the powerset monad $\mathbb{P} = (P, \mu, \eta)$ the lax extension $\widetilde{\mathbb{P}} = (\widetilde{P}, \mu, \eta)$ is described by the lax functor $\widetilde{P}$: **V**-**Rel** $\to$ **V**-**Rel** given by

$$\widetilde{P}r(A,B) = \bigvee \{ v \in \mathbf{V} | \forall x \in A \quad \exists y \in B : v \leq r(x,y) \quad \& \quad \forall y \in B \quad \exists x \in A : v \leq r(x,y) \},$$

for $A \subseteq X$, $B \subseteq Y$ and every **V**-relation $r : X \to\!\!\!\!\!\!\mapsto Y$.

In all the examples above, to have the uniform extension to **V**-**Rel**, we only need **V** ccd, since all the **Set**-functors involved satisfy BC and all the corresponding Barr extensions to **Rel** preserve the $\bot$-relation.

2.2.3 Lax extensions in terms of algebra structures

Let $\mathbb{T} = (T, \mu, \eta)$ be a monad on **Set** and let $\mathbf{V} = (V, \otimes, k)$ be a quantale. Assume that **V** is ccd and that both T and μ satisfy BC. In [13] M.M. Clementino and D. Hofmann showed the link between the uniform extension $\widetilde{T}$ of T, defined by (2.14), and the $\mathbb{T}$-algebra structure on V

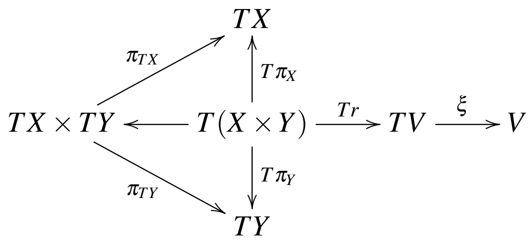
$$\xi : TV \to V, \quad \mathfrak{v} \mapsto \bigvee \{ v \in \mathbf{V} : \mathfrak{v} \in T(\uparrow v) \},$$

where $\uparrow v = \{ u \in \mathbf{V} : v \leq u \}$, introduced by Manes in [40].

Proposition 2.2.14 [13, Proposition 4.1] *For any* **V**-*relation* $r : X \to\!\!\!\!\!\!\mapsto Y$*, each* $\mathfrak{x} \in TX$ *and* $\mathfrak{y} \in TY$*,*

$$\widetilde{T} r(\mathfrak{x}, \mathfrak{y}) = \bigvee_{\substack{\mathfrak{w} \in T(X \times Y): \\ T\pi_X(\mathfrak{w}) = \mathfrak{x} \\ T\pi_Y(\mathfrak{w}) = \mathfrak{y}}} \xi \cdot Tr(\mathfrak{w}). \tag{2.16}$$

The extension above can be pictured by the following diagram



where the map $T(X \times Y) \to TX \times TY$, induced by the universal property of the product, is surjective since T has BC (see Proposition 2.2.8). Formula (2.16) actually establishes a more general link between lax extensions of a **Set**-monad $\mathbb{T}$ and $\mathbb{T}$-algebra structures, in the sense that each $\mathbb{T}$-algebra structure $\xi : TV \to V$ can define a lax extension to **V**-**Rel** of $\mathbb{T}$ via formula (2.16). Examples of lax extensions in terms of algebra structures are given when we consider the free-monoid monad $\mathbb{W} = (W, \mu, \eta)$. By Proposition 2.2.14, the uniform extension $\widetilde{W}$ given by (2.15) in Example 2.2.13 is then described in terms of the $\mathbb{W}$-algebra structure

$$\xi^{\wedge} : WV \to V, \quad (u_1, \dots, u_n) \mapsto \bigwedge_{i=1}^{n} u_i.$$

In [21] M.M. Clementino and W. Tholen presented a lax extension $W^{\otimes}$ to **V**-**Rel** of the free-monoid monad described by

$$W^{\otimes} r((x_1,\dots,x_n),(y_1,\dots,y_m)) = \begin{cases} \bigotimes_{i=1}^{n} r(x_i,y_i), & \text{if } m = n, \\ \bot_{\mathbf{V}}, & \text{otherwise,} \end{cases}$$

for $(x_1,\dots,x_n) \in WX$, $(y_1,\dots,y_m) \in WY$ and every **V**-relation $r : X \relbar\!\!\!\mapsto Y$. This extension can be given as well in terms of the $\mathbb{W}$-algebra structure

$$\xi^{\otimes} : WV \to V, \quad (u_1, \dots, u_n) \mapsto \bigotimes_{i=1}^{n} u_i$$

using the formula (2.16). In order to keep in mind which $\mathbb{W}$-algebra structure we refer to, we denote the extension $\widetilde{W}$ by $W^{\wedge}$. In the next chapter we will see the relation occurring between these two extensions.

2.3 $(\mathbb{T}, \mathbf{V})$-categories

Let $\mathbf{V} = (V, \otimes, k)$ be a quantale and let $\widehat{\mathbb{T}} = (\widehat{T}, \mu, \eta)$ be a lax extension to **V**-**Rel** of a **Set**-monad $\mathbb{T} = (T, \mu, \eta)$. By a $(\mathbb{T}, \mathbf{V})$-*relation* we mean just a **V**-relation $a : TX \relbar\!\!\!\mapsto X$, where the domain is given by the image TX of the set X, i.e., a map $a : TX \times X \to \mathbf{V}$. A $(\mathbb{T}, \mathbf{V})$-relation $a : TX \relbar\!\!\!\mapsto X$ is said to be *reflexive* if

$$1_X \leq a \cdot \eta_X \quad \text{or equivalently} \quad \eta_X^{\mathrm{op}} \leq a,$$

which in pointwise notation can be expressed by

$$k \le a(\eta_X(x), x)$$

for all $x \in X$. A $(\mathbb{T},\mathbf{V})$-relation $a : TX \relbar\joinrel\mapstochar\joinrel\rightarrow X$ is *transitive* if

$$a \cdot \widehat{T}a \le a \cdot \mu_X \quad \text{or equivalently} \quad a \cdot \widehat{T}a \cdot \mu_X^{\mathrm{op}} \le a,$$

which in pointwise notation is given by

$$\widehat{T}a(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \le a(\mu_X(\mathfrak{X}), x)$$

for all $\mathfrak{X} \in T^2X$, $\mathfrak{x} \in TX$ and $x \in x$. For a $(\mathbb{T},\mathbf{V})$-relation $a : TX \relbar\joinrel\mapstochar\joinrel\rightarrow X$, reflexivity and transitivity can be expressed, in a compact way, by the following diagram

$$\begin{array}{ccccc} X & \xrightarrow{\eta_X} & TX & \xleftarrow{\widehat{T}a} & T^2X \\ & \searrow^{1_X} \; \le & \downarrow a & \le & \downarrow \mu_X \\ & & X & \xleftarrow[a]{} & TX \end{array} \tag{2.17}$$

where the left-side triangle represents the reflexivity property, while the right-side square the transitivity property. In case $\mathbf{V} = \mathbf{2}$, we will often use the *arrow notation* $\mathfrak{x} \to x$ to express $a(\mathfrak{x}, x) = \top$, for $\mathfrak{x} \in TX$ and $x \in X$. Therefore, for instance, the reflexivity and the transitivity properties can be given, respectively, by

$$\eta_X(x) \to x, \qquad \mathfrak{X} \to \mathfrak{x} \to x \Rightarrow \mu_X(\mathfrak{X}) \to x,$$

for all $\mathfrak{X} \in T^2X, \mathfrak{x} \in TX$ and $x \in X$.

Definition 2.3.1 A $(\mathbb{T},\mathbf{V})$-*category* is a pair (X, a) where X is a set and $a : TX \relbar\joinrel\mapstochar\joinrel\rightarrow X$ is a reflexive and transitive $(\mathbb{T},\mathbf{V})$-relation.

We remark that in the literature one may refer to $(\mathbb{T},\mathbf{V})$-categories also as *lax algebras* or $(\mathbb{T},\mathbf{V})$-*algebras* since conditions (2.17) represent lax conditions for an Eilenberg-Moore algebra. Also, in case $\mathbf{V} = \mathbf{2}$, one can refer to *relational algebras*. We also point out the fact that Definition 2.3.1 depends, for what concerns the transitivity part, on the lax extension $\widehat{\mathbb{T}}$, so that, in some cases, we allow us to speak of $(\mathbb{T},\mathbf{V},\widehat{\mathbb{T}})$-categories when we want to stress which lax extension we refer to.

Definition 2.3.2 A morphism $f : (X, a) \to (Y, b)$ between $(\mathbb{T},\mathbf{V})$-categories is said to be a $(\mathbb{T},\mathbf{V})$-*functor* if

$$f \cdot a \le b \cdot Tf \quad \text{or equivalently} \quad a \le f^{\mathrm{op}} \cdot b \cdot Tf,$$

which in pointwise notation is given by

$$a(\mathfrak{x}, x) \le b(Tf(\mathfrak{x}), f(x))$$

for all $\mathfrak{x} \in TX$ and $x \in X$.

The condition above can be expressed by the following diagram

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle a}\big\downarrow\!\!\!\!\!+ & \leq & +\!\!\!\!\!\big\downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y, \end{array} \tag{2.18}$$

which represents a lax condition for a morphism to be a homomorphism of (lax) Eilenberg-Moore algebras. In case $\mathbf{V} = \mathbf{2}$, using the arrow notation, the condition expressed by diagram (2.18) is given by

$$\mathfrak{x} \to x \Rightarrow Tf(\mathfrak{x}) \to f(x),$$

for all $\mathfrak{x} \in TX$ and $x \in X$. Having $(\mathbb{T},\mathbf{V})$-categories as objects and $(\mathbb{T},\mathbf{V})$-functors as morphisms, one forms a category denoted by

$$(\mathbb{T},\mathbf{V})\text{-}\mathbf{Cat}.$$

In this way one can define a (full) subcategory of $(\mathbb{T},\mathbf{V})$-**Cat**: the subcategory $(\mathbb{T},\mathbf{V})$-**Gph** of $(\mathbb{T},\mathbf{V})$-*graphs* given by reflexive $(\mathbb{T},\mathbf{V})$-relations and $(\mathbb{T},\mathbf{V})$-functors.

Example 2.3.3

(1) *Identity monad* - For the lax extension to **V**-**Rel** given by the identity functor $I : \mathbf{V}\text{-}\mathbf{Rel} \to \mathbf{V}\text{-}\mathbf{Rel}$, $(\mathbb{I},\mathbf{V})$-categories and $(\mathbb{I},\mathbf{V})$-functors give the category **V**-**Cat** (**V**-**Gph**) of **V**-*categories* (**V**-graphs) and **V**-*functors*. If $\mathbf{V} = \mathbf{2}$ then **2**-**Cat** is isomorphic to the category **Ord** of *(pre)ordered sets* and *monotone maps.* When $\mathbf{V} = \mathbf{R}_+$ then, as pointed out in the general paper [35] of F.W. Lawvere, $\mathbf{R}_+$-**Cat** is the category of *(generalized) metric spaces* and *non-expansive maps*, while for $\mathbf{V} = \mathbf{I}_{\text{inf}}$, $\mathbf{I}_{\text{inf}}$-**Cat** is isomorphic to the category of *(generalized) ultrametric spaces* and *non-expansive maps.*

(2) *Ultrafilter monad* - For $\mathbf{V} = \mathbf{2}$ M. Barr in [1] proved that $(\mathbb{U},\mathbf{2},\overline{\mathbb{U}})$-**Cat**, where $\overline{\mathbb{U}}$ is the Barr extension given in Example 2.2.11, is isomorphic to the category **Top** of topological spaces continuous maps (see diagram (2.2) in the beginning of the chapter). When $\mathbf{V} = \mathbf{R}_+$, for the uniform extension $\widetilde{\mathbb{U}}$ given in Example 2.2.13, we have that $(\mathbb{U},\mathbf{R}_+,\widetilde{\mathbb{U}})$-**Cat** is isomorphic to the category **App** of *approach spaces* and *non-expansive maps* (the description of approach spaces as lax algebras was established in [10] by M.M. Clementino and D. Hofmann).

(3) *M-ordered monad* - For a given monoid $M = (M,\cdot,1_M)$, the Barr extension $\overline{\mathbb{M}}$ of the M-ordered monad $\mathbb{M}$ given in Example 2.2.11 gives rise to the category $(\mathbb{M},\mathbf{2},\overline{\mathbb{M}})$-**Cat** of *M-ordered sets* and *equivariant maps*, usually denoted simply by M-**Ord**. For a $(\mathbb{M},\mathbf{2})$-relation $a : (M \times X) \relbar\!\!\!+\!\!\!\rightarrow X$ one can write $x \xrightarrow{m} y$ instead of $a((m,x),y) = \top$, i.e., x is in relation with y with weight m.

(4) *Free-monoid monad* - The reflexivity and the transitivity properties of a $(\mathbb{W},\mathbf{2},\overline{\mathbb{W}})$-category (X,a), where $\overline{\mathbb{W}}$ is the Barr extension described in Example 2.2.11, are given, respectively, by

$$(x) \to x,$$

for all $x \in X$, and

$$((x_1^1,\ldots,x_{m_1}^1),\ldots,(x_1^n,\ldots,x_{m_n}^n)) \to (x_1,\ldots,x_n) \to x_0 \Rightarrow (x_1^1,\ldots,x_{m_1}^1,\ldots,x_1^n,\ldots,x_{m_n}^n) \to x_0,$$

for all $((x_1^1,\ldots,x_{m_1}^1),\ldots,(x_1^n,\ldots,x_{m_n}^n)) \in W^2X$, $(x_1,\ldots,x_n) \in WX$ and $x_0 \in X$. A $(\mathbb{W},\mathbf{2},\overline{\mathbb{W}})$-functor $f:(X,a)\to(Y,b)$ is a map $f:X\to Y$ such that

$$(x_1,\ldots,x_n) \to x_0 \Rightarrow (f(x_1),\ldots,f(x_n)) \to f(x_0),$$

for all $(x_1,\ldots,x_n) \in WX$ and $x_0 \in X$. The category $(\mathbb{W},\mathbf{2},\overline{\mathbb{W}})$-**Cat** is usually denoted by **MultiOrd**.

(5) *Powerset monad* - For $\mathbf{V}=\mathbf{2}$, in Example 2.2.6, we introduced the (non-flat) lax extensions $\hat{\mathbb{P}}$ and $\check{\mathbb{P}}$ of the powerset monad $\mathbb{P}=(P,\mu,\eta)$. We have the following isomorphisms

$$(\mathbb{P},\mathbf{2},\check{\mathbb{P}})\text{-}\mathbf{Cat} \cong \mathbf{Ord} \quad (\mathbb{P},\mathbf{2},\hat{\mathbb{P}})\text{-}\mathbf{Cat} \cong \mathbf{Cls},$$

where **Cls** is the category of *closure spaces* and *continuous maps.*

(6) For the lax extension $\mathbb{T}^{\top}$ given in Example 2.2.6 one has

$$(\mathbb{T},\mathbf{V},\mathbb{T}^{\top})\text{-}\mathbf{Cat} \cong \mathbf{Set},$$

for every **Set**-monad $\mathbb{T}$ and every quantale $\mathbf{V}$.

2.3.1 Unitary $(\mathbb{T},\mathbf{V})$-relations

Definition 2.3.4 Given a lax extension $\widehat{\mathbb{T}}=(\widehat{T},\mu,\eta)$ of a **Set**-monad $\mathbb{T}=(T,\mu,\eta)$, the *Kleisli convolution* $s\circ r: TX \to\!\!\!\!\!\!+\,\, Z$ of $(\mathbb{T},\mathbf{V})$-relations $r:TX \to\!\!\!\!\!\!+\,\, Y$ and $s:TY \to\!\!\!\!\!\!+\,\, Z$ is the $(\mathbb{T},\mathbf{V})$-relation defined by

$$s\circ r := s\cdot \widehat{T}r\cdot \mu_X^{\mathrm{op}}.$$

If $\mathbb{T}$ is the identity monad $\mathbb{I}$ then $s\circ r = s\cdot r$, i.e., the Kleisli convolution is just the usual relational composition of $\mathbf{V}$-relations.

Definition 2.3.5 A $(\mathbb{T},\mathbf{V})$-relation $r:TX \to\!\!\!\!\!\!+\,\, Y$ is *right-unitary* if it satisfies

$$r\circ \eta_X^{\mathrm{op}} \le r,$$

and it is *left-unitary* if

$$\eta_Y^{\mathrm{op}}\circ r \le r$$

holds. The $(\mathbb{T},\mathbf{V})$-relation r is *unitary* if it is both right and left unitary.

The conditions for a $(\mathbb{T},\mathbf{V})$-relation r to be right and left unitary can be expressed in terms of the relational composition respectively by

$$r\cdot \widehat{T}1_X \le r \quad \text{and} \quad \eta_Y^{\mathrm{op}}\cdot \widehat{T}r\cdot \mu_X^{\mathrm{op}} \le r.$$

In terms of the Kleisli convolution, a $(\mathbb{T},\mathbf{V})$-category (X,a) is reflexive if

$$\eta_X^{\mathrm{op}} \leq a$$

and it is transitive if

$$a \circ a \leq a.$$

Observe that the inequality above is actually an identity since $a \leq a \circ \eta_X^{\mathrm{op}} \leq a \circ a \leq a$. For what concerns $(\mathbb{T},\mathbf{V})$-functors we have the following: recall that a $(\mathbb{T},\mathbf{V})$-functor $f:(X,a) \to (Y,b)$ is a map $f : X \to Y$ such that

$$f \cdot a \leq b \cdot Tf$$

but, since the $(\mathbb{T},\mathbf{V})$-relation b is right unitary, it satisfies $b \cdot \widehat{T}1_Y = b \circ \eta_Y^{\mathrm{op}} = b$. Hence $b \cdot \widehat{T}f = b \cdot \widehat{T}1_Y \cdot Tf = b \cdot Tf$ so that the $(\mathbb{T},\mathbf{V})$-functor condition can be equivalently given using the lax extension of T, i.e.,

$$f \cdot a \leq b \cdot \widehat{T}f.$$

One can then consider full subcategories of $(\mathbb{T},\mathbf{V})$-**Gph**, namely

$$(\mathbb{T},\mathbf{V})\text{-}\mathbf{UGph} \quad \text{and} \quad (\mathbb{T},\mathbf{V})\text{-}\mathbf{RGph},$$

of unitary $(\mathbb{T},\mathbf{V})$-graphs (X,a) and right unitary $(\mathbb{T},\mathbf{V})$-graphs (X,a), respectively. The following diagram of inclusions follows

$$(\mathbb{T},\mathbf{V})\text{-}\mathbf{Cat} \hookrightarrow (\mathbb{T},\mathbf{V})\text{-}\mathbf{UGph} \hookrightarrow (\mathbb{T},\mathbf{V})\text{-}\mathbf{RGph} \hookrightarrow (\mathbb{T},\mathbf{V})\text{-}\mathbf{Gph}.$$

Definition 2.3.6 A lax extension $\widehat{\mathbb{T}}$ to **V**-**Rel** of a monad $\mathbb{T} = (T,\mu,\eta)$ on **Set** is *associative* whenever the Kleisli convolution of unitary $(\widehat{\mathbb{T}},\mathbf{V})$-relations is associative, i.e.,

$$t \circ (s \circ r) = (t \circ s) \circ r,$$

or, equivalently,

$$t \cdot \widehat{T}(s \cdot \widehat{T}r \cdot \mu_X^{\mathrm{op}}) \cdot \mu_X^{\mathrm{op}} = t \cdot \widehat{T}s \cdot \mu_Y^{\mathrm{op}} \cdot \widehat{T}r \cdot \mu_X^{\mathrm{op}}$$

for all unitary $(\mathbb{T},\mathbf{V})$-relations $r : TX \rightarrow\!\!\!\!\!\!\mapsto Y$, $s : TY \rightarrow\!\!\!\!\!\!\mapsto Z$ and $t : TZ \rightarrow\!\!\!\!\!\!\mapsto W$.

Example 2.3.7

(1) The identity extension to **V**-**Rel** of the identity monad $\mathbb{I}$ is associative. In fact in this case, as we previously remark, the Kleisli convolution coincides with the usual composition of **V**-relations, which is associative.

(2) The largest lax extension $\mathbb{T}^{\top}$ to **V**-**Rel** of a monad $\mathbb{T}$ on **Set** is associative.

(3) The lax extensions $\hat{\mathbb{P}}$ and $\check{\mathbb{P}}$ to **Rel** of the powerset monad $\mathbb{P}$ defined in Example 2.2.6 are both associative.

(4) The Barr extension $\overline{\mathbb{T}}$ of a **Set**-monad $\mathbb{T}=(T,\mu,\eta)$ is associative if both T and μ satisfy BC ([29, Corollary III.1.12.2]).

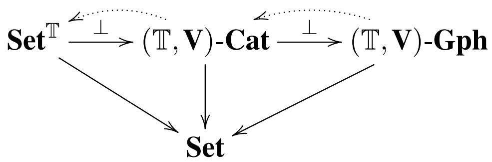
Remark 2.3.8 A flat associative lax extension is always right-whiskering and left-whiskering. Hence, if both T and μ satisfies BC, the Barr extension is right-whiskering and left-whiskering.

2.3.2 Properties of the categories of lax algebras

Let $\mathbf{V}$ be a quantale and let $\widehat{\mathbb{T}}$ be a lax extension to $\mathbf{V}$-**Rel** of a **Set**-monad $\mathbb{T}=(T,\mu,\eta)$. Denote by $\mathbf{Set}^{\mathbb{T}}$ the category of Eilenberg-Moore algebras with respect to $\mathbb{T}$. As it is proved in [10], in case $\widehat{\mathbb{T}}$ is flat, and in [29, III.4] in the non-flat case, the full embeddings

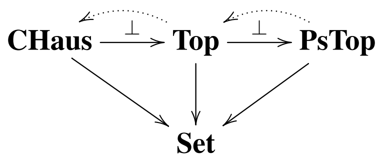
$$\mathbf{Set}^{\mathbb{T}} \hookrightarrow (\mathbb{T},\mathbf{V})\text{-}\mathbf{Cat} \hookrightarrow (\mathbb{T},\mathbf{V})\text{-}\mathbf{Gph}$$

are reflective. Moreover, the canonical forgetful functors from $\mathbf{Set}^{\mathbb{T}}$, $(\mathbb{T},\mathbf{V})$-**Gph** and $(\mathbb{T},\mathbf{V})$-**Cat** into **Set** are topological functors. The above situation can be pictured by the following commutative diagram

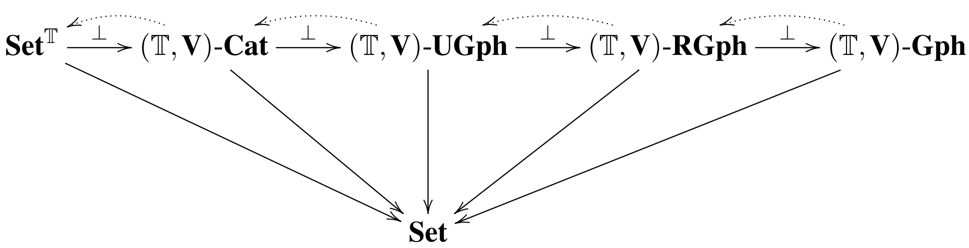


(2.19)

where the vertical arrows are the forgetful functors, the horizontal arrows are the full embeddings and the dotted arrows are the reflections whose underlying morphisms are given by the identities. It follows that the categories $(\mathbb{T},\mathbf{V})$-**Cat** and $(\mathbb{T},\mathbf{V})$-**Gph**, as well as $\mathbf{Set}^{\mathbb{T}}$, are complete and cocomplete. In case of the ultrafilter monad and its Barr extension to **Rel** the diagram (2.19) reduces to



where **CHaus** is the (full) subcategory of **Top** of compact Hausdorff spaces, **PsTop** is the category of *pseudo-topological spaces* and the reflection **Top** $\to$ **CHaus** is given by the well-known *Čech-Stone compactification*.
Diagram (2.19) can be made larger if we consider the categories $(\mathbb{T},\mathbf{V})$-**UGph** and $(\mathbb{T},\mathbf{V})$-**RGph** of unitary graphs and right unitary graphs, respectively. In the diagram



the forgetful functor $(\mathbb{T},\mathbf{V})$-**RGph** $\to$ **Set** is topological (see [29, Corollary III.4.1.5]). As it has been proved in [29, Proposition III.4.2.1], $(\mathbb{T},\mathbf{V})$-**Cat** is reflective in $(\mathbb{T},\mathbf{V})$-**RGph** if $\mu^{\mathrm{op}}:\widehat{T}\to\widehat{T}^2$ is a

natural transformation, while [29, Proposition III.4.2.2] says that the reflector $(\mathbb{T},\mathbf{V})$-**RGph** $\to$ $(\mathbb{T},\mathbf{V})$-**UGph** has an easy one-step construction if the lax extension $\widehat{\mathbb{T}}$ of $\mathbb{T}$ to $\mathbf{V}$-**Rel** is associative.
The following result for the category $(\mathbb{T},\mathbf{V})$-**Gph** will play a key role concerning the study of effective descent morphisms in categories of lax algebras.

Proposition 2.3.9 [19, Theorem 4.6, Remark 4.7] *The category* $(\mathbb{T},\mathbf{V})$-**Gph** *is a quasitopos provided that* $\mathbf{V}$ *is cartesian closed and* T *satisfies BC.*

A quasitopos is, in particular, a locally cartesian closed category and this is precisely the condition we need in the next chapter to study effective descent morphisms in $(\mathbb{T},\mathbf{V})$-categories. An analogous result of Proposition 2.3.9 can be obtained for $(\mathbb{T},\mathbf{V})$-**RGph**.

Proposition 2.3.10 [29, Theorem III.4.6.7] *Let* $\mathbf{V}$ *be cartesian closed and integral and let* $\mathbb{T} = (T,\mu,\eta)$ *be a* **Set** *monad such that* T *satisfies BC. Let* $\widehat{\mathbb{T}} = (\widehat{T},\mu,\eta)$ *be a lax extension of* $\mathbb{T}$ *to* $\mathbf{V}$-**Rel**. *If* $\widehat{\mathbb{T}}$ *is associative and* $\widehat{T}(f^{\mathrm{op}}) = \widehat{T}1_X \cdot (Tf)^{\mathrm{op}}$ *for all maps* $f: X \to Y$*, then the category* $(\mathbb{T},\mathbf{V})$-**RGph** *is locally cartesian closed.*

Another property, in particular concerning the Van Kampen Theorem, is that the category $(\mathbb{T},\mathbf{V})$-**Cat** is an extensive category. This has been deduced in [14] by the fact that coproducts in $(\mathbb{T},\mathbf{V})$-**Cat** are disjoint and universal: disjointness follows from the fact that the forgetful functor $(\mathbb{T},\mathbf{V})$-**Cat** $\to$ **Set** preserves coproducts, while universality follows from the characterization of coproducts, stated in theorem below, and pullback stability of open embeddings.

Theorem 2.3.11 ([38], [14, Theorem 1.3]) *For* $(\mathbb{T},\mathbf{V})$*-categories* (X_i,a_i)*,* $i \in I$*, and* (X,a)*, the following conditions are equivalent:*

(i) (X,a) *is the coproduct of* $(X_i,a_i)_{i\in I}$ *in* $(\mathbb{T},\mathbf{V})$-**Cat***;*

(ii) (a) X *is the coproduct of* $(X_i)_{i\in I}$ *in* **Set***;*

(b) *for each* $i \in I$*, the inclusion* $(X_i,a_i) \hookrightarrow (X,a)$ *is open.*

Corollary 2.3.12 [14, Corollary 1.4] $(\mathbb{T},\mathbf{V})$-**Cat** *is an extensive category.*

Of course, since it is complete, $(\mathbb{T},\mathbf{V})$-**Cat** is also a lextensive category.

Chapter 3

Effective global-descent morphisms in categories of lax algebras

In [11] M.M. Clementino and D. Hofmann investigate effective descent morphisms in the general context of $(\mathbb{T},\mathbf{V})$-categories, in particular proving that open and proper surjection are effective for descent, extending the already known results in **Top**. Their investigation is conducted with respect to flat lax extensions to **V**-**Rel** of the **Set**-monad, in fact in [11] flatness in included in the definition of lax extension. Our investigation starts by generalizing the results obtained in [11] to the case of non-necessarily flat lax extensions. We follow the same method based on the result given by Corollary 1.1.7 which we can re-write in the following way.

Theorem 3.0.1 *Let* **C** *and* **D** *be categories such that*

(a) **D** *has pullbacks and coequalizers and* **C** *is a full subcategory of* **D** *closed under pullback,*

(b) every regular epimorphism in **D** *is an effective descent morphism.*

Then a morphism $p : E \to B$ *in* **C***, which is of effective descent in* **D***, is an effective descent morphism in* **C** *if and only if*

$$E \times_B A \in \mathbf{C} \Rightarrow A \in \mathbf{C}$$

holds for every pullback

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[p]{} & B \end{array}$$

in **D**.

For a lax extension $\widehat{\mathbb{T}}$ to **V**-**Rel** (not necessarily flat) of a given **Set**-monad $\mathbb{T} = (T, \mu, \eta)$, the theorem above can be applied when $\mathbf{C} = (\mathbb{T},\mathbf{V})$-**Cat** and $\mathbf{D} = (\mathbb{T},\mathbf{V})$-**Gph**. In fact, in Section 2.3.2, we saw that $(\mathbb{T},\mathbf{V})$-**Cat** is a full reflective subcategory of $(\mathbb{T},\mathbf{V})$-**Gph**, so that it is closed under pullback, and also that $(\mathbb{T},\mathbf{V})$-**Gph** is cocomplete (and complete), so that the existence of coequalizers is guaranteed. Moreover, by Proposition 2.3.9, if T satisfies BC and **V** is cartesian closed, the category $(\mathbb{T},\mathbf{V})$-**Gph** is a quasitopos and so, in particular, locally cartesian closed. Hence, by Corollary 1.1.5, we know that the effective global-descent morphisms in $(\mathbb{T},\mathbf{V})$-**Gph** are precisely the (necessarily universal) regular epimorphisms. Regular epimorphisms in $(\mathbb{T},\mathbf{V})$-**Gph** have been characterized in [10, Proposition 5.1] as those $(\mathbb{T},\mathbf{V})$-functors $f:(X,a)\to(Y,b)$ such that the underlying map $f:X\to Y$ is surjective and

$$b = f\cdot a\cdot (Tf)^{\mathrm{op}}. \tag{3.1}$$

A $(\mathbb{T},\mathbf{V})$-functor $f:(X,a)\to(Y,b)$ satisfying the condition above is usually called *final*, since (3.1) represents the final structure for b in $(\mathbb{T},\mathbf{V})$-**Gph**. If $\mathbf{V}=\mathbf{2}$ condition (3.1) becomes

$$\begin{array}{ccc} X & \qquad & \mathfrak{x} \dashrightarrow x \\ {\scriptstyle f}\downarrow & & \downarrow \qquad \downarrow \\ Y & & \mathfrak{y} \longrightarrow y, \end{array}$$

i.e., a $(\mathbb{T},\mathbf{2})$-functor $f:(X,a)\to(Y,b)$ is final if for each $\mathfrak{y}\in TY$ and $y\in Y$ with $\mathfrak{y}\to y$, there exist $\mathfrak{x}\in(Tf)^{-1}(\mathfrak{y})$ and $x\in f^{-1}(y)$ such that $\mathfrak{x}\to x$. The following theorem holds.

Theorem 3.0.2 *Let* $\mathbf{V}$ *be a cartesian closed quantale and let* $\widehat{\mathbb{T}}$ *be a lax extension to* $\mathbf{V}$-**Rel** *of a* **Set**-*monad* $\mathbb{T}=(T,\mu,\eta)$*, where* T *satisfies BC. For a class* $\mathfrak{F}$ *of morphisms in* $(\mathbb{T},\mathbf{V})$-**Gph**, $\mathfrak{F}\cap(\mathbb{T},\mathbf{V})$-**Cat** *is a class of effective global-descent morphisms in* $(\mathbb{T},\mathbf{V})$-**Cat** *provided that:*

(1) each f *in* $\mathfrak{F}\cap(\mathbb{T},\mathbf{V})$-**Cat** *is a regular epimorphism in* $(\mathbb{T},\mathbf{V})$-**Cat**;

(2) $\mathfrak{F}$ *is stable under pullback;*

(3) $(\mathbb{T},\mathbf{V})$-**Cat** *is closed under* $\mathfrak{F}$*-images in* $(\mathbb{T},\mathbf{V})$-**Gph**.

Following the definition given in [11], based on the characterization of effective descent maps in **Top** (see Theorem 1.4.6), we introduce the notion of **-quotient morphisms*.

Definition 3.0.3 A $(\mathbb{T},\mathbf{V})$-functor $f:(X,a)\to(Y,b)$ in $(\mathbb{T},\mathbf{V})$-**Gph** is said to be **-quotient* if

$$\forall\mathfrak{Y}\in T^2Y,\quad \forall\mathfrak{y}\in TY,\quad \forall y\in Y$$

$$\widehat{T}b(\mathfrak{Y},\mathfrak{y})\otimes b(\mathfrak{y},y) = \bigvee_{\substack{\mathfrak{X}\in(T^2f)^{-1}(\mathfrak{Y})\\ \mathfrak{x}\in(Tf)^{-1}(\mathfrak{y})\\ x\in f^{-1}(y)}} \widehat{T}a(\mathfrak{X},\mathfrak{x})\otimes a(\mathfrak{x},x)$$

In case $\mathbf{V}=\mathbf{2}$, using the arrow notation, a $(\mathbb{T},\mathbf{2})$-functor $f:(X,a)\to(Y,b)$ is a *-quotient map if for every $\mathfrak{Y}\in T^2Y$, $\mathfrak{y}\in TY$ and $y\in Y$ with $\mathfrak{Y}\to\mathfrak{y}\to y$, there exist $\mathfrak{X}\in(T^2f)^{-1}(\mathfrak{Y})$, $\mathfrak{x}\in(Tf)^{-1}(\mathfrak{x})$ and $x\in f^{-1}(y)$ such that $\mathfrak{X}\to\mathfrak{x}\to x$, i.e.,

$$\begin{array}{ccc} X & \qquad & \mathfrak{X} \dashrightarrow \mathfrak{x} \dashrightarrow x \\ {\scriptstyle f}\downarrow & & \downarrow \qquad \downarrow \qquad \downarrow \\ Y & & \mathfrak{Y} \longrightarrow \mathfrak{y} \longrightarrow y. \end{array}$$

Proposition 3.0.4 *If $f:(X,a)\to(Y,b)$ is a *-quotient morphism in $(\mathbb{T},\mathbf{V})$-**Cat**, then it is a regular epimorphism in $(\mathbb{T},\mathbf{V})$-**Gph**.*

Proof
We have to show that $b=f\cdot a\cdot(Tf)^{\mathrm{op}}$. We already know that $b\geq f\cdot a\cdot(Tf)^{\mathrm{op}}$, since f is, in particular, a $(\mathbb{T},\mathbf{V})$-functor. Hence, it remains to show that $b\leq f\cdot a\cdot(Tf)^{\mathrm{op}}$. Since

$$1_{TY}=T1_Y\leq\widehat{T}1_Y\leq\widehat{T}(b\cdot e_Y)=\widehat{T}b\cdot\widehat{T}e_Y=\widehat{T}b\cdot Te_Y,$$

for $\mathfrak{y}\in TY$ and $y\in Y$, we then have

$$\begin{aligned}
b(\mathfrak{y},y)\leq\widehat{T}b(Te_Y(\mathfrak{y}),\mathfrak{y})\otimes b(\mathfrak{y},y)&=\bigvee_{\substack{\mathfrak{X}\in T^2X:T^2f(\mathfrak{X})=Te_Y(\mathfrak{y})\\ \mathfrak{x}\in TX:Tf(\mathfrak{x})=\mathfrak{y}\\ x\in X:f(x)=y}}\widehat{T}a(\mathfrak{X},\mathfrak{x})\otimes a(\mathfrak{x},x)\quad\text{(f is *-quotient)}\\
&\leq\bigvee_{\substack{\mathfrak{X}\in T^2X:T^2f(\mathfrak{X})=Te_Y(\mathfrak{y})\\ x\in X:f(x)=y}}a(m_X(\mathfrak{X}),x)\quad(a\text{ is transitive})\\
&\leq\bigvee_{\substack{\mathfrak{x}\in TX:Tf(\mathfrak{x})=\mathfrak{y}\\ x\in X:f(x)=y}}a(\mathfrak{x},x),
\end{aligned}$$

since, for $\mathfrak{X}\in T^2X$ such that $T^2f(\mathfrak{X})=Te_Y(\mathfrak{y})$, one has

$$Tf(m_X(\mathfrak{X}))=m_Y(T^2f(\mathfrak{X}))=m_Y(Te_Y(\mathfrak{y}))=\mathfrak{y}.$$

□

Proposition 3.0.5 $(\mathbb{T},\mathbf{V})$-**Cat** *is closed under *-quotient images in* $(\mathbb{T},\mathbf{V})$-**Gph**.

Proof
Let $f:(X,a)\to(Y,b)$ be a *-quotient morphism in $(\mathbb{T},\mathbf{V})$-**Gph** with $(X,a)\in(\mathbb{T},\mathbf{V})$-**Cat**. We want to prove that (Y,b) is transitive. For each $\mathfrak{Y}\in T^2Y$, $\mathfrak{y}\in Y$ and $y\in Y$ we have

$$\widehat{T}b(\mathfrak{Y},\mathfrak{y})\otimes b(\mathfrak{y},y)=\bigvee_{\substack{\mathfrak{X}\in(T^2f)^{-1}(\mathfrak{Y})\\ \mathfrak{x}\in(Tf)^{-1}(\mathfrak{y})\\ x\in f^{-1}(y)}}\widehat{T}a(\mathfrak{X},\mathfrak{x})\otimes a(\mathfrak{x},x)\leq\bigvee_{\substack{\mathfrak{X}\in(T^2f)^{-1}(\mathfrak{Y})\\ x\in f^{-1}(y)}}a(m_X(\mathfrak{X}),x)\leq b(m_Y(\mathfrak{Y}),y).$$

The equality follows from the fact that f is a *-quotient morphism, while the first inequality follows from the fact that a is transitive. □

Finally, by Theorem 3.0.2, and combining the previous results, the following holds.

Theorem 3.0.6 *Let* $\mathbf{V}$ *be a cartesian closed quantale and let* $\widehat{\mathbb{T}}$ *be a lax extension to* $\mathbf{V}$-**Rel** *of a* **Set**-*monad* $\mathbb{T}=(T,\mu,\eta)$*, where* T *satisfies BC. A morphism in* $(\mathbb{T},\mathbf{V})$-**Cat** *is an effective global-descent morphism provided that it is a pullback stable *-quotient morphism in* $(\mathbb{T},\mathbf{V})$-**Gph**.

□

The converse is not true in general, as we will se in Section 3.3.2 in the case of **MultiOrd**. For what concerns descent morphisms in $(\mathbb{T},\mathbf{V})$-**Cat**, in case the lax extension $\widehat{\mathbb{T}}$ is flat, the following characterization holds.

Theorem 3.0.7 [14, Theorem 2.4] *Let $\widehat{\mathbb{T}}$ be a flat lax extension to* $\mathbf{V}$-**Rel** *of a* **Set**-*monad* $\mathbb{T}$. *Assume that every naturality square of η with respect to* $\mathbf{V}$-*relations with finite fibres is a BC-square. Then the following conditions are equivalent for a morphism $f:(X,a)\to(Y,b)$ in* $(\mathbb{T},\mathbf{V})$-**Cat**:

(i) *f is final;*

(ii) *f is a pullback stable regular epimorphism in* $(\mathbb{T},\mathbf{V})$-**Gph**;

(iii) *f is a pullback stable regular epimorphism in* $(\mathbb{T},\mathbf{V})$-**Cat**;

(iv) *f is a descent morphism in* $(\mathbb{T},\mathbf{V})$-**Cat**.

Remark 3.0.8

(1) In general the natural transformation η does not satisfy BC but, in some cases, as in **Top** for instance (see Example 2.2.11), the naturality diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & TX \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle \widehat{T}r} \\ Y & \xrightarrow[\eta_Y]{} & TY \end{array}$$

is a BC-square, i.e., $(\widehat{T}r)^{\mathrm{op}}\cdot\eta_Y=\eta_X\cdot r^{\mathrm{op}}$, provided that the relation r has *finite fibres* in the sense that, for every $y\in Y$, the set $\{x\in X : r(x,y)>\bot\}$ is finite.

(2) In case $\mathbf{V}=\mathbf{2}$, for the ultrafilter monad and its Barr extension to **Rel**, one obtains the characterization in **Top** of descent maps (see Theorem 1.4.5). Accordingly we can then state that a continuous map $f:X\to Y$ in **Top** is a descent map if and only if for each ultrafilter $\mathfrak{y}\in UY$ and each point $y\in Y$ with $\mathfrak{y}\to y$, there exist an ultrafilter $\mathfrak{x}\in(Uf)^{-1}(\mathfrak{y})$ and a point $x\in f^{-1}(y)$ such that $\mathfrak{x}\to x$

$$\begin{array}{c} X \\ {\scriptstyle f}\downarrow \\ Y \end{array} \qquad\qquad \begin{array}{ccc} \mathfrak{x} & \dashrightarrow & x \\ \downarrow & & \downarrow \\ \mathfrak{y} & \longrightarrow & y. \end{array}$$

The description of the map above is a concrete example of the role played by convergence (see Proposition 1.4.12 describing descent maps of finite topological spaces).

An analogous result of Theorem 3.0.6 can be obtained replacing the category $(\mathbb{T},\mathbf{V})$-**Gph** by $(\mathbb{T},\mathbf{V})$-**RGph**, the category of right unitary graphs. In fact, by Proposition 2.3.10, if in addition one has:

(i) $\widehat{\mathbb{T}}$ is associative;

(ii) $\widehat{T}(f^{\mathrm{op}})=\widehat{T}1_X\cdot(Tf)^{\mathrm{op}}$ for all maps $f:X\to Y$, then $(\mathbb{T},\mathbf{V})$-**RGph** is locally cartesian closed. Therefore effective descent morphisms coincide with regular epimorphisms. According to [29, Corollary III.4.1.5], a $(\mathbb{T},\mathbf{V})$-functor $f:(X,a)\to(Y,b)$ in $(\mathbb{T},\mathbf{V})$-**RGph** is a regular epimorphism if and only if $f:X\to Y$ is surjective and

$$b=f\cdot a\cdot \widehat{T}(f^{\mathrm{op}}).$$

Condition (ii) gives

$$f\cdot a\cdot \widehat{T}(f^{\mathrm{op}})=f\cdot a\cdot \widehat{T}1_X\cdot (Tf)^{\mathrm{op}}=f\cdot a\cdot (Tf)^{\mathrm{op}},$$

where the last equality follows from the fact that the structure a is right unitary, so that the description of regular epimorphisms in $(\mathbb{T},\mathbf{V})$-**RGph** coincides with the one given in $(\mathbb{T},\mathbf{V})$-**Gph**. Therefore Theorem 3.0.2 and Theorem 3.0.6, assuming in addition $\mathbf{V}$ integral and conditions (i) and (ii), still hold replacing $(\mathbb{T},\mathbf{V})$-**Gph** by $(\mathbb{T},\mathbf{V})$-**RGph**.

3.1 Open and proper surjections

We study open and proper surjections in $(\mathbb{T},\mathbf{V})$-categories and prove them to be, under some conditions, effective for descent. This was first proved to be true in **Top** $\cong(\mathbb{U},\mathbf{2},\overline{\mathbb{U}})$-**Cat** (see [42], [51] and [53]) and, thanks to the convergence description of such maps, it has been possible, first to generalize the notion of open and proper morphisms in the context of $(\mathbb{T},\mathbf{V})$-categories, and second to generalize the result concerning Descent Theory. In [11] positive answers are given for flat lax extensions while here we study the more general case where lax extensions are not necessarily flat. The idea is the same, that is to use Theorem 3.0.6.

Definition 3.1.1 A $(\mathbb{T},\mathbf{V})$-functor $f:(X,a)\to(Y,b)$ is *open* if

$$f^{\mathrm{op}}\cdot b\le a\cdot (Tf)^{\mathrm{op}},$$

and $f:(X,a)\to(Y,b)$ is *proper* if

$$b\cdot Tf\le f\cdot a.$$

Observe that, in both cases, the inequality is actually an equality, since $f:(X,a)\to(Y,b)$ is a $(\mathbb{T},\mathbf{V})$-functor.

In elementwise notation, a $(\mathbb{T},\mathbf{V})$-functor $f:(X,a)\to(Y,b)$ is open if

$$\forall \mathfrak{y}\in TY\quad \forall x\in X\qquad b(\mathfrak{y},x)=\bigvee_{\mathfrak{x}\in TX:Tf(\mathfrak{x})=\mathfrak{y}} a(\mathfrak{x},x)$$

and proper if

$$\forall \mathfrak{x}\in TX\quad \forall y\in Y\qquad b(Tf(\mathfrak{x}),y)=\bigvee_{x\in X:f(x)=y} a(\mathfrak{x},x).$$

For $\mathbf{V} = \mathbf{2}$ the conditions above become, respectively,

$$\begin{array}{ccc} X & \qquad & \exists\mathfrak{x} \cdots\cdots> x \\ {\scriptstyle f}\downarrow & & \downarrow \qquad\quad \downarrow \\ Y & & \mathfrak{y} \longrightarrow f(x) \end{array}$$

$$\begin{array}{ccc} X & \qquad & \mathfrak{x} \cdots\cdots> \exists x \\ {\scriptstyle f}\downarrow & & \downarrow \qquad\quad \downarrow \\ Y & & Tf(\mathfrak{x}) \longrightarrow y, \end{array}$$

i.e., a $(\mathbb{T},\mathbf{2})$-functor $f:(X,a)\to(Y,b)$ is open if and only if for every $x\in X$ and $\mathfrak{y}\in TY$ with $\mathfrak{y}\to f(x)$, there exists $\mathfrak{x}\in TX$ with $\mathfrak{x}\to x$ and $Tf(\mathfrak{x})=\mathfrak{y}$, and f is proper if and only if for every $\mathfrak{x}\in TX$ and $y\in Y$ with $Tf(\mathfrak{x})\to y$, there exists $x\in X$ with $\mathfrak{x}\to x$ and $f(x)=y$.

Remark 3.1.2 For the ultrafilter monad $\mathbb{U}$ one obtains the descriptions of open and proper maps in **Top**. Comparing with the description of open and proper maps of finite topological spaces (see diagrams (1.20) and (1.21)) one can see another example of the role of convergence.

First of all we need to guarantee the pullback stability of such maps.

Proposition 3.1.3 [29, Proposition V.3.1.4] *Let* $\mathbf{V}$ *be a cartesian closed quantale.*

1. *Proper morphisms are pullback stable in* $(\mathbb{T},\mathbf{V})$-**Gph**.
2. *If T satisfies BC, open morphisms are pullback stable in* $(\mathbb{T},\mathbf{V})$-**Gph**.

Lemma 3.1.4 *Let* $\widehat{\mathbb{T}}=(\widehat{T},\mu,\eta)$ *be a lax extension to* $\mathbf{V}$-**Rel** *of a* **Set**-*monad* $\mathbb{T}$ *and let* $f:(X,a)\to(Y,b)$ *be a* $(\mathbb{T},\mathbf{V})$-*functor.*

(1) *If $\widehat{T}$ is left-whiskering, then $Tf:(TX,\widehat{T}a)\to(TY,\widehat{T}b)$ is a proper $(\mathbb{T},\mathbf{V})$-functor provided that $f:(X,a)\to(Y,b)$ is proper.*

(2) *If $\widehat{T}$ is right-whiskering, then $Tf:(TX,\widehat{T}a)\to(TY,\widehat{T}b)$ is an open $(\mathbb{T},\mathbf{V})$-functor provided that $f:(X,a)\to(Y,b)$ is open.*

Proof

$$(1)\quad Tf\cdot\widehat{T}a=\widehat{T}(f\cdot a)=\widehat{T}(b\cdot Tf)=\widehat{T}b\cdot\widehat{T}(Tf)=\widehat{T}b\cdot T^2f.$$

The first equality follows from the fact that $\widehat{T}$ is left-whiskering, while the second one follows from the fact that $f:(X,a)\to(Y,b)$ is a proper $(\mathbb{T},\mathbf{V})$-functor.

$$(2)\quad (Tf)^{\mathrm{op}}\cdot\widehat{T}b=\widehat{T}(f^{\mathrm{op}})\cdot\widehat{T}b=\widehat{T}(f^{\mathrm{op}}\cdot b)=\widehat{T}(a\cdot(Tf)^{\mathrm{op}})=\widehat{T}a\cdot(T^2f)^{\mathrm{op}}.$$

The third equality follows from the fact that $f:(X,a)\to(Y,b)$ is an open $(\mathbb{T},\mathbf{V})$-functor, while the last one follows from the fact that $\widehat{T}$ is right-whiskering. $\square$

Proposition 3.1.5 *Let* $\widehat{\mathbb{T}}=(\widehat{T},\mu,\eta)$ *be a lax extension to* $\mathbf{V}$-**Rel** *of a* **Set**-*monad* $\mathbb{T}$.

*1. If $\widehat{T}$ is left-whiskering, then proper surjections are *-quotient morphisms.*

*2. If $\widehat{T}$ is right-whiskering, then open surjections are *-quotient morphisms.*

Proof
1. Let $f : (X,a) \to (Y,b)$ be a proper and surjective $(\mathbb{T},\mathbf{V})$-functor. Let $\mathfrak{Y} \in T^2Y$, $\mathfrak{y} \in TY$ and $y \in Y$. Since f is surjective, and since the functor $T : \mathbf{Set} \to \mathbf{Set}$ preserves surjections, there exist $\mathfrak{X} \in T^2X$ and $\mathfrak{x} \in TX$ such that $T^2f(\mathfrak{X}) = \mathfrak{Y}$ and $Tf(\mathfrak{x}) = \mathfrak{y}$. Since f is proper, $b(\mathfrak{y},x)) = \bigvee_{x\in X:f(x)=y} a(\mathfrak{x},x)$ and since, by Lemma 3.1.4, also Tf is proper, we have $\widehat{T}b(\mathfrak{Y},\mathfrak{y}) = \bigvee_{\mathfrak{x}\in TX:Tf(\mathfrak{x})=\mathfrak{y}} \widehat{T}a(\mathfrak{X},\mathfrak{x})$. Therefore,

$$\widehat{T}b(\mathfrak{Y},\mathfrak{y}) \otimes b(\mathfrak{y},y) = \bigvee_{\mathfrak{x}\in TX:Tf(\mathfrak{x})=\mathfrak{y}} \widehat{T}a(\mathfrak{X},\mathfrak{x}) \otimes \bigvee_{x\in X:f(x)=y} a(\mathfrak{x},x) = \bigvee_{\substack{\mathfrak{x}\in TX:Tf(\mathfrak{x})=\mathfrak{y}\\ x\in X:f(x)=y}} \widehat{T}a(\mathfrak{X},\mathfrak{x}) \otimes a(\mathfrak{x},x)$$

which of course implies that

$$\widehat{T}b(\mathfrak{Y},\mathfrak{y}) \otimes b(\mathfrak{y},y) \leq \bigvee_{\substack{\mathfrak{X}\in T^2X:T^2f(\mathfrak{X})=\mathfrak{Y}\\ \mathfrak{x}\in TX:Tf(\mathfrak{x})=\mathfrak{y}\\ x\in X:f(x)=y}} \widehat{T}a(\mathfrak{X},\mathfrak{x}) \otimes a(\mathfrak{x},x).$$

The other inequality is trivially satisfied.
2. An analogous proof can be given for open surjections. □

Combining the previous results, and by Theorem 3.0.6, we get:

Theorem 3.1.6 *Let* $\mathbf{V}$ *be a cartesian closed quantale. Let* $\mathbb{T} = (T,\mu,\eta)$ *be a monad on* **Set** *such that* T *satisfies BC, and let* $\widehat{\mathbb{T}} = (\widehat{T},\mu,\eta)$ *be a lax extension of* $\mathbb{T}$ *to* $\mathbf{V}$-**Rel**.

1. If $\widehat{T}$ is left-whiskering, then proper surjections are effective for descent in $(\mathbb{T},\mathbf{V})$-**Cat**.

2. If $\widehat{T}$ is right-whiskering, then open surjections are effective for descent in $(\mathbb{T},\mathbf{V})$-**Cat**.

□

As we mentioned in Example 2.2.6, the lax extension $\hat{P}$ to **Rel** of the powerset functor P is left-whiskering but not right-whiskering, while $\check{P}$ behaves conversely. Hence, by Theorem 3.1.6, proper surjections are effective for descent in $(\mathbb{P},\mathbf{2},\hat{\mathbb{P}})$-**Cat** and open surjections are effective for descent in $(\mathbb{P},\mathbf{2},\check{\mathbb{P}})$-**Cat**. Moreover the following two propositions hold.

Proposition 3.1.7 *Open surjections are effective for descent in* $(\mathbb{P},\mathbf{2},\hat{\mathbb{P}})$-**Cat**.

Proof
Let $f : (X,a) \to (Y,b)$ be an open surjection in $(\mathbb{P},\mathbf{2},\hat{\mathbb{P}})$-**Cat**. By Theorem 3.0.6 and Proposition 3.2.8, it suffices to show that f is a *-quotient morphism. Let then

$$\mathfrak{B} \longrightarrow B \longrightarrow y_0$$

be a chain of convergence in Y, i.e., $\mathfrak{B} \in P^2Y$, $B \in PY$ and $y_0 \in Y$ with $\hat{P}b(\mathfrak{B},B) = \top$ and $b(B,y_0) = \top$. Since f is surjective, there exists $x_0 \in X$ such that $f(x_0) = y_0$. Since f is open, there exists then

$A \in PX$ such that $Pf(A) = B$ and $A \longrightarrow x_0$. By definition of the extension, $\mathfrak{B} \longrightarrow B$ means that for each $y \in B$, there exists $B' \in \mathfrak{B}$ such that $B' \longrightarrow y$. Hence, since f is open, there exist $A' \in PX$ such that $Pf(A') = B'$ and $A' \longrightarrow x$, where $x \in A$ with $f(x) = y$. With $\mathfrak{A} \in P^2X$ given by all these A', we get $\mathfrak{A} \longrightarrow A$ but, in general, $P^2f(\mathfrak{A}) = \mathfrak{B}$ does not hold since there might exist elements in $\mathfrak{B}$ not in relation to any point of B. For such elements, since f is surjective, take their counter-images via f, and add them to $\mathfrak{A}$ to get an element in P^2X converging to A and mapped to $\mathfrak{B}$ by P^2f. □

Proposition 3.1.8 *Proper surjections are effective for descent in* $(\mathbb{P}, \mathbf{2}, \check{\mathbb{P}})$-**Cat**.

Proof
Let $f : (X, a) \to (Y, b)$ be a proper surjection in $(\mathbb{P}, \mathbf{2}, \check{\mathbb{P}})$-**Cat**. As before, by Theorem 3.0.6 and Proposition 3.2.8, it suffices to show that f is a *-quotient morphism. Let then

$$\mathfrak{B} \longrightarrow B \longrightarrow y_0$$

be a chain of convergence in Y. By definition of the extension, $\mathfrak{B} \longrightarrow B$ means that for each $B' \in \mathfrak{B}$ there exists $y \in B$ such that $B' \longrightarrow y$. Since f is surjective, for each one of these B' there exists $A' \in PX$ with $Pf(A') = B'$. Let $\mathfrak{A} \in P^2X$ be given by all such elements A'. Since f is proper, for each one of these A' there exists a point $x \in X$ such that $f(x) = y$ and $A' \longrightarrow x$. Let $A \in PX$ be the join of all these points $x \in X$ satisfying the property above. Of course $\mathfrak{A} \longrightarrow A$ but, in general, $Pf(A) = B$ does not hold since there might be points in B not in relation to any element of $\mathfrak{B}$. For those points consider their counter-images via f, which are non-empty since f is surjective, and add them to A. With this technique we get an element in PX mapped to B by Pf and in relation with $\mathfrak{A}$. To conclude the proof, apply again to the convergence $B \longrightarrow y_0$ the fact that f is proper. □

3.2 Triquotient maps

Triquotient maps were introduced by E. Michael in [41]. The notion is purely topological but thanks to the characterization of such maps in terms of ultrafilter convergence, given by M.M. Clementino and D. Hofmann in [9], it is possible to define them in the more general context of $(\mathbb{T}, \mathbf{2})$-categories. We start recalling the definition in **Top** and see how we can move to $(\mathbb{T}, \mathbf{2})$-**Cat**. Then we explore the role they play in Descent Theory, based on the key result given by Theorem 3.2.8.
For a topological space X let OX denote its topology. For $x \in X$, $O(x)$ denotes the set of open subsets of X containing x.

Definition 3.2.1 A continuous function $f : X \to Y$ is a *triquotient map* if there exists a map

$$(-)^\sharp : OX \to OY$$

such that:

(T1) $(\forall U \in OX)\ U^\sharp \subseteq f(U)$,

(T2) $X^\sharp = Y$,

(T3) $(\forall U, V \in OX)\ U \subseteq V \Rightarrow U^\sharp \subseteq V^\sharp$,

(T4) $(\forall U \in OX)$ $(\forall y \in U^\sharp)$ $(\forall \Sigma \subseteq OX$ directed$)$ $f^{-1}(y) \cap U \subseteq \bigcup \Sigma \Rightarrow \exists S \in \Sigma : y \in S^\sharp$.

In the finite case, the following characterization was established (see [30] and [7]).

Theorem 3.2.2 *A continuous map $f : X \to Y$ between finite topological spaces is a triquotient map if and only if for every natural number n and every sequence $b_n \to b_{n-1} \to \cdots \to b_0$ in Y, there exists a sequence $x_n \to x_{n-1} \to \cdots \to x_0$ in E such that $f(x_i) = y_i$ for $i = 1, \ldots, n$,*

$$\begin{array}{ccc} X & \quad & x_n \dashrightarrow x_{n-1} \dashrightarrow \cdots \dashrightarrow x_0 \\ {\scriptstyle f}\downarrow & & \downarrow \qquad \downarrow \qquad\qquad \downarrow \\ Y & & y_n \longrightarrow y_{n-1} \longrightarrow \cdots \longrightarrow y_0. \end{array}$$

In [9] M.M. Clementino and D. Hofmann characterize triquotient maps as those that are surjective on chains of ultrafilter convergence, extending the result given by the theorem above for the finite case. The technique they used can be generalized to the context of $(\mathbb{T},\mathbf{2})$-categories, leading to a $(\mathbb{T},\mathbf{2})$-categorical definition of *triquotient maps*. We follow their method where the ultrafilter monad they consider is replaced any **Set**-monad $\mathbb{T}$.

Let $\mathbb{T} = (T, \mu, \eta)$ be a monad on **Set**.

Definition 3.2.3 A $\mathbb{T}$-*relation* on a set X is a subset $r \subseteq TX \times X$. A $\mathbb{T}$-*relational algebra* is a set X equipped with an ultrarelation r on X. A morphism $f : (X,r) \to (Y,s)$ of $\mathbb{T}$-relational algebras (X,r) and (Y,s) (also called $\mathbb{T}$-*relational morphism*) is a map $f : X \to Y$ such that if $(\mathfrak{x},x) \in r$ then $(Tf(\mathfrak{x}), f(x)) \in s$.

We denote by $(\mathbb{T},\mathbf{2})$-**Conv** the category of $\mathbb{T}$-relational algebras and morphisms between them. Often we could use the notation $\mathfrak{x} \to x$ instead of $(\mathfrak{x},x) \in r$. The category $(\mathbb{T},\mathbf{2})$-**Conv** is equipped with a canonical faithful functor $U : (\mathbb{T},\mathbf{2})\text{-}\mathbf{Conv} \to \mathbf{Set}$ sending (X,r) to X.

Proposition 3.2.4 *The functor $U : (\mathbb{T},\mathbf{2})\text{-}\mathbf{Conv} \to \mathbf{Set}$ is topological.*

Proof
Let $(X \xrightarrow{f_i} U(Y_i,s_i))_{i\in I}$ be a U-structured source. Consider $f_i : (X,r) \to (Y_i,s_i)$, where

$$r = \{(\mathfrak{x},x) \in TX \times X \quad \text{such that} \quad (Tf_i(\mathfrak{x}), f_i(x)) \in s_i \quad \text{for every} \quad i \in I\}.$$

Then $((X,r) \xrightarrow{f_i} (Y_i,s_i))_{i\in I}$ is the unique U-initial lift of $(X \xrightarrow{f_i} U(Y_i,s_i))_{i\in I}$. □

Hence, by the proposition above, the category $(\mathbb{T},\mathbf{2})$-**Conv** is complete and cocomplete.
Observe that, if we consider a lax extension $\widehat{\mathbb{T}}$ to **Rel** of the monad $\mathbb{T}$, the category $(\mathbb{T},\mathbf{2})$-**Conv** contains $(\mathbb{T},\mathbf{2})$-**Cat** (and $(\mathbb{T},\mathbf{2})$-**Gph**) as a full subcategory: each $(\mathbb{T},\mathbf{2})$-category (X,a) defines a $\mathbb{T}$-relation $r_{(X,a)}$ by

$$r_{(X,a)} = \{(\mathfrak{x},x) \in TX \times X \quad \text{such that} \quad a(\mathfrak{x},x) = \top\}.$$

For a $\mathbb{T}$-relational algebra (X,r), we consider the projection map

$$p_{(X,r)}: r \to X, \quad (\mathfrak{x},x) \mapsto x$$

and define the following $\mathbb{T}$-relation $R_{(X,r)}$ on r:

$$R_{(X,r)} = \{(\mathfrak{X},(\mathfrak{x},x)) \in Tr \times r \quad \text{such that} \quad Tp_{(X,r)}(\mathfrak{X}) = \mathfrak{x}\}.$$

The map $p_{(X,r)}: (r, R_{(X,r)}) \to (X,r)$ is a morphism of $\mathbb{T}$-relational alegbras, by definition of the structure on r.

We can now define a functor

$$\text{Conv}: (\mathbb{T},\mathbf{2})\text{-}\mathbf{Conv} \to (\mathbb{T},\mathbf{2})\text{-}\mathbf{Conv}$$

in the following way: on objects, $\text{Conv}(X,r) = (r, R_{(X,r)})$, and for a $\mathbb{T}$-relational morphism $f:(X,a) \to (Y,s)$, define

$$\text{Conv}(f): r \to s, \qquad (\mathfrak{x},x) \mapsto (Tf(\mathfrak{x}), f(x)).$$

Since the diagram

$$\begin{array}{ccc} r & \xrightarrow{\text{Conv}(f)} & s \\ {\scriptstyle p_{(X,r)}}\downarrow & & \downarrow{\scriptstyle p_{(Y,s)}} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is commutative, $\text{Conv}(f): \text{Conv}(X,r) \to \text{Conv}(Y,s)$ is a $\mathbb{T}$-relational morphism. Moreover, the equalities

$$\text{Conv}(1_{(X,r)}) = 1_{\text{Conv}(X,r)} \quad \text{and} \quad \text{Conv}(f \cdot g) = \text{Conv}(f) \cdot \text{Conv}(g)$$

hold. Hence $\text{Conv}: (\mathbb{T},\mathbf{2})\text{-}\mathbf{Conv} \to (\mathbb{T},\mathbf{2})\text{-}\mathbf{Conv}$ is a functor and

$$(p_{(X,r)})_{(X,r)\in(\mathbb{T},2)\text{-}\mathbf{Conv}}: \text{Conv} \to 1_{(\mathbb{T},\mathbf{2})\text{-}\mathbf{Conv}}$$

is a natural transformation. In particular, since $\text{Conv}(p_{(X,r)}) = p_{\text{Conv}(X,r)}$ for each $\mathbb{T}$-relational algebra (X,r), the pair (Conv, p) is a well copointed endofunctor. The functor Conv, in case $\mathbb{T}$ is the ultrafilter monad $\mathbb{U}$, is precisely the functor Ult in [9].

We can now define endofunctors Conv^{α} and natural transformations p^{α}_{β} for ordinal numbers α,β with $\beta \leq \alpha$, by:

- $\text{Conv}^0 = 1_{(\mathbb{T},\mathbf{2})\text{-}\mathbf{Conv}}$ and $p^0_0 = 1_{\text{Conv}^0}$;
- $\text{Conv}^{\alpha+1} = \text{Conv}(\text{Conv}^{\alpha})$, $p^{\alpha+1}_{\beta} = p^{\alpha}_{\beta} \cdot p\text{Conv}^{\alpha}$ and $p^{\alpha+1}_{\alpha+1} = 1_{\text{Conv}^{\alpha+1}}$, for $\beta \leq \alpha$;
- $\text{Conv}^{\lambda} = \lim_{\beta\leq\alpha<\lambda} p^{\alpha}_{\beta}$, $p^{\lambda}_{\beta} =$ the limit projection and $p^{\lambda}_{\lambda} = 1_{\text{Conv}^{\lambda}}$, for every limit ordinal λ and every $\beta < \lambda$.

Following the notation of [9], From now on, for a $\mathbb{T}$-relational algebra we write just X instead of (X,r), and we denote $\text{Conv}^{\alpha}(X)$ simply by X_{α} and $\text{Conv}^{\alpha}(f)$ by f_{α}, for every $\mathbb{T}$-relational morphism

$f: X \to Y$ between $\mathbb{T}$-relational algebras.
The transfinite construction can be described in the following way: for each $\mathbb{T}$-relational algebra X and each ordinal α,

$$X_\alpha = \{((\mathfrak{x}_\beta)_{\beta\in\alpha}, x) \in \prod_{\beta\in\alpha} T(X_\beta) \times X \quad | \quad \mathfrak{x}_0 \to x \quad \text{and} \quad (\forall \gamma \leq \beta < \alpha) \quad T(p^\beta_\gamma)_X(\mathfrak{x}_\beta) = \mathfrak{x}_\gamma\},$$

where for each $\beta \leq \alpha$, the projection $(p^\alpha_\beta)_X : X_\alpha \to X_\beta$ is defined by

$$(p^\alpha_\beta)_X((\mathfrak{x}_\gamma)_{\gamma\in\alpha}, x) = ((\mathfrak{x}_\gamma)_{\gamma\in\beta}, x),$$

and the $\mathbb{T}$-relational structure on X_α is defined by

$$\mathfrak{x}_\alpha \to ((\mathfrak{x}_\beta)_{\beta\in\alpha}, x) \quad \iff \quad (\forall \beta \in \alpha) \quad T(p^\alpha_\beta)_X(\mathfrak{x}_\alpha) = \mathfrak{x}_\beta.$$

Finally, if $f: X \to Y$ is a $\mathbb{T}$-relational morphism, then, for each ordinal α and each $((\mathfrak{x}_\beta)_{\beta\in\alpha}, x) \in X_\alpha$,

$$f_\alpha((\mathfrak{x}_\beta)_{\beta\in\alpha}, x) = (Tf_\beta(\mathfrak{x}_\beta)_{\beta\in\alpha}, f(x)).$$

For each ordinal α, and each $\mathbb{T}$-relational space X, an element of $X_{\alpha+1}$ is given by an element $\mathfrak{x}_\alpha \in T(X_\alpha)$ and an element $x \in X$ such that $T(p^\alpha_0)_X(\mathfrak{x}_\alpha) \to x$. The map $f_{\alpha+1} : X_{\alpha+1} \to Y_{\alpha+1}$ is surjective if and only if, for each element $\mathfrak{y}_\alpha$ in $T(Y_\alpha)$ and each $y \in Y$ such that $T(p^\alpha_0)_Y(\mathfrak{y}_\alpha) \to y$, there exist an element $\mathfrak{x}_\alpha$ in $T(X_\alpha)$ and an $x \in f^{-1}(y)$ such that $T(p^\alpha_0)_X(\mathfrak{x}_\alpha) \to x$ and $Tf_\alpha(\mathfrak{x}_\alpha) = \mathfrak{y}_\alpha$.

Definition 3.2.5 If α is an ordinal number, a $\mathbb{T}$-relational morphism $f: X \to Y$ is said to be α-*surjective* if $f_\beta : X_\beta \to Y_\beta$ is surjective for each $\beta \in \alpha$. The morphism f is called Ω-*surjective* if f_α is surjective for every ordinal α.

It immediately follows that 1-surjective maps are just surjective maps, while 2-surjective maps are the final $(\mathbb{T}, \mathbf{2})$-functors.

Definition 3.2.6 A $(\mathbb{T}, \mathbf{2})$-functor $f : (X, a) \to (Y, b)$ is called *triquotient map* if f is Ω-surjective, that is,

$$\begin{array}{ccccccccccc}
X & & \cdots & \dashrightarrow & \mathfrak{x}_{\alpha+1} & \dashrightarrow & \mathfrak{x}_\alpha & \dashrightarrow \cdots \dashrightarrow & \mathfrak{x}_1 & \longrightarrow & x_0 \\
{\scriptstyle f}\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y & & \cdots & \longrightarrow & \mathfrak{y}_{\alpha+1} & \longrightarrow & \mathfrak{y}_\alpha & \longrightarrow \cdots \longrightarrow & \mathfrak{y}_1 & \longrightarrow & y_0.
\end{array}$$

If we consider the ultrafilter monad $\mathbb{U} = (U, \mu, \eta)$, the definition above of triquotient map gives the notion of triquotient map in **Top**, thanks to the following characterization theorem.

Theorem 3.2.7 [9, Theorem 6.4] *Let $f : X \to Y$ be a continuous map between topological spaces. The following conditions are equivalent:*

(i) f is a triquotient map;

(ii) f is Ω-surjective;

(iii) f is λ_Y-surjective;

where λ_Y is the least cardinal larger than the cardinal of Y.

The fact that, in **Top**, triquotient maps are effective for descent was first proved by T. Plewe in [46]. Later on, in [9], M.M. Clementino and D. Hofmann improved the result characterizing the effective descent maps as those morphisms which are 3-surjective. In particular,

Theorem 3.2.8 [9, Theorem 5.2] *A topological continuous map $f : X \to Y$ is effective for descent if and only if it is 3-surjective.*

We apply Theorem 3.0.6 to prove that triquotient maps in $(\mathbb{T}, \mathbf{2}, \overline{\mathbb{T}})$-**Cat**, where $\overline{\mathbb{T}}$ is the Barr extension of a **Set**-monad $\mathbb{T}$, are effective for descent.

Proposition 3.2.9 *If T satisfies BC, then 3-surjective maps are pullback stable in $(\mathbb{T}, \mathbf{2})$-**Gph**.*

Proof
Let $f : X \to Y$ be a 3-surjective map between $(\mathbb{T}, \mathbf{2})$-graphs. In the following pullback diagram in $(\mathbb{T}, \mathbf{2})$-**Gph**,

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{\pi_2} & Z \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

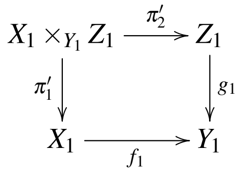
we want to prove that $\pi_2 : X \times_Y Z \to Z$ is a 3-surjective map, i.e., that $(\pi_2)_2 : (X \times_Y Z)_2 \to Z_2$ is surjective. Let then $\mathfrak{z}_1 \in T(Z_1)$ and $z \in Z$ such that $T(p_0^1)_Z(\mathfrak{z}_1) \to z$. Since p_0^1 is a natural transformation, $T(p_0^1)_Y(Tg_1(\mathfrak{z}_1)) \to g(z)$. Hence, since $f_2 : X_2 \to Y_2$ is surjective, there exist $\mathfrak{x}_1 \in T(X_1)$ and $x \in X$ such that $Tf_1(\mathfrak{x}_1) = Tg_1(\mathfrak{z}_1)$, $f(x) = g(z)$ and $T(p_0^1)_X(\mathfrak{x}_1) \to x$. Therefore the pair $(\mathfrak{x}_1, \mathfrak{z}_1)$ belongs to $T(X_1) \times_{T(Y_1)} T(Z_1)$.
Since

$$\begin{array}{ccc} (X \times_Y Z)_1 & \xrightarrow{(\pi_2)_1} & Z_1 \\ {\scriptstyle (\pi_1)_1}\downarrow & & \downarrow{\scriptstyle g_1} \\ X_1 & \xrightarrow[f_1]{} & Y_1 \end{array}$$

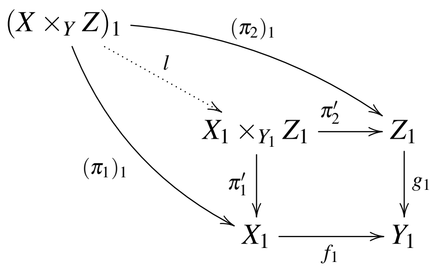
is a commutative diagram, there exists a unique map k

$$\begin{array}{ccccc} T((X \times_Y Z)_1) & & \xrightarrow{T(\pi_2)_1} & & \\ & \overset{k}{\searrow} & & & \\ & & T(X_1) \times_{T(Y_1)} T(Z_1) & \xrightarrow{pr_2} & T(Z_1) \\ & {\scriptstyle T(\pi_1)_1}\searrow & {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle Tg_1} \\ & & T(X_1) & \xrightarrow[Tf_1]{} & T(Y_1). \end{array}$$

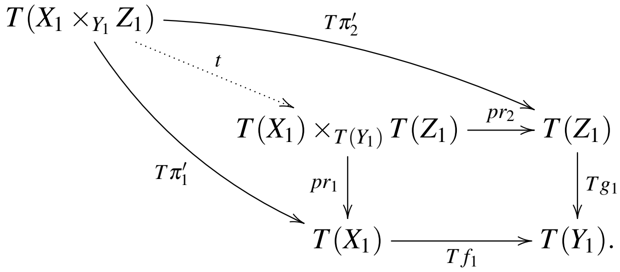
such that $pr_1 \cdot k = T(\pi_1)_1$ and $pr_2 \cdot k = T(\pi_2)_1$. We want to show that k is surjective. Consider the pullback diagram in **Set**



By the universal property, there exists a unique map l



such that $\pi_1' \cdot l = (\pi_1)_1$ and $\pi_2' \cdot l = (\pi_2)_1$. Since T satisfies BC, l is surjective.
Moreover, always by the universal property, there exists a unique map t



such that $pr_1 \cdot t = T\pi_1'$ and $pr_2 \cdot t = T\pi_2'$. Since T satisfies BC, t is surjective.
Since the composition $t \cdot Tl$ satisfies

$$pr_1 \cdot (t \cdot Tl) = T\pi_1' \cdot Tl = T(\pi_1' \cdot l) = T(\pi_1)_1, \quad pr_2 \cdot (t \cdot Tl) = T\pi_2' \cdot Tl = T(\pi_2' \cdot l) = T(\pi_2)_1$$

we conclude that $k = t \cdot Tl$, so that k is, in particular, a surjective map as claimed. Hence, there exists an element $\mathfrak{w}_1 \in T(X \times_Y Z)_1$ such that $k(\mathfrak{w}_1) = t(Tl(\mathfrak{w}_1)) = (\mathfrak{x}_1, \mathfrak{z}_1)$. Therefore, $T(\pi_2)_1(\mathfrak{w}_1) = pr_2(k(\mathfrak{w}_1)) = \mathfrak{z}_1$. It remains to show that $T(p_0^1)_{X \times_Y Z}(\mathfrak{w}_1) \to (x, z)$, i.e.,

$$T\pi_1(T(p_0^1)_{X \times_Y Z}(\mathfrak{w}_1)) \to x \quad \text{and} \quad T\pi_2(T(p_0^1)_{X \times_Y Z}(\mathfrak{w}_1)) \to z.$$

Since p_0^1 is a natural transformation,

$$T\pi_1(T(p_0^1)_{X \times_Y Z}(\mathfrak{w}_1)) = T(p_0^1)_X(T(\pi_1)_1)(\mathfrak{w}_1) = T(p_0^1)_X(pr_1(k(\mathfrak{w}_1))) = T(p_0^1)_X(\mathfrak{x}_1) \to x,$$

$$T\pi_2(T(p_0^1)_{X \times_Y Z}(\mathfrak{w}_1)) = T(p_0^1)_Z(T(\pi_2)_1(\mathfrak{w}_1)) = T(p_0^1)_Z(pr_2(k(\mathfrak{w}_1))) = T(p_0^1)_Z(\mathfrak{z}_1) \to z. \quad \square$$

We may now ask whether 3-surjective maps are effective for descent. The fact that 3-surjective in $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat** maps are *-quotient morphisms has been proved in [20] by M.M. Clementino and G. Janelidze.

Proposition 3.2.10 [20, Problem 2.2] *A* $(\mathbb{T},\mathbf{2})$*-functor* $f:(X,a)\to(Y,b)$ *in* $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat** *is a *-quotient morphism provided that it is 3-surjective.*

Therefore the following theorem holds.

Theorem 3.2.11 *A* $(\mathbb{T},\mathbf{2})$*-functor* $f:(X,a)\to(Y,b)$ *in* $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat** *is an effective descent morphism provided that it is a triquotient map.*

Proof
It immediately follows from this chain of implications:

$$\begin{gathered}\text{triquotient}\\ \Downarrow\\ \text{3-surjective}\\ \Downarrow\\ \text{pullback stable *-quotient in } (\mathbb{T},\mathbf{2})\text{-}\mathbf{Gph}\\ \Downarrow\\ \text{effective descent}\end{gathered} \tag{3.2}$$

where the first one, starting from the top-side, follows by Definition 3.2.6, the second by Propositions 3.2.9 and 3.2.10, and the last one by Theorem 3.0.6. □

In the chain of implications given in the proof of the theorem above, the converse of the last implication is not true in general, as we already mentioned after having stated Theorem 3.0.6. A counter-example is given in **MultiOrd**, where a monotone map is of effective descent if and only if it is a *weak *-quotient map* (see Section 3.3.2). But it is true in **Top**, as it has been proved by M.M. Clementino and G. Janelidze in [20].

Theorem 3.2.12 [20, Theorem 3.3] *For a continuous map* $f:X\to Y$ *between topological spaces, the following conditions are equivalent:*

(i) *f is of effective descent in* **Top**;

(ii) *f is a pullback stable *-quotient map in* **PsTop**.

The converse of the second implication in (3.2) is true, accordingly to [20, Theorem 2.3], always considering the Barr extension, if the canonical map

$$T(A\times B)\to TA\times TB \tag{3.3}$$

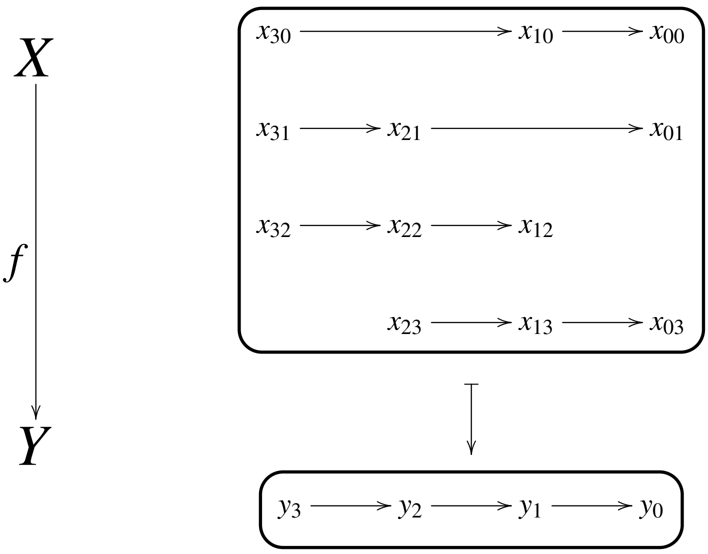
is injective for every two sets A and B, since *-quotient morphisms, assuming the condition above, are proved to be 3-surjective. In case of the Barr extension to **Rel** of a given **Set**-monad $\mathbb{T}=(T,\mu,\eta)$, assuming that every naturality square of η with respect to relations with finite fibres is a BC-square, the following diagram of implications summarizes the general situation in $(\mathbb{T},\mathbf{2})$-**Cat**.

triquotient				
$\Downarrow$				
3-surjective	$\Rightarrow$	pullback stable *-quotient in $(\mathbb{T},\mathbf{2})$-**Gph**	$\Rightarrow$	effective global-descent
$\Downarrow$		$\Downarrow$		$\Downarrow$
2-surjective	$\Leftrightarrow$	pullback stable regular epimorphism	$\Leftrightarrow$	global-descent
$\Downarrow$		$\Downarrow$		
1-surjective	$\Leftarrow$	regular epimorphism.		

Counter-examples of most of the one-direction implications can be given if we take $\mathbb{T}=\mathbb{I}$, the identity monad. In this case $(\mathbb{T},\mathbf{2})$-**Cat** $\cong$ **Ord** and here the notion of 3-surjective map coincides with the notion of *-quotient morphism. In Section 1.4.4 we already gave an example of a pullback stable regular epimorphism not *-quotient (see Example 1.4.20) and an example of a regular epimorphism not pullback stable (see Example 1.4.21). The 1-surjective morphisms are simply the surjective $(\mathbb{T},\mathbf{2})$-functors, so that Example 1.4.21 gives a 1-surjective map not 2-surjective. Of course the 1-surjective morphisms need not be regular epimorphisms. An easy counter-example can be given in **Top**, simply considering the identity map of a set X, with at least two points, equipped for the domain with the discrete topology and for the codomain with indiscrete topology. The map is continuous and surjective but it fails to be quotient. The converse of the implication

$$\text{3-surjetive} \Rightarrow \text{pullback stable *-quotient in } (\mathbb{T},\mathbf{2})\text{-}\mathbf{Gph}$$

is still an open problem, in the sense we do not know if it is true in general (although we know to be true in **Top** and for all **Set**-monads $\mathbb{T}=(T,\mu,\eta)$ for which the canonical map (3.3) is injective for any two sets A and B). As we mentioned previously, an example of an effective descent morphism not pullback stable *-quotient in $(\mathbb{T},\mathbf{2})$-**Gph** can be given in **MultiOrd**. By Theorem 3.2.2 we can easily construct a 3-surjective map not triquotient (we omit the composition arrows in the picture):



All the sequences in Y of length 2 and 1 can be lifted in X but for $n = 3$ the sequence $y_3 \to y_2 \to y_1 \to y_0$ has no lifting in X. In case of **Top** one has

3-surjective $\Leftrightarrow$ effective descent $\Leftrightarrow$ pullback stable *-quotient in **PsTop**.

One may now ask: does Theorem 3.2.11 remain true for extensions different from the Barr extensions? We start remarking that Proposition 3.2.9 is true for any lax extension of the **Set**-monad $\mathbb{T} = (T, \mu, \eta)$, as long as T satisfies BC. Therefore pullback stability of 3-surjective maps is guaranteed. It remains to ask whether Proposition 3.2.10 still holds for different kinds of extensions. It is the case for instance, as we show next, if we consider the two lax extensions $\check{\mathbb{P}}$ and $\hat{\mathbb{P}}$ of the powerset monad $\mathbb{P}$ given in Example 2.2.6.

Proposition 3.2.13 *A* $(\mathbb{P}, \mathbf{2}, \check{\mathbb{P}})$*-functor* $f : (X, a) \to (B, b)$ *is an effective descent morphism provided that it is a triquotient map.*

Proof

We only need to show that if f is 3-surjective then it is a *-quotient morphism. Let $\mathfrak{B} \in P^2Y$, $B \in PY$ and $y_0 \in Y$ such that

$$\mathfrak{B} \to B \to y_0, \tag{3.4}$$

where we recall that $\mathfrak{B} \to B$ means that for every $B' \in \mathfrak{B}$ there exists $y \in B$ such that $B' \to y$. By assumption, we know that $f_2 : X_2 \to Y_2$ is surjective, that is, for every $\mathfrak{b}_1 \in PY_1$ and $y_0 \in Y$ such that $P(p_0^1)_Y(\mathfrak{b}_1) \to y_0$ there exist $\mathfrak{a}_1 \in PX_1$ and $x_0 \in f^{-1}(y_0)$ such that $P(p_0^1)_X(\mathfrak{a}_1) \to x$ and $Pf_1(\mathfrak{a}_1) = \mathfrak{b}_1$, where

$$X_1 = \{(A, x) \in PX \times X : A \to x\}, \quad Y_1 = \{(B, y) \in PY \times Y : B \to y\}$$

and $f_1 : X_1 \to Y_1$ is defined by $(A, x) \mapsto (Pf(A), f(x))$. Hence, given the chain of convergence (3.4), let

$$\mathfrak{b}_1 = \{(B', y) \in PY \times Y : B' \in \mathfrak{B}, y \in B \quad \text{with} \quad B' \to y\} \cup \{(\{\tilde{y}\}, \tilde{y}) \in PY \times Y : \tilde{y} \in \tilde{B}\},$$

where $\tilde{B} = B - \{y \in B : \exists B' \in \mathfrak{B} \quad \text{with} \quad B' \to y\}$. Basically in the definition of $\mathfrak{b}_1$ we are taking, on the left-side of the join, all the elements $B' \in \mathfrak{B}$. Only this part does not guarantee the presence of all elements of B, because of the definition of the convergence, so that we need to add them and this is precisely the right-side part of the join. Now $\mathfrak{b}_1 \in PY_1$ and $P(p_0^1)_Y(\mathfrak{b}_1) = B \to y_0$. Therefore, since f_2 is surjective, there exist $\mathfrak{a}_1 \in PX_1$, $x_0 \in f^{-1}(y_0)$ such that

$$P(p_0^1)_X(\mathfrak{a}_1) \to x_0 \quad \text{and} \quad Pf_1(\mathfrak{a}_1) = \mathfrak{b}_1.$$

Since p_0^1 is a natural transformation,

$$Pf(P(p_0^1)_X)(\mathfrak{a}_1) = P(p_0^1)_Y(Pf_1(\mathfrak{a}_1)) = P(p_0^1)_Y(\mathfrak{b}_1) = B.$$

The element $\mathfrak{a}_1 \in PX_1$ is given by pairs $(A,x) \in PX \times X$ such that $A \to x$. Taking

$$\mathfrak{A} = \{A \in \mathfrak{a}_1\} - \{\tilde{A} \in \mathfrak{a}_1 : Pf(\tilde{A}) = \{\tilde{y}\}\}$$

we get an element $\mathfrak{A} \in P^2X$ such that $P^2f(\mathfrak{A}) = \mathfrak{B}$ and $\mathfrak{A} \to P(p_0^1)_X(\mathfrak{a}_1)$. □

Proposition 3.2.14 *A* $(\mathbb{P},\mathbf{2},\hat{\mathbb{P}})$*-functor* $f : (X,a) \to (B,b)$ *is an effective descent morphism provided that it is a triquotient map.*

Proof
As one can expect, the proof is going to be similar to the previous one with the only difference that we have to be careful with the different extension we are considering. Let $\mathfrak{B} \in P^2Y$, $B \in PY$ and $y_0 \in Y$ such that

$$\mathfrak{B} \to B \to y_0,$$

where we recall that $\mathfrak{B} \to B$ means that for every $y \in B$ there exists $B' \in \mathfrak{B}$ such that $B' \to y$. Let $\mathfrak{b}_1 \in PY_1$ be given by

$$\mathfrak{b}_1 = \{(B',y) \in PY \times Y : B' \in \mathfrak{B}, y \in b \quad \text{with} \quad B' \to y\}.$$

Then $P(p_0^1)_Y(\mathfrak{b}_1) = B \to y_0$. Since $f_2 : X_2 \to Y_2$ is surjective, there exist $\mathfrak{a}_1 \in PX_1$, $x_0 \in f^{-1}(y_0)$ such that $P(p_0^1)_X(\mathfrak{a}_1) \to x_0$ and $Pf_1(\mathfrak{a}_1) = \mathfrak{b}_1$. Since p_0^1 is a natural transformation

$$Pf(P(p_0^1)_X)(\mathfrak{a}_1) = P(p_0^1)_Y(Pf_1(\mathfrak{a}_1)) = P(p_0^1)_Y(\mathfrak{b}_1) = B.$$

The element $\mathfrak{a}_1 \in PX_1$ is given by pairs $(A,x) \in PX \times X$ such that $A \to x$. This time, starting from the elements A in $\mathfrak{a}_1$, we need to add all the subsets $f^{-1}(B')$ for $B' \in \mathfrak{B}$ such that there are no points $y \in B$ with $B' \to y$, that is, define $\mathfrak{A} \in P^2X$ by

$$\mathfrak{A} = \{A \in \mathfrak{a}_1\} \cup \{f^{-1}(B') : B' \in \tilde{\mathfrak{B}}\},$$

where $\tilde{\mathfrak{B}} = \mathfrak{B} - \{B' \in \mathfrak{B} : \text{there are no elements } y \text{ in } B \text{ for which } B' \to y\}$. Then $P^2f(\mathfrak{A}) = \mathfrak{B}$ and $\mathfrak{A} \to P(p_0^1)_X(\mathfrak{a}_1)$. □

3.3 From Rel to V-Rel: the problem of effective descent morphisms

In Section 2.2.2 we saw how to construct the uniform lax extension $\widetilde{\mathbb{T}} = (\widetilde{T}, \mu, \eta)$ to **V**-**Rel** of a **Set**-monad $\mathbb{T} = (T, \mu, \eta)$ through its Barr extension $\overline{\mathbb{T}}$. In particular, if T satisfies BC, **V** is ccd and $k_{\mathbf{V}} = \top_{\mathbf{V}}$ or $\overline{T}$ preserves the $\perp$-relation, then, for $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$ and every **V**-relation $r : X \nrightarrow Y$,

$$\widetilde{T}r(\mathfrak{x}, \mathfrak{y}) = \bigvee\{v \in \mathbf{V} : \overline{T}r_v(\mathfrak{x}, \mathfrak{y}) = \top\}$$

where $r_v : X \nrightarrow Y$ is the relation defined by

$$r_v(x,y) = \begin{cases} \top, & \text{if } v \leq r(x,y) \\ \perp, & \text{otherwise,} \end{cases}$$

defines a (flat) lax extension of $\mathbb{T}$ to **V**-**Rel**. Since the definition of the lax extension $\widetilde{T}$ is strictly related to the Barr extension $\overline{T}$, and it is actually an extension of it, a question arises: is there any relation between effective descent morphisms in the category $(\mathbb{T}, \mathbf{2}, \overline{\mathbb{T}})$-**Cat** and effective descent morphisms in $(\mathbb{T}, \mathbf{V}, \widetilde{\mathbb{T}})$-**Cat**? In case $(\mathbb{T}, \eta, \mu)$ is the identity monad $(\mathbb{I}, 1, 1)$, we saw in Example 2.3.3 that $(\mathbb{I}, \mathbf{2}, \overline{\mathbb{I}})$-**Cat** $\cong$ **Ord**, the category of (pre)ordered sets, and $(\mathbb{I}, \mathbf{V}, \widetilde{\mathbb{I}})$-**Cat** $=$ **V**-**Cat**, the category of **V**-categories. According to Proposition 1.4.13, in **Ord** effective descent morphisms coincide with *-quotient morphisms, that is a monotone map $f : X \to Y$ in **Ord** is of effective descent if and only if

$$\forall \quad y_2 \to y_1 \to y_0 \quad \text{in } B \quad \exists \quad x_2 \to x_1 \to x_0 \quad \text{in } E : \forall i = 0,1,2 \quad p(x_i) = y_i.$$

As it has been proved in [15] by M.M. Clementino and D. Hofmann, in **V**-**Cat** one can get an analogous result but a condition on **V** is required.

Definition 3.3.1 For a ccd quantale **V** one says that **V** is *cancellable* if for all $u, v \in \mathbf{V} - \{\perp\}$, for all families $(u_i)_{i \in I}, (v_i)_{i \in I}$ in **V** with $u_i \leq u$ and $v_i \leq v$ for every $i \in I$

$$\bigvee_{i \in I}(u_i \otimes v_i) \geq u \otimes v \Rightarrow \forall u' \ll u, v' \ll v \quad \exists i \in I : u' \leq u_i \quad \text{and} \quad v' \leq v_i.$$

Theorem 3.3.2 [15, Proposition 6.3] *If* **V** *is cancellable and cartesian closed, then the following conditions are equivalent, for a morphism* $f : (X,a) \to (Y,b)$ *in* **V**-**Cat**.

(i) f *is effective for descent;*

(ii) f *is a* $*$*-quotient morphism;*

(ii) f *is a* $**$*-quotient morphism.*

For a **V**-functor $f : (X,a) \to (Y,b)$ to be a *-quotient morphism means that

$$\forall y_2, y_1, y_0 \in Y \quad b(y_2,y_1) \otimes b(y_1,y_0) = \bigvee_{x_i \in f^{-1}(y_i)} a(x_2,x_1) \otimes a(x_1,x_0),$$

while f is called ***-quotient* if

$$\forall y_2, y_1, y_0 \in Y, u \ll b(y_2,y_1), v \ll b(y_1,y_0) \quad \exists x_i \in f^{-1}(x_i), \text{for } i = 0,1,2, \text{ such that}$$

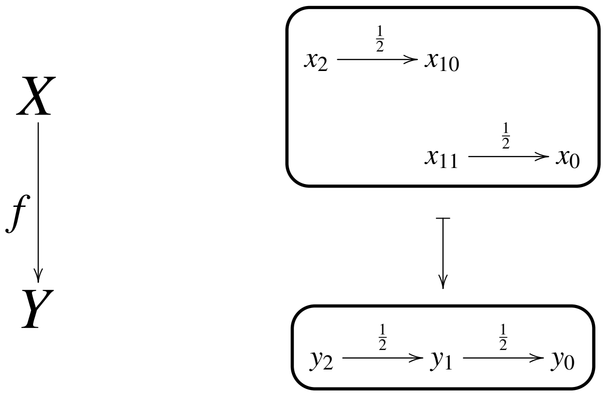
$$u \leq a(x_2, x_1), \quad v \leq a(x_1, x_0).$$

Every **-quotient morphism is *-quotient and the converse is true if **V** is cancellable. In Example 2.1.2, the quantales **2**, $\mathbf{R}_+$ and $\mathbf{I}_*$ are cancellable while $\mathbf{I}_\oplus$ is not cancellable; in fact, as remarked in [15, Example 2.5], for $u_i = 0 = v_i$ and $u = \frac{1}{2} = v$ one has $0 = \bigvee(u_i \oplus v_i) = u \oplus v$ but $\bigvee u_i = 0 \neq u$. This suggests also that the quantale $\mathbf{P} = (P(M), \otimes, k)$ is not cancellable; in fact one can take $M = \mathbf{I}_\oplus$, considered as a monoid, and the elements above giving $\mathbf{I}_\oplus$ not cancellable as one-point set elements. Anyway in [15] the following results are given.

Theorem 3.3.3 [15, Theorem 6.4 (2)] *Let $\otimes$ be a continuous quantale structure on $[0,1]$ with neutral element 1. Then a $[0,1]_\otimes$-functor is effective for descent in $[0,1]_\otimes$-***Cat** *if and only if it is a *-quotient morphism.*

This of course applies for $\otimes = \oplus$ but, as remarked in the same paper, an effective $[0,1]_\otimes$-functor does not need to be **-quotient. The counter-example is given precisely in $\mathbf{I}_\oplus$.

Example 3.3.4 [15, Remark 6.4 (2)] An easy inspection reveals that the following $\mathbf{I}_\oplus$-functor fails to be **-quotient but, since $\frac{1}{2} \oplus \frac{1}{2} = 0$, it is a *-quotient morphism. The class of *-quotient morphisms in $[0,1]_\otimes$-**Cat**, where $\otimes$ is a continuous quantale structure with neutral element 1, is pullback stable (as proved in [15]) and so effective for descent.



Theorem 3.3.5 [15, Theorem 6.4 (4)] *If M is a non-trivial monoid, then:*

(1) *A **P**-functor is effective for descent in **P**-***Cat** *if and only if it is a **-quotient morphism;*

(2) *There are *-quotient morphisms in **P**-***Cat** *which are not effective for descent.*

Inspired by the definition of **-quotient morphisms given above for **V**-functors, and considering a lax extension $\widehat{\mathbb{T}}$ of a **Set**-monad $\mathbb{T}$ to **V**-**Rel**, one can, more generally, call a $(\mathbb{T}, \mathbf{V})$-functor $f : (X,a) \to (Y,b)$ a ***-quotient morphism* if

$$\forall \mathfrak{Y} \in T^2Y, \mathfrak{y} \in TY, y \in Y, u \ll \widehat{T}b(\mathfrak{Y}, \mathfrak{y}), v \ll b(\mathfrak{y}, y) \quad \exists \mathfrak{X} \in (T^2 f)^{-1}(\mathfrak{Y}), \mathfrak{x} \in (Tf)^{-1}(\mathfrak{y}), x \in f^{-1}(y)$$

such that

$$u \leq \widehat{T}a(\mathfrak{X}, \mathfrak{x}), \quad v \leq a(\mathfrak{x}, x).$$

Also in this case every **-quotient morphism is *-quotient and the converse is true if **V** is cancellable.

3.3.1 Effective descent morphisms in $(\mathbb{M},\mathbf{V},\widetilde{\mathbb{M}})$-Cat

In Example 2.2.13 we described the uniform extension $\widetilde{\mathbb{M}}$ to **V**-**Rel** of the M-ordered monad $\mathbb{M} = (M^{\times},\mu,\eta)$. Our aim is to study effective descent morphisms in $(\mathbb{M},\mathbf{V},\widetilde{\mathbb{M}})$-**Cat** and see if there is any relation with the effective descent morphism in M-**Ord** $= (\mathbb{M},\mathbf{2},\overline{\mathbb{M}})$-**Cat**. A complete characterization of effective descent morphisms in M-**Ord** is given in [18] by M.M. Clementino, D. Hofmann and A. Montoli.

Proposition 3.3.6 [18, Theorem 1.8] *An equivariant map $f:(X,a)\to(Y,b)$ in M-**Ord** is of effective descent if and only if it is a *-quotient map, i.e.,*

$$\forall \quad y_2 \xrightarrow{m} y_1 \xrightarrow{n} y_0 \quad \text{in } B \quad \exists \quad x_2 \xrightarrow{m} x_1 \xrightarrow{n} x_0 \quad \text{in } E : \forall i=0,1,2 \quad p(x_i)=y_i.$$

In order to give a complete characterization of the effective descent morphisms in $(\mathbb{M},\mathbf{V},\widetilde{\mathbb{M}})$-**Cat**, we start by describing the category: objects are pairs (X,a) where X is a set and $a:(M\times X)\times X\to\mathbf{V}$ is a map satisfying the following properties

- $k\le a((1_M,x),x)$, $\forall x\in X$
- $a((m,x),x')\otimes a((n,x'),x'')\le a((n\cdot m,x),x'')$, $\forall x,x',x''\in X, m,n\in M$.

A morphism $f:(X,a)\to(Y,b)$ in $(\mathbb{M},\mathbf{V},\widetilde{\mathbb{M}})$-**Cat** is a map $f:X\to Y$ such that

- $a((m,x),x')\le b((m,f(x)),f(x'))$, $\forall x,x'\in X, m\in M$.

Observe that, if $M=1$, the trivial monoid, considered as one-object category, $(\mathbb{M},\mathbf{V},\widetilde{\mathbb{M}})$-**Cat** is isomorphic to the category **V**-**Cat** of **V**-categories and **V**-functors, as it happens for M-**Ord** where, if $M=1$, it is isomorphic to the category **Ord** of (pre)ordered sets and monotone maps. Hence the characterization of the effective descent morphisms in $(\mathbb{M},\mathbf{V},\widetilde{\mathbb{M}})$-**Cat** will be a generalization of Theorem 3.3.2. We base our work on the results given in [11] for **V**-**Cat** and adapt them for our purpose.

Lemma 3.3.7 *Let $f:(X,a)\to(Y,b)$ be an $(\mathbb{M},\mathbf{V})$-functor such that its change of base functor f^* reflects isomorphisms. Then f is a regular epimorphism in $(\mathbb{M},\mathbf{V},\widetilde{\mathbb{M}})$-**Gph**.*

Proof
Let $f:(X,a)\to(Y,b)$ be a $(\mathbb{M},\mathbf{V},\widetilde{\mathbb{M}})$-functor such that its change of base functor f^* reflects isomorphisms. Let $y_1,y_0\in Y$ and $m\in M$. Then

$$\alpha := \bigvee_{x_i\in f^{-1}(y_i)} a((m,x_1),x_0)\le b((m,y_1),y_0) := \beta.$$

We want to show the other inequality. Take the set $2=\{0,1\}$ equipped with the structures b_α and b_β defined as follows:

- $b_\alpha((m,1),0)=\alpha$,
- $b_\alpha((n,1),0)=\bot$, if $n\neq m$,

- $b_\alpha((1_M,0),0) = b_\alpha((1_M,1),1) = k$,
- $b_\alpha((n,0),1) = \bot$, for each $n \in M$,
- $b_\alpha((n,1),1) = b_\alpha((n,0),0) = \bot$, for each $n \neq 1_M$ in M;

while b_β equals b_α except for $((m,1),0)$ where

- $b_\beta((m,1),0) = \beta$.

Then $(2,b_\alpha)$ and $(2,b_\beta)$ are $(\mathbb{M},\mathbf{V},\widetilde{\mathbb{M}})$-categories. Consider the $(\mathbb{M},\mathbf{V})$-functors

$$g_\alpha : (2,b_\alpha) \to (Y,b) \quad g_\beta : (2,b_\beta) \to (Y,b)$$

defined both by $i \mapsto y_i$ for $i = 0,1$. We prove that the image under f^* of the identity map $id_2 : (2,b_\alpha) \to (2,b_\beta)$ is an isomorphism. The morphism we refer to is $id_X \times_Y id_2 : (X \times_Y 2, d_\alpha) \to (X \times_Y 2, d_\beta)$, where the structures $d_\alpha, d_\beta : M \times (X \times_Y 2) \times (X \times_Y 2) \to \mathbf{V}$ are described by

$$d_\alpha(n,(x,i),(x',i')) = a((n,x),x') \wedge b_\alpha((n,i),i')$$

$$d_\beta(n,(x,i),(x',i')) = a((n,x),x') \wedge b_\beta((n,i),i'),$$

for each $n \in M$, $i,i' \in 2$ and $x,x' \in X$ such that $f(x) = y_1$ and $f(x') = y_{i'}$. Since $id_X \times_Y id_2$ is an $(\mathbb{M},\mathbf{V})$-functor, we already have that $d_\alpha \leq d_\beta$, and since it also a **Set**-isomorphism, it remains to show that $d_\beta \leq d_\alpha$. We have

$$a((m,x_1),x_0) \wedge b_\beta((m,1),0) = a((m,x_1),x_0) \leq \bigvee_{x_i \in f^{-1}(y_i)} a((m,x_1),x_0)).$$

The equality follows from the definition of the structure b_β and from the fact that $a((m,x_1),x_0) \leq b((m,y_1)y_0)$, since f is an $(\mathbb{M},\mathbf{V})$-functor. The other cases are trivially satisfied. Hence, since by hypothesis f^* reflects isomorphisms, we conclude that $id_2 : (2,b_\alpha) \to (2,b_\beta)$ is an isomorphism as well, giving rise to the claimed inequality. □

Proposition 3.3.8 *If* $\mathbf{V}$ *is cartesian closed, then every effective descent morphism in* $(\mathbb{M},\mathbf{V},\widetilde{\mathbb{M}})$-**Cat** *is a* $*$*-quotient morphism.*

Proof
Let $f : (X,a) \to (Y,b)$ be an effective descent morphism in $(\mathbb{M},\mathbf{V},\widetilde{\mathbb{M}})$-**Cat**. The pullback functor f^* reflects isomorphisms. By Lemma 3.3.7, f is a regular epimorphism in $(\mathbb{M},\mathbf{V},\widetilde{\mathbb{M}})$-**Gph** and so an effective descent morphism in this category as well. We want to show that f is a *-quotient morphism, that is, for each $y_0,y_1,y_2 \in Y$, $m,n \in M$

$$b((m,y_2),y_1) \otimes b((n,y_1),y_0) = \bigvee_{x_i \in f^{-1}(y_i)} a((m,x_2),x_1) \otimes a((n,x_1),x_0).$$

Assume that there are elements $y_0,y_1,y_2 \in Y$, $m,n \in M$ such that

$$\bigvee_{x_i \in f^{-1}(y_i)} a((m,x_2),x_1) \otimes a((n,x_1),x_0) < b((m,y_2),y_1) \otimes b((n,y_1),y_0) \leq b((n \cdot m,y_2),y_0).$$

Let us call

$$\alpha := \bigvee_{x_i \in f^{-1}(y_i)} a((m,x_2),x_1) \otimes a((n,x_1),x_0).$$

Define on the set $B = \{0,1,2\}$ the following reflexive and non-transitive structure b_α:

- $b_\alpha((m,2),1) = b((m,y_2),y_1)$,
- $b_\alpha((n,1),0) = b((n,y_1),y_0)$,
- $b_\alpha((n \cdot m,2),0) = \alpha$,
- $b_\alpha((1_M,0),0) = b_\alpha((1_M,1),1) = b_\alpha((1_M,2),2) = k$,
- $b_\alpha((l,i),j) = \bot$, elsewhere.

Consider then the $(\mathbb{M},\mathbf{V})$-functor $g : (B,b_\alpha) \to (Y,b)$ defined by $i \mapsto y_i$ for $i = 0,1,2$ and take its pullback along f in $(\mathbb{M},\mathbf{V},\widetilde{\mathbb{M}})$-**Gph**

$$\begin{array}{ccc} (X \times_Y B, d) & \xrightarrow{\pi_2} & (B, b_\alpha) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ (X,a) & \xrightarrow[f]{} & (Y,b), \end{array}$$

where d is the pullback structure on the set $X \times_Y B$ in $(\mathbb{M},\mathbf{V},\widetilde{\mathbb{M}})$-**Gph**. Since f is an effective descent morphism, $(X \times_Y B, d)$ is not transitive, i.e., there exist elements $(x,i),(x',i'),(x'',i'') \in X \times_Y B$ and $\bar{m},\bar{n} \in M$ such that

$$d((\bar{m},(x,i)),(x',i')) \otimes d((\bar{n},(x',i')),(x'',i'')) \nleq d((\bar{n} \cdot \bar{m},(x,i)),(x'',i'')).$$

The only possibility is for $i = 0, i' = 1, i'' = 2$ and $\bar{m} = m, \bar{n} = n$. We have

- $d((n \cdot m,(x_2,2)),(x_0,0)) = a((n \cdot m,x_2),x_0) \wedge b_\alpha((n \cdot m,2),0) = a((n \cdot m,x_2),x_0) \wedge \alpha$,
- $d((m,(x_2,2)),(x_1,1)) \otimes d((n,(x_1,1)),(x_0,0)) \leq a((m,x_2),x_1) \otimes a((n,x_1),x_0)$.

Hence,

$$a((m,x_2),x_1) \otimes a((n,x_1)x_0) \nleq a((n \cdot m,x_2),x_0) \wedge \alpha.$$

But since $a((m,x_2),x_1) \otimes a((n,x_1),x_0) \leq a((n \cdot m,x_2),x_0)$, we conclude that

$$a((m,x_2),x_1) \otimes a((n,x_1),x_0) \nleq \alpha,$$

which contradicts the definition of α. □

For an $(\mathbb{M},\mathbf{V})$-functor $f : (X,a) \to (Y,b)$ to be **-quotient morphism means that

$$\forall y_2, y_1, y_0 \in Y, m, n \in M, u \ll b((n,y_2),y_1), v \ll b((m,y_1),y_0)$$

$$\exists x_i \in f^{-1}(x_i), for\ i = 0,1,2 : u \leq a((n,x_2),x_1), v \leq a((m,x_1),x_0).$$

Theorem 3.3.9 *Let* **V** *be a ccd quantale. If* **V** *is cancellable and cartesian closed, then the following conditions are equivalent for a morphism* $f : (X,a) \to (Y,b)$ *in* $(\mathbb{M}, \mathbf{V}, \widetilde{\mathbb{M}})$-**Cat***:*

(i) f *is effective for descent;*

(ii) f *is a *-quotient morphism;*

(iii) f *is a **-quotient morphism.*

Proof
$(i) \Rightarrow (ii)$ follows from Proposition 3.3.8.
$(ii) \Rightarrow (iii)$ follows from the fact that **V** is cancellable by hypothesis.
$(iii) \Rightarrow (i)$ We only need to show that **-quotient morphisms are pullback stable in $(\mathbb{M}, \mathbf{V}, \widetilde{\mathbb{M}})$-**Gph**. Let

$$\begin{array}{ccc} (X \times_Y Z, d) & \xrightarrow{\pi_2} & (Z,c) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ (X,a) & \xrightarrow[f]{} & (Y,b) \end{array}$$

be a pullback diagram in $(\mathbb{M}, \mathbf{V}, \widetilde{\mathbb{M}})$-**Gph**, where d is the pullback structure of the set $X \times_Y Z$ and f is a **-quotient morphism. We want to show that π_2 is a **-quotient morphism as well. Let $z_2, z_1, z_0 \in Z, m, n \in M$ and $u \ll c((n,z_2),z_1), v \ll c((m,z_1),z_0)$. Since g is an $(\mathbb{M}, \mathbf{V})$-functor,

$$u \ll c((n,z_2),z_1) \leq b((n,g(z_2)),g(z_1)) \Rightarrow u \ll b((n,g(z_2)),g(z_1))$$

$$v \ll c((m,z_1),z_0) \leq b((m,g(z_1)),g(z_0)) \Rightarrow v \ll b((m,g(z_1)),g(z_0)).$$

Since f is a **-quotient morphism, there exist $x_2, x_1, x_0 \in X$ such that $f(x_i) = g(z_i)$, for $i = 0,1,2$, and

$$u \leq a((n,x_2),x_1), \quad v \leq a((m,x_1),x_0).$$

Therefore, $(x_i, z_i) \in X \times_Y Z$, for $i = 0,1,2$, and

$$u \leq a((n,x_2),x_1) \wedge c((n,z_2),z_1) = d((n,(x_2,z_2)),(x_1,z_1))$$

$$v \leq a((m,x_1),x_0) \wedge c((m,z_1),z_0) = d((m,(x_1,z_1)),(x_0,z_0))$$

as wished. □

This generalizes Theorem 3.3.2 and offers the perspective that, under hypotheses on the quantale **V**, relations between effective descent morphisms in $(\mathbb{T}, \mathbf{2}, \overline{\mathbb{T}})$-**Cat** and $(\mathbb{T}, \mathbf{V}, \widetilde{\mathbb{T}})$-**Cat** might exist. In fact, in both cases of the identity monad and the M-ordered monad, the effective descent morphisms, passing from $(\mathbb{T}, \mathbf{2}, \overline{\mathbb{T}})$-**Cat** to $(\mathbb{T}, \mathbf{V}, \widetilde{\mathbb{T}})$-**Cat**, remain the *-quotient morphisms. In the next section we study the case where $\mathbb{T} = \mathbb{W}$, the free-monoid monad.

3.3.2 Effective descent morphisms in $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-Cat and $(\mathbb{W},\mathbf{V},\mathbb{W}^{\wedge})$-Cat

In Sections 2.2.2 and 2.2.3 we introduced two different lax extensions to **V**-**Rel** of the free-monoid monad $\mathbb{W}=(W,\mu,\eta)$, which we denoted by $\mathbb{W}^{\wedge}$ and $\mathbb{W}^{\otimes}$. Recall that $\mathbb{W}^{\wedge}$ comes from the uniform construction of lax extensions so that ccd of **V** is required. Because of this, when we talk about the extension $\mathbb{W}^{\wedge}$, we automatically assume **V** ccd, in order to make the reading smoother. As we did for the M-ordered monad, we want to study effective descent morphisms in the context of $(\mathbb{W},\mathbf{V})$-categories and see if there is any possible connection with the effective descent morphisms in **MultiOrd** $=(\mathbb{W},\mathbf{2},\overline{\mathbb{W}})$-**Cat**. In particular, we are mainly interested in the case of $(\mathbb{W},\mathbf{V},\mathbb{W}^{\wedge})$-**Cat** because of its direct relation with the Barr extension $\overline{\mathbb{W}}$. Effective descent morphisms in **MultiOrd** have been characterized by M.M. Clementimo, D. Hofmann and A. Montoli in [18].

Theorem 3.3.10 [18, Theorem 1.10] *A monotone map in* **MultiOrd** *is of effective descent if and only if it is a weak *-quotient map.*

A $(\mathbb{W},\mathbf{2})$-functor $f:(X,a)\to(Y,b)$ is said to be a *weak *-quotient map* if for each $y_0,y_1,\dots,y_n$, $y^i_1,\dots,y^i_m\in Y$, $i\in\{1,\dots,n\}$, such that

$$b((y^i_1,\dots,y^i_m),y_i)=\top \quad\text{and}\quad b((y_1,\dots,y_n),y_0)=\top,$$

there exist $x_j\in f^{-1}(y_j)$, $j=1,\dots,n$, and $x^i_j\in f^{-1}(x^i_j)$, $j=1,\dots,m$, such that

$$a((x^i_1\dots,x^i_m),x_i)=\top \quad\text{and}\quad a((x_1,\dots,x_n),x_0)=\top.$$

As remarked in [18], with a counter-example induced by the proof of the theorem above, weak *-quotient morphisms need not to be *-quotient. This implies that the converse of Theorem 3.0.6 is not true in general. One of the steps to reach the characterization above is given by an equivalent formulation of transitivity for a $(\mathbb{W},\mathbf{2},\overline{\mathbb{W}})$-graph. An analogous result can be obtained for a $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-graph. Observe that the categories of $(\mathbb{W},\mathbf{V})$-graphs, for both the extensions $\mathbb{W}^{\otimes}$ and $\mathbb{W}^{\wedge}$, coincide, since in the description of the reflexivity property the extension is not involved. But we will anyway stress the notation if needed, in particular to remark in which context we are interested in getting the result.

Lemma 3.3.11 *For an* $(X,a)\in(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-**Gph***, the following conditions are equivalent:*

(i) (X,a) *is transitive;*

(ii) $\forall x_0,x_1,\dots,x_n,x^i_1,\dots,x^i_m\in X$, *with* $i\in\{1,\dots,n\}$

$$a((x^i_1,\dots,x^i_m),x_i)\otimes a((x_1,\dots,x_n),x_0)\le a((x_1,\dots,x^i_1,\dots,x_{i-1},x^i_m,x_{i+1},\dots,x_n),x_0).$$

Proof

$(i)\Rightarrow(ii)$ Let $x_0,x_1,\dots,x_n,x^i_1,\dots,x^i_m\in X$ with $i\in\{1,\dots,n\}$. Consider

$$((x_1),\dots,(x_{i-1}),(x^i_1,\dots,x^i_m),(x_{i+1}),\dots,(x_n))\in W^2X.$$

Since (X,a) is transitive,

$$a((x_1),x_1)\otimes\cdots\otimes a((x_1^i,\ldots,x_m^i),x_i)\otimes\cdots\otimes a((x_n),x_n)\otimes a((x_1,\ldots,x_n),x_0)\leq$$

$$\leq a((x_1,\ldots,x_{i-1},x_1^i,\ldots,x_m^i,x_{i+1}\ldots,x_n),x_0).$$

but since

$$k\leq a((x_j),x_j)\quad\text{for each}\quad j=1,\ldots,i-1,i+1,\ldots,n$$

we get the wished inequality of condition (ii).

$(ii)\Rightarrow(i)$ Let $((x_1^1,\ldots,x_{m_1}^1),\ldots,(x_1^n,\ldots,x_{m_n}^n))\in W^2X$, $(x_1,\ldots,x_n)\in WX$ and $x_0\in X$.
For $i=1$, by condition (ii), we have

$$a((x_1^1,\ldots,x_{m_1}^1),x_1)\otimes a((x_1,\ldots,x_n),x_0)\leq a((x_1^1,\ldots,x_{m_1}^1,x_2,\ldots,x_n),x_0).$$

For $i=2$

$$a((x_1^2,\ldots,x_{m_2}^2),x_2)\otimes a((x_1^1,\ldots,x_{m_1}^1,x_2,\ldots,x_n),x_0)\leq$$

$$\leq a((x_1^1,\ldots,x_{m_1}^1,x_1^2,\ldots,x_{m_2}^2,x_3,\ldots,x_n),x_0).$$

Iterating the process, for $i=n$ we get

$$a((x_1^n,\ldots,x_{m_n}^n),x_n)\otimes a((x_1^1,\ldots,x_{m_1}^1,\ldots,x_1^{n-1},\ldots,x_{m_{n-1}}^{n-1},x_n),x_0)\leq$$

$$\leq a((x_1^1,\ldots,x_{m_1}^1,\ldots,x_1^n,\ldots,x_{m_n}^n),x_0).$$

Starting from the last inequality, and coming backwards to $i=1$, we get

$$(\bigotimes_{i=1}^{n}a((x_1^i,\ldots,x_{m_i}^i),x_i))\otimes a((x_1,\ldots,x_n),x_0)\leq a((x_1^1,\ldots,x_{m_1}^1,\ldots,x_1^n,\ldots,x_{m_n}^n),x_0)$$

proving that (X,a) is transitive. □

A connection between the categories $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-**Cat** and $(\mathbb{W},\mathbf{V},\mathbb{W}^{\wedge})$-**Cat** is given when **V** is integral.

Proposition 3.3.12 *If* **V** *is integral, then* $(\mathbb{W},\mathbf{V},\mathbb{W}^{\wedge})$-**Cat** *is contained in* $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-**Cat**, *i.e., there is a full embedding*

$$I:(\mathbb{W},\mathbf{V},\mathbb{W}^{\wedge})\text{-}\mathbf{Cat}\hookrightarrow(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})\text{-}\mathbf{Cat}.$$

Proof

Let (X,a) be an object in $(\mathbb{W},\mathbf{V},\mathbb{W}^{\wedge})$-**Cat**. We want to prove that condition (ii) of Lemma 3.3.11 is satisfied. Let $x_0,x_1,\ldots,x_n,x_1^i,\ldots,x_m^i\in X$ with $i\in\{1,\ldots,n\}$. Consider

$$((x_1),\ldots,(x_{i-1}),(x_1^i,\ldots,x_m^i),(x_{i+1}),\ldots,(x_n))\in W^2X.$$

Since (X,a) is transitive,

$$(a((x_1),x_1)\wedge\cdots\wedge a((x_1^i,\ldots,x_m^i),x_i)\wedge\cdots\wedge a((x_n),x_n))\otimes a((x_1,\ldots,x_n),x_0)\le$$
$$\le a((x_1,\ldots,x_{i-1},x_1^i,\ldots,x_m^i,x_{i+1}\ldots,x_n),x_0).$$

But

$$(a((x_1),x_1)\wedge\cdots\wedge a((x_1^i,\ldots,x_m^i),x_i)\wedge\cdots\wedge a((x_n),x_n))=a((x_1^i,\ldots,x_m^i),x_i),$$

since $k=\top\le a((x_j),x_j)$ for $j=1,\ldots,i-1,i+1,\ldots,n$. □

Inspired by the notion of weak *-quotient morphisms for $(\mathbb{W},\mathbf{2})$-functors, we have the following definitions.

Definition 3.3.13 A $(\mathbb{W},\mathbf{V})$-functor $f:(X,a)\to(Y,b)$ is called *weak *-quotient* if for each $y_0,y_1,\ldots,y_n$, $y_1^i,\ldots,y_m^i\in Y$ with $i\in\{1,\ldots,n\}$

$$b((y_1^i,\ldots,y_m^i),y_i)\otimes b((y_1,\ldots,y_n),y_0)=\bigvee_{\substack{x_j^i\in f^{-1}(y_j^i)\\ x_j\in f^{-1}(y_j)}} a((x_1^i,\ldots,x_m^i),x_i)\otimes a((x_1,\ldots,x_n),x_0).$$

Definition 3.3.14 A $(\mathbb{W},\mathbf{V})$-functor $f:(X,a)\to(Y,b)$ is called *weak **-quotient* if for each $y_0,y_1,\ldots,y_n$, $y_1^i,\ldots,y_m^i\in Y$ with $i\in\{1,\ldots,n\}$ and for each $u\ll b((y_1^i,\ldots,y_m^i),y_i)$, $v\ll b((y_1,\ldots,y_n),y_0)$ there exist $x_j\in f^{-1}(y_j)$, for $j=1,\ldots,n$, and $x_j^i\in f^{-1}(y_j^i)$, for $j=1,\ldots,m$ such that

$$u\le a((x_1^i,\ldots,x_m^i),x_i),\quad v\le a((x_1,\ldots,x_n),x_0).$$

It follows that every weak **-quotient morphism is weak *-quotient while the converse is true if $\mathbf{V}$ is cancellable.

Lemma 3.3.15 *Weak **-quotient morphisms are pullback stable in* $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-**Gph**.

Proof
Let $f:(X,a)\to(Y,b)$ be a weak **-quotient morphism in $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-**Gph** and let

$$\begin{array}{ccc}(X\times_Y Z,d) & \xrightarrow{\pi_2} & (Z,c)\\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle g}\\ (X,a) & \xrightarrow[f]{} & (Y,b)\end{array}$$

be a pullback diagram in $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-**Gph**, where d is the pullback structure on the set $X\times_Y Z$. We want to show that π_2 is a weak **-quotient morphism as well. Let then $z_0,z_1,\ldots,z_n,z_1^i,\ldots,z_m^i\in Z$, with $i\in\{1,\ldots,n\}$, and let $u\ll c((z_1^i,\ldots,z_m^i),z_i)$, $v\ll c((z_1,\ldots,z_n),z_0)$. Since g is a $(\mathbb{W},\mathbf{V})$-functor,

$$u\ll b((g(z_1^i),\ldots,g(z_m^i)),g(z_i)),\quad v\ll b((g(z_1),\ldots,g(z_n)),g(z_0)).$$

Since f is a weak **-quotient morphism, there exist $x_0, x_1, \ldots, x_n, x_1^i, \ldots, x_m^i \in X$ such that $f(x_j) = g(z_j)$, for $j = 1, \ldots, n$, $f(x_j^i) = g(z_j^i)$, for $j = 1, \ldots, m$, and

$$u \leq a((x_1^i, \ldots, x_m^i), x_i), \quad v \leq a((x_1, \ldots, x_n), x_0).$$

Therefore $(x_0, z_0) \in X \times_Y Z$, $((x_1, z_1), \ldots, (x_n, z_n)) \in W(X \times_Y Z)$ and $((x_1^i, z_1^i), \ldots, (x_m^i, z_m^i)) \in W(X \times_Y Z)$. Hence

$$u \leq a((x_1^i, \ldots, x_m^i), x_i) \wedge c((z_1^i, \ldots, z_m^i), z_i) = d(((x_1^i, z_1^i), \ldots, (x_m^i, z_m^i)), (x_i, z_i)),$$

$$v \leq a((x_1, \ldots, x_n), x_0) \wedge c((z_1, \ldots, z_n), z_0) = d(((x_1, z_1), \ldots, (x_n, z_n)), (x_0, z_0)).$$

□

Lemma 3.3.16 *Weak *-quotient morphisms in* $(\mathbb{W}, \mathbf{V}, \mathbb{W}^{\otimes})$-**Cat** *are final.*

Proof
Let $f : (X, a) \to (Y, b)$ be a weak *-quotient morphism in $(\mathbb{W}, \mathbf{V}, \mathbb{W}^{\otimes})$-**Cat**. We want to show that for each $y_0 \in Y$ and $(y_1, \ldots, y_n) \in WX$

$$b((y_1, \ldots, y_n), y_0) = \bigvee_{x_j \in f^{-1}(y_j)} a((x_1, \ldots, x_n), x_0).$$

Let $y_0 \in Y$ and $(y_1, \ldots, y_n) \in WY$. We have

$$b((y_1, \ldots, y_n), y_0) = k \otimes b((y_1, \ldots, y_n), y_0) \leq b((y_i), y_i) \otimes b((y_1, \ldots, y_n), y_0),$$

for $i \in \{1, \ldots, n\}$. Since f is a weak *-quotient morphism,

$$\begin{aligned} b((y_i), y_i) \otimes b((y_1, \ldots, y_n), y_0) = &\bigvee_{x_j \in f^{-1}(y_j)} a((x_i), x_i) \otimes a((x_1, \ldots, x_n), x_0) \leq \\ \leq &\bigvee_{x_j \in f^{-1}(y_j)} a((x_1, \ldots, x_n), x_0). \end{aligned}$$

The last inequality follows from the transitivity of the $(\mathbb{W}, \mathbf{V}, \mathbb{W}^{\otimes})$-algebra (X, a). □

Theorem 3.3.17 *If* **V** *is cancellable and cartesian closed, then a morphism in* $(\mathbb{W}, \mathbf{V}, \mathbb{W}^{\otimes})$-**Cat** *is of effective descent if and only if it is a weak *-quotient morphism.*

Proof
($\Leftarrow$) Let $f : (X, a) \to (Y, b)$ be a weak *-quotient morphism in $(\mathbb{W}, \mathbf{V}, \mathbb{W}^{\otimes})$-**Cat**. By Lemma 3.3.16, f is final and so it is an effective descent morphism in $(\mathbb{W}, \mathbf{V}, \mathbb{W}^{\otimes})$-**Gph**. Let

$$\begin{array}{ccc} (X \times_Y Z, d) & \xrightarrow{\pi_2} & (Z, c) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ (X, a) & \xrightarrow[f]{} & (Y, b) \end{array}$$

be the pullback in $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-**Gph** of f along a morphism $g:(Z,c)\to(Y,b)$, where $(X\times_Y Z,d)$ is in $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-**Cat**. We want to show that (Z,c) is in $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-**Cat** as well. Let then $z_0,z_1,\dots,z_n,z_1^i,\dots,z_m^i\in Z$, with $i\in\{1,\dots,n\}$. By Lemma 3.3.15, π_2 is a weak *-quotient morphism. Hence

$$c((z_1^i,\dots,z_m^i),z_i)\otimes c((z_1,\dots,z_n),z_0)=$$
$$=\bigvee_{\substack{x_j^i\in f^{-1}(g(z_j^i))\\ x_j\in f^{-1}(g(z_j))}} d(((x_1^i,z_1^i),\dots,(x_m^i,z_m^i)),(x_i,z_i))\otimes d(((x_1,z_1),\dots,(x_n,z_n)),(x_0,z_0)).$$

Since $(X\times_Y Z,d)$ is transitive,

$$\bigvee_{\substack{x_j^i\in f^{-1}(g(z_j^i))\\ x_j\in f^{-1}(g(z_j))}} d(((x_1^i,z_1^i),\dots,(x_m^i,z_m^i)),(x_i,z_i))\otimes d(((x_1,z_1),\dots,(x_n,z_n)),(x_0,z_0))\leq$$
$$\leq\bigvee_{\substack{x_j^i\in f^{-1}(g(z_j^i))\\ x_j\in f^{-1}(g(z_j))}} d(((x_1,z_1),\dots,(x_1^i,z_1^i),\dots,(x_m^i,z_m^i)\dots,(x_n,z_n)),(x_0,z_0))\leq$$
$$\leq c((z_1,\dots,z_{i-1},z_1^i,\dots,z_m^i,z_{i+1},\dots,z_n),z_0).$$

$(\Rightarrow)$ Let $f:(X,a)\to(Y,b)$ be an effective descent morphism in $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-**Cat**. Assume there exist $y_0,y_1,\dots,y_n,y_1^i,\dots,y_m^i\in Y$, with $i\in\{1,\dots,n\}$ such that

$$\bigvee_{\substack{x_j^i\in f^{-1}(y_j^i)\\ x_j\in f^{-1}(y_j)}} a((x_1^i,\dots,x_m^i),x_i)\otimes a((x_1,\dots,x_n),x_0)<b((y_1^i,\dots,y_m^i),y_i)\otimes b((y_1,\dots,y_n),y_0).$$

Let us call

$$\alpha:=\bigvee_{\substack{x_j^i\in f^{-1}(y_j^i)\\ x_j\in f^{-1}(y_j)}} a((x_1^i,\dots,x_m^i),x_i)\otimes a((x_1,\dots,x_n),x_0)$$

for simplicity.
On the set $B=\{0\}\cup\{1,\dots,n\}\cup\{(i,1),\dots,(i,m)\}$, define the following reflexive but non-transitive structure $b_\alpha:WB\times B\to\mathbf{V}$:

- $b_\alpha((j),j))=k$, for each $j\in B$,
- $b_\alpha((1,\dots,n),0)=b((y_1,\dots,y_n),y_0)$,
- $b_\alpha(((i,1),\dots,(i,m)),i)=b((y_1^i,\dots,y_m^i),y_i)$,
- $b_\alpha((1,\dots,i-1,(i,1),\dots,(i,m),i+1,\dots,n),0)=\alpha$,
- $b_\alpha((\dots),j)=\perp$, elsewhere.

Take $g:(B,b_\alpha)\to(Y,b)$ the $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-functor defined by

$$j\mapsto y_j,\quad j=0,\dots,n,\qquad (i,j)\mapsto y_j^i,\quad j=1,\dots,m,$$

and consider the pullback in $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-**Gph**

$$\begin{array}{ccc} (X\times_Y B,d) & \xrightarrow{\pi_2} & (B,b_\alpha) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ (X,a) & \xrightarrow[f]{} & (Y,b) \end{array}$$

of f along g. The $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-algebra $(X\times_Y B,d)$ is not transitive. Hence there are elements $(x_0,i_0),(x_1,i_1),\dots,(x_n,i_n),(x_1^i,j_1^i),\dots,(x_m^i,j_m^i)\in X\times_Y B$, with $i\in\{i_1,\dots,i_n\}$ such that

$$d(((x_1^i,j_1^i),\dots,(x_m^i,j_m^i)),(x_i,i_i))\otimes d(((x_1,i_1),\dots,(x_n,i_n)),(x_0,i_0))\nleq$$

$$\nleq d(((x_1,i_1),\dots,(x_{i-1},i_{i-1}),(x_1^i,j_1^i),\dots,(x_m^i,j_m^i),(x_{i+1},i_{i+1}),\dots,(x_n,i_n)),(x_0,i_0)),$$

which is only possible if $i_0=0,i_1=1,\dots,i_n=n,j_1^i=(i,1),\dots,j_m^i=(i,m)$. We have

- $d(((x_1,1),\dots,(x_1^i,(i,1)),\dots,(x_m^i,(i,m)),\dots,(x_n,n)),(x_0,0))=$
  $=a((x_1,\dots,x_1^i,\dots,x_m^i,\dots,x_n),x_0)\wedge\alpha,$
- $d(((x_1^i,(i,1)),\dots,(x_m^i,(i,m))),(x_i,i))\otimes d(((x_1,1),\dots,(x_n,n)),(x_0,0))\leq$
  $\leq a((x_1^i,\dots,x_m^i),x_i)\otimes a((x_1,\dots,x_n),x_0).$

This implies that

$$a((x_1^i,\dots,x_m^i),x_i)\otimes a((x_1,\dots,x_n),x_0)\nleq a((x_1,\dots,x_1^i,\dots,x_m^i,\dots,x_n),x_0)\wedge\alpha.$$

Hence

$$a((x_1^i,\dots,x_m^i),x_i)\otimes a((x_1,\dots,x_n),x_0)\nleq\alpha$$

which contradict the definition of α. □

If **V** is a (cancellable) frame, the characterization of effective descent morphisms in $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-**Cat** is also a characterization for effective descent morphisms in $(\mathbb{W},\mathbf{V},\mathbb{W}^{\wedge})$-**Cat**, since the two categories coincide. By Theorem 3.3.17 and Lemma 3.3.12, we have

Proposition 3.3.18 *If* **V** *is cancellable, integral and cartesian closed, then a morphism* $f:(X,a)\to(Y,b)$ *in* $(\mathbb{W},\mathbf{V},\mathbb{W}^{\wedge})$-**Cat** *which is weak *-quotient in* $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-**Cat***, is of effective descent in* $(\mathbb{W},\mathbf{V},\mathbb{W}^{\wedge})$-**Cat** *if and only if for each pullback diagram in* $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-**Cat**

$$\begin{array}{ccc} (X\times_Y Z,d) & \xrightarrow{\pi_2} & (Z,c) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ (X,a) & \xrightarrow[f]{} & (Y,b), \end{array}$$

$$(X\times_Y Z,d)\in(\mathbb{W},\mathbf{V},\mathbb{W}^{\wedge})\text{-}\mathbf{Cat}\Rightarrow(Z,c)\in(\mathbb{W},\mathbf{V},\mathbb{W}^{\wedge})\text{-}\mathbf{Cat}.$$

□

Therefore, also in the case of the free-monoid monad, as we already saw for the identity monad and the M-ordered monad, a relation between the effective descent morphisms in $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat** and in $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat** exists, although this time one requires **V** to be a frame. In fact, both in **MultiOrd** and $(\mathbb{W},\mathbf{V},\mathbb{W}^{\wedge})$-**Cat**, the effective descent morphisms are the weak *-quotient morphisms. The following table summarizes what we got for what concerns the effective descent morphisms in the examples studied so far. The quantale **V** is assumed to be cartesian closed and cancellable (and so also ccd) in case of the identity monad $\mathbb{I}$ and of the M-ordered monad $\mathbb{M}$, while for the free-monoid monad $\mathbb{W}$, in addition, **V** is also a frame.

effective descent	**V**-**Cat**	$(\mathbb{M},\mathbf{V},\widetilde{\mathbb{M}})$-**Cat**	$(\mathbb{W},\mathbf{V},\mathbb{W}^{\wedge})$-**Cat**
Ord	*-quotient		
M-**Ord**		*-quotient	
MultiOrd			weak *-quotient

3.3.3 Reflecting and preserving effective descent morphisms

The examples given so far, in **Ord**, M-**Ord** and **MultiOrd**, and their corresponding categories **V**-**Cat**, $(\mathbb{M},\mathbf{V},\widetilde{\mathbb{M}})$-**Cat** and $(\mathbb{W},\mathbf{V},\mathbb{W}^{\wedge})$-**Cat**, show that, under hypotheses on **V**, a possible relation between the effective descent morphisms in $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat** and in $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat** might exist. In this section we try to analyze the general case, in particular studying how suitable functors can preserve and reflect effective descent morphisms. In [21] M.M. Clementino and W. Tholen showed how a monoidal functor $\alpha:\mathbf{V}\to\mathbf{V}'$, between monoidal-closed categories **V** and $\mathbf{V}'$, may induce a functor

$$F_\alpha:(\mathbb{T},\mathbf{V},\widehat{\mathbb{T}})\text{-}\mathbf{Cat}\to(\mathbb{T},\mathbf{V}',\widehat{\mathbb{T}}')\text{-}\mathbf{Cat}, \tag{3.5}$$

where $\widehat{\mathbb{T}}$ and $\widehat{\mathbb{T}}'$ are (flat) lax extensions to **V**-**Rel** and $\mathbf{V}'$-**Rel**, respectively, of a **Set**-monad $\mathbb{T}=(T,\mu,\eta)$. Of course this applies to the case when **V** and $\mathbf{V}'$ are simply quantales. For a given $(\mathbb{T},\mathbf{V},\widehat{\mathbb{T}})$-algebra (X,a) the idea is just to compose the map $a:TX\times X\to\mathbf{V}$ with α

$$TX\times X\xrightarrow{a}\mathbf{V}\xrightarrow{\alpha}\mathbf{V}'$$

and, under some hypotheses, one gets a $(\mathbb{T},\mathbf{V}',\widehat{\mathbb{T}}')$-algebra $(X,\alpha\cdot a)$. We give an explicit proof when the lax extension is induced by a $\mathbb{T}$-algebra structure, as done in Section 2.2.3. A lax quantale homorphism $\alpha:\mathbf{V}\to\mathbf{V}'$ between quantales $\mathbf{V}=(V,k_{\mathbf{V}},\otimes_{\mathbf{V}})$ and $\mathbf{V}'=(V',k_{\mathbf{V}'},\otimes_{\mathbf{V}'})$ may induce a functor

$$F_\alpha:(\mathbb{T},\mathbf{V},\widehat{\mathbb{T}}_\xi)\text{-}\mathbf{Cat}\to(\mathbb{T},\mathbf{V}',\widehat{\mathbb{T}}_{\xi'})\text{-}\mathbf{Cat},$$

where $\widehat{\mathbb{T}}_\xi$ is the lax extension of $\mathbb{T}$ to **V**-**Rel** given by a $\mathbb{T}$-algebra structure $\xi:TV\to V$ and $\widehat{\mathbb{T}}_{\xi'}$ is the lax extension of $\mathbb{T}$ to $\mathbf{V}'$-**Rel** given by a $\mathbb{T}$-algebra structure $\xi':TV'\to V'$.
Recall first that by a lax quantale homomorphism $\alpha:\mathbf{V}\to\mathbf{V}'$ we mean a monotone map between V and V' satisfying

$$\alpha(u)\otimes_{\mathbf{V}'}\alpha(v)\le\alpha(u\otimes_{\mathbf{V}}v)\quad\text{and}\quad k_{\mathbf{V}'}\le\alpha(k_{\mathbf{V}}),$$

for all $u,v\in V$.

Proposition 3.3.19 *Let $\mathbb{T}=(T,\eta,\mu)$ be a* **Set**-*monad. Let* **V** *and* **V**$'$ *be quantales, $\xi : TV \to V$ and $\xi' : TV' \to V'$ be $\mathbb{T}$-algebras structures of* **V** *and* **V**$'$*, respectively. Let $\alpha : \mathbf{V} \to \mathbf{V}'$ be a lax quantale homomorphism satisfying*

$$\alpha\cdot\xi \geq \xi'\cdot T\alpha.$$

Then

$$F_\alpha : (\mathbb{T},\mathbf{V},\widehat{\mathbb{T}}_\xi)\text{-}\mathbf{Cat} \to (\mathbb{T},\mathbf{V}',\widehat{\mathbb{T}}_{\xi'})\text{-}\mathbf{Cat}$$

is a functor induced by α where $\widehat{T}_\xi : \mathbf{V}\text{-}\mathbf{Rel} \to \mathbf{V}\text{-}\mathbf{Rel}$ and $\widehat{T}_{\xi'} : \mathbf{V}'\text{-}\mathbf{Rel} \to \mathbf{V}'\text{-}\mathbf{Rel}$ are the lax extensions of T

$$\widehat{T}_\xi r(\mathfrak{x},\mathfrak{y}) = \bigvee_{\substack{\mathfrak{w}\in T(X\times Y):\\ T\pi_X(\mathfrak{w})=\mathfrak{x}\\ T\pi_Y(\mathfrak{w})=\mathfrak{y}}} \xi\cdot Tr(\mathfrak{w}),\quad \widehat{T}_{\xi'} r(\mathfrak{x},\mathfrak{y}) = \bigvee_{\substack{\mathfrak{w}\in T(X\times Y):\\ T\pi_X(\mathfrak{w})=\mathfrak{x}\\ T\pi_Y(\mathfrak{w})=\mathfrak{y}}} \xi'\cdot Tr(\mathfrak{w}),$$

for any **V**-*relation $r : X \nrightarrow Y$ and each $\mathfrak{x}\in TX$, $\mathfrak{y}\in TY$, induced respectively by ξ and ξ'.*

Proof

The functor F_α is defined as in the case of (3.5). Let (X,a) be a $(\mathbb{T},\mathbf{V},\widehat{\mathbb{T}}_\xi)$-algebra, that is, a set X equipped with a map $a : TX\times X \to \mathbf{V}$ satisfying the reflexivity and the transitivity properties. The assignment

$$F_\alpha(X,a) = (X,\alpha\cdot a).$$

defines a reflexive and transitive $(\mathbb{T},\mathbf{V}',\widehat{\mathbb{T}}_{\xi'})$-algebra.
Reflexivity: for each $x\in X$,

$$k_{\mathbf{V}'} \leq \alpha(k_{\mathbf{V}}) \leq \alpha(a(\eta_X(x),x)).$$

Transitivity: for each $\mathfrak{X}\in T^2X$, $\mathfrak{x}\in TX$ and $x\in X$,

$$\begin{aligned}
\widehat{T}_{\xi'}(\alpha\cdot a)(\mathfrak{X},\mathfrak{x})\otimes_{\mathbf{V}'}(\alpha\cdot a)(\mathfrak{x},x) &= \bigvee_{\substack{\mathfrak{w}\in T(TX\times X):\\ T\pi_{TX}(\mathfrak{w})=\mathfrak{X}\\ T\pi_X(\mathfrak{w})=\mathfrak{x}}} \xi'\cdot T(\alpha\cdot a)(\mathfrak{w})\otimes_{\mathbf{V}'}(\alpha\cdot a)(\mathfrak{x},x) \leq \\
&\leq \bigvee_{\substack{\mathfrak{w}\in T(TX\times X):\\ T\pi_{TX}(\mathfrak{w})=\mathfrak{X}\\ T\pi_X(\mathfrak{w})=\mathfrak{x}}} \alpha\cdot(\xi\cdot Ta(\mathfrak{w}))\otimes_{\mathbf{V}'}(\alpha\cdot a)(\mathfrak{x},x) \leq \\
&\leq \bigvee_{\substack{\mathfrak{w}\in T(TX\times X):\\ T\pi_{TX}(\mathfrak{w})=\mathfrak{X}\\ T\pi_X(\mathfrak{w})=\mathfrak{x}}} \alpha((\xi\cdot Ta(\mathfrak{w}))\otimes_{\mathbf{V}} a(\mathfrak{x},x)) \leq \\
&\leq \alpha(\bigvee_{\substack{\mathfrak{w}\in T(TX\times X):\\ T\pi_{TX}(\mathfrak{w})=\mathfrak{X}\\ T\pi_X(\mathfrak{w})=\mathfrak{x}}} (\xi\cdot Ta)(\mathfrak{w})\otimes_{\mathbf{V}} a(\mathfrak{x},x)) \leq \\
&\leq \alpha(a(\mu_X(\mathfrak{X}),x)).
\end{aligned}$$

Each $(\mathbb{T},\mathbf{V},\widehat{\mathbb{T}}_\xi)$-functor $f : (X,a)\to(Y,b)$ gives rise to a morphism

$$F_\alpha f : (X,\alpha\cdot a) \to (Y,\alpha\cdot b)$$

in $(\mathbb{T},\mathbf{V}',\widehat{\mathbb{T}}_{\xi'})$-**Cat**. In fact, for each $\mathfrak{x}\in TX$ and $x\in x$, since $a(\mathfrak{x},x)\leq b(Tf(\mathfrak{x}),f(x))$, we have

$$\alpha(a(\mathfrak{x},x))\leq\alpha(b(Tf(\mathfrak{x}),f(x))).$$

One can easily verify that F_α is a functor. □

Remarks 3.3.20 If the lax quantale homomorphism $\alpha:\mathbf{V}\to\mathbf{V}'$ is a full embedding, then the induced functor $F_\alpha:(\mathbb{T},\mathbf{V},\widehat{\mathbb{T}}_\xi)\text{-}\mathbf{Cat}\to(\mathbb{T},\mathbf{V}',\widehat{\mathbb{T}}_{\xi'})\text{-}\mathbf{Cat}$ is a full embedding as well.

This covers the situation studied in Proposition 3.3.12 for the free-monoid monad $\mathbb{W}=(W,\mu,\eta)$. In fact, considering the extensions $\mathbb{W}^\wedge$ and $\mathbb{W}^\otimes$, the lax quantale homomorphism $\alpha:\mathbf{V}\to\mathbf{V}$ is the identity morphism $1_\mathbf{V}:\mathbf{V}\to\mathbf{V}$ and, for $\mathbf{V}$ integral,

$$\alpha\cdot\xi^\wedge\geq\xi^\otimes\cdot W\alpha,$$

since the following diagram

$$\begin{array}{ccc}\bigotimes_{i=1}^n u_i & & \\ \vdots & \searrow & \\ \bigwedge_{i=1}^n u_i & \xrightarrow[\pi_{u_j}]{} & u_j\end{array}$$

commutes by the universal property of the meet.

The situation given in Proposition 3.3.19 interests mostly in the case of the full embedding

$$\iota:\mathbf{2}\hookrightarrow\mathbf{V}$$

defined by

$$\top\mapsto k_\mathbf{V}\quad\bot\mapsto\bot_\mathbf{V},$$

which is nothing but the composition $\mathbf{2}\xrightarrow{E}\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}\xrightarrow{L}\mathbf{V}$ in Section 2.2.2. The functor

$$F_\iota:(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})\text{-}\mathbf{Cat}\hookrightarrow(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})\text{-}\mathbf{Cat}$$

induced by ι, where $\overline{\mathbb{T}}$ is the Barr extension of the **Set**-monad $\mathbb{T}=(T,\mu,\eta)$, with T satisfying BC, and $\widetilde{\mathbb{T}}$ is the uniform extension described in Section 2.2.2, is still a full embedding. Of course we need to assume that $\mathbf{V}$ is a ccd quantale such that $k_\mathbf{V}=\top_\mathbf{V}$ or $\overline{T}$ preserves the $\bot$-relation. Having in mind our goal, that is to investigate possible relations between effective descent morphisms in $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat** and effective descent morphisms $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat**, the first question is to ask whether such a full embedding F_ι preserves and reflects effective descent morphisms.

Lemma 3.3.21 *The full embedding*

$$F_\iota:(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})\text{-}\mathbf{Cat}\hookrightarrow(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})\text{-}\mathbf{Cat}$$

preserves pullbacks.

Proof

Just recall first how the functor F_ι is defined. To each $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-category (X,a),

$$F_\iota(X,a) = (X, \iota\cdot a).$$

For each $\mathfrak{x}\in TX$ and $x\in X$, the map $\iota\cdot a : TX\times X\to \mathbf{V}$ is defined by

$$\iota\cdot a(\mathfrak{x},x) = \begin{cases} k_{\mathbf{V}}, & \text{if } a(\mathfrak{x},x)=\top,\\ \bot_{\mathbf{V}}, & \text{if } a(\mathfrak{x},x)=\bot. \end{cases}$$

Let $f:(X,a)\to(Y,b)$ and $g:(Z,c)\to(Y,b)$ be morphisms in $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat**. Take the pullback in $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat**

$$\begin{array}{ccc} (X\times_Y Z,\tilde{d}) & \xrightarrow{\pi_2} & (Z,\iota\cdot c)\\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle F_\iota g}\\ (X,\iota\cdot a) & \xrightarrow[F_\iota f]{} & (Y,\iota\cdot b). \end{array}$$

For each $\mathfrak{w}\in T(X\times_Y Z)$ and $(x,z)\in X\times_Y Z$, the map $\tilde{d}: T(X\times_Y Z)\times(X\times_Y Z)\to\mathbf{V}$ is described by

$$\tilde{d}(\mathfrak{w},(x,z)) = \begin{cases} k_{\mathbf{V}}, & \text{if } a(T\pi_1(\mathfrak{x}),x)=\top \quad \& \quad c(T\pi_2(\mathfrak{w}),z)=\top,\\ \bot_{\mathbf{V}}, & \text{otherwise.} \end{cases}$$

The structure $\tilde{d}$ is precisely the structure $\iota\cdot\bar{d}$ on $X\times_Y Z$, where $\bar{d}: T(X\times_Y Z)\times(X\times_Y Z)\to\mathbf{2}$ is the pullback structure in $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat**. □

Proposition 3.3.22 *Let* **V** *be a ccd and integral quantale. Let $\mathbb{T}=(T,\eta,\mu)$ be a* **Set***-monad such that T satisfies BC, and let $\widetilde{\mathbb{T}}=(\widetilde{T},\widetilde{\mu},\widetilde{\eta})$ be its uniform extension to* **V**-**Rel**. *Assume that every naturality square of $\widetilde{\eta}$ with respect to* **V***-relations with finite fibres is a BC-square. Then the full embedding*

$$F_\iota : (\mathbb{T},\mathbf{2},\overline{\mathbb{T}})\text{-}\mathbf{Cat}\hookrightarrow(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})\text{-}\mathbf{Cat}$$

reflects effective descent morphisms.

Proof

Let $f:(X,a)\to(Y,b)$ be a morphism in $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat** such that $F_\iota:(X,\iota\cdot a)\to(Y,\iota\cdot b)$ is an effective descent morphism in $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat**. Since by Lemma 3.3.21 the functor F_ι preserves pullbacks, we need to show that for each pullback diagram

$$\begin{array}{ccc} (X\times_Y Z,\tilde{d}) & \xrightarrow{\pi_2} & (Z,c)\\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle g}\\ (X,\iota\cdot a) & \xrightarrow[F_\iota f]{} & (Y,\iota\cdot b) \end{array}$$

in $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat**,

$$(X\times_Y Z,\tilde{d})\in(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})\text{-}\mathbf{Cat}\Rightarrow(Z,c)\in(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})\text{-}\mathbf{Cat}.$$

The fact that $F_\iota f$ is an effective descent morphism implies that $F_\iota f$ is, in particular, a final morphism. Therefore also π_2 is final, since final morphisms are pullback stable (if T satisfies BC), and, since $(X \times_Y Z, \tilde{d})$ is in $(\mathbb{T}, \mathbf{2}, \overline{\mathbb{T}})$-**Cat**, it follows that (Z, c) belongs to $(\mathbb{T}, \mathbf{2}, \overline{\mathbb{T}})$-**Cat** as well. □

Proposition 3.3.23 *The full embedding*

$$F_\iota : (\mathbb{T}, \mathbf{2}, \overline{\mathbb{T}})\text{-}\mathbf{Cat} \hookrightarrow (\mathbb{T}, \mathbf{V}, \widetilde{\mathbb{T}})\text{-}\mathbf{Cat}$$

*preserves *-quotient morphisms.*

Proof
Let $f : (X, a) \to (Y, b)$ be a *-quotient morphism in $(\mathbb{T}, \mathbf{2}, \overline{\mathbb{T}})$-**Cat**. We want to show that $F_\iota : (X, \iota \cdot a) \to (Y, \iota \cdot b)$ is a *-quotient morphism in $(\mathbb{T}, \mathbf{V}, \widetilde{\mathbb{T}})$-**Cat**, i.e., for each $\mathfrak{Y} \in T^2Y$, $\mathfrak{y} \in TY$ and $y \in y$,

$$\widetilde{T}(\iota \cdot b)(\mathfrak{Y}, \mathfrak{y}) \otimes (\iota \cdot b)(\mathfrak{y}, y) = \bigvee_{\substack{\mathfrak{X} \in T^2X : T^2 f(\mathfrak{X}) = \mathfrak{Y} \\ \mathfrak{x} \in TX : Tf(\mathfrak{x}) = \mathfrak{y} \\ x \in X : f(x) = y}} \widetilde{T}(\iota \cdot a)(\mathfrak{X}, \mathfrak{x}) \otimes (\iota \cdot a)(\mathfrak{x}, x).$$

It immediately follows from the fact that the diagram

$$\begin{array}{ccc} \mathbf{Rel} & \xrightarrow{\overline{T}} & \mathbf{Rel} \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota} \\ \mathbf{V}\text{-}\mathbf{Rel} & \xrightarrow[\widetilde{T}]{} & \mathbf{V}\text{-}\mathbf{Rel} \end{array}$$

is commutative, where the functor $\iota : \mathbf{Rel} \to \mathbf{V}\text{-}\mathbf{Rel}$ is induced by the full embedding $\iota : \mathbf{2} \hookrightarrow \mathbf{V}$. □

This proposition turns out to be particularly useful when *-quotient morphisms coincide with effective descent morphisms in $(\mathbb{T}, \mathbf{2}, \overline{\mathbb{T}})$-**Cat**, as it is, for example, the case of the identity monad and the M-ordered monad.

3.3.4 The relational method for effective descent morphisms in $(\mathbb{T}, \mathbf{V}, \widetilde{\mathbb{T}})$-Cat

Throughout this section the quantale $\mathbf{V}$ is assumed to be a ccd frame. We present a method, which we call *relational method*, to study effective descent morphisms in $(\mathbb{T}, \mathbf{V}, \widetilde{\mathbb{T}})$-**Cat**. Let $\mathbb{T} = (T, \eta, \mu)$ be a **Set**-monad such that T has BC. Let $\overline{T} : \mathbf{Rel} \to \mathbf{Rel}$ be the Barr extension of T and let $\widetilde{T} : \mathbf{V}\text{-}\mathbf{Rel} \to \mathbf{V}\text{-}\mathbf{Rel}$ be the uniform lax extension of T to **V**-**Rel**. Each morphism $f : (X, a) \to (Y, b)$ in $(\mathbb{T}, \mathbf{V}, \widetilde{\mathbb{T}})$-**Cat** defines a family of morphisms

$$(f_u : (X, a_u) \to (Y, b_u))_{u \in \mathbf{V}}$$

in $(\mathbb{T}, \mathbf{2}, \overline{\mathbb{T}})$-**Cat**, where $a_u : TX \times X \to \mathbf{2}$ and $b_u : TY \times Y \to \mathbf{2}$ are defined, respectively, by

$$a_u(\mathfrak{x}, x) = \top \Leftrightarrow u \leq a(\mathfrak{x}, x) \quad \text{and} \quad b_u(\mathfrak{y}, y) = \top \Leftrightarrow u \leq b(\mathfrak{y}, y).$$

We check first that, for each $u \in \mathbf{V}$, $(X, a_u) \in (\mathbb{T}, \mathbf{2}, \overline{\mathbb{T}})$-**Cat**. Let $x \in X$. Then

$$a_u(\eta_X(x), x) = \top, \quad \text{i.e.,} \quad u \leq a(\eta_X(x), x),$$

since $\top = k \leq a(\eta_X(x), x)$ from the fact that (X,a) is reflexive. Hence (X,a_u) is itself reflexive.
To prove that (X,a_u) is also transitive, let $\mathfrak{X} \in T^2X$, $\mathfrak{x} \in TX$ and $x \in X$ such that $\overline{T}a_u(\mathfrak{X},\mathfrak{x}) = \top$ and $a_u(\mathfrak{x},x) = \top$. The last one means of course, by definition, that $u \leq a(\mathfrak{x},x)$. Since

$$\widetilde{T}a(\mathfrak{X},\mathfrak{x}) \otimes a(\mathfrak{x},x) = \bigvee\{u \in \mathbf{V} : \overline{T}a_u(\mathfrak{X},\mathfrak{x}) = \top\} \otimes a(\mathfrak{x},x),$$

we have

$$u = u \otimes u \leq \widetilde{T}a(\mathfrak{X},\mathfrak{x}) \otimes a(\mathfrak{x},x) \leq a(\tilde{\mu}_X(\mathfrak{X}),x),$$

where the last inequality follows from the transitivity of the $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-algebra (X,a). Hence,

$$u \leq a(\mu_X(\mathfrak{X}),x) \Rightarrow a_u(\mu_X(\mathfrak{X}),x) = \top$$

as wished. It remains to show that, for each $\mathfrak{x} \in TX$ and $x \in X$,

$$a_u(\mathfrak{x},x) = \top \Rightarrow b_u(Tf(\mathfrak{x}), f(x)) = \top.$$

Let then $\mathfrak{x} \in TX$ and $x \in x$ such that $a_u(\mathfrak{x},x) = \top$, i.e., $u \leq a(\mathfrak{x},x)$. Since $f : (X,a) \to (Y,b)$ is a morphism in $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat**, $u \leq a(\mathfrak{x},x)$ implies that $u \leq b(Tf(\mathfrak{x}), f(x))$, proving that $b_u(Tf(\mathfrak{x}), f(x)) = \top$.

The idea is to use informations in $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat**, where there are several examples involving the characterization of the effective descent morphisms, to obtain results in $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat**. To do that, starting from a morphism $f : (X,a) \to (Y,b)$ in $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat**, we split it into *slices* $(f_u : (X,a_u) \to (Y,b_u))_{u\in\mathbf{V}}$ in $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat** (i.e., morphisms of relational algebras), as we described above, in order to get sufficient conditions for the $(\mathbb{T},\mathbf{V})$-functor f to be an (effective) descent morphism.

Remark 3.3.24 To define for each $u \in \mathbf{V}$ a morphism $f_u : (X,a_u) \to (Y,b_u)$ in $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat**, starting from a morphism $f : (X,a) \to (Y,b)$ in $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat**, we need **V** to be integral ad idempotent. Since all the examples of quantales given in Example 2.1.2 are commutative (except for $P(M)$ unless the monoid M is itself commutative), we assume **V** to be a frame for a smoother reading. Recall that frames can be identified as those commutative quantales which are integral and idempotent.

Proposition 3.3.25 *Let* **V** *be a ccd frame and let* $f : (X,a) \to (Y,b)$ *be a morphism in* $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat**. *If for each* $u \in \mathbf{V}$, $f_u : (X,a_u) \to (Y,b_u)$ *is final, then* f *is final.*

Proof
Let $f : (X,a) \to (Y,b)$ be a $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-functor and let $(f_u : (X,a_u) \to (Y,b_u))_{u\in\mathbf{V}}$ be the family of $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-functors induced by f such that f_u is final for each $u \in \mathbf{V}$. We want to show that for each $\mathfrak{y} \in TY$ and $y \in Y$,

$$b(\mathfrak{y},y) = \bigvee_{\substack{\mathfrak{x}\in TX:Tf(\mathfrak{x})=\mathfrak{y}\\ x\in X:f(x)=y}} a(\mathfrak{x},x).$$

But

$$b(\mathfrak{y},y) = \bigvee\{u \in \mathbf{V} : u \leq b(\mathfrak{y},y)\} = \bigvee\{u \in \mathbf{V} : b_u(\mathfrak{y},y) = \top\}.$$

Let $u \in \mathbf{V}$ such that $u \leq b(\mathfrak{y},y)$. Since f_u is final, there exist $\mathfrak{x} \in TX$ and $x \in X$ such that $a_u(\mathfrak{y},y) = \top$, i.e., $u \leq a(\mathfrak{x},x)$. □

Corollary 3.3.26 *Let* $\mathbf{V}$ *be a ccd frame and let* $\mathbb{T} = (T, \eta, \mu)$ *be a* **Set**-*monad such that* T *satisfies BC and every naturality square of* η *with respect to* $\mathbf{V}$-*relations with finite fibres is a BC-square. A morphism* $f : (X,a) \to (Y,b)$ *in* $(\mathbb{T}, \mathbf{V}, \widetilde{\mathbb{T}})$-**Cat** *is a descent morphism provided that* $f_u : (X,a_u) \to (Y,b_u)$ *is a descent morphism in* $(\mathbb{T}, \mathbf{2}, \overline{\mathbb{T}})$-**Cat** *for each* $u \in \mathbf{V}$.

□

Theorem 3.3.27 *If the ccd frame* $\mathbf{V}$ *is totally ordered, then a morphism* $f : (X,a) \to (Y,b)$ *in* $(\mathbb{T}, \mathbf{V}, \widetilde{\mathbb{T}})$-**Cat** *is of effective descent provided that* $f_u : (X,a_u) \to (Y,b_u)$ *is a pullback stable *-quotient morphism in* $(\mathbb{T}, \mathbf{2}, \overline{\mathbb{T}})$-**Gph** *for each* $u \in \mathbf{V}$.

Proof

Let $f : (X,a) \to (Y,b)$ be a morphism in $(\mathbb{T}, \mathbf{V}, \widetilde{\mathbb{T}})$-**Cat**. We prove that f is a pullback stable *-quotient morphism in $(\mathbb{T}, \mathbf{V}, \widetilde{\mathbb{T}})$-**Gph**. We first prove that f is a *-quotient morphism, that is, for each $\mathfrak{Y} \in T^2Y$, $\mathfrak{y} \in Y$ and $y \in Y$

$$\widetilde{T}b(\mathfrak{Y},\mathfrak{y}) \otimes b(\mathfrak{y},y) = \bigvee_{\substack{\mathfrak{X} \in T^2X : T^2f(\mathfrak{X}) = \mathfrak{Y} \\ \mathfrak{x} \in TX : Tf(\mathfrak{x}) = \mathfrak{y} \\ x \in X : f(x) = y}} \widetilde{T}a(\mathfrak{X},\mathfrak{x}) \otimes a(\mathfrak{x},x).$$

Let $\mathfrak{Y} \in T^2Y$, $\mathfrak{y} \in TY$ and $y \in Y$. We have

$$\widetilde{T}b(\mathfrak{Y},\mathfrak{y}) \otimes b(\mathfrak{y},y) = \bigvee\{u \in \mathbf{V} : \overline{T}b_u(\mathfrak{Y},\mathfrak{y}) = \top\} \otimes \bigvee\{v \in \mathbf{V} : v \leq b(\mathfrak{y},y)\} =$$

$$= \bigvee\{u \in \mathbf{V} : \overline{T}b_u(\mathfrak{Y},\mathfrak{y}) = \top\} \otimes \bigvee\{v \in \mathbf{V} : b_v(\mathfrak{y},y) = \top\} =$$

$$= \bigvee\{u \otimes v : \overline{T}b_u(\mathfrak{Y},\mathfrak{y}) = \top \quad \& \quad b_v(\mathfrak{y},y) = \top\}.$$

Let $u, v \in \mathbf{V}$ with $\overline{T}b_u(\mathfrak{Y},\mathfrak{y}) = b_v(\mathfrak{y},y) = \top$. Since $\mathbf{V}$ is totally ordered, $u \leq v$ or $v \leq u$.

- *Case* $u \leq v$.
  If $u \leq v$, then $u \otimes v = u \wedge v = u$ and, since $u \leq v \leq b(\mathfrak{y},y)$, we conclude that $b_u(\mathfrak{y},y) = \top$. Hence, since $f_u : (X,a_u) \to (Y,b_u)$ is a *-quotient morphism, there exist $\mathfrak{X} \in T^2X$, $\mathfrak{x} \in TX$ and $x \in X$, with $T^2f(\mathfrak{X}) = \mathfrak{Y}$, $Tf(\mathfrak{x}) = \mathfrak{y}$, $f(x) = y$, such that $\overline{T}a_u(\mathfrak{X},\mathfrak{x}) = \top$ and $a_u(\mathfrak{x},x) = \top$.

- *Case* $v \leq u$.
  If $v \leq u$, then $u \otimes v = u \wedge v = v$ and, since

  $$b_u \leq b_v \Rightarrow \overline{T}b_u \leq \overline{T}b_v,$$

  we conclude that $\overline{T}b_v(\mathfrak{Y},\mathfrak{y}) = \top$. Hence, since $f_v : (X,a_v) \to (Y,b_v)$ is a *-quotient morphism, there exist $\mathfrak{X} \in T^2X$, $\mathfrak{x} \in TX$ and $x \in X$, with $T^2f(\mathfrak{X}) = \mathfrak{Y}$, $Tf(\mathfrak{x}) = \mathfrak{y}$, $f(x) = y$, such that $\overline{T}a_v(\mathfrak{X},\mathfrak{x}) = \top$ and $a_v(\mathfrak{x},x) = \top$.

It remains to show the pullback stability. Let

$$\begin{array}{ccc} (X \times_Y Z, d) & \xrightarrow{\pi_2} & (Z,c) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ (X,a) & \xrightarrow[f]{} & (Y,b) \end{array}$$

be a pullback diagram in $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Gph**, where f is a *-quotient morphism. We want to show that π_2 is a *-quotient morphism as well. We show that the morphism

$$(\pi_2)_u : (X \times_Y Z, d_u) \to (Z, c_u)$$

in $(T,\mathbf{2},\overline{T})$-**Gph**, induced by π_2, is a *-quotient morphism for each $u \in \mathbf{V}$. Recall that the structure $d_u : T(X \times_Y Z) \times (X \times_Y Z) \to \mathbf{2}$ is given by

$$d_u(\mathfrak{w},(x,z)) = \begin{cases} \top, & \text{if } u \leq d(\mathfrak{w},(x,z)) = a(T\pi_1(\mathfrak{w}),x) \wedge c(T\pi_2(\mathfrak{w}),z) \\ \bot, & \text{otherwise,} \end{cases}$$

for each $\mathfrak{w} \in T(X \times_Y Z)$ and $(x,z) \in X \times_Y Z$. We want to show that each of these $(\pi_2)_u$, for $u \in \mathbf{V}$, is the pullback in $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Gph** of $f_u : (X,a_u) \to (Y,b_u)$ which, by hypothesis, are pullback stable *-quotient morphisms. So, for each $u \in \mathbf{V}$, consider the pullback diagram

$$\begin{array}{ccc} (X \times_Y Z, d^u) & \xrightarrow{\pi_2^u} & (Z, c_u) \\ {\scriptstyle \pi_1^u}\downarrow & & \downarrow{\scriptstyle g_u} \\ (X, a_u) & \xrightarrow[f_u]{} & (Y, b_u) \end{array}$$

in $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Gph**, where f_u is the (pullback stable *-quotient) morphism induced by f, g_u is the morphism induced by g, and $d^u : T(X \times_Y Z) \times (X \times_Y Z) \to \mathbf{2}$ is the pullback structure in $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Gph** on the set $X \times_Y Z$, given by

$$d^u(\mathfrak{w},(x,z)) = \begin{cases} \top, & \text{if } a_u(T\pi_1^u(\mathfrak{w}),x) = \top \;\&\; c_u(T\pi_2^u(\mathfrak{w}),z) = \top \\ \bot, & \text{otherwise,} \end{cases}$$

for each $\mathfrak{w} \in T(X \times_Y Z)$ and $(x,z) \in X \times_Y Z$. Since

$$a_u(T\pi_1^u(\mathfrak{w}),x) = \top \quad \Rightarrow \quad u \leq a(T\pi_1(\mathfrak{w}),x)$$

$$c_u(T\pi_2^u(\mathfrak{w}),z) = \top \quad \Rightarrow \quad u \leq c(T\pi_2(\mathfrak{w}),z),$$

we conclude that $(X \times_Y Z, d_u) \cong (X \times_Y Z, d^u)$. Hence, for each $u \in \mathbf{V}$, $(\pi_2)_u : (X \times_Y Z, d_u) \to (Z, c_u)$ is a *-quotient morphism. $\square$

The following example shows that the converse of Proposition 3.3.25 and of Theorem 3.3.27 is not true.

Example 3.3.28 Let $\mathbf{V} = ([0,\infty],\leq,\wedge)$ be the ccd totally ordered frame giving rise to the category **V-Cat** of (generalized) ultrametric spaces and non-expansive maps. Let

$$f : (\mathbb{N} \cup \{\infty\}, a) \to (\{y,\infty\}, b)$$

be the non-expansive map defined by

$$f(\infty) = \infty \quad \text{and} \quad f(n) = y, \forall n \in \mathbb{N}.$$

The structure $a : (\mathbb{N} \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\}) \to \mathbf{V}$ is defined as follows

- $a(n,\infty) = 1 - \frac{1}{n}$, $\forall n \in \mathbb{N}$,
- $a(n,n) = a(\infty,\infty) = 1$, $\forall n \in \mathbb{N}$,
- $a(n,m) = 0$, $\forall n \neq m$,
- $a(\infty,n) = 0$, $\forall n \in \mathbb{N}$,

while the structure $b : \{y,\infty\} \times \{y,\infty\} \to \mathbf{V}$ is given by

- $b(y,\infty) = b(y,y) = b(\infty,\infty) = 1$,
- $b(\infty,y) = 0$.

One can easily check that f is a **-quotient morphism, and therefore effective for descent, since **-quotient morphisms are pullback stable in **V**-**Gph** (a first direct proof of that is given in [15] but it can be immediately deduced from Theorem 3.3.9, applied to $M = 1$, the trivial monoid). But, for $u = 1$, the **2**-functor

$$f_u : (X, a_u) \to (Y, b_u)$$

is not an effective descent map in **Ord**. Actually it is not even a regular epimorphism. In fact,

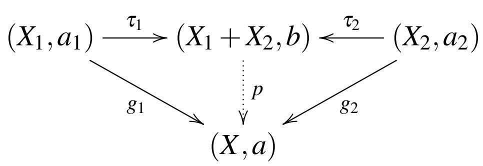
$$1 \leq b(y,\infty) = 1 \quad \Rightarrow \quad b_1(y,\infty) = \top.$$

But none of the elements $n \in \mathbb{N}$ is in relation with ∞ in (X, a_1) since

$$1 \nleq 1 - \frac{1}{n} = a(n,\infty), \quad \forall n \in \mathbb{N}.$$

3.4 A Van Kampen Theorem in categories of lax algebras

In Section 1.5.6 a categorial version of the Van Kampen Theorem is given, showing its relation with Descent Theory (see Lemma 1.5.4 and Theorem 1.5.5). Therefore one can be interested in studying it in the context of lax algebras since, as it is mentioned in Section 2.3.2, the category $(\mathbb{T}, \mathbf{V})$-**Cat** is a lextensive category. Let $p : (X_1 + X_2, b) \to (X, a)$ be the $(\mathbb{T}, \mathbf{V})$-functor from the coproduct of (X_1, a_1) and (X_2, a_2) into (X, a) induced by the embeddings $g_1 : X_1 \hookrightarrow X$ and $g_2 : X_2 \hookrightarrow X$



where τ_1 and τ_2 are the canonical injections. The structure $b : T(X_1+X_2)\times(X_1+X_2)\to \mathbf{V}$ on $X_1+X_2=(X_1\times\{1\})\cup(X_2\times\{2\})$ is described by

$$b(\mathfrak{y},(x,i)) = \begin{cases} a_i(\mathfrak{y}_i,(x,i)), & \text{if } \mathfrak{y}=T\tau_i(\mathfrak{y}_i) \text{ for } \mathfrak{y}_i\in TX_i,\\ \bot, & \mathfrak{y}\notin T\tau_i(X_i),\end{cases}$$

for $\mathfrak{y}\in T(X_1+X_2)$ and $(x,i)\in X_1\times X_2$. In [14] M.M. Clementino and D. Hofmann give a Van Kampen Theorem in the context of lax algebras with respect to the class of morphisms given by all the $(\mathbb{T},\mathbf{V})$-functors. It includes all the categories of $(\mathbb{T},\mathbf{V})$-categories where the **Set**-monad $\mathbb{T}=(T,\mu,\eta)$ and its (flat) lax extension $\widehat{\mathbb{T}}=(\widehat{T},\mu,\eta)$ to **V**-**Rel** satisfy the following conditions:

$(C0)$ every naturality square of η with respect to **V**-relations with finite fibres is a BC-square;

$(C1)$ $\widehat{T}$ is left-whiskering;

$(C2)$ μ satisfies BC;

$(C3)$ T preserves coproducts.

Theorem 3.4.1 [14, Theorem 3.5] *Let* **V** *be a cartesian closed quantale and let* $\widehat{\mathbb{T}}$ *be a flat lax extension to* **V**-**Rel** *of a* **Set**-*monad* $\mathbb{T}=(T,\mu,\eta)$*, where* T *satisfies BC and conditions* $(C0)-(C3)$ *hold. Denoting by* **C** *the category* $(\mathbb{T},\mathbf{V},\widehat{\mathbb{T}})$-**Cat***, if the following diagram*

$$\begin{array}{ccc} (X_0,a_0) & \xrightarrow{f_1} & (X_1,a_1)\\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1}\\ (X_2,a_2) & \xrightarrow[g_2]{} & (X,a)\end{array}$$

is a pullback, with g_1 *and* g_2 *embeddings, then the functor*

$$K_{g_1,g_2} : \mathbf{C}\downarrow X \to (\mathbf{C}\downarrow X_1)\times_{\mathbf{C}\downarrow X_0}(\mathbf{C}\downarrow X_2)$$

is an equivalence of categories if and only if the morphism $p:(X_1+X_2,b)\to(X,a)$*, induced by* g_1 *and* g_2*, is a final morphism.*

This is an immediate consequence of the Theorem below.

Theorem 3.4.2 [14, Theorem 3.3] *Let* **V** *be a cartesian closed quantale and let* $\widehat{\mathbb{T}}$ *be a flat lax extension to* **V**-**Rel** *of a* **Set**-*monad* $\mathbb{T}=(T,\mu,\eta)$*, where* T *satisfies BC and conditions* $(C0)-(C3)$ *hold. If* (X_1,a_1) *and* (X_2,a_2) *are* $(\mathbb{T},\mathbf{V})$-*subcategories of* (X,a) *and* $p:(X_1+X_2,b)\to(X,a)$ *is the* $(\mathbb{T},\mathbf{V})$-*functor induced by their embeddings* g_1 *and* g_2*, then the following conditions are equivalent in* $(\mathbb{T},\mathbf{V})$-**Cat***:*

(i) *p is a pullback stable *-quotient morphism in* $(\mathbb{T},\mathbf{V})$-**Gph***;*

(ii) *p is of effective descent;*

(iii) *p is a descent morphism;*

(iv) *p is final;*

(v) *for any* $\mathfrak{x} \in TX$ *and* $x \in X$*, either there exists* $i \in \{1,2\}$ *such that* $\mathfrak{x} \in Tg_i(TX_i)$ *and* $x \in g_i(X_i)$*, or* $a(\mathfrak{x},x) = \bot$.

The two theorems above include for instance the identity monad $\mathbb{I}$ and the ultrafilter monad $\mathbb{U}$ with corresponding suitable extensions as it is, if **V** is ccd, the uniform extension. Recall that in **Top** a version of the Van Kampen Theorem, with respect to the class of all continuous maps, was already given (see Theorem 1.5.6). The M-ordered monad $\mathbb{M}$ is not mentioned in the paper but we remark that, considering its uniform extension to **V**-**Rel**, it satisfies the conditions above, so that it can be included in the list. Although the free-monoid functor W does not preserve coproducts, also in $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-**Cat** a version of the Van Kampen Theorem is given. In fact, in [14, Theorem 3.4], it is proved that the same result holds for p in $(\mathbb{W},\mathbf{V},\mathbb{W}^{\otimes})$-**Cat**. In general observe that $(i) \Rightarrow (ii)$ follows from Theorem 3.0.6 and $(ii) \Rightarrow (iii)$ is always true by definition. The implication $(iii) \Rightarrow (iv)$ is where condition $(C0)$ and the flatness of the extension are required while $(iv) \Leftrightarrow (v)$ is proved to be always true. If we assume an extra condition on **V**, namely

$(C4)$ $u \otimes v = \bot \Rightarrow u = \bot$ or $v = \bot$, for all $u,v \in \mathbf{V}$,

then Theorem 3.4.2, and so also Theorem 3.4.1, are true in $(\mathbb{W},\mathbf{V},\mathbb{W}^{\wedge})$-**Cat**. The proof is similar to the case where the extension is given by $\mathbb{W}^{\otimes}$.

Theorem 3.4.3 *Let* **V** *be a ccd and cartesian closed quantale such that condition* $(C4)$ *is satisfied. If* (X_1,a_1) *and* (X_2,a_2) *are* $(\mathbb{W},\mathbf{V},\mathbb{W}^{\wedge})$*-subcategories of* (X,a) *and* $p : (X_1+X_2,b) \to (X,a)$ *is the* $(\mathbb{W},\mathbf{V},\mathbb{W}^{\wedge})$*-functor induced by their embeddings* g_1 *and* g_2*, then the following condition are equivalent in* $(\mathbb{W},\mathbf{V},\mathbb{W}^{\wedge})$-**Cat***:*

(i) *p is a pullback stable *-quotient morphism in* $(\mathbb{W},\mathbf{V})$-**Gph***;*

(ii) *p is of effective descent;*

(iii) *p is a descent morphism;*

(iv) *p is final;*

(v) *for any* $\mathfrak{x} \in TX$ *and* $x \in X$*, either there exists* $i \in \{1,2\}$ *such that* $\mathfrak{x} \in Tg_i(TX_i)$ *and* $x \in g_i(X_i)$*, or* $a(\mathfrak{x},x) = \bot$.

Proof
We start observing that conditions $(C0)-(C2)$ are satisfied even if condition $(C1)$ is not longer required. The implications $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$, $(iii) \Rightarrow (iv)$ and $(iv) \Leftrightarrow (v)$ follow as before. So what one really needs to show is $(iv) \Rightarrow (i)$. Recall that, since W satisfies BC, final morphisms are pullback stable in $(\mathbb{W},\mathbf{V})$-**Gph**. Let then p be a final morphism. We need only to show that p is a *-quotient morphism. Let $\mathfrak{X} = ((x_1^1,\dots,x_{m_1}^1),\dots,(x_1^n,\dots,x_{m_n}^n)) \in W^2X$, $\mathfrak{x} = (x_1,\dots,x_n) \in WX$ and $x \in X$ such that

$$W^{\wedge}a(\mathfrak{X},\mathfrak{x}) \otimes a(\mathfrak{x},x) \neq \bot.$$

Since p is final, there exists $j \in \{1,2\}$ such that $\mathfrak{x} \in WX_j$ and $x \in X_j$. Assume, without loss of generality, that $j=1$. Transitivity of a guarantees that $a(\mu_X(\mathfrak{X}),x) \geq W^\wedge a(\mathfrak{X},\mathfrak{x}) \otimes a(\mathfrak{x},x) \neq \bot$ and, since p is final, there exists $i \in \{1,2\}$ such that $\mu_X(\mathfrak{X}) \in WX_i$ and $x \in X_i$. The diagram

$$\begin{array}{ccc} W^2X_i & \xrightarrow{\mu_{X_i}} & WX_i \\ {\scriptstyle W^2g_i}\downarrow & & \downarrow{\scriptstyle Wg_i} \\ W^2X & \xrightarrow[\mu_X]{} & WX \end{array}$$

is a BC-square, so that $\mathfrak{X} \in W^2X_i$.

- if $i=1$ the proof is complete, since $\mu_X(\mathfrak{X}) \in WX_1 \Rightarrow \mathfrak{X} \in W^2X_1$.

- if $i=2$ we have that $\mathfrak{X} \in Wg_2(WX_2) - Wg_1(WX_1)$ and $x \in g_2(X_2)$. The proof is then complete if $\mathfrak{x} \in Wg_2(WX_2)$. If this is not the case, i.e., there exists $l \in \{1,\ldots,n\}$ such that $x_l \notin X_2$, one can consider $\mathfrak{X}_l \in W^2X$ defined by

  $$\mathfrak{X}_l = ((x_1^1,\ldots,x_{m_1}^1),\ldots,(x_l),\ldots,(x_1^n,\ldots,x_{m_n}^n)).$$

  Basically $\mathfrak{X}_l$ is given by the same words of $\mathfrak{X}$ with the only difference that in the position l the word $(x_1^l,\ldots,x_{m_l}^l)$ is replaced by the word (x_l). Since $\mathbf{V}$ is integral,

  $$W^\wedge a(\mathfrak{X}_l,x) = \bigwedge_{i\in\{1,\ldots,n\}-\{l\}} a((x_1^i,\ldots,x_{m_i}^i),x_i) \neq \bot.$$

  Then, by condition $(C4)$,

  $$a(\mu_X(\mathfrak{X}_l),x) \geq (\bigwedge_{i\in\{1,\ldots,n\}-\{l\}} a((x_1^i,\ldots,x_{m_i}^i),x_i)) \otimes a((x_1,\ldots,x_n),x) \neq \bot.$$

  But, under our assumptions, $\mu_X(\mathfrak{X}_l) \notin Wg_1(WX_1) \cup Wg_2(WX_2)$, which contradicts finality of p.

□

If we now consider the powerset monad $\mathbb{P} = (P,\mu,\eta)$, and its uniform extension $\widetilde{\mathbb{P}}$ to $\mathbf{V}$-**Rel**, we have the following result.

Theorem 3.4.4 *Let* $\mathbf{V}$ *be a ccd and cartesian closed quantale such that condition* $(C4)$ *is satisfied. If* (X,a_1) *and* (X,a_2) *are* $(\mathbb{P},\mathbf{V},\widetilde{\mathbb{P}})$*-subcategories of* (X,a) *and* $p:(X_1+X_2,b)\to(X,a)$ *is the* $(\mathbb{P},\mathbf{V},\widetilde{\mathbb{P}})$*-functor induced by their embeddings, then for the statements*

(i) *p is final;*

(ii) *for any* $A\in PX$ *and* $x\in X$*, either there exists* $i\in\{1,2\}$ *such that* $A\in Pg_i(PX_i)$ *and* $x\in g_i(X_i)$*, or* $a(A,x)=\bot$*;*

(iii) *p is a pullback stable *-quotient morphism in* $(\mathbb{P},\mathbf{V})$-**Gph**;

(iv) *p is effective for descent;*

(v) *p is a descent morphism;*

one has the implications $(i) \Leftrightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v)$.

Proof

From the observations we did previously, one has the implications $(i) \Leftrightarrow (ii)$, $(iii) \Rightarrow (iv)$ and $(iv) \Rightarrow (v)$. Therefore we only need to show $(ii) \Rightarrow (iii)$. Final morphisms are pullback stable in $(\mathbb{P}, \mathbf{V})$-**Gph**, since P satisfies BC and $\mathbf{V}$ is assumed to be cartesian closed. Therefore we have to show only that p is a *-quotient morphism. Let $\mathfrak{A} \in P^2X$, $A \in PX$ and $x_0 \in X$ such that

$$\widetilde{P}a(\mathfrak{A}, A) \otimes a(A, x_0) \neq \bot.$$

By definition of the uniform extension, $\widetilde{P}a(\mathfrak{A}, A) \neq \bot$ implies the existence of an element $u \in \mathbf{V}$ $(u \neq \bot)$ such that $\overline{P}a_u(\mathfrak{A}, A) = \top$, i.e.,

$$\forall A^{\mathfrak{A}} \in \mathfrak{A} \quad \exists x \in A : u \leq a(A^{\mathfrak{A}}, x) \quad \text{and} \quad \forall x \in A \quad \exists A^{\mathfrak{A}} \in \mathfrak{A} : u \leq a(A^{\mathfrak{A}}, x).$$

Since p is final, there exists $j \in \{1, 2\}$ such that $A \in PX_j$ and $x_0 \in X_j$. Assume, without loss of generality, that $j = 1$. Transitivity of a guarantees that $a(\mu_X(\mathfrak{A}), x_0) \geq \widetilde{P}a(\mathfrak{A}, A) \otimes a(A, x_0) \neq \bot$ and, since p is final, there exists $i \in \{1, 2\}$ such that $\mu_X(\mathfrak{A}) \in PX_i$ and $x \in X_i$. The diagram

$$\begin{array}{ccc} P^2X_i & \xrightarrow{\mu_{X_i}} & PX_i \\ {\scriptstyle P^2g_i}\downarrow & & \downarrow{\scriptstyle Pg_i} \\ P^2X & \xrightarrow[\mu_X]{} & PX \end{array}$$

is a BC-square, so that $\mathfrak{A} \in P^2X_i$.

- if $i = 1$ the proof is complete, since $\mu_X(\mathfrak{A}) \in PX_1 \Rightarrow \mathfrak{A} \in P^2X_1$.

- if $i = 2$ we have that $\mathfrak{A} \in Pg_2(PX_2) - Pg_1(WX_1)$ and $x_0 \in g_2(X_2)$. The proof is then complete if $A \in Pg_2(WX_2)$. If this is not the case, i.e., there exists $x^* \in A$ such that $x^* \notin X_2$, one can consider $\mathfrak{A}_{x^*} \in P^2X$ defined by

  $$\mathfrak{A}_{x^*} = \mathfrak{A} \cup \{\{x^*\}\}.$$

  We prove that $\widetilde{P}(\mathfrak{A}_{x^*}, A) \neq \bot$. We already know that there is an element $u \in \mathbf{V}$ such that $\overline{P}a_u(\mathfrak{A}, A) = \top$. For the same element u one has, moreover, $\overline{P}a_u(\mathfrak{A}_{x^*}, A) = \top$. In fact, for $\{x^*\} \in \mathfrak{A}_{x^*}$, one has $u \leq a(\{x^*\}, x^*) = \top$, since $\mathbf{V}$ is integral. Then, by condition $(C4)$,

  $$a(\mu_X(\mathfrak{A}_{x^*}), x_0) \geq \widetilde{P}a(\mathfrak{A}_{x^*}, A) \otimes a(A, x_0) \neq \bot.$$

  But, under our assumptions, $\mu_X(\mathfrak{A}_{x^*}) \notin Pg_1(PX_1) \cup Pg_2(PX_2)$, which contradicts finality of p.

□

We remark that the theorem above does not accomplish completely our goal, since we do not know whether implication $(v) \Rightarrow (i)$ is true, as it is in the other cases analyzed before. The reason comes from the fact that the powerset monad does not satisfy condition $(C0)$. By Theorem 3.4.4, we can anyway state the following result.

Theorem 3.4.5 *Let* $\mathbf{V}$ *be a ccd and cartesian closed quantale such that condition* $(C4)$ *is satisfied. Let* $\widetilde{\mathbb{P}}$ *be the uniform extension to* $\mathbf{V}$-$\mathbf{Rel}$ *of the powerset monad* $\mathbb{P} = (P, \mu, \eta)$*. Denoting by* $\mathbf{C}$ *the category* $(\mathbb{P}, \mathbf{V}, \widetilde{\mathbb{P}})$-$\mathbf{Cat}$*, if the following diagram*

$$\begin{array}{ccc} (X_0, a_0) & \xrightarrow{f_1} & (X_1, a_1) \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ (X_2, a_2) & \xrightarrow[g_2]{} & (X, a) \end{array}$$

is a pullback, with g_1 *and* g_2 *embeddings, then the functor*

$$K_{g_1,g_2} : \mathbf{C} \downarrow X \to (\mathbf{C} \downarrow X_1) \times_{\mathbf{C} \downarrow X_0} (\mathbf{C} \downarrow X_2)$$

is an equivalence of categories if the morphism $p : (X_1 + X_2, b) \to (X, a)$*, induced by* g_1 *and* g_2*, is a final morphism, i.e., for any* $A \in PX$ *and* $x \in X$*, either there exists* $i \in \{1, 2\}$ *such that* $A \in Pg_i(PX_i)$ *and* $x \in g_i(X_i)$*, or* $a(A, x) = \bot$*.*

□

Chapter 4

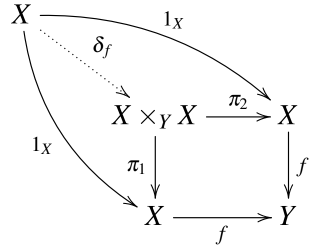
Effective étale-descent morphisms in categories of lax algebras

In Chapter 1 we saw several developments concerning the study of (effective) étale-descent maps in **Top**, including the complete characterization of the effective étale-decent morphisms in **Ord**, given by G. Janelidze and M. Sobral, which solves the problem of the characterization for finite topological spaces. Our first contribution concerning the more general problem of étale-descent in categories of lax alegbras is given by the complete characterization of the effective étale-descent morphisms in M-**Ord** in [2], which we are going to recall in Section 4.2.

4.1 Étale morphisms

In [16] M.M. Clementino, D. Hofmann and G. Janelidze give a characterization of local homeomorphisms between topological spaces in terms of ultrafilter convergence. This characterization suggests a definition of local homeomorphisms (*étale morphisms*) in the context of lax algebras. We recall that a *local homeomorphism* between topological spaces is a continuous map $f : X \to Y$ such that each $x \in X$ has an open neighbourhood U_x with $f(U_x)$ open and the restriction $f_{U_x} : U_x \to f(U_x)$ of f to U_x a homeomorphism. Equivalent formulations are given.

Proposition 4.1.1 *For a continuous map $f : X \to Y$, consider the following commutative diagram*



where the square is a pullback and the map $\delta_f : X \to X \times_Y X$, $x \mapsto (x,x)$, is the one induced by the universal property. The following are equivalent:

(i) *f is a local homeomorphism;*

(ii) *f is open and locally injective;*

(iii) *both f and δ_f are open.*

Definition 4.1.2 A continuous map $f : X \to Y$ of topological spaces is a *discrete fibration* if for each $x \in X$ and each ultrafilter $\mathfrak{y}$ with $\mathfrak{y} \to f(x)$ in Y, there exists a unique ultrafilter $\mathfrak{x}$ such that $\mathfrak{x} \to x$ in X and $f(\mathfrak{x}) = \mathfrak{y}$,

$$\begin{array}{ccc} X & \qquad & \exists!\mathfrak{x} & \dashrightarrow & x \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\ Y & & \mathfrak{y} & \longrightarrow & f(x). \end{array}$$

A local homeomorphism is a discrete fibration while the converse is not true in general, as proved in [16] where the following counter-example is given.

Example 4.1.3 [16, Example 2] Let $\mathfrak{x}$ be a non-principal ultrafilter on the set $\mathbb{N}$ of natural numbers equipped with the topology $\{A \subseteq \mathbb{N} : 0 \in A \Rightarrow A \in \mathfrak{x}\}$. Let $f : \mathbb{N} \to \{0,1\}$ defined by

$$n \mapsto \begin{cases} 0, & \text{if } n = 0, \\ 1, & \text{otherwise,} \end{cases}$$

where $\{0,1\}$ is the Sierpiński space with non-trivial open subset $\{1\}$. Then f is not a local homeomorphism, since it is not injective at any neighbourhood of 0, but it is a discrete fibration.

The following characterization holds.

Theorem 4.1.4 [16, Theorem 2] *For a continuous map $f : X \to Y$, the following conditions are equivalent:*

(i) *f is a local homeomorphism;*

(ii) *f is a pullback stable discrete fibration;*

(iii) *both f and δ_f are discrete fibrations.*

Based on the above results in **Top**, definitions of discrete fibrations and étale morphisms, in the context of $(\mathbb{T},\mathbf{2})$-categories, have been introduced in [17] by M.M. Clementino, D. Hofmann and G. Janelidze. Let $\mathbb{T} = (T,\mu,\eta)$ be a **Set**-monad where T satisfies BC.

Definition 4.1.5 A $(\mathbb{T},\mathbf{2})$-functor $f : (X,a) \to (Y,b)$ is said to be a *discrete fibration* if for each $x \in X$ and each $\mathfrak{y} \in TY$ with $b(\mathfrak{y}, f(x)) = \top$, there exists a unique $\mathfrak{x} \in TX$ such that $a(\mathfrak{x},x) = \top$ and $Tf(\mathfrak{x}) = \mathfrak{y}$,

$$\begin{array}{ccc} X & \qquad & \exists!\mathfrak{x} & \dashrightarrow & x \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow \\ Y & & \mathfrak{y} & \longrightarrow & f(x). \end{array}$$

Following the same arguments given in [16] in the case of the ultrafilter monad, for each $(X,a) \in (\mathbb{T},\mathbf{2})$-**Cat**, one denotes by $\mathrm{Conv}(X,a)$ the set of pairs $(\mathfrak{x},x)$, where $x \in X$ and $\mathfrak{x} \in TX$ with $\mathfrak{x} \to x$, i.e., $a(\mathfrak{x},x) = \top$. The set $\mathrm{Conv}(X,a)$ has a canonical convergence structure, giving rise to the functor Conv studied in Section 3.2. Each $(\mathbb{T},\mathbf{2})$-functor $f : (X,a) \to (Y,b)$ induces a map

$$\mathrm{Conv}(f) : \mathrm{Conv}(X,a) \to \mathrm{Conv}(Y,b), \quad (\mathfrak{x},x) \mapsto (Tf(\mathfrak{x}), f(x)).$$

A $(\mathbb{T},\mathbf{2})$-functor $f : (X,a) \to (Y,b)$ is a discrete fibration if and only if the diagram

$$\begin{array}{ccc} \mathrm{Conv}(X,a) & \xrightarrow{\mathrm{Conv}(f)} & \mathrm{Conv}(Y,b) \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a pullback in **Set**, where π_X and π_Y are the projection maps.

Proposition 4.1.6

(1) *Let $f : (X,a) \to (Y,b)$ and $g : (Y,b) \to (Z,c)$ be $(\mathbb{T},\mathbf{2})$-functors. If $g \cdot f$ and g are discrete fibrations, then so is f.*

(2) *Let*

$$\begin{array}{ccc} W & \xrightarrow{k} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[g]{} & Y \end{array}$$

*be a pullback diagram in $(\mathbb{T},\mathbf{2})$-**Cat**, such that k is an injective map. If T : **Set** $\to$ **Set** satisfies BC, then if f is a discrete fibration, then so is h. In particular, the class of discrete fibrations is stable under pullback along injective $(\mathbb{T},\mathbf{2})$-functors.*

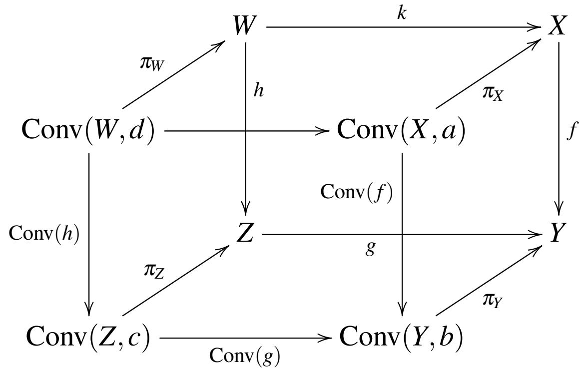
Proof

(1) Consider the following commutative diagram in **Set**

$$\begin{array}{ccccc} \mathrm{Conv}(X,a) & \xrightarrow{\mathrm{Conv}(f)} & \mathrm{Conv}(Y,b) & \xrightarrow{\mathrm{Conv}(g)} & \mathrm{Conv}(Z,c) \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi_Z} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z. \end{array}$$

Since, by hypothesis, the outer diagram and the right-hand square are pullback diagrams, also the left-hand square is a pullback diagram, and the result follows.

(2) In the commutative diagram



we only need to show that the maps $\pi_W : \mathrm{Conv}(W,d) \to W$ and $\mathrm{Conv}(h) : \mathrm{Conv}(W,d) \to \mathrm{Conv}(Z,c)$ are jointly monic. This is true since π_X and $\mathrm{Conv}(f)$ are jointly monic, because f is a discrete fibration and $\mathrm{Conv}(k)$ is injective, (since k is) by hypothesis.

□

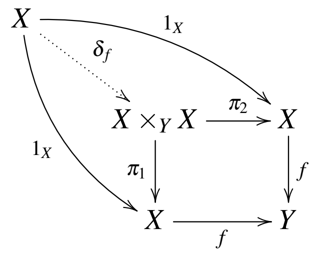
Observe that the description of the set $\mathrm{Conv}(X,a)$ can be done also starting from an object $(X,a) \in (\mathbb{T},\mathbf{2})$-**Gph** so that the proposition above remains true in $(\mathbb{T},\mathbf{2})$-**Gph**.

Definition 4.1.7 A $(\mathbb{T},\mathbf{2})$-functor $f : (X,a) \to (Y,b)$ is an *étale morphism* if it is a pullback stable discrete fibration.

Remark 4.1.8 If the functor $T : \mathbf{Set} \to \mathbf{Set}$ is cartesian then discrete fibrations are pullback stable, so that the class of étale morphisms coincides with the class of discrete fibrations, as it happens, for instance, in **Ord**.

The following characterization holds.

Proposition 4.1.9 *For a $(\mathbb{T},\mathbf{2})$-functor $f : (X,a) \to (Y,b)$, consider the following commutative diagram*



where the square is a pullback and the $(\mathbb{T},\mathbf{2})$-functor $\delta_f : X \to X \times_Y X$ is the one induced by the universal property. The following conditions are equivalent:

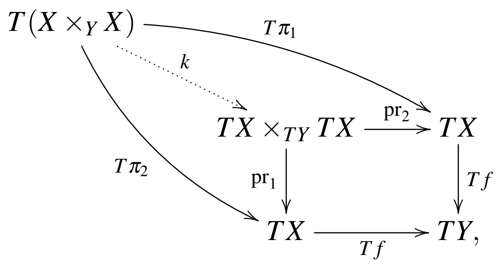
(i) *f is an étale morphism;*

(ii) *both f and δ_f are discrete fibrations;*

(iii) *both f and δ_f are open.*

Proof

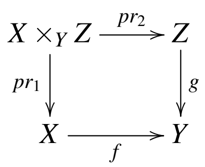
$(i) \Rightarrow (ii)$ By definition, f is a discrete fibration. Let $x \in X$ and $\mathfrak{w} \in T(X \times_Y X)$ such that $\mathfrak{w} \to (x,x)$, i.e., $T\pi_1(\mathfrak{w}) \to x$ and $T\pi_2(\mathfrak{w}) \to x$. Since π_2 is a discrete fibration, $T\pi_2(\mathfrak{w})$ is the only element converging to x and mapped by $T\delta_f$ into $\mathfrak{w}$. Therefore also δ_f is a discrete fibration.
$(ii) \Rightarrow (iii)$ Follows by definition of discrete fibration.
$(iii) \Rightarrow (i)$ We start proving that f is a discrete fibration. We already know that f is open so let $x \in X$ and $\mathfrak{y} \in TY$ with $\mathfrak{y} \to f(x)$. Let us suppose that there exist $\mathfrak{x}$ and $\mathfrak{x}'$ in TX such that $\mathfrak{x} \to x$, $\mathfrak{x}' \to x$ and $Tf(\mathfrak{x}) = Tf(\mathfrak{x}') = \mathfrak{y}$. Therefore the pair $(\mathfrak{x},\mathfrak{x}')$ belongs to $TX \times_{TY} TX$. In the following diagram induced by the universal property of the pullback



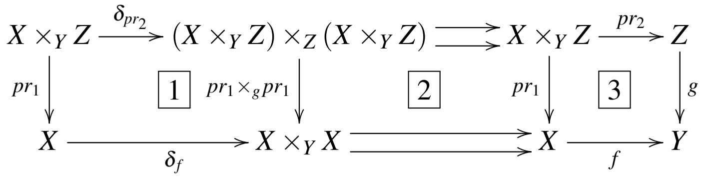
since T satisfies BC, there exists an element $\mathfrak{w} \in T(X \times_Y X)$ such that $k(\mathfrak{w}) = (\mathfrak{x},\mathfrak{x}')$. Moreover $\mathfrak{w} \to (x,x)$, since $T\pi_1(\mathfrak{w}) = \mathfrak{x} \to x$ and $T\pi_2(\mathfrak{w}) = \mathfrak{x}' \to x$. But, since δ_f is open, there exists an element $\bar{\mathfrak{x}} \in TX$ such that $T\delta_f(\bar{\mathfrak{x}}) = \mathfrak{w}$ and $\bar{\mathfrak{x}} \to x$. By the equalities

$$(T\pi_1 \cdot T\delta_f)(\bar{\mathfrak{x}}) = T\pi_1(\bar{\mathfrak{w}}) = \mathfrak{x}, \quad (T\pi_1 \cdot T\delta_f)(\bar{\mathfrak{x}}) = T(\pi_1 \cdot \delta_f)(\bar{\mathfrak{x}}) = T1_X(\bar{\mathfrak{x}}) = 1_{TX}(\bar{\mathfrak{x}}) = \bar{\mathfrak{x}}$$

we conclude that $\bar{\mathfrak{x}} = \mathfrak{x}$. The same holds for $\mathfrak{x}'$ so that $\mathfrak{x} = \bar{\mathfrak{x}} = \mathfrak{x}'$.
It remains to show the pullback stability. Consider the following pullback diagram



where f is a discrete fibration. We want to show that pr_2 is a discrete fibration as well. Since T satisfies BC, we know that pr_2 is open. It is enough to show that δ_{pr_2} is open. Consider the following commutative diagram



where $\boxed{1}\boxed{2}\boxed{3} = \boxed{3}$ and $\boxed{2}\boxed{3}$ are pullbacks. Therefore, by general properties of pullback squares, also $\boxed{1}$ is a pullback and, since δ_f is open, also δ_{pr_2} is open. □

The notion of étale morphism for a **V**-functor has been introduced in [15] by M.M. Clementino and D. Hofmann. In particular,

Definition 4.1.10 A **V**-functor $f:(X,a)\to(Y,b)$ is an *étale morphism* if both f and δ_f are open.

One can verifies that $\delta_f:(X,a)\to(X,a)\times_{(Y,b)}(X,a)$ is open if and only if for all x_1,x_1',x_0 in X with $f(x_1)=f(x_1')$ and $x_1\neq x_1'$,

$$a(x_1,x_0)\wedge a(x_1',x_0)=\bot.$$

This suggests a more general definition for étale morphism for an arbitrary $(\mathbb{T},\mathbf{V})$-functor.

Definition 4.1.11 A $(\mathbb{T},\mathbf{V})$-functor $f:(X,a)\to(Y,b)$ is an *étale morphism* if both f and δ_f are open.

This definition covers both cases where $\mathbf{V}=\mathbf{2}$ (see Definition 4.1.7 and Proposition 4.1.9) and $\mathbb{T}=\mathbb{I}$ (see Definition 4.1.10). In case of **V**-functors, by Definition 4.1.10, if **V** is totally ordered one has that a **V**-functor $f:(X,a)\to(Y,b)$ is an étale morphism if and only if the following condition holds:

$$\forall y_1\in Y,x_0\in X \quad \exists! x_1\in X: f(x_1)=y_1 \quad \& \quad b(y_1,f(x_0))=a(x_1,x_0),$$

with $a(x_1',x_0)=\bot$ for each x_1' in X such that $x_1'\neq x_1$ and $f(x_1')=y_1$.
On the other hand, if the quantale **V** is ccd, then one has the following characterization.

Proposition 4.1.12 *Let* **V** *be a ccd quantale. For the* **V***-functor* $f:(X,a)\to(Y,b)$ *the following statements are equivalent:*

(i) *f is an étale morphism;*

(ii) $\forall y_1\in Y,x_0\in X,u\ll b(y_1,f(x_0))(u\neq\bot) \quad \exists! x_1\in X: f(x_1)=y_1 \quad \& \quad u\ll a(x_1,x_0)$;

(iii) $\forall y_1\in Y,x_0\in X,u\ll b(y_1,f(x_0))(u\neq\bot) \quad \exists! x_1\in X: f(x_1)=y_1 \quad \& \quad u\leq a(x_1,x_0)$.

Proof
$(i)\Rightarrow(ii)$ Let $y_1\in Y,x_0\in X,u\in\mathbf{V}$ such that $u\neq\bot$ and $u\ll b(y_1,f(x_0))$. Since f is open,

$$b(y_1,f(x_0))=\bigvee_{x_1\in X:f(x_1)=y_1}a(x_1,x_0).$$

Hence,

$$u\ll\bigvee_{x_1\in X:f(x_1)=y_1}a(x_1,x_0)$$

implies, since **V** is ccd, that there exists $x_1\in X$ such that $f(x_1)=y_1$ and $u\ll a(x_1,x_0)$. Let us suppose now there exists $x_1'\in X$, with $x_1'\neq x_1$, such that $f(x_1')=y_1$ and $u\ll a(x_1',x_0)$. Since δ_f is open, $a(x_1,x_0)\wedge a(x_1',x_0)=\bot$. Therefore, $u\leq a(x_1,x_0)\wedge a(x_1',x_0)\Rightarrow u=\bot$, giving rise to a contradiction.
$(ii)\Rightarrow(iii)$ It immediately follows.
$(iii)\Rightarrow(i)$ We start by proving f to be open. Let $y_1\in Y$ and $x_0\in X$. We want to show that

$$b(y_1,f(x_0))\leq\bigvee_{x_1\in X:f(x_1)=y_1}a(x_1,x_0),$$

since the other inequality is trivially satisfied. Since $\mathbf{V}$ is ccd,

$$b(y_1, f(x_0)) = \bigvee\{u \in \mathbf{V} : u \ll b(y_1, f(x_0))\}.$$

Since (iii) holds, for each $u \ll b(y_1, f(x_0))$, with $u \neq \bot$, there exists a unique $x_1 \in X$ such that $f(x_1) = y_1$ and $u \leq a(x_1, x_0)$. Hence,

$$b(y_1, f(x_0)) = \bigvee\{u \in \mathbf{V} : u \ll b(y_1, f(x_0))\} \leq \bigvee_{x_1 \in X : f(x_1) = y_1} a(x_1, x_0)$$

as claimed. It remains to show that δ_f is open. Let $x_0 \in X$ and let $x_1, x_1' \in X$ with $x_1 \neq x_1'$ and $f(x_1) = f(x_1')$. We want to show that $a(x_1, x_0) \wedge a(x_1', x_0) = \bot$. If this is not the case, since $\mathbf{V}$ is ccd, there would exist an element $u \in \mathbf{V}$, with $u \neq \bot$, such that $u \ll a(x_1, x_0) \wedge a(x_1', x_0)$. But the implication

$$\bot \neq u \ll a(x_1, x_0) \wedge a(x_1', x_0) \leq b(f(x_1), f(x_0)) = b(f(x_1'), f(x_0)) \Rightarrow u \ll b(f(x_1), f(x_0))$$

gives rise to a contradiction. □

4.2 Effective étale-descent morphisms in M-Ord

In [2] we give a complete characterization of the effective étale-descent morphisms in the category M-**Ord** of M-ordered sets and monotone maps. It has been inspired by the results given in [31] by G. Janelidze and M. Sobral, where effective étale-descent morphisms in **Ord** are characterized (see Section 1.4.3), and in [50] by M. Sobral, where a characterization of effective descent morphisms in **Cat**, with respect to the class of discrete (co)fibrations, is given. As we will see, the result in M-**Ord** represents a consequence of the one in **Cat**, so that we decided to dedicate a section to show how the characterization in **Cat** of the effective descent morphisms, with respect to the class of discrete fibrations, is obtained.

4.2.1 Effective descent morphisms in Cat with respect to the class of discrete fibrations

We start pointing out that **Cat** is a category of $\mathbf{V}$-categories; in fact, although in Chapter 2 we defined $(\mathbb{T}, \mathbf{V})$-categories with respect to **Set**-monads $\mathbb{T}$ and quantales $\mathbf{V}$, we remarked that more general settings can be considered such us, for instance, where $\mathbf{V}$ is a monoidal closed category (see [21]). In this context the category **Cat** of small categories and functors turns out to be (up to isomorphism) a category of $\mathbf{V}$-categories, where $\mathbf{V} = \mathbf{Set}$. The work in **Cat** represents a generalization of the work in **Ord**, since a (pre)ordered set X can be identified as a small (thin) category where the object class is the set X and there is a morphism $x_1 \to x_0$ from x_1 to x_0 if and only if $x_1 \leq x_0$. The notion of discrete fibration presented in Section 4.1 is a translation to the context of $(\mathbb{T}, \mathbf{V})$-categories of the notion of discrete fibration in **Cat**. In fact, when applied to **Ord** $\hookrightarrow$ **Cat**, it coincides with discrete fibration in **Cat**. Recall that a functor $F : X \to Y$ in **Cat** is called a *discrete fibration* if for every object $x_0 \in X$ and every morphism in Y of the form $g : y_1 \to F(x_0)$ there exists a unique morphism $f : x_1 \to x_0$ in X

such that $F(f) = g$,

$$\begin{array}{ccc} X & & x_1 \overset{\exists! f}{\dashrightarrow} x_0 \\ {\scriptstyle F}\downarrow & & \downarrow \qquad \downarrow \\ Y & & y_1 \xrightarrow[g]{} F(x_0). \end{array}$$

The strategy is the same as in the case of **Ord** (see diagram (3.3.25)). Let $\mathbb{E}$ be the class of discrete fibrations in **Cat** and let $P : E \to B$ be a functor between small categories E and B. The standard equivalence $\mathbb{E}(X) \simeq \mathbf{Set}^{X^{\mathrm{op}}}$, for a small category X, is an equivalence of **Cat**-indexed categories $\mathbb{A} : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$ and $\mathbb{B} : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$ defined respectively by

$$\begin{array}{ccc} E \longmapsto \mathbb{E}(E) & \quad & E \longmapsto \mathbf{Set}^{E^{\mathrm{op}}} \\ {\scriptstyle P}\uparrow \qquad \downarrow{\scriptstyle P^*} & & {\scriptstyle P}\uparrow \qquad \downarrow{\scriptstyle \mathbf{Set}^{P^{\mathrm{op}}}} \\ B \longmapsto \mathbb{E}(B) & & B \longmapsto \mathbf{Set}^{B^{\mathrm{op}}}. \end{array}$$

The category $\mathrm{Des}_{\mathbb{E}}(P)$ of descent data (with respect to $\mathbb{A}$) is then equivalent to the category $\mathrm{Des}_{\mathbb{B}}(P)$ of descent data with respect to the pseudo-functor $\mathbb{B}$ (we denoted this category by **X** in the case of **Ord**). This is given by pairs (X, ξ) where $X : E^{\mathrm{op}} \to \mathbf{Set}$ is a functor and ξ is a family of functions $\xi_{x,x'} : X(x) \to X(x')$, defined for $x, x' \in E$ with $P(x) = P(x')$, and satisfying the following conditions:

- $\xi_{x,x} = 1_{X(x)}$, for each $x \in E$;
- $\xi_{x',x''} \cdot \xi_{x,x'} = \xi_{x,x''}$, for each $x, x', x'' \in E$ with $P(x) = P(x') = P(x'')$;
- the diagram

$$\begin{array}{ccc} X(x_0) & \xrightarrow{\xi_{x_0,x_0'}} & X(x_0') \\ {\scriptstyle X(f)}\downarrow & & \downarrow{\scriptstyle X(f')} \\ X(x_1) & \xrightarrow[\xi_{x_1,x_1'}]{} & X(x_1') \end{array}$$

  is commutative for all $f : x_1 \to x_0$ and $f' : x_1' \to x_0'$ in E such that $P(f) = P(f')$.

The pair (X, ξ) can be seen as a double functor from the (double) category $\mathrm{Eq}(p)$, the internal category in **Cat** given by the kernel pair of P, to the (double) category $S(\mathbf{Set})$ of commutative squares in **Set** described by:

- horizontal arrows, i.e., elements $x, x' \in E$ such that $P(x) = P(x')$

$$x \longrightarrow x' \qquad \mapsto \qquad X(x) \xrightarrow{\xi_{x,x'}} X(x')$$

- vertical arrows, i.e., morphisms $f : x_1 \to x_0$ in E

$$\begin{array}{ccc} x_0 & & X(x_0) \\ {\scriptstyle f}\uparrow & \mapsto & \downarrow{\scriptstyle X(f)} \\ x_1 & & X(x_1) \end{array}$$

- squares in E

$$\begin{array}{ccc} x_0 & \longrightarrow & x'_0 \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f'} \\ x_1 & \longrightarrow & x'_1 \end{array} \quad \mapsto \quad \begin{array}{ccc} X(x_0) & \xrightarrow{\xi_{x_0,x'_0}} & X(x'_0) \\ {\scriptstyle X(f)}\downarrow & & \downarrow{\scriptstyle X(f')} \\ X(x_1) & \xrightarrow[\xi_{x_1,x'_1}]{} & X(x'_1) \end{array}$$

preserving horizontal and vertical compositions as well as all identities. Therefore one can use again the adjunction (1.26) to identify double functors from $\mathrm{Eq}(P)$ to $S(\mathbf{Set})$ with functors from $Z(\mathrm{Eq}(P))$ to **Set**. In [50] the category $Z(\mathrm{Eq}(P))$ of zigzags induced by the internal category $\mathrm{Eq}(P)$ in **Cat** is described as the quotient category $\widetilde{E}/\sim$, where $\widetilde{E}$ is the free category generated by the disjoint union $E_1 + (E_0 \times_{B_0} E_0)$, where E_1 and E_0 (B_0) are the discrete categories of the morphisms and of the objects of the category E (B), respectively, and $\sim$ is the smallest equivalence relation containing

- all pairs $(\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot, \cdot \xrightarrow{gf} \cdot)$ if $f,g \in E_1$;

- all pairs $(x \to x' \to x'', x \to x'')$ if $x \to x', x' \to x'' \in E_0 \times_{B_0} E_0$, meaning that $P(x) = P(x') = P(x'')$;

- and $(f' \cdot (x_1,x'_1), (x_0,x'_0) \cdot f)$

$$\begin{array}{ccc} x_1 & \longrightarrow & x'_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ x_0 & \longrightarrow & x'_0 \end{array}$$

  if $(x_i,x'_i) \in E_0 \times_{B_0} E_0$, for $i = 0,1$, and $P(f) = P(f')$.

Therefore objects in $Z(\mathrm{Eq}(P))$ are the same as in E while morphisms are equivalent classes of morphisms in $\widetilde{E}$ of the form

$$\begin{array}{cccccccc} x'_n & & & & & & & \\ \downarrow{\scriptstyle f_n} & & & & & & & \\ x_{n-1} & \longrightarrow & x'_{n-1} & & & & & \\ & & \downarrow{\scriptstyle f_{n-1}} & & & & & \\ & & x_{n-2} & \longrightarrow & x'_{n-2} & & & \\ & & & & \downarrow & & & \\ & & & & \cdot & \dashrightarrow & \cdot & \\ & & & & & & \vdots & \\ & & & & & & \cdot & \dashrightarrow & x'_1 \\ & & & & & & & & \downarrow{\scriptstyle f_1} \\ & & & & & & & & x_0 \end{array}$$

with $P(x_i) = P(x'_i)$, for $i = 1,\dots,n-1$, $f_j \in E_1$, for $j = 1,\dots,n$, which, as in **Ord**, one calls *n-zigzags* and denotes by $z = (x_0,x'_0)f_1 \cdots f_n(x_n,x'_n$. The factorization (1.25) of P through $Z(\mathrm{Eq}(p))$ still holds

$$\begin{array}{ccc} E & \xrightarrow{\quad P \quad} & B \\ & {\scriptstyle\psi}\searrow \quad \nearrow{\scriptstyle\varphi} & \\ & Z(\mathrm{Eq}(p)) & \end{array}$$

with $\psi(f) = [f]$, the equivalent class of $f \in E_1$ in $\widetilde{E}$, $\varphi(x) = P(x)$, for each $x \in E$ and $\varphi([z]) = P(f_1)\cdots P(f_{n-1})\cdot P(f_n)$, for each zigzag z as above.

Theorem 4.2.1 [50, Theorem 2] *The functor $P : E \to B$ is an effective $\mathbb{E}$-descent morphism if and only if φ is a full and faithful lax epimorphism.*

Corollary 4.2.2 [50, Corollary 3] *The functor $P : E \to B$ is an effective $\mathbb{E}$-descent morphism if and only if*

(i) *for each morphism $g : P(x_1) \to P(x_0)$ in B there exists a unique (up to equivalence) zigzag $z = (x_0,x'_0)f_1 \cdots f_n(x_n,x'_n)$ in $Z(\mathrm{Eq}(p))$ with $g = P(f_1)\cdots P(f_{n-1})\cdot P(f_n)$;*

(ii) *every object $y \in B$ is a retract of an object in $P(E)$, i.e., for each $y \in B$ there exists an object $x \in E$ such that $1_y = t \cdot s$, for $s : y \to P(x)$ and $t : P(x) \to y$ in B.*

Theorem 1.4.14 is an immediate consequence of the Corollary above.

Corollary 4.2.3 [50, Corollary 4] *If B is a (pre)ordered set considered as a category, a functor $P : E \to B$ is an effective $\mathbb{E}$-descent morphisms if and only if the functor $\varphi : Z(\mathrm{Eq}(P)) \to B$ is an equivalence of categories.*

Remarks 4.2.4

- A *discrete cofibration* is the dual notion of discrete fibration. The results given in Theorem 4.2.1 and in Corollary 4.2.2 are self-dual so that they represent also a characterization of the effective descent morphisms in **Cat**, with respect to the class of discrete cofibrations.

- Two 1-zigzags in $Z(\mathrm{Eq}(P))$ with the same image by φ are equivalent.

4.2.2 The characterization of the effective étale-descent morphisms in the category of M-ordered sets

Let $\mathbb{M} = (M^\times,\mu,\eta)$ be the M-ordered monad, where $M = (M,\cdot,1_M)$ is a monoid, and let $\overline{\mathbb{M}}$ be its Barr extension to **Rel**. In Example 2.3.3 we already saw how for a relation $a : M \times X \relbar\joinrel\mapstochar\joinrel\rightarrow X$ one can write $x \xrightarrow{m} y$ instead of $a((m,x),y) = \top$. As remarked in [29, Section V.1.4], this arrow notation for the structure of an $(\mathbb{M},\mathbf{2})$-category (X,a) emphasizes that X is actually the object set of a small category, denoted again by X, with hom-sets

$$X(x,y) = \{(x,m,y) \mid m \in M \quad \text{and} \quad x \xrightarrow{m} y\}$$

for $x,y \in X$; moreover this small category comes equipped with a faithful functor

$$\nu_X : X \to M, \quad (x,m,y) \mapsto m,$$

with M considered as a one-object category. Under this perspective identity morphisms and composition in an M-ordered set X are given by

$$x \xrightarrow{1_M} x \quad \text{and} \quad (x \xrightarrow{m} y \quad \& \quad y \xrightarrow{n} z \Rightarrow x \xrightarrow{n \cdot m} z),$$

while a morphism $f : X \to Y$ must satisfy

$$x \xrightarrow{m} y \Rightarrow f(x) \xrightarrow{m} f(y)$$

for all $x,y \in X$ and $m \in M$. Defining an M-*norm* to be a functor from the small category X to the category M, we have a full embedding

$$E : (\mathbb{M}, \mathbf{2})\text{-}\mathbf{Cat} \hookrightarrow \mathbf{Cat} \downarrow M \tag{4.1}$$

which sends each $(\mathbb{M}, \mathbf{2})$-category (X,a) to the pair (X, ν_X).

Proposition 4.2.5 [29, Proposition V.1.4.2] *The functor E is reflective and identifies* $(\mathbb{M}, \mathbf{2})$*-categories as those small categories over M whose norm is faithful.*

Remark 4.2.6 [29, Remarks V.1.4.3]

(1) In the trivial case where $M = 1$, the trivial monoid, $(\mathbb{M}, \mathbf{2})$-**Cat** in nothing but **2**-**Cat** $\cong$ **Ord**, also identified as the full subcategory of **Cat** given by small categories X for which the functor $X \to \mathbf{1}$ is faithful;

(2) The category $(\mathbb{M}, \mathbf{2})$-**Cat** is isomorphic to $P(M)$-**Cat**: for a $P(M)$-category (X,a), one assigns the $(\mathbb{M}, \mathbf{2})$-category (X, a_M) defined by

$$a_M((m,x),y) = \top \Leftrightarrow m \in a(x,y);$$

on the other hand, for an $(\mathbb{M}, \mathbf{2})$-category (X,a) one defines the $P(M)$-category $(X, a_{P(M)})$ by

$$a_{P(M)}(x,y) = \{m \in M : a((m,x),y) = \top\}.$$

Now let $\mathbb{E}$ be the class of étale morphisms in M-**Ord**. Since the monad $\mathbb{M}$ is cartesian, the class of étale morphisms coincides with the class of discrete fibrations. Using the arrow notation, an equivariant map $f : X \to Y$ in M-**Ord** is an étale morphism if and only if

$$\forall x_0 \in X, \quad \forall y_1 \in Y, \quad \forall m \in M : y_1 \xrightarrow{m} f(x_0) \Rightarrow \exists! x_1 \in f^{-1}(y_1) : x_1 \xrightarrow{m} x_0,$$

$$\begin{array}{ccc} X & & \exists! x_1 \overset{m}{\dashrightarrow} x_0 \\ {\scriptstyle f}\downarrow & & \downarrow \qquad\quad \downarrow \\ Y & & y_1 \underset{m}{\longrightarrow} f(x_0). \end{array}$$

Therefore the notion of discrete fibration for a morphism in M-**Ord** coincides with the notion of discrete fibration for functors in **Cat** when we consider M-ordered sets as (M-normed) small categories. This tells us that the arguments used in **Cat** for the characterization of effective descent morphisms, with respect to the class of discrete fibrations, can be applied in this context. By Theorem 4.2.1 we have the following.

Theorem 4.2.7 [2, Theorem 3.1] *An equivariant map $p : E \to B$ is an effective étale-descent morphism in M-**Ord** if and only if $\varphi : Z(\mathrm{Eq}(p)) \to B$ is a full and faithful lax epimorphism in **Cat**.*

Proof

Let $p : E \to B$ be an equivariant map in M-**Ord**. Consider the following diagram

$$(\mathbb{M}, \mathbf{2})\text{-}\mathbf{Cat} \overset{E}{\to} \mathbf{Cat} \downarrow M \overset{U}{\to} \mathbf{Cat},$$

where E is the full embedding (4.1) and U is the obvious forgetful functor. Since E preserves pullbacks, the pullback functor $p^* : \mathbb{E}(B) \to \mathbb{E}(E)$ is described by the following diagram

$$\begin{array}{ccc} E \times_B A & \overset{\pi_2}{\longrightarrow} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \overset{p}{\longrightarrow} & B \\ {\scriptstyle \nu_E}\downarrow & \swarrow{\scriptstyle \nu_B} & \\ M & & \end{array}$$

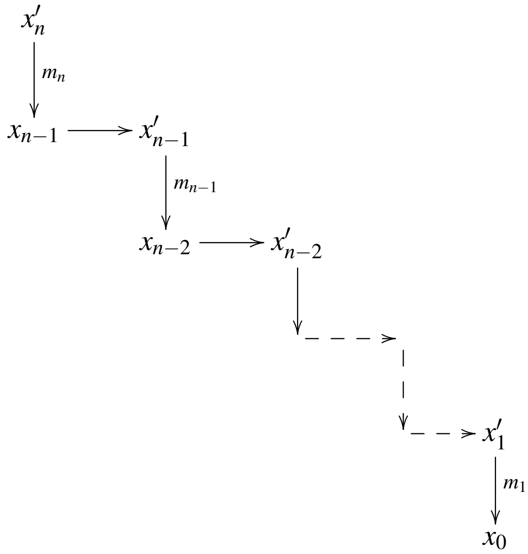
where the square is in **Cat**. In fact for a discrete fibration $\alpha : A \to B$ of small categories, being in particular a faithful functor, the composition

$$A \overset{\alpha}{\to} B \overset{\nu_B}{\to} M$$

gives an M-valued norm for A making $\alpha : A \to B$ an object in $\mathbb{E}(B)$. Hence the arguments in **Cat** of Section 4.2.1, leading to the characterization of effective descent morphisms in **Cat** with respect to the class of discrete fibrations, can be used to get a characterization of the effective étale-descent morphisms in M-**Ord**. Following those arguments, the equivariant map $p : E \to B$ can be then factorized in **Cat** in the following way

$$\begin{array}{ccc} E & \overset{p}{\longrightarrow} & B \\ & {\scriptstyle \psi}\searrow \quad \nearrow{\scriptstyle \varphi} & \\ & Z(\mathrm{Eq}(p)), & \end{array}$$

where $Z(\mathrm{Eq}(p))$ is the category of zigzags with E as object-set and morphisms given by equivalent classes of zigzags of the form



where $p(x_i) = p(x'_i)$ for $i = 1, \cdots, n-1$, and $x'_i \xrightarrow{m_i} x_{i-1}$ in E for $i = 1, \cdots, n$. The notation for such an n-zigzag as above will be

$$z = (x_0, x'_0)m_1 \cdots m_n(x_n, x'_{0n}).$$

The morphism ψ is defined as the identity on objects and $\psi(x_1 \xrightarrow{m} x_0) = [x_1 \xrightarrow{m} x_0]$ on morphisms, while φ on objects acts as p and the image of an equivalent class of an n-zigzag z via φ is

$$\varphi([z]) = p(x_n) \xrightarrow{m_n} p(x'_n) \xrightarrow{m_{n-1}} \cdots \xrightarrow{m_1} p(x'_0) = p(x_n) \xrightarrow{m_1 \cdots m_{n-1} \cdot m_n} p(x'_0).$$

By Theorem 4.2.1, we can then state that the equivariant $p : E \to B$ is an effective étale-descent morphism if and only if φ is a full and faithful lax epimorphism in **Cat**. □

Corollary 4.2.8 [2, Corollary 3.2] *An equivariant map $p : E \to B$ is an effective étale-descent morphism in M-*__Ord__* if and only if*

(i) For each $p(x) \xrightarrow{l} p(x')$ in B with $l \in M$ there exists a (unique up to equivalence) zigzag $z = (x_0, x'_0)m_1 \cdots m_n(x_n, x'_n)$ in $Z(\mathrm{Eq}(p))$ with $l = m_1 \cdots m_{n-1} \cdot m_n$;

(ii) every point $y \in B$ is in relation to a point of the image via a right-invertible element of the monoid, i.e., for each $y \in B$ there exist $x \in E$, $n, m \in M$ such that $p(x) \xrightarrow{n} y$ and $y \xrightarrow{m} p(x)$ with $n \cdot m = 1_M$.

□

Remarks 4.2.9 [2, Remark 3.3]

- As we already mentioned in Remarks 4.2.6, when $M = 1$, 1-**Ord** $\cong$ **Ord**. Hence the characterization of effective étale-descent morphisms in M-**Ord** generalizes the characterization in **Ord**.

- As in **Ord** uniqueness of zigzags encodes the fact that $Z(\mathrm{Eq}(p))$ is a (pre)ordered set, here it encodes the property that $Z(\mathrm{Eq}(p))$ is an M-ordered set, with the norm given by

$$Z(\mathrm{Eq}(p)) \xrightarrow{\varphi} B \xrightarrow{\nu_B} M.$$

- As in **Cat**, the results given by Theorem 4.2.7 and by Corollary 4.2.8 are self-dual, so that they also represent a characterization of the effective descent morphisms in M-**Ord**, with respect to the class of discrete cofibrations.

4.3 Towards effective étale-descent morphisms in V-Cat

In Chapter 3 we introduced (what we called) the relational method for which, knowing data about (effective) descent morphisms in $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat**, one can get informations to study the problem of descent in $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat**. This has been possible since a morphism $p:(E,e)\to(B,b)$ in $(\mathbb{T},\mathbf{V},\widetilde{\mathbb{T}})$-**Cat**, where $\widetilde{\mathbb{T}}$ is the uniform extension to **V**-**Rel** of $\mathbb{T}$, can be split up in a family of morphisms $(p_u:(E,e_u)\to(B,b_u))_{u\in\mathbf{V}}$ in $(\mathbb{T},\mathbf{2},\overline{\mathbb{T}})$-**Cat**, where $\overline{\mathbb{T}}$ is the Barr extension of $\mathbb{T}$. Of course one has to assume some conditions on T and **V**, such as T satisfies BC and **V** is a ccd frame. In this section we want to apply the same method for what concerns effective étale-descent morphisms. In particular we will consider the case when $\mathbb{T}=\mathbb{I}$, the identity monad, so that the categories involved are **Ord**, where effective étale-descent morphisms are characterized (see Section 1.4.3), and **V**-**Cat**, where we want to obtain informations.

4.3.1 Regular epimorphisms in V-Cat

The example in **Ord** suggests that for a morphism to be effective étale-descent, the property of being a regular epimorphism might play a key role, in particular for what concerns its description in terms of convergence. In [28] D. Hofmann obtained a characterization of regular epimorphisms in categories of lax algebras. The main motivation was to obtain a characterization using convergence based on the (already) known characterization of quotient maps in the category **Ord** of (pre)ordered sets and monotone maps.

Theorem 4.3.1 [28, Theorem 10] *Let $\widehat{\mathbb{T}}$ be a flat lax extension to* **V**-**Rel** *of a* **Set**-*monad* $\mathbb{T}=(T,\mu,\eta)$ *such that μ extends to a (strict) natural tranformation and $\widehat{\mathbb{T}}$ is left-whiskering. Then a morphism $f:(X,a)\to(Y,b)$ in $(\mathbb{T},\mathbf{V})$-***Cat** *is a regular epimorphism if and only if there exists an ordinal γ such that*

$$b=f\cdot a_f^{\gamma}\cdot T^{\gamma}f^{\mathrm{op}}\cdot(\mu_Y^{\gamma})^{\mathrm{op}}=f\cdot a_f^{\gamma}\cdot(\mu_X^{\gamma})^{\mathrm{op}}\cdot Tf^{\mathrm{op}},$$

where a_f^{γ} is the "zigzag" structure $a_f^{\gamma}:T^{\gamma}X\,{\to\!\!\!\!\!\!\mapsto}\,X$ (see [28, Section 4]).

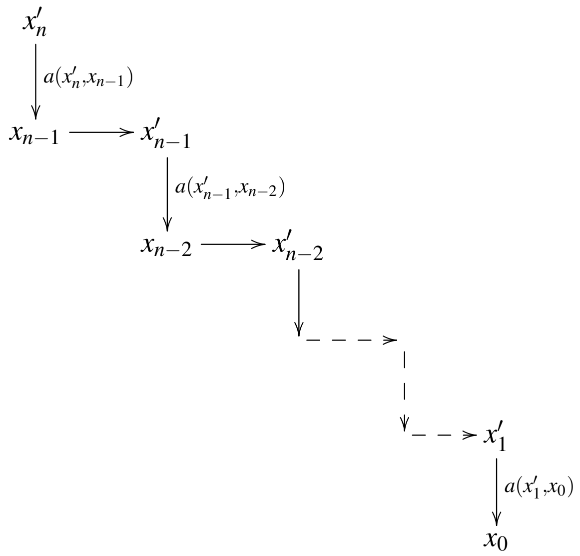
In case of the ultrafilter monad $\mathbb{U}=(U,\mu,\eta)$ and its Barr extension to **Rel** one has the following characterization.

Theorem 4.3.2 [28, Theorem 12] *A continuous map $f : X \to Y$ in* **Top** *is a regular epimorphism if and only if there exists an ordinal γ such that, for any $\mathfrak{y} \in UY$ and $y_0 \in Y$,*

$$\mathfrak{y} \to y_0 \iff \begin{cases} \text{there exist } \mathfrak{X} \in U^{\gamma}X \text{ and } x_0 \in X \text{ such that} \\ Uf \cdot \mu_X^{\gamma}(\mathfrak{X}) = \mathfrak{y} \quad \& \quad f(x_0) = y_0 \quad \& \quad \mathfrak{X} a_f^{\gamma} x_0. \end{cases}$$

For our purpose, we are mainly interested in the case of the identity monad.

Theorem 4.3.3 [28, Theorem 11] *A morphism $f : (X,a) \to (Y,b)$ in* **V**-**Cat** *is a regular epimorphism if and only if, for each y_1 and y_0 in Y, $b(y_1,y_0)$ is the supremum of all $a(x'_n,x_{n-1}) \otimes a(x'_{n-1},x_{n-2}) \otimes \cdots \otimes a(x'_1,x_0)$ obtained from zigzags*



(4.2)

in (X,a) $(n \in \mathbb{N})$ with $f(x'_n) = y_1$, $f(x_0) = y_0$ and $f(x_i) = f(x'_i)$, for $i = 1,\ldots,n-1$.

Observe that if $\mathbf{V} = \mathbf{2}$ one obtains the characterization of regular epimorphisms in **Ord** (see diagram (1.23)). When **V** is ccd we can get equivalent formulations which turn out to be useful to study effective étale-descent morphisms.

Proposition 4.3.4 *Let* **V** *be a ccd quantale. For a* **V***-functor $f : (X,a) \to (Y,b)$ the following conditions are equivalent:*

(i) *f is a regular epimorphism;*

(ii) *for each $y_1, y_0 \in Y$ and $u \ll b(y_1,y_0)$, there exists a zigzag* (4.2) *in (X,a) with $f(x'_n) = y_1$, $f(x_0) = y_0$, $f(x_i) = f(x'_i)$, for $i = 1,\ldots,n-1$, and such that*

$$u \ll a(x'_n,x_{n-1}) \otimes a(x'_{n-1},x_{n-2}) \otimes \cdots \otimes a(x'_1,x_0);$$

(iii) *for each $y_1, y_0 \in Y$ and $u \ll b(y_0,y_1)$, there exists a zigzag* (4.2) *in (X,a) with $f(x'_n) = y_1$, $f(x_0) = y_0$, $f(x_i) = f(x'_i)$, for $i = 1,\ldots,n-1$, and such that*

$$u \leq a(x'_n,x_{n-1}) \otimes a(x'_{n-1},x_{n-2}) \otimes \cdots \otimes a(x'_1,x_0).$$

Proof
$(i) \Rightarrow (ii)$ Let $f:(X,a)\to(Y,b)$ be a regular epimorphism in **V**-**Cat** and let $y_1,y_0\in Y$ and $u\ll b(y_1,y_0)$. By Theorem 4.3.3,

$$b(y_1,y_0)=\bigvee\{a(x'_n,x_{n-1})\otimes a(x'_{n-1},x_{n-2})\otimes\cdots\otimes a(x'_1,x_0)\quad\text{obtained from zigzags (4.2)}\}.$$

Since **V** is ccd, there exists a zigzag such that

$$u\ll a(x'_n,x_{n-1})\otimes a(x'_{n-1},x_{n-2})\otimes\cdots\otimes a(x'_1,x_0).$$

$(ii)\Rightarrow(iii)$ It immediately follows.
$(iii)\Rightarrow(i)$ Let y_1 and y_0 in Y. Since **V** is ccd,

$$b(y_1,y_0)=\bigvee\{u\in\mathbf{V}:u\ll b(y_1,y_0)\}.$$

By hypothesis, for each $u\ll b(y_1,y_0)$, there exists a zigzag (4.2) in (X,a) with $f(x'_n)=y_1$, $f(x_0)=y_0$, $f(x_i)=f(x'_i)$, for $i=1,\ldots,n-1$, and such that

$$u\leq a(x'_n,x_{n-1})\otimes a(x'_{n-1},x_{n-2})\otimes\cdots\otimes a(x'_1,x_0).$$

Hence the set $\{u\in\mathbf{V}:u\ll b(y_1,y_0)\}$ has the supremum of all $a(x'_n,x_{n-1})\otimes a(x'_{n-1},x_{n-2})\otimes\cdots\otimes a(x'_1,x_0)$, obtained from zigzags (4.2), as upper bound. Therefore,

$$b(y_1,y_0)\leq\bigvee\{a(x'_n,x_{n-1})\otimes a(x'_{n-1},x_{n-2})\otimes\cdots\otimes a(x'_1,x_0)\quad\text{obtained from zigzags (4.2)}\}.$$

It remains to show the other inequality. For a zigzag (4.2) in (X,a) with $f(x'_n)=y_1$, $f(x_0)=y_0$, $f(x_i)=f(x'_i)$, for $i=1,\ldots,n-1$, we have

$$a(x'_n,x_{n-1})\otimes a(x'_{n-1},x_{n-2})\otimes\cdots\otimes a(x'_1,x_0)\leq b(y_1,f(x_{n-1}))\otimes\cdots\otimes b(f(x_1),y_0)\leq b(y_1,y_0).$$

Hence,

$$\bigvee\{a(x'_n,x_{n-1})\otimes a(x'_{n-1},x_{n-2})\otimes\cdots\otimes a(x'_1,x_0)\quad\text{obtained from zigzags (4.2)}\}\leq b(y_1,y_0).$$

□

4.3.2 The relational method for effective étale-descent morphisms in V-Cat

Let $p:(E,e)\to(B,b)$ be a **V**-functor where **V** is assumed to be a frame. Let

$$(p_u:(E,e_u)\to(B,b_u))_{u\in\mathbf{V}}$$

be the family of monotone maps in **Ord** defined accordingly and let $\mathbb{E}$ be the class of étale morphisms in **V**-**Cat**. We claim that, if **V** is ccd and totally ordered, then p is an effective étale-descent morphism in **V**-**Cat** provided that p_u is an effective étale-descent morphism in **Ord** for each $u\in\mathbf{V}$. We show, under our assumptions, that the (comparison) functor

$$\Phi^p_{\mathbb{E}} : \mathbb{E}(B,b) \to \mathrm{Des}_{\mathbb{E}}(p)$$

is an equivalence of categories. We split the proof in three lemmas, showing, in each step, that the comparison functor is faithful, full and essentially surjective on objects.

Lemma 4.3.5 *Let* $\mathbf{V}$ *be a ccd frame. The functor* $\Phi^p_{\mathbb{E}}$ *is faitfhful provided that there exists an* $u \in \mathbf{V}$ *(*$u \neq \bot$*) such that* $p_u : (E, e_u) \to (B, b_u)$ *satisfies condition* (iii) *of Corollary 1.4.15.*

Proof
Let

$$f, g : ((A,a),\alpha) \to ((A',a'),\alpha')$$

be morphisms in $\mathbb{E}(B,b)$ such that $\Phi^p_{\mathbb{E}}(f) = \Phi^p_{\mathbb{E}}(g)$, i.e., $1_E \times_B f = 1_E \times_B g$. This condition implies that for each $z \in A$ with $\alpha(z) \in p(E)$ we have $f(z) = g(z)$. Hence, in order to show that $f = g$, let z be an element in A such that $\alpha(z) \notin p(E)$. By hypotheses there exists an $u \in \mathbf{V}$ (with $u \neq \bot$) such that the monotone map $p_u : (E, e_u) \to (B, b_u)$ in **Ord** satisfies condition (iii) of Corollary 1.4.15. Therefore there exists $x \in E$ such that

$$u \leq b(\alpha(z), p(x)) \quad \text{and} \quad u \leq b(p(x), \alpha(z)).$$

Let $\bar{u} \ll u$ such that $\bar{u} \neq \bot$. Since α is an étale morphism, there exists a unique $z_x \in A$ such that $\alpha(z_x) = p(x)$ and $\bar{u} \leq a(z_x, z)$. The fact that both f and g are $\mathbf{V}$-functors implies that $\bar{u} \leq a'(f(z_x), f(z))$ and $\bar{u} \leq a'(g(z_x), g(z))$, where $f(z_x) = g(z_x)$. Now, since α is an étale morphism, there exists a unique element $\tilde{z} \in A$ such that $\alpha(\tilde{z}) = \alpha(z)$ and $\bar{u} \leq a(\tilde{z}, z_x)$. But the fact that $\bar{u} \leq a(\tilde{z}, z_x) \wedge a(z_x, z) \leq a(\tilde{z}, z)$ implies $\tilde{z} = z$, since α is étale. Hence $\bar{u} \leq a'(f(z), f(z_x))$ and $\bar{u} \leq a'(g(z), g(z_x))$ give that $f(z) = g(z)$, since α' is an étale morphism. □

Lemma 4.3.6 *Let* $\mathbf{V}$ *be a ccd and totally ordered frame. The functor* $\Phi^p_{\mathbb{E}}$ *is full provided that* $p_u : (E, e_u) \to (B, b_u)$ *is effective for étale-descent in* **Ord** *for each* $u \in \mathbf{V}$*.*

Proof
We start considering $p : (E, e) \to (B, b)$ surjective. Let $((A,a),\alpha)$ and $((A',a'),\alpha')$ be objects in $\mathbb{E}(B,b)$ and let

$$f : ((E \times_B A, d), \mathrm{pr}_1, 1_E \times_B \mathrm{pr}_2) \to ((E \times_B A', d'), \mathrm{pr}'_1, 1_E \times_B \mathrm{pr}'_2)$$

be a morphism in $\mathrm{Des}_{\mathbb{E}}(p)$ where $((E \times_B A, d), \mathrm{pr}_1, 1_E \times_B \mathrm{pr}_2)$ and $((E \times_B A', d'), \mathrm{pr}'_1, 1_E \times_B \mathrm{pr}'_2)$ are the images by $\Phi^p_{\mathbb{E}}$ of $((A,a),\alpha)$ and $((A',a'),\alpha')$, respectively. We want to define a morphism

$$\bar{f} : ((A,a),\alpha) \to ((A',a'),\alpha')$$

such that $\Phi^p_{\mathbb{E}}(\bar{f}) = f$. Let $z \in A$. Since we are assuming p surjective, there exists $x \in X$ such that $\alpha(z) = p(x)$. Hence the pair (x,z) is an elements in $E \times_B A$ so that we can map it by f. Let $(x, z') = f(x,z)$. Define then $\bar{f}$ by

$$z \mapsto z'. \tag{4.3}$$

Notice that the choice of such z' might depend on the choice of the element $x \in X$ such that $\alpha(z) = p(x)$. But, since f is a morphism in $\mathrm{Des}_{\mathbb{E}}(p)$, we have that the diagram

$$\begin{array}{ccc} E \times_B (E \times_B A) & \xrightarrow{1_E \times_B f} & E \times_B (E \times_B A') \\ {\scriptstyle 1_E \times_B \mathrm{pr}_2} \downarrow & & \downarrow {\scriptstyle 1_E \times_B \mathrm{pr}'_2} \\ E \times_B A & \xrightarrow[f]{} & E \times_B A' \end{array}$$

is commutative. Hence, the map $\bar{f}$ given by (4.3) is well-defined, making the **Set**-diagram

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}} & A' \\ & {\scriptstyle \alpha} \searrow \quad \swarrow {\scriptstyle \alpha'} & \\ & B & \end{array}$$

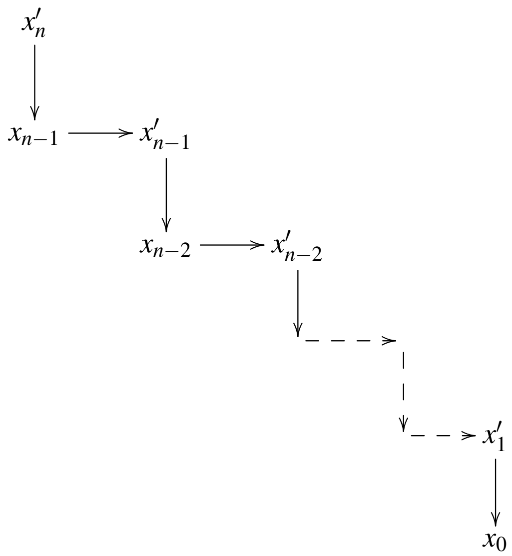
commutative. It remains to show that $\bar{f} : (A, a) \to (A', a')$ is a **V**-functor, that is, for each $z_1, z_0 \in A$

$$a(z_1, z_0) \leq a'(\bar{f}(z_1), \bar{f}(z_0)).$$

To prove the inequality above, we show that for each $u \ll a(z_1, z_0)$ we have $u \leq a'(\bar{f}(z_1), \bar{f}(z_0))$. Let $u \in \mathbf{V}$ with $u \ll a(z_0, z_1)$. We can assume that $u \neq \perp$. Since $\alpha : (A, a) \to (B, b)$ is a **V**-functor,

$$u \ll a(z_1, z_0) \leq b(\alpha(z_1), \alpha(z_0)) \Rightarrow u \ll b(\alpha(z_1), \alpha(z_0)).$$

Hence, since $p_u : (E, e_u) \to (B, b_u)$ is an effective étale-descent morphism in **Ord**, there exists a unique (up to equivalence) zigzag



in (E,e_u), with $n \in \mathbb{N}$, such that $p(x_0) = \alpha(z_0)$, $p(x_n) = \alpha(z_1)$, $p(x_i) = p(x_i')$, for $i = 1,\ldots,n-1$. The fact that the horizontal arrows in the zigzag above are in (E,e_u) means that

$$u \leq e(x_i', x_{i-1}), \quad \text{for } i = 1,\ldots,n.$$

Let us assume that $n = 2$ and that the zigzag is then of the form

$$\begin{array}{ccc} x_2 & & \\ \downarrow & & \\ x_1 & \longrightarrow & x_1' \\ & & \downarrow \\ & & x_0, \end{array}$$

with $p(x_0) = \alpha(z_0)$, $p(x_2) = \alpha(z_1)$, $p(x_1) = p(x_1')$ and $u \leq e(x_2,x_1)$, $u \leq e(x_1',x_0)$.
Let $\bar{u} \ll u$ such that $\bar{u} \neq \bot$. Since α is an étale morphism, there exists a unique $\tilde{z}_1 \in A$ such that $\alpha(\tilde{z}_1) = p(x_1) = p(x_1')$ and $\bar{u} \leq a(\tilde{z}_1, z_0)$. Moreover, there exists a unique $\bar{z}_1 \in A$ such that $\alpha(\bar{z}_1) = \alpha(z_1) = p(x_2)$ and $\bar{u} \leq a(\bar{z}_1, \tilde{z}_1)$. But, since α is étale, we must have $\bar{z}_1 = z_1$. Hence, since $f : (E \times_B A, d) \to (E \times_B A', d')$ is a **V**-functor,

$$\bar{u} \leq e(x_2,x_1) \wedge a(z_1,\tilde{z}_1) \leq e(x_2,x_1) \wedge a'(\bar{f}(z_1),\bar{f}(\tilde{z}_1)) \leq a'(\bar{f}(z_1),\bar{f}(\tilde{z}_1))$$

and

$$\bar{u} \leq e(x_1',x_0) \wedge a(\tilde{z}_1,z_0) \leq e(x_1',x_0) \wedge a'(\bar{f}(\tilde{z}_1),\bar{f}(z_0)) \leq a'(\bar{f}(\tilde{z}_1),\bar{f}(z_0)).$$

Therefore,

$$\bar{u} \leq a'(\bar{f}(z_1),\bar{f}(\tilde{z}_1)) \wedge a'(\bar{f}(\tilde{z}_1),\bar{f}(z_0)) \leq a'(\bar{f}(z_1),\bar{f}(z_0)).$$

Since $u = \bigvee\{\bar{u} \in \mathbf{V} : \bar{u} \ll u\}$, we conclude that $u \leq a'(\bar{f}(z_1),\bar{f}(z_0))$.
Let us now assume $p : (E,e) \to (B,b)$ not surjective. Let $z \in A$ such that $\alpha(z) \notin p(E)$ and let $u \in \mathbf{V}$ with $u \neq \bot$. Since $p : (E,e_u) \to (B,b_u)$ is effective for étale-descent, there exists $x \in E$ such that

$$u \leq b(\alpha(z),p(x)) \quad \text{and} \quad u \leq b(p(x),\alpha(z)).$$

Let $\bar{u} \ll u$. Since α is an étale morphism, there exists a unique $z_x \in A$ such that $\alpha(z_x) = p(x)$ and $\bar{u} \leq a(z_x, z)$. Since $(x,z_x) \in E \times_B A$, we know how to map the point z_x by $\bar{f}$. Let $\bar{f}(z_x) = z_x' \in A'$. Now, since α' is étale, there exists a unique $z' \in A'$ such that $\alpha'(z') = \alpha(z)$ and $\bar{u} \leq a'(z', z_x')$. Define then

$$\bar{f}(z) := z'.$$

We must now verify that the definition we gave does not depend on the choice of $u \in \mathbf{V}$ and on the choice of $\bar{u} \ll u$. We start with the latter. Let $\bar{\bar{u}} \ll u$ with $\bar{\bar{u}} \neq \bot$ and $\bar{\bar{u}} \neq \bar{u}$. Since α is an étale morphism, there would exist a unique $\tilde{z}_x \in A$ such that $\alpha(\tilde{z}_x) = p(x) = \alpha(z_x)$ and $\bar{\bar{u}} \leq a(z, \tilde{z}_x)$. But, since **V** is totally ordered, $\tilde{z}_x = z_x$. Hence, in A', $\bar{f}(\tilde{z}_x) = \bar{f}(z_x)$. Now, since α' is an étale morphism, there would exist a unique $\tilde{z}' \in A'$ such that $\alpha'(\tilde{z}') = \alpha(z) = \alpha(z')$ and $\bar{\bar{u}} \leq a(\tilde{z}', \bar{f}(z_x))$. Since **V** is

totally ordered, we conclude that $\tilde{z}' = z'$.
Let now $\tilde{u} \in \mathbf{V}$ with $\tilde{u} \neq \bot$ and $\tilde{u} \neq u$. We then use that fact that $p_{\tilde{u}} : (E, e_{\tilde{u}}) \to (B, b_{\tilde{u}})$ is effective for étale-descent in **Ord**. Hence there might exist $\tilde{x} \in X$, with $\tilde{x} \neq x$, such that

$$\tilde{u} \leq b(\alpha(z), p(\tilde{x})) \quad \text{and} \quad \tilde{u} \leq b(p(\tilde{x}), \alpha(z)).$$

Let $\bar{\tilde{u}} \ll \tilde{u}$. Since α is an étale morphism, there exists a unique $z_{\tilde{x}} \in A$ such that $\alpha(z_{\tilde{x}}) = p(\tilde{x})$ and $\bar{\tilde{u}} \leq a(z_{\tilde{x}}, z)$. Observe that $\bar{\tilde{u}} \wedge \bar{u} \leq a(z_{\tilde{x}}, z_x)$. Now, since α' is an étale morphism, there exists a unique $\tilde{z}' \in A'$ such that $\alpha'(\tilde{z}') = \alpha(z)$ and $\bar{\tilde{u}} \leq a'(\tilde{z}', \bar{f}(z_{\tilde{x}}))$. Hence,

$$\bot \neq \bar{\tilde{u}} \wedge (\bar{\tilde{u}} \wedge \bar{u}) \leq a'(\tilde{z}', \bar{f}(z_{\tilde{x}})) \wedge a'(\bar{f}(z_{\tilde{x}}), \bar{f}(z_x)) \leq a(\tilde{z}', \bar{f}(z_x)).$$

Since $\mathbf{V}$ is totally ordered, we conclude that $\tilde{z}' = z'$. From the construction we made, it follows that $\bar{f} : (A, a) \to (A', a')$ is a $\mathbf{V}$-functor such that $\alpha' \cdot \bar{f} = \alpha$. □

Lemma 4.3.7 *Let* $\mathbf{V}$ *be a ccd and totally ordered frame. The functor* $\Phi^p_{\mathbb{E}}$ *is essentially surjective on objects provided that* $p_u : (E, e_u) \to (B, b_u)$ *is effective for étale-descent in* **Ord** *for each* $u \in \mathbf{V}$.

Proof
Let $((C, c), \gamma, \xi)$ be an object in $\mathrm{Des}_{\mathbb{E}}(p)$. We want to find an object $((A, a), \alpha) \in \mathbb{E}(B)$ such that

$$\Phi^p_{\mathbb{E}}((A, a), \alpha) \cong ((C, c), \gamma, \xi).$$

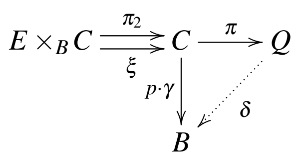
We use the technique studied in Chapter 1 to construct the left adjoint $\Psi^p_{\mathbb{E}}$ of the comparison factor. Consider (Q, q) the object part of the coequalizer $((Q, q), \pi)$ in $\mathbf{V}$-**Cat** of the pair

$$E \times_B C \underset{\xi}{\overset{\pi_2}{\rightrightarrows}} C \xrightarrow{\ \pi\ } Q.$$

The quotient set Q is given by the following equivalence relation $\sim_\xi$ defined on the set C: for $z, z' \in C$,

$$z \sim_\xi z' \iff p(\gamma(z)) = p(\gamma(z')) \quad \text{and} \quad z' = \xi(\gamma(z'), z),$$

while the structure $q : Q \times Q \to \mathbf{V}$ is the final structure of the morphism $\pi : (C, c) \to Q$. Since $p \cdot \gamma \cdot \pi_2 = p \cdot \gamma \cdot \xi$, by the universal property of the coequalizer, there exists a unique morphism $\delta : Q \to B$ such that the diagram



commutes. In Section 1.1 we referred to the diagram above as the descent situation describing Q. We claim $((Q, q), \delta)$ is an object in $\mathbb{E}(B)$ such that $\Phi^p_{\mathbb{E}}((Q, q), \delta) \cong ((C, c), \gamma, \xi)$.
We start assuming $p : (E, e) \to (B, b)$ surjective. We show first that $\delta : Q \to B$ is open. Let $[z_0] \in Q$, $y_1 \in B$ and $u \in \mathbf{V}$ such that $u \neq \bot$ and $u \ll b(y_1, \delta[z_0])$. Since $p_u : (E, e_u) \to (B, b_u)$ is an effective

étale-descent morphism in **Ord**, there exists a unique (up to equivalence) zigzag

$$
\begin{array}{ccccccc}
x'_n & & & & & & \\
\downarrow & & & & & & \\
x_{n-1} & \longrightarrow & x'_{n-1} & & & & \\
 & & \downarrow & & & & \\
 & & x_{n-2} & \longrightarrow & x'_{n-2} & & \\
 & & & & \downarrow & & \\
 & & & & \cdot & \dashrightarrow & \cdot \\
 & & & & & & \vdots \\
 & & & & & & \cdot \dashrightarrow x'_1 \\
 & & & & & & \qquad \downarrow \\
 & & & & & & \qquad x_0
\end{array}
\tag{4.4}
$$

in (E, e_u), with $n \in \mathbb{N}$, such that $p(x_0) = \delta[z_0]$, $p(x'_n) = y$, $p(x_i) = p(x'_i)$ for $i = 1, \ldots, n-1$. The fact that the horizontal arrows of the zigzag above are in (E, e_u) means that

$$u \leq e(x'_i, x_{i-1}) \quad \text{for} \quad i = 1, \ldots n.$$

Assume $n = 2$ so that the zigzag is of the form

$$
\begin{array}{ccc}
x_2 & & \\
\downarrow & & \\
x_1 & \longrightarrow & x'_1 \\
 & & \downarrow \\
 & & x_0,
\end{array}
$$

where $p(x_0) = \delta[z_0]$, $p(x_2) = y$, $p(x_1) = p(x'_1)$ and $u \leq e(x_2, x_1)$, $u \leq e(x'_1, x_0)$. Let $\bar{u} \ll u$ with $\bar{u} \neq \bot$ and let $z_0 \in \pi^{-1}[z_0]$. We can assume that $\gamma(z_0) = x_0$ since, if it is not the case, we just take $\xi(x_0, z_0)$. This is a point in C such that $\gamma(\xi(x_0, z_0)) = x_0$ and $\pi(\xi(x_0, z_0)) = [z_0]$. Since γ is an étale morphism, there exists a unique $z'_1 \in C$ such that $\gamma(z'_1) = x'_1$ and $\bar{u} \ll c(z'_1, z_0)$. Take now the point $z_1 = \xi(x_1, z'_1)$. This is a point in C such that $\gamma(z_1) = x_1$ and $z_1 \sim_\xi z'_1$. Since γ is étale, there exists a unique $z_2 \in C$ such that $\gamma(z_2) = x_2$ and $\bar{u} \leq c(z_2, z_1)$. Now, since $\pi : C \to Q$ is a **V**-functor, we conclude that

$$\bar{u} \leq q(\pi[z_2], \pi[z'_1]) \wedge q(\pi[z_1], z_0]) \leq q(\pi[z_2], z_0]).$$

This proves that $\delta : Q \to B$ is open, in fact $u \leq q(\pi[z_2], [z_0])$, since $u = \bigvee\{\bar{u} \in \mathbf{V} : \bar{u} \ll u\}$.
We now show that $\delta : Q \to B$ is étale. Let $[\zeta] \in Q$ such that $[\zeta] \neq \pi[z_2]$, $\delta[\zeta] = y_1$ and $u \leq q([\zeta], [z_0])$.

Let $\bar{u} \ll u$. Since $\pi : C \to Q$ is a regular epimorphism, there exists a zigzag

$$\begin{array}{ccccccc}
\zeta'_m & & & & & & \\
\big\downarrow {\scriptstyle c(\zeta_m,\zeta_{m-1})} & & & & & & \\
\zeta_{m-1} & \longrightarrow & \zeta'_{m-1} & & & & \\
& & \big\downarrow {\scriptstyle c(\zeta'_{m-1},\zeta_{m-2})} & & & & \\
& & \zeta_{m-2} & \longrightarrow & \zeta'_{m-2} & & \\
& & & & \big\downarrow & & \\
& & & & \cdots & \dashrightarrow & \\
& & & & & \vdots & \\
& & & & & \dashrightarrow & \zeta'_1 \\
& & & & & & \big\downarrow {\scriptstyle c(\zeta'_1,\zeta_0)} \\
& & & & & & \zeta_0
\end{array} \tag{4.5}$$

in (C,c), with $m \in \mathbb{N}$, such that $\pi(\zeta_0) = [z_0]$, $\pi(\zeta'_m) = [\zeta]$, $\pi(\zeta_j) = \pi(\zeta'_j)$ for $j = 1, \ldots, m-1$, and

$$\bar{u} \leq c(\zeta_m, \zeta_{m-1}) \wedge c(\zeta'_{m-1}, \zeta_{m-2}) \wedge \cdots \wedge c(\zeta'_1, \zeta_0).$$

This implies that $\bar{u} \leq c(\zeta'_j, \zeta_{j-1})$ for $j = 1, \ldots, m$. The image by γ of the above zigzag in (C,c) will be a zigzag in (E,e) of the form

$$\begin{array}{ccccccc}
\gamma(\zeta'_m) & & & & & & \\
\big\downarrow {\scriptstyle \bar{u} \leq e(\gamma(\zeta'_m),\gamma(\zeta_{m-1}))} & & & & & & \\
\gamma(\zeta_{m-1}) & \longrightarrow & \gamma(\zeta'_{m-1}) & & & & \\
& & \big\downarrow {\scriptstyle \bar{u} \leq e(\gamma(\zeta'_{m-1}),\gamma(\zeta_{m-2}))} & & & & \\
& & \gamma(\zeta_{m-2}) & \longrightarrow & \gamma(\zeta'_{m-2}) & & \\
& & & & \big\downarrow & & \\
& & & & \cdots & \dashrightarrow & \\
& & & & & \vdots & \\
& & & & & \dashrightarrow & \gamma(\zeta'_1) \\
& & & & & & \big\downarrow {\scriptstyle \bar{u} \leq e(\gamma(\zeta'_1),\gamma(\zeta_0))} \\
& & & & & & \gamma(\zeta_0).
\end{array}$$

Since $p_{\bar{u}} : (E, e_{\bar{u}}) \to (B, b_{\bar{u}})$ is an effective étale-descent morphism in **Ord**, this zigzag of length m must be equivalent to the zigzag (4.4) of length n we got to prove that δ is open. We assumed that $n = 2$. Hence, we study the cases where $m = 1$ and $m = 2$.

Case $m = 1$

In (C,c) we have

$$z_2 \xrightarrow{\bar{u}\leq c(z_2,z_1)} z_1$$

$$z'_1 \xrightarrow{\bar{u}\leq c(z'_1,z_0)} z_0$$

$$\zeta_1 \xrightarrow{\bar{u}\leq c(\zeta_1,\zeta_0)} \zeta_0,$$

while in (E,e) we have

$$x_2 \xrightarrow{\bar{u}\ll e(x_2,x_1)} x_1$$

$$x'_1 \xrightarrow{\bar{u}\ll e(x'_1,x_0)} x_0$$

$$\gamma(\zeta_1) \xrightarrow{\bar{u}\leq e(\gamma(\zeta_1),\gamma(\zeta_0))} \gamma(\zeta_0).$$

Since the two zigzags above must be equivalent with respect to $(E,e_{\bar{u}})$, there exists a chain in (E,e) of the form

$$\chi_2 \xrightarrow{\bar{u}\leq e(\chi_2,\chi_1)} \chi_1 \xrightarrow{\bar{u}\leq e(\chi_1,\chi_0)} \chi_0$$

with $p(\chi_2)=y$, $p(\chi_1)=p(x_1)=p(x'_1)$ and $p(\chi_0)=\delta[z_0]$.
Let $\bar{\bar{u}}\ll\bar{u}$ with $\bar{\bar{u}}\neq\bot$ and let $t_0=\xi(\chi_0,z_0)$. Then, in (C,c), we have that

$$t_0\sim_\xi z_0\sim_\xi \zeta_0.$$

Since γ is an étale morphism, there exists a unique $t_1\in C$ such that $\gamma(t_1)=\chi_1$ and $\bar{\bar{u}}\leq c(t_1,t_0)$. Consider now the points (x_0,t_0) and (x'_1,t_1) in $E\times_B C$. Since $\xi: E\times_B C\to C$ is a **V**-functor,

$$\bar{\bar{u}}\leq e(x'_1,x_0)\wedge c(t_1,t_0)\leq c(\xi(x'_1,t_1),\xi(x_0,t_0))=c(\xi(x'_1,t_1),z_0).$$

Since γ is an étale morphism, we conclude that $z'_1=\xi(x'_1,t_1)$. Therefore,

$$t_1\sim_\xi z'_1\sim_\xi z_1.$$

The same construction gives the existence of a unique element $t_2\in C$ such that $\gamma(t_2)=\chi_2$, $\bar{\bar{u}}\leq c(t_2,t_1)$ and

$$t_2\sim_\xi z_2.$$

Since $\xi: E\times_B C\to C$ is a **V**-functor,

$$\bar{\bar{u}}\leq e(\gamma(\zeta_1),\gamma(\zeta_0))\wedge c(t_2,t_0)\leq c(\xi(\gamma(\zeta_1),t_2),\xi(\gamma(\zeta_0),t_0))=c(\xi(\gamma(\zeta_1),t_2),\zeta_0).$$

Since γ is an étale morphism, we conclude that

$$z_2\sim_\xi t_2\sim_\xi \zeta_1,$$

which contradicts the fact that $\pi[z_2] \neq [\zeta]$.

Case $m = 2$
In (C,c) we have

$$z_2 \xrightarrow{\bar{u} \leq c(z_2,z_1)} z_1$$

$$z'_1 \xrightarrow{\bar{u} \leq c(z'_1,z_0)} z_0$$

$$\zeta_2 \xrightarrow{\bar{u} \leq c(\zeta_2,\zeta_1)} \zeta_1$$

$$\zeta'_1 \xrightarrow{\bar{u} \leq c(\zeta'_1,\zeta_0)} \zeta_0,$$

while in (E,e) we have

$$x_2 \xrightarrow{\bar{u} \ll e(x_2,x_1)} x_1$$

$$x'_1 \xrightarrow{\bar{u} \ll e(x'_1,x_0)} x_0$$

$$\gamma(\zeta_2) \xrightarrow{\bar{u} \leq e(\gamma(\zeta_2),\gamma(\zeta_1))} \gamma(\zeta_1)$$

$$\gamma(\zeta'_1) \xrightarrow{\bar{u} \leq e(\gamma(\zeta'_1),\gamma(\zeta_0))} \gamma(\zeta_0).$$

In fact, since the two zigzags above must be equivalent with respect to $(E, e_{\bar{u}})$, we have that

$$p(x_1) = p(x'_1) = p(\gamma(\zeta_1)) = p(\gamma(\zeta'_1)).$$

Since the pairs (x'_1, ζ'_1), (x_0, ζ_0) are in $E \times_B C$, and since $\xi : E \times_B C \to C$ is a $\mathbf{V}$-functor, we have

$$\bar{u} \leq e(x'_1, x_0) \wedge c(\zeta'_1, \zeta_0) \leq c(\xi(x'_1, \zeta'_1), \xi(x_0, \zeta_0)) = c(\xi(x'_1, \zeta'_1), z_0),$$

by the fact that $z_0 \sim_\xi \zeta_0$. Since γ is an étale morphism, we have $\xi(x'_1, \zeta'_1) = z'_1$, so that,

$$z_1 \sim_\xi z'_1 \sim_\xi \zeta'_1 \sim_\xi \zeta_1.$$

An analogous argument leads to the equivalence

$$z_2 \sim_\xi \zeta_2,$$

which contradicts the fact that $\pi[z_2] \neq [\zeta]$.
It remains to show that $\Phi^p_{\mathbb{E}}((Q,q),\delta) \cong ((C,c),\gamma,\xi)$. The image of $((Q,q),\delta)$ by $\Phi^p_{\mathbb{E}}$ is the triple

$((E\times_B Q,d),\mathrm{pr}_1,1_E\times_B \mathrm{pr}_2)$, where

$$\begin{array}{ccc} (E\times_B Q,d) & \xrightarrow{\mathrm{pr}_2} & (Q,q) \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \delta} \\ (E,e) & \xrightarrow[p]{} & (B,b) \end{array}$$

is the pullback in **V**-**Cat** of δ along p, with d the pullback structure. The isomorphism

$$i:(C,c)\to(E\times_B Q,d)$$

in **V**-**Cat** is given by

$$z\mapsto(\gamma(z),[z]).$$

The map i is injective; in fact, given $z_0,z_1\in C$ such that $(\gamma(z_0),[z_0])=(\gamma(z_1),[z_1])$,i.e., $\gamma(z_0)=\gamma(z_1)$ and $z_0\sim_\xi z_1$, we conclude that $z_0=z_1$, since

$$z_0=\xi(\gamma(z_0),z_0)=\xi(\gamma(z_1),z_0)=z_1,$$

by the properties of the descent data ξ.
Surjectivity of i also follows: given an element $(x,[z])\in E\times_B Q$, let $\xi(x,z)\in C$. Then,

$$i(\xi(x,c))=(\gamma(\xi(x,z)),[\xi(x,z)])=(x,[z]),$$

again using properties of the descent data ξ. So far we proved that $i:C\to E\times_B Q$ is a **Set**-isomorphism. Since i is also a **V**-functor, in fact $c(z_1,z_0)\le e(\gamma(z_1),\gamma(z_0))\wedge q([z_1],[z_0])$ for each $z_0,z_1\in C$, it remains to show that

$$e(x_1,x_0)\wedge q([z_1],[z_0])\le c(\xi(x_1,z_1),\xi(x_0,z_0)),$$

for each $(x_0,[z_0])$ and $(x_1,[z_1])$ in $E\times_B Q$.
Let $u\le e(x_1,x_0)\wedge q([z_1],[z_0])$. We want to show that $u\le c(\xi(x_1,z_1),\xi(x_0,z_0))$. Let $\bar{u}\ll u$ with $\bar{u}\neq\bot$. Since $\pi:C\to Q$ is a regular epimorphism, there exists a zigzag as in (4.5) in (C,c), with $m\in\mathbb{N}$, such that $\pi(\zeta'_m)=[z_1]$, $\pi(\zeta_0)=[z_0]$, $\pi(\zeta_j)=\pi(\zeta'_j)$, for $j=1,\dots,m-1$, and

$$\bar{u}\le c(\zeta'_m,\zeta_{m-1})\wedge c(\zeta'_{m-1},\zeta_{m-2})\wedge\cdots\wedge c(\zeta'_1,\zeta_0).$$

Assume $m=2$. Since $\bar{u}\ll e(x_1,x_0)$, and since $\gamma:C\to E$ is an étale morphism, there exists a unique $t_1\in C$ such that $\gamma(t_1)=x_1$ and $\bar{u}\le c(t_1,\xi(x_0,z_0))$. We want to show that

$$t_1\sim_\xi \xi(x_1,z_1)$$

since this would imply, by the properties of the descent data ξ, that $t_1 = \xi(x_1, z_1)$. The situation is quite similar to the case where we had to construct the étale morphism $\delta : Q \to B$. In (E, e) we have

$$\begin{array}{llll}
\gamma(\zeta_2) \xrightarrow{\bar{u} \ll e(\gamma(\zeta_2),\gamma(\zeta_1))} \gamma(\zeta_1) & & & \\
& \gamma(\zeta_1') \xrightarrow{\bar{u} \ll e(\gamma(\zeta_1'),\gamma(\zeta_0))} \gamma(\zeta_0) & & \\
x_1 \xrightarrow{\qquad\qquad \bar{u} \leq e(x_1,x_0) \qquad\qquad} x_0 & & &
\end{array}$$

and, since $p_{\bar{u}} : (E, e_{\bar{u}}) \to (B, b_{\bar{u}})$ is an effective étale-descent morphism, the two zigzags above must be equivalent. Hence, there exists in (E, e) a zigzag of the form

$$\chi_2 \xrightarrow{\bar{u} \leq e(\chi_2,\chi_1)} \chi_1 \xrightarrow{\bar{u} \leq e(\chi_1,\chi_0)} \chi_0$$

with $p(\chi_2) = p(x_1) = \delta[z_1]$, $p(\chi_1) = p(\gamma(\zeta_1)) = p(\gamma(\zeta_1'))$ and $p(\chi_0) = p(x_0) = \delta[z_0]$.
Taking in (C, c) the point $\mathfrak{z}_0 = \xi(\chi_0, z_0)$, since γ is an étale morphism, we have a zigzag in (C, c) of the form

$$\mathfrak{z}_2 \xrightarrow{\bar{u} \leq c(\mathfrak{z}_2,\mathfrak{z}_1)} \mathfrak{z}_1 \xrightarrow{\bar{u} \leq c(\mathfrak{z}_1,\mathfrak{z}_0)} \mathfrak{z}_0$$

such that $\gamma(\mathfrak{z}_2) = \chi_2$ and $\gamma(\mathfrak{z}_1) = \chi_1$.
Summing up, in (C, c) we have the following situation

$$\begin{array}{ll}
\zeta_2 \xrightarrow{\bar{u} \leq c(\zeta_2,\zeta_1)} \zeta_1 & \\
& \zeta_1' \xrightarrow{\bar{u} \leq c(\zeta_1',\zeta_0)} \zeta_0 \\
\mathscr{Z}_1 \xrightarrow{\qquad \bar{u} \leq c(\mathscr{Z}_1,\xi(x_0,z_0)) \qquad} \xi(x_0, z_0) & \\
\mathfrak{z}_2 \xrightarrow{\bar{u} \leq c(\mathfrak{z}_2,\mathfrak{z}_1)} \mathfrak{z}_1 \xrightarrow{\bar{u} \leq c(\mathfrak{z}_1,\mathfrak{z}_0)} \mathfrak{z}_0, &
\end{array}$$

where

$$\mathscr{Z}_1 \sim_\xi \mathfrak{z}_2 \sim_\xi \zeta_2 \sim_\xi z_1 \sim_\xi \xi(x_1, z_1)$$

as claimed. The same argument holds for each $\bar{u} \ll u$ and, since $u = \bigvee\{\bar{u} \in \mathbf{V} : \bar{u} \ll u\}$, the result follows.
To conclude the proof, it remains to show that the following diagram

$$\begin{array}{ccc}
E \times_B C & \xrightarrow{1_E \times_B i} & E \times_B (E \times_B Q) \\
{\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle 1_E \times_B \mathrm{pr}_2} \\
C & \xrightarrow[i]{} & E \times_B Q
\end{array}$$

is commutative. Let (x,z) be an element in $E\times_B C$. Then,

$$(i\cdot\xi)(x,z)=(x,[\xi(x,z)])=(x,[z])=(1_E\times_B \mathrm{pr}_2\cdot 1_E\times_B i)(x,z),$$

which proves that the diagram above is commutative.
To prove that $\Phi^p_{\mathbb{E}}:\mathbb{E}(B,b)\to \mathrm{Des}_{\mathbb{E}}(p)$ is essentially surjective on objects, we assumed $p:(E,e)\to(B,b)$ surjective. If it is not the case, the construction of the étale morphism $\delta:Q\to B$ has the following modification: to the set Q, constructed in the same way as in the surjective case, we need to add the points of $B-p(E)$ with values

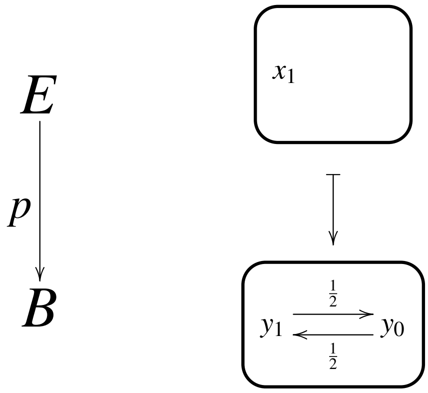
$$q(y,[z])=b(y,\delta[z])$$

for each $y\in B-p(E)$ and $[z]\in Q$. □

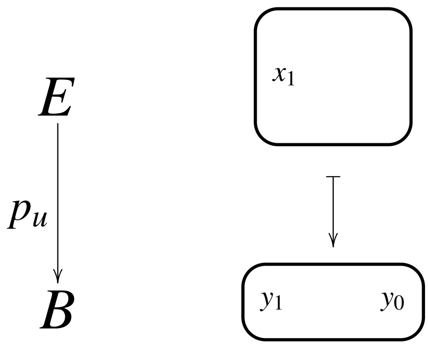
The three lemmas give the following result.

Theorem 4.3.8 *If* **V** *is a ccd totally ordered frame, then a morphism* $p:(E,e)\to(B,b)$ *in* **V**-**Cat** *is of effective étale-descent provided that* $p_u:(E,e_u)\to(B,b_u)$ *is an effective étale-descent morphism in* **Ord** *for each* $u\in$ **V**. □

Remark 4.3.9 The relational method turns out to be not much useful in case p is not surjective. To study this situation we consider the following **V**-functor $p:(E,e)\to(B,b)$, where $\mathbf{V}=\mathbf{I}_{\mathrm{inf}}$.

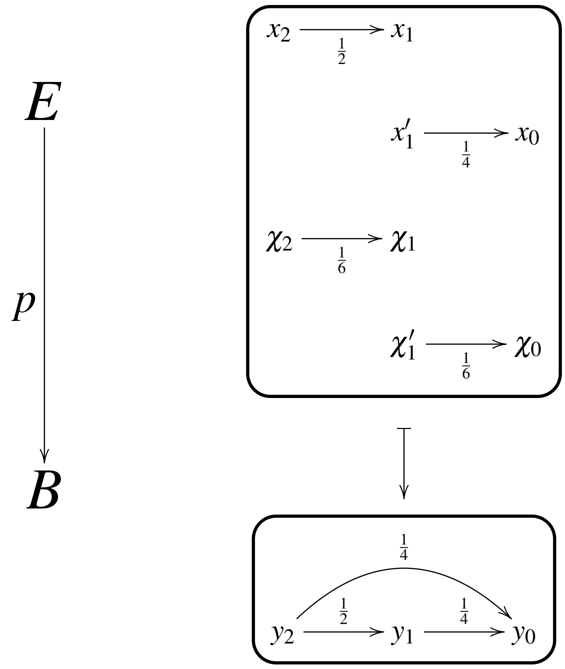


Although p restricted to its image is the one-element set identity, Theorem 4.3.8 can not be applied since for $\frac{1}{2}<u\leq 1$ the monotone map $p_u:(E,e_u)\to(B,b_u)$



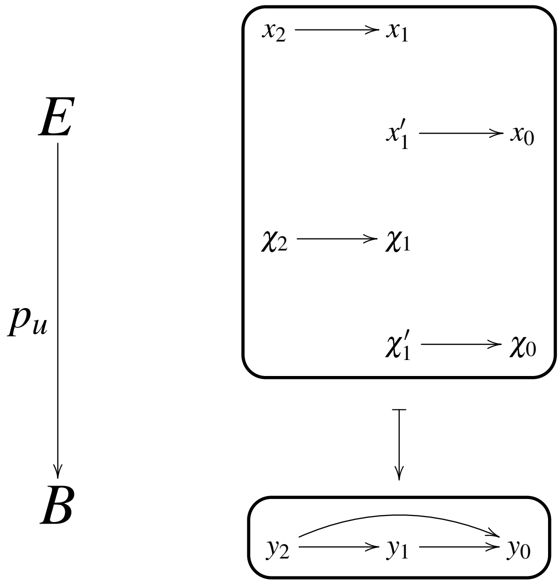
is not effective for étale-descent in **Ord**. In fact condition (*iii*) of Corollary 1.4.15 is not satisfied. Observe that it is the case only if we replace the value $\frac{1}{2}$ with 1. With this replacement each p_u, for all $u \in \mathbf{V}$, is an effective étale-descent morphism in **Ord** and the theorem can be applied to conclude that p is effective for étale-descent in $\mathbf{I}_{\text{inf}}$-**Cat**. Observe anyway that Lemma 4.3.5 can be applied, concluding that the comparison functor $\Phi^p_{\mathbb{E}}$ is faithful. If p is surjective, the relational method for the problem of étale-descent in **V**-**Cat** turns out to more useful, as the following example suggests.

Example 4.3.10 Consider the following morphism $p : (E,e) \to (B,b)$ in $\mathbf{I}_{\text{inf}}$-**Cat**.

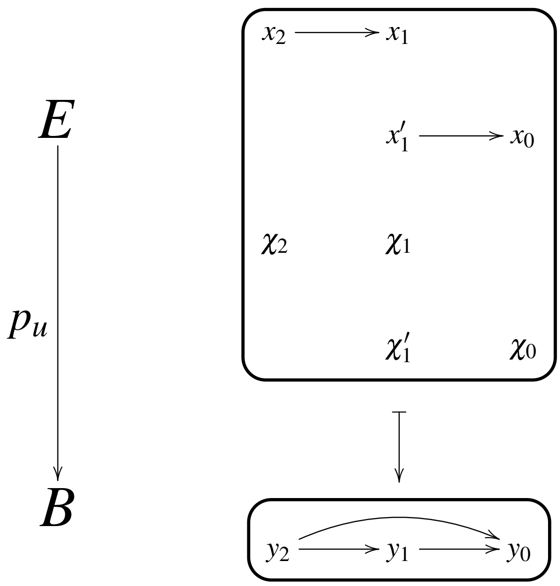


Observe that p is not effective for descent, since it is not a *-quotient morphism (see Theorem 3.3.3). In Section 4.4 we will show that a **V**-functor, for an arbitrary quantale **V**, is an effective étale-descent morphism provided that it is effective for global-descent. Therefore this can not be applied in our case to conclude that p is an effective étale-descent morphism but we can check if the hypotheses of Theorem 4.3.8 are satisfied. Accordingly, we split p into suitable slices $p_u : (E,e_u) \to (B,b_u)$ in **Ord**.

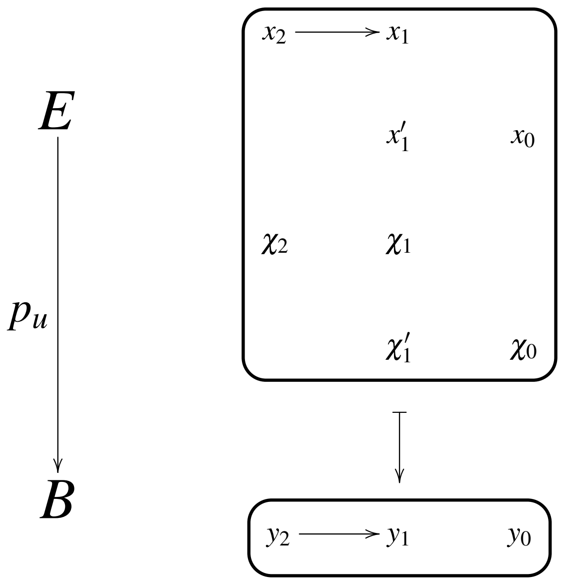
Case $0 \leq u \leq \frac{1}{6}$:



Case $\frac{1}{6} < u \leq \frac{1}{4}$:



Case $\frac{1}{4} < u \leq \frac{1}{2}$:



In all the cases pictured above the monotone map p_u is effective for étale-descent in **Ord**, as an easy inspection of Corollary 1.4.15 reveals. In the case $\frac{1}{2} < u \leq 1$ no arrows are involved for p_u, neither in the domain E nor in the codomain B, so that all the conditions of Corollary 1.4.15 are trivially satisfied. Therefore, by Theorem 4.3.8, we conclude that p is an effective étale-descent morphism in $\mathbf{I}_{\text{inf}}$-**Cat**.

Remark 4.3.11 The converse of Theorem 4.3.8 in not true in general, as Example 3.3.28 reveals. The $\mathbf{I}_{\text{inf}}$-functor $p : (E,e) \to (B,b)$ is effective for descent and so, by Theorem 4.4.6, an effective étale-descent morphism, but the monotone map $p_1 : (E,e_1) \to (Y,b_1)$ is not even a regular epimorphism.

4.3.3 A sufficient condition for effective étale-descent in V-Cat

In the previous section a criterion to study effective étale-descent morphisms in **V**-**Cat** is given. The sufficient condition we stated in Theorem 4.3.8 is quite strong, as Remark 4.3.11 suggests. Also the condition on **V** to be a frame is restrictive since, for instance, only three cases in Example 2.1.2 are included. Therefore, in this section, we try to give a another sufficient condition for effective étale-descent in **V**-**Cat**, not requiring **V** to be a frame. Let $\mathbf{V} = (V, \otimes, k)$ be a ccd totally ordered quantale satisfying the condition (called $(C4)$ in Section 3.4)

$$u \otimes v = \bot \Rightarrow \quad u = \bot \quad \text{or} \quad v = \bot, \quad \text{for all} \quad u, v \in \mathbf{V}. \tag{4.6}$$

This condition makes the map

$$q : \mathbf{V} \to \mathbf{2}, \quad q(u) = \top \Leftrightarrow u > \bot_{\mathbf{V}}$$

a quantale homomorphism so that the induced morphism

$$q : \mathbf{V}\text{-}\mathbf{Cat} \to \mathbf{Ord}, \tag{4.7}$$

defined by $(X,a) \to (X, q \cdot a)$, is a functor. We remark that the quantale homomorphism $q : \mathbf{V} \to \mathbf{2}$ is the left adjoint of the full embedding $\iota : \mathbf{2} \hookrightarrow \mathbf{V}$

$$u \mapsto \begin{cases} k_{\mathbf{V}}, & \text{if } u = \top, \\ \bot_{\mathbf{V}}, & \text{if } u = \bot, \end{cases}$$

if and only if **V** is integral but this is not a condition we need. Let $p : (E,e) \to (B,b)$ be a **V**-functor, $\mathbb{E}$ the class of étale morphisms in **V-Cat** and $\Phi^p_{\mathbb{E}} : \mathbb{E}(B,b) \to \mathrm{Des}_{\mathbb{E}}(p)$ the comparison functor. As done in the previous section, we show that $\Phi^p_{\mathbb{E}}$ is faithful, full and essentially surjective on objects.

Lemma 4.3.12 *Let* **V** *be a ccd and totally ordered quantale such that condition* (4.6) *is satisfied. The functor* $\Phi^p_{\mathbb{E}}$ *is faithful provided that*

$$\forall y \in B \quad \exists x \in E : \quad b(y, p(x)) \otimes b(p(x), y) \neq \bot. \tag{4.8}$$

Observe that the condition above, since (4.6) holds, is equivalent to the fact that both $b(y, p(x))$ and $b(p(x), y)$ are different from the bottom element $\bot$.

Proof
Let

$$f, g : ((A,a),\alpha) \to ((A',a'),\alpha')$$

be morphisms in $\mathbb{E}(B,b)$ such that $\Phi^p_{\mathbb{E}}(f) = \Phi^p_{\mathbb{E}}(g)$, i.e., $1_E \times_B f = 1_E \times_B g$. This equality implies that for each $z \in A$ such that $\alpha(z) \in p(E)$ we have $f(z) = g(z)$. Hence, in order to show that $f = g$, let z be an element in A such that $\alpha(z) \notin p(E)$. By (4.8), there exists $x \in E$ such that

$$b(\alpha(z), p(x)) \neq \bot \quad \text{and} \quad b(p(x), \alpha(z)) \neq \bot.$$

Since α is an étale morphism, there exists a unique $z_x \in A$ such that $\alpha(z_x) = p(x)$, $a(z_x, z) = b(p(x), \alpha(z))$ and $a(\tilde{z}_x, z) = \bot$ for each $\tilde{z}_x \neq z_x$ such that $\alpha(\tilde{z}_x) = p(x)$. The fact that both f and g are **V**-functors gives

$$a(z_x, z) \leq a'(f(z_x), f(z)) \wedge a'(g(z_x), g(z)),$$

where $f(z_x) = g(z_x)$. Now, since α is an étale morphism, there exists a unique element $\bar{z} \in A$ such that $\alpha(\bar{z}) = \alpha(z)$, $a(\bar{z}, z_x) = b(\alpha(z), p(x))$ and $a(\bar{\bar{z}}, z_x) = \bot$ for each $\bar{\bar{z}} \neq \bar{z}$ such that $\alpha(\bar{\bar{z}}) = \alpha(z)$. Moreover, with condition (4.6), we have that $\bar{z} = z$. Hence in A' we get $f(z) = g(z)$ as wished. □

Lemma 4.3.13 *Let* **V** *be a ccd and totally ordered quantale such that condition* (4.6) *is satisfied. Assume* $p : (E,e) \to (B,b)$ *surjective. The functor* $\Phi^p_{\mathbb{E}}$ *is full provided that for each* $u \ll b(y_1, y_0)$ *there* *exists a unique (up to equivalence) zigzag*

$$\begin{array}{ccccccccc}
x'_n & & & & & & & & \\
\downarrow{\scriptstyle e(x_n,x_{n-1})} & & & & & & & & \\
x_{n-1} & \longrightarrow & x'_{n-1} & & & & & & \\
 & & \downarrow{\scriptstyle e(x'_{n-1},x_{n-2})} & & & & & & \\
 & & x_{n-2} & \longrightarrow & x'_{n-2} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \cdot & \dashrightarrow & \cdot & & \\
 & & & & & & \vdots & & \\
 & & & & & & \cdot & \dashrightarrow & x'_1 \\
 & & & & & & & & \downarrow{\scriptstyle e(x'_1,x_0)} \\
 & & & & & & & & x_0
\end{array} \tag{4.9}$$

in (E,e) *with* $n \in \mathbb{N}$, $p(x'_n) = y_1$, $p(x_0) = y_0$, $p(x_i) = p(x'_i)$, *for* $i = 1,\dots,n-1$, *and*

$$u \ll e(x'_n, x_{n-1}) \otimes e(x'_{n-1}, x_{n-2}) \otimes \cdots \otimes e(x'_1, x_0).$$

We did not define yet the equivalence class for a zigzag as above and here is where the functor $q : \mathbf{V}\text{-}\mathbf{Cat} \to \mathbf{Ord}$ of (4.7) is needed. In Section 1.4.3 an equivalence relation for zigzags in **Ord** is defined so that we say that the equivalence class of a zigzag (4.9) is given by the equivalence class of the zigzag in **Ord** obtained by mapping (4.9) via q.

Proof

Let $((A,a),\alpha)$ and $((A',a'),\alpha')$ be objects in $\mathbb{E}(B,b)$ and let

$$f : ((E \times_B A, d), \mathrm{pr}_1, 1_E \times_B \mathrm{pr}_2) \to ((E \times_B A', d'), \mathrm{pr}'_1, 1_E \times_B \mathrm{pr}'_2)$$

be a morphism in $\mathrm{Des}_{\mathbb{E}}(p)$, where $((E \times_B A, d), \mathrm{pr}_1, 1_E \times_B \mathrm{pr}_2)$ and $((E \times_B A', d'), \mathrm{pr}'_1, 1_E \times_B \mathrm{pr}'_2)$ are the images by $\Phi^p_{\mathbb{E}}$ of $((A,a),\alpha)$ and $((A',a'),\alpha')$, respectively. We want to define a morphism

$$\bar{f} : ((A,a),\alpha) \to ((A',a'),\alpha')$$

such that $\Phi^p_{\mathbb{E}}(\bar{f}) = f$.

Let $z \in A$. Since p is surjective, there exists $x \in X$ such that $\alpha(z) = p(x)$. Hence the pair (x,z) is an element in $E \times_B A$ so that we can map it by f. Let $(x,z') = f(x,z)$. Define then $\bar{f}$ by

$$z \mapsto z'. \tag{4.10}$$

Notice that the choice of such z' might depend on the choice of the element $x \in X$ such that $\alpha(z) = p(x)$. But, since f is a morphism in $\mathrm{Des}_{\mathbb{E}}(p)$, we have that the diagram

$$\begin{array}{ccc} E \times_B (E \times_B A) & \xrightarrow{1_E \times_B f} & E \times_B (E \times_B A') \\ {\scriptstyle 1_E \times_B \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle 1_E \times_B \mathrm{pr}_2'} \\ E \times_B A & \xrightarrow[f]{} & E \times_B A' \end{array}$$

is commutative. Hence, the map $\bar{f}$ given by (4.10) is well-defined, making the **Set**-diagram

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}} & A' \\ & {\scriptstyle \alpha}\searrow \quad \swarrow{\scriptstyle \alpha'} & \\ & B & \end{array}$$

commutative. It remains to show that $\bar{f} : (A,a) \to (A',a')$ is a **V**-functor, that is, for each $z_1, z_0 \in A$

$$a(z_1,z_0) \leq a'(\bar{f}(z_1), \bar{f}(z_0)).$$

To prove the inequality above, we show that for each $u \ll a(z_1,z_0)$ we have $u \leq a'(\bar{f}(z_0), \bar{f}(z_1))$. Let $u \in \mathbf{V}$, with $u \neq \bot$, such that $u \ll a(z_1,z_0)$. Since $\alpha : (A,a) \to (B,b)$ is a **V**-functor,

$$u \ll a(z_1,z_0) \leq b(\alpha(z_1), \alpha(z_0)) \Rightarrow u \ll b(\alpha(z_1), \alpha(z_0)).$$

By hypotheses, there exists a unique (up to equivalence) zigzag (4.9) in (E,e), with $n \in \mathbb{N}$, such that $p(x_0) = \alpha(z_0)$, $p(x_n') = \alpha(z_1)$, $p(x_i) = p(x_i')$, for $i = 1, \dots, n-1$, and

$$u \ll e(x_0', x_1) \otimes \cdots \otimes e(x_{n-1}', x_n).$$

Let us assume that $n = 2$ and that the zigzag is then of the form

$$\begin{array}{ccc} x_2 & & \\ \downarrow{\scriptstyle e(x_2,x_1)} & & \\ x_1 & \longrightarrow & x_1' \\ & & \downarrow{\scriptstyle e(x_1',x_0)} \\ & & x_0, \end{array}$$

with $p(x_0) = \alpha(z_0)$, $p(x_2) = \alpha(z_1)$, $p(x_1) = p(x_1')$ and $u \ll e(x_2,x_1) \otimes e(x_1',x_0)$. Since α is an étale morphism, there exists a unique $\tilde{z}_1 \in A$ such that $\alpha(\tilde{z}_1) = p(x_1) = p(x_1')$ and $a(\tilde{z}_1, z_0) = b(p(x_1), \alpha(z_0))$, with $a(\bar{z}_1, z_0) = \bot$ for each $\bar{z}_1 \neq \tilde{z}_1$ with $\alpha(\bar{z}_1) = p(x_1) = p(x_1')$. Moreover, since α is étale, $a(z_1, \tilde{z}_1) = b(\alpha(z_2), p(x_1))$. Hence, since $f : (E \times_B A, d) \to (E \times_B A', d')$ is a **V**-functor,

$$e(x_2,x_1) = e(x_2,x_1) \wedge a(z_2, \tilde{z}_1) \leq e(x_2,x_1) \wedge a'(\bar{f}(z_2), \bar{f}(\tilde{z}_1)) \leq a'(\bar{f}(z_2), \bar{f}(\tilde{z}_1))$$

and

$$e(x_1',x_0) = e(x_1',x_0) \wedge a(\tilde{z}_1,z_0) \leq e(x_1',x_0) \wedge a'(\bar{f}(\tilde{z}_1),\bar{f}(z_0)) \leq a'(\bar{f}(\tilde{z}_1),\bar{f}(z_0)).$$

Therefore,

$$u \ll e(x_2,x_1) \otimes e(x_1',x_0) \leq a'(\bar{f}(z_2),\bar{f}(\tilde{z}_1)) \otimes a'(\bar{f}(\tilde{z}_1),\bar{f}(z_0)) \leq a'(\bar{f}(z_2),\bar{f}(z_0))$$

as claimed. □

Remark 4.3.14 In case $p:(E,e) \to (B,b)$ is not surjective, one has to define $\bar{f}:(A,a) \to (A',a')$ for elements $z \in A$ such that $\alpha(z) \notin p(E)$. For that one needs condition (4.8). In fact, for an element $z \in A$ such that $\alpha(z) \notin p(E)$, by condition (4.8), there exists an element $x \in E$ such that

$$b(\alpha(z),p(x)) \otimes b(p(x),\alpha(z)) \neq \bot.$$

Since α is étale, there exists a unique element $z_x \in A$ such that $\alpha(z_x) = p(x)$, $a(z_x,z) = b(p(x),\alpha(z))$ and $a(\tilde{z}_x,z) = \bot$ for each $\tilde{z}_x \neq z_x$ with $\alpha(\tilde{z}_x) = p(x)$. Consider $z_x' = \bar{f}(z_x) \in A'$. Since α' is an étale morphism, there exists a unique element $z' \in A'$ such that $\alpha'(z') = \alpha(z)$, $a'(z',z_x') = b(\alpha(z),p(x))$ and $a'(\tilde{z}',z_x') = \bot$ for each $\tilde{z}' \neq z'$ with $\alpha'(\tilde{z}') = \alpha(z)$. The assignment

$$z \mapsto z' \tag{4.11}$$

makes $\bar{f}:(A,a) \to (A',a')$ a well-defined **V**-functor (in fact (4.11) does not depend on the choice of $x \in E$) such that $\alpha' \cdot \bar{f} = \alpha$.

Lemma 4.3.15 *Let* **V** *be a ccd and totally ordered quantale such that condition* (4.6) *is satisfied. Assume* $p:(E,e) \to (B,b)$ *surjective. The functor* $\Phi^p_{\mathbb{E}}$ *is essentially surjective on objects provided that for each* $u \ll b(y_1,y_0)$ *there exists a unique (up to equivalence) zigzag* (4.9) *in* (E,e) *with* $n \in \mathbb{N}$, $p(x_n') = y_1$, $p(x_0) = y_0$, $p(x_i) = p(x_i')$, *for* $i = 1,\ldots,n-1$, *and*

$$u \ll e(x_n',x_{n-1}) \otimes e(x_{n-1}',x_{n-2}) \otimes \cdots \otimes e(x_1',x_0).$$

Proof
Let $((C,c),\gamma,\xi)$ be an object in $\mathrm{Des}_{\mathbb{E}}(p)$. We want to find an object $((A,a),\alpha) \in \mathbb{E}(B)$ such that

$$\Phi^p_{\mathbb{E}}((A,a),\alpha) \cong ((C,c),\gamma,\xi).$$

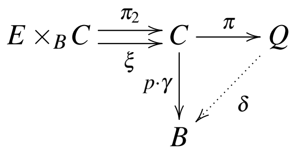
As we did in the previous section, to construct such an element $((A,a),\alpha)$ in $\mathbb{E}(B)$ we use the general arguments for the construction of the left adjoint of the comparison functor (see diagram (1.5) in Section 1.1). Consider the object part (Q,q) of the coequalizer $((Q,q),\pi)$ in **V**-**Cat** of the pair

$$E \times_B C \underset{\xi}{\overset{\pi_2}{\rightrightarrows}} C \xrightarrow{\pi} Q.$$

The quotient set Q is given by the equivalence relation $\sim_\xi$ defined on the set C: for $z,z' \in C$,

$$z \sim_\xi z' \iff p(\gamma(z)) = p(\gamma(z')) \quad \text{and} \quad z' = \xi(\gamma(z'),z),$$

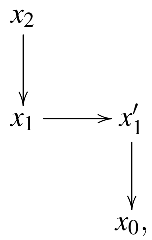
while the structure $q : Q \times Q \to \mathbf{V}$ is the final structure of the morphism $\pi : (C,c) \to Q$. Since $p \cdot \gamma \cdot \pi_2 = p \cdot \gamma \cdot \xi$, by the universal property of the coequalizer, there exists a unique morphism $\delta : Q \to B$ such that the diagram



commutes. We claim that $((Q,q),\delta)$ is an object in $\mathbb{E}(B)$ such that $\Phi^p_{\mathbb{E}}((Q,q),\delta) = ((C,c),\gamma,\xi)$. We show first that $\delta : Q \to B$ is open. Let $[z_0] \in Q$, $y_1 \in B$ and $u \in \mathbf{V}$ with $u \ll b(y_1, \delta[z_0])$. By hypothesis, there exists a unique (up to equivalence) zigzag (4.9) in (E,e), with $n \in \mathbb{N}$, such that $p(x_0) = \delta[z_0]$, $p(x'_n) = y_1$, $p(x_i) = p(x'_i)$ for $i = 1,\dots,n-1$, and

$$u \ll e(x'_n, x_{n-1}) \otimes e(x'_{n-1}, x_{n-2}) \otimes \cdots \otimes e(x'_1, x_0).$$

Assume $n = 2$ so that the zigzag is of the form



where $p(x_0) = \delta[z_0]$, $p(x_2) = y_1$, $p(x_1) = p(x'_1)$ and

$$u \ll e(x_2, x_1) \otimes e(x'_1, x_0).$$

Let $z_0 \in \pi^{-1}[z_0]$. We can assume that $\gamma(z_0) = x_0$ since, if it is not the case, we just take $\xi(x_0, z_0)$. This is a point in C such that $\gamma(\xi(x_0,z_0)) = x_0$ and $\pi(\xi(x_0,z_0)) = [z_0]$. Since γ is an étale morphism, there exists a unique element $z'_1 \in C$ such that $\gamma(z'_1) = x'_1$, $c(z'_1, z_0) = e(x'_1, x_0)$ and $c(\tilde{z}'_1, z_0) = \bot$ for each $\tilde{z}'_1 \neq z'_1$ with $\gamma(\tilde{z}'_1) = x'_1$. Take now the point $z_1 = \xi(x_1, z'_1)$. This is a point in C such that $\gamma(z_1) = x_1$ and $z_1 \sim_\xi z'_1$. Since γ is étale, there exists a unique element $z_2 \in C$ such that $\gamma(z_2) = x_2$, $c(z_2, z_1) = e(x_2, x_1)$ and $c(\tilde{z}_2, z_1) = \bot$ for each $\tilde{z}_2 \neq z_2$ with $\gamma(\tilde{z}_2) = x_2$. Now, since $\pi : C \to Q$ is a $\mathbf{V}$-functor, we conclude that

$$u \ll c(z_2, z_1) \otimes (z'_1, z_0) \leq q(\pi[z_2], \pi[z'_1]) \otimes q(\pi[z_1], z_0]) \leq q(\pi[z_2], z_0]).$$

which implies that $u \ll q(\pi[z_2], z_0])$ proving that $\delta : Q \to B$ is open.
We now show that $\delta : Q \to B$ is étale. Let then $[\zeta] \in Q$ such that $[\zeta] \neq \pi[z_2]$, $\delta[\zeta] = y_1$ and $u \ll$

$q([\zeta],[z_0])$. Since $\pi : C \to Q$ is a regular epimorphism, there exists a zigzag

$$\begin{array}{cccccccc}
\zeta'_m & & & & & & & \\
\big\downarrow{\scriptstyle c(\zeta_m,\zeta_{m-1})} & & & & & & & \\
\zeta_{m-1} & \longrightarrow & \zeta'_{m-1} & & & & & \\
 & & \big\downarrow{\scriptstyle c(\zeta'_{m-1},\zeta_{m-2})} & & & & & \\
 & & \zeta_{m-2} & \longrightarrow & \zeta'_{m-2} & & & \\
 & & & & \big\downarrow & & & \\
 & & & & \cdot & \dashrightarrow & \cdot & \\
 & & & & & & \vdots & \\
 & & & & & & \cdot & \dashrightarrow \zeta'_1 \\
 & & & & & & & \big\downarrow{\scriptstyle c(\zeta'_1,\zeta_0)} \\
 & & & & & & & \zeta_0
\end{array} \tag{4.12}$$

in (C,c), with $m \in \mathbb{N}$, such that $\pi(\zeta_0) = [z_0]$, $\pi(\zeta'_m) = [\zeta]$, $\pi(\zeta_j) = \pi(\zeta'_j)$ for $j = 1,\ldots,m-1$, and

$$u \ll c(\zeta'_m,\zeta_{m-1}) \otimes c(\zeta'_{m-1},\zeta_{m-2}) \otimes \cdots \otimes c(\zeta'_1,\zeta_0).$$

The image by γ of the above zigzag in (C,c) will be a zigzag in (E,e) of the form

$$\begin{array}{cccccccc}
\gamma(\zeta'_m) & & & & & & & \\
\big\downarrow{\scriptstyle e(\gamma(\zeta'_m),\gamma(\zeta_{m-1}))} & & & & & & & \\
\gamma(\zeta_{m-1}) & \longrightarrow & \gamma(\zeta'_{m-1}) & & & & & \\
 & & \big\downarrow{\scriptstyle e(\gamma(\zeta'_{m-1}),\gamma(\zeta_{m-2}))} & & & & & \\
 & & \gamma(\zeta_{m-2}) & \longrightarrow & \gamma(\zeta'_{m-2}) & & & \\
 & & & & \big\downarrow & & & \\
 & & & & \cdot & \dashrightarrow & \cdot & \\
 & & & & & & \vdots & \\
 & & & & & & \cdot & \dashrightarrow \gamma(\zeta'_1) \\
 & & & & & & & \big\downarrow{\scriptstyle e(\gamma(\zeta'_1),\gamma(\zeta_0))} \\
 & & & & & & & \gamma(\zeta_0).
\end{array}$$

By hypothesis, this zigzag of length m must be equivalent to the zigzag of the form (4.9) of length n we got to prove δ open. We assumed that $n = 2$. Hence, we study the cases where $m = 1$ and $m = 2$.

Case $m = 1$

In (C,c) we have

$$z_2 \xrightarrow{c(z_2,z_1)} z_1$$

$$z'_1 \xrightarrow{c(z'_1,z_0)} z_0$$

$$\zeta_1 \xrightarrow{u \ll c(\zeta_1,\zeta_0)} \zeta_0,$$

while in (E,e) we have

$$x_2 \xrightarrow{e(x_2,x_1)} x_1$$

$$x'_1 \xrightarrow{e(x'_1,x_0)} x_0$$

$$\gamma(\zeta_1) \xrightarrow{u \ll e(\gamma(\zeta_1),\gamma(\zeta_0))} \gamma(\zeta_0).$$

Since the two zigzags above must be equivalent, there exists a chain in (E,e) of the form

$$\chi_2 \xrightarrow{e(\chi_2,\chi_1)\neq\perp} \chi_1 \xrightarrow{e(\chi_1,\chi_0)\neq\perp} \chi_0$$

with $p(\chi_2) = y_1$, $p(\chi_1) = p(x_1) = p(x'_1)$ and $p(\chi_0) = \delta[z_0]$. Let $t_0 = \xi(\chi_0, z_0)$. Then, in (C,c), we have that

$$t_0 \sim_\xi z_0 \sim_\xi \zeta_0.$$

Since γ is an étale morphism, there exists a unique element $t_1 \in C$ such that $\gamma(t_1) = \chi_1$, $c(t_1,t_0) = e(\chi_1,\chi_0)$ and $c(\tilde{t}_1,t_0) = \perp$ for each $\tilde{t}_1 \neq t_1$ with $\gamma(\tilde{t}_1) = \chi_1$. Consider now the points (x_0,t_0), (x'_1,t_1) in $E \times_B C$. Since $\xi : E \times_B C \to C$ is a **V**-functor,

$$\perp \neq e(x'_1,x_0) \wedge c(t_1,t_0) \leq c(\xi(x'_1,t_1), \xi(x_0,t_0)) = c(\xi(x'_1,t_1), z_0).$$

Since γ is an étale morphism, we conclude that $z'_1 = \xi(x'_1,t_1)$. Therefore,

$$t_1 \sim_\xi z'_1 \sim_\xi z_1.$$

The same construction gives the existence of an element $t_2 \in C$ such that $\gamma(t_2) = \chi_2$, $c(t_2,t_1) = e(\chi_2,\chi_1)$ and

$$t_2 \sim_\xi z_2.$$

Since $\xi : E \times_B C \to C$ is a **V**-functor,

$$\perp \neq e(\gamma(\zeta_1),\gamma(\zeta_0)) \wedge c(t_2,t_0) \leq c(\xi(\gamma(\zeta_1),t_2), \xi(\gamma(\zeta_0),t_0)) = c(\xi(\gamma(\zeta_1),t_2),\zeta_0).$$

Since γ is an étale morphism, we conclude that

$$z_2 \sim_\xi t_2 \sim_\xi \zeta_1,$$

which contradicts the fact that $\pi[z_2] \neq [\zeta]$.

Case m $= 2$ In (C,c) we have

$$z_2 \xrightarrow{c(z_2,z_1)} z_1$$

$$z_1' \xrightarrow{c(z_1',z_0)} z_0$$

$$\zeta_2 \xrightarrow{c(\zeta_2,\zeta_1)} \zeta_1$$

$$\zeta_1' \xrightarrow{c(\zeta_1',\zeta_0)} \zeta_0,$$

while in (E,e) we have

$$x_2 \xrightarrow{e(x_2,x_1)} x_1$$

$$x_1' \xrightarrow{e(x_1',x_0)} x_0$$

$$\gamma(\zeta_2) \xrightarrow{e(\gamma(\zeta_2),\gamma(\zeta_1))} \gamma(\zeta_1)$$

$$\gamma(\zeta_1') \xrightarrow{e(\gamma(\zeta_1'),\gamma(\zeta_0))} \gamma(\zeta_0).$$

In fact the two zigzags above must be equivalent, so that

$$p(x_1) = p(x_1') = p(\gamma(\zeta_1)) = p(\gamma(\zeta_1')).$$

Since the pairs (x_1', ζ_1'), (x_0, ζ_0) are in $E \times_B C$, and since $\xi : E \times_B C \to C$ is a $\mathbf{V}$-functor, we have

$$\bot \neq e(x_1', x_0) \wedge c(\zeta_1', \zeta_0) \leq c(\xi(x_1', \zeta_1'), \xi(x_0, \zeta_0)) = c(\xi(x_1', \zeta_1'), z_0),$$

by the fact that $z_0 \sim_\xi \zeta_0$. Since γ is an étale morphism, we have $\xi(x_1', \zeta_1') = z_1'$, so that,

$$z_1 \sim_\xi z_1' \sim_\xi \zeta_1' \sim_\xi \zeta_1.$$

An analogous argument leads to the equivalence

$$z_2 \sim_\xi \zeta_2,$$

which contradicts the fact that $\pi[z_2] \neq [\zeta]$.
It remains to show that

$$\Phi^p_{\mathbb{E}}((Q,q),\delta) \cong ((C,c),\gamma,\xi).$$

The image of $((Q,q),\delta)$ by $\Phi^p_{\mathbb{E}}$ is the triple $((E\times_B Q,d),\mathrm{pr}_1,1_E\times_B \mathrm{pr}_2)$, where

$$\begin{array}{ccc} (E\times_B Q,d) & \xrightarrow{\mathrm{pr}_2} & (Q,q) \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \delta} \\ (E,e) & \xrightarrow[p]{} & (B,b) \end{array}$$

is the pullback in **V**-**Cat** of δ along p, with d the pullback structure. The isomorphism

$$i:(C,c)\to(E\times_B Q,d)$$

in **V**-**Cat** is given by

$$z\mapsto(\gamma(z),[z]).$$

The map i is injective; in fact, given $z_0,z_1\in C$ such that $(\gamma(z_0),[z_0])=(\gamma(z_1),[z_1])$,i.e., $\gamma(z_0)=\gamma(z_1)$ and $z_0\sim_\xi z_1$, we conclude that $z_0=z_1$, since

$$z_0=\xi(\gamma(z_0),z_0)=\xi(\gamma(z_1),z_0)=z_1,$$

by the properties of the descent data ξ.
Surjectivity of i also follows: given an element $(x,[z])\in E\times_B Q$, let $\xi(x,z)\in C$. Then,

$$i(\xi(x,c))=(\gamma(\xi(x,z)),[\xi(x,z)])=(x,[z]),$$

again using properties of the descent data ξ. So far we proved that $i:C\to E\times_B Q$ is a **Set**-isomorphism. Since i is also a **V**-functor, in fact $c(z_1,z_0)\leq e(\gamma(z_1),\gamma(z_0))\wedge q([z_1],[z_0])$ for each $z_0,z_1\in C$, it remains to show that

$$e(x_1,x_0)\wedge q([z_1],[z_0])\leq c(\xi(x_1,z_1),\xi(x_0,z_0)),$$

for each $(x_0,[z_0])$ and $(x_1,[z_1])$ in $E\times_B Q$.
Let $u\ll e(x_1,x_0)\wedge q([z_1],[z_0])$. We want to show that $u\leq c(\xi(x_1,z_1),\xi(x_0,z_0))$. Since $\pi:C\to Q$ is a regular epimorphism, there exists a zigzag (4.12) in (C,c), with $m\in\mathbb{N}$, such that $\pi(\zeta'_m)=[z_1]$, $\pi(\zeta_0)=[z_0]$, $\pi(\zeta_j)=\pi(\zeta'_j)$, for $j=1,\ldots,m-1$, and

$$u\ll c(\zeta_m,\zeta_{m-1})\otimes c(\zeta'_{m-1},\zeta_{m-2})\otimes\cdots\otimes c(\zeta'_1,\zeta_0).$$

Assume $m=2$. Since $\gamma:C\to E$ is an étale morphism, there exists a unique element $t_1\in C$ such that $\gamma(t_1)=x_1$, $c(t_1,\xi(x_0,z_0))=e(x_1,x_0)$ and $c(\tilde{t}_1,\xi(x_0,z_0))=\perp$ for each $\tilde{t}_1\neq t_1$ with $\gamma(\tilde{t}_1)=x_1$. We want to show that

$$t_1\sim_\xi \xi(x_1,z_1)$$

since this would imply, by the properties of the descent data ξ, that $t_1 = \xi(x_1, z_1)$.
The situation is quite similar to the case where we had to construct the étale morphism $\delta : Q \to B$. In (E,e) we have

$$\begin{array}{lll} \gamma(\zeta_2) \xrightarrow{e(\gamma(\zeta_2),\gamma(\zeta_1))} \gamma(\zeta_1) & & \\ & \gamma(\zeta_1') \xrightarrow{e(\gamma(\zeta_1'),\gamma(\zeta_0))} \gamma(\zeta_0) & \\ x_1 \xrightarrow{\qquad\qquad e(x_1,x_0) \qquad\qquad} x_0 & & \end{array}$$

and, by hypotheses, the two zigzags above must be equivalent. Hence, there exists in (E,e) a zigzag of the form

$$\chi_2 \xrightarrow{e(\chi_2,\chi_1)\neq\bot} \chi_1 \xrightarrow{e(\chi_1,\chi_0)\neq\bot} \chi_0$$

with $p(\chi_2) = p(x_1) = \delta[z_1]$, $p(\chi_1) = p(\gamma(\zeta_1)) = p(\gamma(\zeta_1'))$ and $p(\chi_0) = p(x_0) = \delta[z_0]$.
Taking in (C,c) the point $\mathfrak{z}_0 = \xi(\chi_0, z_0)$, since γ is an étale morphism, we have a zigzag in (C,c) of the form

$$\mathfrak{z}_2 \xrightarrow{c(\mathfrak{z}_2,\mathfrak{z}_1)} \mathfrak{z}_1 \xrightarrow{c(\mathfrak{z}_1,\mathfrak{z}_0)} \mathfrak{z}_0$$

such that $\gamma(\mathfrak{z}_2) = \chi_2$, $\gamma(\mathfrak{z}_1) = \chi_1$ and $c(\mathfrak{z}_2,\mathfrak{z}_1) = e(\chi_2,\chi_1)$, $c(\mathfrak{z}_1,\mathfrak{z}_0) = e(\chi_1,\chi_0)$. Summing up, in (C,c) we have the following situation

$$\begin{array}{lll} \zeta_2 \xrightarrow{c(\zeta_2,\zeta_1)} \zeta_1 & & \\ & \zeta_1' \xrightarrow{c(\zeta_1',\zeta_0)} \zeta_0 & \\ \mathscr{Z}_1 \xrightarrow{\qquad c(t_1,\xi(x_0,z_0)) \qquad} \xi(x_0,z_0) & & \\ \mathfrak{z}_2 \xrightarrow{c(\mathfrak{z}_2,\mathfrak{z}_1)} \mathfrak{z}_1 \xrightarrow{c(\mathfrak{z}_1,\mathfrak{z}_0)} \mathfrak{z}_0, & & \end{array}$$

where

$$t_1 \sim_\xi \mathfrak{z}_2 \sim_\xi \zeta_2 \sim_\xi z_1 \sim_\xi \xi(x_1,z_1).$$

To conclude the proof, it remains to show that the following diagram

$$\begin{array}{ccc} E \times_B C & \xrightarrow{1_E \times_B i} & E \times_B (E \times_B Q) \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle 1_E \times_B \mathrm{pr}_2} \\ C & \xrightarrow[i]{} & E \times_B Q \end{array}$$

is commutative. Let (x,z) be an element in $E \times_B C$. Then,

$$(i \cdot \xi)(x,z) = (x, [\xi(x,z)]) = (x, [z]) = (1_E \times_B \mathrm{pr}_2 \cdot 1_E \times_B i)(x,z),$$

which proves that the diagram above is commutative. □

Remark 4.3.16 If $p:(E,e)\to(B,b)$ is not surjective, the construction of the étale morphism $\delta : Q\to B$ needs only a little extra work: to the set Q, constructed in the same way as in the surjective case, we need just to add the points of $B-p(E)$ with values

$$q(y,[z]) = b(y,\delta[z])$$

for each $y\in B-p(E)$ and $[z]\in Q$.

By Lemmas 4.3.12, 4.3.13 and 4.3.15, the following theorem holds.

Theorem 4.3.17 *Let* **V** *be a ccd and totally ordered quantale such that condition* (4.6) *is satisfied. The* **V***-functor* $p:(E,e)\to(B,b)$ *is an effective étale descent morphism provided that*

(i) *for each* $u \ll b(y_1,y_0)$ *there exists a unique (up to equivalence) zigzag* (4.9) *in* (E,e) *with* $n\in\mathbb{N}$, $p(x'_n)=y_1$, $p(x_0)=y_0$, $p(x_i)=p(x'_i)$, *for* $i=1,\dots,n-1$, *and*

$$u \ll e(x'_n,x_{n-1})\otimes e(x'_{n-1},x_{n-2})\otimes\cdots\otimes e(x'_1,x_0);$$

(ii) $\forall y\in B \quad \exists x\in E: \quad b(y,p(x))\otimes b(p(x),y)\neq\bot.$

□

4.4 (Effective) global-descent versus (effective) étale-descent

In Section 1.4.4 we already studied the relation between the (effective) global-descent morphisms and the (effective) étale-descent morphisms in the category **Top** of topological spaces and continuous maps. Here we analyze the more general case concerning $(\mathbb{T},\mathbf{V})$-categories. Let $\widehat{\mathbb{T}}$ be a lax extension to **Rel** of a **Set**-monad $\mathbb{T}=(T,\mu,\eta)$. To show that the effective descent morphisms are effective for étale-descent in $(\mathbb{T},\mathbf{2})$-**Cat** we need first to recall the following result concerning discrete fibrations.

Proposition 4.4.1 [18, Proposition 5.2] *If T satisfies BC then, for a pullback diagram in* $(\mathbb{T},\mathbf{2})$-**Cat**

$$\begin{array}{ccc} X\times_Y Z & \xrightarrow{\pi_2} & Z \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[p]{} & Y, \end{array}$$

with p a final morphism, one has that if π_1 *is a discrete (co)fibration, then g is a discrete (co)fibration.*

Theorem 4.4.2 *Suppose that the following conditions are satisfied:*

(i) $\widehat{T}$ *is flat;*

(ii) *T satisfies BC;*

(iii) *every naturality square of* η *with respect to relations with finite fibres is a BC-square.*

Then an effective descent morphism $p : (E,e) \to (B,b)$ *in* $(\mathbb{T},\mathbf{2})$-**Cat** *is effective for étale-descent.*

Proof

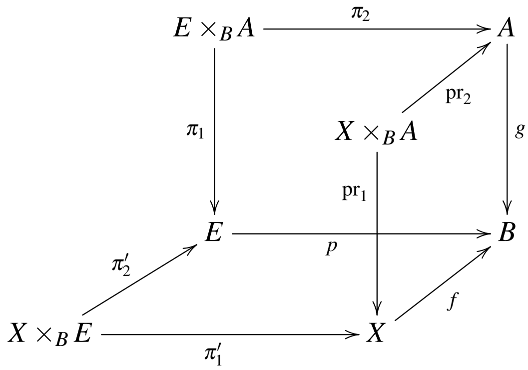
Let $p : E \to B$ be an effective descent morphism in $(\mathbb{T},\mathbf{2})$-**Cat**. We use Proposition 1.4.14 to show that p is also effective with respect to the class of étale morphisms, i.e., pullback stable discrete fibrations. Consider a pullback diagram in $(\mathbb{T},\mathbf{2})$-**Cat**

$$\begin{array}{ccc} E\times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ E & \xrightarrow[p]{} & B, \end{array}$$

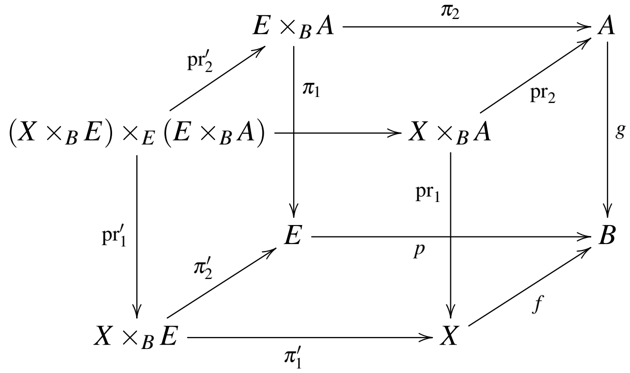
where π_1 is an étale morphism. The relational structure on $E\times_B A = \{(x,z) \in E\times A \mid p(z) = \alpha(z)\}$ is defined by

$$\mathfrak{w} \to (x,z) \Leftrightarrow T\pi_1(\mathfrak{w}) \to x \quad \text{and} \quad T\pi_2(\mathfrak{w}) \to z,$$

for any $\mathfrak{w} \in T(E\times_B A)$ and $(x,z) \in E\times_B A$. We want to prove that g is an étale morphism as well. By Theorem 3.0.7 and Proposition 4.4.1, g is a discrete fibration. To prove that every pullback of g is a discrete fibration we consider the following diagram



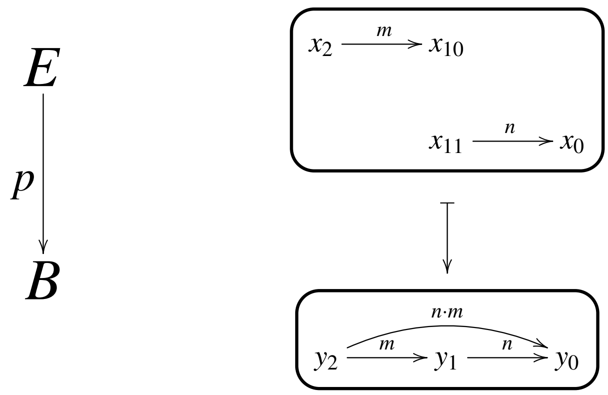
where the three faces are pullbacks. We want to prove that pr_1 is a discrete fibration. First of all observe that since effective descent morphisms are pullback stable π_1' is an effective descent morphism. Building the pullback on the left-side, i.e., the pullback of π_1 along π_2', by universality we get a cube such that all faces are pullbacks.



Now, since π_1 is an étale morphism, pr_1' is a discrete fibration, and, using the same argument that we used to prove that g is a discrete fibration, we conclude that pr_1 is a discrete fibration as well. □

The theorem is a generalization of Theorem 1.4.17 (where the same technique to get the proof is used) since the ultrafilter monad and its Barr extension to **Rel** satisfy conditions (i), (ii) and (iii). Other monads are included (with corresponding Barr extensions), namely the identity monad (we already know from Section 1.4.4), the M-ordered monad and the free-monoid monad. For what concerns the Barr extension of the powerset monad, condition (iii) is not satisfied and we do not know whether the theorem remains true. The converse of Theorem 4.4.2 is not true in general, as we already know by Section 1.4.4 where an example of an effective étale-descent morphism not effective for descent (even in the surjective case) is given in **Ord** (see Example 1.4.21). This gives also a counter-example in M-**Ord** since, as we saw, if $M = 1$, the trivial monoid, then 1-**Ord** $\cong$ **Ord**. Anyway we can also exhibit a counter-example in the non-trivial case.

Example 4.4.3 [2, Remark 3.3] Thanks to Proposition 3.3.6 and Corollary 4.2.8, an easy inspection of the picture below reveals that the monotone map p is an effective étale-descent morphism in M-**Ord** but not effective for global-descent.



Summing up, denoting the category $(\mathbb{T},\mathbf{2})$-**Gph** by **D**, the following diagram of implications holds in $(\mathbb{T},\mathbf{2})$-**Cat**

pullback stable *-quotient in **D**	$\Rightarrow$	effective global-descent	$\Rightarrow$	effective étale-descent
		$\Downarrow$		$\Downarrow$
$\Downarrow$		global-descent	$\Rightarrow$	étale-descent
		$\Updownarrow$		
final	$\Leftrightarrow$	universal regular epimorphism		

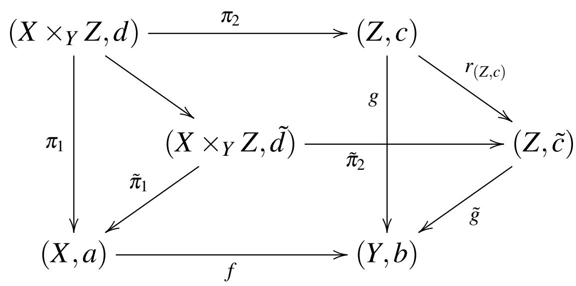
if conditions (i), (ii) and (iii) of Theorem 4.4.2 are satisfied. The diagram above is a generalization of diagram (1.27), so that most of the counter-examples of the one-direction implications given in Section 1.4.4 still work in this case. The only one where the case of the identity monad does not help is for a morphism effective for descent not a pullback stable *-quotient morphism in $(\mathbb{T},\mathbf{2})$-**Gph**. In this case, as we already saw from Theorem 3.3.10, a counter-example can be given for the free-monoid, that is, in **MultiOrd**.

We give now a criterion of a final morphism in $(\mathbb{T},\mathbf{2})$-**Cat** to be effective for étale-descent. Consider the following two classes of morphisms

- $\mathbb{E}_0$={pullback stable discrete fibrations in $(\mathbb{T},\mathbf{2})$-**Cat**}

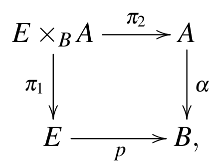
- $\mathbb{E}_1$={pullback stable discrete fibrations in $(\mathbb{T},\mathbf{2})$-**Gph**}.

We show that $\mathbb{E}_0 \subseteq \mathbb{E}_1$ in order to apply Proposition 1.4.14. Consider the following diagram



where $f \in \mathbb{E}_0$, g is in $(\mathbb{T},\mathbf{2})$-**Gph**, $r_{(Z,c)}$ is the reflection of the object (Z,c) via the reflector $r : (\mathbb{T},\mathbf{2})$-**Gph** $\to (\mathbb{T},\mathbf{2})$-**Cat**, the bottom square is a pullback diagram in $(\mathbb{T},\mathbf{2})$-**Cat** and the upper square is a pullback diagram in $(\mathbb{T},\mathbf{2})$-**Gph**. Since $f \in \mathbb{E}_0$, $\tilde{\pi}_2$ is a discrete fibration. Let us suppose that T : **Set** $\to$ **Set** satisfies BC. By Proposition 4.1.6, since $r_{(Z,c)}$ is an injective map, π_2 is a discrete fibration, proving that $f \in \mathbb{E}_1$ as claimed. Therefore, by Proposition 1.4.14, we then get the following result.

Proposition 4.4.4 *Let $p : (E,e) \to (B,b)$ be a morphism in $(\mathbb{T},\mathbf{2})$-**Cat** which is effective for étale-descent in $(\mathbb{T},\mathbf{2})$-**Gph**, where T : **Set** $\to$ **Set** satisfies BC. Then p is effective for étale-descent in $(\mathbb{T},\mathbf{2})$-**Cat** if and only if for each pullback diagram in $(\mathbb{T},\mathbf{2})$-**Gph***



where α is a pullback stable discrete fibration,

$$E \times_B A \in (\mathbb{T},\mathbf{2})\text{-}\mathbf{Cat} \Rightarrow A \in (\mathbb{T},\mathbf{2})\text{-}\mathbf{Cat}.$$

□

The proof of Theorem 4.4.2 still holds in $(\mathbb{T},\mathbf{2},\widehat{\mathbb{T}})$-**Gph**, where $\widehat{\mathbb{T}}$ is any lax extension (not necessarily flat) to **Rel** of the **Set**-monad $\mathbb{T}$. In this case one requires only that T : **Set** $\to$ **Set** satisfies BC. Therefore, since final morphisms in $(\mathbb{T},\mathbf{2})$-**Cat** are effective for étale-descent in $(\mathbb{T},\mathbf{2})$-**Gph**, the following result holds.

Corollary 4.4.5 *Let $p : (E, e) \to (B, b)$ be a final morphism in $(\mathbb{T}, \mathbf{2})$-**Cat**, with $T : \mathbf{Set} \to \mathbf{Set}$ satisfying BC. Then p is effective for étale-descent in $(\mathbb{T}, \mathbf{2})$-**Cat** if and only if for each pullback diagram in $(\mathbb{T}, \mathbf{2})$-**Gph***

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B, \end{array}$$

where α is a pullback stable discrete fibration,

$$E \times_B A \in (\mathbb{T}, \mathbf{2})\text{-}\mathbf{Cat} \Rightarrow A \in (\mathbb{T}, \mathbf{2})\text{-}\mathbf{Cat}.$$

□

If we assume that every naturality square of η with respect to relations with finite fibres is a BC-square, the corollary above gives a criterion for descent morphisms to be effective for étale-descent in $(\mathbb{T}, \mathbf{2})$-**Cat**.

Also in **V**-**Cat** we can prove that effective descent morphisms are effective for étale-descent.

Theorem 4.4.6 *A **V**-functor $p : (E, e) \to (B, b)$ is an effective étale-descent morphism provided that it is effective for descent.*

Proof
We use again Proposition 1.4.14. Let

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ E & \xrightarrow[p]{} & B, \end{array}$$

be a pullback diagram in **V**-**Cat**, where π_1 is an étale morphism and p is an effective descent morphism. We split the proof of g étale in two parts, that is, g open and δ_g open:

- *g open:* let $y_1 \in B$ and $z_0 \in A$. Since p is, in particular, final, we have that

$$b(y_1, g(z_0)) = \bigvee_{\substack{x_1 \in E : p(x_1) = y_1 \\ x_0 \in E : p(x_0) = g(z_0)}} e(x_1, x_0).$$

Since π_1 is open, each element $e(x_1, x_0)$ of the join above satisfies

$$e(x_1, x_0) = \bigvee_{z_1 \in A : g(z_1) = p(x_1)} d((x_1, z_1), (x_0, z_0)) \leq \bigvee_{z_1 \in A : g(z_1) = y_1 = p(x_1)} a(z_1, z_0).$$

Hence

$$b(y_1, g(z_0)) \leq \bigvee_{z_1 \in A : g(z_1) = y_1 = p(x_1)} a(z_1, z_0).$$

The other inequality is trivially satisfied.

- δ_g *open:* consider the following diagram

$$\begin{array}{ccccccc}
E\times_B A & \xrightarrow{\delta_{\pi_1}} & (E\times_B A)\times_A (E\times_B A) & \rightrightarrows & E\times_B A & \xrightarrow{\pi_1} & E \\
\downarrow{\scriptstyle \pi_2} & \boxed{1} & \downarrow{\scriptstyle \pi_2\times_p \pi_2} & \boxed{2} & \downarrow{\scriptstyle \pi_2} & \boxed{3} & \downarrow{\scriptstyle p} \\
A & \xrightarrow[\delta_g]{} & A\times_B A & \rightrightarrows & A & \xrightarrow[g]{} & B
\end{array}$$

where $\boxed{1}\boxed{2}\boxed{3} = \boxed{3}$ and $\boxed{2}\boxed{3}$ are pullbacks. By general properties of pullback squares, also the square $\boxed{1}$ is a pullback. Therefore, since p is final, also $\pi_2 \times_p \pi_2$ is final and, since δ_{π_1} is open, we get that δ_g is open as well. □

Therefore in **V**-**Cat**, for **V** cancellable, the following diagram of implications holds

**-quotient	⇔	effective global-descent	⇒	effective étale-descent
⇕		⇓		⇓
*-quotient		global-descent	⇒	étale-descent
⇓		⇕		
final	⇔	universal regular epimorphism.		

The converse of all the one-direction implications is not true in general, as the case for $\mathbf{V} = \mathbf{2}$ (i.e., in **Ord**) reveals (see Example 1.4.20 and Example 1.4.21 in Section 1.4.3).

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