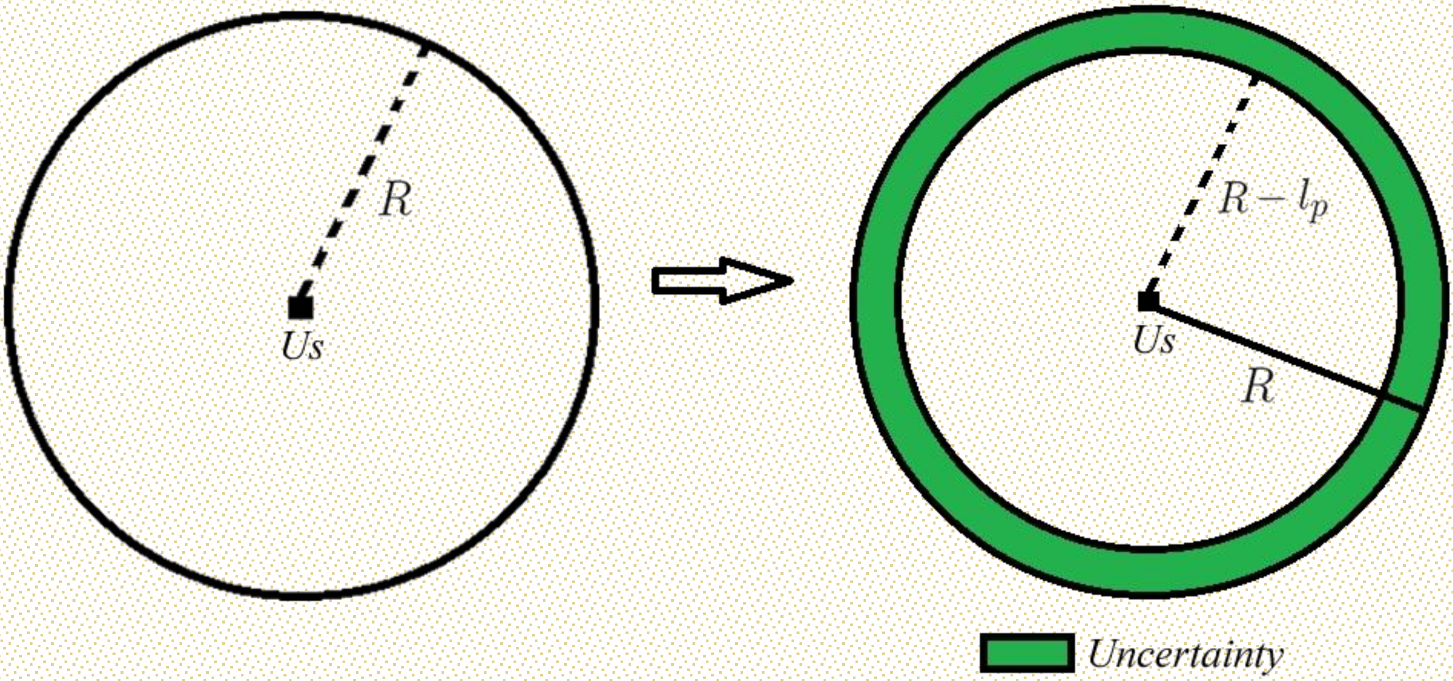

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CALCULATION OF THE COSMOLOGICAL CONSTANT USING THE HOLOGRAPHIC PRINCIPLE

Dissertation presented to the Physics Department at
University of Coimbra to obtain the Master's degree in Physics

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MASTER THESIS

Calculation of the Cosmological Constant Using the Holographic Principle

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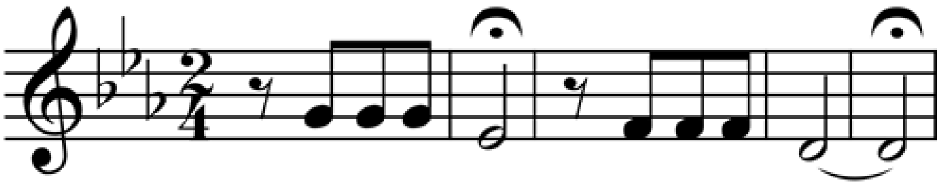
Para o meu avô,
que estará sempre comigo.



The beginning of Beethoven's symphony No. 5,
a universal symbol of fight, resistance and standing up for your beliefs.

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Abstract

The aim of this Thesis is to study the effect of an Event Horizon on the entanglement of the Quantum Vacuum and how entanglement, together with the Holographic Principle, may explain the current value of the Cosmological Constant, in light of recent theories. Entanglement is tested for vacuum states very near and very far from the Horizon of a de Sitter Universe, using the Peres-Horodecki (PPT) criterion. The states are averaged inside two boxes of volume V so that they acquire the structure of a bipartite Quantum Harmonic Oscillator, for which the PPT criterion is a necessary but not sufficient condition of separability.

The first chapters are an introduction to: the Quantum Vacuum and its physical manifestation, quantum entanglement and its application to Gaussian States and Quantum Harmonic Oscillators, the Holographic Principle, the relation between entanglement and the Holographic Principle, theories developed to calculate the Cosmological Constant, and the experimental methods used to measure it.

Entanglement was found between states averaged inside spherical shells with thickness of the order of one Planck distance (l_p), when one of the states is near the Horizon, and the other state is anywhere in the Universe. Entanglement disappears when the distance of the state near the horizon and the Horizon increases to a value somewhere between $10l_p$ and $60l_p$, or, in other words, when the state is at distances larger than $O(10l_p)$ to the Horizon. If we consider the Horizon not as a surface but as a spherical shell of thickness l_p, then this means that there is entanglement between the states in the Horizon and the rest of the Universe.
When both states are at distances larger than $\sim 60l_p$ from the Horizon, no entanglement was found.

Resumo

O objetivo desta Tese é estudar o efeito de um Horizonte no entrelaçamento do Vácuo Quântico e como esse entrelaçamento, junto com o Principio Holográfico, pode explicar (tendo em conta teorias recentes) o valor atual da Constante Cosmológica. O entrelaçamento é testado para estados de vácuo muito perto e muito longe do Horizonte de um Universo de de Sitter, usando o critério de Peres-Horodecki (PPT). É feita uma média dos estados dentro de duas caixas de volume V, de modo a que eles adquiram a estrutura de um sistema bipartidario de osciladores harmónicos quânticos, para os quais o critério PPT é uma condição necessária (mas não suficiente) de separabilidade.

Os primeiros capítulos da Tese são uma introdução a: o Vácuo Quântico e a sua manifestação física, entrelaçamento quântico e a sua aplicação a Estados Gaussianos e osciladores harmónicos quânticos, o Principio Holográfico, a relação entre o entrelaçamento e o Principio Holográfico, teorias desenvolvidas para calcular a Constante Cosmológica e métodos experimentais usadas para medi-la.

Entrelaçamento foi encontrado para estados cuja média foi feita dentro de casca esféricas com espessura da ordem da distancia de Planck (l_p), quando um dos estados se encontra perto do Horizonte, e o outro se encontra em qualquer posição do Universo. O entrelaçamento desaparece quando a distancia do estado mais próximo do Horizonte e o Horizonte aumenta para valores algures entre $10l_p$ e $60l_p$, ou, noutras palavras, quando o estado está a distancias maiores do que $O(10l_p)$ em relação ao Horizonte. Se considerarmos o Horizonte não como uma superfície, mas sim como uma casca esférica de espessura l_p, então isto significa que existe entrelaçamento entre os estados no Horizonte e o resto do Universo.
Quando ambos os estados estão a distancias maiores que $\sim 60l_p$ do Horizonte nunca se encontra entrelaçamento.

Contents

Part I

Invitation

Chapter 1

Introduction

1.1 Purpose and Motivation

Note: Throughout this Thesis, natural units ($\hbar = c = k_B = 1$) will be used unless stated otherwise.

In 1998, evidence was found that suggested that our Universe was expanding at an accelerating rate, fueled by a mysterious Dark Energy. [1, 2] Since then, numerous attempts have been made to explain the physical meaning of such energy, but up to date, no complete explanation has been found for the phenomenon. What is worse, many of the attempts were met with extreme inconsistencies.
There exist many theories of Dark Energy. There are, however, three main interpretations about the origin of the Dark Energy. The first two assume that it must, at least partially, come from a constant that naturally appears in the Einstein equations, the Cosmological Constant.

The simplest explanation is that it is only a constant, just like the Gravitational constant or the electron's mass. It is just a constant that appears in an equation and it must be calculated through observations. [3]

The second explanation, and probably the most widely accepted one, is that the Cosmological Constant comes from (or, at least, is partially accounted for, by) the energy of the vacuum. As will be shown in later chapters, the vacuum energy can have physical manifestations (Casimir effect) and it seems to behave, in a cosmological sense, just like a Cosmological Constant.
Often it is assumed that the origin of the Cosmological Constant is due both to the vacuum energy, and the existence of a constant that appears in the Einstein equations. They, together, form what is measured as the Dark Energy. In that case, it is usual to refer to the sum of both contributions as Cosmological Constant.
When calculations of the vacuum energy in our Universe where made, the resulting Cosmological Constant was 123 orders of magnitude greater than the experimental value. Not exactly the definition of success!

The third type of theory that is usually proposed is that the Dark Energy originates from a scalar field ϕ, that, at the present time, varies so slowly that it appears constant, but it may have varied at different rates in the past. It is usually suggested that models of this kind can also explain the early inflation of the Universe. The first model of this type was called the Quintessential model. There are, however, countless variations of this theory, that can be found on the literature. [4–6]

None of these explanations are fully established, and in many cases they aren't even successful in describing the observed data. [4] In the context of this thesis, the terms Cosmological Constant and Dark Energy will be used interchangeably, they will have the same meaning unless expressed otherwise: the Cosmological Constant is whatever explains the accelerating expansion of our Universe.

A few years before the evidence for the Cosmological Constant was found, it was proposed by G. 'tHooft in [7], and L. Susskind in [8], that the number of degrees of freedom in our universe is being over counted. The universe must have entropy smaller than or equal to the entropy of a Black Hole of the same size. Such entropy grows with the area of its boundary, not with its volume. This means that the information inside a physical system is not proportional to its volume, but, at the most, to its surface. Our 3-dimensional Universe must have the same amount of degrees of freedom as the 2-dimensional surface that surrounds it. This is called the Holographic Principle.
Therefore the problem arises: If we cannot count every state inside a volume of a system, if there is some kind of dilution of states as a consequence of the Holographic Principle, then, when we calculate the vacuum energy of all the states inside our Universe we are over counting the number of states. Hence, one ends up with an energy that does not correspond to the true value - a value that respects the Holographic Principle. The problem is, then, how exactly should we obtain the right amount of states? Some attempts were made with a good degree of success [9–13] however they usually rely on using a vacuum energy that is chosen to fit the holographic bound, or choosing a certain energy cutoff that is not necessarily natural.

Hence, the first question is: How can one calculate a vacuum energy in a way that respects the Holographic Principle, but doesn't need to introduce unnatural energy cutoffs?

The answer to this question was the starting point for this Thesis. A working proposal [14] that assumes there exists some distribution function $f(\mathbf{r}, \mathbf{k})$ that dilute the states accessible in our Universe, in a similar way as the fermi-dirac distribution $f = \frac{1}{e^{(\epsilon-\mu)\beta}+1}$ sets the average number of particles a given state can have. Using some simplified arguments, a shape for these functions was proposed, $f(r, k) = \frac{l_p}{R}\frac{1}{rk}$ [14], that gives back the same results obtained by the other holographic Cosmological Constant theories.

Now the attention turns to the next question: How to determine the shape of the function $f(r, k)$? What is the mechanism that permits the dilution of states?

It was suggested in several papers [15–18, 18–20] that the entropy of entanglement of a scalar field, when space is separated by a spherical boundary, is proportional to the area of the boundary, just like in the Holographic Principle. In [15] the possibility is suggested that entanglement might explain the entropy of a Black Hole, and in [10] it is suggested that it is the energy of the entanglement itself, originated by the spherical boundary of our visible Universe, that creates the accelerated expansion of the Universe. This last theory gives the same result as the holographic Cosmological Constant theories and the vacuum energy with the functions $f(r,k)$.

Hence, three more questions arise: What is the role of quantum entanglement in the Holographic Principle? Can quantum entanglement explain the functions $f(r,k)$? What, exactly, is the energy of entanglement?

To solve this, we propose the study of entanglement between vacuum states in a de Sitter Universe, a Universe dominated by the vacuum energy, to which our present Universe will evolve in the future. To be able to calculate entanglement of a scalar field $\hat{\phi}$, which is a continuous variable system, the field will be averaged inside two boxes (each centered on different points in space) such that we reduce the problem to the study of bipartite Quantum Harmonic Oscillators (each box is a state). [21] In this way, we will be able to use the PPT criterion to test for the existence of entanglement between vacuum states in the de Sitter Universe. Of course, the field $\hat{\phi}$ will need to be calculated for a curved spacetime, which in itself is a challenge. If the averaged states turn out to be Gaussian States, then we will also be able to calculate the degree of entanglement between vacuum state (using the Negativity[1]), which means we will know exactly which regions of space contribute more, or less, to the entanglement of the Universe, if entanglement is the mechanism responsible for Holographic Principle, this would lead us one step closer to find the shape of $f(r,k)$. However, it should be noted that, in this case, results may be different from the ones in [15–18, 18–20], since the Negativity and the Entropy of entanglement are both entanglement measurements, but are not equivalent.

1.2 Discussion Layout

The structure of this Thesis is as follows:

Part II is where we present the theoretical background necessary the rest of the Thesis.

In chapter 2, we introduce the concept of the Quantum Vacuum and vacuum energy. We also show how the quantization of a vector and a scalar field is done in a flat spacetime. Finally, we deduce the Casimir effect to show that the Quantum Vacuum can have a physical manifestation, and discuss the validity of the effect as a proof of that.

In chapter 3, we define what quantum entanglement is and explain how it can be tested and quantified in the context of quantum mechanics. The von Neumann entropy, mutual information and Negativity are introduced as entanglement measurements, and the Schmidt decomposition and PPT criterion as separability criteria.

[1]Negativity is a measure of the magnitude of entanglement. It will be introduced on chapter 3.

Next, we introduce the concept of Gaussian States, and how entanglement is defined for these states. The PPT criterion and Negativity are deduced for Gaussian States. Finally, a way of generalizing the PPT criterion for Gaussian States, to other infinite variable systems is presented.

In chapter 4, we introduce the Holographic Principle, explain what a Black Hole is and how that relates to the degrees of freedom of a region with limited gravity. Next we present a conjecture that permits the application of the Holographic Principle to general spacetimes, and how that can be applied to our Universe. Finally, we introduce a theory that relates quantum entanglement to the entropy of a Black Hole and to the degrees of freedom inside the Horizon of the Black Hole.

In chapter 5 the Cosmological Constant is explained more in depth. Its effect on the behavior of the Universe, and several theories about its physical meaning, are studied in more detail. The possible relation between the Cosmological Constant and vacuum energy, Spontaneous Symmetry Breaking, the Holographic Principle and the quantum entanglement, are explored. Finally, we explain the physics behind the observations that originally suggested the existence of the Cosmological Constant, discuss the validity of these conclusions and mention other experiments that support these observations.

Part III is were we develop the original content of the Thesis, and comment the results.

In chapter 6 we explain and discuss more in depth how the dilution of states can occur using $f(r,k)$ and how that can give the right Cosmological Constant. We discuss which surface should be chosen as the holographic boundary of the Universe. We also discuss how the energy of entanglement can be defined and how much energy a pair of entangled vacuum states can have.

In chapter 7 we calculate a scalar field operator $\hat{\phi}$ in a de Sitter space. Then we average the field inside two boxes and calculate the respective Hadamard functions and Covariance Matrices. The presence of entanglement is tested for vacuum states throughout space. We experiment with two different types of boxes, and for states very near and very far from the Horizon. An analysis and discussion of the results obtained is presented.

In chapter 8 some concluding remarks are made, followed by some possible improvements of the work presented in this Thesis.

Some of these chapters and sections are important because they present ideas that play a significant role on the justification of our methods and the interpretation of the results. Others are relevant in a more direct manner, they are explicitly used in the original content of this Thesis.
The chapters whose content is necessary for the understanding of the calculations made in Part III are: section 2.1, chapter 3, section 4.1, section 4.2.2 and section 5.1 (especially subsection 5.1.2).

Part II

Theoretical Background

Chapter 2

Quantum Vacuum

The purpose of this chapter will be to explain the origin of the Quantum Vacuum and also to calculate its most important physical manifestation - the Casimir Effect. The study of this effect is very important as it will be useful later in this thesis. Quantum Vacuum as the cause of this effect will be subjected to critical analysis in light of Jaffe's article which derives the Casimir force without the use of the Quantum Vacuum [22].

2.1 The origin of the Quantum Vacuum

This section will follow closely the Electromagnetic Quantum Vacuum derivation of *Peter Milonni* [23]. We will start by considering the classical vector field $\mathbf{A}(\mathbf{r}, t)$, the magnetic vector potential, which obeys the equation

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

If we are in a region of space where there are no electric charges, $\rho = 0$ and with the electric potential $\phi = 0$ we will get the following Maxwell equations,

$$\nabla \cdot \mathbf{E} = 0, \tag{2.1a}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.1b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.1c}$$

$$\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}. \tag{2.1d}$$

With the last two equations we get,

$$\nabla \times (\nabla \times \mathbf{A}) = -\frac{\partial^2 \mathbf{A}}{\partial t^2} \tag{2.2}$$

using the property

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{2.3}$$

and the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ we get that,

$$\nabla^2 \mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0. \tag{2.4}$$

Using the separation of variables we get that one solution of (2.4) is

$$\mathbf{A}(\mathbf{r}, t) = \alpha(0) e^{-i\omega t} \mathbf{A}_0(\mathbf{r}).$$

The complex conjugate of this solution is also a solution of (2.4), so by combining the two solutions we get

$$\mathbf{A}(\mathbf{r}, t) = \alpha(t) \mathbf{A}_0(\mathbf{r}) + \alpha^*(t) \mathbf{A}_0^*(\mathbf{r}) \tag{2.5}$$

with $\alpha(t) = \alpha(0)e^{-i\omega t}$. Equation (2.5) is also a solution of (2.4) because it is a linear differential equation.$\alpha(t)$ and $\mathbf{A}_0(\mathbf{r})$ are the solutions to the following differential equations

$$\frac{\partial^2 \alpha(t)}{\partial t^2} = -\omega^2 \alpha(t) \tag{2.6a}$$

$$\nabla^2 \mathbf{A}_0(\mathbf{r}) = -\mathbf{k}^2 \mathbf{A}_0(\mathbf{r}) \tag{2.6b}$$

with $\|\mathbf{k}\| = \omega$. As we can see, $\alpha(t)$ and $\mathbf{A}_0(\mathbf{r})$ both obey the equation of a harmonic oscillator, one in time and the other in space. This gives us the formula of a monochromatic magnetic vector $\mathbf{A}$, which is different for each vector k. We will now denote this $\mathbf{A}_k$ and call it a field mode. $\mathbf{A}_\mathbf{k}$ is also equivalent to a harmonic oscillator, since it is the sum of two solutions of harmonic oscillators. To get a field independent of ω and $\mathbf{k}$ we have to sum over all possible $\mathbf{k}$.

$$\mathbf{A}(\mathbf{r}, t) = \sum_\mathbf{k} \mathbf{A}_\mathbf{k}(\mathbf{r}, t) = \sum_\mathbf{k} \left[\alpha(0) e^{-i\omega_\mathbf{k} t} \mathbf{A}_0(\mathbf{r}) + \alpha^*(0) e^{i\omega_\mathbf{k} t} \mathbf{A}_0^*(\mathbf{r})\right] \tag{2.7}$$

We can calculate the electric and magnetic fields using the Maxwell Equations ,

$$\mathbf{E}(\mathbf{r}, t) = -i \sum_\mathbf{k} \omega_\mathbf{k} \left[-\alpha(0) e^{-i\omega_\mathbf{k} t} \mathbf{A}_0(\mathbf{r}) + \alpha^*(0) e^{i\omega_\mathbf{k} t} \mathbf{A}_0^*(\mathbf{r})\right] \tag{2.8a}$$

$$\mathbf{B}(\mathbf{r}, t) = \sum_\mathbf{k} \left[\alpha(0) e^{-i\omega_\mathbf{k} t} \nabla \times \mathbf{A}_0(\mathbf{r}) + \alpha^*(0) e^{i\omega_\mathbf{k} t} \nabla \times \mathbf{A}_0^*(\mathbf{r})\right]. \tag{2.8b}$$

Now we will determine $\mathbf{A}_0$. We will consider a field bounded inside a box of volume L^3 that repeats itself throughout the space. In a sense we divide the space into boxes, were the field is equal in every box. We can later consider a free field in space by making $V \to \infty$, or, in practice, transforming the sum in (2.7) into an integral. This transforms equation (2.6b) into three equations with periodic boundary conditions

$$\nabla^2 A_{0i} + \mathbf{k}^2 A_{0i}.$$

So we get that $(k_x, k_y, k_z) = \frac{2\pi}{L}(n_x, n_y, n_z)$. We will consider $\mathbf{A}_{0i} = C_i e^{i\mathbf{k}\cdot\mathbf{r}} \hat{e}_\mathbf{k}$, which is the equation of a plane wave (normally used to characterize a free field, which is our objective). Also $C_x = C_y = C_z = C$ for symmetry reasons. Now, because V will be taken to infinity, $\mathbf{A}_0$ must be normalized inside the box. This gives that $C = \frac{1}{\sqrt{V}}$. So in the end we get

$$\mathbf{A}_\mathbf{k}(\mathbf{r}, t) = \frac{1}{\sqrt{V}} \left[\alpha(0) e^{-i(\omega_\mathbf{k} t - \mathbf{k}\cdot\mathbf{r})} + \alpha^*(0) e^{i(\omega_\mathbf{k} t - \mathbf{k}\cdot\mathbf{r})}\right] \hat{e}_\mathbf{k} \tag{2.9}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \left[\alpha(0) e^{-i(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{r})} + \alpha^*(0) e^{i(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{r})}\right] \hat{e}_{\mathbf{k}}. \tag{2.10}$$

In the limit $V \to \infty$ we get the result wanted, a free field. In order to do that we make the change

$$\sum_{\mathbf{k}} \longrightarrow \frac{V}{(2\pi)^3} \int d^3\mathbf{k}, \tag{2.11}$$

This comes from the definition of the Riemann sum,

$$\lim_{\Delta k \to 0} \sum_{l} f(k_l) \Delta k = \int dk f(k). \tag{2.12}$$

For a triple integral we have,

$$\sum_{lmn} f(\mathbf{k}_{lmn}) \Delta k_l^x \Delta k_m^y \Delta k_n^z. \tag{2.13}$$

We know that $\Delta k_l^i = k_{l+1}^i - k_l^i = \frac{2\pi}{L}$, so 2.13 becomes

$$\sum_{lmn} f(\mathbf{k}_{lmn}) \frac{(2\pi)^3}{V}. \tag{2.14}$$

So, taking the Riemann sum limit is equivalent to take $V \to \infty$,

$$\lim_{V\to\infty} \sum_{lmn} f(\mathbf{k}_{lmn}) \frac{(2\pi)^3}{V} = \int d^3\mathbf{k} f(\mathbf{k}) \tag{2.15}$$

Which is equivalent to

$$\lim_{V\to\infty} \sum_{lmn} f(\mathbf{k}_{lmn}) = \lim_{V\to\infty} \frac{V}{(2\pi)^3} \int d^3\mathbf{k} f(\mathbf{k}), \tag{2.16}$$

and so we get the limit 2.11.
With 2.11 we get

$$\mathbf{A}(\mathbf{r},t) = \frac{\sqrt{V}}{(2\pi)^3} \int d^3\mathbf{k} \left[\alpha(0) e^{-i(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{r})} + \alpha^*(0) e^{i(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{r})}\right] \hat{e}_{\mathbf{k}}. \tag{2.17}$$

Until now, we have considered classical fields. Now we want to make a jump to Quantum Field Theory. To do that we use the formalism of the second quantization and transport everything to Fock space, where fields become operators. Consequently, we can define two operators $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$ that have the following comutation properties [23]

$$[a_{\mathbf{k}\lambda}, a_{\mathbf{k}'\lambda'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'}, \quad [a_{\mathbf{k}}, a_{\mathbf{k}'}] = [a_{\mathbf{k}}^\dagger, a_{\mathbf{k}'}^\dagger] = 0 \tag{2.18}$$

These two operators, $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}^\dagger$, are called the annihilation and creation operators (or ladder operators, because they climb a "ladder" defined by the particle number), respectively, because they add or subtract particles with momentum $\mathbf{k}$ (and polarization λ) from a Fock quantum state, as shown in equations (2.19a) and (2.19b).

$$\hat{a}_{\mathbf{k}\lambda} |n_{\mathbf{k}\lambda}\rangle = \sqrt{n_{\mathbf{k}\lambda}}\, |(n-1)_{\mathbf{k}\lambda}\rangle, \ \hat{a}_{\mathbf{k}\lambda} |0_{\mathbf{k}\lambda}\rangle = 0 \tag{2.19a}$$

$$\hat{a}_{\mathbf{k}\lambda}^\dagger |n_{\mathbf{k}\lambda}\rangle = \sqrt{(n+1)_{\mathbf{k}\lambda}}\, |(n+1)_{\mathbf{k}\lambda}\rangle \tag{2.19b}$$

where $n_{\mathbf{k}}$ is the number of particles with $\mathbf{k}$ momentum in a state.

We want to create a field operator that creates a field state out of a vacuum state,

$$\hat{\mathbf{A}}(\mathbf{r},t)\left|0\right\rangle = \left|\mathbf{A}(\mathbf{r},t)\right\rangle, \tag{2.20}$$

that is proportional to the creation and annihilation operators. What we do to achieve this is to consider $\alpha = \sqrt{\frac{2\pi}{\omega(\mathbf{k})}}a$ and $\alpha^* = \sqrt{\frac{2\pi}{\omega(\mathbf{k})}}a^\dagger$. In the case of an EM field, the particles corresponding to the field modes are the photons. For this reason, we must consider a new quantum number λ, since a photon can have two polarizations. All things considered, equation (2.17) becomes

$$\mathbf{A}(\mathbf{r},t) = \sum_\lambda \int d^3\mathbf{k} \left(\frac{V}{(2\pi)^5\omega(\mathbf{k})}\right)^{1/2} \left[\hat{a}_{\mathbf{k}\lambda}e^{-i(\omega(\mathbf{k})t-\mathbf{k}\cdot\mathbf{r})} + \hat{a}^\dagger_{\mathbf{k}\lambda}e^{i(\omega(\mathbf{k})t-\mathbf{k}\cdot\mathbf{r})}\right]\mathbf{e}_{\mathbf{k}\lambda}. \tag{2.21}$$

Equations (2.8a) and (2.8b), using the properties in (2.18), become

$$\mathbf{E}(\mathbf{r},t) = i\sum_\lambda \int d^3\mathbf{k} \left(\frac{V\omega(\mathbf{k})}{(2\pi)^5}\right)^{1/2} \left[\hat{a}_{\mathbf{k}\lambda}e^{-i(\omega(\mathbf{k})t-\mathbf{k}\cdot\mathbf{r})} - \hat{a}^\dagger_{\mathbf{k}\lambda}e^{i(\omega(\mathbf{k})t-\mathbf{k}\cdot\mathbf{r})}\right]\mathbf{e}_{\mathbf{k}\lambda} \tag{2.22a}$$

$$\mathbf{B}(\mathbf{r},t) = i\sum_\lambda \int d^3\mathbf{k} \left(\frac{V}{(2\pi)^5\omega(\mathbf{k})}\right)^{1/2} \left[\hat{a}_{\mathbf{k}\lambda}e^{-i(\omega(\mathbf{k})t-\mathbf{k}\cdot\mathbf{r})} - \hat{a}^\dagger_{\mathbf{k}\lambda}e^{i(\omega(\mathbf{k})t-\mathbf{k}\cdot\mathbf{r})}\right]\mathbf{k}\times\mathbf{e}_{\mathbf{k}\lambda}. \tag{2.22b}$$

Now we calculate the Hamiltonian,

$$\hat{H}_{\mathbf{k}\lambda} = \frac{1}{4\pi}\int d^3\mathbf{r}\left(\mathbf{E}^2_{\mathbf{k}\lambda} + \mathbf{B}^2_{\mathbf{k}\lambda}\right), \tag{2.23}$$

for a field mode. Here we just have to substitute equations (2.22a) and (2.22b) and we get,

$$\hat{H}_{\mathbf{k}\lambda} = \frac{\omega(\mathbf{k})}{2}(\hat{a}_{\mathbf{k}\lambda}\hat{a}^\dagger_{\mathbf{k}\lambda} + \hat{a}^\dagger_{\mathbf{k}\lambda}\hat{a}_{\mathbf{k}\lambda}) = \omega(\mathbf{k})\left(\hat{a}^\dagger_{\mathbf{k}\lambda}\hat{a}_{\mathbf{k}\lambda} + \frac{1}{2}\right), \tag{2.24}$$

where the last step follows from the commutation relations in (2.18). Of course, the generalization for particles with any momentum $\mathbf{k}$ in continuous space is,

$$\hat{H} = \frac{V}{(2\pi)^3}\sum_\lambda \int d^3\mathbf{k}\omega(\mathbf{k})\left(\hat{a}^\dagger_{\mathbf{k}\lambda}\hat{a}_{\mathbf{k}\lambda} + \frac{1}{2}\right). \tag{2.25}$$

Using the eingevalues of $\hat{a}_{\mathbf{k}\lambda}$ and $\hat{a}^\dagger_{\mathbf{k}\lambda}$ we can see that the energy of a state with $n_{\mathbf{k}\lambda}$ particles, $E_{n_{\mathbf{k}\lambda}} = \left\langle n_{\mathbf{k}\lambda}\right|\hat{H}_{\mathbf{k}\lambda}\left|n_{\mathbf{k}\lambda}\right\rangle$ is the same as a Quantum Harmonic oscillator,

$$E_{n_{\mathbf{k}\lambda}} = \sum_\lambda\left(n_{\mathbf{k}\lambda} + \frac{1}{2}\right)\omega(\mathbf{k}), \tag{2.26}$$

just as we suspected.

The same happens in the classical case, if we take the canonical momentum and canonical coordinate to be

$$p(t) = \frac{k}{\sqrt{4\pi}}[\alpha(t) + \alpha^*(t)] \tag{2.27a}$$

$$q(t) = \frac{i}{\sqrt{4\pi}}[\alpha(t) - \alpha^*(t)] \tag{2.27b}$$

we get the Hamiltonian of the Classical Harmonic oscillator [23],

$$H_C = \frac{1}{2}(p^2 + w^2 q^2) = \frac{k^2}{2\pi}|\alpha(t)|^2. \tag{2.28}$$

This means that a field mode is mathematically equivalent to a harmonic oscillator.

What is interesting in the quantum field case is that if we consider the concept of vacuum as being the state with no particles (and with minimum energy field, or Zero Point Energy) we get, from (2.26), that the energy of a vacuum state is not zero, but $E_0 = \sum_\lambda \frac{\omega(\mathbf{k})}{2}$, for the field mode. Also, this state has certain statistical fluctuations in its field $\hat{\mathbf{A}}$, resulting from the random nature of quantum mechanics. This means that although $\langle 0| \hat{\mathbf{A}} |0\rangle = \left\langle \hat{\mathbf{A}} \right\rangle = 0$ (the field is zero in the vacuum state on average) it doesn't actually need to be zero all the time. We can see this the following way. A fluctuation of a variable around its average value is given by the formula

$$\delta E^2 = \left\langle \hat{\mathbf{A}}^2 \right\rangle - \left\langle \hat{\mathbf{A}} \right\rangle^2. \tag{2.29}$$

Since $\left\langle \hat{\mathbf{A}} \right\rangle = 0$, $\left\langle \hat{\mathbf{A}} \right\rangle^2 = 0$. But $\left\langle \hat{\mathbf{A}}^2 \right\rangle$ doesn't need to be zero and in fact, it won't be. This means that $\hat{\mathbf{A}}$ will have variations regardless of it being in a vacuum state with zero particles or a state full of electrons. An interpretation of this phenomenon is that these fluctuations are actually virtual particles created from nothing that exist during a time so small (compared to their own energy) that they violate the uncertainty principle, $\delta E \delta t \geq \frac{\hbar}{2}$, such that they cannot be detected and so they don't exist for all practical purposes and can violate fundamental laws, such as the conservation of energy. All space is filled with such particles. Finally for all possible $\mathbf{k}$ we get

$$E_0 = \frac{V}{(2\pi)^3} \sum_\lambda \int d^3\mathbf{k} \frac{\omega(\mathbf{k})}{2} \tag{2.30}$$

This means that the energy density of the vacuum is infinite since the integral in $\mathbf{k}$ goes from $-\infty$ to ∞ (all momentum space).

Of course the solution for the real scalar field can be calculated using the same procedure. Take

$$\hat{\phi}(\mathbf{r}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \hat{\phi}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{r}} \tag{2.31}$$

Here, we define the ladder operators the following way,

$$\hat{\phi}(\mathbf{k}, t) = \frac{1}{\sqrt{2\omega(\mathbf{k})}} \left(\hat{a}(\mathbf{k}) e^{-i\omega(\mathbf{k})t} + \hat{a}^\dagger(-\mathbf{k}) e^{i\omega(\mathbf{k})t} \right) \tag{2.32}$$

which is basically equivalent to what we did in (2.5). This works because even though $\hat{\phi}(\mathbf{r}, t)$ is a real field, $\hat{\phi}(\mathbf{k}, t)$ can be complex [24].
Then we can define

$$\hat{\phi}(\mathbf{r}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega(\mathbf{k})}} \left(\hat{a}(\mathbf{k}) e^{-i(\omega(\mathbf{k})t + \mathbf{k}\cdot\mathbf{r})} + \hat{a}^\dagger(\mathbf{k}) e^{i(\omega(\mathbf{k})t - \mathbf{k}\cdot\mathbf{r})} \right) \tag{2.33}$$

where, in the second term, we have made the variable change $\mathbf{k} \to -\mathbf{k}$. Finally, for the Hamiltonian we get,

$$\hat{H} = \frac{V}{(2\pi)^3} \int d^3\mathbf{k}\omega(\mathbf{k}) \left(\hat{a}^\dagger_\mathbf{k} \hat{a}_\mathbf{k} + \frac{1}{2} \right), \tag{2.34}$$

which gives the energy

$$E = \frac{V}{(2\pi)^3} \int d^3\mathbf{k}\omega(\mathbf{k}) \left(\hat{a}^\dagger_\mathbf{k} \hat{a}_\mathbf{k} + \frac{1}{2} \right), \tag{2.35}$$

and, finaly, the vacuum energy,

$$E_0 = \frac{V}{(2\pi)^3} \int d^3\mathbf{k} \frac{\omega(\mathbf{k})}{2}. \tag{2.36}$$

2.2 Casimir Effect: Physical Manifestation of the Vacuum

The most common setting of the Casimir effect consists of two perfectly conducting parallel (in the z direction by convention) plates that are infinite in the x and y directions and infinitesimal in the z direction (see fig. 2.1) or, in practice, plates that are much larger in the x and y directions than in the z direction. If you isolate the plates from any interaction they will still experience an attractive force out of nowhere! What happens is, in a pictorial way, virtual particles that appear from the vacuum states collide with the two parallel plates creating a pressure on them [23]. Since there are more vacuum states outside the plates than inside[1], there will be more virtual particles outside and hence the pressure will create the attractive force.

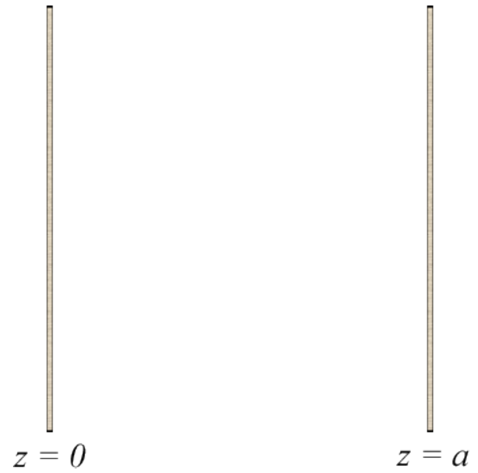


Figure 2.1: Position of the plates in the Casimir effect experiment

Of course the plates don't need to be perfectly conducting, they don't even need to be parallel or have this geometry (see [25] for other examples). Even the first experimental successful verification of this effect used a large sphere and a (close to) infinite plate (both almost perfect conductors) [26]. The effect is general, but it is easier to calculate it for parallel perfectly conducting plates.
The importance of this effect is, of course, to demonstrate that the Quantum Vacuum and, more specifically, the energy associated to it, is to be taken seriously, and not just be ignored as a mathematical eccentricity of a physical theory. There are other effects that are explained by the existence of the Quantum Vacuum [23] but the Casimir Effect is by far the most famous and cited.

In the rest of the section we shall calculate the Casimir pressure, the pressure that both plates experience. We will follow closely one of the proofs in [25]. It is not the easiest, but it is by far the most adaptable and the mathematical technique will be useful for the rest of the thesis.

[1] Of course there is no outside or inside since we don't have a closed surface, but we will understand inside as in between the plates and the outside as the reciprocal.

2.2.1 Derivation of the Casimir pressure

In order to calculate the pressure originated by the Quantum Vacuum, we introduce the symmetric Energy-Momentum tensor,

$$T^{\mu\nu} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}, \tag{2.37}$$

where T^{00} is the energy density; T^{01}, T^{02} and T^{03} are the momentum density in the x, y and z directions (respectively); T^{11}, T^{12} and T^{23} represent the shear pressure; and T^{11}, T^{22} and T^{33} terms represent the pressure, again in the x, y and z directions. Now, there are two paths we can take, we could calculate the Zero Point Energy inside and outside the plates and then calculate the force and the pressure in the plates, or we can calculate directly the pressure. Both methods are valid and not too different in terms of difficulty, but since the aim is the pressure, we shall calculate it directly from (2.37). We want to know T^{33} since we want the pressure only in the z direction (the other components will be equal inside and outside and hence they will cancel each other). In quantum mechanics language we intend to calculate the average of the operator ($\hat{T}^{33}$) that represents the physical quantity of pressure in the vacuum state, or:

$$P_z^{CAS} = \langle 0| \hat{T}^{33} |0\rangle . \tag{2.38}$$

For an electromagnetic field, (2.37) takes the form [27],

$$T^{\mu\nu} = \begin{pmatrix} \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) & [\mathbf{E}\times\mathbf{B}]_x & [\mathbf{E}\times\mathbf{B}]_y & [\mathbf{E}\times\mathbf{B}]_z \\ [\mathbf{E}\times\mathbf{B}]_x & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xy} \\ [\mathbf{E}\times\mathbf{B}]_y & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ [\mathbf{E}\times\mathbf{B}]_z & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{pmatrix}, \tag{2.39}$$

where σ_{ij} is the Maxwell stress tensor and it takes the form,

$$\sigma_{ij} = E_i E_j + B_i B_j - \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right)\delta_{ij}. \tag{2.40}$$

From (2.40) we get that,

$$\langle 0| \hat{T}^{33} |0\rangle = \langle 0| \frac{\mathbf{E}^2 + \mathbf{B}^2}{2} - E_z^2 - B_z^2 |0\rangle . \tag{2.41}$$

The fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ obey the Maxwell equations with a field source (a current $\mathbf{J}(\mathbf{r}, t)$),

$$\nabla \cdot \mathbf{E} = \rho, \tag{2.42a}$$
$$\nabla \cdot \mathbf{B} = 0, \tag{2.42b}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.42c}$$
$$\nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}. \tag{2.42d}$$

When we are in empty space with no source for the electromagnetic fields we get equations (2.1a) to (2.1d). But for now we are going to consider a mysterious source far away from our system, because for the formalism that we are going to use the mathematical structure of the source (even if it is zero) won't matter. Using equations (2.42c) and (2.42d) we get that,

$$(\nabla \times \nabla \times +\partial_t)\, \mathbf{E}(\mathbf{r}, t) = \partial_t \mathbf{J}(\mathbf{r}, t) \tag{2.43a}$$

$$(\nabla \times \nabla \times +\partial_t)\, \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{J}(\mathbf{r}, t). \tag{2.43b}$$

We obtained two differential equations, one for each field. To solve these differential equations we are going to use the formalism of the Green functions. For a scalar function $\phi(x)$ it works this way: if we have a linear differential operator $\hat{L}$ such that,

$$\hat{L}\phi(x) = J(x), \tag{2.44}$$

then there exists a funcion $G(x, x')$, called the Green function, that obeys a similar differential equation,

$$\hat{L}G(x, x') = \delta(x - x') \tag{2.45}$$

or in other words, it acts as a kind of inverse of the operator $\hat{L}$ when $x = x'$ (if we integrate (2.45) the result will be one). By knowing this function we can calculate $\phi(x)$ using,

$$\phi(x) = \int dx G(x, x') J(x'). \tag{2.46}$$

The same concept can be generalized for equations 2.43a and 2.43b, but now the Green functions are tensors in order to obey the equations,

$$\mathbf{E}(x) = \int d^4x \; \mathbf{G}_\mathbf{E}(x, x') \cdot (\partial_t \mathbf{J}(x')), \tag{2.47a}$$

$$\mathbf{B}(x) = \int d^4x \; \mathbf{G}_\mathbf{B}(x, x') \cdot (\nabla \times \mathbf{J}(x')), \tag{2.47b}$$

where here the x stands for the four-vector $x = (t, \mathbf{r})$. In our case, however, it is more useful to consider a different source, the polarization vector $\mathbf{P}$ such that,

$$\mathbf{J} = \partial_t \mathbf{P}. \tag{2.48}$$

where we are assuming that only the polarization current exists (there are no free charges). We can do this as long as we change the Green tensors and differential equations accordingly.

Let us start from the following equation,

$$\mathbf{E}(x) = \int d^4x \; \mathbf{G}_\mathbf{E}(x, x') \cdot \mathbf{P}(x'), \tag{2.49}$$

then, using equations (2.42c) and (2.42d) we get,

$$\nabla \times \mathbf{G}_\mathbf{E}(x, x') = -\partial_t \mathbf{\Psi}(x, x') \tag{2.50a}$$

$$\nabla \times \mathbf{\Psi}(x, x') = \mathbf{I}\partial_t \delta(x - x') + \partial_t \mathbf{G}_\mathbf{E}(x, x') \tag{2.50b}$$

where $\mathbf{\Psi}(x, x')$ is not the Green function of the magnetic field, just a tensor that we are calculating in reference to the source of the electric field (mathematically the sources of **E** and **B** are different, like in equations (2.47a) and (2.47b)). Also notice we have substituted **J** by the derivative of the delta function, that way we get the Green tensor in relation to the **P** source.

Knowing the Green tensor we can calculate **E** and **B**. But it can do more than that. Making the jump to quantum field theory, we regard a Green function as a propagator, it propagates a wave function from point x to point x' such that if we have a field operator $\hat{\phi}(x)$,

$$\langle 0| \hat{\phi}(x)\hat{\phi}(x') |0\rangle = \frac{G(x,x')}{i}, \tag{2.51}$$

as long as $t' > t$ [2]. This specific definition yields the Feynman propagator. This is also true for the electromagnetic field,

$$\langle 0| \mathbf{E}(x) \otimes \mathbf{E}(x') |0\rangle = \frac{\mathbf{G_E}(x,x')}{i} \tag{2.52a}$$

$$\langle 0| \mathbf{B}(x) \otimes \mathbf{B}(x') |0\rangle = \frac{\mathbf{G_B}(x,x')}{i} \tag{2.52b}$$

where $\mathbf{E}(x) \otimes \mathbf{E}(x')$ is the tensor product between the vectors, also called outer product or dyadic product, and it is equal to:

$$\mathbf{E}(x) \otimes \mathbf{E}(x') = \begin{pmatrix} E_x E_{x'} & E_x E_{y'} & E_x E_{z'} \\ E_y E_{x'} & E_y E_{y'} & E_y E_{z'} \\ E_z E_{x'} & E_z E_{y'} & E_z E_{z'} \end{pmatrix}. \tag{2.53}$$

Hence we get from (2.41) that

$$P_z^{CAS} = \lim_{x' \to x} \left(Tr\left[\mathbf{G_E}(x,x')\right] + Tr\left[\mathbf{G_B}(x,x')\right] - \left[\mathbf{G_E}(x,x')\right]_{33} - \left[\mathbf{G_B}(x,x')\right]_{33} \right). \tag{2.54}$$

Now we take the Fourier transform in the time coordinate[3],

$$\mathbf{G_E}(x,x') = \int d\omega e^{-i\omega(t-t')} \mathbf{G_E}(\mathbf{r},\mathbf{r}') \tag{2.55a}$$

$$\mathbf{\Psi}(x,x') = \int d\omega e^{-i\omega(t-t')} \mathbf{\Psi}(\mathbf{r},\mathbf{r}') \tag{2.55b}$$

and we get, from (2.50a) and (2.50b),

$$\nabla \times \mathbf{G_E}(\mathbf{r},\mathbf{r}') = i\omega \mathbf{\Psi}(\mathbf{r},\mathbf{r}') \tag{2.56a}$$

$$\nabla \times \mathbf{\Psi}(x,x') = -i\omega \mathbf{I}\delta(\mathbf{r}-\mathbf{r}') - i\omega \mathbf{G_E}(\mathbf{r},\mathbf{r}'). \tag{2.56b}$$

We now make the change in Green tensor, $\mathbf{G'_E} = \mathbf{G_E} + \mathbf{I}\delta(\mathbf{r}-\mathbf{r}')$ such that we get,

$$\nabla \cdot \mathbf{G'_E} = 0 \tag{2.57a}$$

$$\nabla \cdot \mathbf{\Psi} = 0, \tag{2.57b}$$

[2] It is called the time ordered product.

[3] $\mathbf{G_E}(\mathbf{r},\mathbf{r}')$ and $\mathbf{\Psi}(\mathbf{r},\mathbf{r}')$ also depend on ω, but these dependencies will remain implicit, since we will be applying more Fourier transforms such that leaving these dependencies explicit would become too cumbersome.

which leads to,

$$(\nabla^2+\omega^2)\mathbf{G'_E}(\mathbf{r},\mathbf{r'}) = -\nabla\times(\nabla\times\mathbf{I})\delta(\mathbf{r},\mathbf{r'}) \tag{2.58a}$$
$$(\nabla^2+\omega^2)\mathbf{\Psi}(\mathbf{r},\mathbf{r'}) = i\omega\nabla\times\mathbf{I}\delta(\mathbf{r},\mathbf{r'}), \tag{2.58b}$$

analogous to equations (2.43a) and (2.43b) where we used (2.3), which in this case gives $\nabla\times\nabla\times\mathbf{G'_E} = -\nabla^2\mathbf{G'_E}$. This means that we can calculate the Electric field's Green tensor and we will get the Magnetic field also. Since the separation on the plates is in the z direction and (as we stated) only the pressure in z will matter, we will take one final Fourier transform such that we will be left only with a dependence in z (and z'),

$$\mathbf{G'_E}(\mathbf{r},\mathbf{r'}) = \int \frac{d^2\mathbf{k}_\perp}{(2\pi)^2} e^{i\mathbf{k}_\perp\cdot(\mathbf{r}-\mathbf{r'})_\perp}\mathbf{g}(z,z'), \tag{2.59}$$

where $\mathbf{r}_\perp = (x,y)$ and similarly for $\mathbf{k}_\perp$ and the primed coordinates.

Finally, since we only know how to calculate scalar Green functions we devide (2.58a) into all the components of the $\mathbf{g}'(z,z')$ tensor. Fortunately the tensor we get from the equation (though, as we will see, not the $\mathbf{g}(z,z')$ tensor itself) is symmetric. For example, the equation for $g_{xy}(z,z')$ will be the same as for $g_{yx}(z,z')$, such that instead of nine equations we will have six! These equations are:

$$\left(\partial_z^2 - k^2 + \omega^2\right) g_{xx} = (\partial_z - k_y^2)\delta(z-z') \tag{2.60a}$$
$$\left(\partial_z^2 - k^2 + \omega^2\right) g_{yy} = (\partial_z - k_x^2)\delta(z-z') \tag{2.60b}$$
$$\left(\partial_z^2 - k^2 + \omega^2\right) g_{zz} = -k^2\delta(z-z') \tag{2.60c}$$
$$\left(\partial_z^2 - k^2 + \omega^2\right) g_{xy} = k_x k_y\delta(z-z') \tag{2.60d}$$
$$\left(\partial_z^2 - k^2 + \omega^2\right) g_{zx} = -ik_x\partial_z\delta(z-z') \tag{2.60e}$$
$$\left(\partial_z^2 - k^2 + \omega^2\right) g_{zy} = -ik_y\partial_z\delta(z-z') \tag{2.60f}$$

with $\lambda = k^2 - \omega^2$ and $k^2 = k_x^2 + k_y^2$ [4]. We can also get equation (2.54) just in terms of the $\mathbf{g}$ components by using (2.55a) and (2.59),

$$\begin{aligned} P_z^{CAS} = \lim_{z'\to z}\int\frac{d\omega}{2\pi}\int\frac{d^2\mathbf{k}_\perp}{(2\pi)^2}\frac{1}{2i\omega^2}[-(\omega^2-k^2)g_{zz} + (\omega^2+k_y^2)g_{xx} + (\omega^2-k_x^2)g_{yy} \\ +ik_y(\partial_z g_{yz} - \partial_{z'}g_{zy}) + ik_x(\partial_z g_{xz} - \partial_{z'}g_{zx}) + k_xk_y(g_{xy}+g_{yx}) + \partial_z\partial_{z'}(g_{xx}+g_{yy}) \\ -(\omega^2 - k^2 + ik_x + ik_y)\delta(z-z')]. \end{aligned} \tag{2.61}$$

Up until now everything was general, now we will need to input the information that there are parallel plates in order to get the boundary conditions necessary to solve the equations. That is precisely why we have chosen our plates to be perfect conductors, because it implies that the Electromagnetic field will be zero inside them or, in this case, it will be zero in $z=0$ and $z=a$. This can be translated formally into the following equation,

$$\mathbf{n}\times\mathbf{G_E}(\mathbf{r},\mathbf{r'})|_{z=0,a} = 0, \tag{2.62}$$

where $\mathbf{n} = (0,0,1)$ is the unit vector normal to the plates (pointing towards the interior of a plate). This means that the field will be zero in the directions orthogonal

[4] Here we have omitted the index $\perp$ so that we don't have a heavy notation.

to $\mathbf{n}$ but not in the directions parallel to $\mathbf{n}$, because close to the surface of a plate the field is not zero and since our plates are only a surface (remember they are infinitesimal in width) the field perpendicular to the plates won't be zero.

For $\mathbf{G'_E}(\mathbf{r},\mathbf{r'})$ equation (2.62) becomes

$$\mathbf{n}\times\mathbf{G'_E}(\mathbf{r},\mathbf{r'})|_{z=0,a} = \mathbf{n}\times\mathbf{I}\delta(\mathbf{r}-\mathbf{r'}). \tag{2.63}$$

We now calculate (2.63) explicitly and in terms of the tensor $\mathbf{g}(z,z')$,

$$\begin{pmatrix} -g_{yx} & -g_{yy} & -g_{yz} \\ g_{xx} & g_{xy} & g_{xz} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\delta(z-z') & 0 \\ \delta(z-z') & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.64}$$

For the remaining terms (the ones that are not in (2.64)) we will use the condition (2.57a), which is equivalent to

$$\begin{cases} ik_x g_{xx} + ik_y g_{yx} + \partial_z g_{zx} = 0 \\ ik_x g_{xy} + ik_y g_{yy} + \partial_z g_{zy} = 0 \\ ik_x g_{xz} + ik_y g_{yz} + \partial_z g_{zz} = 0 \end{cases} \tag{2.65}$$

together with (2.64) and evaluated at the surface of the plates gives

$$\begin{cases} \partial_z g_{zx}|_{z=0,a} = -ik_x\delta(z-z')|_{z=0,a} \\ \partial_z g_{zy}|_{z=0,a} = -ik_y\delta(z-z')|_{z=0,a} \\ \partial_z g_{zz}|_{z=0,a} = 0. \end{cases} \tag{2.66}$$

The boundary conditions of, for example, g_{zy} and g_{yz} are different and that's why $\mathbf{g}$ is not a symmetric tensor. The method we will use to solve the differential equations can be found in [28]. To illustrate the method we will solve the component xy,

$$(\partial_z^2+\lambda^2)g_{xy} = k_x k_y \delta(z-z'). \tag{2.67}$$

If we redefine the function such that $f(z,z') = \frac{g_{xy}(z,z')}{k_x k_y}$ then we have

$$(\partial_z^2+\lambda^2)f = \delta(z-z'). \tag{2.68}$$

We will assume that f is a step function of the form:

$$f = \begin{cases} Af_L(z,z') & \text{if } z<z' \\ Bf_R(z,z') & \text{if } z>z' \end{cases} \tag{2.69}$$

where f_L and f_R are the solutions of the homogeneous equation

$$(\partial_z^2+\lambda^2)f = 0, \tag{2.70}$$

where f_L is the solution calculated with the left boundary condition (in this case $f|_{z=0} = f_L|_{z=0} = 0$) and f_R is the solution with the boundary conditions in the right plate ($f|_{z=a} = f_R|_{z=a} = 0$). So we get

$$f = \begin{cases} A\sin\lambda z & \text{if } z<z' \\ B\sin\lambda(z-a) & \text{if } z>z' \end{cases} \tag{2.71}$$

The constants A and B are calculated by using the continuity conditions of f and $\partial_z f$,

$$\begin{cases} \lim_{\epsilon\to 0}\left[f(z'+\epsilon,z')-f(z'-\epsilon,z')\right]=0 \\ \lim_{\epsilon\to 0}\left[\partial_z f(z'+\epsilon,z')-\partial_z f(z'-\epsilon,z')\right]=1. \end{cases} \tag{2.72}$$

Solving the two equations we get,

$$\begin{cases} A=\frac{\sin\lambda(z'-a)}{\lambda\sin\lambda a} \\ B=\frac{\sin\lambda z'}{\lambda\sin\lambda a}. \end{cases} \tag{2.73}$$

and g_{xy} is

$$g_{xy}(z,z')=\begin{cases} \frac{k_x k_y}{\lambda\sin\lambda a}\sin\lambda z\sin\lambda(z'-a) & \text{if } z<z' \\ \frac{k_x k_y}{\lambda\sin\lambda a}\sin\lambda z'\sin\lambda(z-a) & \text{if } z>z' \end{cases} \tag{2.74a}$$

The rest of the components are derived in a similar fashion,

$$g_{xx}(z,z')=\begin{cases} \frac{k_x^2-\omega^2}{\lambda\sin\lambda a}\sin\lambda z\sin\lambda(z'-a) & \text{if } z<z' \\ \frac{k_x^2-\omega^2}{\lambda\sin\lambda a}\sin\lambda z'\sin\lambda(z-a) & \text{if } z>z' \end{cases}+\delta(z-z') \tag{2.74b}$$

$$g_{yy}(z,z')=\begin{cases} \frac{k_y^2-\omega^2}{\lambda\sin\lambda a}\sin\lambda z\sin\lambda(z'-a) & \text{if } z<z' \\ \frac{k_y^2-\omega^2}{\lambda\sin\lambda a}\sin\lambda z'\sin\lambda(z-a) & \text{if } z>z' \end{cases}+\delta(z-z') \tag{2.74c}$$

$$g_{zz}(z,z')=\begin{cases} \frac{-k^2}{\lambda\sin\lambda a}\cos\lambda z\cos\lambda(z'-a) & \text{if } z<z' \\ \frac{-k^2}{\lambda\sin\lambda a}\cos\lambda z'\cos\lambda(z-a) & \text{if } z>z' \end{cases} \tag{2.74d}$$

$$g_{xz}(z,z')=\begin{cases} \frac{ik_x}{\sin\lambda a}\sin\lambda z\cos\lambda(z'-a) & \text{if } z<z' \\ \frac{ik_x}{\sin\lambda a}\cos\lambda z'\sin\lambda(z-a) & \text{if } z>z' \end{cases} \tag{2.74e}$$

$$g_{yz}(z,z')=\begin{cases} \frac{ik_y}{\sin\lambda a}\sin\lambda z\cos\lambda(z'-a) & \text{if } z<z' \\ \frac{ik_y}{\sin\lambda a}\cos\lambda z'\sin\lambda(z-a) & \text{if } z>z' \end{cases} \tag{2.74f}$$

$$g_{zx}(z,z')=\begin{cases} \frac{ik_x}{\sin\lambda a}\cos\lambda z\sin\lambda(z'-a) & \text{if } z<z' \\ \frac{ik_x}{\sin\lambda a}\sin\lambda z'\cos\lambda(z-a) & \text{if } z>z' \end{cases} \tag{2.74g}$$

$$g_{zy}(z,z')=\begin{cases} \frac{ik_y}{\sin\lambda a}\cos\lambda z\sin\lambda(z'-a) & \text{if } z<z' \\ \frac{ik_y}{\sin\lambda a}\sin\lambda z'\cos\lambda(z-a) & \text{if } z>z' \end{cases}. \tag{2.74h}$$

Substituting all the components of $\mathbf{g}$ in (2.61) we get the pressure contribution from the inside, the space between the plates, by taking $z=0,a$ and we get,

$$P_{in}=\int\frac{d\omega}{2\pi}\int\frac{d^2\mathbf{k}_\perp}{(2\pi)^2}i\lambda\cot\lambda a, \tag{2.75}$$

where the $\delta(0)$ terms were ignored because they will cancel out since they are equal inside and outside the plates. Now we need to calculate the outside vacuum contribution. For the outside boundary conditions we will use the fact that far away from the plates the electromagnetic field will be approximately a free field, or in mathematical terms

$$\lim_{z\to+\infty} g_{ij}(z,z')=e^{i\lambda z}. \tag{2.76}$$

Doing all the calculations again and substituting in (2.61) we get that

$$P_{out} = \int \frac{d\omega}{2\pi} \int \frac{d^2\mathbf{k}_\perp}{(2\pi)^2} \lambda. \tag{2.77}$$

The pressure we measure in experiments is the difference between the two pressures (that's why the delta terms cancel out, they are equal inside and outside)

$$P_{CAS} = P_{in} - P_{out} = \int \frac{d\omega}{2\pi} \int \frac{d^2\mathbf{k}_\perp}{(2\pi)^2} i\lambda(\cot \lambda a + i). \tag{2.78}$$

Changing the variables to $\omega = i\zeta$ and $\lambda = il$ gives

$$P_{CAS} = -\int \frac{d\zeta}{2\pi} \int \frac{d^2\mathbf{k}_\perp}{(2\pi)^2} l(\coth la - 1). \tag{2.79}$$

In spherical coordinates, if we consider $\mathbf{l} = (\zeta, \mathbf{k}_\perp)$ we get

$$P_{CAS} = -\frac{8\pi}{(2\pi)^3} \int_0^{+\infty} dl \frac{l^3}{e^{2la} - 1}, \tag{2.80}$$

where the equality $\coth(la) - 1 = \frac{2}{e^{2la}-1}$ was used. The integral in (2.80) is known and gives $\frac{\pi^4}{120a^3}$. In the end we get,

$$P_{CAS} = -\frac{\pi^2}{240a^4}, \tag{2.81}$$

an attractive pressure in agreement with the literature [23,25,29] as well as with the experiments [26,30].

2.2.2 Casimir Effect and the reality of the Quantum Vacuum

The Casimir Effect still stands as the most known (and possibly the most important) proof of the physical implications of the Quantum Vacuum. R.L. Jaffe in [22] proposed that the Casimir Effect is only a consequence of the van der Waals interactions between the plates. He proposed an interaction to modulate the interaction between the plates of the form (in one dimension)

$$\mathcal{L}_{\text{int}} = \frac{1}{2} g\sigma(x)\phi^2(x), \tag{2.82}$$

where $\sigma(x) = \delta(x) + \delta(x + a)$ represents the position of the plates (or points in this case), g is the strength of the interaction (in the case of the van der Waals force it represents the fine structure constant α) and $\phi(x)$ is the field that creates the interaction between the plates. The interesting thing is that, when there is an interaction term like (2.82) in the Lagrangian of the system, equation (2.51) takes the form

$$\langle\Omega| \hat{\phi}(x)\hat{\phi}(x') |\Omega\rangle = \frac{G(x, x')}{i}, \tag{2.83}$$

where $|\Omega\rangle \neq |0\rangle$ unless there's no interaction [31]. And in [22] the Casimir pressure is calculated by using (2.83) and subtracting all the vacuum contributions in (2.51),

$$\frac{G_{\text{used}}(x, x')}{i} = \langle\Omega| \hat{\phi}(x)\hat{\phi}(x') |\Omega\rangle - \langle 0| \hat{\phi}(x)\hat{\phi}(x') |0\rangle \,, \tag{2.84}$$

and taking the limit $g \to +\infty$ (so that we have the boundary conditions $\phi(0) = \phi(a) = 0$),

$$P_{CAS} = -\lim_{g\to+\infty} \frac{\partial}{\partial a} E = -\frac{\pi}{24a^2}, \tag{2.85}$$

which is the correct value for the one dimensional Casimir Effect [24] [5]. The limit in g is equivalent to saying that [22] (using the simplistic Drude model)

$$\alpha \gg \frac{m}{4\pi n a^2} \tag{2.86}$$

where m and n are the effective mass and the total number of electrons in the plates. This equation remains valid for all experiments made for the Casimir Effect [22], so it isn't known if the effect would cease if (2.86) would not hold. The lesson here is that Jaffe calculates the Casimir Pressure without reference o the Vacuum Energy although there are still a few issues with the theory. For example, how good is (2.82) in modeling the van der Waals interaction the autor claims to be the real origin of the Casimir Effect and what exactly is the nature of the used field $\phi(x)$. The field $\phi(x)$ could be interpreted as the vacuum fluctuations (remember that even for vacuum the field isn't always zero) and that by subtracting the vacuum terms in (2.84) we are only subtracting the contribution from the vacuum without the plates, which is a standard technique to renormalize the Casimir Pressure - it is what we did in (2.78), the outside vacuum is mathematically equivalent to the vacuum with no plates. Jaffe's theory is also incompatible with the calculation of the Casimir effect of perfectly conducting spherical shells as done in [25] (see [22]). Jaffe is then far from disproving the Quantum Vacuum origin of the Casimir Effect but his article is certainly serious enough to be taken in consideration when discussing the reality and physical meaning of the Zero Point Energy.

[5] The standard result for 3D can be found, using the same method, in [32].

Chapter 3

Quantum Entanglement and Gaussian States

In this chapter we shall study how states in both discrete and continuous variable quantum system can be entangled. We will also study how to detect and measure entanglement on a theoretical level. The study of entanglement is very relevant for Cosmology today as the phenomenon is at the heart of many theories that try to explain both the Holographic Principle (chapter 4) and the Cosmological Constant (chapter 5). Special emphasis will be made on entanglement in Gaussian States and Quantum Harmonic Oscillator for they are the states relevant to this Thesis.

3.1 Introduction to Quantum Entanglement

Imagine two particles named a and b, with quantum states:

$$|\psi_a\rangle = c_1^a |1_a\rangle + c_2^a |2_a\rangle , \tag{3.1a}$$

$$|\psi_b\rangle = c_1^b |1_b\rangle + c_2^b |2_b\rangle , \tag{3.1b}$$

where the eigenstates $\{|j_k\rangle\}$ form a basis for the Hilbert space, H_k, of the subsystem (particle) k.

We will assume that both bases (a and b) are normalized,

$$c_1^k c_1^{k*} + c_2^k c_2^{k*} = 1, \tag{3.2}$$

and orthogonal,

$$\langle i_k | \, j_k\rangle = \delta_{ij}. \tag{3.3}$$

Now suppose we want to describe a system with both particles. Then we would have a Hilbert space of the form,

$$H_{ab} = H_a \otimes H_b, \tag{3.4}$$

of dimension $N = N_a \times N_b$, where $N_k = \text{dimension}(H_k)$.

The eigenstates of this space would be

$$|i_a j_b\rangle = |i_a\rangle \, |j_b\rangle \, . \tag{3.5}$$

Here, $|A\rangle|B\rangle$ denotes a tensor product, but not like the one described in chapter 2. This tensor product doesn't produce a tensor but a vector of higher dimension. From now on we will denote the tensor product in (3.5) just *tensor product* and the tensor product in (2.52a) as *dyadic product*.
If we give a matrix representation for some vectors $|u\rangle = \binom{u_1}{u_2}$ and $|v\rangle = \binom{v_1}{v_2}$ then:

$$\text{Tensor product: } |u\rangle|v\rangle = \begin{pmatrix} u_1v_1 & u_1v_2 & u_2v_1 & u_2v_2 \end{pmatrix}^T \tag{3.6a}$$

$$\text{Dyadic product: } |u\rangle\langle v| = \begin{pmatrix} u_1v_1^* & u_1v_2^* \\ u_2v_1^* & u_2v_2^* \end{pmatrix} \tag{3.6b}$$

Here we define the dyadic product with the conjugate of the second state because the states in Quantum Mechanics can represent complex vectors.

Now, to construct a state $|\phi_{ab}\rangle$ that describes the whole system, one might just try to perform a tensor product of (3.1a) and (3.1b):

$$\begin{aligned} |\phi_{ab}\rangle &= |\psi_a\rangle|\psi_b\rangle = (c_1^a|1_a\rangle + c_2^a|2_a\rangle) \otimes (c_1^b|1_b\rangle + c_2^b|2_b\rangle) \\ &= c_1^ac_1^b|1_a1_b\rangle + c_1^ac_2^b|1_a2_b\rangle + c_2^ac_1^b|2_a1_b\rangle + c_2^ac_2^b|2_a2_b\rangle\,. \end{aligned} \tag{3.7}$$

This is a perfectly valid state for the ab system. It doesn't, however, represent all the possible choices of $|\phi_{ab}\rangle$. The most general way we can write a state $|\phi_{ab}\rangle$ is

$$|\phi_{ab}\rangle = \sum_{ij} p_{ij}|i_aj_b\rangle\,. \tag{3.8}$$

For example, the state

$$|\phi_{ab}\rangle = p_1|1_a1_b\rangle + p_2|2_a2_b\rangle\,, \tag{3.9}$$

cannot be writen as (3.7) as long as $p_1, p_2 \neq 0$, no combination of constants c_i^a, c_j^b can give this expression.
States that can be written as a tensor product of the states of the subsystems (as in (3.7)) are called *separable states*. States that cannot be represented by (3.7) are *entangled states*.

In an entangled system, the composite state cannot be described by the properties of the subsystems alone. That is because the system a cannot be fully described without the system b (and vice versa) for they are now part of a larger system ab through which they are somewhat "connected".

A famous example is that of two electrons that are made to interact with one another so that they form a larger system. Because of the Pauli Exclusion Principle we know that they cannot have the same spin if there is nothing else that can distinguish them.[1] Then, by measuring the spin of one of the electrons we know automatically the spin of the other - they are entangled. The total system is represented by the state

$$|\psi\rangle = l_1|ud\rangle + l_2|du\rangle\,. \tag{3.10}$$

Notice how it is quite clear that this state cannot be described by a state of the form (3.7).

[1] No two identical fermions can occupy the same quantum state.

It is easy to see, at least for the case of two 2-dimensional subsystems, why a measurement on one of the subsystems would have such a "predictive" power on the other. In this case an entangled state must always be of the form $l_1 |ab\rangle + l_2 |cd\rangle$ with $a \neq c$ and $b \neq d$, otherwise it would be automatically separable (we would just factor the common substate). For instance: $l_1 |ab\rangle + l_2 |bb\rangle = (l_1 |a\rangle + l_2 |b\rangle) \otimes |b\rangle =$ separable.

Judging separability by eye is feasible in these simple examples, but for more complex cases there is a more formal way of defining separability. For this we use the Schmidt decomposition. It states that any state $|\phi_{ab}\rangle \in H_{ab}$ can be defined according to a basis of states $\{|\psi_{ki}\rangle\} \in H_k$ (where $k = a, b$) the following way [33, 34]

$$|\phi_{ab}\rangle = \sum_i^r p_i |\psi_{ai}\rangle |\psi_{bi}\rangle , \tag{3.11}$$

where $\{p_i\}$ are real and normalized, $\sum_i p_i^2 = 1$. The states $|\psi_{ki}\rangle$ are not necessarily the states in (3.1a), (3.1b), or in (3.5). The Hilbert spaces of the subsystems (H_k) can have different dimensions. In that case r will be equal to or smaller than the smallest dimension of the two subspaces. [33]
If (3.11) can be written as in (3.7) then we can decompose it using only one coefficient p_i,

$$|\phi_{ab}\rangle = p_i |\psi_{ai}\rangle |\psi_{bi}\rangle = |\psi_a\rangle |\psi_b\rangle . \tag{3.12}$$

The rule is then:

If the Schmidt decomposition can be written with only one coefficient $p_i = 1$ then the total state of the system is separable.
If there are more than one coefficients different than zero, the state is entangled.
If all the coefficients are different than zero and $p_i = \frac{1}{\sqrt{r}}$, for any i, then the state is said to be maximally entangled.

For a system of 2×2 dimensions, the maximally entangled states are known as the Bell states and are:

$$|B_\pm\rangle = \frac{1}{\sqrt{2}} (|1_a 1_b\rangle \pm |2_a 2_b\rangle) , \tag{3.13a}$$

$$|S_\pm\rangle = \frac{1}{\sqrt{2}} (|1_a 2_b\rangle \pm |2_a 1_b\rangle) . \tag{3.13b}$$

To write, for example, $|S_+\rangle$ in terms of the Schmidt decomposition we have

$$|S_+\rangle = \frac{1}{\sqrt{2}} (|1_a 2_b\rangle + |2_a 1_b\rangle) =$$

$$= \frac{1}{\sqrt{2}} |1_a\rangle (|2_b\rangle + |1_b\rangle) + \frac{1}{\sqrt{2}} (|2_a\rangle - |1_a\rangle) |1_b\rangle =$$

$$= \frac{1}{\sqrt{2}} |1_a\rangle |\psi_{b1}\rangle + \frac{1}{\sqrt{2}} |\psi_{a2}\rangle |1_b\rangle .$$

Hence it is maximally entangled.

Now it is useful to introduce the concept of density matrices. A density matrix of a state described by $|\psi\rangle$ is an operator defined by

$$\hat{\rho}_\psi = |\psi\rangle \langle\psi| \,, \tag{3.14}$$

which can also be used to describe the system (just as we would use $|\psi\rangle$).

It is easy to see, from the following property of the trace,

$$Tr(ABC) = Tr(CAB), \tag{3.15}$$

that the density matrix is normalized if

$$Tr(\hat{\rho}_\psi) = 1, \tag{3.16}$$

and that we can calculate the expectation value of an observable $\hat{O}$ the following way,

$$\left\langle \hat{O} \right\rangle = \langle\psi| \hat{O} |\psi\rangle = Tr(\hat{\rho}_\psi \hat{O}). \tag{3.17}$$

There are states in Quantum Mechanics that can only be described by a density matrix. This happens when there is a "classical"[2] uncertainty associated with lack of knowledge in which states the system at hand might be. Because of, say, the interaction of the system with the surroundings we may not know its exact state but only the probability with which it may be in one element of a set of states. This way, we cannot construct a state to represent the system, but we can construct a density operator

$$\hat{\rho} = \sum_i^M p_i |\phi_i\rangle \langle\phi_i| \,, \tag{3.18}$$

where $\{|\phi_i\rangle\}$ is the set of the possible states the system can be in (not necessarily orthogonal) and p_i are the respective probabilities.
We associate a "classic" probability to each state to get a probability distribution, the same way we do with an ensemble of states in Statistical Physics.
The states that can be described by a state $|\psi\rangle$ are called *pure states*, and states that cannot are called *mixed states*. Similarly to the entangled states, we say a system is maximally mixed if all the coefficients are of the form $p_i = \frac{1}{\sqrt{M}}$ - we have absolutely no way of guessing, even probabilistically, in what state the system will be in.

To elaborate more on the difference between pure and mixed states let's look again at the spin of an electron [3]. Let's say we have an electron in a state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|u\rangle + |d\rangle\right). \tag{3.19}$$

If measured, the spin of the electron will be, half of the times, $\frac{1}{2}$ (state $|u\rangle$) and, the other half, it will be $-\frac{1}{2}$ (state $|d\rangle$). It is not that the electron will half of the

[2]Here classic means that it doesn't come from the intrinsic randomness of Quantum Mechanics but simply from our lack of information about the whole system.

[3]When we talk about the spin of the electron we are talking about the z component of the total spin, $S_z = \pm\frac{1}{2}$.

times be in the state $|u\rangle$, no. It will always be in the state $|\psi\rangle$, it just so happens that, because of the inherent random nature of Quantum Mechanics, this state will give two different observations of the spin if repeatedly probed, or, equivalently, N different electrons in the state (3.19) will all be measured in either $|u\rangle$ or $|d\rangle$.

Now imagine we have another electron in the state $|u\rangle$. Now we want to measure its spin but we don't have a very good machine. The experimental setup will interact (even if slightly) with the electron and it will have a probability of altering its state (for example, a photon can collide with the electron and change its spin from $\frac{1}{2}$ to $-\frac{1}{2}$). Because of this, after a while, we don't know exactly what the state of the electron is. Let's assume that it can only be in either $|u\rangle$ or $|d\rangle$, for simplicity. It is not that it has a Quantum Mechanical probability of being in either of them, if that was the case we would know exactly its state, it would be something like (3.19). We just do not know in which state the electron is because of its interactions with the environment, which is very different.
We can measure the spin many times and see how often it has spin $\frac{1}{2}$ and how often it has spin $-\frac{1}{2}$ and calculate the probability of it being in each state. That way we can construct the density matrix,

$$\hat{\rho}_e = p_u |u\rangle \langle u| + p_d |d\rangle \langle d| . \tag{3.20}$$

An easy way to know if a system is pure or mixed is to see if $Tr(\hat{\rho}^2) = 1$. For pure states it is easy to see that $\hat{\rho} = \hat{\rho}^2$. Then if the density matrix is normalized both traces will be equal to one. For a mixed state this doesn't happen, and so $Tr(\hat{\rho}) < 1$. [35]

The definition of separable states in terms of the density matrix is similar: If we can write the density matrix as

$$\hat{\rho_{ab}} = \sum_i \omega_i \hat{\rho}_{ai} \otimes \hat{\rho}_{bi}, \tag{3.21}$$

with $\omega_i \geq 0$ and $\sum_i \omega_i = 1$, then it is separable, otherwise, it is entangled. [34]

For a system with $H_{ab} = H_a \otimes H_b$, the most general form of (3.18) is [36]

$$\hat{\rho}_{ab} = \sum_i p_i \left|\phi_{(ab)i}\right\rangle \left\langle\phi_{(ab)i}\right| , \tag{3.22}$$

or, writing in terms of the eigenstates (3.5),

$$\hat{\rho}_{ab} = \sum_{ijkl} c_{ijkl} \left(|i_a\rangle \langle k_a|\right) \otimes \left(|j_b\rangle \langle l_b|\right) . \tag{3.23}$$

The generalization for the case of $H = \bigotimes_i^N H_i$ should be easy enough.

Suppose we only want to deal with one of the subspaces (for example, subspace H_a), either because it is useful or we just don't have the necessary information about the other one. As we have seen, we can't always just ignore one of the subsystems since it may have influence over the other. What we can do is trace out all the degrees of freedom associated to the subspace we want to "ignore", that is, we do a partial trace over one of the subspaces (in this case H_b),

$$\hat{\rho}_{ab}^{T_b} = Tr_b(\hat{\rho}_{ab}) = \sum_j \langle j_b | \, \hat{\rho}_{ab} \, | j_b \rangle \,, \tag{3.24}$$

and we use $\hat{\rho}_{ab}^{T_b}$ to describe subsystem a. It is easy to show that, for the case of separable states, using (3.21),

$$Tr_b(\hat{\rho}_{ab}) = \hat{\rho}_a, \tag{3.25}$$

and the same goes for the Partial Trace over a. However, for entangled pure states [34, 35]

$$Tr_b(\hat{\rho}_{ab}) = \text{Mixed Density Matrix} \neq \hat{\rho}_a, \tag{3.26}$$

where here $\hat{\rho}_a$ denotes the density matrix constructed using the state $|\psi_a\rangle$ of a. Equation (3.26) tells us that though we may know all that we can know about the system itself, we don't have complete information about the subsystems. That translates to the reduced matrices being mixed matrices.

3.1.1 Entanglement Criteria and Measurements

We can measure the amount hidden/inaccessible information of our system by using the von Neumann entropy

$$S = -Tr\left(\hat{\rho} \log \hat{\rho}\right). \tag{3.27}$$

By hidden information we are talking about the degree of mixedness of our system. If it is a pure state the entropy is zero (we know the maximum we can know about the system), and if it is a maximally mixed state the entropy has its maximum value (we have no idea in which of the possible set of states the system might be - all probabilities are the same). [37]

From property (3.26), we can see that for an entangled pure state, the reduced density matrices of the corresponding subsystems will have a von Neumann entropy greater than zero. A maximally entangled state will maximize the "reduced" entropy [33, 34]:

$$S_{a,b} = -Tr\left(\hat{\rho}^{\,r}_{a,b} \log \hat{\rho}^{\,r}_{a,b}\right) \Rightarrow \begin{cases} S^{\text{Pure}}_{a,b} = 0 \\ S^{\text{Ent.}}_{a,b} > 0 \end{cases}, \tag{3.28}$$

where $\hat{\rho}^{\,r}_{a,b} = Tr_{a,b}(\hat{\rho})$.

Hence, with the von Neumann entropy we have constructed an entanglement measurement for a pure state. The von Neumann entropy can be related to the concept of mutual information (I) which measures the mutual dependence of the subsystems. For the case of a bipartite system ρ_{AB}, we have [38]

$$I = S(\hat{\rho}_A) + S(\hat{\rho}_B) - S(\hat{\rho}_{AB}). \tag{3.29}$$

For pure states, the mutual information becomes

$$I = S(\hat{\rho}_A) + S(\hat{\rho}_B) = 2S(\hat{\rho}_i), \tag{3.30}$$

which is just double the entropy in (3.28) and as such, it can also used to measure entanglement. The usefulness of the mutual information lies in that fact that, for example, it can be related also to observables other than the von Neumann entropy. [20]

But what if they are mixed states? In that case one of the most used entanglement criterion is the Peres-Horodecki (also called Partial Positive Transpose or PPT). To be able to present the criterion let us introduce the concept of a partial transpose. This basically consists in applying a transpose on only one of the subsystems.
Using (3.23) we can define the partial transpose:

$$\hat{\rho}_{ab}^{T_a} = \sum_{ijkl} c_{ijkl} \left(|i_a\rangle \langle k_a| \right)^T \otimes \left(|j_b\rangle \langle l_b| \right) = \sum_{ijkl} c_{ijkl} \left(|k_a\rangle \langle i_a| \right) \otimes \left(|j_b\rangle \langle l_b| \right), \tag{3.31}$$

and similarly for $\hat{\rho}_{ab}^{T_b}$.

Now, what the PPT criterion tells us is [33, 34]:

If a state is separable, then $\hat{\rho}_{ab}^{T_i} \geq 0$, with $i = a$ or b.

This criterion is only a necessary but not sufficient condition of separability. It proves if there is entanglement, but it doesn't disprove it. However it has been proven that if the system has dimensions of 2×2 or 2×3, then if the state is entangled $\hat{\rho}_{ab}^{T_i} \leq 0$. [33, 34]

Finally, an entanglement measure can be defined through the PPT criterion.
The negativity tells us how much an entangled state fails to have a positive $\hat{\rho}_{ab}^{T_i}$, and is defined the following way [34]

$$N = \frac{||\hat{\rho}^{T_i}||_1 - 1}{2}. \tag{3.32}$$

The norm $||A||_1 = Tr\left(\sqrt{A^\dagger A}\right)$ is known as the Schatten 1-norm.

The measurements and criteria presented here are the ones most relevant to this thesis. For more information on the subject, the reader is referred to [33].

3.2 Entanglement in Continuous Variable Systems

3.2.1 Gaussian states

In this subsection we shall introduce the concept of Gaussian States as well as ways to detect and measure their entanglement. It is useful to study these states since they are the simplest continuous variable states with the most developed study of entanglement. Also, Gaussian States are very common in physics. Coherent States (e.g. LASERS), Thermal States (e.g. black body radiation) and Squeezed States (e.g. non-linear optical phenomena) are all Gaussian States. [38,39] Even the vacuum of a free field is a Gaussian States.

Before we define what a Gaussian State is, we need to introduce some important concepts.

Suppose that we have a system with n modes, each with a annihilation and creation operators, $\hat{a}_k$ and $\hat{a}_k^\dagger$. We define the canonical conjugate operators

$$\hat{x}_k = \frac{1}{\sqrt{2}}\left(\hat{a}_k^\dagger + \hat{a}_k\right) \tag{3.33a}$$

$$\hat{p}_k = \frac{i}{\sqrt{2}}\left(\hat{a}_k^\dagger - \hat{a}_k\right). \tag{3.33b}$$

We also define the system vector ξ which contains every canonical operator of the system,

$$\xi = (x_1, p_1, ..., x_n, p_n)^T. \tag{3.34}$$

From (2.18) we can deduce that

$$[\xi_\alpha, \xi_\beta] = i\Omega_{\alpha\beta}, \tag{3.35}$$

where

$$\Omega = \bigoplus_{i=1}^{n} J = \left.\begin{pmatrix} J & 0 & 0 & 0 \\ 0 & J & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & J \end{pmatrix}\right\} n \qquad , \tag{3.36}$$

with

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.37}$$

and the 0's in (3.36) represent 2×2 matrices with all elements being zero.
For example, for $n = 2$, (3.36) becomes

$$\Omega = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3.38}$$

In (2.19a) and (2.19b) the states of the Fock space were defined. In here we are going to define other types of states, the Coherent States, which are eigenstates of the annihilation operator

$$\hat{a}_k \left|\alpha_k\right\rangle = \alpha_k \left|\alpha_k\right\rangle, \tag{3.39}$$

with $\alpha_k \in \mathbb{C}$. We can define the Weyl Displacement Operator [38, 39]

$$\hat{D}(\alpha_k) = e^{\alpha_k \hat{a}_k^\dagger - \alpha_k^* \hat{a}_k} \tag{3.40}$$

that creates a Coherent State from the vacuum state

$$\left|\alpha_k\right\rangle = \hat{D}(\alpha_k)\left|0_k\right\rangle. \tag{3.41}$$

With ξ we can define a general Displacement Operator

$$\hat{D}(X) = e^{i\xi^T \Omega X} \tag{3.42}$$

were $X \in \mathbb{R}^{2n}$ and $|X\rangle$ is a generalized state obtained by applying the new operator to the global vacuum:

$$|X\rangle = \hat{D}(X) |0\rangle . \tag{3.43}$$

This state will also be and eigenvector of the annihilation operators [38, 39]

$$\hat{a}_k |X\rangle = (X_k + iX_{k+1}) |X\rangle = \alpha_k |X\rangle . \tag{3.44}$$

This means that $X = (\langle \hat{x}_1 \rangle_\alpha , \langle \hat{p}_1 \rangle_\alpha , ..., \langle \hat{x}_n \rangle_\alpha , \langle \hat{p}_n \rangle_\alpha)^T$, where $\langle \hat{x}_k \rangle_\alpha = \langle \alpha_k | \hat{x}_k | \alpha_k \rangle$.
Finally we shall define the Characteristic function

$$\chi(X) = Tr(\hat{\rho}\hat{D}(X)), \tag{3.45}$$

whose importance will be shown in a minute.

One more concept we need to define: the Wigner distribution (or Wigner function), $W(\mathbf{r}, \mathbf{p})$. The Wigner distribution is a quasiprobability distribution, in the sense that it is normalized,

$$\int d\mathbf{r} \int d\mathbf{p}\ W(\mathbf{r}, \mathbf{p}) = 1, \tag{3.46}$$

but it is not necessarily always positive. It basically represents the Quantum Mechanical particle in the phase space, where a picture of Quantum Mechanics, alternative to (for example) the Schrödinger picture, can be constructed.
For example, the probability of a particle being inside a box B can be calculated through [40]:

$$P(B) = \int_B d\mathbf{r} \int d\mathbf{p}\ W(\mathbf{r}, \mathbf{p}). \tag{3.47}$$

We can also define the expectation value of an operator $\hat{O}$ in this picture,

$$\left\langle \hat{O} \right\rangle = \int d\mathbf{r} \int d\mathbf{p}\ O(\mathbf{r}, \mathbf{p}) W(\mathbf{r}, \mathbf{p}), \tag{3.48}$$

where $O(\mathbf{r}, \mathbf{p})$ is obtained through the Wigner transformation

$$O(\mathbf{r}, \mathbf{p}) = \int d\mathbf{u} \, \langle \mathbf{r} - \mathbf{u} | \, \hat{O} \, | \mathbf{r} + \mathbf{u} \rangle \, e^{2i\mathbf{p}\cdot\mathbf{u}}. \tag{3.49}$$

For a pure state with wave function $\psi(\mathbf{r})$, the Wigner distribution takes the form [38, 40]

$$W(\mathbf{r}, \mathbf{p}) = \frac{1}{\pi^n} \int d\mathbf{u}\ \psi(\mathbf{r} - \mathbf{u}) \psi^*(\mathbf{r} + \mathbf{u}) e^{2i\mathbf{p}\cdot\mathbf{u}}, \tag{3.50}$$

which, for a state represented by $\hat{\rho}$ generalizes to

$$W(\mathbf{r}, \mathbf{p}) = \frac{1}{\pi^n} \int d\mathbf{u} \, \langle \mathbf{r} + \mathbf{u} | \, \hat{\rho} \, | \mathbf{r} - \mathbf{u} \rangle \, e^{2i\mathbf{p}\cdot\mathbf{u}}. \tag{3.51}$$

We can also use the Wigner distribution for our continuous variable system, expressed in terms of ξ and X [38, 39]

$$W(X) = \frac{1}{\pi^n} \int dY\ \chi(Y) e^{iY^T \Omega X}, \tag{3.52}$$

where $Y \in \mathbb{R}^{2n}$.

Finally, let us define what a Gaussian State is [38]:

A Gaussian State is a state whose Wigner distribution and Characteristic function are Gaussian functions.

An example of this is, as already mentioned, the vacuum state of a field in free space (in this case 1-dimensional) [38]

$$\psi_0(x) = \frac{1}{\pi^{\frac{1}{4}}} e^{-\frac{x^2}{2}}. \tag{3.53}$$

From (3.50) we get

$$W_0(x,p) = \frac{1}{\pi} e^{-x^2-p^2}, \tag{3.54}$$

which is clearly Gaussian.

An important feature that comes from the definition is that Gaussian States are completely described by what is called as the *first* and *second canonical moments* [38, 39],

$$d_\alpha = \langle \xi_\alpha \rangle, \tag{3.55a}$$

$$V_{\alpha\beta} = \frac{1}{2} \langle \{\xi_\alpha, \xi_\beta\} \rangle - \langle \xi_\alpha \rangle \langle \xi_\beta \rangle, \tag{3.55b}$$

where $\{A, B\} = AB + BA$ is the anticommutator.

That is because the general form of the Characteristic function and the Wigner distribution depend on the first and second moments and the variable vector X:

$$\chi(Y) = e^{-\frac{1}{4}Y^T \Omega \mathbf{V} \Omega^T Y - i(\Omega d)^T Y} \tag{3.56a}$$

$$W(X) = \frac{1}{\pi^n} \frac{1}{\sqrt{\det(\mathbf{V})}} e^{(X-d)^T \mathbf{V}^{-1} (X-d)}. \tag{3.56b}$$

The matrix $\mathbf{V}$, formed by the second moments, is called the Covariance Matrix. For it to be a physically valid Covariance Matrix it must respect the inequality [38, 39]

$$\mathbf{V} + \frac{i}{2}\Omega \geq 0, \tag{3.57}$$

which represents the uncertainty principle for Gaussian States.

Finally, because the first moments cannot affect properties related to the state being entangled or separable, since they can be arbitrarily adjusted by *local unitary operations* [41], and since for the rest of the thesis we will only be concerned with the systems in the vacuum state, when $\langle \xi \rangle = \langle 0 | \xi | 0 \rangle = 0$ [42], we will consider, from now on, $d_\alpha = 0$, and

$$V_{\alpha\beta} = \frac{1}{2} \langle \{\xi_\alpha, \xi_\beta\} \rangle. \tag{3.58}$$

Peres-Horodecki criterion for a bipartite state

The Peres-Horodecki (PPT) criterion was first implemented to the formalism of Gaussian States by R. Simon in [43] and it will be primarily this article that we will use as the basis for this section.

In a system described by a Wigner distribution, the transpose of a density matrix is equivalent to a mirror reflection in the phase space [43],

$$\hat{\rho} \rightarrow \hat{\rho}^{T} \Leftrightarrow W(q,p) \rightarrow W(q,-p). \tag{3.59}$$

Consequently, a partial positive transform on one of the substates of the system will result on a mirror reflection on its phase space alone, or in the case of a bipartite system,

$$\hat{\rho} \rightarrow \hat{\rho}^{PT_2} \Leftrightarrow W(q_1,p_1,q_2,p_2) \rightarrow W(q_1,p_1,q_2,-p_2), \tag{3.60}$$

and similarly for $\hat{\rho}^{PT_1}$.

So here PPT criterion can be stated as:

For a separable two-state system, performing a mirror reflection on the phase space of one of the states must yield a Wigner function that is physically valid.

As we have already seen, for a Gaussian State to be physical, it is necessary and sufficient that it obeys the uncertainty relation (3.57). The mirror reflection on, for example, state 2 will be equivalent to performing the following transformation,

$$\xi \rightarrow \Lambda\xi \tag{3.61}$$

with

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.62}$$

With this transformation we will obtain a new pseudo-Covariance Matrix that we will denote $\tilde{\mathbf{V}} = \Lambda\mathbf{V}\Lambda$. It will only be a true Covariance Matrix if it obeys (3.57), that is, if $\mathbf{V}$ represents a separable state.

We have now translated the PPT criterion to the language of Gaussian States, but we have not yet found a practical way to use it. For that we need to introduce Sympletic Groups.

For an n-dimensional vector ξ if there exists a transformation S such that

$$\xi \rightarrow \xi' = S\xi \Rightarrow \left[\xi'_\alpha, \xi'_\beta\right] = \left[\xi_\alpha, \xi_\beta\right], \tag{3.63}$$

then S is called a Sympletic transformation. All of the possible transformations of the type S form the Sympletic Group $S_p(2n,\mathbb{R})$ and they are represented by (n×n)-dimensional real matrices. It can also be proved that if $S \in S_p(2n,\mathbb{R})$ then $S^T \in S_p(2n,\mathbb{R})$ and that $\mathrm{Det}(S) = 1$. [44]

For a Gaussian State the first part of (3.63) is equivalent to

$$\mathbf{V} \rightarrow \mathbf{V}' = S\mathbf{V}S^T \tag{3.64}$$

and the second part is equivalent to

$$S\Omega S^T = \Omega. \tag{3.65}$$

The consequence of (3.65) is that

$$S\left(\mathbf{V} + \frac{i}{2}\Omega \geq 0\right)S^T \Leftrightarrow \mathbf{V}' + \frac{i}{2}\Omega \geq 0, \tag{3.66}$$

which means that a Sympletic transformation of a valid Covariance Matrix gives us back another valid Covariance Matrix.

If we perform a partial mirror reflection such that $\mathbf{V} \rightarrow \tilde{\mathbf{V}}$ then the uncertainty relation will only be respected for transformation of the form $S \in S_p(n, \mathbb{R}) \otimes S_p(n, \mathbb{R}) \subset S_p(2n, \mathbb{R})$ [43]. That is because only separable states will respect the relation and as such we can only perform sympletic transformations on each of the subsystems in separate to preserve the structure of $\mathbf{V}$. Such transformations are called local transformations and we denote them by S_{local}. For a bipartite system,

$$S_{\text{local}} = \begin{pmatrix} S_1 & 0 \\ 0 & S_1 \end{pmatrix}, \tag{3.67}$$

where $S_1 J S_1^T = S_2 J S_2^T = J$.

Using the matrix $\mathbf{V}$ in the form

$$\mathbf{V} = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix},$$

then,

$$\mathbf{V} \rightarrow S_{\text{local}}\mathbf{V}S_{\text{local}}^T \Leftrightarrow \begin{cases} A \rightarrow S_1 A S_1^T \\ B \rightarrow S_2 B S_2^T \\ C \rightarrow S_1 C S_2^T \end{cases} \tag{3.68}$$

The following quantities are sympletic invariants: $\text{Det}(A)$, $\text{Det}(B)$, $\text{Det}(C)$ and $Tr(AJCJBJC^T)$ (and, consequently, $\text{Det}(\mathbf{V})$). Proof:

$$\text{Det}(A') = \text{Det}(S_1 A S_1^T) = \text{Det}(S_1)\text{Det}(A)\text{Det}(S_1^T) = \text{Det}(A),$$

and the same for B and C.

$$Tr(A'JC'JB'JC^{T'}) = Tr(S_1 A S_1^T J S_1 C S_2^T J S_2 B S_2^T J S_2 C^T S_1^T) = Tr(AJCJBJC^T),$$

because $S_i J S_i^T = J$. Finally, $\text{Det}(S_i) = 1$ is a consequence of $\text{Det}(S) = 1$.

Now, let us try to derive the PPT criterion for a specific easy case. Any Covariance Matrix can through a sympletic transformation become a diagonal matrix of the form [39],

$$\mathbf{V} = S^T \boldsymbol{\nu} S, \tag{3.69}$$

where,

$$\boldsymbol{\nu} = \bigoplus_{i=1}^{n} \begin{pmatrix} \nu_i & 0 \\ 0 & \nu_i \end{pmatrix} \tag{3.70}$$

and, for the case of two states,

$$\boldsymbol{\nu} = \begin{pmatrix} \nu_1 & 0 & 0 & 0 \\ 0 & \nu_1 & 0 & 0 \\ 0 & 0 & \nu_2 & 0 \\ 0 & 0 & 0 & \nu_2 \end{pmatrix}. \tag{3.71}$$

Since a sympletic transformation was used to transform $\mathbf{V}$ into ν, it means that ν is also a valid Covariance Matrix of the system. The values ν_i are called the sympletic eigenvalues and, because of (3.57), $\nu_i \geq 0$.

An equivalent statement to saying $\mathbf{V} + \frac{i}{2}\Omega \geq 0$ is saying that the eigenvalues of $\mathbf{V} + \frac{i}{2}\Omega$ are all greater than zero. In the case of $\boldsymbol{\nu}$ this becomes $\nu_i \geq \frac{1}{2}$.
We can also relate the sympletic eigenvalues to any general Covariance Matrix $\mathbf{V}$. The eigenvalues of $|i\Omega\mathbf{V}|$[4] will be equal to the eigenvalues of $\boldsymbol{\nu}$. The proof of this statement is the following [39]:

The eigenvalues of $i\Omega\boldsymbol{\nu}$ are $\pm\nu_i$. This is easy to show for a two-state system (the case of interest) by just multiplying the matrices and calculating the resulting eigenvalues. Now, because (3.69) and the definition of a sympletic transformation,

$$i\Omega\boldsymbol{\nu} = i\Omega S^T \mathbf{V} S = S^{-1} i\Omega \mathbf{V} S. \tag{3.72}$$

But it is a known result that any two matrices A and B that have a transformation S between themselves,

$$S^{-1}AS = B, \tag{3.73}$$

are called similar matrices and share the same eigenvalues. (As a matter of fact they will share many of the properties that define a matrix.) [45]
Hence, the eigenvalues of $|i\Omega\mathbf{V}|$ will be equal to the eigenvalues of $\boldsymbol{\nu}$.

For a two-state system, it is easy to show that the two sympletic eigenvalues are,

$$\nu_{\pm} = \sqrt{\frac{\Delta \pm \sqrt{\Delta^2 - 4\mathrm{Det}\left(\mathbf{V}\right)}}{2}}, \tag{3.74}$$

where $\Delta = \mathrm{Det}\left(A\right) + \mathrm{Det}\left(B\right) + 2\mathrm{Det}\left(C\right)$.

Using (3.74) in $\nu_i \geq \frac{1}{2}$ and noting that $\nu_+ \geq \nu_-$, we get that the uncertainty relation (3.57) is equivalent to,

$$\mathrm{Det}(A) + \mathrm{Det}(B) + 2\mathrm{Det}(C) - \left(\frac{1}{4} + 4\mathrm{Det}(\mathbf{V})\right) \leq 0. \tag{3.75}$$

Since everything in this equation is sympletic invariant, it will hold true for any Covariance Matrix (of a bipartite state of course).

[4] Here the notation $|A|$ denotes the a matrix that whose eigenvalues are the absolute value of the eigenvalues of A.

Finally, to get the PPT criterion, we just need to find an equivalent relation for a state after the partial transpose ($\tilde{\mathbf{V}}$). It is easy to show that from the quantities in (3.75), the only one altered by the transformation (3.61) is $\text{Det}(\tilde{C}) = -\text{Det}(C)$. So, under that transformation (3.75) becomes

$$F = \text{Det}(A) + \text{Det}(B) - 2\text{Det}(C) - \left(\frac{1}{4} + 4\text{Det}(\mathbf{V})\right) \leq 0, \tag{3.76}$$

where we now define the function F such that when it is positive the states are entangled.
It was proven in [43] that for the bipartite case this condition is both necessary and sufficient condition to test the separability of $\mathbf{V}$ (as was in the original case in the original PPT criterion).

In [21] it was proven that, for a bipartite system of Quantum Harmonic Oscillator, (3.76) is still a valid test for the entanglement. But now $F \leq 0$ is only a necessary (but no sufficient) condition of separability, meaning that we can use (3.76) to prove that there is entanglement, but we cannot use it to prove that the states are separable. This is because for non-Gaussian States, the Simon condition in (3.76) is only one of several (infinite) conditions the system needs to obey in order to be separable, which also means that, although we can still say that, according to Simon criterion, state $|\psi_a\rangle$ is more entangled than state $|\psi_b\rangle$, this could very well be the reverse in another test of [21].

Negativity for a bipartite state

We can also define the Negativity associated with (3.76), but to do so, we first need to calculate the density matrix ρ for a Gaussian States.

The most general way we can describe a Gaussian States is: [39, 46]

$$\hat{\rho} = \bigotimes_i \hat{\rho}_i. \tag{3.77}$$

where $\hat{\rho}_i$ are thermal density matrices corresponding to each mode of the Gaussian States. A thermal density matrix is defined as

$$\hat{\rho}_{\text{thermal}} = \frac{e^{-\beta\hat{H}}}{Tr\left[e^{-\beta\hat{H}}\right]}, \tag{3.78}$$

or, in a Fock space with a basis $\{|n\rangle\}$

$$\hat{\rho}_{\text{thermal}} = \left(1 - e^{-\beta\omega}\right)\sum_{n=0}^{\infty} e^{-\beta\omega n}\,|n\rangle\langle n|\,. \tag{3.79}$$

What we can do now, is to write it in terms of the Covariance Matrix $\boldsymbol{\nu}$ so that we can define the Negativity for Gaussian States.

The average number of particles in a Thermal State is,

$$\bar{n} = Tr(\hat{\rho}n) =$$

$$= (1 - e^{-\beta\omega}) \sum_{n=0}^{\infty} n e^{-\beta\omega n} =$$

$$= (1 - e^{-\beta\omega}) \left(-\frac{1}{\omega} \frac{d}{d\beta} \right) \sum_{n=0}^{\infty} e^{-\beta\omega n}, \tag{3.80}$$

where it is easy to prove that $\sum_{n=0}^{\infty} e^{-\beta\omega n} = \frac{1}{1-e^{-\beta\omega n}}$.

Equation (3.80) is then simplified to,

$$\bar{n} = \frac{1}{e^{\beta\omega n} - 1}. \tag{3.81}$$

This means that,

$$e^{-\beta\omega} = \frac{\bar{n}}{1 + \bar{n}}, \tag{3.82}$$

and (3.79) becomes,

$$\hat{\rho} = \frac{1}{\bar{n}+1} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{\bar{n}+1} \right)^n |n\rangle \langle n| \,. \tag{3.83}$$

Now, going back to the matrix $\boldsymbol{\nu}$, since it is a valid Covariance Matrix for the system, (3.58) still applies [46, 47] which means that

$$\nu_i = \frac{1}{2} \langle n| \{\hat{x}_i, \hat{x}_i\} |n\rangle = \frac{1}{2} \langle n| \{\hat{p}_i, \hat{p}_i\} |n\rangle \,, \tag{3.84}$$

where annihilation and creation operators are not necessarily the same as for $\mathbf{V}$ since the sympletic transformation perform a change of basis on matrices. Since $\hat{x}_i = \frac{1}{\sqrt{2}} \left(\hat{a}_i + \hat{a}_i^\dagger \right)$, (3.84) becomes,

$$\nu_i = \frac{1}{2} \langle n| \left\{ \hat{a}_i, \hat{a}_i^\dagger \right\} |n\rangle = \bar{n}_i + \frac{1}{2}, \tag{3.85}$$

where we used the property $\left[\hat{a}_i, \hat{a}_j^\dagger \right] = \delta_{ij}$. The same result would have been obtained if we had used the canonical momentum $p_i = \frac{1}{\sqrt{-2}} \left(\hat{a}_i - \hat{a}_i^\dagger \right)$.

Now, if we input (3.85) into (3.79), we get the density matrix for each state of the two-state system

$$\hat{\rho}_i = \frac{2}{2\nu_i + 1} \sum_{n=0}^{\infty} \left(\frac{2\nu_i - 1}{2\nu_i + 1} \right)^n |n\rangle_i \langle n|_i. \tag{3.86}$$

It is easy to show that for a diagonal matrix (which our ρ is), the norm in (3.32) is equal to,

$$||A_{\text{diagonal}}||_1 = \begin{pmatrix} |a_1| & 0 & \cdots & 0 \\ 0 & |a_2| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |a_n| \end{pmatrix}. \tag{3.87}$$

In the case of $\hat{\rho}$,

$$||\hat{\rho}^{T_i}||_1 = \prod_i ||\hat{\rho}_i^{T_i}||_1, \tag{3.88}$$

where,

$$||\hat{\rho}_i^{T_i}||_1 = \frac{2}{2\tilde{\nu}_i + 1} \sum_{n=0}^{\infty} \left(\frac{|2\tilde{\nu}_i - 1|}{2\tilde{\nu}_i + 1} \right)^n, \tag{3.89}$$

and $2\tilde{\nu}_i$ denotes the sympletic eigenvalues after the PPT was performed on the Covariance Matrix $\mathbf{V}$.

If $\nu_i \geq \frac{1}{2}$ (which, if you remember from last section, is the PPT condition of separability) then $||\hat{\rho}_i^{T_i}||_1 = 1$, and if $\nu_i < \frac{1}{2}$ then $||\hat{\rho}_i^{T_i}||_1 = \frac{1}{\tilde{\nu}_i}$.

Finally, it can be proved that $\tilde{\nu}_+ \geq \frac{1}{2}$ always [41], which means that,

$$N = \max \left[0, \frac{1 - 2\tilde{\nu}_-}{4\tilde{\nu}_-} \right]. \tag{3.90}$$

3.2.2 Infinite Mode Generalization using Detector Box Model

Although the techniques described in the last subsection are only applicable to Gaussian States, there is still a "trick" we can do to try to apply them to more general types of states. A method was developed in [48] (that we shall call Detector Box model) were a field operator $\hat{\phi}(x)$[5] is averaged over two boxes of volume V,

$$\hat{\phi}_V(\mathbf{r}, t) = \frac{1}{V} \int_V d\mathbf{u}\, \hat{\phi}(\mathbf{r} + \mathbf{u}, t), \tag{3.91}$$

thus reducing the number of degrees of freedom from infinite field modes to two modes, $\phi_V(x)$ and $\phi_V(x')$ (one for each box).

The canonical momentum field, $\hat{\pi} = \partial_t \hat{\phi}$, is integrated over the box,

$$\hat{\pi}_V(\mathbf{r}, t) = \int_V d\mathbf{u}\, \hat{\pi}(\mathbf{r} + \mathbf{u}, t), \tag{3.92}$$

since momentum is additive. [42] These operator ($\hat{\phi}_V$ and $\hat{\pi}_V$) are called the collective operators.
With (3.91) and (3.92) vector ξ becomes

$$\xi = (\phi_V(x), \pi_V(x), \phi_V(x'), \pi_V(x'))^T. \tag{3.93}$$

The equal time commutation relations of $\hat{\phi}$ and $\hat{\pi}$ are [24]

$$\left[\hat{\phi}(\mathbf{r}, t), \hat{\pi}(\mathbf{r}', t)\right] = i\delta(\mathbf{r} - \mathbf{r}'), \tag{3.94a}$$

$$\left[\hat{\phi}(\mathbf{r}, t), \hat{\phi}(\mathbf{r}', t)\right] = [\hat{\pi}(\mathbf{r}, t), \hat{\pi}(\mathbf{r}', t)] = 0. \tag{3.94b}$$

This means that the equal time relations of ξ, because of the average in (3.92), will be equal to (3.35). It is easy to see that the integral over V won't affect (3.94b). For (3.94a) we have

$$\left[\hat{\phi}_V(\mathbf{r}, t), \hat{\pi}_V(\mathbf{r}', t)\right] = \frac{1}{V} \int_V d\mathbf{u} \int_V d\mathbf{u}' \left[\hat{\phi}(\mathbf{r} + \mathbf{u}, t), \hat{\pi}(\mathbf{r}' + \mathbf{u}', t)\right] =$$

[5] A scalar massless field will be used in this example.

$$= \frac{i}{V} \int_V d\mathbf{u} \int_V d\mathbf{u}' \delta(\mathbf{r} + \mathbf{u} - \mathbf{r}' - \mathbf{u}') = \begin{cases} i & \text{if } \mathbf{r} = \mathbf{r}' \\ 0 & \text{if } \mathbf{r} \neq \mathbf{r}' \end{cases} . \tag{3.95}$$

Because we had to use the equal time commutation relations in order to obtain (3.95), this method can only test the existence of entanglement between states at the same time, $t = t'$. This type of entanglement is known as spatial entanglement.

Notice that (3.95) are the same relations of a 2-dimensional Quantum Harmonic Oscillator (and of a two mode Gaussian State). Following this similarity we can define a creation ($\hat{A}$) and annihilation ($\hat{A}^\dagger$) operators, such that,

$$\hat{\phi}_V(x) = \frac{1}{\sqrt{2}} \left(\hat{A}(x) + \hat{A}^\dagger(x) \right), \tag{3.96}$$

where, of course,

$$\hat{A}(x) = \frac{1}{V} \int_V d\mathbf{u} \int \frac{d\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{\omega}} e^{-i\omega t} e^{i\mathbf{k}\cdot(\mathbf{r}+\mathbf{u})} \hat{a}_{\mathbf{k}}. \tag{3.97}$$

We can, with ξ, construct a Covariance Matrix $\mathbf{V}$, the same way we would construct one for a normal Gaussian State,

$$V_{\alpha\beta} = \frac{1}{2} \langle \{ \xi_\alpha, \xi_\beta \} \rangle .$$

If the Covariance Matrix constructed this way respects the uncertainty relation (3.57) then the system constructed will constitute a valid Gaussian State. [38] In that case we can apply the entanglement measurements deduced in last section.

The physical meaning of averaging over a box V can interpreted as the simulation of an hypothetical entanglement detector and its characteristics such as a finite spatial resolution. Following this interpretation, in [49] (3.91) was generalized to

$$\hat{\phi}_V(x) = \int dx \; g_V(x) \phi(x), \tag{3.98}$$

where $g_V(x)$ is be the detector profile, it defines how our detector averages the states inside the box. The specific case in (3.91) would be the case of a top-hat detector - it averages the states inside V with equal probability, and outside V everything is zero.

Different types of profiles might give different results of entanglement since the way we average the states can camouflage or destroy the entanglement. For example if entanglement is only found for very low momentum k, then the equal averaging of the top hat might hide the entanglement, as is the case in [49]. There, the entanglement is found only for a Gaussian detector profile that gives lower weight to high k. In the same article it is argued that one can only prove the separability of two modes ϕ_V and ϕ'_V if one is to try every possible $g_V(x)$, thus rendering the PPT condition for Gaussian States, when evaluated with the method worked in this section, a de facto necessary but not sufficient condition for separability. There is, however, no standard way to determine which detector profiles give acceptable interpretations of reality. A $g_V(x)$ might detect entanglement but distort the system so much that one might ask if the information gained is of any significant use.

Finally, if we want to use the smallest boxes possible either to use the maximum spatial definition and try to alter the original system the least possible, or just because the integral in (3.91) might not be solvable, we can always take the volume of the boxes V to zero, as was originally done in [42] for a top hat detector.
The detector model is not defined for $V = 0$, but we can always take $V \to 0$,

$$\lim_{V\to 0} \hat{\phi}_V(x) = \lim_{V\to 0} \frac{1}{V} \int_V d\mathbf{u}\; \hat{\phi}(\mathbf{r}+\mathbf{u}, t) =$$

$$= \lim_{V\to 0} \frac{1}{V} \left(V\hat{\phi}(\mathbf{r}, t) + O(V^2) \right) = \hat{\phi}(x). \tag{3.99}$$

In a similar way, we get that

$$\lim_{V\to 0} \hat{\pi}_V(x) = V\hat{\pi}(x). \tag{3.100}$$

This means that for V very small, the Covariance Matrix becomes,

$$\mathbf{V} = \begin{pmatrix} \left\{\hat{\phi}, \hat{\phi}\right\} & V\left\{\hat{\phi}, \hat{\pi}\right\} & \left\{\hat{\phi}, \hat{\phi}'\right\} & V\left\{\hat{\phi}, \hat{\pi}'\right\} \\ V\left\{\hat{\pi}, \hat{\phi}\right\} & V^2\left\{\hat{\pi}, \hat{\pi}\right\} & V\left\{\hat{\pi}, \hat{\phi}'\right\} & V^2\left\{\hat{\pi}, \hat{\pi}'\right\} \\ \left\{\hat{\phi}', \hat{\phi}\right\} & V\left\{\hat{\phi}', \hat{\pi}\right\} & \left\{\hat{\phi}', \hat{\phi}'\right\} & V\left\{\hat{\phi}', \hat{\pi}'\right\} \\ V\left\{\hat{\pi}', \hat{\phi}\right\} & V^2\left\{\hat{\pi}', \hat{\pi}\right\} & V\left\{\hat{\pi}', \hat{\phi}'\right\} & V^2\left\{\hat{\pi}', \hat{\pi}'\right\} \end{pmatrix}, \tag{3.101}$$

where $(\hat{\phi}', \hat{\pi}') = (\hat{\phi}(x'), \hat{\pi}(x'))$. The generalization for two different boxes is easy enough. With this Covariance Matrix we can test the existence of entanglement either using (3.76) or (3.90).

As was done in [42] the Covariance Matrix can be constructed using the Hadamard function,

$$H(x, x') = \left\langle \left\{ \hat{\phi}(x), \hat{\phi}(x') \right\} \right\rangle. \tag{3.102}$$

For example:

$$\left\langle \left\{ \hat{\phi}(x), \hat{\phi}(x) \right\} \right\rangle = \lim_{x'\to x} H(x, x'), \tag{3.103}$$

and,

$$\left\langle \left\{ \hat{\phi}(x), \hat{\pi}(x) \right\} \right\rangle = \lim_{x'\to x} \partial_{t'} H(x, x'), \tag{3.104}$$

and so on.
This is especially useful for the cases in which we can calculate the Hadamard function directly (for example, using the method of images [42]).

Finally, even if the states turn out to be non-Gaussian, we can still use (3.76) as a necessary condition of separability (as discussed at the end of section 3.2.1), since $\hat{\phi}_V(x)$ and $\hat{\phi}_V(x')$ will still have the structure of Quantum Harmonic Oscillator. We loose, however, the ability to calculate the degree of entanglement of the system (e.g. Negativity).

Chapter 4

Holographic Principle

4.1 Introduction to the Holographic Principle

In this section we will show the conceptual origin of Holography to build a certain intuition and in the next section a more general formalism will be shown so that we can apply it to cosmology (the Covariant Entropy Conjecture [50]).

But what is the Holographic Principle? In the words of physicist Leonard Susskind [8]:

"According to 't Hooft [in [7]] the combination of quantum mechanics and gravity requires the three dimensional world to be an image of data that can be stored on a two dimensional projection much like a holographic image."

In other words, the number of distinct orthogonal quantum states in a region of volume V is not

$$N(V) \propto e^V \tag{4.1}$$

as one would expect if the entropy of a region was proportional to its volume because

$$S = ln(N). \tag{4.2}$$

$N(V)$ is actually related to the area of the boundary of the region and it behaves that way because of the physics of black holes.

A Black Hole occurs when an object (typically a star) is sufficiently massive to overcome any stabilizing forces and collapse gravitationally (shrink to zero volume [27]) and ending up as a singularity that has a gravitational pull so strong that not even light will escape if it gets near enough. The most simple Black Hole (without electrical charge or angular momentum) is described with the Schwarzschild metric,

$$ds^2 = \left(1 - \frac{2Ml_p^2}{r}\right) dt^2 - \left(1 - \frac{2Ml_p^2}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \tag{4.3}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

This metric is the only vacuum solution[1] of the Einstein's equations that is spherically symmetric and asymptotically flat (for $r \to \infty$ it becomes the Minkowski metric) [51]. It can be used the describe stars and Black Holes but in both cases it assumes that they don't have any angular momentum (otherwise it would ruin spherical symmetry). The main difference between a star and a Black Hole is that the radius of the star is larger than $2Ml_p^2$ [2] whereas the mass distribution of a Black Hole lies within the radius $2Ml_p^2$. This means that if a photon (ds=0) was at a distance $r = 2Ml_p^2$ of the center of the Black Hole it would have a velocity (assuming that $d\Omega = 0$)

$$\frac{dr}{dt} = v_r = 1 - \frac{2Ml_p^2}{r} = 0 \tag{4.4}$$

which means that the contribution from the singularity at $r = 2Ml_p^2$ suppresses any outward photon flux towards a Minkowski observer at infinity. In other words, at $r \leq 2Ml_p^2$ one wouldn't be able to escape the Black Hole even if one had a velocity equal to the velocity of light. $R_S = 2Ml_p^2$ is called the Schwarzschild radius and past this point no information can be retrieved from the Black Hole directly (it will be later radiated from the Black Hole in the form of Hawking's radiation) because nothing carrying the information can escape. The surface that R_S generates (in this case a spherical surface) is called the Event Horizon of the Black Hole [3]. The Schwarzschild radius is also usually taken to be the radius of the Black Hole itself.

Although R_S is important for a far away observer (the observer in (4.3)) since it represents a singularity in (4.3) it is only an apparent singularity since it will disappear for an observer near the Event Horizon. To show this we take coordinates near the horizon, $r^* = r - 2M$ and $r^* << 2Ml_p^2$ such that (4.3) will become

$$ds^2 \approx \frac{r^*}{2Ml_p^2} dt^2 - \frac{2Ml_p^2}{r^*} dr^{*2} - r^2 d\Omega^2 \tag{4.5}$$

now we simplify with $\rho = \sqrt{8Ml_p^2 r^*}$ and $\sigma = \frac{t}{4M}$ and we end up with,

$$ds^2 = \rho^2 d\sigma^2 - d\rho^2 - r^2 d\Omega^2 \tag{4.6}$$

which is the metric of an accelerated referential as seen by an inertial observer. It has no singularity whatsoever and so R_S is just and apparent singularity, it appears to observers far away, at $r \to \infty$.

Incidentally, the fact that near the horizon we have a metric of the form (4.6) suggests that we'll have a temperature associated with the horizon that will be proportional to the acceleration necessary for an observer not to fall into the Black Hole [24, 53]. Such temperature was proven to exist by Hawking and Gibbons in [54] and has the form,

$$T_H = \frac{a}{2\pi}, \tag{4.7}$$

[1] In this spacetime there is only a body of mass M that creates the spacetime distortion but since the metric represents the outside of that body we say there is no matter in its region of validity and, hence, it is a vacuum solution.

[2] Here l_p is the Planck size, the smallest possible size that a quantum system can have without turning into a Black Hole, according to current theories [7].

[3] Although to be completly correct it would be the boundary of the horizon, horizons are lightlike hypersurfaces [52].

where $a = \frac{1}{2R_S}$ here is the surface gravity of the horizon. This means that there are thermal states associated with the horizon that explain the existence of the Hawking radiation (see the last section of this chapter).

But what does this all have to do with the Holographic Principle? If we can associate a temperature to a Black Hole it means that we can associate entropy, and such an entropy was found to be proportional to the area of the horizon [54],

$$S_{BH} = \frac{A_{BH}}{4l_p^2}. \tag{4.8}$$

Equation 4.8 gives an entropy that was expected from a two-dimensional lattice with spacing l_p [7]. Now imagine a piece of space with volume V, energy E and entropy S. Imagine also an hypothetical black whole with the same volume, and energy and entropy, E_{BH} and S_{BH}. There are two possibilities for the entropy S, it can either be $S \leq S_{BH}$ or $S > S_{BH}$. Let us assume that it is the latter. E must be smaller than E_{BH} otherwise the system would collapse to a Black Hole. We can always take the system to the case in which $E_2 \simeq E_{BH}$ (but still smaller) by dropping small bits of matter slowly enough so that there's approximately no work being applied to the system and, by the second law of thermodynamics, $S_2 \geq S$. Then if we drop a little more energy into the system, just enough that $E_2 = E_{BH}$, a Black Hole will be formed and by then we will have that the final entropy of the system is S_{BH} but, by definition $S \geq S_{BH}$ and $S_2 \geq S_{BH}$ which contradicts the second law of thermodynamics. So we are left with the possibility that $S \leq S_{BH}$ always. But that means that,

$$S(V) \leq \frac{A}{4l_p^2}, \tag{4.9}$$

and consequently the number of distinct orthogonal quantum states in V will be,

$$N(V) \leq \text{const } e^{A}. \tag{4.10}$$

The maximum number of distinct states a system with V can have is proportional to the number of states there can be in a two-dimensional lattice with spacing l_p and area A.

This type of reasoning is, of course, only valid in a handful of spacetimes. The volume V must be a sphere (because the Black Hole is a sphere) and the spacetime where V exists must permit the formation of BHs. Also the region of space in consideration must be gravitationally stable[4] [50, 55].
In reality, we can consider volumes that are not spherical in the sense that if a spherical volume V_1 obeys (4.9) then a volume of general form, $V \in V_1$, must also obey (4.9) but with the area of the boundary of V_1. It just produces a weaker condition.

Now we will show an example where this formulation of the Holographic Principle fails.

This example is a cosmological one and was proposed by Susskind and Fischler in [56]. It stresses the importance of having a more general Holographic Principle

[4] For example, a star collapsing into a Black Hole is not a stable system. In general, we should consider system with weak gravity. [50]

that can be applied to any kind of spacetime. Today's universe is well modeled by the Friedmann metric,

$$ds^2 = dt^2 - a(t)^2 \left[dr^2 + r^2 d\Omega^2\right], \tag{4.11}$$

where a(t) is the scale factor that accounts for the way the Universe expands. This metric assumes that our universe is homogeneous and isotropic, and that there is a constant entropy density (σ) for comoving volumes. There is a more general way in which we can write this is [51]

$$ds^2 = dt^2 - a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \tag{4.12}$$

where $k = 1, 0 - 1$ for spherical (closed), flat and hyperbolic (open) universes (respectively), but lets take $k = 0$ for now.
A comoving distance (d_c) is a distance that doesn't account for the expansion of the universe and a proper distance (d) is a distance that does. The relation between them is

$$d = a(t)d_c. \tag{4.13}$$

Now take a spherical region of space with volume V and area of boundary A, such that $V = a^3 V_c$ and $A = a^2 A_c$ because of (4.13). If we apply the Holography Principle to such region we'll get

$$S \leq \frac{A}{4l_p^2} = \frac{\pi(aR)^2}{l_p^2} \tag{4.14}$$

but because of the constant entropy density we have

$$S = \sigma V_c \tag{4.15}$$

which means that

$$S = \sigma \frac{4}{3}\pi R^3 \tag{4.16}$$

so our Holographic Principle only holds if

$$R \leq \frac{3a^2}{4\sigma l_p^2} \tag{4.17}$$

which obviously will be violated for R large enough.

So, if this doesn't work, how can we define a valid boundary to apply the Holographic Principle? Next section we will go over a more general formalism proposed by Raphael Bousso in [50] and [57].

4.2 Holographic Principle in general spacetimes

4.2.1 Hypersurface, Horizons and the Raychaudhuri equation

Before introducing the Covariant Entropy Conjecture let us formalize some definitions. In a space with n dimensions an object with $n-1$ dimensions equal to the set of solutions of an equation of the form $f(x_1, x_2, ..., x_n) = 0$ is a hypersurface. It is the generalization of the concept of a 2 dimensional surface in euclidean space. For example, a spherical surface with radius R is $2D$ and formed by the set of solutions of equation $x^2 + y^3 + z^2 - R^2 = 0$, it is then a $(3-1)$ hypersurface. We can also define a hypersurface with a vector field were each vector is normal to each point (of the hypersurface)and its vicinity (as we see in fig. fig.4.1).

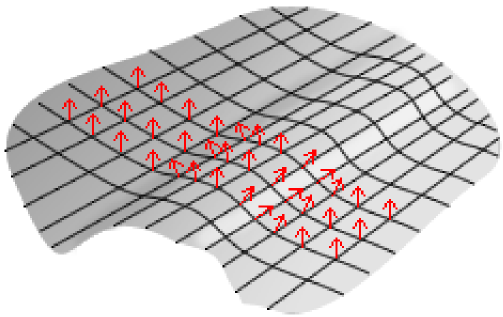


Figure 4.1: A two dimensional hypersurface in a 3 dimensional space. Here the red arrows represent the vectors from the vector field said to be normal to the hypersurface.

Let us call such vector field $\xi(x)$ where for instance $\xi(x_a)$ is a vector in point x_a and $\xi(x_b)$ is a different vector, in point x_b. This way we can define three different types of hypersurfaces:

- **Timelike hypersurfaces** were $\xi(\mathbf{x}) \cdot \xi(\mathbf{x}) < 0$;
- **Lightlike** or **null hypersurfaces** were $\xi(\mathbf{x}) \cdot \xi(\mathbf{x}) = 0$;
- **Spacelike hypersurface** were $\xi(\mathbf{x}) \cdot \xi(\mathbf{x}) > 0$.

Meaning that a spacelike vector field is normal to a timelike hypersurface and vice-versa. For example, a physical volume is a spacelike hypersurface, all its points are existing at the same time and so they are united by spacelike (unphysical) geodesics, they are not causally connected. Geodesics are the generalization of a straight line for general (curved) spacetimes. They give the shortest distance between two points and represent the possible trajectories that particles can have in a spacetime diagram (see fig. fig.4.2). Since they are one dimensional objects they depend on only one parameter τ (usually, but not necessarily, the proper time of the particle described by it) and are denoted as $x^\mu(\tau)$. Physically geodesics represent curves (in $3+1$ dimensions) of particles with constant velocity (special relativity) or free-falling (general relativity). Timelike geodesics represent spacetime trajectories of massive particles, particles that move at speeds smaller than the speed of light, lightlike geodesics represent particles that are moving at the speed of light and spacelike geodesics represent particles (or would represent if they were to exist) that are moving at speeds bigger than the speed of light. One can define a geodesic as the solution of the equation

$$\eta^\mu \eta^\mu{}_{;\nu} = 0, \tag{4.18}$$

where η represents the vector field tangential to the geodesic and is defined by

$$\eta^{\mu} = \frac{dx^{\mu}}{d\tau}. \tag{4.19}$$

$\eta^{\mu}{}_{;\nu}$ is the covariant derivative. The covariant derivative is a generalization of the normal derivative that is invariant by a change of basis or, in other words, it doesn't depend on the metric of a spacetime. It is normally denoted by $A_{;\beta}$ or $\nabla_{\beta}A$.
The normal derivative can also be denoted in a similar way: $A_{,\mu} = \partial_{\mu}A$. For a scalar s, since scalars are quantities that don't depend of the choice of basis, then

$$s_{;\beta} = s_{,\beta}. \tag{4.20}$$

For a vector ξ, we have that

$$\xi^{\nu}{}_{;\beta} = \xi^{\nu}{}_{,\beta} + \xi^{\lambda}\Gamma^{\nu}_{\mu\lambda} = g^{\alpha\nu}\xi_{\alpha;\beta}, \tag{4.21}$$

where

$$\Gamma^{\beta}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right), \tag{4.22}$$

are called the Christofell symbols.
For a tensor (**T**) (specifically a tensor with two parameters, a 2-form [51]) the covariant derivative is

$$T^{\mu\nu}{}_{;\beta} = T^{\mu\nu}{}_{,\beta} + T^{\alpha\nu}\Gamma^{\mu}_{\alpha\beta} + T^{\mu\alpha}\Gamma^{\nu}_{\alpha\beta} = g^{\mu\omega}g^{\nu\gamma}T_{\omega\gamma;\beta}. \tag{4.23}$$

Of course that, for a flat Minskowski metric, both definitions of derivatives are equivalent.

If η is spacelike, lightlike or timelike then it generates a spacelike, lightlike or timelike geodesic, respectively.
We now define a congruence; a congruence is a set of curves that never disappear nor intersect each other, only one can pass through each point. If these curves are geodesics then we have a congruence of geodesics. Since a hypersurface has a vector field ξ normal to it, we can define a vector field η that will be tangent to the hypersurface, such that $\xi \cdot \eta = 0$. This means that a hypersurfuce will be filled by a congruence of curves generated by this vector field. Null hypersurfaces will have null curves because since $\xi \cdot \xi = 0$, ξ is tangent to itself. It can also be proven that these curves, in null hypersurfaces, are geodesics [52]. Thus, we can say that null hypersurfaces are generated by these surface orthogonal null geodesics [50].

But why is all this relevant? It is because an Event Horizon is a null hypersurface, it is generated by the geodesics of light that enters the Black Hole but cannot leave. In fact an Event Horizon (H) has the three following properties [52]:

1. Any two points of H can only be connected by a light signal. Or, in other words, they can't be connected by a timelike curve. This is true because (H) is a null hypersruface;
2. Any null geodesics in H can have beggining points outside H;
3. Any null geodesics in H cannot have end points outside H.

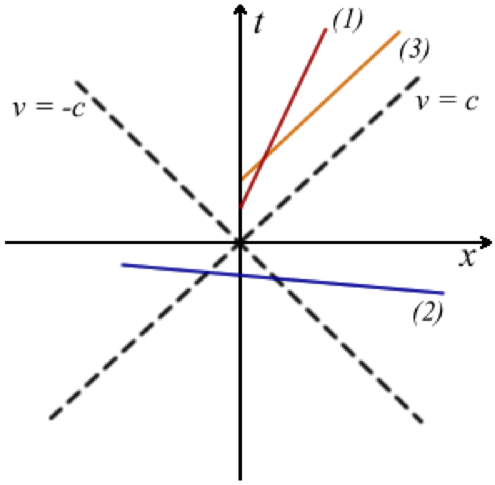


Figure 4.2: An example of the three possible types of geodesics in a two dimensional Minskowski spacetime diagram. The red curve (1) represents a timelike geodesic, the blue curve (2) represents a spacelike geodesic and the orange curve (3) represents a lightlike geodesics.

2 and 3 tell us that null geodesics can enter but they cannot leave. Finally, what is normally called the Event Horizon, is actually the (two dimensional) boundary of the (three dimensional) null hypersurface H that is the "actual" Event Horizon.

One last useful concept must be defined, the Raychaudhuri equation. The Raychaudhuri equation tells us how the "cross section" of a congruence of curves behaves, that is, if the congruence has a positive, negative of null expansion. In the case of null geodesics it can tell us how a null hypersurface behaves, which will prove rather useful for the next subsection.

The equation for a congruence of null geodesics is [58]

$$\frac{d\theta}{d\tau} = -\frac{1}{2}(\theta^2 + \sigma^2 - \omega^2) - R_{\mu\nu}\eta^\mu\eta^\nu. \tag{4.24}$$

θ tells us how an infinitesimal area element of a cross section of a congruence behaves and it is called the expansion rate(fig.4.3 a)). It is defined by the equation([52,59])

$$\frac{d\delta A}{d\tau} = \theta dA \tag{4.25}$$

where τ is the parameter of the geodesics passing through δA, and dA denotes the coordinates parallel to the surface the infinitesimal area [59]. σ is the shear stress and it tells us how much δA is compressed or distended (fig.4.3 b)) and ω tells us how much the geodesics through δA rotate (fig.4.3 c)). $R_{\mu\nu}$ is the Ricci tensor and it tells us how curved is the spacetime we are working in (in Minkowski space the tensor is zero) and η is the vector tangent to the geodesics through δA [5]. From the Frobenius theorem([60]) we know that if the vectors that generate the congruence are orthogonal to a hypersurface then $\omega = 0$, they can't rotate. Hence,

[5]The geodesics through δA have the same τ and the same tangent vector because it is an infinitesimal area element and all the geodesics are infinitesimally close there.

if our congruence of null geodesics is generated by the vector field normal to a null hypersurface, equation 4.24 becomes:

$$\frac{d\theta}{d\tau} = -\frac{1}{2}(\theta^2 + \sigma^2) - R_{\mu\nu}\eta^{\mu}\eta^{\nu}. \tag{4.26}$$

We are now equipped with all the necessary concepts to introduce the Covariant Entropy Conjecture.

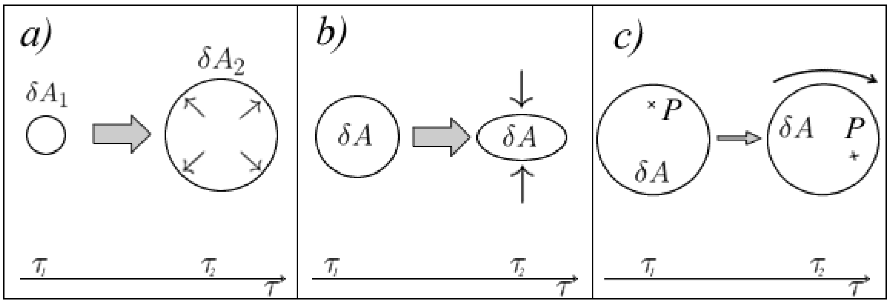


Figure 4.3: Examples of a) the expansion of the infinitesimal area along the parameter τ that is expressed in the parameter θ (in the figure $\delta A_1 < \delta A_2$); b) the shear that the area δA can be subjected to, expressed by the σ; c) the rotation of the curves through the area δA which is express in variable ω.

4.2.2 The Covariant Entropy Conjecture

In [50] a theory is put forward on how to find the Holographic surface[6] for a certain system - the Covariant Entropy Conjecture. What was explained in the last section is that, basically, a physical volume (a spacelike hypersurface) cannot have more entropy than a Black Hole of the same size. But, as discussed before, this type of formulation of the Holographic Principle, with that we shall call Bekenstein's Bound (because he was the first to propose it), is very limited in application.

Covariant Entropy Conjecture tries to state a more general formulation of the Holographic Principle, and it does so by shifting the attention from trying to find a boundary whose area bounds the entropy of the system to trying to find which system has an entropy bounded by the area of a specific boundary. That is, for a surface B with area A, we ask which system with entropy S satisfies (4.9). The bound itself, (4.9) is maintained in the conjecture.

Another departure of Covariant Entropy Conjecture from the normal formulation of Holographic Principle is that we do not use the entropy of a spacelike hypersurface in (4.9) but the entropy of a null hypersurface. This is done in order to formulate a covariant theory (that does not depend on the observer) [50].

What is done is that a surface B (specifically at surface contained in a spacelike hypersurface - a space surface) with area A is chosen. That surface will have four

[6]We define Holographic surface as the surface of a system with entropy S whose area respects (4.9).

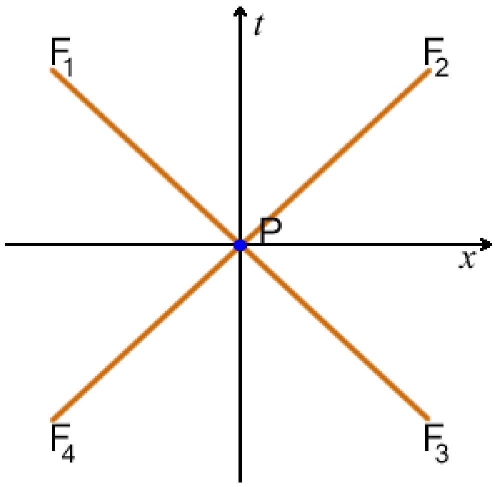


Figure 4.4: Example of a surface B, in a two-dimensional spacetime, in this case represented by a point P. The congruences F_i are represented by the orange lines orthogonal to P. How we define the name of each congruence $(F_1, ..., F_4)$ is arbitrary, in the image we show a possible configuration of the names.

orthogonal congruences of null geodesics, two in each side of the surface (one going to the future and another going to the past). We shall call these congruences F_i with $i = 1, 2, 3, 4$.
To illustrate this consider again a 4-dimensional spacetime, where we represent only two of the coordinates (t, x) (see fig.4.4). Here a surface corresponds to a point[7] (or a finite set of points) and a hypersurface corresponds to a line. We can see that a point P has four null geodesics coming out of it [8], two in $t > 0$ and two in $t < 0$, or, two in $x > 0$ and two in $x < 0$, hence, two at each "side" of the surface, and two pointing to the future and two to the past.
Next, we follow one of the F_i whose expansion is zero or negative ($\theta \leq 0$) until the expansion becomes positive (or until it reaches a boundary or singularity of the spacetime). Now, let's call the null hypersurface generated by F_i while $\theta \leq 0$, L. In other words, L is a null hypersurface generated by F_i such that, at every point of L, F_i has $\theta \leq 0$. Finally, what is proposed by the Covariant Entropy Conjecture is that, the entropy contained on L respects (4.9).

The reason why we follow the congruences with nonpositive expansion is that for a general L and a general (not necessarily closed) surface B, the concept of inside and outside is replaced by the negative and nonnegative expansion of the congruences [50]. We can see this intuitively in fig.4.5. If we have a family of lines, l_1, orthogonal to each point of B pointing to the center, and another family pointing towards the infinite (l_2), then the inside of the sphere is defined as the side of B were the lines tend to become closer to each other (in this case it is l_1). The same concept is used in Covariant Entropy Conjecture except generalized for null hypersurfaces. Also, we need not have a closed surface B to define the "inside" and "outside". This

[7] By choosing a point in (t, x), y and z can still vary, and if they do we have a 2-dimensional surface. Notice that we use different names of surface (2D) and hypersurface (that has one less dimension than the spacetime, in this case is 4D).

[8] They actually represent the congruence of geodesics that are orthogonal to P, in this case, they all must obey equation $t = x$.

happens because the presence of gravity itself alters the behavior of the geodesics and, hence, it can alter the expansion of the cross-section of the congruences. We can have positive and negative expansions of F_i even if B is an open surface.

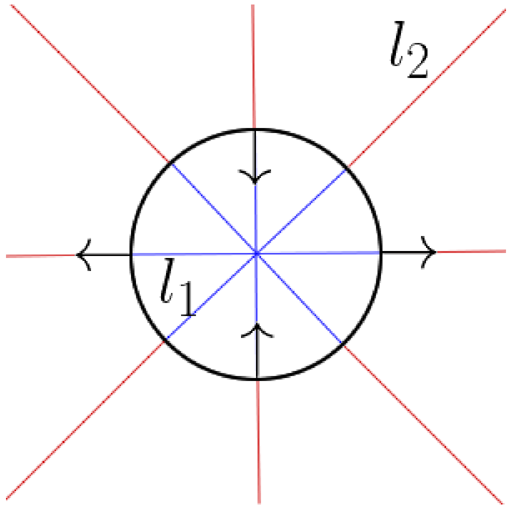


Figure 4.5: As it can be seen, the lines that go to the inside of the circle get closer together, while the lines that go into the outward direction get separates, their cross section gets larger. We call l_1 to the lines pointing to the inside of the circle, and l_2 to the lines pointing to the outside.

Let us, then, state, in a more formal way, the Covariant Entropy Conjecture, using the words of its creator, R. Bousso:

"Let M be a four-dimensional spacetime on which Einstein's equation is satisfied with the dominant energy condition holding for matter[9]. Let A be the area of a connected[10] two-dimensional spatial surface B contained in M. Let L be a hypersurface bounded by B and generated by one of the four null congruences orthogonal to B. Let S be the total entropy contained on L. If the expansion of the congruence is non-positive (measured in the direction away from B) at every point on L, then $S \leq \frac{A}{4l_p^2}$."

Being a conjecture, Covariant Entropy Conjecture is, of course, not proven[11], but it was tested in a variety of systems (for example, a collapsing star) where the Bekenstein Bound fails or is inconclusive (see [50, 55, 57]).
Even if Covariant Entropy Conjecture ends up not being applicable in all possible spacetimes, it gives us a very general way of choosing possible Holographic boundaries for our systems. We only need to test if (4.9) is respected in each case, after we have chosen our B.

Finally, the Covariant Entropy Conjecture wouldn't be very helpful if it couldn't, under the right circumstances, recover the Beckenstein's Bound. Imagine closed surface B with area A having at least one future oriented L, with nonpositive ex-

[9]This means that, for all timelike vectors v_μ then $T^{\mu\nu}v_\mu v_\nu \geq 0$ and $T^{\mu\nu}v_\mu$ is nonspacelike which basically means that every type o matter in the Universe has to have nonnegative energy density and a velocity smaller or equal to the speed of light. [50]

[10]Connected means that any two points of the surface can be connected by a path on the surface.

[11]Although proof for spacetimes under some conditions has been produced in [61].

pansion at every point (so that we can use the Covariant Entropy Conjecture). For example, a spherical surface B inside a Black Hole doesn't have a future oriented hypersurface going to the outside, since gravity will be so strong that all the null geodesics generating this hypersurface will all go to the singularity of the Black Hole. In the case of a spherical surface (fig.4.5), we have one L with $\theta < 0$, whose direction represents the inside of the surface.
Now, we will also assume that the only boundary of L is B. Then, for a volume V enclosed by B in the causal past of L, every bit of information in V will eventually pass through L (see fig.4.6). For example, in the case of a sphere inside a spherical shell, a family F_i of null geodesics coming out of B in direction to the center will pass through every bit of information inside the sphere. This is why we impose that L can't have any other boundaries, so that no entropy in V can escape through them.
By the second law of thermodynamics, the entropy that passes through L (S_L) can only be equal to, or greater than, the entropy in V (S_V). Finally, since, by the Covariant Entropy Conjecture, $S_L \leq \frac{A}{4l_p^2}$ which means that $S_V \leq \frac{A}{4l_p^2}$.

> Hence, for a closed surface B with area A, that has at least one future directed null hypersurface L (with $\theta \leq 0$ at every point) with no boundary other than B, it follows that for a spatial region V bounded by B and in the causal past of L (or, in other words, that it is on the same side as L), the relation $S_V \leq \frac{A}{4l_p^2}$ is valid.

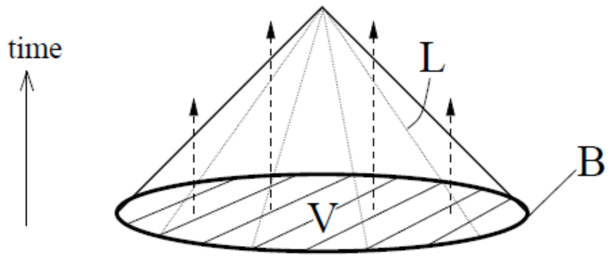
This is called the *Spacelike Projection Theorem*, and it was proposed and deduced in [50] and is a generalization of the Bekenstein bound for more general spacetimes. The Bekenstein bound is recovered from the *Spacelike Projection Theorem* for weakly gravitational (Bekenstein) systems in asymptotically flat spacetimes [50, 55].

Cosmology and the Holographic Principle

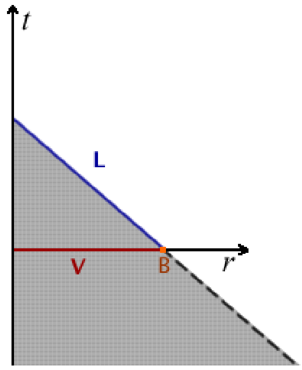
Now let's apply our knowledge of the Holographic Principle to Cosmology. It was suggested in [56] that a good holographic boundary for the Friedmann metric would be the radius of the visible Universe (also known as particle Horizon, or past Horizon),

$$R_{\text{past}} = a(t) \int_{t_B}^{t} \frac{d\tilde{t}}{a(\tilde{t})}, \tag{4.27}$$

where t_B is the time when the Big Bang occurred (usually taken to be zero), and t is the present time. This radius tells us the maximum distance that we can see when we look at the Universe, no matter how good our telescope is. It is the distance traveled by light since the Big Bang.
In fact, it was proved in [56] that the Holographic Principle holds for this radius in flat ($k = 0$) and open ($k = 1$) Universes. However, when the article was published, the existence of a Cosmological Constant hadn't been substantiated by evidence, which means that the proof does not hold for future times ($t \to \infty$). It will be proven in chapter 6 that R_{past} is indeed a good holographic boundary for a Friedmann Universe with $k = 0$.



(a) Here we have a three dimensional representation of what would happen with a spherical boundary (here represented by the circle B). The null hypersurface, L, formed by the congruence coming out of B has the area V in its causal past. This image was taken from [55].



(b) Here we represent the radial coordinate r vs the times coordinate t. A sphere B of a certain radius is located at the boundary of both L and V (which are represented by lines). The shaded region represents the causal past of L, where L is formed by the null geodesics that go from B into the center of the sphere. Note that, since $r > 0$, every geodesic that starts in V and goes into the future will eventually pass through L.

Figure 4.6: The volume V is in the causal past of L, which means that every timelike or null like geodesics starting from V will pass through L eventually. Thus, as long as we are dealing with matter that is not faster than the speed of light, every entropy on V will eventually pass through L.

There are, however, other possible holographic boundaries. Here are other two often suggested:

The future Horizon,

$$R_f = a(t) \int_t^{\infty} \frac{d\tilde{t}}{a(\tilde{t})}, \tag{4.28}$$

events that happen at a distance greater than this radius will never be observed by us, it marks the maximum distance we will ever be able to see.
The Hubble Horizon (HH),

$$R_H = \frac{a}{\dot{a}} = \frac{1}{H}, \tag{4.29}$$

where H is the Hubble parameter, is the distance beyond which (for $k = 0$) the expansion of the Universe exceeds the speed of light. [62, 63]

It will be shown in chapter 5 that the future Horizon and the HH respect the Holographic Principle for $k = 0$. So we have three possible Horizons for $k = 0$. Indeed the Holographic Principle and the Covariant Entropy Conjecture gives us the Horizons that respect the Holographic Entropy Bound, but is there a way to know which of them is closer to saturate the bound, so that some kind of prediction could be made as to the possible entropy of the system?
Notice that, if we want to find a regions that obeys the Holographic Principle, any of these Horizons is perfectly fine. However, if we want to make some kind of prediction about the entropy of a system, for example, assume that a certain region of space has entropy equal (or approximately equal) to the maximum allowed by the Holographic Principle, then, the boundary that is closest to saturate the Holographic Principle bound will be the best one. Note that this assumption of saturation (or almost saturation) will be made in the following chapters, both by us and by other authors. It was proven in [50] that the Horizon for which the corresponding system is closer to saturate the Entropy Bound is the HH (for $k = 0$) and so, at least from the Horizons formed by the future-directed null congruences, it would be the preferred one.

4.3 Black Hole Entropy and Quantum Entanglement

Although a complete theory of Quantum Gravity doesn't exist yet, some attempts at describing a Black Hole in the frame of Quantum Mechanics have been made. In this last section of the chapter, two of those attempts will the presented, because they relate the concept of quantum entanglement with the entropy of a Black Hole, thus setting a bridge between Holography and entanglement that will play an important role later in the Thesis. Since this kind of theories can get conceptually very heavy and can easily develop into less accepted/tested areas of physics, we shall only attempt to present the most important concepts here (supported, of course, by the respective literature), leaving the option of a more complete research on the subject up to the reader.

The metric in (4.6) is very similar to the metric of an observer moving with uniform acceleration (with $a = \frac{1}{\rho}$) in a Minkowski spacetime in the x direction, with Rindler coordinates: [34]

$$ds^2 = \rho^2 d\sigma^2 - d\rho^2 - dy^2 - dz^2, \tag{4.30}$$

and

$$\begin{aligned} t &= \rho \sinh \sigma, \\ x &= \rho \cosh \sigma, \\ y &= y', \\ z &= z'. \end{aligned} \tag{4.31}$$

This accelerated observer (or Rindler observer) will have trajectories inside either region I ($a > 0$) or region II ($a < 0$) as seen in fig.4.7. For $a \to \infty$ the trajectory of a Rindler observer will approach a null geodesic. Because of this, a Rindler observer cannot cross the boundary formed by the null geodesics with origin in $x = 0$. Also, no observer in Minkowski space can cross these boundaries to the Rindler regions, since they would need to have spacelike geodesics (hence, velocities greater than the velocity of light). Then arises what is called an apparent Horizon. It does not exist physically (unlike, say, the Event Horizon of a Black Hole, which exists explicitly in the metric), but it appears when there is a Rindler observer.
A Rindler observer will observe black-body radiation coming from the vacuum, with an associated temperature of (4.7), that an inertial observer won't be able to see. This is called the Unruh effect and it essentially happens because the number of particles in a given state can change with the referential, hence we can have zero particles in an inertial frame and thermal bath in an accelerating one. [24, 53]

We can interpret (4.6) as the referential of an observer very near the Horizon, with an acceleration a opposing the gravitational pull of the Black Hole. Then, the Black Hole Horizon is the Rindler Horizon with the same associated temperature[12], and the Hawking radiation coming out of the Black Hole can be explained as a variation of the Unruh effect.

Since the particle content depends on the referential, the vacuum state of a referential in Minkowski space ($|0_M\rangle$) won't be the same as the vacuum state of a referential in Rindler space. Also, we won't have only one Rindler vacuum, but two ($|0_I\rangle$ and $|0_{II}\rangle$), since the two regions where the Rindler observer can be are causally disconnected, since to go from, say, I to II forces the observer to go through the apparent Horizons in fig.4.7. Hence $H_{\text{Rindler}} = H_I \otimes H_{II}$. [24, 34, 53]

The Minkowski vacuum can be related to Rindler states through [24, 34, 53]

$$|0_{kM}\rangle = \sum_n C_n |n_{kI}\rangle |n_{kII}\rangle, \tag{4.32}$$

and

$$C_n = \frac{1}{\sqrt{Z}} e^{-\frac{\beta E_n}{2}}, \tag{4.33}$$

with $Z = \sum_n e^{-\beta E_n}$ and $\beta = \frac{2\pi}{a}$. Notice that (4.32) is equivalent to a Schmidt decomposition, which means that if more than one coefficient C_n is nonzero, the states of I and II are entangled with each other.

[12]Although the Black Hole Horizon exists in the Schwarzschild metric, it cannot be seen by an in falling observer, consequently it makes sense that for (4.6), the Event Horizon is only apparent.

In the case of a Black Hole, the role of the Minkowski referential is played by an observer falling through the Black Hole's Event Horizon. Region I, for example, are the states near the Horizon, where the metric becomes (4.6). Since there is this relation between the Rindler metric and a Black Hole, it is suggestive to assume that the vacuum state of a Black Hole, $|\psi\rangle$, can be written as something like

$$|\psi\rangle = \frac{1}{\sqrt{Z}} \sum_i e^{-\frac{\beta E_i}{2}} |n_i\rangle |h_i\rangle , \tag{4.34}$$

where n denotes the states near the Horizon. Now we just need to find which states correspond to the region II of the Rindler space, or, in this case, to h.
It is proposed in [15] that the Event Horizon of the Black Hole should work as a mirror of the structure of the near Horizon region, and, as such, it should be considered the region II. [64] A possible explanation for this comes from one of the proposed mechanisms of the creation of Hawking's radiation[13], in which a vacuum fluctuation occurs near the Event Horizon, where a pair of virtual particles is created. Then, the strong gravitational pull of the Black Hole separates the pair, where one particle can escape the Black Hole, and the other is caught. [66] This way, excitations in the near regions of the Event Horizon would correspond to matter falling into the Horizon, which, in turn, would correspond to excitations in the Horizon [64].
Thus, it is proposed that $|h_i\rangle$ correspond to the state in the Event Horizon, which are actually states distributed in a region between the actual Horizon ($r = R_S$) and the stretched horizon (r_{st}) [15,64]. The stretched horizon is the distance after which ($r > r_{st}$) the physics of the Black Hole can be described, to a good approximation, by known physics [67]. In [15, 64] the stretched horizon is defined as being at one l_p from R_S for the observer in (4.6), which means that for an observer in (4.3), $r_{st} = R_S + O\left(\frac{1}{M}\right)$ (this can be seen using the relations between the coordinates in (4.5) and in (4.6)). The spreading of the Horizon states through the region between r_{st} and R_S can thus be understood as the uncertainty of space having a minimum of l_p. Smaller regions are not well described by current theories, since, as already stated, states smaller than that would turn into black holes. The Planck distance is, thus, a natural cutoff, which, as will be shown in chapter 5, is often used in physics.

The region near a Black Hole is defined in [15,64] as the region with $r \leq \frac{3}{2}R_S$ (which corresponds limit of the region where stable circular orbits around the Black Hole become impossible).

Notice that, since $|n_i\rangle$ tells us the particle content near the Horizon, these states represent the Hawking radiation being emitted from the Black Hole (and also the near vacuum states). Hence, in (4.34), by the Schmidt decomposition, we have entanglement between the Horizon and the radiation being emitted.
The Hawking's radiation that was emitted at earlier times[14] is represented in the state $|r\rangle$, which represents the rest of the system that is not the Horizon or the near region (everything further than $\frac{3}{2}R_S$). [15] A distant observer will see a system composed by the Black Hole and the rest of the system (particularly Black Hole and early Hawking's radiation), [15]

$$|\phi\rangle = |\psi\rangle |r\rangle . \tag{4.35}$$

[13]For more detailed possible explanations about this "mirror" property see [64, 65].

[14]That it was emitted at earlier times means that it had time to escape the near region of the Black Hole, into the far region, $r > \frac{3}{2}R_S$.

The entropy of entanglement between the states n and the Horizon can be calculated using (3.28), [17, 65]

$$S_{\text{Horizon}} = S_{\text{near}} \simeq \frac{A}{4l_p^2}, \tag{4.36}$$

which is approximately equal to the entropy of the Black Hole. This seems to strongly suggest the entanglement originated by the Unruh effect could give the Black Hole its characteristic entropy.

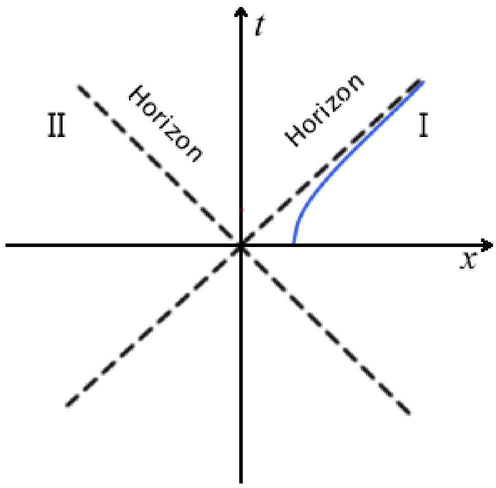


Figure 4.7: A diagram of the Rindler regions. I corresponds to the regions with positive acceleration, and II to the negative acceleration. They are causally disconnected from each other and from the Minkowski referential because of the Horizons that appear. The curve in blue is an example of a possible trajectory, of an accelerated observer, that approaches a lightlike geodesic.

However, another possibility for the Black Hole entropy, presented in [15], relates the entanglement between the Black Hole and the early Hawking radiation with S_{BH}. In that case S_{BH} arises from the existence of degrees of freedom on the stretched Horizon[15]. These degrees of freedom originate from quantum gravity, since for a remote observer the more "classical" description of the Black Hole is sufficient, then if there is a quantum gravity number k that distinguishes different Black Hole states $|\psi_k\rangle$, they must all seem the same (they must be degenerate) for a distant observer. [15, 64]

The number of microstates of the Black Hole is $e^{S_{\text{vac}}}$ and so, $k = 1, ..., e^{\sim\frac{A}{4l_p^2}}$. Here, S_{vac} stands for the entropy of the Black Hole without accounting for the entropy of the matter that falls into it. It is argued in [15, 64] that $S_{\text{vac}} \simeq \frac{A}{4l_p^2}$.

The Black Hole state $|\psi_k\rangle$ is still written has

$$|\psi_k\rangle = \frac{1}{\sqrt{Z}} \sum_i e^{-\frac{\beta E_i}{2}} |n_i\rangle |h_{ik}\rangle . \tag{4.37}$$

The difference here is that now it is suggested that, what a distant observer sees

[15] In [64] it is argued that these degrees of freedom are also distributed in the near region. Regardless, the majority of the degrees of freedom could be in the region between the Horizon and the stretched Horizon, as it is the region where quantum gravity becomes dominant.

is [15, 64]

$$|\phi\rangle = \sum_{k=1}^{e^{\sim \frac{A}{4l_p^2}}} d_k \, |\psi_k\rangle \, |r_k\rangle \, , \tag{4.38}$$

where there is also information in $|r_k\rangle$ about the quantum gravity degrees of freedom, because that information is carried by the early Hawking radiation.

For a maximally entangled system, the partial traced states will be maximally mixed which means that, for (4.38),

$$S_{\text{Ent}} = S_{\text{vac}} \simeq \frac{A}{4l_p^2}. \tag{4.39}$$

Entanglement seems to play a significant role in Black Hole thermodynamics. The entanglement of states near ((4.34)) and far ((4.38)) from the Event Horizon, with the Event Horizon itself, can be used to explain the S_{BH}. In [15, 64] it is argued that there is no apparent reason as to why these theories wouldn't be applicable to other spacetimes with boundaries, de Sitter Universes (see chapter 5), thus making entanglement a plausible mechanism to describe the Holographic Principle. Of course, the precise details of the relation between the entanglement and Holography will only be fully understood with the development of a complete theory of quantum gravity.
The relation of entanglement with the Holographic Principle (more specifically, the entropy of bounded regions in Minkowski and Friedmann spacetimes), and also its possible relation with the Cosmological Constant will be explored in greater detail in section 5.1.2.

Chapter 5

Cosmological Constant: What has been done

5.1 Old and New Ideas

The Universe is believed to be expanding at an accelerating rate (as indicated by experiments such as [1, 2, 5, 6, 68]). Originally, it made sense to physicists that the Universe's expansion would tend to zero and eventually become negative, because of the gravitational attraction between the galaxies, similar to a reverse Big Bang (the Big Crunch). When measurements appeared that contradicted this statement it was such a big mystery that the concept of "Dark Energy" - a mysterious energy that "makes" the universe continue to expand - had to be created to explain it all. The most common and accepted explanation for the Dark Energy is the Cosmological Constant, a constant that appears in the Einstein equation (5.1) which is (apparently) independent of the content of matter/energy present in a spacetime. [4, 5]

To talk about the Cosmological Constant we must first explain its origin. Gravity is described by the Einstein equation [27, 51]

$$\mathbf{G} = 8\pi G\mathbf{T}, \tag{5.1}$$

where $\mathbf{G}$ is the Einstein tensor (which gives us information about the geometry of the spacetime) and $\mathbf{T}$ is the energy-momentum tensor (introduced in chapter 2).

What this equation tells us is basically that

Presence of energy/matter = Deformation of the structure of spacetime.

The most simple form $\mathbf{G}$ can have is [27, 51],

$$\mathbf{G} = \mathbf{R} - \frac{1}{2}R\mathbf{g} - \Lambda\mathbf{g}, \tag{5.2}$$

where $\mathbf{R}$ and R are called the Ricci tensor and scalar (respectively) and are also measures of how curved the spacetime is. They are related to the metric tensor $\mathbf{g}$ through [51]

$$R_{\mu\nu} = \Gamma^{\beta}_{\nu\mu,\beta} - \Gamma^{\beta}_{\beta\mu,\nu} + \Gamma^{\beta}_{\beta\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\beta}_{\nu\lambda}\Gamma^{\lambda}_{\beta\mu} \tag{5.3}$$

and

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{5.4}$$

Finally, Λ is the famous Cosmological Constant. It appears in (5.2) because there is an ambiguity in one of the properties of $\mathbf{G}$: $\nabla \cdot \mathbf{G} = 0$, where here the ∇ stands for the covariant derivative. This property comes from the conservation of the energy-momentum tensor, $\nabla \cdot \mathbf{T} = 0$. We can add a constant multiplied by the metric to $\mathbf{G}$ and it won't change the result, $\nabla \cdot \mathbf{G} = \nabla \cdot (\mathbf{G} + \Lambda \mathbf{g}) = 0$. There is no theoretical way to specify what this constant is but we will dedicate the last part of this chapter to the discussion its measurements. Notice also that because of the presence of Λ in (5.2) the Einstein Tensor is not zero for a flat spacetime (when the metric is Minkowski). There is a minimum value of curvature even for a Minkowski Universe.

As stated in chapter 4, our universe is well described by the Friedmann metric (Eq.(4.11))

$$ds^2 = dt^2 - a(t)^2 \left[dr^2 + r^2 d\Omega^2 \right].$$

For mathematical simplicity and since evidence suggests that our Universe is flat [5], we will always take $k = 0$ in the context of this thesis.

At large scales we can see the galaxies in our universe just as if they were particles of a fluid and if we choose the average rest frame as the frame of (4.11) (or better yet, if we take the approximation that the galaxies are all stationary in relation to each other) we get a perfect fluid. If we take these assumptions, the energy-momentum tensor takes the form [27, 51]

$$\mathbf{T} = (\rho + p)\mathbf{u} \otimes \mathbf{u} - p\, \mathbf{g} \tag{5.5}$$

where $\mathbf{u}$ is the proper velocity. For (4.11) we have (here in Cartesian coordinates)

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & a(t)p & 0 & 0 \\ 0 & 0 & a(t)p & 0 \\ 0 & 0 & 0 & a(t)p \end{pmatrix}. \tag{5.6}$$

From $\nabla \cdot \mathbf{T} = 0$ the following equation follows [27, 51]

$$\frac{d}{dt}(\rho V) = -p \frac{dV}{dt} \tag{5.7}$$

which is the first law of thermodynamics.[1]

We can divide the energy density into its contributions,

$$\rho = \rho_m + \rho_r + \rho_\Lambda, \tag{5.8}$$

where ρ_m is the energy density of matter, ρ_r is the energy density of radiation and ρ_Λ is the energy density of the Cosmological Constant.

[1] The equation should have derivatives in order of the proper time τ but for a Friedmann universe $\tau = t$ and so derivatives in order to t were used.

Here we are introducing the Cosmological Constant energy density. If we take Λ from $\mathbf{G}$ in (5.1) and put it in the right side of the equation, we can define the energy density attributed to the influence of the cosmological constant $\rho_\Lambda = \frac{\Lambda}{8\pi G}$. This means that the pressure will also have a similar form

$$p = p_m + p_r + p_\Lambda. \tag{5.9}$$

but since $p_m << p_r, p_\Lambda$ at all times [27], this reduces to,

$$p = p_r + p_\Lambda. \tag{5.10}$$

For radiation, the relation between the density and the pressure is well known [27,51]

$$p_r = \frac{1}{3}\rho_r. \tag{5.11}$$

It is also easy to see from (5.6) that $p_\Lambda = -\rho_\Lambda$. Hence, the total pressure of the Universe will be

$$p = \frac{1}{3}\rho_r - \rho_\Lambda. \tag{5.12}$$

Finally, equation (5.7) becomes

$$\frac{d}{dt}(\rho_m V) + \frac{d}{dt}(\rho_r V) + \frac{d}{dt}(\rho_\Lambda V) = -\frac{1}{3}\rho_r \frac{dV}{dt} + \rho_\Lambda \frac{dV}{dt}. \tag{5.13}$$

If we consider the exchange between radiation and matter to be negligible then we can separate the equation into three:

$$\begin{cases} \frac{d}{dt}(\rho_m V) = 0 \\ \frac{d}{dt}(\rho_r V) + \frac{1}{3}\rho_r \frac{dV}{dt} = 0 \\ \frac{d}{dt}(\rho_\Lambda) - \rho_\Lambda \frac{dV}{dt} = 0 \end{cases} \Leftrightarrow \begin{cases} \rho_m V = C_m \\ \rho_r V^{\frac{4}{3}} = C_r \\ \rho_\Lambda = C_\Lambda \end{cases}. \tag{5.14}$$

Finally, since $V \propto a^3$ and defining: $C_i/V_0 = \rho_{i0}$ (were ρ_{i0} is the value of ρ_i today), then

$$\begin{cases} \rho_m = \frac{\rho_{m0}}{a^3} \\ \rho_r = \frac{\rho_{r0}}{a^4} \\ \rho_\Lambda = \rho_\Lambda \end{cases} \tag{5.15}$$

for any time t. Finally, from the trace of the equation (5.1) for the Friedmann metric, we get a differential equation for $a(t)$ [27]

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho. \tag{5.16}$$

Putting (5.15) into (5.16) we get,

$$H^2 = H_0^2 \left(\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_\Lambda\right), \tag{5.17}$$

where $H = \left(\frac{\dot{a}}{a}\right)$ is the Hubble parameter and H_0 is it's value today.
The Ω_i are defined by the ratio of the energy density (ρ_i) and the critical energy density $\rho_c = \frac{3H^2}{8\pi G}$, today ($\frac{\rho_{i0}}{\rho_{c0}}$). The critical energy density is the total energy density

that the Universe needs to have for it to be flat ($k = 0$).
For the remainder of this Thesis, we will consider the following values for the density parameters [5]

$$\begin{cases} \Omega_{m0} = 0.25(\sim 0.3) \ , \\ \Omega_{r0} = 0.05(\sim 0) \ , \\ \Omega_\Lambda = 0.7 \ . \end{cases} \tag{5.18}$$

We can see now the importance of the Cosmological Constant in Cosmology. Since the Universe is expanding, $a(t \to \infty) \to \infty$ and so in the future

$$H^2(t \to \infty) = H_0^2 \Omega_\Lambda, \tag{5.19}$$

or, in other words, the Universe will be dominated by the Cosmological Constant. Also, the expansion of the Universe will become exponentially big

$$a(t \to \infty) = e^{H_0 \sqrt{\Omega_\Lambda}(t - t_{\text{now}})}. \tag{5.20}$$

This is called the de Sitter Universe, which is dominated by a Cosmological Constant with a negative pressure. It has a radius $R = \frac{1}{H_0\sqrt{\Omega_\Lambda}} = \frac{1}{H}$ associated with an Event Horizon that appears in Eq.(4.11) with a suitable change of coordinates [69]:

$$ds^2 = \left(1 - \frac{v^2}{R^2}\right) du^2 - \left(1 - \frac{v^2}{R^2}\right)^{-1} dv^2 + v^2 d\Omega^2, \tag{5.21}$$

with

$$r = \frac{v}{\sqrt{1 - \frac{v^2}{R^2}}} e^{-\frac{u}{R}}$$

$$t = u + \frac{R}{2} \ln \left(1 - \frac{v^2}{R^2}\right) .$$

Notice that the de Sitter radius (R_{dS}) is equal to the Hubble Horizon studied last chapter.

It is because of (5.20) that the Cosmological Constant can explain why the Universe is expanding at an accelerating rate. Without it, $\dot{a} \to 0$ and the Universe would eventually stop expanding, as originally predicted by physicists. However, regardless of the success of the Cosmological Constant in predicting the behavior of our Universe, its origin and physical interpretation (as well as possible theoretical calculation) remain a mystery. Some developments on this front will be presented in the remainder of this chapter but since the number of proposed theories to explain the Cosmological Constant is so big and their nature so diverse only the models based on the theories and ideas relevant to this thesis will be shown. To get a greater understanding of the different Cosmological Constant and Dark Energy models out there the reader is referred to [4–6, 70].

5.1.1 Vacuum Energy and the Cosmological Constant

The most accepted interpretation of the Cosmological Constant is that it comes from the vacuum energy. [27] Since the energy of a system has a contribution from the vacuum energy,[2]

$$E = :E: + E_{ZPE}, \tag{5.23}$$

then the energy-momentum tensor will also have a contribution due to the Zero Point Energy (E_{ZPE}),

$$\mathbf{T} = :\mathbf{T}: + \mathbf{T}_{ZPE}. \tag{5.24}$$

From now on we will adopt the following notation: $\boldsymbol{\eta}$ stand for the Minkowski metric,

$$\boldsymbol{\eta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{5.25}$$

It would be more intuitive to have in (5.1) a curvature tensor $\bar{\mathbf{G}}$ that is zero if $\mathbf{g} = \boldsymbol{\eta}$. Indeed this was one of the original assumptions used by Einstein to derive his original equation,

$$\mathbf{R} - \frac{1}{2}R\mathbf{g} = 8\pi G\mathbf{T}, \tag{5.26}$$

before he introduced the Cosmological Constant to try to save the concept of a static universe [27].

It is also safe to assume that since there is only curvature when there's matter/energy present, then if $\mathbf{g} = \boldsymbol{\eta}$ then $\mathbf{T} = 0$, hence we change (5.1) to

$$\bar{\mathbf{G}} = 8\pi G\mathbf{T}. \tag{5.27}$$

The simplest $\bar{\mathbf{G}}$ is, of course,

$$\bar{\mathbf{G}} = \mathbf{R} - \frac{1}{2}R\mathbf{g}. \tag{5.28}$$

From (5.24), we have that

$$\bar{\mathbf{G}} = 8\pi G\,(:\mathbf{T}: + \mathbf{T}_{ZPE}) \tag{5.29}$$

which, if we assume that,

$$\mathbf{T}_{ZPE} = \rho_{ZPE}\mathbf{g}, \tag{5.30}$$

becomes

$$\bar{\mathbf{G}} - \rho_{ZPE}\mathbf{g} = 8\pi G{:}\mathbf{T}{:}, \tag{5.31}$$

We recover (5.1) if $\rho_{ZPE} = \rho_\Lambda$ and if $\mathbf{T}$ in the original equation doesn't account for the energy of the vacuum. The last assumption makes sense if we remember that the Friedmann equation (5.16) only needs the information that: a) our Universe is isotropic, b) our Universe is homogeneous, and c) the majority of the content of our Universe (galaxies) can, at large scales, be described as a perfect fluid. It doesn't take into account, at least directly, the Quantum Vacuum. The only place where it could appear is precisely the energy density attributed to the Cosmological Constant.

So we cannot prove that the Cosmological Constant comes from the Zero Point Energy or that the Dark Energy comes from either one of them, but we can see

[2] The notation $:A:$ denotes a quantity A after the vacuum contribution has been removed (not necessarily subtracted).

how it is a very attractive explanation. Indeed it was seen in chapter 2 that the Zero Point Energy can indeed have a physical manifestation and as such it makes sense that it should be able to affect the curvature of the spacetime just as regular matter. But unlike regular matter, the Zero Point Energy is always present just like the Cosmological Constant term. It also makes more sense to have an Einstein tensor $\bar{\mathbf{G}}$ that is zero when there is no curvature. From now on we will just refer to $\mathbf{G}$ and write the Cosmological Constant always explicitly.

Finally, it could always happen that there is an intrinsic curvature Λ_b (a bare Cosmological Constant) associated with spacetime *and* that the Zero Point Energy also curves space. In that case we can always write,

$$\rho_\Lambda^{\text{exp}} = \rho_{ZPE} + \rho_{\Lambda_b}, \tag{5.32}$$

where $\rho_\Lambda^{\text{exp}}$ is the density of the total Cosmological Constant, the one measured by experiments.

Let us now try to calculate ρ_Λ by calculating the Zero Point Energy of a massless scalar field. Since our Universe is locally flat (as is any Universe described by (5.1)) we will ignore the effects of curvature on the energy of the vacuum. Consequently, from (2.35) we know that,

$$\rho_\Lambda = \int_0^\infty \frac{dk}{(2\pi)^2} k^3 \tag{5.33}$$

which diverges. But this formula is only valid for energies where our description of reality is valid. For example, at energies of the order of l_p^{-1}, our states would start to form Black Holes, which would mean that, at least above this energy, our physical description of reality (Quantum Field Theory) would stop being effective. So, to account for that, we can insert a UV cutoff (ω_m) that represents the value of energy where above which reality cannot be described by our formalism. Then

$$\rho_\Lambda = \int_0^{\omega_m} \frac{dk}{(2\pi)^2} k^3 = \frac{\omega_m^4}{4(2\pi)^2}. \tag{5.34}$$

Now the problem is to choose a cutoff which is both meaningful in a physical sense and gives the measured value of ρ_Λ. One of the first attempts at this was by famous physicist Wolfgang Pauli in the 1930's [4]. He chose the inverse of the classical radius of the electron as the cutoff, $\omega_m = \frac{2\pi m_e}{\alpha}$, where m_e is the mass of the electron and α is the fine structure constant. Using the fact that $R_{dS} = (H_0\sqrt{\Omega_\Lambda})^{-1}$ he then calculated what would the radius of our Universe would be for his value of the Cosmological Constant and it gave a result much smaller than the radius of the earth.
This famous anecdote is used in cosmology to illustrate the dangers and difficulty of finding a meaningful cutoff that gives the correct results. Of course by now we know that there isn't anything particularly meaningful about the radius of the electron.
Of the cutoffs used the one that seems to make the most sense and at the same time gives the worst result is the Planck energy ($\frac{2\pi}{l_p}$). The Planck length is $l_p = 1.616229 \times 10^{-35} m$ [71] and Zero Point Energy for this cutoff is then,

$$\rho_{l_p} = \frac{(2\pi)^2}{4 l_p^4} = 21.9 \times 10^{112} eV^4 \tag{5.35}$$

The density associated with the Cosmological Constant in units of the critical density is $\Omega_\Lambda = 0.734$ [72] were the critical density is the density that the Universe has to have in order to be a flat Friedmann Universe, and it is $\rho_c = 4.061 \times 10^{-11} eV^4$ [73]. So the energy density associated with the Cosmological Constant is $\rho_\Lambda = 2.981 \times 10^{-11} eV^4$. This means that the ratio of the Zero Point Energy density and the Cosmological Constant density is

$$\frac{\rho_{l_p}}{\rho_\Lambda} \sim 10^{123}. \tag{5.36}$$

The vacuum energy density is 123 orders of magnitude bigger than what it should be. This has been called one of the greatest problems in theoretical physics today [66].

Another cutoff often suggested is the neutrino mass (m_ν) on the grounds that it gives a value very close to ρ_Λ but there are no physical arguments to suggest why [4, 74], it would only be reverse engineering. Even if the use of a cutoff as a means of renormalizing (5.33) may seem a little too crude and ignoring the curvature of the Universe to careless, more complete methods of renormalization were used to calculate ρ_{ZPE} (with and without the influence of curvature) with no better results [4].

There is one final comment to be made regarding the relation between the Cosmological Constant and the Zero Point Energy - what about the spontaneous symmetry breaking? When it occurs it changes the position of the minimum of energy and that must have an effect on the Cosmological Constant. To show how this can happen let us use one of the simplest quantum field theories with a potential, the ϕ^4-theory. It is described by the following Lagrangian

$$\mathcal{L}_\phi = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4 = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V[\phi], \tag{5.37}$$

with $\lambda > 0$. The lagrangian that gives the Einstein equation (5.1) is [75]

$$\mathcal{L}_E = -\frac{1}{16\pi G} R - \rho_\Lambda \tag{5.38}$$

so that the total Lagrangian of a system (field plus curvature) is

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V[\phi] - \frac{1}{16\pi G} R - \rho_\Lambda. \tag{5.39}$$

Let us redefine the field Lagrangian so that it accounts for the Cosmological Constant (as it would if it is the Zero Point Energy),

$$\mathcal{L}'_\phi = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V[\phi] - \rho_\Lambda. \tag{5.40}$$

Now we calculate the energy-momentum tensor of the field ϕ

$$T_{\mu\nu} = 2 \frac{\partial \mathcal{L}'_\phi}{\partial g^{\mu\nu}} - g_{\mu\nu} \mathcal{L}'_\phi = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\alpha \phi \tag{5.41}$$

Finally, the vacuum energy-momentum tensor is

$$\langle 0 | T_{\mu\nu} | 0 \rangle = g_{\mu\nu} \left(\rho_\Lambda + \langle 0 | V[\phi] | 0 \rangle \right) \tag{5.42}$$

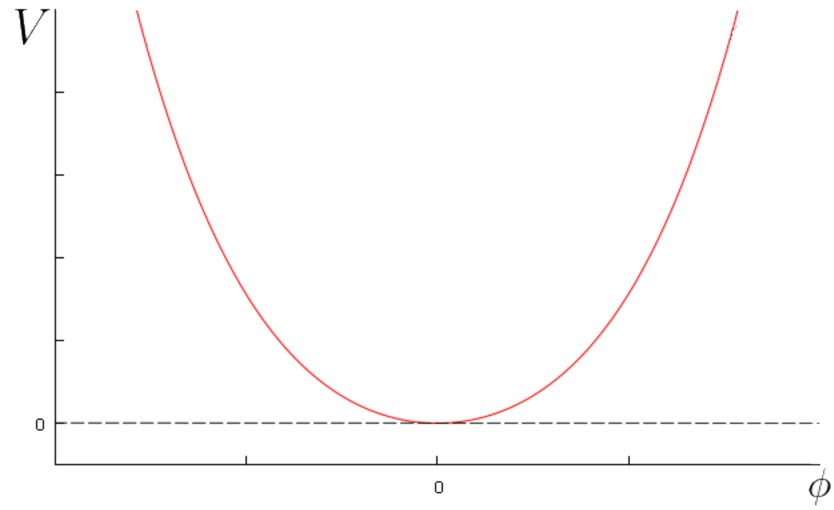


Figure 5.1: Shape of the potential V in the case of $m^2 > 0$.

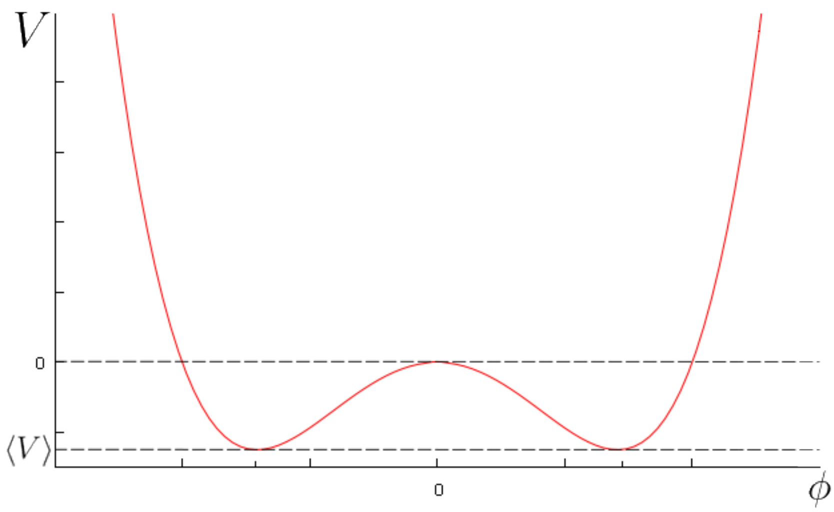


Figure 5.2: Shape of the potential V in the case of $m^2 < 0$ where the global symmetry of the system was broken.

Now, since

$$V[\phi] = \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\phi^4, \tag{5.43}$$

then, if $m^2 \geq 0$ the potential will have the form of fig. 5.1 with a minimum $\langle V \rangle = 0$ in $\phi = 0$. If $m^2 < 0$ then the potential will have the form shown in fig. 5.2, then the potential will have two minima $\langle V \rangle = \frac{-3m^4}{2\lambda}$ in $\phi = \pm\sqrt{\frac{-6m^2}{\lambda}}$. For the Higgs boson, the new minimum has the following value [4]

$$\rho_{\text{Higgs}} = -m^2_{\text{Higgs}} \frac{m^2_W}{g^2_W} \simeq -1.2 \times 10^{44} eV^4 \tag{5.44}$$

where m_{Higgs} is the mass of the Higgs, m_W is the mass of the gauge bosons $W^{\pm}$ and g_W is the weak gauge coupling. Finally, the total vacuum density (which would correspond to the measured value of the Cosmological Constant energy density) of the system is

$$\rho_{ZPE} \simeq \rho_\Lambda - 1.2 \times 10^{44}. \tag{5.45}$$

This means that because of the spontaneous symmetry breaking and regardless of the origin of Λ, ρ_Λ would have to be

$$\rho_\Lambda = \rho_{ZPE} - \rho_{\text{Higgs}} = 1.2 \times 10^{44} + 2.98 \times 10^{-11} eV^4$$

in order for the measured density ρ_{ZPE} to be the correct value. It would need to have a precision of 55 decimal places to cancel out the contribution of ρ_{Higgs} in just the right way. An incredible precision to ask for, especially if one is to consider Λ just a parameter of (5.1) without no physical meaning. Although not necessarily wrong, this result is so strange it was called the "fine-tuning problem" [4].

5.1.2 Cosmological Constant from Holography

As we have seen in chapter 4, the maximum number of states in the (visible) Universe must be smaller than its surface. This means that if the Cosmological Constant comes from the Zero Point Energy then in (5.33) we are over-counting states:

$$N_s = \int d\mathbf{r} \int d\mathbf{k} \sim \frac{V}{\epsilon^3}, \tag{5.46}$$

where $\omega_m \sim \frac{1}{\epsilon}$ is the UV energy cutoff.

Here a clarification is needed: in (5.46) we are counting the number of states in which a particle can be (N_s) while in (4.1) we are counting the number of states the system can be in, or in other words, the number of quantum degrees of freedom of the system (N).
For example, if we have a discrete chain with four sites were we can put a fermion with spin $\frac{1}{2}$, then we have 2×4 states were the fermion can be. Is the case of n sites we have $2 \times n$ states. If we have, however, four o those fermions, one in each site, then the system has 2^4 possible configurations or states (2^n for n sites). For many lattice systems we will have something like $N \sim e^{CN_s}$ [76] where, in the example given, $C = \frac{\log 2}{2}$.

Since a system in a box V with a UV cutoff ω_m is similar to a lattice system with the same size and with separation ϵ, we can use these relations. [76] This is so because of the uncertainty principle: $\delta E \delta r \sim 1$. Since $E \leq \omega_m$ then $\delta E \leq \omega_m \Leftrightarrow \delta E \geq \epsilon^{-1}$ which means that $\delta r \sim \frac{1}{\delta E} \geq \epsilon$, there will be a minimum distance between the particles of the system.

Finally, (5.46) implies that

$$S \sim \frac{V}{\epsilon^3} \tag{5.47}$$

instead of

$$S_{BH} \sim \frac{A}{l_p^2} \tag{5.48}$$

unless we choose an ϵ that somehow nullifies the extra dimension of V.
So Lets try to do just that,

$$R^3\omega_m^3 \sim \frac{R^2}{l_p^2} \Leftrightarrow \omega_m = (Rl_p^2)^{-\frac{1}{3}}. \tag{5.49}$$

This means that, since the energy density will be of the order of $\rho \sim \omega_m^4$,

$$\rho \sim (Rl_p^2)^{-\frac{4}{3}} \sim 10^{31} eV^4 \tag{5.50}$$

which is 42 orders of magnitude bigger than ρ_Λ. Better than the last result, but still a disaster. There is, however, something else that can be done in the context of the Holographic Principle.
Following the reasoning in [76] we can add another Holographic limit if we say that no system can have more energy than a Black Hole of the same size (otherwise it would become a Black Hole). The energy of a Black Hole is just the relativistic rest energy,

$$E_{BH} = M_{BH} = \frac{R_S}{2G}. \tag{5.51}$$

Hence

$$E_{\text{syst}} = \frac{4}{3}\pi R^3 \omega_m^4 \leq \frac{R}{2G} \tag{5.52}$$

or, in terms of the energy density,

$$\rho \sim \frac{1}{R^2 l_p^2} \sim 10^{-10} eV^4, \tag{5.53}$$

which is just one order of magnitude above the Cosmological Constant value!
This means that the UV cutoff would be (approximately),

$$\omega_m^H \sim (Rl_p)^{-\frac{1}{2}}. \tag{5.54}$$

Note that this means that the entropy of the Universe will never saturate the Holographic Entropy Bound,

$$R^3\omega_m^3 = S_{\text{syst}} \sim S_{BH}^{\frac{3}{4}}, \tag{5.55}$$

consequently, the Holographic Energy Bound is stronger than the Entropy Bound.
Finally, after accounting all constants we have that the Holographic Cosmological Constant energy density is

$$\rho_H = \frac{3}{8\pi l_p^2 R^2} \tag{5.56}$$

Next step is to see if the density in (5.53) behaves as the Cosmological Constant energy density in equations (5.7) and (5.17). That behavior has been studied in [77, 78] and we will dedicate the rest of this subsection to present it.

First, let us assume that the entropy of the Universe doesn't completely saturate the Energy Bound (5.52), such that

$$R^3\omega_m^4 = d^2\frac{R}{2l_p^2} \tag{5.57}$$

where d^2 is just a proportionality constant. Obviously, $d^2 \leq 1$. Then (5.56) becomes

$$\rho_H = \frac{3d^2}{8\pi l_p^2 R^2}. \tag{5.58}$$

Let us also assume that this Dark Energy has the following equation of state,

$$p = \mu\rho, \tag{5.59}$$

where μ doesn't need to be -1.

For now, we will take the size of the Universe to be the size of its Event Horizon,

$$R_{eh}(t) = a(t)\int_t^\infty \frac{d\tilde{t}}{a(\tilde{t})} = a(t)\int_{a(t)}^\infty \frac{d\tilde{a}}{\tilde{a}^2\tilde{H}}. \tag{5.60}$$

Regardless of how we define the size, in principle, (5.58) will depend on time through R. In the de Sitter limit, $R_{eh}(t\to\infty)\to R_{dS}$, the Event Horizon is constant and we recover (5.15): $\rho_H(t\to\infty)\to$ constant.
From (5.7) we can get a differential equation for ρ_H:

$$\dot{\rho}_H = -3H(\rho_H + p_H) = -3H(\mu+1)\rho_H. \tag{5.61}$$

Since

$$\dot{\rho}_H = \frac{d}{da}(\rho_H)Ha = -\frac{2Ha}{R}\rho_H\left(\frac{R}{a} - \frac{1}{aH}\right) \tag{5.62}$$

then (5.61) becomes

$$\mu = -\frac{1}{3}\left(1 + \frac{2}{RH}\right). \tag{5.63}$$

Defining

$$\bar{\Omega}_i(t) = \frac{\rho_i(t)}{\rho_c(t)} \tag{5.64}$$

such that $\bar{\Omega}_i(t=0) = \Omega_i$, then, from (5.58) we have that

$$\mu = -\frac{1}{3}\left(1 + \frac{2}{d}\sqrt{\bar{\Omega}_H}\right). \tag{5.65}$$

It is easy to see that for $d > 0$ that $\mu < -\frac{1}{3}$. Since R_{eh} will be getting bigger until it reaches R_{dS}, then $\dot{R}_{eh} \geq 0$, and so,

$$\dot{A} \geq 0 \tag{5.66a}$$

$$\dot{V} \geq 0 \tag{5.66b}$$

Equation (5.66a) gives us the following inequality

$$\dot{A} = 8\pi R\dot{R} = 8\pi R(HR-1) =$$

$$= 8\pi R\left(\frac{d}{\sqrt{\bar{\Omega}_H}} - 1\right) \geq 0 \tag{5.67}$$

which is only true if

$$d \geq \bar{\Omega}_H. \tag{5.68}$$

This, together with (5.65) gives us another condition for μ,

$$\mu \geq -1. \tag{5.69}$$

Equation (5.66b) gives us the exact same condition.
Curiously enough, the comoving volume V_c will shrink,

$$\dot{V}_c = 4\pi R_c^2 \dot{R}_c = -\frac{4\pi}{a^3} R_{eh}^2, \leq 0, \tag{5.70}$$

since R_{eh} and a will always be positive in an expanding Universe. This means that, since the comoving entropy density $\sigma_c = a^3\sigma$ is constant [56] then the entropy of the Universe must also decrease with time. So, if the Holographic Entropy Bound was obeyed in earlier times it will be obeyed forever [56,76]. Consequently it is true that, for a Universe of size R_{eh}, the Entropy Bound is a weak bound, as argued in [76].

Finally, the bounds determined for μ and d^2 by [77] are,

$$\begin{cases} \bar{\Omega}_H \leq d^2 \leq 1 \\ -1 \leq \mu \leq -\frac{1}{3} \end{cases} \tag{5.71}$$

We can easily see that, for $\bar{\Omega}_H = 1$ this reduces to,

$$\begin{cases} d^2 = 1 \\ \mu = -1 \end{cases} \tag{5.72}$$

The Energy Bound is saturated and the equation of state $p = -\rho$ finally has the intended behavior.

It turns out that the limit $\bar{\Omega}_H = 1$ corresponds to a flat de Sitter Universe. Since this type of Universe is dominated by the Cosmological Constant, then, if $k = 0$, $\rho_H = \rho_c$ for all times. Hence, we recover, in this limit, the exact behavior of the Cosmological Constant - it is truly constant and creates an accelerating expansion, with $p = -\rho$.
But why doesn't the holographic theory presented this section accounts for the correct pressure of the Cosmological Constant without having to take the limit of our Universe to the de Sitter Universe ($t \to \infty$)? Maybe the reason if that since we start with the assumption that the Quantum Vacuum constitutes all of the states of the observable Universe (such as in (5.33)) then it makes sense that we have only the correct behavior of the Cosmological Constant when that assumption is fulfilled at a cosmological level. Meaning that, we are assuming right away the we are in a de Sitter Universe, and that is possibly why the holographic Cosmological Constant only behaves like the real Cosmological Constant in this limit. Also, if the Cosmological Constant density is really constant (as in (5.15)) then by deducing the correct Cosmological Constant in the de Sitter spacetime we will have the correct Cosmological Constant for any time.

Using similar arguments as the ones presented in here, it was determined in [78] that the past Horizon:

$$R_{\text{past}} = a(t) \int_0^t \frac{dt'}{a(t')},$$

used in (5.58), does not give a Cosmological Constant with the correct equation of state, such that the Universe wouldn't accelerate.

Cosmological Constant from Entanglement

In [10] the possible role of the Quantum Entanglement as the origin of the dilution of states necessary to obtain (5.58) was explored.

Generally the entropy of the entanglement of a scalar field ϕ in the ground state on a system with a spherical boundary (of radius R) takes the form [16–19]

$$S_{\text{Ent}} = \beta \frac{R^2}{\epsilon^2}, \tag{5.73}$$

where β is a proportionality constant. Notice the striking similarity of the entanglement entropy and the entropy of a Black Hole, (4.8). This may indicate a significant role of entanglement in determining the degrees of freedom of a Black Hole and, through the Holographic Principle, of our Universe.
In a nutshell, (5.73) was calculated by constructing the density matrix of a scalar field ϕ in a lattice with spacing ϵ and then tracing out the states outside the boundary in order to calculate the von Neumann entropy. This way, the field could be (approximately) considered as discrete, which makes the detection of the entanglement possible.[3]
In particular, Müller in [19] calculates this relation for a field in a spherical lattice of separation ϵ in a Friedmann Universe, with $R = R_{eh}$. In both [16, 19] the proportionality constant was determined to be $\beta = 0.3$ (it would have to be $\beta = \pi$ to saturate (4.8)).
Finally, if there are a number N_{dof} of fields in the Universe then each one of them can contribute an entropy of the type of (5.73) to the total entanglement entropy. In this case, if we assume that the contribution of each field is more of less similar, then [10, 19]:

$$S_{\text{Ent}} = N_{dof} \beta \frac{R^2}{\epsilon^2}. \tag{5.74}$$

The First Law of Black Hole thermodynamics (for a Schwarzschild Black Hole) reads [51]

$$dE_{BH} = T_H dS_{BH}. \tag{5.75}$$

Because (5.73), a similar definition for the entanglement "thermodynamics" may be adopted [10, 17]

$$dE_{\text{Ent}} = T_{\text{Ent}} dS_{\text{Ent}}. \tag{5.76}$$

[3] This approximation is similar to what was done in order to relate (5.46) with the entropy of the system - the cutoff ϵ produces a sort of lattice.

In accordance with the theory discussed at the end of chapter 4, there seems to be indications that the majority of the contributions to the quantum correlations accounted in (5.73) come from states near the Horizon [18, 20]. It makes sense, then, to consider T_{Ent} as the Gibbons-Hawking temperature of our Universe's Event Horizon: $T_{\text{Ent}} = \frac{1}{2\pi R_{eh}}$ [54]. From this and (5.73), (5.76) gives us an entanglement energy:

$$E_{\text{Ent}} = \frac{N_{dof}\beta R_{eh}}{\pi\epsilon^2}. \tag{5.77}$$

This means, of course, that the energy density of the entanglement will be

$$\rho_{\text{Ent}} = \frac{3\beta N_{dof}}{4\pi\epsilon^2 R_{eh}^2}. \tag{5.78}$$

If we take $\epsilon = l_p$ then we get something very similar to (5.58). Assuming just that we can get a relation between β and d:

$$d = \frac{\sqrt{\beta N_{dof} l_p^2}}{2\pi\epsilon} = \frac{\sqrt{\beta N_{dof}}}{2\pi}. \tag{5.79}$$

For $\beta = 0.3$ and $N_{dof} = 118$ [4] we would have $d = 0.95$ and, using (5.65) we get $\mu(t = 0) = -0.93$, which is very close to the intended behavior.
Of course for a more complete description of d we would need to account for different β_i for every field i [10],

$$\sum_i \beta_i N_{dof}^i = 4\pi^2 d^2, \tag{5.80}$$

and, of course, calculate β in the continuous limit.

Energies similar to (5.77),

$$E_{\text{Ent}} \propto \frac{R}{\epsilon^2}, \tag{5.81}$$

were obtained in [17] after a gravitational correction $\sqrt{g_{tt}} = \frac{\epsilon}{R}$ was multiplied to the original entanglement energy, because of the proximity of the entangled states to the Horizon.

Although it might seem a little bit shady to consider a thermodynamic equation for the entanglement entropy as in (5.76), it is also very elegant how it relates the Entropy Bound with the Energy Bound. The results obtained by [10] seem to indicate that forcing the vacuum to obey (5.57) and (5.48) is equivalent and the fact that the entropy of the Universe is apparently far from saturating the Entropy Bound happens because we are not accounting for the existence of entanglement - its entropy can't be accounted for by classical thermodynamic considerations.

To end this section, we will just mention that there is also another theory that relates entanglement with the Cosmological Constant, developed in [11], in which the equation of state of the Cosmological Constant is deduced from the evolution of entanglement inside our Universe.

[4] This is the number of particles in the Standard Model. For the Minimal Supersymmetric Standard Model it would be $N_{dof} = 244$. [10]

5.1.3 The Quintessential Model

Now, we shall briefly talk about the Quintessential model which is one of the big candidates to explain the Dark Energy. The idea behind it is to treat the Dark Energy not as a Cosmological Constant that appears in (5.1), but as a real scalar field ϕ, with a Lagrangian [4]

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V[\phi], \tag{5.82}$$

which, in a Friedmann metric, leads to the following field equation [4]

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{dV}{d\phi} = 0. \tag{5.83}$$

Here, V is a potential that depends on the field ϕ.

From (5.6) and from the formula of $\mathbf{T}$ for a scalar field in a general spacetime [6],

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left[\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V[\phi] \right], \tag{5.84}$$

we get that

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V[\phi], \tag{5.85a}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V[\phi]. \tag{5.85b}$$

For a very slowly varying Dark Energy, $\dot{\phi}^2 \ll V[\phi]$ and we recover the right equation of state, $p_\phi = -\rho_\phi$.

But why a scalar field? Contrary to other theories, where Dark Energy is not time dependent, in the Quintessential Model time dependence is an important feature, for it also uses the field ϕ ([5]) to explain the inflation period (a period of exponential expansion that followed the Big Bang). This way, the field can vary in order to explain the inflation and how the Universe evolved to its state today. [5, 6] The potential $V[\phi]$ is often chosen to accommodate the features that we want the theory to have, such as how the field should vary during the Universe's history. Some examples o potentials used are [6]

$$V_1[\phi] = V_0 e^{-A\phi}, \tag{5.86}$$

and,

$$V_2[\phi] = \frac{B^{4+n}}{\phi^n}, \tag{5.87}$$

where V_0, A and B are constants that need to be determined in order for ρ_ϕ to have the correct value (often a formula similar to (5.58)). [6]

Although the theory relates two concepts of Cosmology in an elegant way, it still seems to have some problems. It also has some fine-tuning problems [4], the nature and interpretation of the field ϕ is a mystery, and it depends a lot on choosing the right potential and accompanying parameters (which often relates to the fine-tuning problems [6]).

For a deeper understanding and variations of the Quintessential Model, and similar theories, the reader is referred to [6].

5.2 Experimental Evidence

In this section we shall briefly talk about the most important methods that prove the existence and calculate the value of the Cosmological Constant. Special emphasis will be made on the method used in [1, 2] since these were the first observations to measure the Ω_Λ. In those observations, supernova explosions at different distances from the Earth where measured and then, using the redshift of light coming from the explosion, the expansion of the Universe could be measured which showed that the Universe was accelerating.

There are several types of supernovae, the ones used in [1, 2] where type Ia supernovae, which are believed[5] to originate from a binary system of two stars where at least one of them is a white dwarf. The white dwarf acquires matter from the other star (for possible mechanisms of this, see [79]) and when it reaches a certain critical mass, it explodes - the explosion is called the supernova.

The luminosity of an astronomical object is defined as the total amount of energy emitted per unit of time,

$$L = \frac{dE}{dt}. \tag{5.88}$$

The brighter the object is, the higher luminosity it has. The maximum of L can be predicted because there is a strong correlation between the maximum and the rate of decrease of L (see fig.5.3 for an example of a curve of L). Also, L is a characteristic of the supernova, they all emit more or less the same amount of energy, since the white dwarf is believed to explode always at (approximately) the same critical mass. Hence in principle, the luminosity does not depend on which region of space the supernova is. That, and because of their very intense brightness, makes them ideal to calculate the distances in our Universe.

Even though the luminosity is a characteristic of the astronomical object, the further away it is, the less bright we see it. That is because, as observers, we do not see the luminosity, what we see is the flux of energy, F, (or, in other words, the radiation) that arrives to us. The brightness of a distant astronomical object is, then, measured in terms of its apparent magnitude m (which is proportional to the logarithm of F, see (5.89)). The bigger m is, the less bright the object seems. Usually the apparent magnitude is compared to the absolute magnitude M which is the magnitude that the astronomical object would have if it was situated at a distance of 10 parsecs from earth. We have [6, 79]

$$m - M = \mu = -2.5 \log_{10}\left(\frac{F}{F'}\right), \tag{5.89}$$

[5] There is no standard model for the origin of this type of supernovae [68].

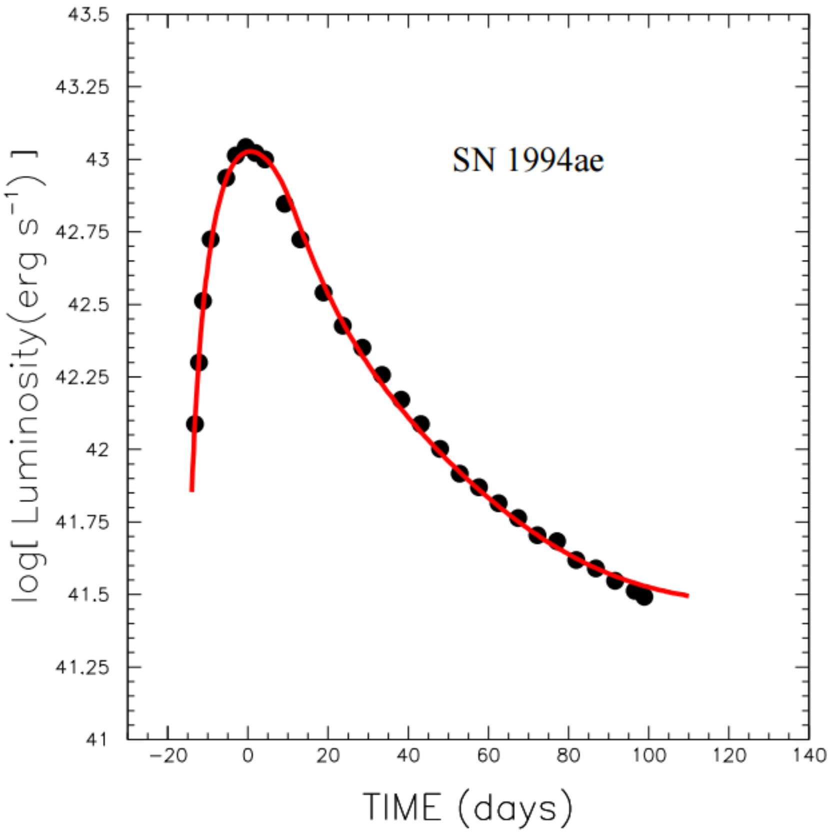


Figure 5.3: The evolution of a type Ia supernova's luminosity (in this case, $\log L$) as time passes. The red line is the prediction according to a analytic model (see [80]), and the black dots are the measurements from [81]. Notice that the fitting between theory and experiment is very good. The graphic was taken from [80].

where F is the flux of energy observed on earth, and F' is the energy flux that would be observed if the object was at 10 parsecs from earth.

But how is F related to L? A flux is defined as

$$F = \frac{dI}{dA}, \tag{5.90}$$

where $I = \frac{dq}{dt}$ is the current of the quantity q, and A is the area through which this quantity flows. In our case, the quantity q is energy and $I = L$. For objects emitting the same quantity of radiation in all directions, we have that [6, 79]

$$F = \frac{L}{A} = \frac{L}{4\pi D_L^2}, \tag{5.91}$$

where D_L is our distance to the object, and it is called the luminosity distance. Without expansion, D_L would just be the Euclidean distance, but this is not the case, since our Universe is expanding and there is redshift of the light being observed, which, of course, results on a difference in the measured quantity m (and F). In this case, D_L relates to two other distances: the area distance d_A measured by the observer here on earth (see fig.5.4), and the transverse comoving distance d_G measured by the body emitting the radiation (see fig.5.5). [82, 83]

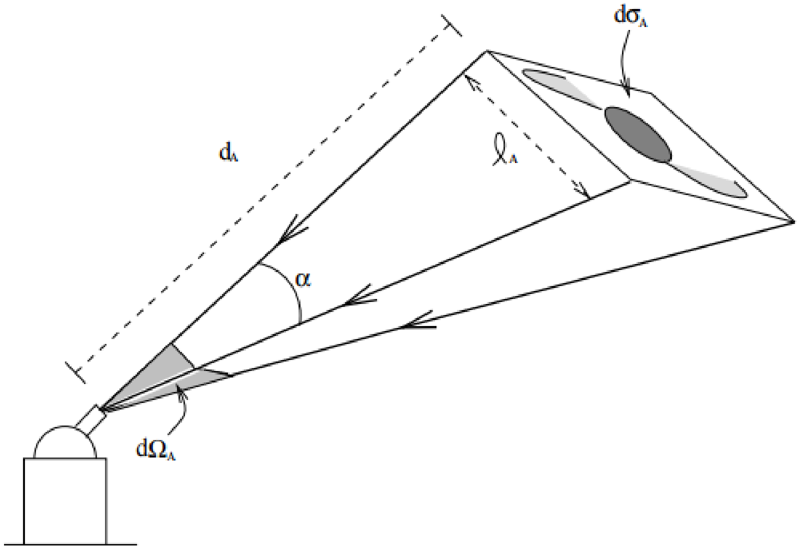


Figure 5.4: In this figure we have a graphical representation of the area distance, d_A. This distance can be defined as the distance traveled by photons in a past directed congruence of null geodesics that start at t_0 from the observer, with a solid angle $d\Omega_A$, and end in the astronomical object with cross-section $d\sigma_A$. In first approximation, it can be determined as $d_A \simeq \frac{l_A}{\alpha}$ for very distant (or very small) astronomical objects. This figure was taken from [82]

The relation between the distances is called the *Etherington's reciprocity theorem* [82, 83],

$$D_L = (1+z)d_G = (1+z)^2 d_A, \tag{5.92}$$

where

$$\frac{1}{1+z} = \frac{a(t_{\text{obs}})}{a(t_{\text{em}})}, \tag{5.93}$$

and z is the redshift factor.

The time when we observe the radiation emitted by the astronomical object, corresponds to the present time $t_{\text{obs}} = t_0 \Leftrightarrow a(t_0) = 1$. For that reason, we will denote the time of emission of the radiation, t_{em}, as just t.
The relation between the transverse comoving distance, d_G, and the comoving distance between the observer and the emitter is, for a flat ($k = 0$) Friedmann universe, [84]

$$d_G = a(t_0)d_c = d_c, \tag{5.94}$$

where,

$$d_c = \int_t^{t_0} \frac{dt}{a(t)}. \tag{5.95}$$

Then, equation (5.91) becomes

$$F = \frac{L}{4\pi d_c^2(1+z)^2}. \tag{5.96}$$

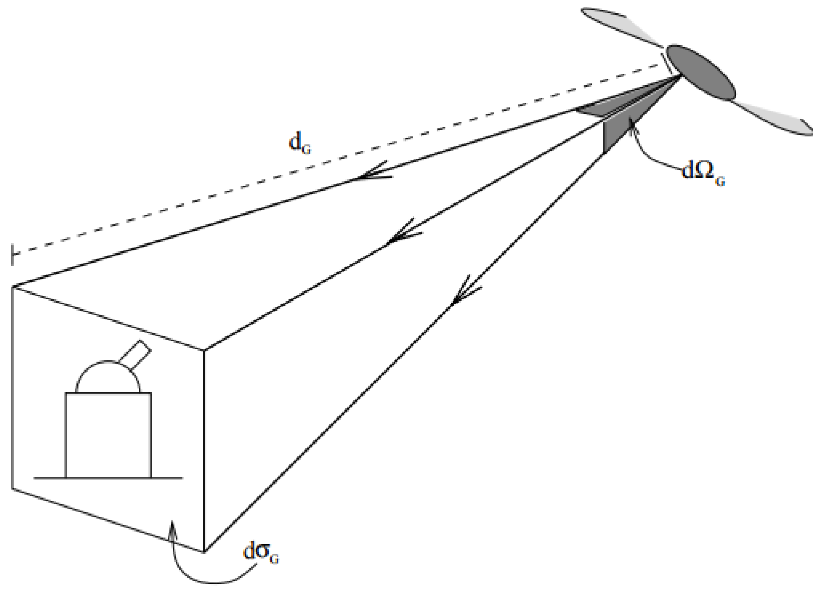


Figure 5.5: In this figure we have a graphical representation of the transverse comoving distance, d_G. This distance can be defined as the distance traveled by photons in a future directed congruence of null geodesics that start from the astronomical object, with a solid angle $d\Omega_G$, and stops when it reaches the observer, at t_0, with a cross-section $d\sigma_G$. This figure was taken from [82]

From (5.91) and (5.89), we get

$$\mu = 5\log_{10}\left(\frac{D_L}{Mpc}\right) + 25. \tag{5.97}$$

Also, from (5.95), we have that

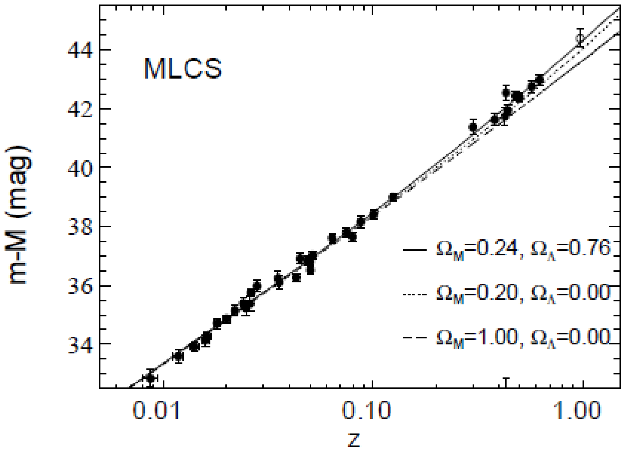
$$D_L = (1+z)\int_a^1 \frac{da}{a^2 H} = -(1+z)\int_0^z \frac{dz}{H} = \\ = \frac{1+z}{H_0}\int_0^z \frac{H_0}{H}dz, \tag{5.98}$$

hence, we can get a relation between the measured quantity $m-M$ and the curvature codified in the Hubble parameter, H, and the redshift parameter z,

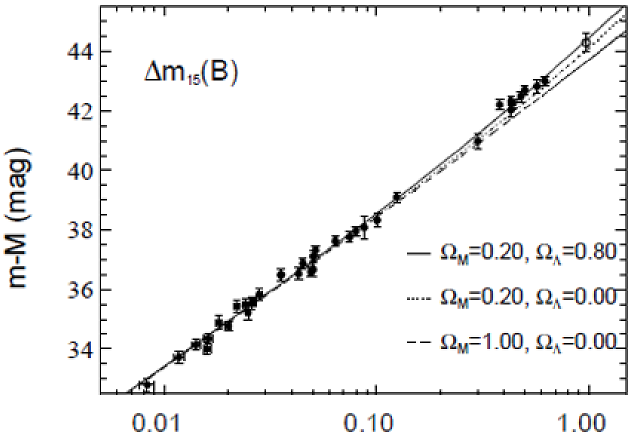
$$10^{\frac{\mu-25}{5}} = \frac{1+z}{H_0}\int_0^z \frac{H_0}{H}dz. \tag{5.99}$$

This means that the magnitude of an astronomical object is only dependent on z (since H varies with a which, in turn, depends on z). We can now test which combination of parameters in (5.17) gives us the best fitting for the graphic μ vs z.

Note that, what has been described until here is just a simplified version how the observations of the brightness of type Ia supernovae are made. Of course, in observations of this complexity, we have the detectors involved, other possible objects that can exist between us and the supernovae observed (like, for example, dust, or



(a) The parameters that best fit the Supernovae data, using the MCLS, are $H_0 = 65.2 \pm 1.3\ km\ s^{-1}\ Mpc^{-1}$, $\Omega_{m0} = 0.24$ and $\Omega_\Lambda = 0.76$.



(b) The parameters that best fit the Supernovae data, using the template fitting method (Δm_{15}), are $H_0 = 63.8 \pm 1.3\ km\ s^{-1}\ Mpc^{-1}$, $\Omega_{m0} = 0.2$ and $\Omega_\Lambda = 0.8$.

Figure 5.6: These μ vs z plots show the data taken from the type Ia supernovae (dots) [1], and also the theoretical predictions obtained using several values for the parameters H_0, Ω_{m0} and Ω_Λ (lines). The Ω_{r0} was assumed to be zero in the article. The two graphics correspond to two different methods used to analyze the results of the observation of the supernovae. They were taken from [1]. Notice the best fit corresponds to $\Omega_{m0} \sim 0.24$ and $\Omega_\Lambda \sim 0.8$ which are very close to the most recent accepted values [5]. The data observed suggests the existence of a Cosmological Constant that dominates the Universe's density and originates the accelerated expansion.

objects that distort the radiation emitted, with gravity), etc. The analysis of the results can get more involved. That is why, for example, two graphs are shown in fig. 5.6, they correspond to two different methods of analyzing the results [1]. However, all the important concepts where described here.

As mentioned above, type Ia supernovae where used in [1, 2]. The data obtained in [1] are shown in fig.5.6, with some plots of μ_{teor} using different values of the parameters H_0, Ω_{m0} and Ω_Λ in (5.17) (Ω_{r0} was taken to be zero since at the present time it is much smaller than the other densities [1, 6]). Notice that the best fit is achieved for $\Omega_{m0} \simeq 0.2$ and $\Omega_\Lambda \simeq 0.8$ (see the original articles for a more in depth analysis of the statistical significance of this values). Since those articles were published, more measurements have been made (for example, fig.5.7) that reinforce the conclusion that there is a Cosmological Constant, and its value has been refined to $\Omega_\Lambda \sim 0.7 \neq 0$, which means that the Universe does seem to be expanding at an accelerating rate. [5, 6, 68]

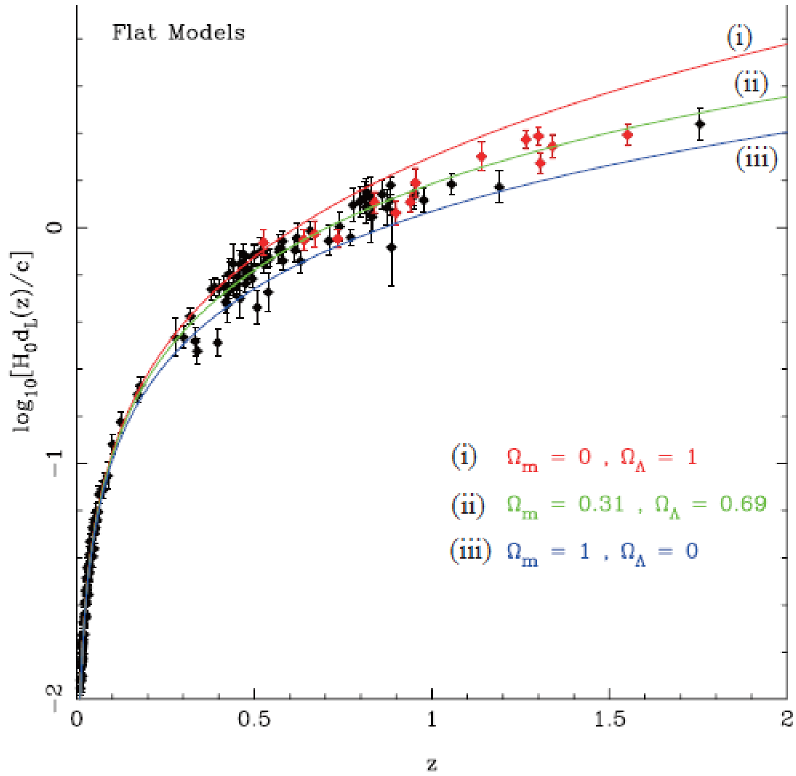


Figure 5.7: Here we show the plot of the logarithm of the luminosity distance (the c is just the velocity of light) vs z. Minus a few constants, this graphic is equivalent to the ones in fig. 5.6, but for bigger values of z. With the new data, it is easier to notice that best model of the Universe corresponds to a Cosmological Constant with $\Omega_\Lambda \sim 0.7$. The image was taken from [6]

There is, however, some room for other values of $(\Omega_m, \Omega_\Lambda)$, when regarding the data obtained from the supernovae. [68] For example, different methods are used to analyze the results from the supernovae, and those will often give different results, just like in fig. 5.6. Also, there is still some uncertainty in the values of the Hubble constant H_0 obtained from other observations. Current estimates of the Hubble constant vary between 60 and 75 $km\ s^{-1}\ Mpc^{-1}$ [68,85], and of course, the value we use for H_0 will affect the choice of parameters Ω_m, Ω_Λ) that best fit the observations. In [68], the possibility of other values of $(H_0, \Omega_{m0}, \Omega_\Lambda)$ is explored, in particular the possibility of a Universe with constant expansion (Milne Universe).

Finally, what really gives us confidence about the existence of a Cosmological Constant is that the results obtained through the study of the type Ia supernovas have also been confirmed by a number of other methods. For example, a Universe without Ω_Λ does not have an age consistent with the known age of the Universe, although this fact does not specify the value of Ω_Λ since, for example, both a Milne Universe and an accelerating Universe give the correct age. [68] More importantly there are two methods which confirm the existence of a Cosmological Constant with $\Omega_\Lambda \sim 0.7$. [68] The first is that, $\Omega_\Lambda \sim 0.7$ is consistent with the independent measures made of Ω_m. Since evidence suggests that $k = 0$ [5], and $\Omega_r \ll \Omega_m$, then, if $\Omega_\Lambda = 0$, then $\Omega_m = \Omega = 1$. However, since $\Omega_m \sim 0.3$, that means there must be something in the Universe with a density of about $1 - \Omega_m = 0.7$ to explain how the Universe is flat ($k = 0$) with such a low density of matter. The second source of evidence for this Cosmological Constant is the measurements of the angular spectrum of the *Cosmological Microwave Background*, whose description is beyond the scope of this Thesis. For more details about these latter measurements, and also on other possible interpretations of the data supporting the Cosmological Constant, the reader is referred to [68].

Part III

Cosmological Constant, Entanglement and Event Horizons

Chapter 6

Cosmological Constant: A Holographic Vacuum Energy

6.1 Zero Point Energy and the Holographic Principle

Several method for calculating the Cosmological Constant using the Holographic Principle were presented in chapter 5. All of them used Thermodynamics related arguments to compute the Cosmological Constant, giving (approximately) the same results.

To try to enhance our understanding of how this dilution of possible states in the Universe works, a mechanism is proposed in order to unite these types of Holographic theories with the calculation of the Cosmological Constant through the Zero Point Energy [14]. Here, we take the number of states of a scalar field, but now we add a function $f(\mathbf{r},\mathbf{k})$ that dilutes the number of states in the Universe to the correct value ($N_{hol.}$),

$$N_{hol.} = \int d\mathbf{r} \int \frac{d\mathbf{k}}{(2\pi)^3} f(\mathbf{r},\mathbf{k}) \tag{6.1}$$

But how many states will there be? It is argued in [14, 15, 64] that we must account not for the number of states in a surface, as in section 4.1, but the number of states on a spherical shell with the depth of l_p (see 6.1), as was discussed at the end of section 4.3.

Since the best spatial definition we can have is of order of l_p then certainly the closest shape we can have to a spherical surface is a spherical shell and so, the information of any state in our Universe will be projected to a state somewhere inside this shell. Hence, it is reasonable to consider that the maximum number of states our Universe can have is proportional to this Holographic shell. Since for a massless scalar field, $\omega = k = \frac{2\pi}{\lambda}$, we will take the energy cutoff as $\omega_m = \frac{2\pi}{l_p}$, since $\lambda_{\min} \sim \delta r_{\min} \sim l_p$. We must keep in mind that ω_m (or $\lambda_{\min}$) cannot be defined with that much precision. The physics right before we reach that energy is no better described by our current understanding than the energies right above, and a few factors can always be argued

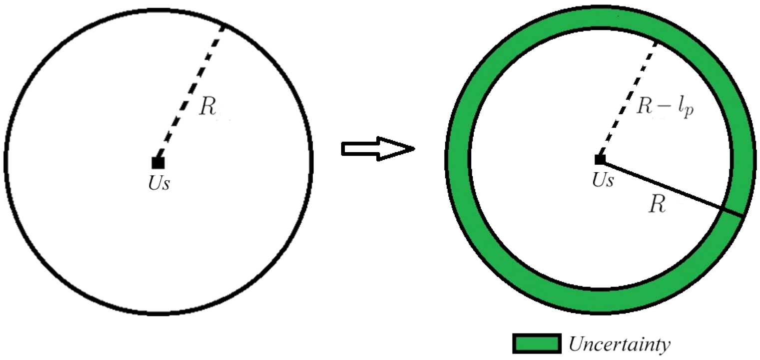


Figure 6.1: Instead of taking the number of holographic states as the states in the surface of radius R (left) we take the number of states in a spherical shell of outer radius R and inner radius $R - l_p$ according to the uncertainty principle [14, 15, 64].

to be missing in the definition of the energy cutoff. Our arguments only intend to be right in the order of magnitude and so there is no need to worry too much about being too precise with this kind of definitions.
We can also define a minimum energy cutoff, inversely proportional to the radius of the holographic surface we are using, R, such that $\omega_{\text{min}} = \frac{2\pi}{R}$.
Notice that since we are considering that the number of states in the visible Universe[1] fits inside a spherical shell, we have to consider also a shell in the k-space, with radius $k_{max} = \frac{2\pi}{l_p}$ and thickness $\frac{2\pi}{R}$.

Now we can calculate the maximum number of states,

$$N_{max} = (4\pi)^2 \int_{R-l_p}^{R} drr^2 \int_{\frac{2\pi}{l_p}-\frac{2\pi}{R}}^{\frac{2\pi}{l_p}} \frac{dk}{(2\pi)^3} k^2 \simeq 16\pi^2 \frac{R}{l_p}, \tag{6.2}$$

were we used the fact that $R \gg l_p$. Then equation (6.1) will become,

$$N_{hol.} = \int d\mathbf{r} \int \frac{d\mathbf{k}}{(2\pi)^3} f(\mathbf{r}, \mathbf{k}) \sim 16\pi^2 \frac{R}{l_p}, \tag{6.3}$$

assuming the number of states in our Universe saturates the Holographic (shell) bound.

Let us discuss the function $f(\mathbf{r}, \mathbf{k})$. It will be impossible to get the exact form of this function without a quantum Holographic theory, but we can try and guess some of its possible characteristics. For example, since quantum physics works perfectly well in small system even by over counting states, for small enough $|\mathbf{r}|$, $f(\mathbf{r}, \mathbf{k}) \sim 1$. Also, for too large energies we start getting closer to the formation of a Black Hole or, in other words, it is for large energies that the $f(\mathbf{r}, \mathbf{k})$ must dilute more. So, for small enough $|\mathbf{k}|$ we should also have $f(\mathbf{r}, \mathbf{k}) \sim 1$.For these reasons, we can guess that, at

[1]When we talk about a visible Universe we are talking about the inside of the Event Horizon we are considering. It can be the inside of the visible Universe radius or any other Event Horizon in consideration.

least in some form of approximation, $f(\mathbf{r},\mathbf{k}) \propto \frac{l_p R^{-1}}{rk}$ [14], this way, $f(\mathbf{r},\mathbf{k}) \sim 1$ at least for the smallest states ($\sim l_p$) with least energy ($\sim \frac{1}{R}$) the Universe can have. This could be, for example, the first term in a series of the form

$$f(r,k) = \sum_{nm} c_{nm} \frac{1}{r^n k^m}. \tag{6.4}$$

In order for (6.3) to be satisfied, we have that $f(\mathbf{r},\mathbf{k}) = f(r,k) = \frac{2\pi l_p R^{-1}}{rk}$, which is not too troublesome since the extra factor of 2π might come from the somewhat arbitrary definition of l_p and ω_m.

The next step is to use these functions to calculate the Cosmological Constant. We start with the Zero Point Energy of a massless scalar field, but this time accounting for the dilution of states,

$$E_{ZPE} = (4\pi)^2 \int_{l_p}^{R} dr \int_{2\pi/R}^{2\pi/l_p} \frac{dk}{(2\pi)^3}\; r^2 k^2\; \frac{\omega(\mathbf{k})}{2}\; \frac{2\pi l_p R^{-1}}{rk} \simeq \frac{8}{3}\pi^3 \frac{R}{l_p^2}. \tag{6.5}$$

Dividing the energy by the volume of the Universe we get the energy density,

$$\rho_{ZPE} = \frac{2\pi^2}{(Rl_p)^2} \tag{6.6}$$

Notice that equation (6.6) gives a very similar result to the theories presented in subsection 5.1.2. This seems to be able to relate the Holographic Energy Bound ($E \leq E_{BH}$) with the Holographic Entropy Bound ($S \leq S_{BH}$), as was done in [10].

The visible Universe has a radius $R_{\text{vis}} = 2.2 \times 10^{33}\ eV^{-1}$ [86], and the future Horizon is approaching the de Sitter Horizon $R_{dS} = \frac{1}{H} \sim 10^{33}\ eV^{-1}$ [72]. This means that, in terms of values today, both Horizons are very similar. Since we are only interested in orders of magnitude, we will use $R \sim 10^{33}\ eV^{-1}$. Then we can calculate the energy density,

$$\rho_{ZPE} = 2.4 \times 10^{-9}\ eV^4. \tag{6.7}$$

As we have seen in chapter 5, the energy density for the Cosmological Constant is $\rho_\Lambda = 2.981 \times 10^{-11}\ eV^4$.

Our value is in remarkable agreement with the experimental value, in comparison with the result obtained by this method but without the function $f(r,k)$. We need to keep in mind that these considerations are not very precise and hence very precise results can't be expected. With the holographic state density function $f(r,k)$ we got a very good reduction of the 123 orders of magnitude that the original method was off by and also recovered a formula for ρ_Λ also obtained by other authors [10, 76].

Other types of functions where tried that give back similar results to (6.6), such as $f(r,k) = \frac{l_p^2}{r^2}$, and $f(r,k) \sim \frac{1}{r^2 l_p^2}$ (a lot of terms of (6.4) give back the correct energy density). The trick here would be, of course, to try and find the real $f(r,k)$ functions, based on a Holographic Quantum Gravity theory, which is no easy task. But not all hope is lost. Let us remember the possible role that the quantum entanglement might have in the dilution of states and the Black Hole entropy (section 5.1.2 and section 4.3). The study of the entanglement caused by the expansion of our

Universe and the consequence of the existence of an Event Horizon is, then, a good path to follow. But before we concentrate on that question, we have to discuss something very important: which Event Horizon should we choose for the Holographic Principle?

6.1.1 Selecting a Holographic boundary

As we have seen in section 4.2, there can be several Holographic boundaries. Initially, it was proposed by Fischler and Susskind [56] that the radius of the visible Universe could form our boundary. In this subsection we build up from their arguments in order to determine if the radius of the visible Universe is an acceptable Holographic boundary. In the article they proved that the entropy bound is not violated for the past and near future of a Friedmann flat universe. Since [56] was published the same year as the most convincing evidence for the Cosmological Constant was put forward ([1, 2]) they had no way to test for a Universe with the Cosmological Constant which affects their arguments for the preservation of the bound in the future. We shall test the entropy bound in the future of a flat Friedmann universe with a Cosmological Constant in this section.
Even if the radius of the visible Universe works as a Holographic boundary, it might not be the best one. We will at the end of this section argue that the de Sitter radius is the best Holographic Boundary for a Friedmann universe.

Since the universe is expanding, for $t \to \infty$ we will get $a(t) \to \infty$, and equation (5.17) will become

$$\dot{a} \simeq H_0\sqrt{\Omega_\Lambda}a. \tag{6.8}$$

The solution will be something like $a \propto e^{H_0\sqrt{\Omega_\Lambda}t}$. To estimate the value of the proportionality constant we can make the approximation that today's universe is already in this limit (meaning, is already dominated by the Cosmological Constant) which is not true but since Ω_Λ is $\sim 70\%$ of the total density it is not a bad approximation. Thus, solving (6.8) we get,

$$a(t) = a(t_{\text{now}})e^{H_0\sqrt{\Omega_\Lambda}(t-t_{\text{now}})}. \tag{6.9}$$

Since $a(t_{\text{now}}) = 1$ and making $C = e^{H_0\sqrt{\Omega_\Lambda}t_{\text{now}}}$, we get

$$a(t) = Ce^{H_0\sqrt{\Omega_\Lambda}t} \tag{6.10}$$

This way, we can calculate the comoving radius,

$$R_d = \int_{t_{\text{now}}}^{t} \frac{dt'}{a(t')} + R_0 = \frac{1}{CH_0\sqrt{\Omega_\Lambda}}\left(\frac{1}{C} - e^{-H_0\sqrt{\Omega_\Lambda}t}\right) + R_0, \tag{6.11}$$

where R_0 is the radius of the visible Universe in the present. Using the expression for R_d, have that

$$\sigma R_d^3 < \frac{(aR_d)^2}{4G}, \tag{6.12}$$

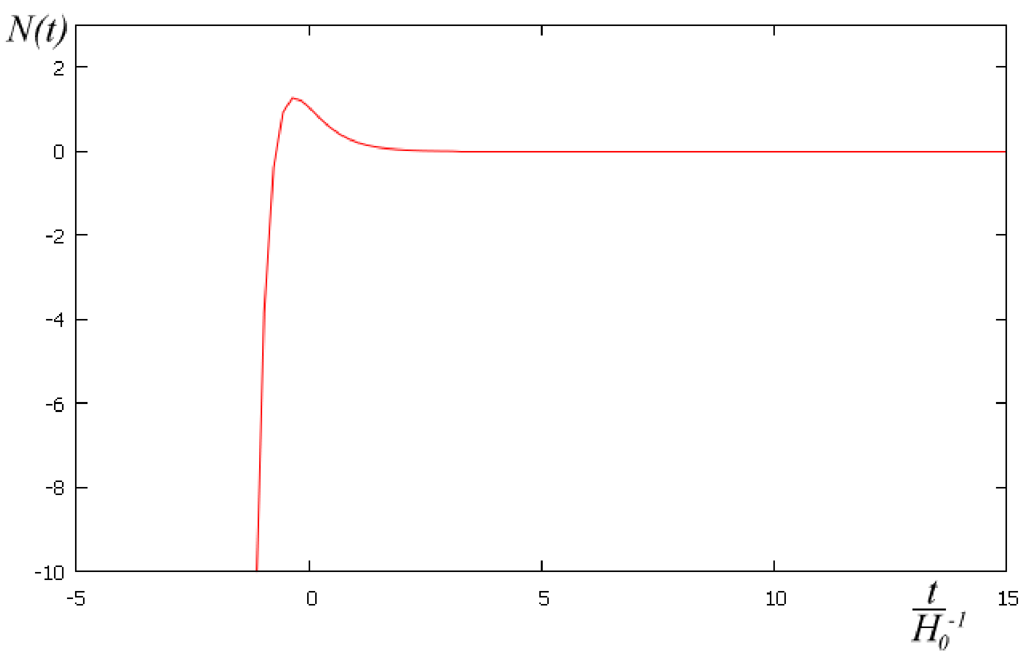


Figure 6.2: Graph of the evolution of $N(t)$ in function of time (in units of H_0^{-1}).

becomes

$$\frac{4G\sigma_c}{C^3 H_0\sqrt{\Omega_\Lambda}}\left(\frac{e^{-2H_0\sqrt{\Omega_\Lambda}t}}{C} - e^{-3H_0\sqrt{\Omega_\Lambda}t}\right) + \frac{4G\sigma_c}{C^2}R_0 e^{-H_0\sqrt{-2\Omega_\Lambda}t} < 1. \tag{6.13}$$

Defining $N(t) = \left(e^{-2H_0\sqrt{\Omega_\Lambda}t}\left[\frac{1}{C} + CR_0H_0\sqrt{\Omega_\Lambda}\right] - e^{-3H_0\sqrt{\Omega_\Lambda}t}\right)$, we can write (6.12) as

$$\frac{4G}{C^3}\frac{\sigma_c N(t)}{H_0\sqrt{\Omega_\Lambda}} < 1. \tag{6.14}$$

For $t \to -\infty$, $N(t) < 0$ [2] and for $t \to \infty$, $N(t) = 0$ (see fig.6.2), so it is clear that in these cases (6.14) will be fulfilled since the other constants are positive. As we can see from the graph in figure (6.2), $N(t)$ has a maximum, so if (6.14) is not violated there it won't be violated anywhere.

$$\frac{d}{dt}N(t) = 3H_0\sqrt{\Omega_\Lambda}e^{-3H_0\sqrt{\Omega_\Lambda}t} - 2H_0\sqrt{\Omega_\Lambda}\left[\frac{1}{C} + CR_0H_0\sqrt{\Omega_\Lambda}\right]e^{-2H_0\sqrt{\Omega_\Lambda}t} = 0, \tag{6.15}$$

which gives,

$$t_m = \frac{\log\left(\frac{3}{2}\left[\frac{1}{C} + CR_0H_0\sqrt{\Omega_\Lambda}\right]^{-1}\right)}{H_0\sqrt{\Omega_\Lambda}}. \tag{6.16}$$

So, the maximum of $N(t)$ is,

$$N(t_m) = \left(\frac{2}{3}\right)^2\left[\frac{1}{C} + CR_0H_0\sqrt{\Omega_\Lambda}\right]^3 - \left(\frac{2}{3}\right)^3\left[\frac{1}{C} + CR_0H_0\sqrt{\Omega_\Lambda}\right]^3 \simeq 1.24 \tag{6.17}$$

[2] Of course, since the Universe as a beginning, this limit does not have a physical meaning, but just to be sure that (6.12) is always fulfilled, we will take it into consideration.

Condition (6.14) then becomes

$$\frac{4G}{C^3}\frac{\sigma_c}{H_0\sqrt{\Omega_\Lambda}} < 0.81 \tag{6.18}$$

The entropy of the universe today in $S_U \sim 10^{102}$ [87], and the present comoving radius is $R_d \sim 10^{32}\ eV^{-1}$(the age of the Universe in natural units), so $\sigma_c \sim 10^6$.The Gravitational Constant is $G \sim 10^{-50}\ eV^{-1}$ [88], and the Hubble constant is $H_0 = 1.45 \times 10^{-33}\ eV$ [89]. Inserting the numbers into (6.18), we get that

$$3.45 \times 10^{-11} < 0.81 \tag{6.19}$$

which means that the entropy of the Universe will always respect the Holographic boundary created using $R_{textuppast}$. In general, for $a = C_i e^{H_0\sqrt{\Omega_\Lambda}t}$ condition (6.18) becomes

$$C_i > 1.29 \times 10^{-\frac{11}{4}}, \tag{6.20}$$

but since $C_i = a(t_i)e^{H_0\sqrt{\Omega_\Lambda}t_i}$, if $t_i > t_{\text{now}}$ then $a(t_i) > a(t_{\text{now}})$ and $C_i > C$ and so (6.20) will always be fulfilled.

Thus we conclude that the radius of the visible Universe can be used as a Holographic Boundary. The question now is if it is the best choice. It was determined in [78] that for a Cosmological Constant energy density of the form (6.6), this radius does not return the correct behavior of the Cosmological Constant or our Universe, unlike the future Horizon, which does [77, 78]. Also in section 4.2.2 it was argued that probably the best Holographic boundary for flat Friedmann spacetime was given by the Hubble Horizon, which is equivalent to the de Sitter Event Horizon. Since the future Horizon tends to the de Sitter horizon, for $t \to \infty$, and since, for the remainder of this Thesis, we will be using the de Sitter Universe, it makes more sense, in our case, to use the de Sitter Event Horizon as our Holographic boundary.

6.2 Entanglement as the origin of the Dilution of states

Some theories have tried to use quantum entanglement to explain the Cosmological Constant [10–13] and, as we have seen from section 4.3 and subsection 5.1.2, entanglement seems to be a good candidate for the origin of the entropy of a Black Hole [15–18] and even the entropy of our Universe and its possible area dependence [15, 16, 19] hence explaining the Holographic Principle.
Some articles indicate that correlations between vacuum quantum states near the boundary of a quantum system seem to be the origin of the Holographic behavior of said system [15, 18, 20]. Indeed we have seen that in [15, 64] the entanglement of the states in near the Event Horizon with the states in the Event Horizon might codify the information of the Black Hole/Universe. Then, if we interpret the entanglement entropy as a way of counting all the entangled pairs (and the degree of entanglement) [90] it is not strange that it will depend on the area of the boundary, even though we are counting states in a volume, all states will be entangled with states in the horizon and thus we will have degrees of freedom proportional to the area [17].

For example, if every pair of entangled states were to be maximally entangled, then because of the monogamy of entanglement (no state can be maximally entangled with more than one other state) [91], the maximum number of entangled pairs would be equal to the number of states in the horizon.
Although many of these articles fail to reproduce the Holographic Entropy, $S_{hol.} = \frac{A}{4l_p^2}$, they are able to get from entanglement the dependence of the entropy on the area of the boundary, $S_{ent} \propto \frac{A}{l_p^2}$. Thus, entanglement seems to play a significant role in determining the degrees of freedom of our Universe and, consequently, determining a Holographic Cosmological Constant.
But can quantum entanglement have a direct effect on the energy similar to what $f(r,k)$ seems to have, and, more surprisingly, can it have an energy of its own like it is assumed in section 5.1.2? What is the energy of entanglement?
Let's first study a simple discrete state: a two spin system.

Consider a state with eingenstates $|u\rangle$ and $|d\rangle$.The most general state that can be created with these two eigenvalues is

$$|\psi\rangle = \alpha\,|uu\rangle + \beta\,|ud\rangle + \gamma\,|du\rangle + \theta\,|dd\rangle\,, \tag{6.21}$$

where $(\alpha, \beta, \gamma, \theta)$ are complex amplitudes.
Then, for example, states

$$|\psi_1\rangle = \alpha\,|uu\rangle + \gamma\,|du\rangle \tag{6.22a}$$
$$|\psi_2\rangle = \beta\,|ud\rangle + \theta\,|dd\rangle \tag{6.22b}$$

will be separable, and states

$$|\psi_3\rangle = \beta\,|ud\rangle + \gamma\,|du\rangle \tag{6.23a}$$
$$|\psi_4\rangle = \alpha\,|uu\rangle + \theta\,|dd\rangle \tag{6.23b}$$

will entangled for $\alpha, \beta, \gamma, \theta \neq 0$, and maximally entangled for $\alpha = \beta = \theta = \frac{1}{\sqrt{2}}$ and $\gamma = \frac{1}{\sqrt{2}}$.
Now we calculate the energy associated with $|\psi\rangle$. The Hamiltonian for a system of two spins is

$$\hat{H} = \frac{\omega}{2}\sigma_1 \cdot \sigma_2 \tag{6.24}$$

where $\sigma_{\mathbf{c}i} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ is the operator that measures the spin in system i, and $\hat{\sigma}_j$ are the components represented by the Pauli matrices. In the matrix representation we have

$$\hat{H} = \frac{\omega}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.25}$$

The eigenstates are

$$|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6.26a}$$

and

$$|uu\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad |ud\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{6.27a}$$

$$|du\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad |dd\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{6.27b}$$

and $|\psi\rangle$ is

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \theta \end{pmatrix} \tag{6.28}$$

The energy of the system is then

$$E = Tr(\rho \hat{H}) \tag{6.29}$$

where ρ is the density matrix and is given by

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \theta \end{pmatrix} \begin{pmatrix} \alpha^* & \beta^* & \gamma^* & \theta^* \end{pmatrix} = \begin{pmatrix} |\alpha|^2 & \alpha^*\beta & \alpha^*\gamma & \alpha^*\theta \\ \beta^*\alpha & |\beta|^2 & \beta^*\gamma & \beta^*\theta \\ \gamma^*\alpha & \gamma^*\beta & |\gamma|^2 & \gamma^*\theta \\ \theta^*\alpha & \theta^*\beta & \theta^*\gamma & |\theta|^2 \end{pmatrix} \tag{6.30}$$

Finally, equation (6.29) becomes

$$E = \frac{\omega}{2}\left(|\alpha|^2 + |\theta|^2 - |\beta|^2 - |\gamma|^2 + 2[\gamma\beta^* + \gamma^*\beta]\right). \tag{6.31}$$

Naturally, the coefficients in (6.21) will influence the energy of the system and, consequently, for an entangled state E will be different from a separable one. For example, states in (6.22a) and (6.22b) will have energies

$$E_1 = \frac{\omega}{2}(|\alpha|^2 - |\gamma|^2) \tag{6.32a}$$

$$E_2 = \frac{\omega}{2}(|\theta|^2 - |\beta|^2) \tag{6.32b}$$

and the entangled states (6.23a) and (6.23a) will have energies

$$E_3 = \frac{\omega}{2}(2[\gamma^*\beta + \beta^*\gamma] - |\beta|^2 - |\gamma|^2) \tag{6.33a}$$

$$E_4 = \frac{\omega}{2}(|\alpha|^2 + |\theta|^2) \tag{6.33b}$$

If we take the case of maximally entangled states

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}\left(|ud\rangle - |du\rangle\right) \tag{6.34a}$$

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}\left(|uu\rangle + |dd\rangle\right) \tag{6.34b}$$

and the separable states

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(|uu\rangle + |du\rangle\right) \tag{6.35a}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|ud\rangle + |dd\rangle\right) \tag{6.35b}$$

then the energies (6.33a),(6.33b),(6.32a) and (6.32b) become

$$\begin{aligned} E_1 &= 0, \\ E_2 &= 0, \\ E_3 &= -\frac{3}{2}\omega, \\ E_4 &= \frac{\omega}{2}, \end{aligned}$$

where the entangled states have different energies than the separable states.

So, if a system of two spins were to evolve, or be changed, from (6.35b) to (6.34b) it would have an energy change of $\delta E = E_4 - E_2 = \frac{\omega}{2}$. For this system, there's also a difference between E_3 and E_4, which correspond to two maximally entangled states and so, we cannot define this way an energy of entanglement. The question is not if entanglement has a specific energy but how the existence of entanglement affects the energy of the system. If the vacuum of our Universe where to be entangled then our calculations for separable states could be giving the wrong value. In simple quantum mechanical systems it is easy to calculate such changes but in quantum field theory it is not so trivial.

Now let's try to study the influence of entanglement on the vacuum energy. The following argument will give us have an idea of a possible "entanglement energy". It was originally formulated by Ted Jacobson [92, 93] and built upon in the context of this Thesis (using the concept of mutual information). Consider a bipartite system of pure states described by the Hamiltonian

$$\hat{H} = \hat{H}_1 + \hat{H}_2. \tag{6.37}$$

where $\hat{H}_1$ is the Hamiltonian of the subsystem 1, and $\hat{H}_2$ is the Hamiltonian of the subsystem 2. The uncertainty of the system will be given by

$$\delta E^2 = \left\langle \hat{H}^2 \right\rangle - \left\langle \hat{H} \right\rangle^2 = \left\langle \hat{H}_1^2 \right\rangle - \left\langle \hat{H}_1 \right\rangle^2 + \left\langle \hat{H}_2^2 \right\rangle - \left\langle \hat{H}_2 \right\rangle^2 + 2\left(\left\langle \hat{H}_1 \hat{H}_2 \right\rangle - \left\langle \hat{H}_1 \right\rangle \left\langle \hat{H}_2 \right\rangle \right) \tag{6.38}$$

The mutual information of systems $|1\rangle$ and $|2\rangle$ is related to their Hamiltonians through [20]

$$I \geq \frac{\left(\left\langle \hat{H}_1 \hat{H}_2 \right\rangle - \left\langle \hat{H}_1 \right\rangle \left\langle \hat{H}_2 \right\rangle \right)^2}{2|\hat{H}_1|^2 |\hat{H}_2|^2}. \tag{6.39}$$

Since the states are pure, I will be zero if there is no entanglement which means that (6.38) will become

$$\delta E_{sep}^2 = \left\langle \hat{H}_1^2 \right\rangle - \left\langle \hat{H}_1 \right\rangle^2 + \left\langle \hat{H}_2^2 \right\rangle - \left\langle \hat{H}_2 \right\rangle^2 = \delta E_1^2 + \delta E_2^2. \tag{6.40}$$

The extra term will only appear if there's entanglement in the system. We can, in a way, call the extra term the energy uncertainty of entanglement,

$$\delta E_{ent}^2 = 2\left(\left\langle \hat{H}_1\hat{H}_2\right\rangle - \left\langle \hat{H}_1\right\rangle\left\langle \hat{H}_2\right\rangle\right). \tag{6.41}$$

The existence of entanglement between the subsystems affects the uncertainty associated with the quantum system. If we consider $|1\rangle$ and $|2\rangle$ to be vacuum states then we can consider the energy of the system E of order of δE. If both subsystems are separated by a distance r then for a massless quantum gas we have that

$$\delta E \sim \frac{1}{r}, \tag{6.42}$$

which means that

$$\delta E_{ent} \sim \frac{1}{r}, \tag{6.43}$$

and we can guess that there's also an entanglement contribution to the energy of the system of the order of

$$E_{ent} \sim \frac{1}{r}. \tag{6.44}$$

All of this is, of course, very approximate and illustrative.

In [17], two definitions of entanglement energy, for entanglement between two regions (1 and 2) separated by a boundary, are proposed: $E^I = Tr(:\hat{H}_{tot}:\rho_1\otimes\rho_2)$ and $E^{II} = Tr((:\hat{H}_1:+:\hat{H}_2:)\rho)$, where $\hat{H}_{tot}$ is the Hamiltonian of the entangled system, $\hat{H}_i$ are the Hamiltonians of each individual region, ρ is the density matrix of the entangled system, and ρ_i are the density matrices of (again) each individual region.

It is then plausible that the presence of entanglement may alter both the entropy and the energy of a system. The next step is to try and test if the vacuum in our Universe is entangled and how that entanglement behaves throughout the space and near the Event Horizon. It needs to be done not only to eventually try to calculate the functions $f(r,k)$, but mainly to shed some light into what is the exact role of the entanglement in Holographic Principle and how it relates to the Cosmological Constant.

Chapter 7

Entanglement due to Event Horizons

To test the existence of entanglement due to the Universe's Event Horizon, we will use the formalism shown in subsection 3.2.2 for Gaussian States). A feature that distinguishes this calculation from the ones cited in section 5.1.2 is that in our case we don't need to consider a discrete quantum field in a lattice as our Universe, to be able to test for the existence of entanglement.
We will test the entanglement between a state in the Event Horizon and a state very near, and then with another very far from it (near our position as observers). Then we will test the entanglement for two states very near the Event Horizon and two states very far. To construct the Covariance Matrix, we will need to calculate the Hadamard function which means that we will need to calculate and quantize the field in an expanding Universe. To do this we'll follow a method presented in [94,95].

7.1 Quantization of a field in de Sitter spacetime

We have already seen that in a flat spacetime the field operator ($\hat{\phi}$) is defined by equation (2.33),

$$\hat{\phi}(x) = \int \frac{d\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega}} \left(e^{-ik_\mu x^\mu} \hat{a}_{\mathbf{k}} + e^{ik_\mu x^\mu} \hat{a}^\dagger_{\mathbf{k}} \right).$$

Since $u_{\text{flat}}(x,k) = Ke^{-ik_\mu x^\mu}$ is the plane wave solution of the Klein-Gordon equation in a Minkowski spacetime, we can analogously define $\hat{\phi}$ in a general spacetime by integrating for all solutions of the general Klein-Gordon equation ($u(x,k)$) in a similar fashion,

$$\hat{\phi}(x) = \int d\mathbf{k} \left(u(x,k)\hat{a}_{\mathbf{k}} + u^*(x,k)\hat{a}^\dagger_{\mathbf{k}} \right). \tag{7.1}$$

The creation and annihilation operators, $a^{\dagger}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$, are defined the following way [94]

$$\hat{a}_{\mathbf{k}} = \left\langle u(x,k), \hat{\phi}(x) \right\rangle, \tag{7.2a}$$

$$\hat{a}^{\dagger}_{\mathbf{k}} = -\left\langle u^{*}(x,k), \hat{\phi}(x) \right\rangle, \tag{7.2b}$$

where,

$$\langle f, g\rangle = \int_{\Sigma} d\Sigma_{\mu} i\sqrt{-g} g^{\mu\nu} \left(f^{*}\partial_{\nu} g - (\partial_{\nu} f^{*}) g\right) \tag{7.3}$$

is the Klein-Gordon inner product, a generalization of the bra-ket inner product for curved spacetimes. Σ is a spacelike hypersurface.

In a curved spacetime $\hat{\phi}$ and $\hat{\pi}$ will have the following equal time commutation relations [96]:

$$\left[\hat{\phi}(x), \hat{\pi}(x')\right] = i\delta(\mathbf{r}; \mathbf{r}'), \tag{7.4a}$$

$$\left[\hat{\phi}(x), \hat{\phi}(x')\right] = \left[\hat{\pi}(x), \hat{\pi}(x')\right] = 0. \tag{7.4b}$$

The $\delta(\mathbf{r};\mathbf{r}')$ is the three dimensional version of the invariant Dirac delta. An invariant delta function need to be defined because in a curved spacetime an infinitesimal volume dV is defined as [96]

$$dV = d^4x\sqrt{-g}, \tag{7.5}$$

where $d^4x = dtd\mathbf{r}$.
Consequently, for the spatial coordinates we have

$$\int dV\ \delta(x,x')f(x) = \int d^4x\ \sqrt{-g}\frac{\delta(x-x')}{\sqrt{-g}} f(x) = \int d^4x\ \delta(x-x')f(x) = f(x'), \tag{7.6}$$

where $\delta(x,x')$ is the invariant delta function.
For a Friedmann universe we will prove[1] that $\delta(\mathbf{r};\mathbf{r}')$ is defined in the same manner as $\delta(x,x')$, that is $\delta(\mathbf{r};\mathbf{r}') = \frac{\delta(\mathbf{r}-\mathbf{r}')}{\sqrt{-g}}$.

The commutation relations for the creation and annihilation operators are defined as in the Minkowski spacetime,

$$\left[\hat{a}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}\right] = \delta(\mathbf{k}-\mathbf{k}'), \tag{7.7a}$$

$$\left[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}\right] = \left[\hat{a}^{\dagger}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}\right] = 0, \tag{7.7b}$$

because the curvature doesn't affect the infinitesimal volume in the momentum space. This means that (7.7a) becomes

$$\langle u(x,k), u(x,k')\rangle = \delta(\mathbf{k}-\mathbf{k}'), \tag{7.8}$$

and (7.4a) becomes

$$\int d\mathbf{k}\left(u(x,k)\partial_t u^{*}(x',k) - u^{*}(x,k)\partial_t u(x',k)\right) = i\delta(\mathbf{r};\mathbf{r}'), \tag{7.9}$$

[1] A similar proof can be found in [95].

with $t = t'$.
Equation (7.8) expresses the normalization and orthogonality of the solutions of the Klein-Gordon equation.

Finally, we can calculate the Hadamard function (3.102),

$$H(x,x') = \langle 0| \left\{ \hat{\phi}(x), \hat{\phi}(x') \right\} |0\rangle ,$$

or, in order of the solutions $u(k,x)$,

$$H(x,x') = \int d\mathbf{k} \int d\mathbf{k}' \left(u(x,k)u^*(x',k') + u^*(x,k)u(x',k') \right) , \tag{7.10}$$

where we have used (7.7a) and (7.7b).

In a general metric, the d'Alembertian takes the form [34]

$$\Box\phi(x) = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi(x)), \tag{7.11}$$

which, for a Friedmann universe, becomes,

$$\Box\phi(x) = \left(\partial_t^2 + 3\frac{\dot{a}(t)}{a(t)}\partial_t - \frac{\nabla^2}{a^2(t)} \right) \phi(x), \tag{7.12}$$

and so, the equation for a massless scalar field is

$$\left(\partial_t^2 + 3\frac{\dot{a}(t)}{a(t)}\partial_t - \frac{\nabla^2}{a^2(t)} \right) \phi(x) = 0. \tag{7.13}$$

For this metric we have that

$$\sqrt{-g} = a^3(t). \tag{7.14}$$

If we consider the Σ in (7.3) to be a hypersurface with $t =$ const, then

$$\langle f, g\rangle = \int d\mathbf{r}\ ia^3(t) \left(f^*\partial_t g - (\partial_t f^*)g \right) . \tag{7.15}$$

Equation (7.8) becomes,

$$\int d\mathbf{r}\ ia^3(t) \left(u^*(x,k)\partial_t u(x,k') - u(x,k')\partial_t u^*(x,k) \right) = \delta(\mathbf{k} - \mathbf{k}'). \tag{7.16}$$

As we have seen in chapter 5, our Universe will evolve to a de Sitter (vacuum dominated) Universe with an exponential scale factor,

$$a(t) = e^{t/R}.$$

Since at present time (we set $t_{now} = 0$) $a = 1$, Ω_m is about 30% of the total content of the Universe. Thus, approximating our Universe to the de Sitter Universe isn't a particularly good approximation but it is acceptable, and it can give us a qualitative understanding of the properties of our Universe. The important feature of the de

Sitter universe is that we can solve the Klein-Gordon equation analytically. For such Universe, equation (7.13) becomes

$$\left(\partial_t^2 + \frac{3}{R}\partial_t - e^{-2t/R}\nabla^2\right) u(x,k) = 0, \tag{7.17}$$

where k denotes the four momentum. Assuming that $u(x,k) = y(t,k)\xi(\mathbf{r},k)$, we get two differential equations,

$$\partial_t^2 y(t,k) + \frac{3}{R}\partial_t y(t,k) + \omega^2 e^{-2t/R} y(t,k) = 0, \tag{7.18a}$$

$$\nabla\xi(\mathbf{r},k) + \omega^2\xi(\mathbf{r},k) = 0, \tag{7.18b}$$

were $\omega = k_0 = \sqrt{\mathbf{k}\cdot\mathbf{k}}$.

The spatial part of the wave equation is not affected by the curvature and the solution is just the free wave (like in the Minkowski Universe)

$$\xi(\mathbf{r},\mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{7.19}$$

The general solution for the time dependent part of the equation is

$$\begin{aligned} y(t,\omega) = C_1[p(t,\omega)\sin p(t,\omega) + \cos p(t,\omega)]+ \\ + C_2[\sin p(t,\omega) - p(t,\omega)\cos p(t,\omega)], \end{aligned} \tag{7.20}$$

where $p(t,\omega) = R\omega e^{-t/R}$.

Now we have to calculate the constants C_1 and C_2 and in order to do that we follow a method presented in [94,95]. For a Friedmann metric, the expansion of the universe can be measured by the quantity [95],

$$C_n(t) = \frac{d^n}{dt^n}\frac{\dot{a}}{a}. \tag{7.21}$$

Notice that $C_0(t) = H(t)$, which is the Hubble parameter here (not to be confused with the Hadamard function).
If $C_n = 0$ for any $n \geq 0$ then the Universe is not expanding, and the vacuum states in it will be equal to the Minkowski vacuum states (there will be no particle creation because of the expansion).
For de Sitter, we have that,

$$\begin{cases} C_n(t) = 0 & \text{if } n > 0 \\ C_n(t) = \frac{1}{R} & \text{if } n = 0 \end{cases} \tag{7.22}$$

which means that there is always expansion. However, if we change the time coordinate to the conformal time η, such that $d\eta = \frac{dt}{a}$, then, the scale factor becomes,

$$a(\eta) = \frac{1}{1 - \frac{\eta}{R}}, \tag{7.23}$$

if we set $\eta = 0 \Leftrightarrow t = 0$ (we can set the origin of the referential anywhere we want). Then, Friedmann metric becomes

$$ds^2 = a^2(\eta)(d\eta^2 - dr^2 + r^2 d\Omega^2). \tag{7.24}$$

With (7.23) we have that

$$C_n(\eta) = \frac{n!}{R^{n+1}\left(1-\frac{\eta}{R}\right)^{n+1}} \tag{7.25}$$

which tends to zero for $\eta \to \pm\infty$. In these limits the expansion becomes constant which means that the field modes must, in these limits, have the same form as the solutions in the Minkowski spacetime. [94, 95]. Notice that since $t = R\log\left(\frac{1}{1-\frac{\eta}{R}}\right)$, the relation between t and η only exists for $-\infty < \eta \leq R$ which means that the values $\eta > R$ are only mathematical definitions with no real physical meaning.
We can calculate the wave equation in the conformal time by changing (7.18a) to the metric in (7.24) [94],

$$\partial_\eta^2 y + \frac{2}{R}\frac{1}{1-\frac{\eta}{R}}\partial_\eta y + \omega^2 y = 0. \tag{7.26}$$

If we then factor out the scale factor,

$$\chi(\eta,\omega) = a(\eta)y(\eta,\omega), \tag{7.27}$$

then (7.26) becomes,

$$\partial_\eta^2 \chi + \omega^2(\eta)\chi = 0, \tag{7.28}$$

with $\omega^2(\eta) = \omega^2 - \frac{\partial_\eta^2 a(\eta)}{a(\eta)}$. For $\eta \to \pm\infty$, the equation becomes,

$$\partial_\eta^2 \chi + \omega^2 \chi = 0, \tag{7.29}$$

which is equal as for the Minkowski spacetime. Then, we have that

$$\chi(\eta \to \pm\infty, \omega) \simeq \frac{1}{\sqrt{2\omega}(2\pi)^{\frac{3}{2}}} e^{-i\omega\eta}. \tag{7.30}$$

From (7.27), we have that

$$\begin{aligned}\chi(\eta,\omega) &= C_1\left[R\omega\sin(\omega(R-\eta)) + \frac{\cos(\omega(R-\eta))}{1-\frac{\eta}{R}}\right] + C_2\left[-R\omega\cos(\omega(R-\eta)) + \frac{\sin(\omega(R-\eta))}{1-\frac{\eta}{R}}\right] \\ &\xrightarrow[\eta\to\pm\infty]{} -C_1 R\omega\sin(\eta\omega) - C_2 R\omega\cos(\eta\omega).\end{aligned} \tag{7.31}$$

For (7.31) to be equal to (7.30), we have that

$$\begin{cases} C_1 = \frac{i}{(2\pi\omega)^{\frac{3}{2}}\sqrt{2R^2}} \\ C_2 = \frac{-1}{(2\pi\omega)^{\frac{3}{2}}\sqrt{2R^2}}. \end{cases} \tag{7.32}$$

In the end we get

$$u(x,k) = \frac{\left(i + \frac{R\omega}{a(t)}\right)}{(2\pi\omega)^{\frac{3}{2}}\sqrt{2R^2}} e^{i\frac{R\omega}{a(t)}} e^{i\mathbf{r}\cdot\mathbf{k}}. \tag{7.33}$$

Now we have got to test (7.33) with (7.9) and (7.16). Equation (7.9) in a Friedmann universe becomes

$$\int d\mathbf{k}\left(u(x,k)\partial_t u^*(x',k) - u^*(x,k)\partial_t u(x',k)\right) = i\frac{\delta(\mathbf{r}-\mathbf{r}')}{a^3}. \tag{7.34}$$

Let us prove this.
The d'Alembertian acting on a Feynman propagator G_F has the following effect [96],

$$\Box G(x,x') = \delta(x,x') = -\frac{\delta(x-x')}{\sqrt{-g}}. \tag{7.35}$$

G can also be defined by the time ordered product of two wave operators (2.51),

$$\frac{G(x,x')}{i} = \langle 0|\, T\hat{\phi}(x)\hat{\phi}(x')\,|0\rangle\,,$$

which is equivalent to,

$$\frac{G(x,x')}{i} = \Theta(t-t')\,\langle 0|\,\hat{\phi}(x)\hat{\phi}(x')\,|0\rangle + \Theta(t'-t)\,\langle 0|\,\hat{\phi}(x')\hat{\phi}(x)\,|0\rangle\,. \tag{7.36}$$

From (7.12) together with (7.35) and (7.36) we have that

$$\begin{aligned} -i\Box G(x,x') = &\left(\partial_t^2 + 3\frac{\dot{a}(t)}{a(t)}\partial_t - \frac{\nabla^2}{a^2(t)}\right)[\Theta(t-t')\,\langle 0|\,\hat{\phi}(x)\hat{\phi}(x')\,|0\rangle + \\ &+ \Theta(t'-t)\,\langle 0|\,\hat{\phi}(x')\hat{\phi}(x)\,|0\rangle] \end{aligned} \tag{7.37}$$

Since the fields commute for equal time, and using the property

$$\delta(t-t') = \partial_t\Theta(t-t') = -\partial_t\Theta(t'-t), \tag{7.38}$$

we get that,

$$\partial_t\,\langle 0|\,T\hat{\phi}(x)\hat{\phi}(x')\,|0\rangle = \langle 0|\,T\hat{\pi}(x)\hat{\phi}(x')\,|0\rangle\,, \tag{7.39}$$

and,

$$\partial_t^2\,\langle 0|\,T\hat{\phi}(x)\hat{\phi}(x')\,|0\rangle = \langle 0|\,T(\partial_t^2\hat{\phi}(x))\hat{\phi}(x')\,|0\rangle + \delta(t-t')[\hat{\pi}(x),\hat{\phi}(x')]. \tag{7.40}$$

Combining (7.39) and (7.40) in (7.37) we get,

$$\begin{aligned} -i\Box G(x,x') = &\,\delta(t-t')[\hat{\pi}(x),\hat{\phi}(x')] + \Theta(t-t')\Box\,\langle 0|\,\hat{\phi}(x)\hat{\phi}(x')\,|0\rangle \\ &+ \Theta(t'-t)\Box\,\langle 0|\,\hat{\phi}(x')\hat{\phi}(x)\,|0\rangle\,, \end{aligned} \tag{7.41}$$

which, because (7.13), is equal to,

$$\Box G(x,x') = -i\delta(t-t')[\hat{\phi}(x),\hat{\pi}(x')]. \tag{7.42}$$

This in combination with (7.35) gives us the commutation relation,

$$[\hat{\phi}(x),\hat{\pi}(x')] = i\frac{\delta(\mathbf{r}-\mathbf{r}')}{\sqrt{-g}}, \tag{7.43}$$

which proves (7.34).

Now going back to testing our solutions, substituting (7.33) into (7.16) we get,

$$\begin{aligned} (2\pi)^3 a^3 i\,(y^*(t,k)\partial_t y(t,k') - y(t,k')\partial_t y^*(t,k)) = 1 \Leftrightarrow \\ \Leftrightarrow -i(2\pi)^3(C_2^*C_1 - C_1^*C_2)R^2\omega^2 = 1, \end{aligned} \tag{7.44}$$

condition which is respected by the constants (7.32).

As for (7.34), substituting (7.33), we get,

$$\int d\mathbf{k}\frac{i}{2(2\pi)^3a^3}\left(e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}+e^{i\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})}\right)=i\frac{\delta(\mathbf{r}-\mathbf{r}')}{a^3}, \tag{7.45}$$

we can immediately see the condition is fulfilled. Hence, (7.33) is a solution consistent with the conditions derived through the quantization of the field operator $\hat{\phi}$.

Finally, we have,

$$\hat{\phi}(x)=\int\frac{d\mathbf{k}}{(2\pi\omega)^{\frac{3}{2}}\sqrt{2R^2}}\left\{\left(\frac{R\omega}{a(t)}+i\right)e^{i\left(\frac{R\omega}{a(t)}+\mathbf{k}\cdot\mathbf{r}\right)}\hat{a}_{\mathbf{k}}+\left(\frac{R\omega}{a(t)}-i\right)e^{-i\left(\frac{R\omega}{a(t)}+\mathbf{k}\cdot\mathbf{r}\right)}\hat{a}^\dagger_{\mathbf{k}}\right\}. \tag{7.46}$$

Since our Universe has spatial spherical symmetry it would make the rest of the calculations easier if we expand the spatial part in spherical harmonics,

$$e^{i\mathbf{k}\cdot\mathbf{r}}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}i^l j_l(\omega r)Y_l^{m*}(\alpha,\beta)Y_l^m(\theta,\phi), \tag{7.47}$$

where (in spherical coordinates) $\mathbf{k}=(\omega,\alpha,\beta)$ and $\mathbf{r}=(r,\theta,\phi)$. Defining,

$$\hat{\phi}(x)=\sum_{lm}\hat{\phi}_{lm}(x), \tag{7.48}$$

then,

$$\begin{aligned}\hat{\phi}_{lm}(x)=4\pi\int d\mathbf{k}\frac{i^l j_l(\omega r)}{(2\pi\omega)^{\frac{3}{2}}\sqrt{2R^2}}\Bigg\{&\left(\frac{R\omega}{a(t)}+i\right)e^{i\frac{R\omega}{a(t)}}Y_l^{m*}(\alpha,\beta)Y_l^m(\theta,\phi)\hat{a}_{\mathbf{k}}+\\ &+(-1)^l\left(\frac{R\omega}{a(t)}-i\right)e^{-i\frac{R\omega}{a(t)}}Y_l^{m}(\alpha,\beta)Y_l^{m*}(\theta,\phi)\hat{a}^\dagger_{\mathbf{k}}\Bigg\},\end{aligned} \tag{7.49}$$

where,

$$\sum_{lm}=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}. \tag{7.50}$$

We can similarly define $H_{lm;l'm'}(x,x')$,

$$H(x,x')=\sum_{lm}\sum_{l'm'}H_{lm;l'm'}(x,x'). \tag{7.51}$$

The states $\hat{\phi}_{lm}$ are solutions of (7.17) and they represent physical states on their own (in analogy with the case of the energy levels in the hydrogen atom).

7.2 PPT criterion in the de Sitter spacetime

We will now test the entanglement between states with zero angular momentum ($l=l'=0$) using the Simon criterion presented in section 3.2.1.

For this case, the wave operator is,

$$\hat{\phi}_{00}(r,t) = \int d\mathbf{k}\frac{\sin(\omega r)}{(2\pi\omega)^{\frac{3}{2}}\sqrt{2R^2}r}\left\{\left(\frac{R\omega}{a(t)}+i\right)e^{i\frac{R\omega}{a(t)}}\hat{a}_{\mathbf{k}} + \left(\frac{R\omega}{a(t)}-i\right)e^{-i\frac{R\omega}{a(t)}}\hat{a}^{\dagger}_{\mathbf{k}}\right\}, \tag{7.52}$$

and the Hadamard function becomes,

$$H_{00;00}(r,t;r',t') = 4\sqrt{2\pi}\int d\omega\int d\omega'\left(\frac{\sin(\omega r)\sin(\omega' r')}{2R^2\sqrt{\omega\omega'}rr'}\left\{\left(\frac{R\omega}{a(t)}+i\right)\times\right.\right.$$

$$\left.\left.\left(\frac{R\omega'}{a(t')}-i\right)e^{iR\left(\frac{\omega}{a(t)}-\frac{\omega'}{a(t')}\right)} + \left(\frac{R\omega}{a(t)}-i\right)\left(\frac{R\omega'}{a(t')}+i\right)e^{-iR\left(\frac{\omega}{a(t)}-\frac{\omega'}{a(t')}\right)}\right\}\right). \tag{7.53}$$

If we now consider the Universe at the present time, such that $a(0) = 1$, the Hadamard function becomes:

$$H_{00;00}(r,r') = 4\sqrt{2\pi}\int d\omega\int d\omega'\left(\frac{\sin(\omega r)\sin(\omega' r')}{2R^2\sqrt{\omega\omega'}rr'}\times\right.$$

$$\left.\times\left\{(R\omega+i)(R\omega'-i)e^{iR(\omega-\omega')} + (R\omega-i)(R\omega'+i)e^{-iR(\omega-\omega')}\right\}\right). \tag{7.54}$$

We can use in the integrals of ω and ω' the same cutoffs we used in (6.5). Let us define the following functions,

$$I_1(r) = \int_{\frac{2\pi}{R}}^{\frac{2\pi}{l_P}} dz\sqrt{z}\sin(rz)\sin(Rz), \tag{7.55a}$$

$$I_2(r) = \int_{\frac{2\pi}{R}}^{\frac{2\pi}{l_P}} dz\sqrt{z}\sin(rz)\cos(Rz), \tag{7.55b}$$

$$I_3(r) = \int_{\frac{2\pi}{R}}^{\frac{2\pi}{l_P}} dz\frac{\sin(rz)}{\sqrt{z}}\sin(Rz), \tag{7.55c}$$

$$I_4(r) = \int_{\frac{2\pi}{R}}^{\frac{2\pi}{l_P}} dz\frac{\sin(rz)}{\sqrt{z}}\cos(Rz). \tag{7.55d}$$

Using the following trigonometric identities,

$$\sin(x-y) = \sin x\cos y - \cos x\sin y, \tag{7.56a}$$

$$\cos(x-y) = \cos x\cos y + \sin x\sin y, \tag{7.56b}$$

we get that,

$$H_{00;00}(r,r') = \frac{4\sqrt{2\pi}}{rr'}\left\{I_1(r)\left(I_1(r')+\frac{I_4(r')}{R}\right)+I_2(r)\left(I_2(r')-\frac{I_3(r')}{R}\right)+\right.$$
$$\left.+\frac{I_3(r)}{R}\left(\frac{I_3(r')}{R}-I_2(r')\right)+\frac{I_4(r)}{R}\left(\frac{I_4(r')}{R}+I_1(r')\right)\right\}. \tag{7.57}$$

Now to calculate the Covariance Matrix we need to calculate $\partial_t H$, $\partial_{t'} H$ and $\partial_t \partial_{t'} H$. From (7.53) we have that,

$$\partial_t H_{00;00}(r,t;r',t') = 4\sqrt{2\pi}\int d\omega \int d\omega' \frac{\sin(r\omega)\sin(r'\omega')}{R^2 r r' a(t)}\sqrt{\frac{\omega}{\omega'}}\times$$

$$\times\left\{\sin\left(R\left[\frac{\omega}{a(t)}-\frac{\omega'}{a(t')}\right]\right)\frac{R^2\omega\omega'}{a(t)a(t')}-\frac{R\omega}{a(t)}\cos\left(R\left[\frac{\omega}{a(t)}-\frac{\omega'}{a(t')}\right]\right)\right\}, \quad (7.58)$$

$$\partial_{t'} H_{00;00}(r,t;r',t') = -4\sqrt{2\pi}\int d\omega \int d\omega' \frac{\sin(r\omega)\sin(r'\omega')}{R^2 r r' a(t')}\sqrt{\frac{\omega'}{\omega}}\times$$

$$\times\left\{\sin\left(R\left[\frac{\omega}{a(t)}-\frac{\omega'}{a(t')}\right]\right)\frac{R^2\omega\omega'}{a(t)a(t')}+\frac{R\omega'}{a(t')}\cos\left(R\left[\frac{\omega}{a(t)}-\frac{\omega'}{a(t')}\right]\right)\right\}, \quad (7.59)$$

and

$$\partial_t\partial_{t'} H_{00;00}(r,t;r',t') = 4\sqrt{2\pi}\int d\omega \int d\omega' \frac{\sin(r\omega)\sin(r'\omega')}{r r' a^2(t) a^2(t')}\sqrt{(\omega\omega')^3}\cos\left(R\left[\frac{\omega}{a(t)}-\frac{\omega'}{a(t')}\right]\right) \quad (7.60)$$

Equations (7.58), (7.59) and (7.60), in a similar way as before, become,

$$\partial_t H_{00;00}(r,r') = -\frac{4\sqrt{2\pi}}{rr'}\left\{\frac{(I_6(r)I_4(r')+I_5(r)I_3(r'))}{R} - I_5(r)I_2(r') + I_6(r)I_1(r')\right\}, \quad (7.61)$$

$$\partial_{t'} H_{00;00}(r,r') = -\frac{4\sqrt{2\pi}}{rr'}\left\{\frac{(I_6(r')I_4(r)+I_5(r')I_3(r))}{R} - I_5(r')I_2(r) + I_6(r')I_1(r)\right\}, \quad (7.62)$$

and,

$$\partial_t\partial_{t'} H_{00;00}(r,r') = \frac{4\sqrt{2\pi}}{rr'}\left\{I_6(r)I_6(r') + I_5(r)I_5(r')\right\}, \quad (7.63)$$

with,

$$I_5(r) = \int_{\frac{2\pi}{R}}^{\frac{2\pi}{l_P}} dz\sqrt{z^3}\sin(rz)\sin(Rz), \quad (7.64a)$$

$$I_6(r) = \int_{\frac{2\pi}{R}}^{\frac{2\pi}{l_P}} dz\sqrt{z^3}\sin(rz)\cos(Rz). \quad (7.64b)$$

Now we have to calculate the functions $I_j(r)$,

$$I_1(r) = \frac{1}{2}\left\{-\frac{\sqrt{\frac{\pi}{2}}S\left(\sqrt{\frac{(R-r)2z}{\pi}}\right)}{(R-r)^{\frac{3}{2}}} + \frac{\sqrt{\frac{\pi}{2}}S\left(\sqrt{\frac{(R+r)2z}{\pi}}\right)}{(R+r)^{\frac{3}{2}}} + \sqrt{z}\left(\frac{\sin([R-r]z)}{R-r} - \frac{\sin([R+r]z)}{R+r}\right)\right\}\Bigg|_{\frac{2\pi}{R}}^{\frac{2\pi}{l_P}}, \quad (7.65)$$

$$I_2(r) = \frac{1}{2}\Bigg\{ -\frac{\sqrt{\frac{\pi}{2}}C\left(\sqrt{\frac{(R-r)2z}{\pi}}\right)}{(R-r)^{\frac{3}{2}}} + \frac{\sqrt{\frac{\pi}{2}}C\left(\sqrt{\frac{(R+r)2z}{\pi}}\right)}{(R+r)^{\frac{3}{2}}}$$

$$- \sqrt{z}\left(-\frac{\cos([R-r]z)}{R-r} + \frac{\cos([R+r]z)}{R+r}\right)\Bigg\}\Bigg|_{\frac{2\pi}{R}}^{\frac{2\pi}{l_P}}, \tag{7.66}$$

$$I_3(r) = \sqrt{\frac{\pi}{2}}\left\{\frac{C\left(\sqrt{\frac{(R-r)2z}{\pi}}\right)}{\sqrt{R-r}} - \frac{C\left(\sqrt{\frac{(R+r)2z}{\pi}}\right)}{\sqrt{R+r}}\right\}\Bigg|_{\frac{2\pi}{R}}^{\frac{2\pi}{l_P}}, \tag{7.67}$$

$$I_4(r) = \sqrt{\frac{\pi}{2}}\left\{-\frac{S\left(\sqrt{\frac{(R-r)2z}{\pi}}\right)}{\sqrt{R-r}} + \frac{S\left(\sqrt{\frac{(R+r)2z}{\pi}}\right)}{\sqrt{R+r}}\right\}\Bigg|_{\frac{2\pi}{R}}^{\frac{2\pi}{l_P}}, \tag{7.68}$$

$$I_5(r) = \frac{1}{2}\Bigg\{ -\frac{3\sqrt{\frac{\pi}{2}}C\left(\sqrt{\frac{(R-r)2z}{\pi}}\right)}{2(R-r)^{\frac{5}{2}}} + \frac{3\sqrt{\frac{\pi}{2}}C\left(\sqrt{\frac{(R+r)2z}{\pi}}\right)}{2(R+r)^{\frac{5}{2}}} +$$

$$+3\sqrt{z}\left(\frac{\cos([R-r]z)}{2(R-r)^2} - \frac{\cos([R+r]z)}{2(R+r)^2}\right) + \tag{7.69}$$

$$+ z^{\frac{3}{2}}\left(\frac{\sin([R-r]z)}{R-r} - \frac{\sin([R+r]z)}{R+r}\right)\Bigg\}\Bigg|_{\frac{2\pi}{R}}^{\frac{2\pi}{l_P}},$$

$$I_6(r) = \frac{1}{2}\Bigg\{\frac{3\sqrt{\frac{\pi}{2}}S\left(\sqrt{\frac{(R-r)2z}{\pi}}\right)}{2(R-r)^{\frac{5}{2}}} - \frac{3\sqrt{\frac{\pi}{2}}S\left(\sqrt{\frac{(R+r)2z}{\pi}}\right)}{2(R+r)^{\frac{5}{2}}} + \frac{\sqrt{z}}{2(R-r)^2}\times$$

$$\times(2(R-r)z\cos([R-r]z) - 3\sin([R-r]z)) - \frac{\sqrt{z}}{2(R+r)^2}\times \tag{7.70}$$

$$\times(2(R+r)z\cos([R+r]z) - 3\sin([R+r]z))\Bigg\}\Bigg|_{\frac{2\pi}{R}}^{\frac{2\pi}{l_P}}.$$

$S(x)$ and $C(x)$ denote the Fresnel integrals,

$$S(x) = \int_0^x \sin(t^2)dt \tag{7.71a}$$

$$C(x) = \int_0^x \cos(t^2)dt \tag{7.71b}$$

Since the minimal spatial definition our system can have is the Planck length, when we talk about states near the Event Horizon we are referring to states with a radial position $r = R - nl_p$ where n is the number of Planck distances to the Event Horizon and is incremented by integer intervals. As discussed in section 4.3, the states said to represent the Event Horizon will be distributed between R and $R - l_p$. Since

$R - r \simeq nl_p$ and $R + r \simeq 2R$ we can calculate the functions $I_j(r)$ as $r \to R$,

$$I_1^1(n) \simeq -\frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{1}{nl_p^{\frac{3}{2}}}\left\{\frac{S(\sqrt{4n})}{\sqrt{n}} - 2\sin(2\pi n)\right\}, \tag{7.72a}$$

$$I_2^1(n) \simeq -\frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{1}{nl_p^{\frac{3}{2}}}\left\{\frac{C(\sqrt{4n})}{\sqrt{n}} - 2\cos(2\pi n)\right\}, \tag{7.72b}$$

$$I_3^1(n) \simeq \sqrt{\frac{\pi}{2nl_p}}C(\sqrt{4n}), \tag{7.72c}$$

$$I_4^1(n) \simeq -\sqrt{\frac{\pi}{2nl_p}}S(\sqrt{4n}), \tag{7.72d}$$

$$I_5^1(n) \simeq -\frac{3}{2}\sqrt{\frac{\pi}{2}}\frac{1}{l_p^{\frac{5}{2}}}\left\{-\frac{C(\sqrt{4n})}{2n^{\frac{5}{2}}} + 3\frac{\cos(2\pi n)}{n^2} - 3\sqrt{\frac{l_p}{R}}\frac{1}{n^2} + 4\pi\frac{\sin(2\pi n)}{n}\right\}, \tag{7.72e}$$

$$I_6^1(n) \simeq \frac{1}{l_p^{\frac{5}{2}}}\left\{\frac{3\sqrt{2}}{8}\frac{S(\sqrt{4n})}{n^{\frac{5}{2}}} + \frac{\sqrt{2\pi}}{4n^2}\left(4\pi n\cos(2\pi n) - 3\sin(2\pi n)\right) - \frac{\sqrt{2\pi}\pi l_p}{2R}\cos\left(\frac{4\pi R}{l_p}\right)\right\}, \tag{7.72f}$$

were we have denoted $I_j^1(n)$ as the approximated $I_j(r)$ functions near the Event Horizon.

As we have seen in chapter 3, the Covariance Matrix is related to the field and the canonical momentum through,

$$V_{\alpha\beta} = \frac{1}{2}\langle 0|\{\xi_\alpha, \xi_\beta\}|0\rangle, \tag{7.73}$$

where $\xi = \left(\hat{\phi}(x), \hat{\pi}(x), \hat{\phi}(x'), \hat{\pi}(x')\right)$. Here, the information that dealing with vacuum states is in the fact that the Covariance Matrix is calculated by averaging $\{\xi_\alpha, \xi_\beta\}$ in the vacuum, $|0\rangle$.

From the equation above and the Hadamard function (3.102), we construct the matrix $\mathbf{V}$,

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix} \lim_{x'\to x} H(x,x') & \lim_{x'\to x}\partial_{t'}H(x,x') & H(x,x') & \partial_{t'}H(x,x') \\ \lim_{x'\to x}\partial_{t'}H(x,x') & \lim_{x'\to x}\partial_t\partial_{t'}H(x,x') & \partial_t H(x,x') & \partial_t\partial_{t'}H(x,x') \\ H(x,x') & \partial_t H(x,x') & \lim_{x\to x'} H(x,x') & \lim_{x\to x'}\partial_{t'}H(x,x') \\ \partial_{t'}H(x,x') & \partial_t\partial_{t'}H(x,x') & \lim_{x\to x'}\partial_{t'}H(x,x') & \lim_{x\to x'}\partial_t\partial_{t'}H(x,x') \end{pmatrix}. \tag{7.74}$$

Following the method presented in chapter 3, we will average $\hat{\phi}(x)$ inside a box of infinitesimal volume V, and $\hat{\phi}(x')$ inside a box V'. Consequently, (7.74) will become

$$\mathbf{V} = \begin{pmatrix} A_1 & VA_2 & G_1 & V'G_2 \\ VA_2 & V^2A_3 & VG_3 & VV'G_4 \\ G_1 & VG_3 & B_1 & V'B_2 \\ V'G_2 & VV'G_4 & V'B_2 & V'^2B_3 \end{pmatrix} = \begin{pmatrix} A & G \\ G^T & B \end{pmatrix} \tag{7.75}$$

where,

$$A = \frac{1}{2}\begin{pmatrix} \lim_{x'\to x} H(x,x') & V\lim_{x'\to x}\partial_{t'}H(x,x') \\ V\lim_{x'\to x}\partial_{t'}H(x,x') & V^2\lim_{x'\to x}\partial_t\partial_{t'}H(x,x') \end{pmatrix}, \tag{7.76a}$$

$$B = \frac{1}{2}\begin{pmatrix} \lim_{x\to x'} H(x,x') & V'\lim_{x\to x'}\partial_{t'}H(x,x') \\ V'\lim_{x\to x'}\partial_{t'}H(x,x') & V'^2\lim_{x\to x'}\partial_t\partial_{t'}H(x,x') \end{pmatrix}, \tag{7.76b}$$

$$G = \frac{1}{2}\begin{pmatrix} H(x,x') & V'\partial_{t'}H(x,x') \\ V\partial_t H(x,x') & VV'\partial_t\partial_{t'}H(x,x') \end{pmatrix}. \tag{7.76c}$$

With the Covariance Matrix, we can finally test for the existence of entanglement using the PPT criteria (3.76),

$$F(x,x') = \mathrm{Det}(A)(x) + \mathrm{Det}(B)(x') - 2\mathrm{Det}(C)(x,x') - \left(\frac{1}{4} + \mathrm{Det}(\mathbf{V})(x,x')\right).$$

We can define a Hadamard function that measures the correlations between two states near the Event Horizon:

$$\begin{aligned} H^{(1)}_{00}(n,m) = \frac{4\sqrt{2\pi}}{R^2}\Bigg\{ & I^1_1(n)\left(I^1_1(m) + \frac{I^1_4(m)}{R}\right) + I^1_2(n)\left(I^1_2(m) - \frac{I^1_3(m)}{R}\right) + \\ & + \frac{I^1_3(n)}{R}\left(\frac{I^1_3(m)}{R} - I^1_2(m)\right) + \frac{I^1_4(n)}{R}\left(\frac{I^1_4(m)}{R} + I^1_1(m)\right)\Bigg\}, \end{aligned} \tag{7.77}$$

where, to save some space, we use H_{00} instead of $H_{00;00}$ for the Hadamard function of states with $l = l' = 0$.
This function is obtained by inserting eqs. (7.72a) to (7.72f) into (7.57). We get its derivatives by inserting eqs. (7.72a) to (7.72f) into eqs. (7.58) to (7.60). Using this we construct the Covariance Matrix for two states near the Event Horizon (that we shall call $\mathbf{V}^{(1)}_{00}$), $\hat{\phi}_V(n, t = 0)$ and $\hat{\phi}_{V'}(m, t' = 0)$, with $l = l' = 0$, using (7.75).

The detector profile we have chosen is the top hat, but we still need to choose the shape of the boxes where we will be averaging the states. The first type of boxes we will try are the cubic ones, with $V = L^3$, were L will be of the order of the Planck length. Lets see if they can be used in the infinitesimal approximation:

$$\hat{\phi}_L(\mathbf{r},t) = \frac{1}{L^3}\int_{-\frac{L}{2}}^{\frac{L}{2}} d\mathbf{u}\ \hat{\phi}(\mathbf{r}+\mathbf{u},t) \simeq$$

$$\simeq \frac{1}{L^3}\int_{-\frac{L}{2}}^{\frac{L}{2}} d\mathbf{u}\left[\hat{\phi}(\mathbf{r},t) + \mathbf{u}\cdot\nabla\hat{\phi}(\mathbf{r},t)\right] =$$

$$= \frac{1}{L^3}\left(L^3\hat{\phi}(\mathbf{r},t) + L^6\ \hat{e}_{\mathbf{u}}\cdot\nabla\hat{\phi}(\mathbf{r},t)\right) \Leftrightarrow$$

$$\Leftrightarrow \lim_{L\to 0}\hat{\phi}_L(\mathbf{r},t) = \hat{\phi}(\mathbf{r},t).$$

Which means that they can be used in (7.75).

7.3 Results and Discussion

Before we start to show the results for the entanglement of our system, it is important for the discussion that will follow to notice right away that regardless of the boxes or approximations that were used (near or far from the Event Horizon), the determinant of $\mathbf{V}$ is always negative for our system, which is inconsistent with (3.57) [43]. Also, when we try to calculate $\boldsymbol{\nu}$, we get a complex matrix, which is also inconsistent with a Covariance Matrix, which should be real. Hence, we are forced to conclude that with the boxes and detector profiles used, our states are not Gaussian States. The Simon criterion is thus (as we have seen in chapter 3) only an implication of the PPT criterion and so, only a necessary but not sufficient condition for separability. Since

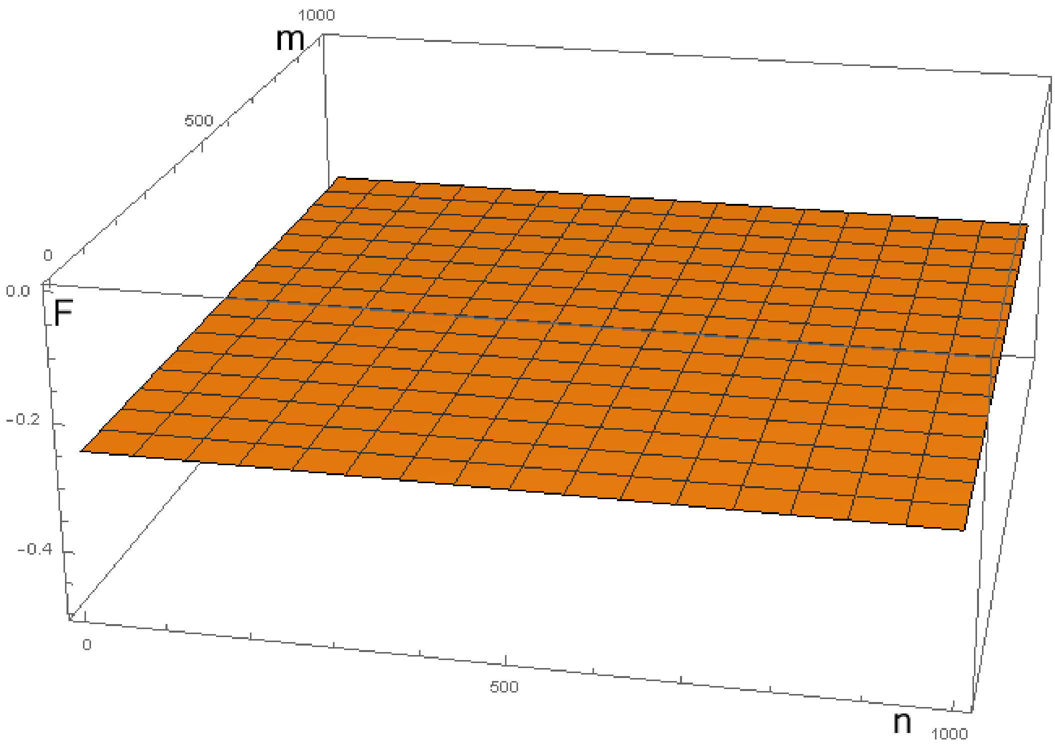


Figure 7.1: Entanglement is tested (using PPT criterion through F) for two states near and in the Event Horizon, averaged over square boxes of volume l_p^3. $F \simeq -\frac{1}{4}$ for all values tested which means that no entanglement is detected near the Event Horizon for these boxes.

the degrees of freedom of the Event Horizon are actually distributed between the Event Horizon itself and the stretched horizon, we will consider $r_H = R_{eh} - \frac{l_p}{2} \Leftrightarrow n = \frac{1}{2}$ for the position of the states "on the Event Horizon". The states corresponding to the Event Horizon will all be averaged inside a box of size $V = l_p^3$, so that the box fits inside the shell mentioned above. For simplicity, from now on we will use Planck units (unless stated otherwise), where $l_p = 1$. Notice that, since the states between the Event Horizon and the stretched Horizon are supposed to be in a region were nature stops being reliably described by known physics, when we take $n = 0.5$ to represent the Event Horizon, we must take into account that that corresponds to an approximation of reality. We are just simulating the states in the de Sitter Horizon this way. In fact, that is more or less what is done by averaging over boxes with dimension of Planck length, ignoring what is inside and accounting all as one state.

For these boxes, there is no entanglement detected, at least for $n, m \in [0.5, 1000]$ [2], as can be seen in fig. 7.1. One has $F \simeq -\frac{1}{4}$ always, which means that the correlations between the two states near the Event Horizon are, with this types of boxes, too weak to produce entanglement to be noticed. Also, there doesn't seem to be any difference in the results when we change box sizes. With values $V \in \{10, 100, 1000\}$ the graphs obtained are equal to fig. 7.1. Since we have a

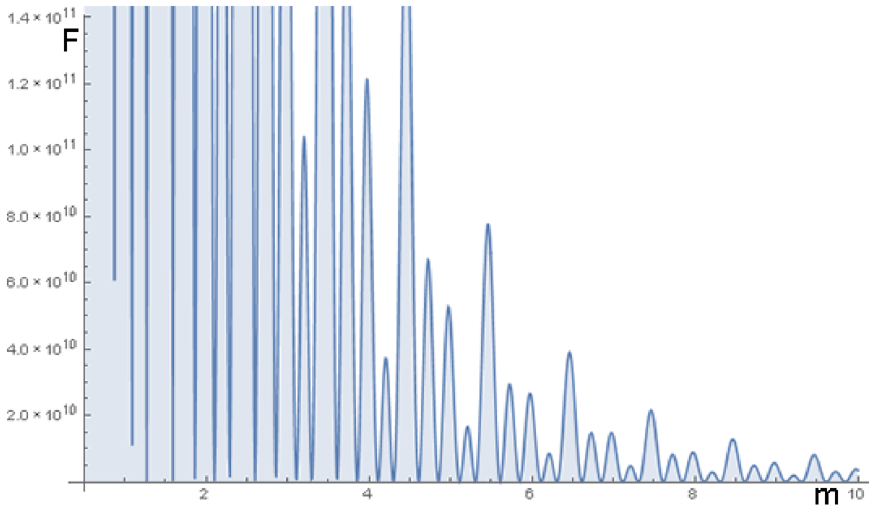


Figure 7.2: The presence of entanglement is tested using F, with one of the states on $n = \frac{1}{2}$ and the other varying is position m. The states here are averaged inside two boxes with the form of a spherical shell of radius $\sim R$ and thickness l_p. There are very big oscillations of F when the states are close, but when m gets bigger the oscillations diminish and F stabilizes. Note that the maximum of the biggest oscillations are cut from the graph for convenience. They are, however, finite and the largest one is $F_{\max} \simeq 5.17 \times 10^{14}$, at $(n, m) = (0.5, 0.66)$ (which would not be physical since both boxes need to be at a distance $m - n > 1$ from each other).

system with spherical symmetry and since the states of the Event Horizon are inside a spherical shell, we can try to average the states over infinitesimal spherical shells. Then, $V = \frac{4}{3}\pi \left[\left(r + \frac{l}{2}\right)^3 - \left(r - \frac{l}{2}\right)^3\right] \simeq 4\pi r^2 l$ as long as $l \ll r$.
Again, we will test the validity of these boxes (this time assuming the state doesn't have any angular dependence, and using $t = 0$):

$$\hat{\phi}_l(r) = \frac{1}{r^2 l} \int_{r-\frac{l}{2}}^{r+\frac{l}{2}} du\ u^2 \hat{\phi}(u).$$

Expanding $\hat{\phi}(u)$ around $u = r$ we get

$$\hat{\phi}(u) = \sum_j \frac{(u - r)^j}{j!} \partial_u^j \hat{\phi}(r),$$

which means that

$$\hat{\phi}_l(r) = \sum_j \frac{1}{r^2 l} \frac{\partial_u^j \hat{\phi}(r)}{j!} \int_{r-\frac{l}{2}}^{r+\frac{l}{2}} du\ u^2 (u - r)^j =$$

[2]For values larger than 1000, the program used to calculate the values of F, Wolfram Mathematica, started giving numerical errors originating from the extremely small numbers being calculated.

$$= \sum_j \frac{1}{r^2 l} \frac{\partial_u^j \hat{\phi}(r)}{j!} \left\{ \frac{(u-r)^{j+1}}{(j+3)(j+2)(j+1)} \left(2r^2 + 2(j+1)ru + (j^2+3j+2)u^2\right) \right\} \Bigg|_{r-\frac{l}{2}}^{r+\frac{l}{2}} \simeq$$

$$\simeq \sum_j \frac{1}{r^2 l} \frac{\partial_u^j \hat{\phi}(r)}{(j+3)!} r^2 (j^2+5j+6) \left\{ \left(\frac{l}{2}\right)^{j+1} - \left(\frac{-l}{2}\right)^{j+1} \right\} =$$

$$= \sum_j \frac{1}{r^2 l} \frac{\partial_u^j \hat{\phi}(r)}{(j+3)!} r^2 (j^2+5j+6) \times \begin{cases} l^{j+1} & \text{if } j \text{ is even} \\ 0 & \text{if } j \text{ is odd} \end{cases} =$$

$$= \sum_j \frac{\partial_u^j \hat{\phi}(r)}{(j+3)!} (j^2+5j+6) l^{2j} \Leftrightarrow \lim_{l \to 0} \hat{\phi}_l(r) = \hat{\phi}(r),$$

where we used $r - l \simeq r$ in the third line.
So we can see that the spherical shell is allowed in our approximation. For states very near the Event Horizon, with spherical shells of the size of a Planck distance we have $V \simeq 4\pi R^2 l_p$, or, again in Planck units, $V \simeq 4\pi R^2$. The interesting thing about these boxes is that, they are infinitesimal (or very small) in the radial direction, which is the only direction on which our fields,$\hat{\phi}_{00}(r)$ depend. But since they are very big in terms of volume, we are able to compensate for the very small values of our Hadamard functions.

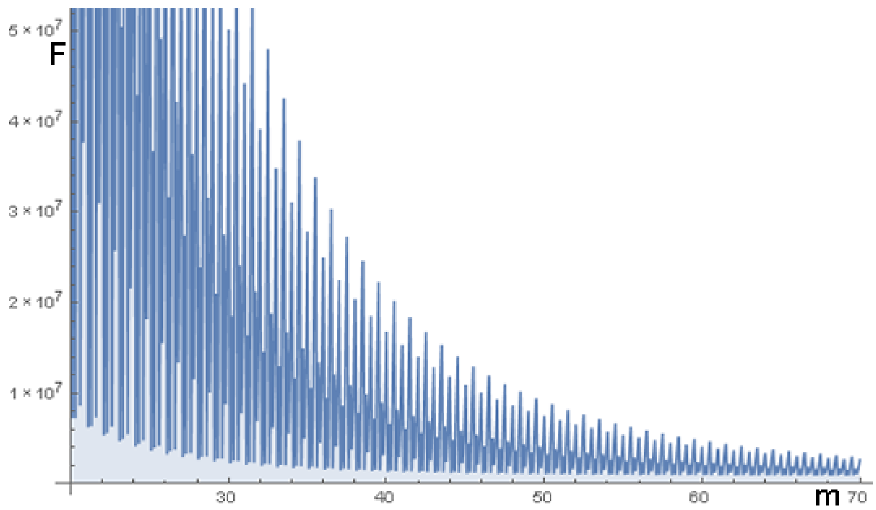


Figure 7.3: The presence of entanglement is tested using F, with one of the states on $n = \frac{1}{2}$ and the other varying is position m. The states here are averaged inside two boxes with the form of a spherical shell of radius $\sim R$ and thickness l_p. F still oscillates quite a lot for $m \sim 70$.

We will start again by placing one of the states at $n = 0.5$ to represent a state of the Event Horizon. Both states will be averaged over a shell with thickness of a Planck length. Notice that our boxes this time are cover all the volume between the Event Horizon and the stretched horizon, which means that this average is better at accounting for all the states in the Event Horizon.
From fig. 7.2 we can see that $F(n = 0.5, m)$ has a lot of oscillations for m near the Event Horizon, but as it gets far the oscillations diminish (fig. 7.3 to fig. 7.5) and F tends to a positive constant (fig. 7.6) $F(0.5, m \to \infty) = 864\ 398$. What is

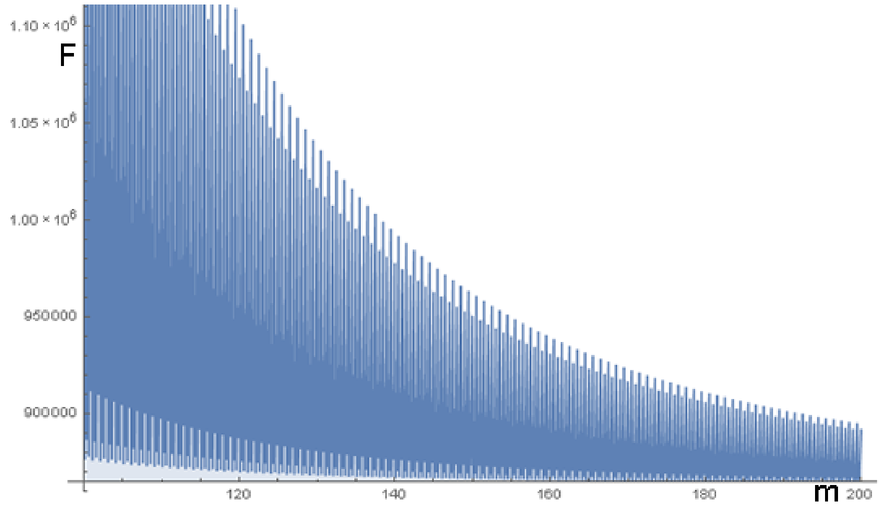


Figure 7.4: The presence of entanglement is tested using F, with one of the states on $n = \frac{1}{2}$ and the other varying is position m. The states here are averaged inside two boxes with the form of a spherical shell of radius $\sim R$ and thickness l_p. Note that the x-axis does not cross the y-axis at $F = 0$, meaning that, even the minima of the oscillations are, at this point, very much above $F = 0$.

happening is that the correlations of the state $\hat{\phi}_{00}(m)$ $(H_{00}(m,m))$, represented in $\mathrm{Det}(B)$, and the correlations between $\hat{\phi}_{00}(n)$ and $\hat{\phi}_{00}(m)$ $(H_{00}(n,m))$, represented in $\mathrm{Det}(G)$ and $\mathrm{Det}(\mathbf{V})$, will tend to zero as m gets further away from the Event Horizon. Hence, only the correlations of $\hat{\phi}_{00}(m)$ with itself $(H_{00}(n,n))$ will survive, which means that $F \to \mathrm{Det}(A)(n = 0.5) = 864\ 398$. The presence of oscillations can be explained by the fact that, in the radial direction, our boxes have a a thickness of one Planck length, and because that is the intrinsic spatial uncertainty, it makes sense that when we analyze distances of that order of magnitude we get some physically doubtful behavior. In fig. 7.10, we se that these oscillations have a period that is approximately $\frac{1}{2}$, which means that if we use discrete values of $n, m = 0.5, 1.5, 2.5, ...$ we get much smoother graphs of F (see fig. 7.8 and fig. 7.9), which makes sense because our minimal spatial resolution is $\delta n = 1$. Boxes closer than that are averaging some of the same states. Then, using the discrete values of (n, m), as in fig. 7.9, is more physically meaningful.
Since $F > 0$ always, the states near the Event Horizon are entangled with the states in the Event Horizon.

In fig. 7.7 we have fixed $m = 10\ 000$, since for that distance $F \simeq \mathrm{Det}(A)(n)$, this way we can see up to which distances of n the rest of the states near the Event Horizon remain entangled with $\hat{\phi}(n)$. The entanglement seems to get smaller the further away n is from the Event Horizon and is detected only until $n \sim O(10)$ ($n \sim 35$ in fig. 7.8). Even with the oscillations, we can see that after $n \sim 60$, no entanglement is detected for states near the Event Horizon. Although the are considering the Event Horizon as a shell of thickness 1, it is not incoherent to expect that entanglement with the Event Horizon would cease to exist abruptly at $n = 1$. Since the cutoff definitions are not precise, and since known physics at those lengths is dubious, the spread out of entanglement for lengths $n \sim O(1)$ is not that strange, especially since F is already very small when it gets to $n \sim 10$, compared to its value for $n = 0.5$.

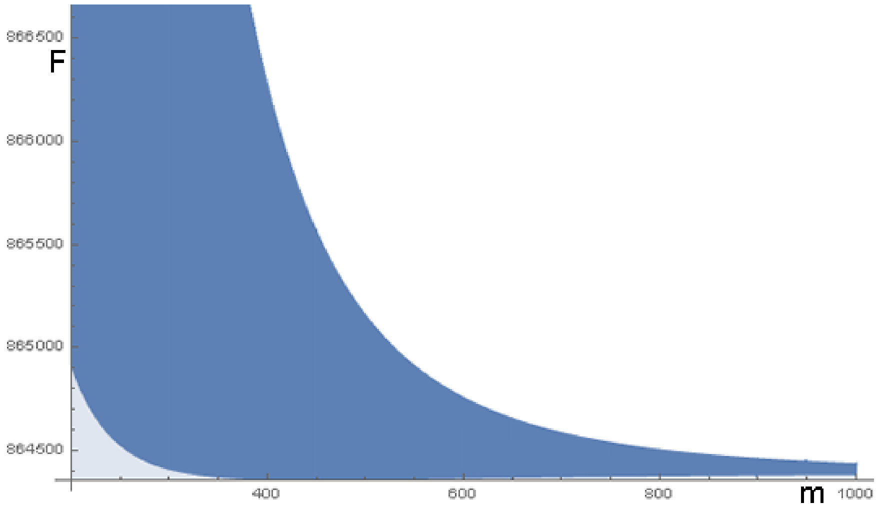


Figure 7.5: The presence of entanglement is tested using F, with one of the states on $n = \frac{1}{2}$ and the other varying is position m. The states here are averaged inside two boxes with the form of a spherical shell of radius $\sim R$ and thickness l_p. For values of $m > 500$, F finally seems to stabilize around a number. The oscillations will continue to increase, but the overall graph will stop decreasing.

Now we are going to test the entanglement of states far away from the Event Horizon (very near our position as observers). For that we need to go back to eqs. (7.65) to (7.70) and make the suitable approximations for states very near the origin of our referential. Repeating what we have done for states near the Event Horizon, we now have that $R \pm r \simeq R$. By substituting this into the equations for $I_j(r)$ we get that they are all zero. However, in the Hadamard function and its time derivatives (eqs. (7.57) to (7.60)) we will have $\frac{1}{rr'}$ multiplying the functions $I_j(r)$ which means that for $r, r' \to 0$ we could have an indetermination. Therefore, we expand $I_j(r)$ in Taylor series and see what happens to the Hadamard function for very small r. We shall denote the functions for $r << R$ as $I_j^2(r)$,

$$I_1^2(r) \simeq \frac{r}{2R^{\frac{5}{2}}}\sqrt{\frac{\pi}{2}}\left\{8\pi - 8\pi\left(\frac{R}{l_p}\right)^{\frac{3}{2}}\cos\left(2\pi\frac{R}{l_p}\right) + 3S(2) - 3S\left(2\sqrt{\frac{R}{l_p}}\right) + 6\sqrt{\frac{R}{l_p}}\sin\left(2\pi\frac{R}{l_p}\right)\right\} \tag{7.78}$$

$$I_2^2(r) \simeq \frac{r}{2R^{\frac{5}{2}}}\sqrt{\frac{\pi}{2}}\left\{-6 + 6\sqrt{\frac{R}{l_p}}\cos\left(2\pi\frac{R}{l_p}\right) + 3C(2) - 3C\left(2\sqrt{\frac{R}{l_p}}\right) + 8\pi\left(\frac{R}{l_p}\right)\sin\left(2\pi\frac{R}{l_p}\right)\right\} \tag{7.79}$$

$$I_3^2(r) \simeq \frac{r}{R^{\frac{3}{2}}}\sqrt{\frac{\pi}{2}}\left\{-2 + 2\sqrt{\frac{R}{l_p}}\cos\left(2\pi\frac{R}{l_p}\right) + C(2) - C\left(2\sqrt{\frac{R}{l_p}}\right)\right\} \tag{7.80}$$

$$I_4^2(r) \simeq \frac{r}{R^{\frac{3}{2}}}\sqrt{\frac{\pi}{2}}\left\{2\sqrt{\frac{R}{l_p}}\sin\left(2\pi\frac{R}{l_p}\right) + S(2) - S\left(2\sqrt{\frac{R}{l_p}}\right)\right\} \tag{7.81}$$

$$I_5^2(r) \simeq \frac{r}{4l_p^2R^{\frac{7}{2}}}\sqrt{\frac{\pi}{2}}\left\{\left(\frac{24}{l_p} - 3\pi^2R^2 + 6l_p^2\right)\sqrt{\frac{R}{l_p}}\cos\left(2\pi\frac{R}{l_p}\right) - 15l_p^2C\left(2\sqrt{\frac{R}{l_p}}\right) + 15l_pC(2) + \right.$$
$$\left. + l_p^2(32\pi - 30) + 40l_p\left(\frac{R}{l_p}\right)^{\frac{3}{2}}\sin\left(2\pi\frac{R}{l_p}\right)\right\} \tag{7.82}$$

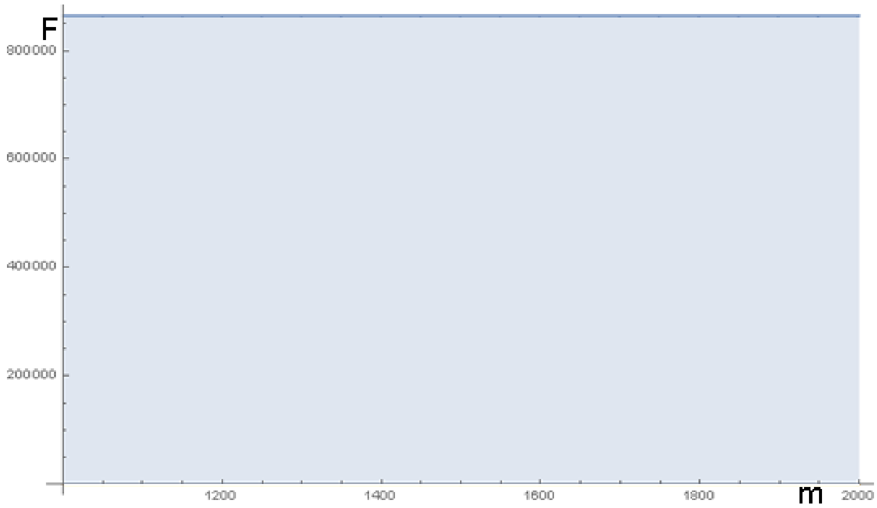


Figure 7.6: The presence of entanglement is tested using F, with one of the states on $n = \frac{1}{2}$ and the other varying is position m. The states here are averaged inside two boxes with the form of a spherical shell of radius $\sim R$ and thickness l_p. In this diagram the vertical axis starts at $F = 0$ (unlike the other figures) and that way we can see the oscillation by now are much smaller than the value of the average of F, which is not decreasing any more and is now a constant with the value $F_{\text{avg}}(m > 1000) \simeq 864\,398$.

$$I_6^2(r) \simeq \frac{r}{4l_p^2 R^{\frac{7}{2}}}\sqrt{\frac{\pi}{2}}\left\{40l_p^2\pi - 40\pi l_p\left(\frac{R}{l_p}\right)^{\frac{3}{2}}\cos\left(2\pi\frac{R}{l_p}\right) + 15l_p^2 S(2) - 15l_p^2 S\left(2\sqrt{\frac{R}{l_p}}\right) + \right.$$
$$\left. + \left[\frac{24}{l_p} + 32R^2 + 6l_p^2\right]\sqrt{\frac{R}{l_p}}\sin\left(2\pi\frac{R}{l_p}\right)\right\} \tag{7.83}$$

Although the terms in these equations are proportional only to r, the expressions were expanded up to second order but the terms proportional to r^2 were zero.

Let us define two more Hadamard functions:

$$H_{00}^{(12)}(n,r) = \frac{4\sqrt{2\pi}}{R^2}\left\{I_1^1(n)\left(I_1^2(r) + \frac{I_4^2(r)}{R}\right) + I_2^1(n)\left(I_2^2(r) - \frac{I_3^2(r)}{R}\right) + \right.$$
$$\left. + \frac{I_3^1(n)}{R}\left(\frac{I_3^2(r)}{R} - I_2^2(r)\right) + \frac{I_4^1(n)}{R}\left(\frac{I_4^2(r)}{R} + I_1^2(r)\right)\right\}, \tag{7.84}$$

$$H_{00}^{(2)}(r,r') = \frac{4\sqrt{2\pi}}{R^2}\left\{I_1^2(r)\left(I_1^2(r') + \frac{I_4^2(r')}{R}\right) + I_2^2(r)\left(I_2^2(r') - \frac{I_3^2(r')}{R}\right) + \right.$$
$$\left. + \frac{I_3^2(r)}{R}\left(\frac{I_3^2(r')}{R} - I_2^2(r')\right) + \frac{I_4^2(r)}{R}\left(\frac{I_4^2(r')}{R} + I_1^2(r')\right)\right\}. \tag{7.85}$$

$H_{00}^{(12)}(n,r)$ measures the correlation between a state near the Event Horizon and another far from it, and $H_{00}^{(2)}(r,r')$ measures correlations between two states far from the Event Horizon. [48]

The fact that the functions $I_j^2(r)$ are proportional to r means that $H_{00}^{(2)}(r,r') = \text{const}$ which in turn means that even if there is entanglement for small r, at least for

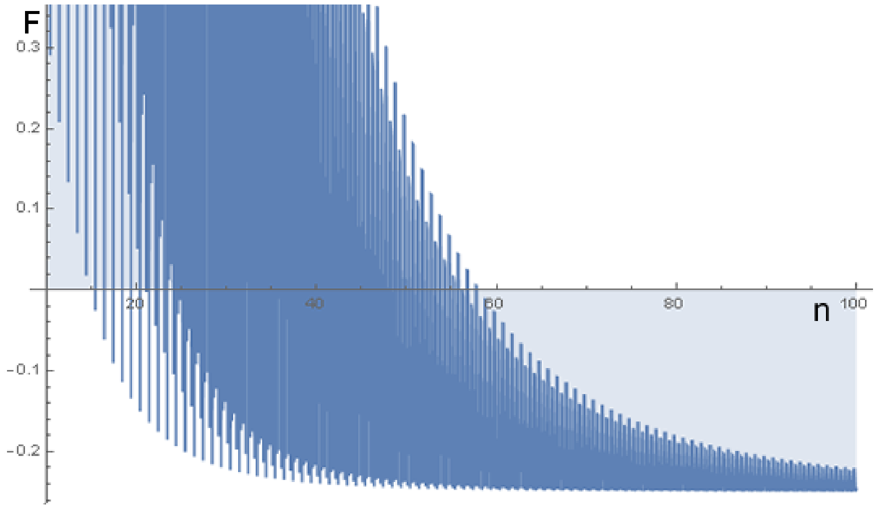


Figure 7.7: We measure when the entanglement between a state far from the Event Horizon ($m > O(1000)$) and one very near the Event Horizon ($n < O(100)$) ceases. The states here are averaged inside two boxes with the form of a spherical shell of radius $\sim R$ and thickness l_p.

infinitesimal boxes (with constant volume), it won't depend on the distance. Thus, the possible dependence of entanglement with distance (seen in, for example, fig. 7.9) comes from the existence of an Event Horizon.

We can construct, in a similar way, the corresponding derivatives, $\partial_t H_{00}^{(i)}$, $\partial_{t'} H_{00}^{(i)}$ and $\partial_t \partial_{t'} H_{00}^{(i)}$, from eqs. (7.58) to (7.60). Then using (7.75) we can construct the Covariance Matrix of the two states far from the Event Horizon ($\mathbf{V}_{00}^{(2)}$). The spherical shell boxes for $R \gg r \gg l_p$ will have volume $V = 4\pi r^2 l_p$.

Finally, the Covariance Matrix for one state near and another far from the Event Horizon is:

$$\mathbf{V}_{00}^{(12)} = \begin{pmatrix} H_{00}^{(1)}(n,n) & V\partial_{t'} H_{00}^{(1)}(n,n) & H_{00}^{(12)}(n,r) & V'\partial_{t'} H_{00}^{(12)}(n,r) \\ V\partial_t H_{00}^{(1)}(n,n) & V^2 \partial_t \partial_{t'} H_{00}^{(1)}(n,n) & V\partial_t H_{00}^{(12)}(n,r) & VV'\partial_t \partial_{t'} H_{00}^{(12)}(n,r) \\ H_{00}^{(12)}(n,r) & V\partial_t H_{00}^{(12)}(n,r) & H_{00}^{(2)}(r,r) & V'\partial_{t'} H_{00}^{(2)}(r,r) \\ V' H_{00}^{(12)}(n,r) & VV'\partial_{t'} H_{00}^{(12)}(n,r) & V'\partial_{t'} H_{00}^{(2)}(r,r) & V'^2 \partial_t \partial_{t'} H_{00}^{(2)}(r,r) \end{pmatrix} \tag{7.86}$$

Using the spherical shells as boxes, we detect entanglement between the Event Horizon and states near Earth, with F having the same value as in fig.7.5. This indicates that entanglement (with this type of box - spherical shell) exists throughout the Universe as long as one of the states is at a $n < O(10)$(see fig. 7.6). Also, only the box belonging to the state near the Event Horizon matters. The other state, has long has it is at a distance $r < R - O(10)$ (or $n > O(10)$), does not play a significant role in the existence of entanglement since his correlations are too weak. Consequently as long as the state in (or very near) the Event Horizon is averaged over a spherical shell, there will always be entanglement between the Event Horizon (or a state with $n < O(10)$) and the rest of the Universe (at least for $m > O(100)$). This holds true for spherical shells of thickness as small as $l \simeq 5 \times 10^{-4}$ (or $l \simeq 5 \times 10^{-4} l_p$ in natural units), because $\mathrm{Det}(A_l) = l^2 \mathrm{Det}(A_{l=1})$ and therefore entanglement will be detected

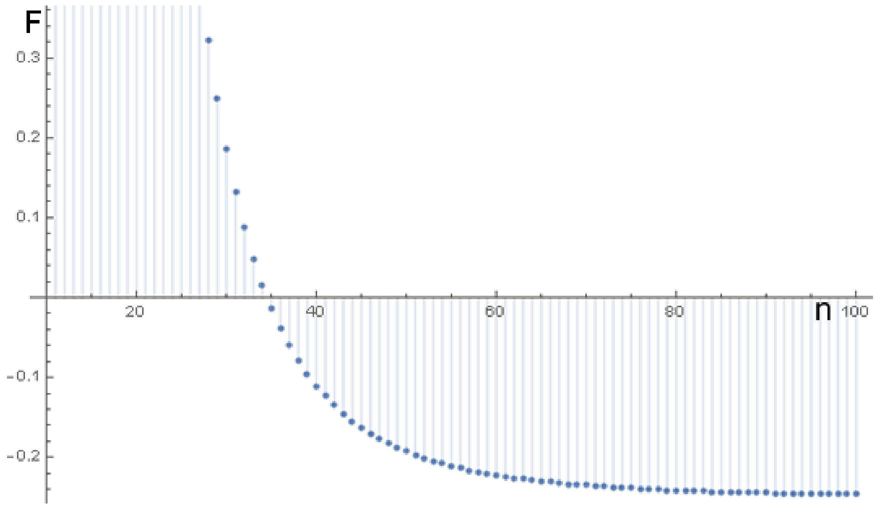


Figure 7.8: The graph in fig. 7.7 is repeated, but now changing the values of n by integers.

for $(n, m) = (0.5, > O\,(100))$ if $l^2\text{Det}(A) = l^2\ 864\ 398 > \frac{1}{4}$. Here A_l stands for the matrix A in (7.75) with a box $V = 4\pi R^2 l$.

In the case of two states very far from the Event Horizon, no entanglement is detected, neither with cubic boxes or with spherical shells (see fig.7.11), although it is worth mentioning here that, since we are making the approximation that $V = 4\pi r^2 l_p$, which is only valid if $r \gg l_p$, then for values $r \sim 0$ the behavior can be different from fig.7.11. But since F, in this case, is so close to $-\frac{1}{4}$ (the correlations of the states near Earth are very small) then, regardless of the box (as long as it can be used in the infinitesimal box approximation), we expect that no entanglement should ever be detected when both states are very far from the Event Horizon, at least with the Simon criterion.[3]

It is worthwhile noting that for distances further than, say, $m \sim O(10)$ or $O(100)$ from the Horizon (all the way up to $r \ll R$), entanglement is caused only by the correlations of $\hat{\phi}_{00}(n = 0.5)$ with iseIf (because the term $Det(A)(n = 0.5)$ dominates F).

We have proven that there is entanglement between states both near and far from the Event Horizon, and a state in the Event Horizon (or very near the Event Horizon). However, we cannot conclude that the states are separable far from the Event Horizon (fig. 7.11), or near the Event Horizon when both boxes are cubic (fig. 7.1). Also, since the states are not Gaussian States, we cannot calculate the negativity, we cannot measure the intensity of the entanglement, and thus, we cannot calculate its characteristics in the de Sitter Universe. Thus, we aren't any closer to finding some kind of formula for the dilution of states of the type (6.1).

That said, entanglement was found in the de Sitter vacuum, which seems to support the theories presented in section 5.1.2. The fact that F is greater near the Event

[3]There could always be a detector profile that could alter the functions $I_j(r)$ so much that correlations for states very far from the Event Horizon would become big enough to dominate the $-\frac{1}{4}$ term in F.

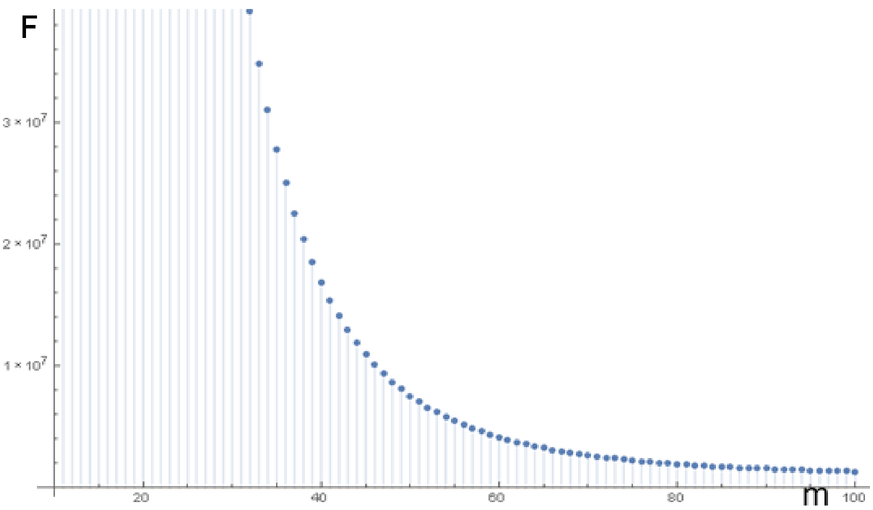


Figure 7.9: The presence of entanglement is tested using F, with one of the states on $n = \frac{1}{2}$ and the other varying in position m. The states here are averaged inside two boxes with the form of a spherical shell of radius $\sim R$ and thickness l_p. Here the values of m are changed by integers and we get a much smoother and well behaved curve, with F becoming smaller for greater values of m until it reaches a constant value.

Horizon, and that states in (and near) the Event Horizon are entangled with the rest of the Universe (which means that there are more entangled states and with stronger entanglement near the Event Horizon) is also consistent with the theories mentioned in section 6.2 and section 5.1.2. This behavior may be interpreted as evidence in favor of a thermodynamics of the entanglement associated with the Event Horizon, such as the one used in the theories presented in section 5.1.2,

$$dE_{\text{ent}} = T_{\text{unruh}} dS_{BH}.$$

Also, since the Event Horizon is entangled with the rest of the Universe, this could give a clue of the mechanism that allows the information of the Universe to fit in the 2-dimensional surface that surrounds it (as theorized in [7, 8]). Entanglement could be the connection between the information throughout the Universe, and the information at the Holographic surface formed by the Event Horizon. Of course this line of reasoning would require more research.
It should be mentioned that by the end of the work on this Thesis, two articles where found which also study the entanglement in the de Sitter Universe [97, 98], using a similar method of averaging states inside boxes to get Gaussian States, although the nature of their boxes and detector profiles are different.

More types of boxes and detector profiles could be used in order to try to get Gaussian States. Also, integrating explicitly the field operators inside the boxes (instead of using the approximation of infinitesimal ones) would give a larger freedom when choosing the form and the size of the boxes, although that would make the expressions for the Covariance Matrix even bigger and harder to handle both in terms of algebraic manipulations and computational power (which is part of the reason why the infinitesimal box approximation was chosen for this thesis). The bottom line would be to try to find a combination of boxes and detector profiles that would give us Gaussian States. That way the magnitude of entanglement could be calculated, which would give us the opportunity of better studying the behavior

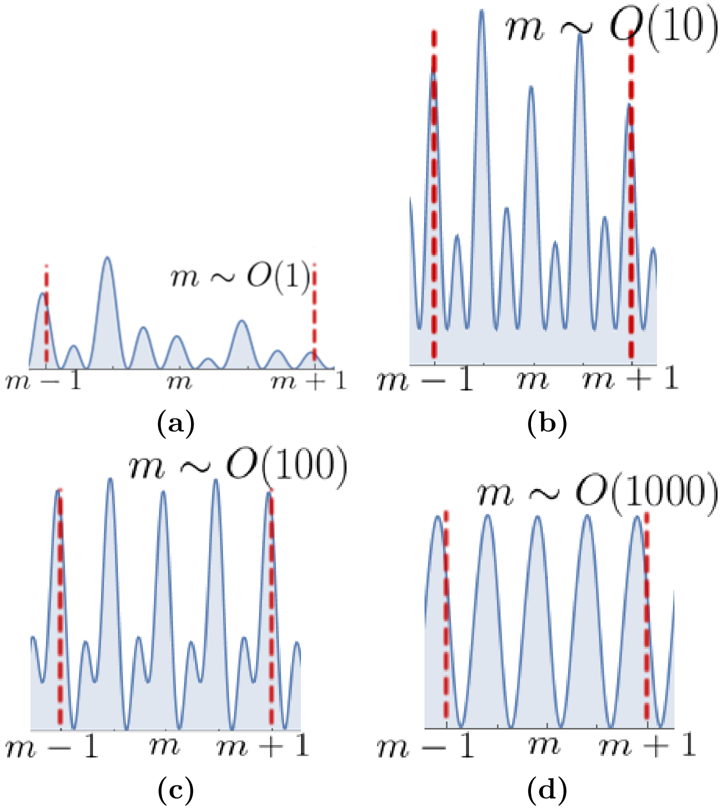


Figure 7.10: Here we show the form of the evolution of the sub-planckian oscillations of F for several orders of m, where the boxes used are the spherical shells and the other state is at $n = 0.5$. Notice that although they change shape somewhat for different values of m, they always have a wavelength $\lambda < 1$.

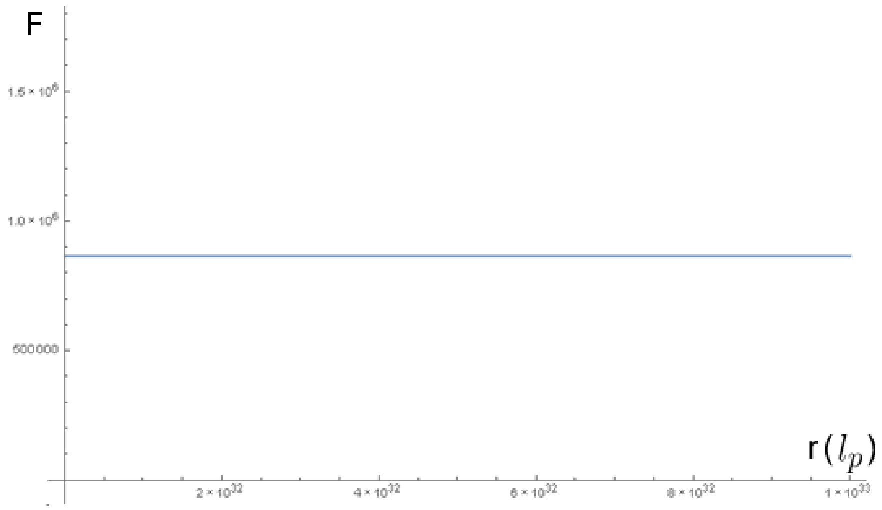


Figure 7.11: Entanglement is tested (using PPT criterion through F) for a state in the Event Horizon and the other in the region of $r \ll R$, averaged over square boxes of volume 1. $F \simeq 864\ 398$ for all values tested which means that there isn't detected any entanglement near the Event Horizon for these boxes.

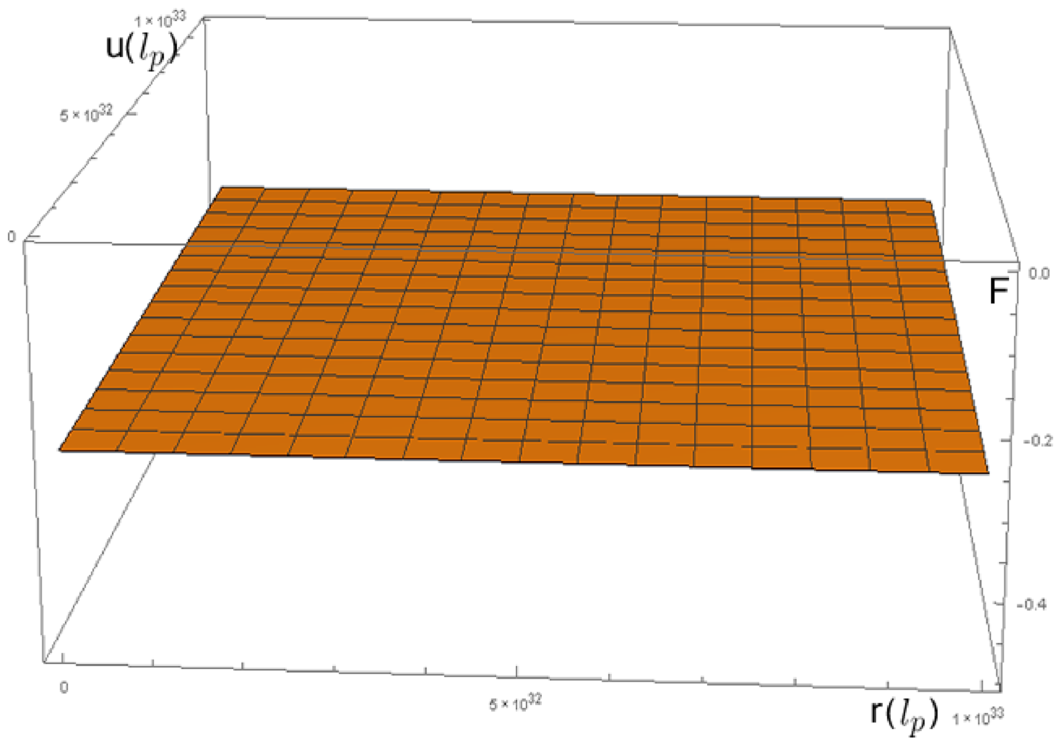


Figure 7.12: Entanglement is tested (using PPT criterion through F) for two states very far from the Event Horizon ($r, u \ll R$), averaged over square boxes of volume 1. $F \simeq -\frac{1}{4}$ for all values tested which means that there isn't detected any entanglement very far from the Event Horizon, for these boxes.

of entanglement of the vacuum in the de Sitter Universe.
Unless the choice of boxes/detector profiles would change drastically the relation between the $\mathbf{V}$ and the functions eqs. (7.65) to (7.70), one wouldn't expect an extreme departure from the results obtained here. Since the system has larger values of the functions $I_j(r)$ for $r \sim R$, more entanglement would be expected near the

Horizon than away from it (and, consequently, more entanglement with the Event Horizon than with the rest of the Universe).

It is also important to notice that, if we consider other values of l, l' and m, m', the total the entanglement of the system is not additive, that is

$$F \neq \sum_{lm;l'm'} F_{lm;l'm'}, \tag{7.87}$$

because the determinants of the matrices A, B, G and $\mathbf{V}$ will have terms similar to

$$F \sim \mathrm{Det}(\mathbf{V}) \sim H(x,x)\partial_t\partial_{t'}H(x,x)H(x',x')\partial_t\partial_{t'}H(x',x') \sim$$
$$\sim \sum_{\{l_i\}\{m_i\}} H_{l_1m_1;l_2m_2}(x,x)\partial_t\partial_{t'}H_{l_3m_3;l_4m_4}(x,x)H_{l'_1m'_1;l'_2m'_2}(x',x')\partial_t\partial_{t'}H_{l'_3m'_3;l'_4m'_4}(x',x'), \tag{7.88}$$

Hence, the introduction of more (or, in the ideal case, all) of the quantum numbers l and m might give different results of entanglement.

Chapter 8

Conclusion

In section 6.1, a method was reviewed that relates the vacuum energy of the Universe to the dilution of states that is associated with the Holographic Principle, by introducing a function $f(\mathbf{r}, \mathbf{k})$ in the integral of the vacuum mode, and considering the Event Horizon of our Universe as a spherical shell of Planck thickness (which was also suggested in [15]. In that regard, although the new Zero Point Energy gives, for some functions $f(\mathbf{r}, \mathbf{k})$, the result $\rho \sim \frac{1}{R^2 l_p^2}$, which is a known result deduced in other theories that relate the Holographic Principle and the Cosmological Constant (see chapter 5), no general proof for the shape of the function $f(\mathbf{r}, \mathbf{k})$ was produced. However, the density obtained by this method is similar to the one obtained in [10] by considering that the Holographic Principle entropy of the Universe comes from the entanglement entropy of vacuum states, and that produces a Holographic Energy, $dE = T_{BH} S_{hol.}$ which also becomes $\rho \sim \frac{1}{R^2 l_p^2}$. This, together with the suggestion in [15] (that entanglement between states near a Black Hole's Event Horizon and the states in the Event Horizon might explain the entropy of a Black Hole), sparked interest in the role that entanglement of the vacuum states in an expanding Universe could have in the Holographic Principle.
To be able to test the entanglement in the de Sitter Universe, we used the method presented in subsection 3.2.2, where the states where averaged inside two very small boxes (which conceptually can be understood as what happens when a detector with a finite spatial definition measure a state) in order to produce bipartite Gaussian States. However, it was discovered that this method does not produce Gaussian States, at least in the de Sitter Universe and for the boxes and the detector profiles used. That limited the applicability of the entanglement criterion that is used for Gaussian States (Simon criterion), but entanglement could still be tested because the averaged states could be regarded as Quantum Harmonic Oscillators. In this limit, however, the Simon criterion becomes only necessary but not sufficient condition of separability and, also, the amount of entanglement between the states cannot be measured using the Negativity. Entanglement can be proved to exist but not disproved, because the Simon criterion (which is a formulation of the PPT criterion for Gaussian States) ceases to be equivalent to the PPT criterion since for Quantum Harmonic Oscillators, other tests exist that, together with the Simon criterion, form the PPT criterion.

To be able to test the existence of entanglement, a scalar field $\hat{\phi}$ was quantized in a de

Sitter Universe. Although some literature exists on the subject of field quantization in general spacetime, such a systematic and complete calculation of $\hat{\phi}$, as was done in this thesis, was not found anywhere else by this Thesis author. For simplicity, the field was expanded in spherical harmonics, and entanglement was tested for states with zero angular momentum ($l = 0$).

Using the Simon criterion, we found that there is entanglement between vacuum states in the de Sitter Universe, when they are averaged inside spherical shells of volume $V \simeq 4\pi R^2 l_p$, thus supporting the claims made in [10]. More than that, entanglement was only found when at least one of the states was at a distance, d, to the Event Horizon, smaller than $d = R - r \sim O(10l_p)$. Entanglement became very small for distances of the order of $d \sim 10l_p$ and it disappeared completely for $d \sim 60l_p$. This seems to suggest that, if we consider the states belonging to Event Horizon of de Sitter space as all the states between the R and $R - l_p$, then most of the entanglement detected is between states of the Event Horizon and the rest of the Universe. Or, if we assume a priori that entanglement should exist between the vacuum states and the rest of the Universe (like in [15]), then these results could be taken as an indication that the states belonging to the Horizon must fit in a spherical shell of thickness $\sim O(l_p)$.
Also, the Simon criterion for separability was more violated when the second state was very near the Horizon, $u \leq R - 100l_p$, (if the first state was at the Horizon, $r = R - \frac{1}{2}l_p$) which means that, not only the entanglement is only detected if one of the states in/near the Event Horizon, it also seems to be greater when both states are very near the Event Horizon. In this case, entanglement starts with its maximum value when both states are together (which corresponds to a separation of $r - u = l_p$) and decrease with the increase of the distance between the states, until it reaches a value $F = \text{const}$, that remains constant no matter how far the second state is from the Event Horizon. This seems to be in accordance with known results [15, 18, 20] that tells us most of the entanglement of a system comes from modes near the boundary. This could be part of the reason why the entanglement energy in [10] is related to the temperature of the Event Horizon, and the maximum entropy allowed in the Holographic Principle. If most of the entanglement happens with the Event Horizon, it makes sense that the entropy and the temperature we should be using in (5.76) are the ones related to the Event Horizon. It should be noted that this is a pure philosophical statement, no mathematical connection was made in this Thesis regarding the nature of the expression (5.76) and our results, except for the confirmation of the existence of entanglement of the vacuum.
Again, since we are not dealing with Gaussian States, no absolute comment can be made about the quantity of entanglement of the system, since other tests equivalent to the Simon criterion could detect more entanglement, for example, in the region very far from the Horizon. States with other angular momentum could also give different results, especially because applying the Simon criterion to a state that corresponds to the sum all the harmonic quantum numbers (l and m) is not the same as applying the criterion for each state with a quantum number and then adding the results.

Another interesting conclusion is that there is dependence of the entanglement on the spatial distance between the states and the Event Horizon, and not, for example,

on the distance between the states themselves.
For states very far from the Event Horizon, if entanglement exists, then eqs. (7.57) to (7.60) lose their dependence on r. That is, if there is entanglement between states far from the Horizon, it is expected that it remains (at least approximately) constant regardless, of the spatial position of the states. This is expected to remain true as long as the boxes used are sufficiently small and have a constant volume (unlike $V = 4\pi r^2 l$ for example), and the detector profile does not alter too much the form of the field $\hat{\phi}$.

Although some confirmation of the behavior of the entanglement in the presence of an Event Horizon, that was suggested (or calculated) by other methods ([15,18,20]), was produced, no precise hint was obtained regarding the dilution of states in the Holographic Principle or any possible shape of the functions $f(\mathbf{r}, \mathbf{k})$. That is partially due to the inability of calculating the quantity of entanglement of the states in the Universe, but also because of the lack of precise knowledge of how entanglement can affect the energy of the system, in quantum field theory.
The fact that every vacuum state in the Universe is entangled with the Event Horizon is interesting as it can be regarded as evidence for the possible relation between entanglement and Holographic Principle. Indeed it could be a future path of research to study if entanglement could be the mechanism that relates the information inside a volume with the information on its surface boundary.

In section 6.2 the nature of the entanglement energy was discussed. We have seen that an entangled system is in a different quantum state from that of a separable system. Thus, the same physical system (say, two electrons) can have different energies if it is entangled or separable. It was then estimated how the entanglement of vacuum states could affect the vacuum energy, building up from arguments due to Ted Jacobson [92, 93]. It was found that an entangled pair of vacuum states should have energy of the form $\omega_{\text{ent}} \sim \frac{1}{r}$, where r is the distance between the states.

Calculating the negativity $N(r)$ of the entanglement between the Event Horizon and any other state in r, one could postulate that $N(r) = f(r, k)$, and that $E_{ZPE} \sim \int dr \int dk\ r^2 k^3 N(r)$, or even $E_{ZPE} \sim \int dr \int dk\ r^2 k^2 N(r) \omega_{\text{ent}}$, but no proof or test of any of these statements could be given.

Future paths would involve trying to obtain Gaussian States, in order to better understand the behavior of entanglement caused by the de Sitter Event Horizon. That could eventually be achieved by trying to use, for example, bigger boxes, or other types of detector profiles. Furthermore, testing for other harmonic quantum numbers (l, m) is also important, as it could eventually produce Gaussian States. Moreover, the study of the relation between the angular momentum of states and the presence of entanglement would, in itself, be interesting. The same way, the study of how entanglement varies with time (or with $a(t)$), using our calculations, would be possible and not too complicated.

Another interesting subject would be to study the effect of entanglement on the energy of a system in quantum field theory. That way, one could understand how the presence of entanglement might explain the dilution of states in Holographic Principle (for example, deducing $f(\mathbf{r}, \mathbf{k})$).

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