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On the reversed hazard rate
of sequential order statistics

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Abstract

Sequential order statistics can be used to describe the lifetime of a system with n components which works as long as k components function assuming that failures possibly affect the lifetimes of remaining units. In this work, the reversed hazard rates of sequential order statistics are examined. Conditions for the reversed hazard rate ordering and the decreasing reversed hazard rate property of sequential order statistics are given.

1   Introduction

Kamps (1995) introduced the concept of sequential order statistics (SOS) as an extension of the order statistics (OS) model. Following Cramer and Kamps (2003), sequential order statistics can be defined as follows: Let $F_1, \ldots, F_n$ be continuous distribution functions with $F_1^{-1}(1) \leq \cdots \leq F_n^{-1}(1)$ and let $B_1, \ldots, B_n$ independent random variables where B_i is beta distributed with parameters $n - i + 1$ and 1, $1 \leq i \leq n$. Then the random variables

$$X_{i:n}^* = F_i^{-1}\Big(1 - B_i\,\overline{F}_i(X_{i-1:n}^*)\Big), \quad \text{for } i = 1, \ldots, n\,,$$

are called sequential order statistics.

Note that OS are contained in the model of SOS via the specific choice $F_1 = \cdots = F_n$. In the reliability context, there exists a relation between SOS and the lifetimes of sequential k-out-of-n systems, in the same way that there exists a connection between OS and the lifetimes of k-out-of-n systems. In this

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case, the $(n - k + 1)$th SOS in a sample of size n represents the lifetime of a sequential k-out-of-n system (see Cramer and Kamps (2001)). A sequential k-out-of-n system is more flexible than a k-out-of-n system in the sense that, after the failure of some component, the distribution of the residual lifetime of the components at work may change.

The model of SOS is closely connected to several other models of ordered random variables. For instance, it is well known that the specific choice of distribution functions $F_i(t) = 1 - (1 - F(t))^{\alpha_i}$, $t \in \mathbb{R}$, $1 \leq i \leq n$, with a continuous distribution function F and positive real numbers $\alpha_1, \ldots, \alpha_n$ leads to the model of generalized order statistics with parameters $\gamma_i = (n - i + 1)\alpha_i$, $1 \leq i \leq n$. Further results about SOS and related models can be found, for instance, in Kamps (1995); Cramer and Kamps (1996); Kamps and Cramer (2001); Cramer (2006); Balakrishnan et al. (2008); Beutner (2008); Beutner and Kamps (2009); Burkschat (2009); Bedbur (2010); Beutner (2010); Burkschat et al. (2010); Balakrishnan et al. (2011) and Bedbur et al. (2012).

In this article, we focus on particular stochastic comparisons and ageing properties of SOS. Some recent articles on these subjects are, e.g., Zhuang and Hu (2007); Burkschat and Navarro (2011); Navarro and Burkschat (2011) and Torrado et al. (2012). We will present some results on the reversed hazard rate ordering and its associated ageing notion, the decreasing reversed hazard rate (DRHR) property (see, e.g., Block et al. (1998); Sengupta and Nanda (1999); Chandra and Roy (2001); Nanda and Shaked (2001); Finkelstein (2002); Nanda et al. (2003); Ahmad and Kayid (2005); Marshall and Olkin (2007); Shaked and Shanthikumar (2007)). Recent results on the DRHR property of some ordered random variables are given in Kundu et al. (2009) and Wang and Zhao (2010).

In Section 2, we recall the definitions of the reversed hazard rate ordering, the DRHR property and give some notations for SOS. The main results are given in Sections 3 and 4. More precisely, we investigate conditions on the underlying distribution functions on which the SOS are based, in order to compare SOS in the reversed hazard rate ordering and to obtain the DRHR property of SOS.

Throughout the article we use the terms increasing and decreasing in the weak sense, that is, a function g is called increasing (decreasing) if $x \leq y$ implies $g(x) \leq (\geq)g(y)$. Furthermore, we assume that the distributions of the occurring random variables have the same support which is given by an interval of the real line.

2 Definitions and notations

Let X be a non-negative random variable describing a lifetime with distribution function F, survival function $\bar{F} = 1 - F$, density function f and reversed hazard rate function $r_X = f/F$. Analogously, let Y be a non-negative random variable with distribution function G, survival function $\bar{G} = 1 - G$, density function g and reversed hazard rate function $r_Y = g/G$. First, we recall the definition of the reversed hazard rate order (see, e.g., Shaked and Shanthikumar (2007), Section 1.B.6).

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Definition 1 *The random variable X is said to be smaller than Y in the reversed hazard rate order (denoted by $X \leq_{rh} Y$) if $r_X(t) \leq r_Y(t)$ for all $t \geq 0$.*

Let the random variable $X_{(t)}$ be distributed as the time elapsed since the failure time X of a unit, given that the unit failed at or before time $t > 0$, i.e., let the distribution theoretical identity

$$X_{(t)} \stackrel{st}{=} [t - X \mid X \leq t]$$

hold. The random variable $X_{(t)}$ is known as the inactivity time or the reversed residual life of X at time t. Its survival function is given by

$$P\left(X_{(t)} > x\right) = \frac{F(t - x)}{F(t)}, \quad 0 \leq x < t.$$

The reversed hazard rate ordering is related to the random variable $X_{(t)}$, since $X \leq_{\mathrm{rh}} Y$ if $X_{(t)} \geq_{\mathrm{st}} Y_{(t)}$ for all $t \geq 0$ (see Shaked and Shanthikumar (2007), Section 1.B.6).

Related to this ordering, the decreasing reversed hazard rate (DRHR) class of life distributions have been introduced and studied in the literature (see, e.g., Sengupta and Nanda (1999)).

Definition 2 *The random variable X is said to have a decreasing reversed hazard rate (denoted by DRHR) if $r_X(t)$ is decreasing in t.*

Finally, we recall some results from the distribution theory of SOS. Let $X_{1:n}^*, \ldots, X_{n:n}^*$ be the SOS based on distribution functions $F_1, \ldots, F_n$ with respective density functions $f_1, \ldots, f_n$. Let $h_i = f_i / \overline{F}_i$, $i = 1, \ldots, n$, denote the hazard rates. Based on the results in Cramer and Kamps (2003), we can assume that

$$X_{1:n}^* = H_1^{-1}(Z_1),$$
$$X_{i:n}^* = H_i^{-1}(Z_i + H_i(X_{i-1:n}^*)), \text{ for } i = 2, 3, \ldots, n,$$

where H_i^{-1} denotes the inverse function of the cumulative hazard function $H_i = -\ln \overline{F}_i$ and $Z_1, \ldots, Z_n$ are independent random variables where Z_i is exponential distributed with parameter $n - i + 1$, $1 \leq i \leq n$. The density function of the first SOS is given by

$$f_{*,1}(t) = n h_1(t) \bar{F}_{*,1}(t),$$

with reversed hazard rate

$$r_{*,1}(t) = n h_1(t) \frac{\bar{F}_{*,1}(t)}{F_{*,1}(t)}.$$

Moreover, for $i = 2, \ldots, n$, the density function of the ith SOS is given by

$$f_{*,i}(t) = (n - i + 1) h_i(t) \left(\bar{F}_{*,i}(t) - \bar{F}_{*,i-1}(t)\right), \tag{1}$$

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and its reversed hazard rate is given by

$$r_{*,i}(t) = (n - i + 1)h_i(t)\left(\frac{\bar{F}_{*,i}(t) - \bar{F}_{*,i-1}(t)}{F_{*,i}(t)}\right)$$
$$= (n - i + 1)h_i(t)\left(\frac{F_{*,i-1}(t) - F_{*,i}(t)}{F_{*,i}(t)}\right). \tag{2}$$

3 Reversed hazard rate ordering of SOS

In this section, we will study conditions on the underlying distribution functions on which the SOS are based, in order to compare SOS in the reversed hazard rate ordering.

Theorem 3 *Let $X_{1:n}^*, \ldots, X_{n:n}^*$ be the SOS based on $F_1, \ldots, F_n$. Let h_i denote the hazard rate function of F_i for $i = 1, 2, \ldots, n$. Let $1 \le k \le n - 1$. If h_i/h_{i+1} is decreasing for $i = 1, 2, \ldots, k$, then $X_{k:n}^* \le_{rh} X_{k+1:n}^*$.*

Proof: The proof is carried out by induction. At first, we want to show that $X_{1:n}^* = H_2^{-1}(H_2(X_{1:n}^*)) \le_{\text{rh}} X_{2:n}^* = H_2^{-1}(Z_2 + H_2(X_{1:n}^*))$. Since the reversed hazard rate order is closed under increasing transformations (see Lemma 1.B.43 in Shaked and Shanthikumar (2007)), it is sufficient to show

$$H_2(X_{1:n}^*) \le_{\text{rh}} Z_2 + H_2(X_{1:n}^*). \tag{3}$$

According to Lemma 1.B.44 in Shaked and Shanthikumar (2007), if $H_2(X_{1:n}^*)$ is DRHR, then (3) holds. Let us introduce the notation $y_i = H_i^{-1}(t)$, for $i = 1, \ldots, n$, which is an increasing function in t. Let $T_2 = H_2(X_{1:n}^*)$. Then its reversed hazard rate function is given by

$$r_{T_2}(t) = \frac{r_{*,1}(y_2)}{h_2(y_2)} = n\frac{h_1(y_2)}{h_2(y_2)} \cdot \left(\frac{1}{F_{*,1}(y_2)} - 1\right).$$

The function in large brackets is decreasing and $\frac{h_1}{h_2}$ is decreasing by assumption. Thus, T_2 is DRHR, and hence, $X_{1:n}^* \le_{\text{rh}} X_{2:n}^*$. Now, let us assume that the assertion is valid for $i - 1(< k)$, that is, $X_{i-1:n}^* \le_{\text{rh}} X_{i:n}^*$ and we will show that $X_{i:n}^* \le_{\text{rh}} X_{i+1:n}^*$. Let $T_{i+1} = H_{i+1}(X_{i:n}^*)$. Then its reversed hazard rate function is given by

$$r_{T_{i+1}}(t) = \frac{r_{*,i}(y_{i+1})}{h_{i+1}(y_{i+1})} = (n - i + 1)\frac{h_i(y_{i+1})}{h_{i+1}(y_{i+1})} \cdot \left(\frac{F_{*,i-1}(y_{i+1})}{F_{*,i}(y_{i+1})} - 1\right).$$

By the induction hypothesis, $\frac{F_{*,i-1}}{F_{*,i}}$ is decreasing since $X_{i-1:n}^* \le_{\text{rh}} X_{i:n}^*$ and $\frac{h_i}{h_{i+1}}$ is decreasing by assumption. Hence, $T_{i+1} = H_{i+1}(X_{i:n}^*)$ is DRHR. Now, again from Lemma 1.B.44, $H_{i+1}(X_{i:n}^*) \le_{\text{rh}} Z_{i+1} + H_{i+1}(X_{i:n}^*)$ and since the reversed hazard rate order is closed under increasing transformations, we have proved that $X_{i:n}^* \le_{\text{rh}} X_{i+1:n}^*$.

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The condition of the above theorem is easy to check, since it is based on the parent distributions $F_1, \ldots, F_n$. If the condition of Theorem 3 is not satisfied, then SOS do not need to be ordered according to the reversed hazard rate order as the following example illustrates.

Example 4 *Let us consider two SOS, $X_{1:2}^*$ and $X_{2:2}^*$, based on F_1, F_2. According to the results in Cramer and Kamps (2003), the survival function of the first SOS is given by*

$$\bar{F}_{*,1}(t) = \bar{F}_1^2(t)$$

with hazard rate $h_{*,1}(t) = 2h_1(t)$, where h_1 is the hazard rate function of F_1. The survival function of the second SOS is given by

$$\bar{F}_{*,2}(t) = \bar{F}_{*,1}(t) + \bar{F}_2(t) \int_0^t \frac{f_{*,1}(z)}{\bar{F}_2(z)} dz,$$

where $f_{*,1}$ is the density function of $X_{1:2}^*$. Note that the density function of $X_{2:2}^*$ defined in (1) can be written as

$$f_{*,2}(t) = f_2(t) I(t),$$

where

$$I(t) = \int_0^t \frac{f_{*,1}(z)}{\bar{F}_2(z)} dz.$$

Now let us assume that $\bar{F}_1(t) = e^{-t}$ and

$$\bar{F}_2(t) = \begin{cases} e^{-2t}, & 0 \leq t \leq \ln(2)/2, \\ 2e^{-2t}(1 - e^{-2t}), & t > \ln(2)/2. \end{cases}$$

Note that $h_1(t) = 1$ and

$$h_2(t) = \begin{cases} 2, & 0 \leq t \leq \ln(2)/2, \\ 2(1 - 2e^{-2t})/(1 - e^{-2t}), & t > \ln(2)/2. \end{cases}$$

In particular, the condition of Theorem 3 is not satisfied. Then, we have

$$\bar{F}_{*,1}(t) = \bar{F}_1^2(t) = e^{-2t}$$

and we get for $0 < t \leq \ln(2)/2$

$$I(t) = \int_0^t \frac{f_{*,1}(z)}{\bar{F}_2(z)} dz = 2t$$

and for $t > \ln(2)/2$

$$I(t) = \ln(2) + \int_{\ln(2)/2}^t \frac{2e^{-2z}}{2e^{-2z}(1 - e^{-2z})} dz = \ln(2) + \frac{1}{2} \ln(1 - e^{-2t}) + t.$$

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Hence, we obtain

$$F_{*,2}(t) = \begin{cases} 1 - e^{-2t} - 2te^{-2t}, & 0 \le t \le \ln(2)/2, \\ 1 - e^{-2t} - 2e^{-2t}(1 - e^{-2t})(\ln(2) + \frac{1}{2}\ln(1 - e^{-2t}) + t), & t > \ln(2)/2. \end{cases}$$

Then, we get

$$\frac{F_{*,2}(0.37)}{F_{*,1}(0.37)} \approx 0.295 > 0.282 \approx \frac{F_{*,2}(0.43)}{F_{*,1}(0.43)}$$

so that $X_{1:2}^*$ and $X_{2:2}^*$ are not ordered according to the reversed hazard rate ordering.

In the following examples, we discuss two distributions that satisfy the assumptions of Theorem 3 (see also Sengupta and Deshpande (1994) and Rowell and Siegrist (1998)). Hence the corresponding sequential order statistics are reversed hazard rate ordered.

Example 5 *Let us consider the Weibull distributions defined by*

$$F_i(t) = 1 - \exp\left(-\theta_i t^{\beta_i}\right)$$

for $t \ge 0$, where $\theta_i, \beta_i > 0$ for $i = 1,\ldots,n$. In this case, the hazard rate functions are given by

$$h_i(t) = \theta_i \beta_i t^{\beta_i - 1}$$

for $t \ge 0$ and $i = 1,\ldots,n$. In particular,

$$\frac{h_i(t)}{h_{i+1}(t)} = \frac{\theta_i \beta_i}{\theta_{i+1} \beta_{i+1}} t^{\beta_i - \beta_{i+1}}$$

is decreasing in t if and only if $\beta_i \le \beta_{i+1}$. Then, the assumptions of Theorem 3 are satisfied and we get that $X_{k:n}^* \le_{rh} X_{k+1:n}^*$ for every $\theta_i > 0$ when $\beta_1 \le \ldots \le \beta_k \le \beta_{k+1}$.

Example 6 *Let us consider the power function distributions defined by*

$$F_i(t) = \left(\frac{t}{c}\right)^{\alpha_i}$$

for $0 < t < c$ and $\alpha_i > 0$ for $i = 1,\ldots,n$. In this case, the hazard rate functions are given by

$$h_i(t) = \frac{\frac{1}{t}\alpha_i \left(\frac{1}{c}t\right)^{\alpha_i}}{1 - \left(\frac{t}{c}\right)^{\alpha_i}},$$

for $t \ge 0$ and $i = 1,\ldots,n$. In particular,

$$\frac{h_i(t)}{h_{i+1}(t)} = \frac{\alpha_i}{\alpha_{i+1}} \left(\frac{c^{\alpha_{i+1}} - t^{\alpha_{i+1}}}{c^{\alpha_i} - t^{\alpha_i}}\right) t^{\alpha_i - \alpha_{i+1}}$$

is decreasing in t if and only if $\alpha_i \le \alpha_{i+1}$. Hence, the assumptions of Theorem 3 are satisfied and we get that $X_{k:n}^* \le_{rh} X_{k+1:n}^*$ for every $c > 0$ when $\alpha_1 \le \ldots \le \alpha_k \le \alpha_{k+1}$.

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In Torrado et al. (2012), sufficient conditions are given for the likelihood ratio ordering of SOS. First, let us recall the definition of a TP_2 function. A positive function h of two variables, x and y, say, is called TP_2 if $h(x', y)/h(x, y)$ is increasing in y whenever $x \leq x'$. Then, the following result for the likelihood ratio ordering of SOS can be proven (see Theorem 5 in Torrado et al. (2012)).

Theorem 7 *Let $X_{1:n}^*, \ldots, X_{n:n}^*$ be SOS based on $F_1, \ldots, F_n$. If $\frac{f_i(t)}{f_{i+1}(t)}$ and $h_i(t)$ are TP_2 in (i, t), and $F_i \leq_{hr} F_{i+1}$ for $i = 1, \ldots, n-1$, then*

$$X_{i:n}^* \leq_{lr} X_{i+1:n}^*,$$

for $i = 1, \ldots, n-1$.

Note that $h_i(t)$ is TP_2 in (i, t) means that $h_{i+1}(t)/h_i(t)$ is increasing in t. Thus, this is the same condition as in Theorem 3. However, this condition is not a sufficient condition for the likelihood ratio ordering as we illustrate in the following example.

Example 8 *Let us consider two sequential order statistics based on F_1, F_2, where $\bar{F}_1(t) = e^{-t}$ (exponential) and $\bar{F}_2(t) = e^{-t^2}$ (Weibull), for $t \geq 0$. Then their hazard rate functions are $h_1(t) = 1$ and $h_2(t) = 2t$, respectively, so h_1/h_2 is decreasing in t, and hence, the sequential order statistics are reversed hazard rate ordered according to Theorem 3. However, it can be shown that $X_{1:2}^*$ and $X_{2:2}^*$ are not hazard rate ordered (see Example 3.1 in Navarro and Burkschat (2011)) and, as an immediate consequence, the sequential order statistics are not ordered according to the likelihood ratio ordering. Note that h_1 and h_2 are not ordered for $t \geq 0$. Therefore, the condition $F_i \leq_{hr} F_{i+1}$ of Theorem 7 does not hold.*

4 DRHR class of SOS

In this section, we will study the DRHR property of SOS. Since $X_{k-1:n}^* \leq_{rh} X_{k:n}^*$ is equivalent to the fact that $\frac{F_{*,k-1}}{F_{*,k}}$ is a decreasing function, we immediately obtain from (2) the relation

$$X_{k-1:n}^* \leq_{rh} X_{k:n}^* \quad \Longleftrightarrow \quad \frac{r_{*,k}(t)}{h_k(t)} \text{ is decreasing in } t. \tag{4}$$

Consequently, we get this sufficient condition for the DRHR property in the presence of a reversed hazard rate ordering of SOS.

Theorem 9 *Let $X_{1:n}^*, \ldots, X_{n:n}^*$ be SOS based on $F_1, \ldots, F_n$ with hazard rate functions $h_1, \ldots, h_n$. Let $2 \leq k \leq n$ and let h_k be decreasing. If $X_{k-1:n}^* \leq_{rh} X_{k:n}^*$, then $X_{k:n}^*$ is DRHR.*

Proof: Since h_k is decreasing, relation (4) yields the assertion.

Combining this result with Theorem 3 from the preceding section, we get a condition only in terms of the underlying hazard rates.

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Theorem 10 *Let* $1 \leq k \leq n$. *If* h_i/h_{i+1} *is decreasing for* $i = 1, 2, \ldots, k-1$ *and* h_k *is decreasing, then* $X_{k:n}^*$ *is DRHR.*

Proof: Let $k = 1$. If h_1 is decreasing, it is well known that $X_{1:n}^*$ is DHR and hence DRHR. If $2 \leq k \leq n$, then the result follows from Theorem 3 and Theorem 9.

Let the underlying distributions $F_1, F_2, \ldots, F_n$ be given by

$$\bar{F}_i = \bar{F}^{\alpha_i}, \quad i = 1, 2, \ldots, n, \tag{5}$$

where F denotes an absolutely continuous distribution function and $\alpha_i > 0, 1 \leq i \leq n$. In this model the underlying distributions possess proportional hazard rates.

Corollary 11 *Let (5) hold. If F is DHR, then $X_{k:n}^*$ is DRHR for $k = 1, \ldots, n$.*

Proof: By assumption, $h_1, h_2, \ldots, h_n$ are decreasing and $h_i/h_{i+1} = \alpha_i/\alpha_{i+1}$ is constant for $i = 1, 2, \ldots, n-1$. Hence, the result follows from Theorem 10.

Remark 12 *It is well known that the DHR property implies the DRHR property. Note that the conclusion of Corollary 11 cannot be strengthened to the DHR property. This can be shown by considering the second sequential order statistic in model (5) based on a standard exponential distribution $F(t) = 1 - e^{-t}, t \geq 0$, and $\alpha_1 = \alpha_2 = 1$, i.e., the usual order statistic $X_{2:2}$ based on F for a sample of size 2. Then F is DHR (and IHR), but it is well known that $X_{2:2}$ is not DHR (but instead IHR).*

Remark 13 *The DHR assumption on F in Corollary 11 cannot be replaced by the DRHR property. This follows from Counterexample 3.1 in Kundu et al. (2009).*

Consider model (5) and let $\gamma_i = (n - i + 1)\alpha_i$ for $i = 1, \ldots, n$. If the parameters $\gamma_1, \ldots, \gamma_n$ are pairwise different, i.e., $\gamma_i \neq \gamma_j$ for $i \neq j$, then the distribution function and the density of the kth sequential order statistic are given by (see Kamps and Cramer (2001))

$$F^{X_{k:n}^*}(t) = 1 - c_{k-1} \sum_{i=1}^{k} \frac{a_{i,k}}{\gamma_i} \left(1 - F(t)\right)^{\gamma_i},$$

$$f^{X_{k:n}^*}(t) = c_{k-1} \sum_{i=1}^{k} a_{i,k} \left(1 - F(t)\right)^{\gamma_i - 1} f(t),$$

with the constants

$$c_{k-1} = \prod_{j=1}^{k} \gamma_j, \quad a_{i,k} = \prod_{\substack{j=1 \\ j \neq i}}^{k} \frac{1}{\gamma_j - \gamma_i}, \quad 1 \leq i \leq k \leq n.$$

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The empty product $\prod_\emptyset$ is defined to be 1.

In the following theorem, we give a transmission property of the DRHR class among the first sequential order statistics when the underlying distributions possess proportional hazard rates.

Theorem 14 *Let (5) hold and* $2\gamma_1 \geq \gamma_2$*. If* $X_{2:n}^*$ *is DRHR, then* $X_{1:n}^*$ *is DRHR.*

Proof: If $\gamma_1 = \gamma_2$ holds, then the result follows from Theorem 3.2 in Kundu et al. (2009). Let $2\gamma_1 \geq \gamma_2$ and $\gamma_1 \neq \gamma_2$. Then the reversed hazard rates of $X_{1:n}^*$ and $X_{2:n}^*$ are

$$r_{*,1}(t) = \frac{f_{*,1}(t)}{F_{*,1}(t)} = \frac{\gamma_1(1 - F(t))^{\gamma_1 - 1}f(t)}{1 - (1 - F(t))^{\gamma_1}}, \tag{6}$$

$$r_{*,2}(t) = \frac{f_{*,2}(t)}{F_{*,2}(t)} = \frac{\frac{\gamma_1\gamma_2}{\gamma_2 - \gamma_1}\left((1 - F(t))^{\gamma_1 - 1} - (1 - F(t))^{\gamma_2 - 1}\right)f(t)}{1 - \frac{1}{\gamma_2 - \gamma_1}\left(\gamma_2(1 - F(t))^{\gamma_1} - \gamma_1(1 - F(t))^{\gamma_2}\right)}. \tag{7}$$

We want to show that $r_{*,1}/r_{*,2}$ is a decreasing function. It is sufficient to consider the uniform distribution and then to show that

$$\eta_{1,2}(x) = \frac{r_{*,1}(1 - x)}{r_{*,2}(1 - x)}, \quad x \in (0, 1),$$

is increasing. After some simplifications, we obtain

$$\eta_{1,2}(x) = \frac{1}{\gamma_2}\left(\frac{\gamma_1 - \gamma_2}{1 - x^{\gamma_1 - \gamma_2}} - \frac{\gamma_1}{1 - x^{\gamma_1}}\right) + 1, \quad x \in (0, 1),$$

with the derivative

$$\eta_{1,2}'(x) = \frac{1}{\gamma_2}\left(\frac{(\gamma_1 - \gamma_2)^2 x^{\gamma_1 - \gamma_2 - 1}}{(1 - x^{\gamma_1 - \gamma_2})^2} - \frac{\gamma_1^2 x^{\gamma_1 - 1}}{(1 - x^{\gamma_1})^2}\right), \quad x \in (0, 1). \tag{8}$$

Consider the function

$$h(x, d) = \frac{d^2 x^d}{(1 - x^d)^2} = \frac{d^2}{(1 - x^d)(x^{-d} - 1)}, \quad x \in (0, 1), d \neq 0.$$

Note that $h(x, d) = h(x, -d)$ for every $x \in (0, 1)$. We want to show that $h(x, d)$ is a decreasing function in $d > 0$ for fixed $x \in (0, 1)$. Then, we obtain

$$h(x, c) \geq h(x, \gamma_1), \quad 0 < |c| \leq \gamma_1,$$

and the assertion follows from (8) and the assumption $-\gamma_1 \leq \gamma_1 - \gamma_2 < \gamma_1$. Thus, we consider the derivative

$$\frac{\partial}{\partial d}h(x, d) = \frac{d}{(1 - x^d)^3}\left\{2x^d - 2x^{2d} + x^d\ln(x^d) + x^{2d}\ln(x^d)\right\}$$

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for $x \in (0,1)$ and $d > 0$. It is sufficient to show that the expression in curly brackets is non-positive. By applying the substitution $-2\ln(x^d) = z > 0$, this can be shown to be equivalent to

$$-4e^{-z/2} + 4e^{-z} + ze^{-z/2} + ze^{-z} \geq 0.$$

The last expression coincides with the density function of the sum $X_1 + X_2$ of random variables (X_1, X_2) that follow McKay's bivariate gamma distribution with parameters $a = b = 2, c = 1$ (see Kotz et al., 2000, p. 432). This yields the assertion.

Remark 15 *It can be shown for $\gamma_1, \gamma_2 > 0, \gamma_1 \neq \gamma_2$, that*

$$\lim_{x \to 1-} \eta_{1,2}(x) = \frac{1}{2}, \quad \lim_{x \to 0+} \eta_{1,2}(x) = \begin{cases} 0, & \gamma_1 > \gamma_2, \\ (\gamma_2 - \gamma_1)/\gamma_2, & \gamma_1 < \gamma_2. \end{cases}$$

In particular, if $2\gamma_1 < \gamma_2$, then the function $\eta_{1,2}$ is not an increasing function.

Remark 16 *In Kundu et al. (2009) and Wang and Zhao (2010) the transmission of the DRHR property is studied for k-records and m-generalized order statistics, in particular usual order statistics. Theorem 14 extends Theorem 3.1 in Kundu et al. (2009) in the case $n = 2, k \in \mathbb{N}$, and Theorem 2.3(i) in Wang and Zhao (2010) in the case $r = 2$ to the model of SOS.*

The following example illustrates that under the assumptions of Theorem 14 the third SOS may not possess the DRHR property although the second (and therefore the first) SOS is DRHR.

Example 17 *Let (5) hold and suppose that F is a uniform distribution over $(0,1)$. Assume that $\gamma_1 = 2$, $\gamma_2 = 4$ and $\gamma_3 = 1/2$. Note that the assumptions of Theorem 14 hold. Then, from (6) and (7), we get the reversed hazard rates of $X_{1:n}^*$ and $X_{2:n}^*$, namely*

$$r_{*,1}(t) = \frac{1}{t} + \frac{1}{t-2} \quad and \quad r_{*,2}(t) = 2r_{*,1}(t),$$

which are decreasing functions for $t \in (0,1)$, so $X_{1:n}^*$ and $X_{2:n}^*$ both are DRHR. However, it is evident from Fig. 1 that $X_{3:n}^*$ is not so.

Clearly, the preceding example does not contradict a possible general result that $X_{k:n}^*$ DRHR implies $X_{k-1:n}^*$ DRHR for $k \geq 2$. However, the proof of Theorem 14 is tailored to the case $k = 2$ and an extension of this particular derivation to $k \geq 3$ is not obvious.

Finally, we give a condition such that the DRHR property of SOS implies their reversed hazard rate order. Let us first present the following lemma which can be straightforwardly proven (for related results, see, e.g., Righter et al. (2009)).
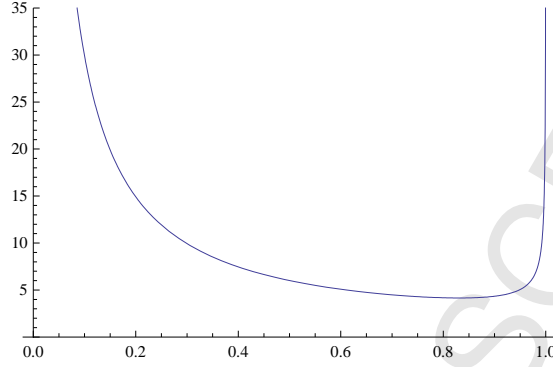
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Figure 1: Plot of the reversed hazard rate function of $X^*_{3:n}$ when (5) holds with $\gamma_1 = 2, \gamma_2 = 4, \gamma_3 = 1/2$ and F is a uniform distribution over $(0,1)$

Lemma 18 *Let X be an absolutely continuous random variable and ϕ be a strictly increasing and convex function. If X is DRHR, then $\phi(X)$ is DRHR.*

Applying the preceding lemma, we have this sufficient condition for the reversed hazard rate order.

Theorem 19 *Let $X^*_{1:n}, \ldots, X^*_{n:n}$ be SOS based on $F_1, \ldots, F_n$ with hazard rate functions $h_1, \ldots, h_n$. Let $2 \le k \le n$ and let h_k be increasing. If $X^*_{k-1:n}$ or $X^*_{k:n}$ is DRHR, then $X^*_{k-1:n} \le_{rh} X^*_{k:n}$.*

Proof: Note that the function H_k is strictly increasing on the support. Moreover, it is convex, because h_k is increasing. Assume that $X^*_{k-1:n}$ is DRHR. Then $H_k(X^*_{k-1:n})$ is DRHR according to Lemma 18. By applying Lemma 1.B.43 and 1.B.44 in Shaked and Shanthikumar (2007), it follows

$$X^*_{k-1:n} = H_k^{-1}(H_k(X^*_{k-1:n})) \le_{\text{rh}} H_k^{-1}\left(Z_k + H_k(X^*_{k-1:n})\right) = X^*_{k:n}.$$

Now assume that $X^*_{k:n}$ is DRHR. Using Lemma 18, we obtain that $W_k = H_k(X^*_{k:n})$ is also DRHR. Because the reversed hazard rate of W_k is given by

$$r_{W_k}(t) = \frac{r_{*,k}}{h_k}\left(H_k^{-1}(t)\right),$$

we conclude that $r_{*,k}(t)/h_k(t)$ is decreasing in t. Thus, the result follows from relation (4).

Remark 20 *It can be seen from the previous proof that $H_k(X^*_{k:n})$ is DRHR if and only if $r_{*,k}(t)/h_k(t)$ is decreasing in t. Taking this into accont, (4) yields the general relation*

$$X^*_{k-1:n} \le_{rh} X^*_{k:n} \quad \Longleftrightarrow \quad H_k(X^*_{k:n}) \text{ is DRHR.}$$

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Acknowledgements

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