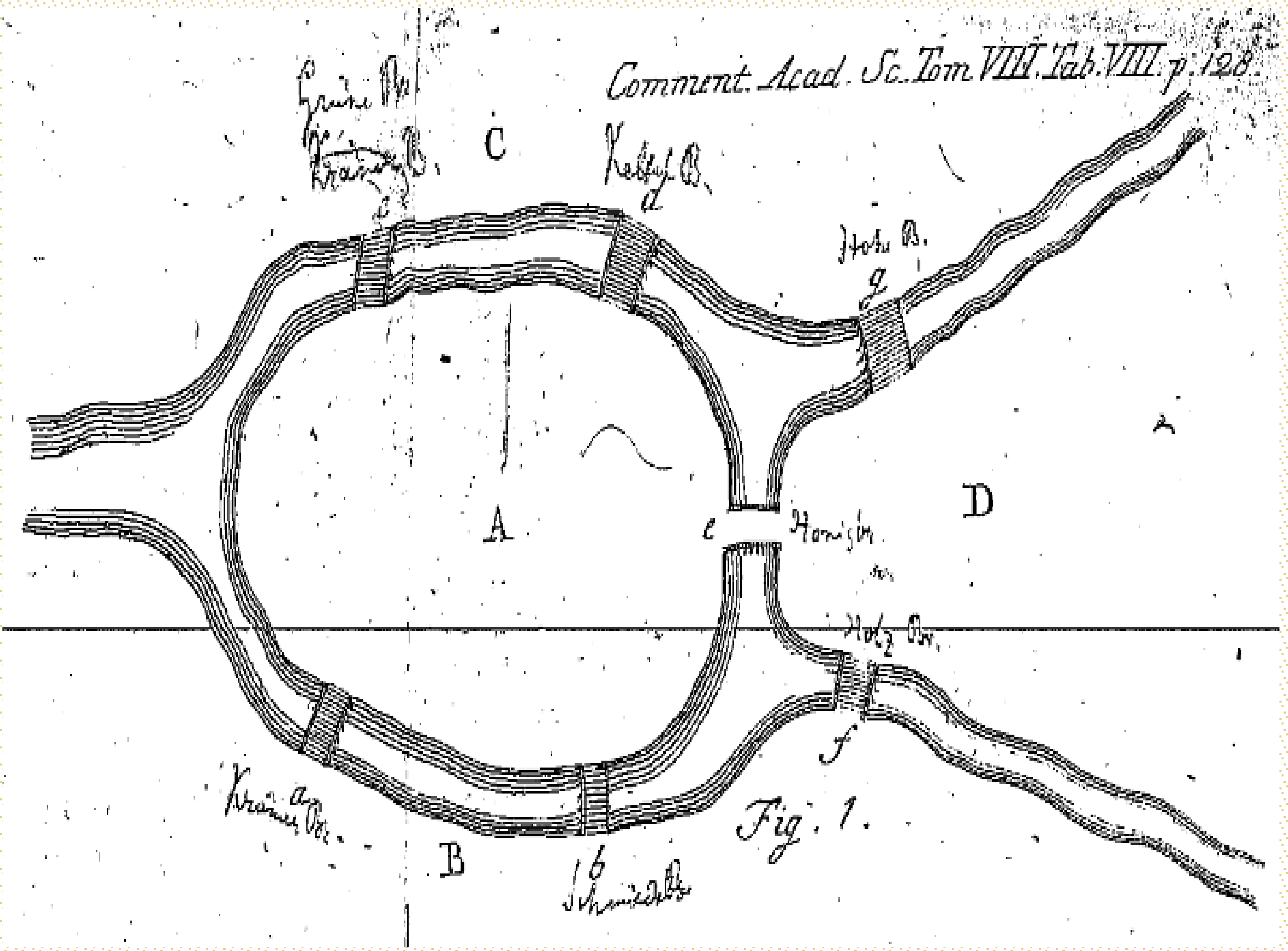

Nuno Álvaro Ferreira Rodrigues

HEURISTICS IN PROBLEM SOLVING FOR THE TEACHING AND LEARNING OF MATHEMATICS

Interuniversity Doctoral Program in Psychology – Educational Psychology Expertise
(association with the Faculty of Psychology at University of Lisbon)
Doctorate thesis with the survey of PhD Professor Ana Cristina Ferreira de Almeida and
PhD Professor Alexander Kovacec, presented to the Faculty of Psychology and
Educational Sciences, University of Coimbra

2015

Universidade de Coimbra

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ABSTRACT

With the purpose of improving success in Mathematics and, concomitantly, the teaching and the learning process, this dissertation is based on heuristics and on theoretical and empirical context to explore examples of mathematical problems solved, as reasoned and effective alternative of experiential learning of problem based Mathematics. The investigation was guided by the questions: *(1) Why the negative conception concerning Mathematics? (2) What are the cognitive processes activated during problem solving tasks? (3) How can teachers support the learning of Mathematics by promoting problem solving ability? (4) In what extent does the systematic practice of problem solving contribute to improve student's performance?* The main targets are teachers and students from Third Cycle (7^{th} to 9^{th} grade) and Secondary Education (10^{th} to 12^{th} grade), including Professional Courses, as well as educational experts that offer workshops in Centers for Instructional Development. For the feasibility of this project we initiated a variety of studies, some more theoretical oriented to explain the problem solving learning processes, and an empirical approach focus on teachers' and their students' performances. Through different methodological approaches the structural axis of this research was the compilation of detailed problem solving statements and resolutions processes with the purpose to create a problem solving *Manual*, to be further tested. The genesis of this work lies in the experiential problem solving context with teachers, mediators of such activity in the classroom with their students, in an accredited training course, which took place in two workshop editions, in Coimbra, between 2011 and 2013. 34 teachers participated and we considered the Kolb experiential learning model (1984) to sustain the training proposed. As characterization elements and efficacy measures we request teachers to answer a questionnaire of attitudes towards problem-solving, strategies for teaching math and problem solving, a written problem solving test evaluation, and problem solving questions to apply to students in the classroom, evaluated according previous established criteria, with the data gathering process ending in critical appraisal concerning students' knowledge and skills in a reflective portfolio. Some students (n = 96) answered to a questionnaire concerning math attitudes and 339 students answered to problem solving questions. This process was designed to help teachers getting aware about the effectiveness of the teaching and learning of heuristic procedures (Ponte, 1992).

We noticed that the workshops of training were considered useful and pertinent. However, when problem solving activity was engaged, we observed compromising behaviors of success among teachers addressing learning and teaching strategies. As for the answers of the 96 students from 9^{th} grade to the questionnaire attitudes towards math subject, were perceived correlations between: *Beliefs/Motivation*, *External Control*, *Mood*, *Resources*, *Heuristics*, and

Knowledge Exhibition which validate Schoenfeld's model about the mechanisms activated in the course of problem solving activity.

Teachers' testimonials, based on the outcome of their students in problem solving activities, suggest that the benefit from using heuristic procedures is the result of a continuous work during an extended period of time. The effect of sporadic intervention doesn't produce consistent changes.

The problem solving activity constitutes a preponderant factor in the teaching and learning process. Teachers and students are aware of its importance. Problem solving implementation in the classroom is compromised by the teacher's duty to carry out the school year program and the student's difficulty to perform autonomous work. Teachers' problem solving continuous training is an opportunity to disseminate mathematical content and effective pedagogical practices.

We suggest, in particular to the 9th grade, because it precedes Secondary Education that requires the choice of an educational pathway, allocated school time to the practice of mathematical problem solving. In this way it would be promoted, in continuous, cognitive processes that can help improve performances, with positive repercussions in various learning contexts. The results suggest that it is latent in the student a learning potential that is eventually curtailed by beliefs of failure and/or inadequate pedagogical practices. We hope that our contribution will help teachers and schools to incorporate problem solving practices in the classroom dynamics.

Keywords:

Heuristics, Learning, Mathematics, Problem Solving, Teachers' continuous training.

RESUMO

Com a intenção de melhorar o sucesso em Matemática e, concomitantemente, os processos de ensino e de aprendizagem, esta dissertação analisa de forma contextualizada o potencial da utilização de heurísticas e exemplos de problemas resolvidos, como alternativa fundamentada e efetiva de aprendizagem experiencial da Matemática. A investigação foi norteada pelas questões: *(1) Porquê a conceção negativa a respeito da disciplina de Matemática? (2) Quais são os processos cognitivos ativados durante tarefas de resolução de problemas? (3) Como podem os professores favorecer a aprendizagem da Matemática pela promoção da capacidade de resolução de problemas? (4) Em que medida a prática sistemática da resolução de problemas contribui para melhorar o desempenho do estudante?* São principais visados professores e alunos dos Ensinos Básico e Secundário, incluindo Cursos Profissionais, assim como formadores que ministram Cursos de Formação. Para a viabilização deste projeto levamos a cabo diferentes estudos, ora de cariz mais teórico-explicativo dos processos de aprendizagem e de resolução de problemas, ora com foco nas práticas do professor e desempenho dos alunos. Através de abordagens metodológicas diversas constituiu eixo estruturante deste trabalho a compilação de enunciados circunstanciados de problemas que resultaram num potencial contributo para a ulterior elaboração de um *Manual* de resolução de problemas, que depois de concebido deve ser testado. O estudo decorreu na sequência de uma experiência de formação acreditada de professores de Matemática, que se desenrolou em duas edições em modalidade de Oficina, em Coimbra, entre 2011 e 2013, tendo subjacente o modelo de aprendizagem experiencial de Kolb (1984). Colaboraram 34 professores de Matemática. Como elementos de caracterização e medidas de eficácia foi pedido aos professores que respondessem a um questionário de atitudes face à resolução de problemas, estratégias de ensino de Matemática e de resolução de problemas, a uma prova escrita de resolução de problemas, e aplicassem provas de resolução de problemas aos seus alunos, as quais avaliaram a partir de critérios partilhados, tendo o processo de recolha de dados culminado na apreciação crítica relativamente aos conhecimentos e competências dos alunos em portefólio reflexivo. Alguns alunos (n = 96) responderam a um questionário de atitudes face à disciplina de Matemática e 339 alunos responderam às provas de resolução de problemas ministradas. Teve este processo a finalidade de auxiliar os professores a consciencializarem-se da eficácia do ensino e da aprendizagem de procedimentos heurísticos (Ponte, 1992).

Verificámos que a oficina de formação foi considerada útil e pertinente. Contudo, em momentos formais, observaram-se entre os professores comportamentos comprometedores do sucesso no ensino e aprendizagem de estratégias de resolução de problemas. Quanto às respostas dos 96 alunos do 9º Ano de Escolaridade ao questionário de atitudes relativamente à

disciplina de Matemática, surgiram correlacionadas as subescalas de: *Crenças/Motivação*, *Controlo Externo*, *Disposição*, *Recursos*, *Heurísticas* e *Conhecimento Manifestado*, o que reflete a validade do modelo de Schoenfeld a respeito dos mecanismos ativados durante o processo de resolução de problemas.

Os depoimentos dos professores, alicerçados nos resultados dos seus alunos nas atividades de resolução de problemas, sugerem que o benefício da utilização de procedimentos heurísticos resulta de um trabalho contínuo durante um período alargado de tempo. O efeito da intervenção esporádica não produz alterações consistentes.

A atividade de resolução de problemas constitui fator preponderante no processo de ensino e aprendizagem da Matemática. Professores e alunos têm noção da sua importância. A implementação em sala de aula é comprometida pelo dever do professor cumprir o programa da disciplina e pela dificuldade do aluno realizar trabalho autónomo. A formação contínua de professores em resolução de problemas constitui uma oportunidade para disseminar conteúdos e práticas pedagógicas eficazes.

Sugere-se, nomeadamente ao nível do 9º Ano de Escolaridade, por se tratar de um ano de transição de ciclo que obriga à escolha de um percurso educativo, tempos letivos alocados à prática da resolução de problemas. Deste modo seriam promovidos, em contínuo, processos cognitivos capazes de ajudar a melhorar desempenhos, com repercussões positivas em diversos contextos de aprendizagem. Os resultados sugerem que está latente no aluno um potencial de aprendizagem que é, eventualmente, cerceado por crenças de insucesso e/ou práticas pedagógicas desajustadas. Esperamos que o nosso contributo ajude os professores e as escolas a incorporar a prática da resolução de problemas nas dinâmicas de sala de aula.

Palavras-Chave:

Aprendizagem, Formação contínua de professores, Heurísticas, Matemática, Resolução de Problemas.

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INTRODUCTION

Problem solving and mathematics has a long lasting connection. Not only because problem solving is a given content in the curricula of mathematics, but also because it is a method that organizes the thinking in learning and teaching of mathematics, and it is a way to achieve wider learning purposes in several areas of knowledge. In an educational context, problem solving is an essential process to learn and to self-regulate learning. On that basis we settle our study on problem-solving heuristics, believing it is crucial that students learn soon how to use those general or specific strategies for better succeed.

All Life is Problem Solving (Popper, 1999), from daily life of each citizen to high level technological and scientific knowledge development.

School is, by definition, a place for different kinds of formative purposes, but it is undoubtedly for formal learning. Enclosed in such rigorousness is the curricular organisation of school subjects, and among them Mathematics. This subject plays a special role in the education system for it provides explanation, understanding and practice which produce consequences in the future personal and professional choices of the students. If the purpose of educational measures is to enhance quality, which is noticeable in students' more solid skills, then the materialisation of such propose is paramount. However, evidence from international evaluation reports and recurrent change in the Mathematics' school syllabus unveil weaknesses, which affect students and families, teachers, school organisation and society. What is in the genesis of unsatisfactory results? Concerning mathematical contents, how can we help to improve the learning and teaching process? There is no single or easy solution, but there are as many solutions as the number of people interested in the subject. As educators looking for reflexivity on our own practices, we are aware of the complexity of this matter and we look at it under a multidimensional perspective, in consonance with Schoenfeld (1992), who prescribed four main categories of individual problem solving activity (knowledge, heuristic strategies, belief systems and metacognition). This way of trying to systematically study the subject, only partially fulfilled our purpose. The data we have gathered reveal that good practices in teaching Mathematics occur on a problem-solving curriculum. For that, teachers should train strategies with diversified mathematical contents combining learning objects with a heuristic script like that based on Pólya's guidelines used to monitor problem solving activities. Instead of a mechanical work the teaching method should appeal to thoughtful and cooperative work, in order to promote learning in a self-regulation process to acquire new reasoning habits and thinking patterns.

The development of logical reasoning, the chance to work as a team, methodology engagement and concentration are some of the attributes, which can be advanced through Mathematics. However, in the transition from the Third Cycle of Basic Education to Secondary School, several factors hinder the choice of an educational path free of beliefs of failure which will probably curtail vocations.

The learning of Mathematics can be promoted by problem solving activity, not only because problem solving "*is the heart of mathematics*" (Halmos, 1980), but also because it is a resource in the field of cognition which can and should be used to systematise thought in diversified tasks. Individuals who possess specific knowledge know how to construct mental schemes, in opposition to novices; differences are visible at the level of memory configurations concerning problem statement understanding, problem solving strategies and problem categorization (Sweller, 1988).

In a classroom context where time to learn mathematical contents is limited, the development of efficient thinking is crucial. Pedagogical strategies are able to endorse learning transfer (Eva, Neville & Norman, 1998), and so educators need to be aware of their role as tutors (Hattie, 2003). More than a style, problem solving is viewed as an instructional methodology (Silver, 1985) which promotes interaction between teacher and students in an environment propitious to think mathematically (Schoenfeld, 1992), learn contents and strategies, and to enhance competences. In addition to problem solving self-competences, other qualities of teachers/tutors are questionable, namely, cognitive and motivational models (Lepper & Woolverton, 2002).

Our study focused on teachers' mediation role on problem solving.

The emphasis on the 9^{th} grade mathematics classes is because it is the end of a study cycle (7^{th} to 9^{th} grade) and a transitional phase determinant for future academic and professional career of the students, by the choices they make.

The aim was to appreciate students' problem solving readiness, analyze problem solving abilities, the perceived attitude towards problem solving, and students' perception before and after examinations take place. In addition, we intended to relate students' problem solving skills with teachers' attitude in face of problem solving. Our proposal arose from the research project drawn up after empirical observation and the review of literature and statistic data on problem solving and learning results, workshops with teachers groups, lectures and informal discussions about learning of problem solving under an open view concerning problem solving transversally in school and extra-scholar environments. We are convinced of the relevance of the theme:

> *The activity of mathematical problem solving in the classroom is viewed (...) as a process that provides students the opportunity to experience the power of mathematics in the world around them. The purpose of teaching problem solving in the classroom is to develop students' problem solving skills, help them acquire ways of thinking, from habits of persistence, and build their confidence in dealing with unfamiliar situations* (Cai & Nie, 2007, p. 471).

We present a conceptual problem solving framework as a challenge to logical mathematical thinking, appealing to students' will to learn by doing (hands-on). Apprentices should be active intervenient elements in mathematical events (Freitas, 2013), and not be confronted with mathematical contents to be assimilated. With an Inquiry-Based Learning approach, we advocate an instructional method which encourages hands-on, active learning centered on the investigation and solving of messy, real-world problems by students practicing scientific methods on authentic problems (Savery, 2006). As other authors recommend (e.g., Kolb, 1984), achievement happens by experiential learning, so that concepts are built on experience and ideas guide new actions into learning cycles, from concrete experience to abstract conceptualization, through reflective observation to active experimentation (planning and trying out the learned objects). Sometimes it is necessary to unlearn to well learn (Low, 2011).

Besides, the continuous cycle of cooperative problem solving work unleashes confidence, promotes self-control, supports solution path and settles conditions to use acquired knowledge in new situations. Knowledge and strategies to solve mathematic problems jointly with belief systems of students and teachers and metacognitive experiences are variables that we sought to study to know better how to increase problem solving skills.

Educators can regulate teaching method by modelling procedures, from the exploration of new situations to abstract ideas, with a friendly comprehensive speech, communication and approach. Such methodologies diminish tension and increase students' capacities. That was what we have tried to test, to reach an advance mode to promote Mathematics by the practice of problem solving. To make this experience replicable and evaluable we have given emphasis to the 9th grade, because of the relevance of the students' status for future academic and professional decisions. So, our ultimate goal is to help students to be cognitively and affectively prepared to successfully pursue their studies in scientific courses. At the same time we have rehearsed and compiled tools desired to make teaching and guided learning more effective. Mainly, our suggestion is to allocate a significant number of lessons of the annual planning of the 9th grade curriculum on problem solving activity.

Besides this work focus on teaching and learning of Mathematics, it requires attentive research on learning processes. Educational Psychology gives us adequate opportunity to equating learning within a supportive consideration of believes and affective involvement as well as by means of a cognitive analyses of students' performances, so as to instruct, monitor and evaluate solving methods, based on developmental and cognitive information processing.

The model that guided our research was heuristic problem-solving model of Pólya (Pólya, 2004) supported by systematic effective learning to promote knowledge transference, hence allowing the teacher to evaluate students' answers and gather quantifiable proficiency data.

The general purpose is to contribute with a fresh methodology that allows students to work contextualized questions with noticeable applicability, in control of the process, namely, comprehension, planning, execution and metacognition (cf. Polya's model, 1945/2004).

Such syllabus aims to stimulate and evaluate the impact of significant learning by discovery without renouncing to the inherent routine procedures required for fluency in mathematics once a framework to effective knowledge acquisition is established, which is essential to master specific contents, identify competences, and optimise learning strategies.

If the mathematics curriculum and curricular goals are established by the Ministry of Education and Science, operationalisation is accomplished at school, and at more refine level in the classroom. Then, it might be done through the work of examples, autonomous labor, self-development of students' capacities, and parents and teachers' expectations, which would help to explain the main core of success in mathematical learning. It is widely known that today's challenge and meaningful learning are decisive both to validate performances and to raise the standard efficacy of self-beliefs. Therefore, in an extended period of time it may amplify the gap between those capable of more complex learning and those who do not achieve minimum standards. Curricular guidelines based on learning theories, empirical evidences, results of education research and the assessment of evaluation educational programmes, together with socio-political will, cultural environments, economical priorities and globalisation, underline this fact.

Our motivation is focused on education and instruction. We do not wish to criticise educational adjustments, reforms or curricular frameworks. But attending to a pendulum movement in many countries in which the curriculum swings to problem solving or to basic competences priorities observed in *"experienced strong shifts back toward curriculum materials' emphasizing basic skills (...) especially due to the alignment of state – and local – level standards and curriculum with the goals of these tests"* (Lesh & Zawojewski, 2007), clearly we defend an integrated view assuming that at the 9th grade students must be involved in problem-solving without disregarding basics in the construction of mathematical problem solving competences.

At key-moments of our research, when choices were to be made, we stuck to a cognitive approach. In spite of the student as an active agent with specific needs, we wish to better understand how he/she copes with learning events, in what concerns motivation and perception of self-attitudes towards Mathematics. Nevertheless, we privileged external agents (the mathematics programme and the teachers, with teaching strategies focussing on heuristics), in order to establish a scenario conducive to better learning Mathematics. We considered the model developed by Bransford, Brown and Cocking (2000) which systematises conceptions of environments propitious to learning, depending on whether the focus is the student (student-centred), knowledge (knowledge-centred), assessment (assessment-centred) or community (community-centred). Under a "psychological" view, the preferred conception is student-centred. That is the reason why in the research data collection, we have noted that to be effective, a problem solving instructive programme should establish strong backup opportunities and promote positive beliefs regarding Mathematics. Mainly, we assume the assertion that self-perceptions should be oriented to succeed and to successfully perform mathematical work. Usually, that is not the case. So, how can we reduce the negative beliefs perpetuated, generation after generation, with negative consequences to personal and professional achievements? How can school change such beliefs of inability through teaching guidance and teachers become agents of students' success, promoting results and efficient expectations of Mathematics learning? Underlying those questions are cultural and social conceptions with deep roots in society. Hence, the rewriting of school practices is vital, as it is the attribution of greater responsibility in the process to parents and society. Nowadays, educational reforms do not seem to consider research on good practices. Students' correct answers are welcome by teachers even if knowledge acquisition is fragile and not permanent. Students' questions are not valued, similarly to problem solving activities. Teaching is exaggeratedly guided towards results and focused on assessment, not in an educational and challenging perspective, but as a chart rank, both for students who wish to enter in a University course, and schools that are evaluated according to the performance of their students in national exams. Vocation and readiness aren't first criteria of progression.

Yoon, Duncan, Lee, Scarloss and Shapley (2007) found that a sustained and intensive programme practice for teachers is related to student achievement. Specialised courses of 14 hours or less showed no effects on student learning, whereas other programmes offering more hours of sustained teacher learning opportunities showed a significant positive outcome on their students' performances. The best achievements were found in programmes offering between 30 and 100 hours spread over 6-12 months.

There is probably no magic formula to increase students' mathematical competence, but it is necessary to present and implement enhanced strategies. We know the Portuguese Education System can't convene all the assumptions regarding the improvement of the learning and teaching process of Mathematics. Nevertheless, our goal is to provide a systematised contribution to the application of Problem Solving by resorting to heuristic procedures.

The general hypothesis inherent to our study is that an intervention programme based on mathematical problem solving supports learning in its cognition, metacognition and motivational components.

Along this work we have tried to collect trigger materials of problem solving aiming ulterior compounding of a guide to teach and learn heuristic procedures in line with the Mathematics programme, with a special focus on the transition to Secondary Education (from 9th to 10th grade).

Due to constraints in the access to students and in the subsequent validation of the study, we have reformulated our initial research plan. So, the sample of participants was teachers and the privileged method the participatory observation in the context of immersive workshops. The participating teachers could experience a programme based on problem-solving, and the teachers' training was the occasion to evaluate the impact on the students' problem-solving processes related with learning results and teachers' reflection about it.

The contribution of this work results on the interconnection of practical and theoretical research, namely, essay of characterisation of teachers' and students' problem-solving abilities; students' expectation status concerning math and mathematical learning; motivation to learn before and after being assessed; teachers' attitudes toward math and problem-solving, and finally, as we considerate it, an essential contribution of this research, it ends with a concrete proposal of a programme of problem-solving suggested for teachers as a guide to approach the subject and method in a contextualised and integrated manner, aiming at facilitate mathematical communication with the students based on problems an defies and its solving process. We wish that this product will be improved and become a guide to foster problem-solving options in schools.

We want to make clear that this incursion of a Mathematics educator into the fields of Learning Psychology meant a supplementary comprehension effort, the reframing of the objective in the individual, his competences and idiosyncrasies, with repercussions on curricular aspects and didactic units. In the course of this project, we believe that we have acquired a more integrative perspective on the teaching and learning process, despite possible gaps in qualitative analysis due to lack of theoretical support. Even so, our efforts to merge theoretical and empirical research produced results which will be presented later. When these results are put together, they help to better understand today's reality and thus analyse tomorrow's possibilities.

In fact, to track our effective proposal, we have taken some questions into account, that gradually help to build the understanding of the relationship between beliefs, content domain and updating strategies for training in mathematical problem solving.

This investment resulted in partial reflections on learning to teach mathematical problem solving, trends observed in the transition to Secondary Education marked by the withdrawal of mathematical learning over time, placing constraints to the vocational curricular paths of the students.

Problem solving and mathematical teaching and learning research are still open to make understandable the hints between the premises and practices under construction. With that in mind, we advocate balanced modifications and adjustments in classroom practice, in line with Lesh and Zawojewski (2007) or English and Sriraman (2010). Although problem solving is a key aspect of teaching and learning education, it has often been a subject of controversial research with impact on the Mathematics curriculum and on curriculum development. In general, the prevalent paradigms were *"based on an assumption that the way students learn to solve problems is to first acquire the mathematical knowledge needed, then acquire the problem solving strategies that will help them decide which already know procedure to deploy, then acquire the metacognitive strategies that will trigger the appropriate use of problem solving strategies and mathematical knowledge, and finally unlearn beliefs and dispositions that prevent effective use of problem solving and metacognitive strategies, while also developing productive beliefs and affect"* (Lesh & Zawojewski, 2007, p. 793).

It does not mean that everything that needs to be learned should be reduced to lists of declarative statements regardless of concept development focussing on processes assumed to function in relatively invariant ways across tasks or situations. Researchers continue to suggest alternative assumptions, and the same happens in relation to cognition, communities of practice, and representational fluency that seems to be converging towards a *models-and-modeling perspective* on mathematical learning and problem solving. Investigators who follow this approach and focus on students' mathematical models and modelling activities study the co-development of mathematical concepts, problem-solving processes, metacognitive functions, and beliefs; use a variety of practical and theoretical approaches – including cognitive and social, mathematical/epistemological, developmental psychology to make predictions and explanations. When they study how modelling and problem-solving skills are learned, they naturally analyse problem-solving processes through a developmental perspective, gravitate to simulations of *real-life* situations, assume that mathematical thinking involves creation and interpretation as much as computation, deductive reasoning and the execution of procedures (cf. Lesh & Zawojewski, 2007, p. 793-4). Mathematical modelling was also our powerful option, simply because the reason pointed out by English and Sriraman *"With the increase in complex systems in today's world, the types of problem solving abilities needed for success beyond school have changed. (...) opportunities for students to generate important constructs themselves (before being introduced to these in regular curriculum) and to create generalizable models"* (p. 283) is essential.

PART 1: LITERATURE REVIEW

Historical notes on a psychological approach to problem solving

Contributions of Cognitive Psychology to mathematical learning

Psychology, the science which studies behaviour and interactions between individuals, contemplates problem solving which is not exclusively confined to mathematical issues. Even to explain mathematical problem solving, the processes activated require the involvement of Psychology focused on the solver. Researchers such as George Pólya brought Psychology and the Cognitive Sciences together with the purpose to validate learning theories (Frederiksen, 1984) and educational problem solving research (Wilson, Fernandez & Hadaway, 1993). The roots of Psychology retrace to the Antiquity, when philosophers put their thoughts in words. However, it was only in the late 19th century that Psychology became independent of Philosophy (Hothersall, 2004). One of the pioneers of this emancipation movement was William James (1842 - 1910). Since the 1890s, his work brought a decisive contribution to Psychology as an autonomous discipline. He was sceptical about animal laboratorial experiments with the purpose of explaining human behaviour. William James emphasised the need to understand the teaching and learning process which occurs in classroom context, considering both teacher and student as active players.

History brought new ideas with the proliferation of studies from different doctrines (Hardin, 2002). Schunk (2012), looking back at the roots of learning and how it occurs, analysed the most important doctrines on the nature of the emergence of knowledge. Empiricism and rationalism postulate experience or reason as the source of knowledge. Through the understanding of these ideologies, theorists and educators can provide meaningful teaching practices which produce learning. Understanding the events that have arisen allows us to understand the need for differential learning comprehension (Edgar, 2012).

Thorndike's eminently experimental doctrine was criticised because we can say that he only brought more rigorous ways to measure less significant educational issues, and that what it was possible to know was limited to what it was possible to measure (Sprinthall & Sprinthall, 1993). Empirical measurable studies prevailed, and no attention was paid to the role of beliefs, in line with naturalistic observations. Thus, the behaviourist approach, as any other theory and field interventions, especially in what concerns learning and problem solving, evolved, and each trend highlighted different but complementary issues.

Cognitive psychologists view problem solving as a process that includes introspection, observation and development of heuristics. Information-processing is based on general problem solving skills and artificial intelligence (Hardin, 2002, p. 227).

> *Basic, automated skills in any domain are those that allow an individual to perform necessary and routine operations without much thought. These skills are overlearned to the point that they become habitual and even unconscious, enabling individuals to operate quickly and accurately without taxing their short-term memories. Automaticity allows individuals to focus their attention on the more complex tasks associated with a specific domain and is a general attribute associated with experts in a domain. Automaticity supports the expert's speed and skill of execution. Unlike basic, automated skills, which occur unconsciously and thus do not tax short-term memory, domain-specific strategies remain under conscious control. They are the processes and procedures in a domain that an individual, even an expert, must consciously think about in order to solve a problem. They are, in other words, the procedural knowledge associated with a domain* (Hardin, 2002, p. 227).

As the understanding of theoretical referential increases, the more consistent the perception of the problem solving process becomes. Each learning theory, in association with its conceptual domain, tries to present a problem solving model.

Behaviourists view problem solving as a process which develops itself through positive and negative reinforcement mechanisms. The methodologies which explain the problem solving process within the framework of the behaviourist learning theory are trial and error and Hull's essential idea that *"where there is a goal to be attained, the organism has at its disposal several different responses (a habit-family) to use to attain that goal. At the same time, however, these habits vary in strength (form a hierarchy). The strength of the habit determines the probability that it will be used"* (Kimble & Kurt, 2014). Stimuli in a problem situation may evoke several different responses, and responses will be produced, one at a time, in order of strength, until either the problem is solved or the organism exhausts its repertoire of responses. In their emphasis on trial-and-error learning and habit strength, behaviourists focused on the role that the stimulus– response interactions might play on problem solving.

Cognitive psychologists view problem solving as a process that includes introspection, observation, and the development of heuristics. The information-processing view of problem solving is based on general problem solving skills and artificial intelligence (Hardin, 2002, p. 227). An early cognitive approach to problem solving was to identify the mental stages a problem solver goes through. Two noted cognitive psychologists, Wallas and Pólya, developed a problem solving four-stage model.

The four stages of problem solving identified by Wallas were *1)* preparation - defining the problem and gathering information relevant to it; *2)* incubation - thinking about the problem at a subconscious level; *3)* inspiration - having a sudden insight into the solution of the problem, and *4)* verification - checking to be certain that the solution was correct. Similarly, Pólya described the following four steps of the problem-solving process: *1)* understand the problem; *2)* devise a plan; *3)* carry out the plan and, finally, *4)* look backwards.

Pólya promoted the idea that the application of general problem-solving strategies was crucial to problem solving expertise and intellectual performance. General problem solving strategies have also been called heuristics. The word *heuristics* comes from the Greek, *heuriskin*, meaning *serving to discover*. A commonly used synonym for heuristics is *rule of thumb*. The heuristics Pólya identifies in mathematical problem solving are discussed within the framework of a four-stage problem solving model, the one we have used to study and teach how to solve mathematical problems. Some of the heuristics applied in this plan include understanding the unknown, drawing a graphic or diagram, thinking of structurally analogous problems, simplifying the problem, and generalizing the problem. If these heuristic methods can be applied to a problem in any context is somewhat controversial, but, in this case, they are considered as general problem-solving skills. As we will see, depending on the type of problem, the heuristics used can be specific, as Alan Schoenfeld tells us.

In addition, other heuristics have been identified. People often have to make decisions in the face of uncertainty, with sketchy information about the situation, on the basis of suggestive but inconclusive evidence. The reasoning processes used to solve uncertainty are often called heuristic judgment. One form of heuristic judgment is similarity judgment, where an instance is evaluated based on prior knowledge of a similar instance. A similar type of judgment is representativeness, where an assumption is made based on the belief that the characteristics of the individual are representative of the group. Another type is the availability heuristic. In this case, judgments are made based on which elements can most easily be retrieved from memory. Analogical reasoning is another heuristic method, where judgment is made by drawing similarities from events that have previously occurred. Still another is the development of a mental model (simulation) to predict the outcome of an event. These heuristics are examples of general purpose thinking skills, which seem applicable to many domains, including mathematical learning.

The heuristics approach emphasises finding a good representation of the problem. While content specific knowledge is required to solve the problem, the belief that general problem solving skills were also valuable was supported by studies in the domains of Mathematics and Computer Science.

As problem solving expert - novice differences are manifest, the information processing theory of learning emerged. This theory emphasises the role of factors such as working memory capacity, organisation of long term memory, and cognitive retrieval of relevant information.

The bulk of current research on problem solving reflects inquiry into the nature of these cognitive processes. Newell's early work on Artificial Intelligence (AI) sustained the *General Problem Solving* theory, as an AI programme uses a finite set of functions to work from the problem state to the solution state. Many simple puzzles and problems within the domain of Logic domain were successfully completed with AI programmes, supporting the idea that success in problem-solving is directly related to general problem-solving skills.

As a consequence of different conceptualizations, research on Mathematical Education and School Psychology would have followed a barren path due to the profusion of experimental studies, many of them with small significance to established knowledge. Hence, fresh theories and instruments started to emerge, whose aim was to accede better understanding of teaching and learning actions (Wilburg, 1995). A classroom scenario where the teacher performed a recital and the students passively listened was criticised, and cognitivism, with the endorsement of Pedagogy and Sociology, brought intelligible alternatives to teaching and learning processes, as well as to the importance of mathematics (Popkewitz, 2004).

The 1960s and 1970s saw many theories pass by. In 1965, Robert Gagne's theory of the conditions of learning was published and learning objectives and their relationship towards appropriate instructional designs were analysed. Cognitivism took roots and started to take the place of behaviourism as advocated by B. F. Skinner through operant conditioning. Albert Bandura, Jerome Bruner, Jean Piaget, Lev Vygotsky, and Robert Gagne's cognitive approaches to learning were all being explored for possible explanations as to how learning should occur (Woolfolk, 2010).

Memorisation was promoted to trigger a swift answer from the student when questioned by the teacher. Willing to change this paradigm, Dewey argued that, in the classroom, the teacher should guide their students into learning.

> *"The teacher is not in the school to impose certain ideas or to form certain habits in the child, but is there as a member of the community to select the influences which shall affect the child and to assist him in properly responding to these influences"* (Dewey, 1897).

An aphorism credited to Dewey, Montessori and Piaget says that *"children learn by doing and by thinking about what they do"* (Papert, 1975, p. 219).

School must be an institution which not only provides knowledge but also stimulates the student to think about how to apply that knowledge. Thus, we urge that teachers and teacher educators become familiar with constructivist views and reflect on them so as to redefine their approaches to teaching and learning, and also research on problem solving.

In *The Child and the Curriculum* (1902), John Dewey writes about two different pedagogical positions; *1)* a curriculum exclusively focused on the contents to be taught, a methodology criticised for promoting student inertia, an opinion that we share because instruction is only truly effective when contents are presented so that students can relate to previously acquired knowledge; *2)* a curriculum focused on the child, advocated by Dewey's disciples, but that Dewey himself didn't subscribe for he considered it a simplified vision about the importance of the teacher and the school subject. Dewey states that the teaching and learning process balances from knowledge transmission and students' interests and experiences, "*the child and the curriculum are simply two limits which define a single process. Just as two points define a straight line, so the present standpoint of the child and the facts and truths of studies define instruction*" (Dewey, 1902, p. 16).

More recently, Freitas (2013) refers that curricular mathematical education is essentially involved with the nature of mathematical objects and less concerned with mathematical events. He advocates that the need for the student to be familiar with the ideal mathematical subject must be revised, and argues that the concept of event better captures the vitality of mathematics and offers new ways of educational mathematical thinking.

Psychology of Development brought a new understanding of human functioning, by recognising the importance of the subject's activity in the construction of their knowledge and, therefore, one's responsibility in the learning process (Veiga Simão, Silva & Sá, 2007).

Constructivist theories have received considerable acceptance in mathematics education. The teacher's responsibility is to introduce situations and contexts within which the learner may construct appropriate knowledge. Even though the constructivist view of mathematics learning is appealing and the theory is the basis of many studies at elementary level, there is still a lack in research at Secondary Education. However, constructivism is consistent with current cognitive theories of problem solving and mathematical views of problem solving involving exploration, pattern finding, and mathematical thinking.

With the advent of Gestaltism, researchers started to think about the brain mechanisms which trigger insight moments leading to complex thought. Research on problem solving activity acquires an increasing significance.

> *The genesis of mathematical discovery is a problem which must inspire the psychologist with the keenest interest. For this is the process in which the human mind seems to borrow least from the exterior world, in which it acts, or appears to act, only by itself and on itself, so that by studying the process of geometric thought we may hope to arrive at what is most essential in the human mind* (Poincaré, 1914, p. 46).

The Wallas model (1926) gathers Henri Poincare's[1] quotations and divides the problem solving activity into four phases: *1)* hard work (the individual makes use of all skills if necessary until exhaustion); *2)* incubation (the individual shifts information from conscious to unconscious levels); *3)* inspiration (the answer suddenly emerges without thinking); *4)* check (solution proof but only with the intention to assure its accuracy).

Gestaltism, with the motto *"The whole is greater than the sum of the parts"* (Hothersall, 2004), gets more supporters, especially mathematicians who write about it, namely George Pólya's *How to Solve It* (1945) and Max Wertheimer posthumous book *Productive Thinking* (1945). Unfortunately, evidence did not fulfil the initial expectations. The emphasis on what was taking place at a subconscious level in the course of the inspiration phase, inaccessible to observation and subsequent study, hindered the interpretation of the processes that lead to coherent reasoning. Teaching guided to effective understanding, as advised by Max Wertheimer, or the division between higher order thinking (*reasoning* or *productive behaviour*) and lower order thinking (*learned behaviour* or *reproductive thinking*) (Maier, 1933) illustrates valid, but indefinite principles, with the exception of George Pólya's heuristic doctrine, whose effectiveness was insufficiently evaluated due to lack of a general implementation.

With the advent of cognitive scientific studies focused on *"…information-processing analyses incorporate both the Gestalt psychologists' interest in internal mental states involved in understanding problems, and the behaviourists' emphasis on actions that are performed in response to specific stimuli"* (Heller & Hungate, 1985, p. 84). Cognitive sciences unleashed the birth of

> *a variety of disciplines and approaches with the aim of providing an integrated scientific account of the mind, its states, processes and functions … cognitive science is, as it were, psychology pursued by novel means; it draws on any potentially relevant discipline (the main contenders being neuroscience, computer science and related modeling techniques from physics and mathematics, linguistics, philosophy and parts of social science* (Andler, 2009, p. 256).

In 1956, the *Symposium on Information Theory*, organised by the Massachusetts Institute of Technology (MIT) and the *Dartmouth Summer Research Project on Artificial Intelligence*, coined the concept of Artificial Intelligence. The proliferation of research centres and the writing of scientific articles and books gave projection to Cognitive Sciences, being the computer a key tool for researchers. As a result, two distinctive lines of action became evident: Cognitive Psychology, which describes processing information models evaluated in an

[1] Mathematician, physicist and philosopher, author of *Science and Method* book where he relates personal examples of sudden inspiration moments.

experimental context, and Artificial Intelligence models based on computational efficiency and programming logical coherence. The association between Cognitive Psychology and Artificial Intelligence allowed the development of the Cognitive Sciences (Aitkenhead & Slack, 1987). Norman (1980) categorises twelve main variables (Belief Systems, Consciousness, Development, Emotion, Interaction, Language and Perception, Learning, Memory, Performance, Skill, and Thought), saying *"The study of Cognitive Science requires a complex interaction among different issues of concern, an interaction that will not be properly understood until all parts are understood, with no part independent of the others, the whole requiring the parts, and the parts the whole"* (p. 1).

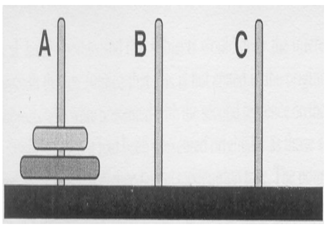
Building an intelligent machine capable of replicating human functional learning is an ancient project that goes back to Antiquity. However, only in the 1950s did it start to consistently materialise. Science fiction writers augured a world dominated by machines. Their visions helped settle the definition of the applications which, nowadays, make Artificial Intelligence: games, problem solving, mathematical proof and automatic language translation (Vignaux, 1991). In 1950, the seminal paper *"Computing machinery and intelligence"* written by Alan Turing (1912 - 1954) referred the possibility of using information processing to construct a non-human entity provided with intelligence.

Scientific development pushed the idea of assembling a machine to mimic human thought forward. In 1948, John von Neumann (1903 - 1957), when questioned about such possibility, answered, *"You insist that there is something a machine cannot do. If you will tell me precisely what it is that a machine cannot do, then I can always make a machine which will do just that!"* (Jaynes, 2003, p. 7).

Effort to organise a consistent thinking theory, which may be implemented and tested, revealed weaknesses since results of algorithmic computational procedures did not necessarily mean simulation of human behaviour. Concerns about thinking rationally emerge. Even though the procedure to codify logical language has its origin in Ancient Greece philosophers, with emphasis on Aristotle's work, 19th century mathematicians were the ones to define symbolic language for concept representation. The *General Problem Solver*, a computational programme developed in 1957 by Alan Newell, J. C. Shaw and Herbert Simon, was conceived to solve any kind of problem that could be converted into symbolic notation, with a division between problem knowledge (set of formulas introduced in the computer) and resolution strategy (solution generated by the machine).

Informal knowledge manipulation, with eventual loss of rigour, its accurate logical codification, as well as the transition from theoretical statement to practical performance are hazards that, without a previously established guide problem solver, could lead to a vortex of computational resources when the task is implemented. The *Travelling Salesman Problem*[2] (TSP), a mathematical analysis from 1930s is a good example. The complexity of optimising a path with a few hundred cities is a challenge to the establishment of more efficient models. Nowadays, better algorithms and last generation microprocessors help delineate in TSP problems with thousands of places to visit nearly the best way, or eventually the best way. The fruitful relation between Mathematics, Philosophy and modern Artificial Intelligence is in the genesis of extraordinary solutions which answer problems that defy the human intellect. Nevertheless, constraints induce a rational behaviour. When acting rationally, the ideal decision is the one that allows the best performance with the available information. No matter the environment and tasks to perform, the goal is focused on the rational agent who receives perceptions and gives back an action that should be the most efficient.

In the book *Human Problem Solving*, Alan Newell and Herbert Simon described in detail problem solving learning processes under the spirit of information processing (Newell & Simon, 1972). The problem of the *Tower of Hanoi*[3] here presented in a simplified version *(Figure 1)*, offers a glance at information processing models. Mentally transfer disks from A to C in three movements, in line with the conditions: *1)* only one disk can be moved at a time; *2)* a larger disk cannot be placed onto a smaller disk.



(Figure 1)

The brain unchains mechanisms capable of selecting chunks of information considered relevant to set in motion an internal model suitable for a given problem, so as to promote logical reasoning. Such internal representation activity feeds on the given information about what is asked, the starting point, restrictions, procedures which may eventually apply and individual experience.

The mental activity developed in the course of the task is typified as sequential and conscious. Questions which present similarities with the problem of the *Tower of Hanoi* allow the individual to verbalise each stage of reasoning and contribute to the understanding of the

[2] Given a list of cities and the distances between them, the task is to establish the shortest way in order to visit, once and only once, each city.

[3] Game invented by François Anatole Lucas (1842 - 1891), with the purpose to evaluate the individual working memory capacity in a problem solving context.

resolution processes that are activated by the individual. The loud reading of statements of problems can be profitable in the way that it shortens the memory work resources which store and control short time information. When submitted to information overload, memory work unchains disorientation and loss of memory that are propitious to error.

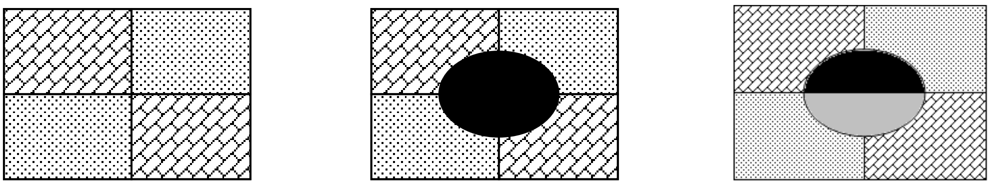
Newell and Simon's model does not establish a qualitative distinction between human and machine, but it considers limitations of information processing: *1)* the amount of active information in the work memory at each moment; *2)* competence in codifying information - potential difficulty in recognising the main aspects of the task; sensorial incapacity in instantaneously codifying the total amount of information; *3)* ability to gather information - stored memories can be adulterated by future memories, or they can be distorted by the effect of previous expectations; *4)* capability to recover information – the human memory is fallible; *5)* aptitude to maintain high motivational levels – at a certain moment, indolence and exhaustion compromise proficient work (Robertson, 2001).

The quest for answers engages the individual in a mental representation, with intermediate stages, which combines knowledge with the purpose to bond the starting point of the problem with the solution. Working memory constrain the number of future actions through eventual omissions of procedures which may jeopardize the success of problem solving activity. As the complexity of the mind became known, the fragilities of cognitive theories gradually started to emerge, since they were only based on information processing actions.

> *To understand a problem, then, the problem solver creates (imagines) objects and relations in his head which correspond to objects and relations in the externally presented problem. These internal objects and relations are the problem solver's internal representation of the problem. Different people may create different internal representations of the same problem. Frequently, problem solvers will make an external representation of some parts of the problem. They do this by drawing sketches and diagrams or by writing down symbols or equations which correspond to parts of the internal representation. Such external representations can be enormously helpful in solving problems* (Hayes, 1989, p. 5).

Even though limited to well-described situations such as the chess game or the problem of the *Tower of Hanoi*, Artificial Intelligence problem solving activity has its most emblematic example in the proof of the *Four Colour Problem*, which allowed demonstrating that four colours are enough to colour any kind of flat map. This task is, under present knowledge, impossible to be accomplished by analytical processes alone. In 1852, Francis Guthrie (1831 - 1899) conjectured whether four colours would be enough to paint a flat map with the minimum number of colours, but his supposition would only become proof in 1976. The authors were Kenneth Appel, Wolfgang Haken and… an IBM 360 computer. In 1994, Paul Seymour, Neil Robertson, Daniel Sanders and Robin Thomas used a more efficient algorithm to make the proof but the computer remained an essential tool in this process. Critics argue that mathematical proof must provide a complete understanding of the outcome, and that does not happen with this proof because the problem was divided in a set of situations tested by a computer.

In the *Four Colour Problem (Figure 2)* we assume that: *1)* neighbour regions cannot be painted with the same colour; *2)* regions with only a point in common are not considered as neighbour regions.



(Figure 2)

The computation power of the machine, explicit in the proof of the *Four Colour Problem*, is simultaneously its most sensitive feature.

The *Turing test*[4] highlights the differences between humans and an artificial intelligence system. In isolated environments, when the question about value 25![5] arises the correct answer comes in a fraction of a second and unveils the source due to the enormous amount of computations that are needed to provide a correct answer.

> *What does it mean when a computer passes the Turing test for some human cognitive function? For example, if an interrogator cannot distinguish between a human and a computer with regard to thinking, reasoning, and problem solving, does that mean that the computer possesses those mental attributes just as humans do? No, say the proponents of* ***week artificial intelligence****, who claim that, at best, a computer can only simulate human mental attributes. Yes, say the proponents of* ***strong artificial intelligence****, who claim that the computer is not merely (in this context) a tool used to study the mind (as the proponents of weak AI claim). Rather, an appropriately programmed computer really is a mind capable of understanding and having mental states* (Hergenhahn & Henley, 2013, p. 595).

[4] Communication between a judge and an individual and a computer in a situation where the machine should reproduce human behaviour in such a way that the judge is unable to identify which of them is the artificial intelligence system.

[5] $n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$ (n is a positive integer).

The brain is a phenomenal processing information system whose activity differs significantly from the performance of the computer. The neuron, as a unit of an intricate net structure, is responsible for complex functions such as pattern recognition and motor control. Its plasticity allows it to evolve. Researchers try hardly to mimic those abilities with the intention to establish neural nets similar to those that are responsible for brain functioning. A set of approaches in the fields of Artificial Intelligence, Cognitive Psychology, Cognitive Science, Neuroscience, and Philosophy of Mind, called Connectionism, started to materialise.

Breakthroughs are often criticised and Artificial Intelligence innovation was no exception. A machine built to copy human thought collides with the theological dogma which states that this is exclusively a human faculty. *Gödel's incompleteness theorems* showed that formal logic was insufficient to build all mathematical theory, and thus exposed computational limitations. Formal law execution makes the computational system to be determinist and, therefore, incapable of predicting human behaviour. The absence of conscience was an argument repeatedly used by Artificial Intelligence detractors.

> *Not until a machine can write a sonnet or compose a concerto because of thoughts and emotions felt, and no: by the chance fall of symbols, could we agree that machine equals brain-that is, not only write it but know that it had written. No mechanism could feel (and not merely artificially signal, an easy contrivance) pleasure at its successes, grief when its valves fuse, be warmed by flattery, be made miserable by its mistakes, be charmed by sex, be angry or depressed when it cannot get what it wants* (Jefferson, 1949, p. 1110).

> *But one of the most interesting applications of artificial intelligence,… , was that of the HAL 9000 introduced by Arthur C. Clark in his novel "2001: A Space Odyssey." HAL was a sentient artificial intelligence that occupied the Discovery spaceship (en route to Jupiter). HAL had no physical form, but instead managed the spaceship's systems, visually watched the human occupants through a network of cameras, and communicated with them in a normal human voice. The moral behind the story of HAL was one of modern-day programming. Software does exactly what one tells it to do, and can make incorrect decisions trying to focus on a single important goal. HAL obviously was not created with Isaac Asimov's three laws of robotics in mind* (Jones, 2008, pp. 2-3).

How can we conciliate inductive reasoning with deductive reasoning? Is it possible to replace an intuitive judgment by a well-defined theorem? Is it possible to replace *ad hoc* procedures by rules exclusively based on elementary rationality criteria to eliminate error?

Plausible thinking (transitory, luck dependent, arguable) performance *"…depend very much on prior information to help us in evaluating the degree of plausibility in a new problem. This reasoning process goes on unconsciously, almost instantaneously, and we conceal how complicated it really is by calling it common sense"* (Jaynes, 2003, p. 6), is connected with the arguing process to extract conclusions and inferences that take place in the unconscious and whose complexity is, in some way, blurred by judgments that are not supported by specialised knowledge but rather by common-sense practice.

Deductive reasoning (controversy-free, definitive, and safe) is intrinsic to computational language and is based on a solid argument with the repeated application of syllogisms (Jaynes, 2003).

If A is true, then B is true.	If A is true, then B is true.
A is true,	B is false,
then B is true.	then A is false.

To make a judgment, we need to have full knowledge of the subject under analysis, but when that is not possible, in alternative we can use less consistent syllogisms.

If A is true, then B is true.	If A is true, then B is true.
B is true,	A is false,
then A becomes more plausible.	then B becomes less plausible.

In the first situation, reasoning does not prove that A is true but the evidence of one of its consequences (B is true) allows to increase the reliability on the veracity of A. In the second case, the information does not prove that B is false, but, as we eliminate an argument which justifies its authenticity, the credit of B diminishes.

The eventuality of engaging on fallacious reasoning justifies the need to understand the inference processes which lead to error in order to identify deficiencies and develop training techniques to eliminate it.

Descartes, Spinoza and Leibniz's deductive reasoning complies with logic principles and consists of taking a particular truth from a general truth in which it is implicit (Galliano, 1979). From a presupposition with unquestionable veracity (larger premise), a relation is established with a specific content (smaller premise) so as to extract a logical consequence (conclusion). There is a decreasing reasoning chain, from general to specific. *"All man are mortal. Socrates is a man. Therefore, Socrates is mortal."*

Bacon, Hobbes, Locke and Hume's inductive method reflects on knowledge as a result from experience, an observation process, a collection of a set of objects, facts or events which allow conclusions that can be applicable to an ampler set or to cases not yet experienced

(Gewandsznajder, 1989). Illations resulting from experience change into independent generalisations of previously established principles (Lakatos & Marconi, 1993). When an experiment is repeated under identical circumstances, equal results are expected. A larger sample of a representative population confers inductive robustness to an argument which is then accepted as valid until it is itself eventually refuted by subsequent research.

If the aim of deductive reasoning is to explain a premise with the safety of precision and the absence of knowledge expansion, inductive reasoning supports the development of knowledge, even though it is vulnerable to inaccuracy. The conceptual models shaped by each individual allow incorporate and understand the information received and keep the consistency between perception and cognition. If inductive reasoning is responsible for nourishing the development of conceptual models, deductive reasoning is essential for its correct use. To be proficient, the process must be cyclical and versatile, because deductions that are not experimentally validated imply, as a consequence, change of thoughts which, in turn, may affect future inductive inferences.

> *A problem arises when a living creature has a goal but does not know how this goal is to be reached. Whenever one cannot go from the given situation to the desired situation simply by action, then there is recourse to thinking… Such thinking has the task of devising some action which may mediate between the existing and the desired situations* (Duncker, 1945, p. 1).

With the multiple contributions of Psychology as a science of cognition capable of explaining mathematical reasoning, some controversy arose that makes us torn between logic formalism and the plasticity of human processes susceptible to emotionally or motivationally influence subjective variables. Mathematical problem solving appears in the middle of this duality of forces, which we will try to understand better.

Problem solving among knowledge acquisition, cognitive processes and learning regulation metacomponents

Meaningful learning and relationship with teaching organisation

Solving problems as a process is the focus of many studies in a hybrid context of Mathematics and Psychology. We have no doubt that learning is an epistemological concept that is paramount for both mathematical educators and those responsible for the development of curricula, as well as psychologists that support individuals to successfully learn and succeed in school.

Now let us talk about significant learning and its relation to the organisation of teaching. David Paul Ausubel (1918 – 2008) studied the intricate processes that lead to the understanding and explanation of the processes which enable knowledge. To make that possible, learning by discovery should be promoted in the classroom, with the vivid intervention of the teacher to guide the student in the discovery of significant events. Many problem solvers find solution strategies but very few use them systematically as an acquired rule (rule acquisition event) (Van Lehn, 1991).

The assimilation theory, also called meaningful learning theory, sustains that the brain stores organised information, so that specific knowledge is assimilated by general contents that are more understandable for the individual. The acquisition of a new skill allows the subject to expand their cognitive structure, a c*ontinuum* process between meaningful learning and memoristic learning. Regarding new knowledge, significant learning occurs when the individual consistently connects fresh data with their inner conceptual framework. The aptitude to explain with a logical reasoning what has been learned, making use of a language with identical significance, substantiates significant learning. Only later is it possible to ensure the strength of the newly acquired skills, when we recover them if they are necessary, and eventually use them with insight in a new situation. If significant learning occurs by discovery, the locus of action is on the student; when learning occurs by reception, the student should act upon those concepts and relate them with the already possessed knowledge.

On the other hand, memoristic learning occurs when the knowledge recently acquired does not find support in the subject's cognitive structure. Randomly accumulated, its sustainability is compromised, as well as its possibility to be used in different contexts, because the individual can only play the new content by repetition. The theory of assimilation is a dynamic model (Faria, 1989, p. 7). Despite the significance of meaningful learning in the educational context, Ausubel identified circumstances in which the occurrence of mechanical learning is inevitable. In Mathematics activity, the use of formulas and rules that come from memorisation is quite usual. Their enunciation is the result of a mechanical process but their easy evocation is particularly meaningful.

Schoenfeld, Smith and Arcavi (1993) used the term *learning event* to refer to changes in the subject's procedural knowledge. Others followed this idea, but later, from further analysis, Van Lehn (1991), observed that learning was an inaccurate label for those events. *Psychologists use the word learning for the storage of any sort of information in memory (…). It is important to emphasize that inferring a rule (i.e., rule acquisition) is different from learning a rule (storing it in a retrievable form). (…) A rule acquisition method corresponds to productions that construct a semantic net representation of a new rule. This network is stored in working memory, and may, with a certain probability, become permanent* (p. 38).

Students' significant learning depends on two main factors, internal and external. Internal factors can be cognitive or social and affective. Regarding students' cognitive structure, we should consider: *1)* the mastery of tools to anchor the subject that the teacher intends to introduce; *2)* new concept differentiation degree comparatively to the prior knowledge in which it is anchored; in case of similarity, the student may eventually mix, confuse or incorporate one in the other; *3)* knowledge robustness to which the new subject will be connected is crucial for the quality of learning. At the internal level, student motivation to learn is a key factor, and in its absence the learning process may be at risk. Albeit essential, motivation is still not enough to succeed for there is risk of mechanical learning. Teaching guided to classroom knowledge reproduction, the teacher's eventual difficulty in capturing students' interest, and repetitive worksheets can lead to this style of learning. In the classroom, as an external agent, the teacher has simultaneously the opportunity and the responsibility of creating adequate conditions for significant learning. Worksheets should promote logical relations, consistent with the knowledge already acquired by the students.

The idea of mechanisms to enhance the learning process was put into practice with the Gradual Differentiation Model under the assumption that in a learning programme, the most general and comprehensive ideas should be presented first and only then should they be gradually differentiated with details and specifications (Ronca & Escobar, 1980).

The structure of the school subject is essential for effective learning. The development of significant connections between what is taught and the students' cognitive legacy should occur. Connection points between the subject and the students' knowledge enhance effective learning. If necessary, the differences between similar, but not identical, concepts should be highlighted, thus allowing the student to differentiate both. This dynamics helps reconcile real or apparent inconsistencies in the students' cognitive structure, which is why it is identified as *Integrative Reconciliation*.

To establish a learning-friendly environment, the teacher must provide worksheets which allow the students to think about previously studied contents. *Advanced Organisers* support interaction, incorporation and retention of new contents to fill the hiatus between what the students already know and what they need to know before the teacher introduces a new subject or manage a classroom activity (Ausubel, Novak & Hanesian, 1980). The definition is generic because *Advanced Organisers*' instruments depend on different factors, with special incidence on the subject, the students' age and their expertise level. When properly organised and implemented, *Advanced Organisers* promote contextualisation and content connection, the link between what is known and what is supposed to be learnt. Those instruments should motivate students and be focused on concrete situations. Regarding the school subject, explicit *Advanced Organisers* should be used when this relation does not exist, and comparative *Advanced Organisers* should be used when students have some previous knowledge where to anchor new concepts.

The classroom must be a dynamic space, open to discussion and debate of ideas.

Crosswhite (1987) refers the danger of a stereotyped environment where teacher and students play previously defined roles. While the teacher explains the *why*, the students wait for the moment when the teacher reduces the activity to basic algorithmic application with a given resolution model.

Metacognition is another crucial issue to understand the conceptualization of both learning and problem solving. Throughout decades, the thought of what facilitates problem solving was focused on procedures and cognitive activities (choice and algorithmic use, heuristics implementation) or results (recording of correct or incorrect answers). The effectiveness of those instruments was questioned because they were considered unsatisfactory to fully explain the behaviour of students in the course of research programmes. Empirical information suggests that metacognitive factors combined with cognitive factors have a crucial influence in student performance.

Significance of Metacognition

Metacognition considered as *"... one's knowledge concerning one's own cognitive processes or anything related to them, e.g., the learning-relevant properties of information or data. For example, I am engaging in metacognition if I notice that I am having more trouble learning A than B; if it strikes me that I should double check C before accepting it as fact"* (Flavell, 1976, p. 232) is essential for the success of learning.

Performance depends significantly on self-monitorisation to regulate and assess cognitive activity. According to Flavell and Wellman's view, metacognitive knowledge comes from the perception of person, task and strategy interactions; individual personality is linked to cognition: intraindividual (knowledge of oneself, faults and qualities, interests, behaviours…), interindividual (knowledge regarding the differences between the self and the others) and universal (general knowledge of society); the task is about the quality and quantity of available information, as well as the criteria which should be respected; information processing depends on the task to perform; the strategy includes knowledge and mastery of instruments, methods and performances to work with efficiency.

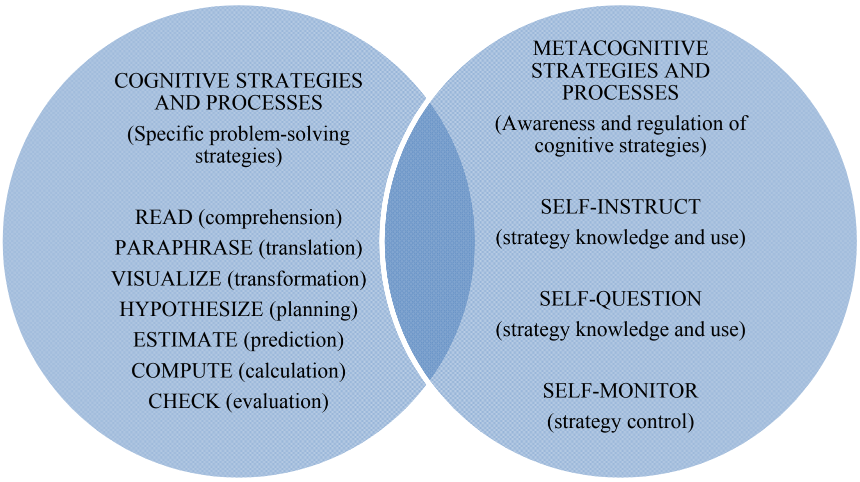
Mathematical problem solving engages metacognition functions: task definition, lower order processes to accomplish what is required, application of a strategy by combining several lower order processes, development or selection of a mental representation about what should be done, allocation of mental resources to perform the task, monitoring of performance, and assessment of results (Kolligian & Sternberg, 1987). Those conscious or unconscious actions help the student move forward and operate both at lower response levels and more advanced abstract levels, in harmony with the nature and complexity of the task to be solved and the knowledge basis of the problem solver. During a think-aloud session and as mathematical problems become more difficult, gifted students verbalised more metacognitive strategies (Montague & Applegate, 1993). In opposition, students with learning disabilities verbalised fewer strategies as problems got more complex. The average achievers, as problems demanded higher cognitive requisites, revealed metacognitive function limitations but no limitations in the use of cognitive strategies. Since metacognition development is a gradual process (Flavell, 1985), the learner must reach both a stage of cognitive maturity and reasoning ability to understand and be familiar with the appropriate learning strategies in order to perform an efficient instruction plan.

Students who do not master problem-solving strategies need explicit instructions concerning cognitive strategies (e.g., visualization, verbal rehearsal, paraphrasing, summarising, estimating) to facilitate their reading, understanding, executing, and evaluating of problems *(Figure 3)* (Wong, 1992). Those students who have a solid collection of problem-solving strategies, but fail in their use, should train metacognitive strategies (e.g., self-instruction, self-monitoring, self-evaluation) to help them activate, select, and monitor strategy use (Graham & Harris, 1994).

The assessment of problem-solving performances should also take into account the students' attitude which seems to play an important role in academic performance (Wong, 1994). When students have a positive perception of their ability and a good attitude towards mathematics, they most likely will approach instruction in a positive manner and will make a commitment to learning. These factors are important in effective strategy instruction (Ellis, 1993).

Formal mathematics is a scientific system, which is coherent, explicit, organised, and logical. This codified body of knowledge, important to intellectual (and aesthetic) achievements, collides with the youngster's informal system, which is spontaneous, intuitive, emotional, and tied to everyday life (Vygotsky, 1986).

Cognitive-metacognitive model of Mathematical Problem Solving (Montague, 1997, p. 168)



(Figure 3)

Concerning the specific domain of learning, Paris and Winograd (1990), ordered metacognition as: *self-appraisal* and *self-management of cognition*, the first *answers questions about what you know, how you think, and when and why to apply knowledge or strategies* and the second is connected to *metacognitions in action, or how metacognition can orchestrate cognitive aspects of problem solving* (p. 8).

The teacher is responsible for training students to be able to plan and control their own performances (Brown, 1987). To achieve this purpose, the teacher must provide problems that require imaginative solutions. Such activities allow students, particularly those with learning difficulties, to more conscientiously guide their cognitive procedures.

Performance can be influenced by the social factors which increase confidence, so students may be tempted to attribute a more favourable opinion to their own hypotheses, to the detriment of outsider opinions. Teacher evaluation should preferentially be focused on performance quality and not on personal appreciation criteria (Nickerson, Perkins & Smith, 1994). Individuals who show high internal control commit themselves to the results of their performance; those who depend on external control justify their performance with luck, chance, or faith. Weiner (1974) considers that both success and failure have origin in an internal or external cause, stable or unstable, manageable or unmanageable by the student *(Chart 1)*.

Success and failures variables / Internal and external causes (Chart 1)

	INTERNAL		EXTERNAL	
	STABLE	**UNSTABLE**	**STABLE**	**UNSTABLE**
NOT MANAGEABLE	Skill	Mood	Task Difficulty	Luck Chance
MANAGEABLE	Usual Effort	Sporadic Effort	Teacher's Influence	Sporadic help from colleagues

Students who feature performance success to internal causes have greater predisposition to autonomously solve mathematical issues. In contrast, students who associate success to external factors tend to support their performance on teacher guidance to reach the solution. Rowe (1983) documented two types of students: those who attribute the responsibility for success or failure to luck or chance, and those who believe that they assume control to increase their chances of success. The first have a non persistent, lethargic attitude, in opposition to those who show strong resilience.

Knowledge is not the only factor to decide behaviour and student performance; therefore, it is important to understand how, when and if students effectively use that specific knowledge. The methodology that leads to the implementation of a good problem solving teaching programme must consider other factors, namely process quality, mainly the *"student belief system"* (Schoenfeld, 1983). Success comes with a combination of at least four types of knowledge: linguistic, schematic, algorithmic and strategic (Meyer, 1982).

Pursuant to the previously delineated ideas, individual availability to problem solving is not a minor issue. In a given context, motivation is fundamental to build a strong bond between the individual and the task to be performed.

Motivational aspects of problem solving and resilience

The ability to overcome adversities is a decisive factor for success. Throughout History, several mathematical problems have defied human intelligence. In some cases, the solution only came after many years, decades or centuries of challenging research.

The *Seven Bridges of Königsberg* is an enigma which allows testing students' resilience. It questions whether it is possible to walk all seven Königsberg bridges, only once. Leonhard Euler (1707 - 1783) proved that the problem did not have a solution.

Fermat's Last Theorem, by Pierre de Fermat (1601 - 1665), suggested that the equation $x^n + y^n = z^n$ where *n* is a natural number greater than two and *x*, *y* and *z* are positive integers, is impossible to solve. *Fermat's Last Theorem* was only properly solved in the 20th century. *"Every even integer greater than 2 can be expressed as the sum of two prime numbers"*. Christian Goldbach's (1690 - 1764) conjecture is easy to verbalise *(Table 1)*, but hard to prove. Mathematicians are still working on a demonstration.

Goldbach's conjecture diagram (Table 1)

EVEN INTEGER (greater than 2)	*4*	*6*	*8*	*10*	*12*	*14*	*16*	*18*	*20*	...
TWO PRIME NUMBERS SUM	*2 + 2*	*3 + 3*	*3 + 5*	*3 + 7* = *5 + 5*	*5 + 7*	*3 + 11* = *7 + 7*	*3 + 13* = *5 + 11*	*5 + 13* = *7 + 11*	*3 + 17* = *7 + 13*	...

Motivation and resilience are main factors to attain mathematical problem solving success, and are also elements of psychological constructs in instructional research leading to the implementation of procedures and subsequent evaluation. Conative constructs, with reference to motivational and volitional aspects of human behaviour, are fusion of cognitive and affective constructs. Despite the distinction between cognition, conation and affection, historically documented in psychology literature, the difference should be considered more in terms of definition rigour and not as a true partition; all human behaviour, in particular the one that includes instructional learning and achievement, engaged some kind of fusion of the three aspects (Hilgard, 1980).

In a formal definition, *conation* corresponds to

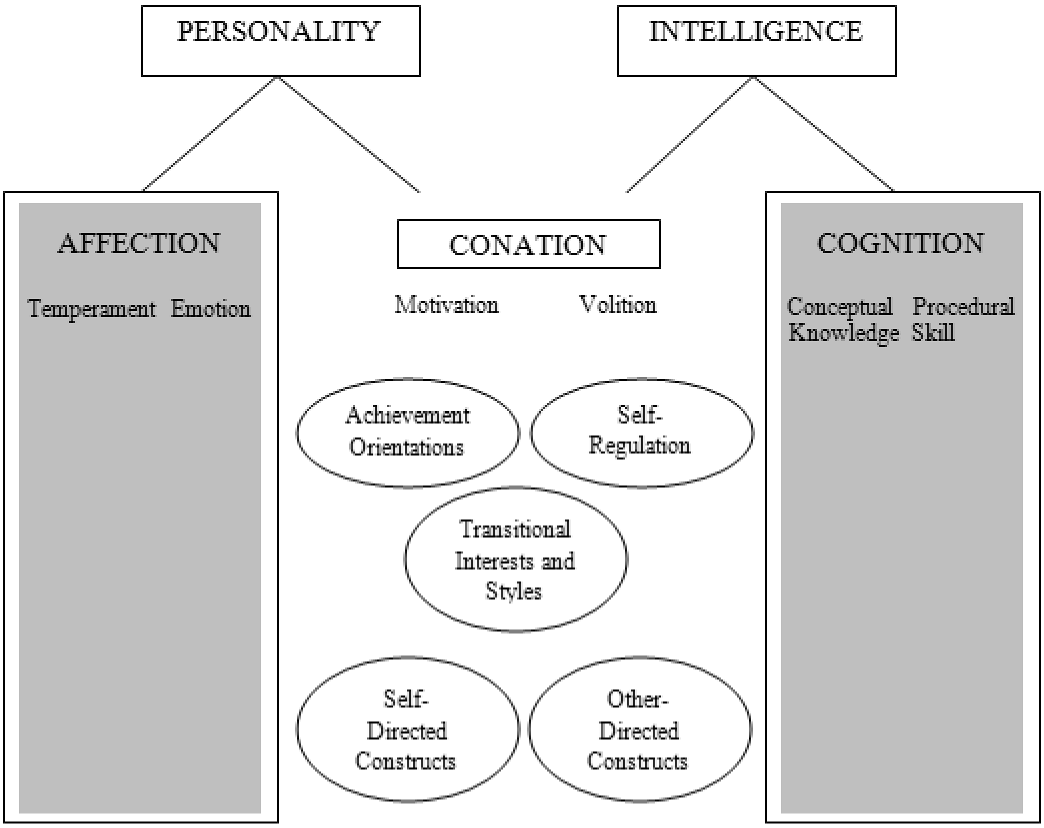
> that aspect of mental process or behavior by which it tends to develop into something else; an intrinsic "unrest" of the organism… almost the opposite of homeostasis. A conscious tendency to act; a conscious striving… Impulse, desire, volition, purposive striving all emphasize the conative aspect (English & English, 1958).

> Among the constructs we place in this category today are several kinds of achievement motivational distinctions, including need for achievement and fear of failure, but also various beliefs about one's own abilities and their use, feelings of self-esteem and self-efficacy, and attitudes and interests concerning particular subject matter learning; volitional aspects pertaining to persistence, academic work ethic, will to learn, mental effort investment, and mindfulness in learning; intentional constructs reflecting control or regulation of actions leading toward chosen goals, attitudes toward the future, and self-awareness about proximal and distal goals and consequences; and many kinds of learning styles and strategies hypothesized to influence cognitive processes and outcomes of instruction. Many other, more traditional personality or style constructs, such as intellectual flexibility, conscientiousness, extraversion, or reflection-impulsivity, could also be added to the list. And many of these constructs and measures may prove extremely useful in understanding student commitment to learning, or lack thereof (Snow & Jackson, 1997).

Kolbe (1990) identifies four conative modes: Fact Finder (instincts to probe, refine and simplify); Follow Thru (instincts to organise, reform and adapt); Quick Start (instincts to improvise, revise and stabilise); and Implementor (instincts to construct, renovate and envision).

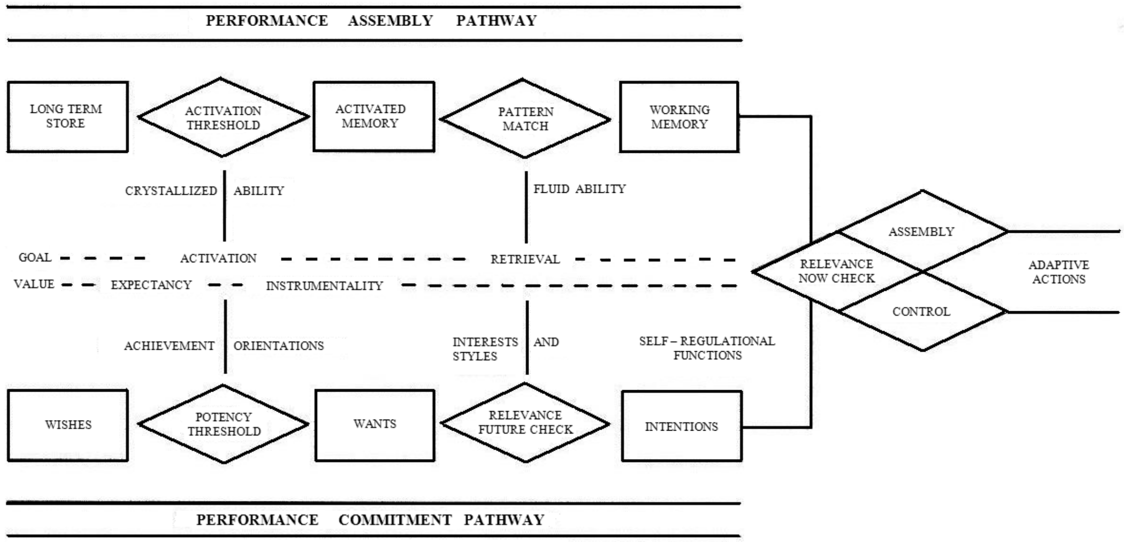
Next *(Figure 4)* is a schematic taxonomy of conative constructs and its location regarding affective and cognitive domains. Within the conation space, and embracing personality and inteligence aspects, motivation and volition establish a dynamic *continuum* from the impetus that came from desire to the action process of making decisions. However, that flow is not extended to the temperament and emotion constructs inside the affective domain, neither to knowledge and the ability construct inside the cognitive domain.

There are conative constructs designed for achievement motivation that are sensitive to individual differences, in wishes or goals, as well as positive or negative expectations related to them. This is where the category of self-directed constructs, self-esteem, and self-efficacy is located. Another category contains volitional, self-regulatory constructs addressing individual differences in intentions, effort and metacognition related to them. Beliefs and perceptions about domains related to school subjects, instructional situations, teachers, and other students, and also theories about persuasion, leadership, social competence or need for social approval are located here.



(Figure 4)

The next diagram *(Figure 5)* illustrates a theoretical course that starts with wishes (values attached to goals that work as incentives about anticipated final states, positive or negative). These wishes are most probably controlled by the individual's basic emotional needs and interests. Subsequently, wishes become wants, when there are enough expectancy and instrumentality (*i.e.*, when surpass a certain threshold of potency). If intentions are considered relevant for the present situation, develops into actions. Differences between motivation and volition emerge along this path. In parallel, we can imagine a course where cognitive performances happen. Available theory and evidence tell us that crystallised cognitive abilities are stored in long term memory units that are activated and retrieved under suitable stimulus conditions. Fluid ability is linked with the working memory because it is thought to reflect more detailed adaptation to the path of person-work interaction and also as inferential reasoning therein. It also exhibits some interaction between crystallised ability and achievement orientations, and fluid ability with some kinds of individual interests and styles, correlations that took place, at least, under some instructional conditions.



(Figure 5)

The individual has expectations about the accomplishment of any particular goal. The theories of Atkinson and Feather (1966) and Heckhausen (1977) predict what goal is selected in a particular situation. Nevertheless, lower order and near future goals diverge, both in hierarchy and instrumentality, from the higher order or more distant future goals. The first ones receive valence and potency from distal or higher goals. Some students want to learn for today's lesson, because they want to do well in the next evaluation test, because they want to achieve a good grade in the final school year, and with that reach a higher level of education. In math problem solving two dimensions of achievement motivation are distinguishable: fear of failure and hope of success. Some students are mainly motivated to avoid failure, while others are primarily motivated to reach success. Both aspects of motivation can lead to effort, investment and success by choosing different kinds of tasks. The first ones choose less difficult tasks where failure is unlikely or very difficult tasks where failure is probable and expected, the second engage in moderately difficult tasks where the outcome is also moderate. McClelland, Atkinson, Clark and Lowell (1953) have evaluated both dimensions of achievement motivation.

Jackson, Ahmed and Heapy (1976) explored the multidimensional structure of achievement motivation. From research they postulated six distinct components.

> Concern for Excellence (motivation for competition with a standard of excellence, to do one's best, as the component originally defined by McClelland et al, 1953); Competitiveness (motivation for competing with others in order to win); Acquisitiveness (motivation based on the reinforcing properties of material rewards); Status With Experts (motivation associated with the rewarding aspects of striving for social recognition with one's peers); Achievement via Independence (motivation to do well in tasks and environments where individual initiative is rewarded) (Snow & Jackson, 1997).

Some researchers do not agree with the single idea of motivation engagement through external reward or to avoid negative consequences, but motivation as intrinsic *stamina*. *"Intrinsically motivated learning is learning that occurs in a situation in which the most narrowly defined activity from which the learning occurs would be done without any external reward or punishment"* (Malone & Lepper, 1987, p. 229). This intrinsic *stamina* comes from the individual's need to act without compulsion and perform well, and arises from an internal locus of causality in which the person assumes a particular behaviour for his internal rewards, including interest (an emotional outcome or reward of intrinsic motivation) and mastery (Deci & Ryan, 1985). Intrinsic motivation occurs in an interstitial individual area, but when it concerns a specific situation, it becomes more difficult to be quantified in a research framework.

Every person's interests, a *"specific form of relationship between a person and an object"* (Schiefele, Krapp & Winteler, 1988), are usually quite durable and stable in relation to specific matters or activities (Schiefele, 1991). An *Educational Interest Theory* should be articulated in cognitive, emotional, and value terms, and attached with subjective meaning and self-intentionality towards the goal. Prenzel (1988) embraces qualities of perseverance (the maintenance of the liaison by repeated engagements with the goal) and selectivity (the stability of content in consecutive engagements over time). We should also distinguish a deep approach from a surface approach to learning. In a deep approach, students consider the subject to be learned (in mathematics it could be a text or a problem) as resources through which they can expand their understanding of the subject foundations. In opposition, in the surface approach, students think about the specific material they were provided with as what is strictly need to be learned, without making any kind of effort to connect it to a wider conceptual framework. Understandably, learners who are intrinsically motivated to learn, and are therefore less worried about performance or other people's judgments about their endeavours to learn, are more predisposed to engage in deep learning. When the locus of motivation is external, *i.e.*, when students are extrinsically motivated to accomplish a goal, learning becomes a process of passive transmission of information from the provided materials to the brain of the individual, a course of action which emphasises memorisation. The dichotomy deep learning versus surface learning is indirectly related to successful performance, but the deep approach will lead to a wider understanding than the surface approach.

Some students show greater difficulty in maintaining intended actions and restraining competing actions or distractions, the volitional control process, according to the action control theory. The difficulty in performing an intention is greater as greater is the strength of competing intentions, the social pressure to engage in alternative activities, and the extent to which the student is currently *state oriented*. State orientation is simultaneously an ability-like and state-like construct hypothesised to influence self-regulatory efficiency. State-oriented persons are characterised by a *"fixation on past, present, or future states, for example, on a past* *failure to attain a goal, on the present emotional consequences of that failure, or on the desired goal state itself"* (Kuhl & Kraska, 1989, p. 366). Another factor which enhances volitional control is the student's perception that they can implement the intention and also observe that the situation/environment is propitious to implement it. Kuhl (1984) itemize six action control strategies (selective attention, emotion control, environmental control, parsimonious information processing, encoding control, and motivation control).

> To illustrate these strategies, suppose that a student needs to support the intention to do homework and inhibit the preference to watch television. The student could use selective attention strategies to try to avoid visual contact and engagement with the television set, or motivational control strategies involving self-reinforcement and punishment to emphasize the sense of satisfaction that comes from completing the homework. Emotion control strategies (such as reassuring self-speech) could also be used to limit anxiety about the difficulty in starting the homework. Environmental control strategies, such as choosing a work environment away from the distraction of the television, could also be used (Snow & Jackson, 1997).

In achievement tasks, individuals exhibit solution-oriented, self-instruction, and sustained performance in intricate circumstances (Diener & Dweck, 1978). Unsolved problems are considered challenges that require strategy and effort to be answered. Performance orientation is typified by avoidance of challenge, weak performance, and negative thoughts facing failure (Elliott & Dweck, 1988). These individuals attempt to maintain positive judgments of their skills and avoid negative opinions (Nicholls & Dweck, 1979).

Challenging mathematical problems such as the *Seven Bridges of Königsberg, Fermat's Last Theorem*, and Christian Goldbach's conjecture underlines the importance of mindful effort investment to engage individuals in heuristics.

> Mindfulness involves intentional, purposeful, metacognitively guided employment of non-automatic, hence effort demanding, mental processes (Salomon, 1987). A learner rarely applies knowledge and skill automatically when needed or appropriate. There must be an intention to mobilize and apply knowledge and skill to a new situation. This intention mobilization is mentally taxing – it demands effort investment in mindful application of knowledge and skill. The difference between what a person can do and what a person actually does in a situation indicates the effect of mindful effort investment. The distinction between mindfulness and mindlessness seems parallel to that between controlled and automatic processing.

Mindfulness is a function of stable individual differences but also of situational, perceptual, and instructional conditions. Persons differ in their tendency to engage in and enjoy effortful cognitive activity versus to minimize mental effort in processing incoming information. Learners high in mindfulness perform better when given loose guidance and enough freedom to work on their own, but react negatively when given unduly specific and continuous guidance. The opposite is seen for learners low in mindfulness. High-minful learners perform better when working alone than in teams. However, in teams that also allow independent activity, highs are unaffected while low-mindful learners tend to loaf. Mindful learners intentionally seek out opportunities to invest mental effort. They are selective – mindful about some aspects of a situation while ignoring others. Mindlessness occurs when a situation is perceived as familiar, undeserving of effort, or too demanding – the sequence of events is passively allowed to unfold without activity it …

Effort avoidance can be distinguished from low need for achievement characterized by laziness and from high fear of failure characterized by striving to achieve. A person motivated by effort avoidance shows active mental or physical escape – that is, mindful avoidance, and no intention to succeed. The causes of effort avoidance seem to be frustrating early experiences in a task domain, so the construct is usually domain specific. But experiencing frustration in many school activities can presumably lead to generalized effort avoidance.

Unsupportive, restrictive intervention styles used by aprents and teachers appear associated with the emergence of effort avoidance. The more teachers or parents use pressure to motivate such persons, the quicker effort avoidance appears. Effort avoiders use their intelligence to convince teachers they are not intelligent enough to cope with tasks given them. They tend to score lower on group tests than on individual intelligence tests. Their strategies for effort avoidance include working very slowly, working very rapidly in slipshod fashion, stopping work when praised, producing feelings of resignation to induce teachers not to push them, and generating excuses for not working (Snow & Jackson, 1997).

Mathematics learning and teaching

Learning to teach: Experiential learning model

Empirical research is implicitly or explicitly based on theoretical models to control observations and data gathering. Kilpatrick (1985) categorised problem solving didactics into osmosis, memorisation, imitation, cooperation and metacognition.

> *This classification of instructional approaches is meant only to be heuristic. Although the research literature in mathematics education is full of studies in which methods of teaching problem solving are compared, we do not have a comprehensive scheme for classifying various features of different methods. Some priority should be given in the next quarter century of research on problem solving in mathematics to delineating instructional approaches and attempting to link their features to their effectiveness* (Kilpatrick, 1985, p. 10).

A positive correlation between the quantity of problems presented to the student and their performance is based on a learning system where the student, when facing a battery of questions, absorbs the resolution techniques as in an osmosis process. The idea that students acquire problem solving tools exclusively by practice depends on what they know and also on their reasoning sophistication. Practice is a necessary condition to improve performance. However, it is not enough because other variables act as performance regulators. *"No instructional program can be successful that does not deal with the effects of students' negative attitudes and beliefs about themselves as problem solvers"* (Kilpatrick, 1985, p. 9).

A methodology based on problem solving elementary partition procedures with algorithmic applications assists the solution (Dahmus, 1970). The success of this methodology, which has previously defined rules sustained in memorisation, loses effectiveness when students are unable to identify situations where they should use those procedures or when they must deal with problems where it is not possible to use them. The certainty that the success of learning depends on the reproduction of behaviours has stimulated research where students compared the differences between their performances with a model student's performance to promote error - correction strategies (Bloom & Broder, 1950). In other research studies, students were involved in a narrative discourse where the teacher shows signs of doubt and appeals to multiple procedures in a continuous dialogue with the students so that they acquire confidence in problem solving decisions, working under the teacher's orientation (Covington & Crutchfield, 1965).

Research methodologies can focus on teaching and learning specific or general contents. Schoenfeld (2007) underlines that questionnaires and information treatment may significantly influence research outcomes. As an example he refers his interest in knowing whether theoretical lessons given to small classes are more efficient than lessons given to a larger group. Conditioned by time management, teachers with large classes evaluate their students with multiple choice tests. A comparative study shows no significant statistical differences between the two forms of instruction; hence we may conclude that they produce equivalent results. Therefore, the economic factor may decisively work in favour of theoretical lessons given to large groups. However, if the evaluation method used to survey educational quality is the proportion of students who pursue their studies in scientific areas, it might be that the best performances come from small classes. Both evaluation models are objective and quantifiable, as well as able to justify educational decisions, even though they may be opposite decisions. There is real danger of generalising results of school population subgroups. Minority behaviours should also be considered.

From Schoenfeld's point of view, credibility, generality and importance are essential vectors of research to answer: *1)* why should readers believe in the author?; *2)* in what context does the research study apply?; *3)* why should it be valid?

When a theoretical orientation is delineated, the empirical research consists of gathering, organising and interpreting data. The researcher should be able to distinguish between correlation, two variables under mutual influence, and a straight attempt to explain in what way two variables relate to one another. To illustrate the difference we refer Gregor Mendel's (1822 - 1884) work, which contributed to establish the rules of heredity. Character transmission from generation to generation had already been observed, but Mendel was the first to understand the mechanism (specifically regarding peas). Data inferences should contribute to scientific knowledge but will not necessarily fully apply to the mathematical science. Galileu Galilei's (1564 - 1642) astronomic observations allowed a more elaborate theory based on the movement of planets around the Sun, in comparison with the geocentric model generally accepted up to that time. Despite flaws in planets' orbits, which are elliptical and not circular, as Johannes Kepler (1571 - 1630) would prove by using data collected by Tycho Brahe (1546 - 1601), Galileu Galilei revolutionised our interpretation of the position of planet Earth in the Solar System.

The relation between the performance of the teacher in the classroom and student learning is a recurrent theme in Mathematical Education. In the 1970s and 1980s, the attention of researchers was focused on the paradigm process/product, and therefore they collected data on: *1)* specific classroom practices (time to execute a task, worksheets and specific questions); *2)* students' performance in normalised tests; *3)* statistical relations between *1)* and *2)*. Results suggested that students perform better when specific procedures are carried out by the teacher and so those practices should be disseminated. In a comparative study with proficient teachers (whose students achieved better school results) and non-proficient teachers, Leinhardt (1990), realised that the first group established, in a univocal direction, routines in the course of the school year. Later, Leinhardt, Putnam, Stein & Baxter (1991) registered that the mathematical knowledge of a large number of teachers from that proficient group was limited to the school curriculum and that the evaluation tests they used were trivial standardised tests. Data reinterpretation suggested that a high control level makes teachers stay in their comfort zone. Hence, students may eventually master procedures but fail to understand the theoretical concepts that lead to mathematical problem practice.

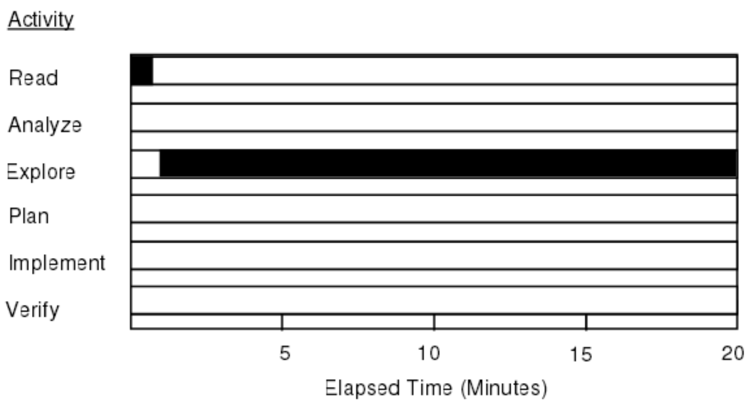
Well defined events take place in previously defined conditions. Predictions allow us to create consistent practical models which support theoretical arguments. The *Periodic Table of the Elements* conceived by Dmitri Mendeleev (1834 - 1907) is a theoretical regulation based on a deterministic prediction of chemical elements still unknown in the 19th century. Meteorological forecasts are the result of complex climatic models that are continually evaluated and perfected. When small variations are introduced in the model, they will most likely unchain different prognostics. Educational Psychology research also involves predictions. Constraints are inversely related to prediction consistency, so researchers should balance that fragile relation in order to achieve the most accurate conclusions.

Each individual has a personality which differentiates them from the group and each classroom has a specific dynamics. Hence, in the educational context procedures are difficult to replicate. However, many studies pass the idea that procedures can be easily generalised. Questionnaires, interviews and observations allow the possibility to challenge, to confirm or to expand a theory. Behaviour that is spontaneous in a specific context may eventually not be exhibited at other moments. Fennema (1977) studied students' performance regarding gender. In a comparative analysis, he concluded that boys more often attribute success to ability, whereas girls more often justify failure as lack of ability. Girls tend to attribute success to effort and boys failure to lack of persistence. If youngsters impute failure to their limited ability to solve problems, then it is unlikely that they will persist. If success is associated to an external factor, they most likely consider problem solving as an activity for which they need the teacher's aid.

Schoenfeld (1992) refers metacognitive processes which highlight students' problem solving strategies and answers as a main issue of Educational Psychology. Students were asked to think aloud, one at a time. The researcher remained in silence with the exception of cases when a student did not make any comments during a predefined period of time. These individual twenty-minute sessions were recorded for posterior data analysis with the purpose to classify crucial moments which lead to success or failure. After an interpretative sign was conceived, preliminary analyses showed absence of students' standard procedures. One individual, in particular, contributed for the advance of the study because he persisted in a specific, yet barren, computation, thus condemning work to failure due to the excessive time spent in the calculations.

Schoenfeld divided each session in periods of activity *(Figure 6)*. Even with a basic subject understanding, if students engage in inadequate strategies, they frequently compromise the successful completion of the task. This underlines the value of promoting strategies which allow students to think of and aim for success.

> *If you want to know why people's attempts to solve challenging (mathematical) problems are successful or not, you need to examine their: 1) knowledge base – just what (mathematics) do they know?; 2) problem-solving strategies, a.k.a. heuristics – what tools or techniques do they have in order to make progress on problems they don't know how to solve?; 3) monitoring and self-regulation – aspects of metacognition concerned with how individuals "manage" the problem-solving resources, including time, at their disposal; and 4) beliefs – individuals' sense of mathematics, of themselves, of the context and more, all of which shape what they perceive and what they choose to do* (2011, pp. 3-4).

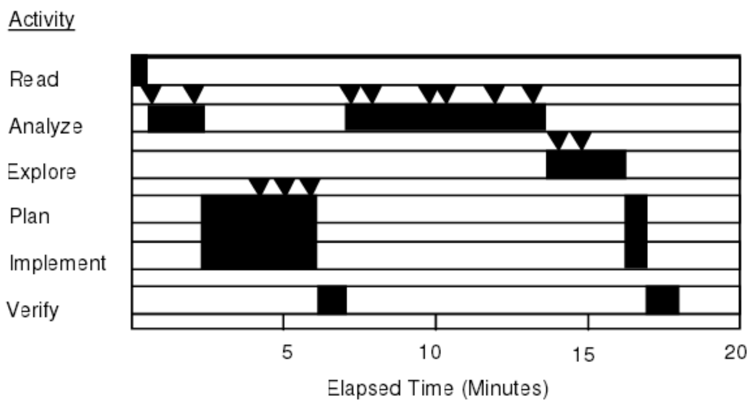


(Figure 6 - A time-line representation of a problem solving attempt)

> *Reflecting on progress while engaged in problem solving, and acting accordingly ("monitoring and self-regulation") is one aspect of what is known as metacognition – broadly, taking one's thinking as an object of inquiry. As the graph above indicates, failing to do so can guarantee failure at problem solving: if one is fully occupied doing things that do not help to solve the problem, one may never get to use the "right" knowledge in the right ways* (2007, pp. 66-67).

Next *(Figure 7)*, Schoenfeld sketch

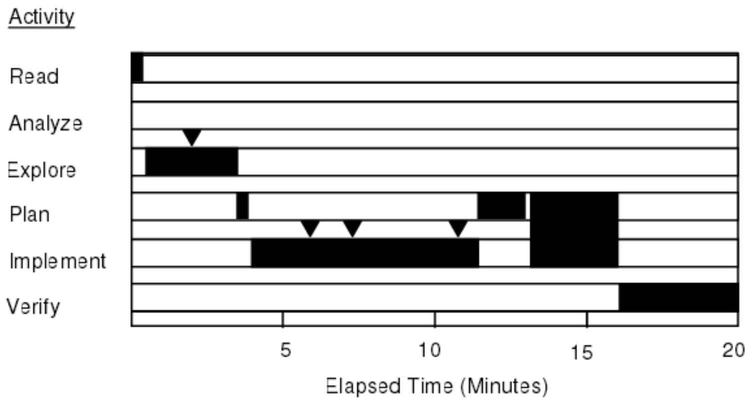
> *the time-line graph of a mathematician working a two-part problem in a content area he had not studied for years. Each of the triangles in the figure represents a time that the mathematician assessed the state of his solution attempt and then acted on the basis of that assessment (sometimes deciding to change direction, sometimes deciding to stay the course – but always with decent reason). There is no question that he solved the problem because of his efficiency. He did not choose the right approaches at first, but, by virtue of not spending too much time on unproductive approaches, managed to find productive ones* (2007, p. 67).



(Figure 7 - A time-line representation of a problem solving attempt)

Subsequently *(Figure 8)*, we have

> *the time-line graph of a pair of students working a problem after having taken my problem solving course. During the course, I focused a great deal on issues of metacognition, acting as a "coach" while groups of students worked problems. (That is, I would regularly intervene to ask students if they could justify the solution paths they had chosen. After a while, the students began to discuss the rationales for their problem solving choices as they worked the problems.)… the students jumped into a solution with little consideration after reading the problem. However, they reconsidered about four minutes into the solution, and chose a plausible solution direction. As it happens, that direction turned out not to be fruitful; about eight minutes later they took stock, changed directions, and went on to solve the problem. What made them effective in this case was not simply that they had the knowledge that enabled them to solve the problem. It is the fact that they gave themselves the opportunity to use that knowledge, by truncating attempts that turned out not to be profitable* (2007, pp. 67-68).

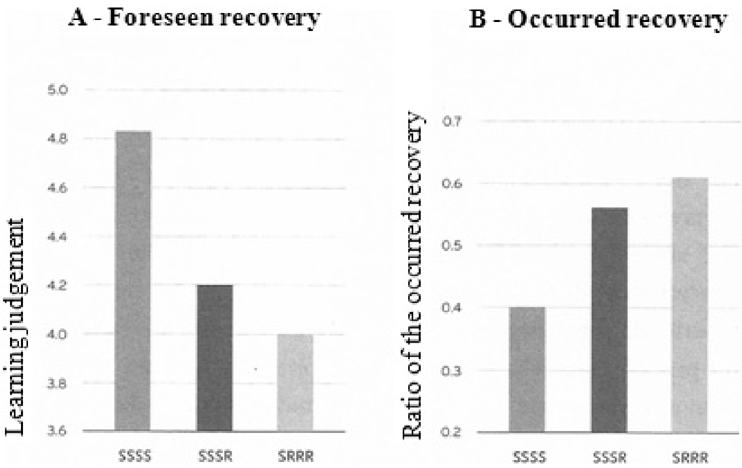


(Figure 8 - A time-line representation of a problem solving attempt)

Students revealed distinct behaviours that Schoenfeld labelled as *episodes*. Concerning the most significant (problem analysis, advice, selection of mathematical knowledge, plan making, plan execution, and answer scrutiny), proficient students spend more time analysing the statement and verifying possible suitable answers than apprentices do. New learners do not spend much time reading the problem nor thinking of a suitable course of action. Attention goes to routine procedures (e.g. computation) without making sure that it is the right path to follow.

Dreyfuss and Eisenberg (1996), Goos (2002) and Teong's (2003) research corroborates Schoenfeld's analysis. Typically, students spend less time reading the statement of the problem. They only superficially read the information given and use their time to explore procedures without establishing or implementing a plan, and without verifying the solution.

Karpicke stated that active recall plays a significant role in learning. This methodology allows knowledge quality assimilation, thus modifying it in an endless renewal cycle (Karpicke & Roediger, 2007). Roediger and Karpicke (2006) evaluated learning consistency over time in situations where participants studied educational contents. The first group repeatedly read for four study periods (SSSS). A second group read the text for three study periods and the last study period was for recalling it (SSSR). A third group first read the text and then recalled it for three study periods (SRRR). When the learning phase was finished, the students made a *judgement of learning* regarding their capacity to remember what they had previously analysed. One week later, the students reviewed the subject to assess if they had effectively consolidated knowledge. Results revealed an inverse proportionality between the students' expectations about their capacity to retrieve information and the number of study periods *(Figure 9)*.



(Figure 9)

The mind as a library where a book repository is stored is an old metaphor that gives the idea of an available physical space ready to be filled, but this idea is inconsistent with reality (Roediger, 1980). Such concept disregards the active recall of the individual's knowledge and the difficulty to accurately repeat the new information. Karpicke, Butler and Roediger (2009) questioned a group of college students on their preferential study method before taking an exam, after reading through their notes and/or the book one time, with the following options; *(a) Go back and restudy either all the material or parts of it*; *(b) Try to recall the material without restudying afterwards*; *(c) do something else*. Later, Karpicke (2009) replaced option *(b)* by *"try to recall the material, and then go back and restudy the text" (Table 2)*. The students who revealed the intention to practice active retrieval increased, but the majority (58%) chose other strategies. The results show the students' reluctance to engage in this new methodology.

Preferential method to take an exam,
after reading through their notes and/or the book one time (Table 2)

ANSWERS	FIRST QUESTIONNAIRE (%)	SECOND QUESTIONNAIRE (%)
Go back and restudy either all the material or parts of it.	57	41
Try to recall the material without restudying afterwards.	18	
Try to recall the material, and then go back and restudy the text.		42
Do something else.	21	17

Karpicke and Roediger (2010) evaluated the effectiveness of active retrieval in a study with educational texts about scientific topics. Three groups worked with differentiated learning strategies; students *(a)* read a text once in a single study period; *(b)* read the text, recalled as much as they could in a recall period, and then reread the text briefly; *(c)* repeatedly recalled the text across a series of eight alternating study/recall periods. One week later, students recalled the material to assess long-term retention. Data showed that group *(a)* recalled 15% of the material, group *(b)* recalled 34% of the material and group *(c)* recalled 80% of the material.

The dissemination of learning methodologies with the aim to measure educational effectiveness and the efficacy of the strategies is a consequence of the educational paradigm which has been in force for the last decades. The study of the relations established in the school microcosm is essential to better understand and act, in order to improve mathematical education.

Beliefs about mathematics

Beliefs are a milestone for educational researchers, as they act as a knowledge regulating agent and model the way students relate to Mathematics.

Students' beliefs can be categorised according to different parameters. McLeod (1992) established four main groups: *1)* beliefs about Mathematics as a school subject; *2)* people's beliefs about themselves and their relation towards Mathematics; *3)* beliefs regarding Mathematics teaching; *4)* beliefs related to social contexts.

BELIEFS ABOUT MATHS AND PROBLEM SOLVING

1. School subjects have little or nothing to do with real life situations (Schoenfeld, 1992).
2. School problem solving techniques do not have any kind of relation with those that are needed to solve daily life problems (Woods, 1987).
3. Formal proof is irrelevant to discover or invent procedures (Schoenfeld, 1992).
4. Mathematics work is a lonely activity, performed by a single individual (Schoenfeld, 1992).
5. Mathematics is only calculation, algorithm and rule memorising (Frank, 1988).
6. The goal is to obtain the right answers. Only the teacher can decide if the answer is right or wrong (Frank, 1988).
7. In the classroom, the role of the student is to acquire mathematical knowledge and exhibit learning. To accomplish that, paying attention in lessons, reading the schoolbook and performing the tasks stipulated by the teacher is enough (Frank, 1988).
8. The role of the teacher is to provide mathematical knowledge and scrutinise students' knowledge acquisition with control exams (Frank, 1988).
9. The difference between a problem and an exercise is in the formal display of these kinds of tasks and does not depend on the level of knowledge of the problem solver (Vila, 2001).

BELIEFS ABOUT PROBLEM SOLVERS

1. Regular students cannot hope to understand Mathematics: they can only aspire to memorise and apply routine procedures without understanding their meaning (Schoenfeld, 1992).
2. When contents are understood, students are able to solve any kind of problem in five minutes or less (Schoenfeld, 1992).
3. Those who performed well in Mathematics are good problem solvers. Those who have difficulties in Mathematics are not good problem solvers (Woods, 1987).

4. Experts only need to consider one approach to find the schematic model to solve the problem (Woods, 1987).
5. Mathematics is for gifted people; others only try to learn contents presented by teachers or books without questioning (Garofalo, 1989).

BELIEFS ABOUT THE PROBLEM SOLVING PROCESS

1. There is only one way to correctly solve each proposed problem; frequently it is the method which the teacher has recently presented in the classroom (Schoenfeld, 1992).
2. There is only one right procedure to solve the problem and only one correct answer (Woods, 1987).
3. Immediately after reading the statement of the problem, we should be able to understand what is asked (Woods, 1987).
4. Each step must be accurate without trial and error experiments (Woods, 1987).
5. To solve a given problem, memorising and using a sequential strategy is enough (Woods, 1987).
6. Mathematical problems are tasks where we use previously learned rules and which consequently can be solved in a few steps (Frank, 1988).
7. Schoolbook exercises can be solved with procedures presented in the manual; indeed, they can be solved with the procedures presented in the paragraph of the schoolbook where that problem is presented (Garofalo, 1989).
8. Attaining the solution is the most important aspect of problem solving activity (Vila, 1995).
9. Problem solving is a linear process (Vila, 1995).

BELIEFS ABOUT LEARNING AND PROBLEM SOLVING OPTIMISATION

1. Improving problem solving skills can be achieved with a single 20-minute brainstorming session (Woods, 1987).
2. Two key aspects to succeed in problem solving activity are learning mathematical techniques and mechanical procedures (Vila, 2001).

> *If the student is not able to do much, the teacher should leave him at least some illusion of independent work. In order to do so, the teacher should help the student discreetly, unobtrusively. The best is, however, to help the student naturally. The teacher should put himself in the student's place, he should see the student's case, he should try to understand what is going on in the student's mind, and ask a question or indicate a step that could have occurred to the student himself* (Pólya, 2004, p. 1).

Mathematics answers theoretical, theoretical-practical or practical questions, specific or not specific to such subject. In *How to Solve It – A New Aspect of Mathematical Method*, published in 1945, George Pólya (1887 - 1985) presented a 4-stage method to successfully solve mathematical problems.

The first stage consists of reading and understanding the statement of the problem. The recognition of the unknown, data and restrictions is critical, because no one reaches the solution without understanding what is given and what is asked. If needed, additional relevant information should be gathered. Then, a working plan must be devised to connect the data with the unknown. If that relation cannot be put into practice, remembering a similar problem and seeing if identical resolution methods can be used is advisable. Drawings, diagrams or content tables can eventually contribute to an idea. For trying to discover a rule, regularity or pattern, the trial and error method is an alternative strategy. All techniques are suitable, but one must be aware of the difficulty of having a good idea. If one has little understanding of the subject or if one knows nothing about what is asked, then the task is not achievable. The third stage is plan execution. The problem solver must verify each step, which should be perfectly clear and error-free. If the delineated strategy leads to a blind alley, then a more straightforward resolution method must be considered or, in alternative, the problem must be looked at under a different perspective, by checking if all relevant information is used. Finally, what has been written should be revisited and the solution checked to consolidate knowledge and optimise problem solving skills. In the execution plan phase, flaws should not be minimised. Therefore, critically analysing the reasonability of the solution is important. Trying to solve the problem through an alternative way is a challenge which helps knowledge consolidation. Eventually, a *Eureka* moment may occur during the problem solving process, but intentionally shortcutting stages is not advisable. The individual must be aware of the uselessness of working a detail or engaging in calculations without fully understanding the problem or without establishing a plan.

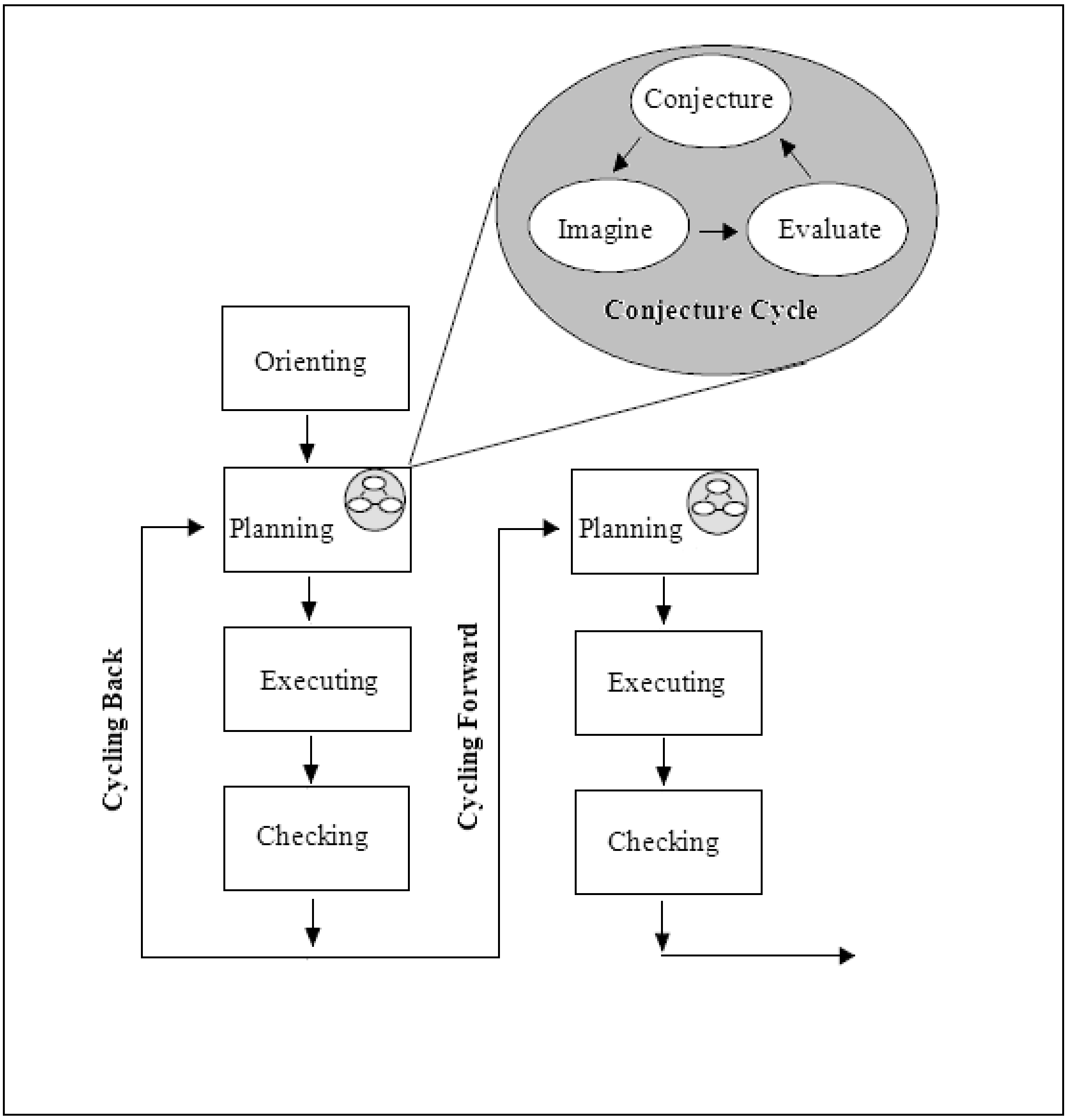
> *The Multidimensional Problem-Solving Framework characterizes the complex interactions between the products and processes of mathematical problem solving. It describes problem solving as a cyclic process, with the problem solver's (a) mathematical knowledge, (b) heuristics, (c) beliefs and emotions, and (d) metacognitive behaviors affecting the effectiveness of their solution approaches* (Bloom, 2008, p. 32).

Pólya's guidelines on mathematical problem solving *(Chart 2)* was further refined by Schoenfeld's research and validated by subsequent studies (Carlson & Bloom, 2005). The mental activities, in which mathematicians engage in, lead to problem understanding. Students' behaviours include reading and rereading the statement of the problem, organising information, defining unknowns, drawing a figure, table or graphic, and any other behaviour which shows attempts to extract some good decisions from the given information. Planning incorporates the mental activities involved in devising an approach to attain a solution. Several strategies and mathematical tools are considered and eventually used, and effectiveness is evaluated. In this phase the solver eventually verbalises an approach to a conjecture or a general solution. Executing procedures is problem solving activity in action. The exhibited behaviours include connecting logical statements, calculating unknowns, executing specific strategies, and performing computations. Then, the correctness of the obtained solution must be evaluated. Analysing behaviours includes spoken reflections on the result or process, correctness calculations, and answer reasonability in line with the statement of the problem. In general, the checking phase comes after execution, in order to verity the correctness of the work. Monitoring should always be present in the course of the problem solving process. Mathematical awareness comes from work, persistence, serendipity and heuristics *"a certain branch of study, not very clearly circumscribed, belonging to logic, or to philosophy, or to psychology, often outlined, seldom presented in detail, and as good as forgotten today. The aim of heuristic is to study the methods and rules of discovery and invention"* (Pólya, 1973, p. 112).

The Multidimensional Problem-Solving Framework (Chart 2)

Phase • Behavior	**Resources**	**Heuristics**	**Affect**	**Monitoring**
• **Orienting** • Sense making • Organizing • Construction	*Mathematical concepts, facts and algorithms are accessed when attempting to make sense of the problem. The solver also scans her/his knowledge base to categorize the problem.*	*The solver often draws pictures, labels unknowns and classifies the problem. (Solvers were sometimes observed saying "this is an X kind of problem.")*	*The curiosity and interest level of the solver affects the solvers motivation to make sense of the problem. If the solver is not interested, he/she often lacks motivation and stalls before starting.*	*Self-talk and reflective behaviors serve to keep the mind engage. The solvers were observed asking "What does this mean?"; "How should I represent this?"; What does that look like?".*
• **Planning** • Conjecturing • Imagining • Evaluating	*Conceptual knowledge and facts are accessed to construct conjectures and make informed decisions about strategies and approaches.*	*Specific computational heuristics were accessed and considered while considering and choosing a solution approach.*	*Beliefs about the methods of mathematics and one's abilities influence conjectures and decisions. Signs of intimacy, anxiety, and frustration are also displayed.*	*Solvers monitor their strategies and plans. They ask themselves "Will this take me where I want to go?"; "How efficient will approach x be?".*
• **Executing** • Computing • Constructing	*Conceptual knowledge, facts and algorithms are accessed when executing, computing and constructing. Without conceptual knowledge, monitoring of constructions is misguided.*	*Fluency with a wide repertoire of heuristics, algorithms and computational approaches are needed for the efficient execution of a solution.*	*Intimacy with the problem, integrity in constructions, frustration, joy, defense mechanisms and concern for aesthetic solutions emerge in the context of constructing and computing.*	*Conceptual understandings and numerical intuitions are employed to monitor both the solution progress and products while constructing statements.*
• **Checking** • Verifying • Decision Making	*Resources, including well-connected conceptual knowledge informed the solved as to the reasonableness or correctness of the solution attained.*	*Computational and algorithmic shortcuts are used to verify the correctness of the answers and to ascertain the reasonableness of the computations.*	*As with the other phases, there were a number of affective behaviors displayed. It is often at this phase that frustration overwhelmed the solver, causing the solver to abandon the task.*	*Reflections on the efficiency, correctness and aesthetic quality of the solution provided useful feedback to the solver.*

The process is dinamic and cyclic *(Figure 10)*.



(Figure 10)

> *The researchers also noticed that during the planning phase, it appeared that the subjects were actively sorting through various possibilities, playing them out in their imagination, before selecting a plan. Often, the mathematicians were observed staring into space at this point in the solving process. When the subjects were asked to verbalize their thoughts during this process, they revealed another mode of cycling thinking. They would make a conjecture, imagine the conjecture being played out in some way, and then decide if the idea was worth pursuing. (...) Influencing all these phases were resources, heuristics, affect, and monitoring* (Bloom, 2008, pp. 33-34).

Philosophers are not immune to taboos which curtail the full development of Science. Bento Caraça (1901 – 1948) (1978) wrote *“Ancient Greek thought was filled with the ghost of change and, as a consequence, the horror of physical movement. The elite rejected manual and mechanical work, and exulted good and virtue, whose quest was the purpose of human beings”*[6] (p. 189). Zeno of Elea (ca. 490 BC – 430 BC) drew attention to the imperfections of Plato and Aristotle’s thoughts based on Geometry’s dominance when compared with Arithmetic’s. In *Achilles and the Tortoise* paradox, the Greek mythological warrior competes with a tortoise. Let us consider the metre as the length unit. Confident of victory, Achilles, who runs ten times faster than the tortoise, grants the tortoise a 100-metre advance. When Achilles reached the tortoise’s starting point, the tortoise had moved ten metres forward. The athlete ran another ten metres, but the slow tortoise was, now, one metre ahead. Achilles got nearer and nearer but apparently he could never reach the tortoise. However, this argument opposes empirical evidence.

Renaissance illuminated the path that leads to knowledge, for *“are vain and error susceptible those sciences that don't grow in experience, the mother of certainty, or that do not end in experience, that is, such sciences that in origin, middle or end doesn’t use any of the five senses”* (Leonardo da Vinci, quoted by Caraça, 1978, p. 201).

René Descartes (1596-1650) influenced modern philosophical thought and scientific knowledge, by questioning what, until then, was accepted, in order to eradicate preconceived beliefs. In *Discourse on the Method of Rightly Conducting One´s Reason and of Seeking Truth in the Sciences*, Descartes categorised four main vectors to understand reality; *1)* never accept anything for true which you do not clearly know to be such, *i.e.*, beware of hasty attitudes and prejudice and consider only what comes to mind with certain clarity without any shadow of doubt; *2)* divide each difficulty in simplified parcels to achieve the solution; *3)* think in a sequential order, beginning with the most simple and easy and, progressively, take on to most complex, ensuring a reasoning order that, eventually, is opposite to what is mind perceived; *4)* enumerate procedures and revisit what has been to ensure error and omission absence.

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[6] Author’s translation: *O pensamento grego dominante aparece invadido pelo horror da transformação, e daí resulta o horror do movimento, do material, do sensível, do manual. O homem de elite rejeita o manual, o mecânico, e exalta o bem e a virtude, de cuja procura faz o fim máximo do homem.*

In addition to the intrinsic difficulties of a problem, success depends on the quantity and quality of individual knowledge, the technical expertise to represent and transform the problem and metacognitive processes to monitor and guide the performance. Individuals may think that the solution comes with the use of procedures which were recently used in class; or they may think that they are being evaluated and then adopt a real or apparent, proactive attitude; or question the teacher in order to obtain hints to reach the solution; or read the teacher's facial expressions to check whether they are working as expected.

> *Studies of expert problem solvers and computer simulation models have shown that the solution of a complex problem requires (1) a rich store of organized knowledge about the content domain, (2) a set of procedures for representing and transforming the problem, and (3) a control system to guide the selection of knowledge and procedures. It is easy to underestimate the deep knowledge of mathematics and extensive experience in solving problems that underlie proficiency in mathematical problem solving. On the other hand, it is equally easy to underestimate the sophistication of the control processes used by experts to monitor and direct their problem-solving activity* (Kilpatrick, 1985, pp. 7-8).

To promote a positive problem solving environment, the teacher should; *1)* be enthusiastic about problem solving activities; *2)* encourage students to carry out their assumptions; *3)* promote perseverance and strengthen students' willingness to find the answer; *4)* emphasise students' verbal presentations of solution; *5)* accept alternative resolution processes from those taught in class; *6)* promote students' correct solutions, with emphasis on resolution strategies; *7)* promote persistence in opposition to resolution swiftness.

> *There is a huge difference between the way that we do mathematics and the way that our students see it. Doing mathematics is a vital, ongoing process of discovery, of coming to understand the nature of particular mathematical objects or systems. First we become familiar with an area. As we do, our intuitions develop. We begin to suspect that something ought to be true. We test it with examples, look for counterexamples, try to get a sense of why it ought to be true. When we think we know what makes it work, we try to prove it. The attempt may or may not succeed. There may be any number of false starts, reverses, retrenchments, and modifications. With perseverance and luck, the result falls into place. Few experiences are so gratifying or exciting: we have charted unknown territory, and enriched ourselves in the process* (Schoenfeld, pp. 5-6).

For Gestalt followers, problem solving was the basis for mathematical learning, although a curriculum supported by this methodology was never presented. Pólya's book, *How to Solve It*, is an excellent guide, with plenty of good practices, but despite its value, it never sustained subsequent systematic implementation. An explanation for it may lie in the difficulty to enlighten mathematical concepts such as elementary axioms, and assess learning consolidation processes by problem solving practice.

Problem solving in the mathematics curriculum

Exercises *vs.* Problem Solving

When problem solving activity is implemented in the classroom, teachers provide the methodological steps and make their students copy the standard solution methods prescribed in the school textbook. Little time is used to teach how to think (Foshay & Kirkley, 2003). Consequently, students reveal a huge difficulty to operate with non-standard problems, even though they may only require common knowledge and routine operations. Educational researchers Garofalo (1985) and Van Streun (2000) hypothesised that the rationale to those low achievements is not the lack of mathematical resources but its inefficient use.

> *There is a very real danger that the type of mathematics instruction we provide students is training them to be rigid in their thinking, not flexible and adaptable, is teaching them how to perform procedures but not when and under what conditions to perform them, and is showing them what to do but not why to do it* (Lester, 1985, p. 43).

Exercise resolution is often associated with the use of algorithms which allow reaching a solution, although, in some cases, the time required for its implementation curtails finding a solution. Heuristic procedures promote thought and strategies which, when isolated, do not ensure success, but contribute to diminish optional optimising procedures.

The diversity of educational paths and the extension of compulsory education are two major consequences of educational reforms whose intent was to adjust school to the progress of society. Regarding the curriculum, we should consider three levels: official intentions written in official documents (the enounced curriculum), the implementation of teacher curricular guidelines (implemented curriculum) and students' effective learning (acquired curriculum). The teacher who implements a problem solving based curriculum guides students to an adventurous discovery. Concept transmission and monotonous algorithmic repetition should be replaced by interactive work, a heuristic promoter of thinking processes which leads to the implementation of strategies to reach the solution to the problem. When compared with standard procedures, it requires a different methodology from what is prescribed for ordinary contents. Teachers must feel confident about lesson contents and be able to differentiate intuitive from formal learning in their students. In the classroom, the concern of attaining the answer to the problem is visible in the students, mainly in formal evaluation moments. Students do not often devote much time to reading the statement of the problem, and therefore perform multiple calculations without a defined strategy, showing satisfaction when they see something they may call a solution on the answer sheet. As a result, the probability of success is minimal, thus unleashing frustration that comes from failure in the student. Then rejection to answer new problems due to anticipation of identical outcomes arises. Students show difficulty to realise that the answer is only a portion of the global logical construction which is problem solving. If teachers succeed in making students act consciously and with method upon each procedure, then they will be able to enthral them for Sciences.

Researchers omit data on what is experienced in the classroom. Eventually, exhaustive statistical analyses intend to enhance significant results from minimum differences between experimental group and control group. By adding a theoretical model which does not consider all the variables, or at least the most significant ones, to quantitative data, the conditions to a non-natural scheme, easier to control but whose quality may be questioned, are gathered. Nontrivial evaluation tests are commonly applied in order to evaluate instruction effectiveness. However, these tests have insufficient parameters to measure efficiency. Students' behaviour is a source of information to inquire the assimilation of heuristic processes during the instruction period. Nevertheless, evidence of instruction efficiency or failure based on a scarce sample or a single experiment should not be validated. The theoretical basis, which is essential for research, is often replaced by preconceived ideas. Difficulty in creating a conjectural model is justified by the necessity to include an instructional programme. Pólya's orientations are a problem solving guide but, in general, the procedures suggested do not differ in substance from what is necessary to solve general problems. The absence of a theoretical model indicates the difficulty in generalising results from a particular study or a set of eventually correlated studies. The concern with individual differences must be clear in the intervention programme, thus allowing each student to develop a critical conscience about their strengths and cognitive weaknesses, as well as promoting competences and diminishing insufficiencies. Students do not have the same natural predisposition to the use of heuristics. In a problem solving diagnostic evaluation test the students who create diagrams, explore specific cases, experiment and generalise can be identified. These elementary discovery procedures are decisive for the heuristic instructional programme. How can we quantify heuristic procedures and extract useful information for subsequent studies?

Bearing in mind students' individual differences, two distinct research approaches emerge: a cognitive correlation in which a general capacity is assessed by problem solving performance; and the evaluation of two groups of students, apprentices and experienced. Skill acquisition has in the mathematics classroom an important and almost infinite research laboratory.

Porto da Silveira (1999) classified students' proficiency as: *1) Inert*: individuals have no knowledge, or residual knowledge, and are unable to decide where to start. They are only able to carry out elementary procedures which should be exhaustively described by the teacher. Students with this profile are confined to solving profusely illustrated exercises which then will be routinely reproduced; *2) Imitator*: after an explanation period, individuals are competent to solve exercises but not able to work independently when difficulties arise. Individuals in this category would beneficially participate in a working group to debate problem solving activities but they are unable to do it by themselves; *3) Capable*: individuals have the ability to solve problems, since they are relatively simple variants of others that they have learned to solve; *4) Advanced*: individuals reveal intelligence and problem solving accuracy in complex and diversified questions. They are able to devise different resolutions from those which have been presented by the teacher; *5) Artist*: higher stage only reached by problem solving expertise capable to engage in more effective methods, alternative paths and elegant solutions.

Differences between exercise practice and problem solving (Chart 3)

EXERCISE ≠	PROBLEM
What is asked and the way to reach the solution are easy to understand.	What to do and frequently the statement of the problem are hard to understand.
Teachers want students to mechanically apply previously acquired and easily identifiable knowledge and routines.	Teachers want students to use intuition, analyse, research, and operate their set of knowledge and previous experiences so as to undertake resolution strategies.
Exercise resolution requires little time to solve.	In general, problem solving requires a period of time that is difficult / impossible to estimate.
There is no fondness in solving an exercise.	Problem solving requires a strong affection investment. In the course of the activity the solver usually explores feelings such as anxiety, trust, frustration, enthusiasm, joy…
Typically, exercises are closed questions.	Open questions that allow variations and generalisations.
Schoolbooks have plenty of exercises.	Schoolbooks have few problem suggestions.

Mathematics Teaching in Portugal (from *Estado Novo* to the year 2000)

In Portugal, throughout the *Estado Novo* period, the mathematics teaching methodology was crystallised in the Single Book, a key object to teaching and learning which focused on mechanisation and memorisation procedures, with the prevalence of computation skills and learning without understanding. By then, the educational system was divided into Primary Education (4 years), First Cycle (1st and 2nd grade), Second Cycle (3rd, 4th and 5th grade) and Third Cycle (6th and 7th grade).

Primary Education deserved special attention from the government, so as to reduce illiteracy by instructing a greater number of children, but without unleashing new social expectations and by curtailing the effects of a hypothetical use of school investment as a factor of social mobility. The purpose of Primary Education was the preparation of the child for experiencing the place that each one should occupy in the social order by an integrative dimension, which promotes moral cohesion and hierarchy respect (Rosas & Brito, 1996).

> *Leave behind encyclopaedic orientation (...) attend to the careful analysis of this teaching and the role that it has in life (...) sometimes - as much as we hate to say it – Primary Education, in our country, is torture for youngsters, and in many cases, a bizarre paradox, and an element to lure illiteracy. Complex study plans were organised in such a way that those programmes could be implemented for all ages except those who are between seven and ten years old - and the result of this guidance was: under the physical point of view, forcing the child to exhaustion with a futile or excessive effort; under the cultural point of view, giving multiple knowledge a superficiality character that caused the greatest confusion; and, under the moral point of view, creating such an aversion concerning school which was, unfortunately, the most dangerous element in favour of the increase of illiteracy.*[7]

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[7] Author's translation: *Abandona-se resolutamente a orientação enciclopedista (...) desce-se à análise cuidada da natureza deste ensino e da função que ele afinal tem a preencher na vida (...) É que por vezes - por mais que nos custe dizê-lo – a instrução primária tem chegado, no nosso País, a ser um verdadeiro martírio para a mocidade e, em muitos casos, a constituir, como que em bizarra exemplificação de um paradoxo, um elemento de aliciamento a favor do próprio analfabetismo. Organizaram-se planos de estudo complexos, constituíram-se com o carácter de obrigatoriedade programas de tal modo complicados que podiam ser exigíveis a todas as idades, menos àquelas que se delimitam entre os sete e os dez anos - e o resultado desta orientação foi: sob o ponto de vista físico, fatigar-se a criança com um esforço inútil ou excessivo; sob o ponto de vista cultural, dar aos seus pretensos multíplices conhecimentos um carácter de superficialidade tão grande que dele logo brotava a confusão maior sobre tudo; e, sob o ponto de vista moral, criar um tal espírito de antipatia e de animadversão pela escola, que constituía infelizmente, no fundo, o mais perigoso elemento a favor do aumento do analfabetismo.* Assembleia Nacional – Diário das Sessões – Diário nº 175, 24-03-1938, pp. 546-554.

The main goal of Primary Education was to teach reading, writing and counting. Such skills were considered sufficient for the individual's social and professional future. Instruction foundations and illiteracy eradication were, according to Congressman Juvenal Aguilar, a simplification of content and procedures.

Educational methodologies applied to subsequent educational school levels were also far from being successful. In a study on computation efficiency of First Cycle students (currently 6th grade), published in *Gazeta de Matemática*, Maria Teodora Alves (1947), referred mediocre performances. The test, with 50 questions divided into 9 categories, required the simplification of the mathematical expressions:

$$2 - 3 - 4 + 7 \qquad\qquad 9 - 2 + 5 - 4$$

Despite the rather low difficulty level for students of this educational level, results show 76.75% of wrong answers. In conclusion, the author writes that students reveal *"serious deficiencies"* (p. 16) in maths computational skills.

The Mathematics curriculum of the Second Cycle (3rd, 4th and 5th grade), in compliance with Law 37112 of 22nd October 1948, was materialised in the referred teaching manual:

> *Algebra Compendium, in one volume, for the 3rd, 4th and 5th grade; Geometry Compendium, in one volume, for the 3rd, 4th and 5th grade. In each chapter, textbooks should offer application exercises, ordered by difficulty degree, with the respective answers. As often as possible, in the Geometry Compendium theorems should be immediately followed by questions, in straightforward graphical or numerical problem presentation or easily deductible theoretical questions. Particular attention should be paid to the graphic look of textbooks, especially in the Geometry Compendium*[8] (GM, 1948, p. 26).

In 1946, in the National Assembly, a proposal to reform Technical and Vocational Education was discussed. In 1925-26, official numbers registered an attendance of 11,272 students, and in 1944-45 the number rose to 36,115 students. Courses were taught at *Industrial and Commercial School* to the sons of middle-class families.

[8] Author's translation: *Compêndio de álgebra, em um volume, para os 3º, 4º e 5º anos; Compêndio de geometria, em um volume, para os 3º, 4º e 5º anos. Em cada capítulo os compêndios deverão apresentar exercícios de aplicação, dispostos segundo ordem crescente de dificuldade, com as respetivas respostas. No compêndio de geometria, e sempre que tal seja possível, os teoremas deverão ser imediatamente seguidos de questões propostas aos alunos, quer sob a forma de pequenos problemas, de natureza gráfica ou numérica, quer sob a forma de questões teóricas de fácil dedução. O aspeto gráfico dos compêndios, principalmente de geometria, deve merecer especial atenção.*

> *Some new industries are already being organized, are already being established, and it turns out that to recruit technical staff, especially factory workers, for these new industries the country struggles with serious difficulties, because we do not have enough workers who are qualified to operate with the complexity that these new industrial facilities require.*[9]

To enlighten mathematical instruction, gifted mathematicians joined the *Movimento Matemático Português*, where Bento de Jesus Caraça was one of the most dynamic experts. Aware of the flaws of the Portuguese education system, he advocated, among other practices, the use of technological instruments in the classroom.

Pires de Lima, Minister of National Education (1947-55), brought further change. He established compulsory education for children between 7 and 13 years old, reinforced social support in schools and set up adult educational programmes - *Plano de Educação Popular, Cursos de Educação de Adultos e da Campanha Nacional de Educação de Adultos*. This was the starting point for the renovation of the Portuguese educational system. The time required to fully implement this reform only generated substantial results in the course of the following decades. Despite the efforts to reduce illiteracy, in the 1960s, Portugal still exhibited a school attendance inferior to 50% among the population between 5 and 14 years old. From the 1950s on, the need to reorganise education was obvious. In 1973, José Veiga Simão (1929 – 2014), Minister of Education (1970-74), presents a global educational reform by promoting kindergarten education, extending compulsory education, reorganising Secondary Education and diversifying the offer in Tertiary Education (Carreira, 1996). Until then, low rates in University admission exams and high fees made it only accessible for the most affluent classes.

The ideals of *Modern Mathematics*[10], introduced in the early 1960s and promoted by the *Bourbaki Society*, were shortly after abandoned, but, in Portugal, in the 1970s, the curriculum was still written under those guidelines.

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[9] Author's translation: *Algumas indústrias novas estão já na fase de organização, estão já a ser montadas, e verifica-se que para recrutar pessoal técnico, sobretudo pessoal operário, para essas novas indústrias se luta com sérias dificuldades, porque não temos, na realidade, abundância de artistas, abundância de operários que possam vir a trabalhar com a complexidade que as instalações dessas novas indústrias requer para serem postas a funcionar.* Assembleia Nacional – Diário das Sessões – Diário nº 81, 30-01-1947, pp. 419-436.

[10] International movement, with a golden era between 1955 and 1975, whose purpose was to produce Literature and introduce Logic, Statistics, Probabilities and Combinatory in the curriculum. Its implementation was supported in Algebra and Geometry axioms to teach contents with more rigour than before.

> *We need to fight the excessive number of exercises that, like cancer, eventually destroys what could be noble and vital in teaching. We must avoid certain artificial or intricate exercises, especially in simple matters. (…) It is more important to allocate time and resources to an interesting question, than solving many different exercises, which have no interest. (…) Among the exercises that may have greater significance are those that can be applied to real-life situations*[11] (Silva, 1965-66).

In 1986, a new educational regulation brings fresh curricular procedures and, in 1991, new programmes of studies. After an experimental design, in 1993, these programmes were finally implemented in Portugal. A more participative intervention of students in the learning process was suggested, as well as more problem solving activities, group work, emphasis on reasoning, references to the History of Mathematics and the use of calculators in the classroom. The excessive length of the Mathematics programme of studies led to an amendment in 1997.

[11] Author's translation: *É preciso combater o excesso de exercícios que, como um cancro, acaba por destruir o que pode haver de nobre e vital no ensino. É preciso evitar certos exercícios artificiosos ou complicados, especialmente em assuntos simples. (…) É mais importante refletir sobre o mesmo exercício que tenha interesse, do que resolver vários exercícios diferentes, que não tenham interesse nenhum. (…) Entre os exercícios que podem ter mais interesse figuram aqueles que se aplicam a situações reais, concretas.*

Programme for International Student Assessment (PISA)

The *Organisation for Economic Cooperation and Development* (OECD) created in 2000 the *Programme for International Student Assessment* (PISA) to evaluate knowledge and skills of 15-year-old students. With triennial periodicity, it aims to assess literacy in three main areas: reading, mathematics and science, considering mathematical literacy as the

> *…individual's capacity to formulate, employ and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts, and tools to describe, explain and predict phenomena. It assists individuals in recognizing the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens* (PISA, 2014, p. 37).

Data gathered from this evaluation allows each country to assess results in order to promote a more effective teaching and learning process. Due to the significance of the project, we reproduce a fragment of a foreword signed by Angel Gurría, the OECD Secretary – General with reference to the 2012 PISA results.

> ***The results from the PISA 2012 assessment, which was conducted at a time when many of the 65 participating countries and economies were grappling with the effects of the crisis, reveal wide differences in education outcomes, both within and across countries.*** *Using the data collected in previous PISA rounds, we have been able to track the evolution of student performance over time and across subjects. Of the 64 countries and economies with comparable data, 40 improved their average performance in at least one subject. Top performers such as Shanghai in China or Singapore were able to further extend their lead, while countries like Brazil, Mexico, Tunisia and Turkey achieved major improvements from previously low levels of performance.*
>
> ***Some education systems have demonstrated that it is possible to secure strong and equitable learning outcomes at the same time as achieving rapid improvements.*** *Of the 13 countries and economies that significantly improved their mathematics performance between 2003 and 2012, three also show improvements in equity in education during the same period, and another nine improved their performance while maintaining an already high level of equity – proving that countries do not have to sacrifice high performance to achieve equity in education opportunities.*

Nonetheless, PISA 2012 results show wide differences between countries in mathematics performance. *The equivalent of almost six years of schooling, 245 score points, separates the highest and lowest average performances of the countries that took part in the PISA 2012 mathematics assessment. The difference in mathematics performances within countries is even greater, with over 300 points – the equivalent of more than seven years of schooling – often separating the highest- and the lowest-achieving students in a country. Clearly, all countries and economies have excellent students, but few have enabled all students to excel.*

The report also reveals worrying gender differences in students' attitudes towards mathematics: *even when girls perform as well as boys in mathematics, they report less perseverance, less motivation to learn mathematics, less belief in their own mathematics skills, and higher levels of anxiety about mathematics. While the average girl underperforms in mathematics compared with the average boy, the gender gap in favour of boys is even wider among the highest-achieving students. These findings have serious implications not only for higher education, where young women are already underrepresented in the science, technology, engineering and mathematics fields of study, but also later on, when these young women enter the labour market. This confirms the findings of the OECD Gender Strategy, which identifies some of the factors that create – and widen – the gender gap in education, labour and entrepreneurship. Supporting girls' positive attitudes towards and investment in learning mathematics will go a long way towards narrowing this gap.*

PISA 2012 also finds that the highest-performing school systems are those that allocate educational resources more equitably among advantaged and disadvantaged schools and that grant more autonomy over curricula and assessments to individual schools. *A belief that all students can achieve at a high level and a willingness to engage all stakeholders in education – including students, through such channels as seeking student feedback on teaching practices – are hallmarks of successful school systems.* (pp. 3-4).

School and Society in the new millennium

Problem solving is a major vector for the teaching and learning of Mathematics which has repercussions in other performances. The *Portuguese Mathematics Olympiad*[12] (PMO) organised by *Sociedade Portuguesa de Matemática* (SPM), and *Canguru Matemático sem Fronteiras*[13], coordinated, in Portugal, by the University of Coimbra - Department of Mathematics, are events devoted to mathematical problem solving, but with different designs. Regarding *Canguru Matemático sem Fronteiras (Table 3)*, there is a significant decrease in the number of participants as we go from *Benjamin* (7th and 8th grade) to Cadet (9th grade) categories. The 9th grade is a pivotal moment in students' educational path. This trend allows us to speculate about the causes of such apparent, or not so apparent, students' lack of interest to participate in this contest. Are they so completely focused on accomplishing Third Cycle studies and on the national final exam in the month of June that they do really not find this competition taking place in the month of March appealing? We don't sustain such point of view.

Number of participants in the Canguru Matemático sem Fronteiras (Table 3)

Year/Category	Scholar Mini I (2nd grade)	Scholar Mini II (3rd grade)	Scholar Mini III (4th grade)	Scholar (5th and 6th grade)	Benjamin (7th and 8th grade)	Cadet (9th grade)	Junior (10th and 11th grade)	Student (12th grade)
2015	13089	14444	13079	26231	14064	4943	2752	535
2014	11555	14344	13037	27623	15440	5878	3040	570
2013	10980	11903	12152	26909	16589	5535	3095	447
2012	9148	10967	13067	31323	17683	6124	2725	697
2011			10873	33913	20538	7004	3363	635
2010				35625	21225	6442	3395	729
2009*				33653	20369	6577	3696	786
2008				32760	19901	7308	3399	638
2007				30686	22513	7962	4272	790
2006**								

* Students from other ten schools participated in the competition but the quantification of all the participants was not possible.

** The available data only report the best classified students of each school and the best twenty by category nationwide.

[12] Problem solving competition for students, from the 1st to the 12th grade, which aims to stimulate mathematical awareness. The PMO is divided into three phases: *1)* the first qualifying round takes place in all the schools that wish to participate and is open to all students; *2)* the second qualifying round is the district final and takes place in some schools throughout the country with students selected according to PMO rules; *3)* the national final, which is organised in a school by invitation of the PMO; thirty students selected according to PMO rules participate in each category. One of the main objectives of this competition is to identify early scientific vocations in Mathematics. Many PMO winners started successful scientific careers.

[13] Competition which promotes mathematics among youngsters and is open to all students. The goals are to stimulate the taste for mathematics; to attract students who have negative beliefs regarding this school subject by allowing them to discover the fun in mathematics work; to increase the national number of participants in each level. This competition consists of a single test.

Teachers find it more difficult to persuade 9th grade students to participate than younger students. Signs of the importance of prescribing significant mathematical contents in this specific school year are therefore clear. Working problem solving heuristics can foster students' motivation and help their performance, allowing them to keep a bond with Math.

In general, Mathematics promotes cognitive development, particularly the ability to solve problems. The curriculum and subsequent teaching suffered deep transformations as a result of the political, social and economic changes which took place in Portugal in the recent decades. Compulsory education enhances the construction of infrastructures to accommodate a growing number of students, the introduction of new technologies and teacher training in Educational Centres and Universities. The improvement in the quality of Education should be translated into students' skills; however, internal and external evaluation quantified by results of national exams consistently unveils scarce outcomes. Progress, in general, has occurred. The Census 2011 (INE) *(Table 4)* should stimulate each educational participant to optimise procedures in order to train youngsters to the challenges of the 21st century.

Census (2001 / 2011) (Table 4)

SCHOLARSHIP RATES IN PORTUGAL	Year 2001	Year 2011
Illiteracy rate	9.03%	5.23%
Children aged 3 to 5 attending kindergarten	52.28%	73.49%
Individuals aged 15 or more who did not complete any scholarship level	18.03%	10.39%
Youngsters aged 18 to 24 who at the most have completed nine school years and are no longer in school	33.37%	22.08%
Youngsters aged 20 to 24 who completed at least Secondary Education	44.04%	60.80%
Ratio of people aged 30 to 34 who completed Tertiary Education	14.22%	28.62%
Ratio of people aged 15 or more who completed at least the 9th grade	37.95%	49.60%
Ratio of people aged 18 or more who completed at least Secondary Education	22.67%	31.69%
Ratio of people aged 23 or more who completed Tertiary Education	8.81%	15.11%

The need to better understand the significance of heuristic procedures as a facilitating agent of success in mathematical learning and an instrumental approach which may be useful for teachers and/or students emerge from this literature review. The importance of Time in some mathematical breakthroughs but also, and even more relevant, the significance of the future in students' perceptions towards Mathematics and subsequently the impact of such perceptions in future curricular choices must also be illustrated.

Time Perspective and Vocational Courses

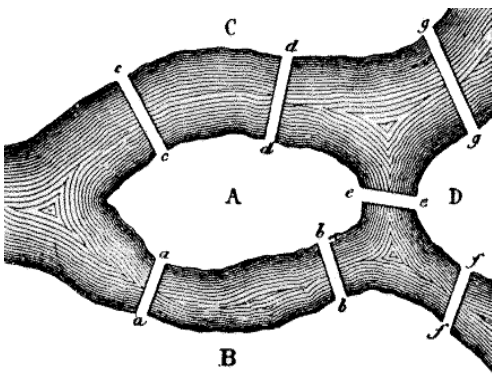
A previous approach embracing maths problem solving and school options

Progress is built with understanding and, therefore, Mathematics is always under construction. As more levels are added to the scientific knowledge we come into a position where we can see further and acquire a better consciousness of the great beyond. If the tools used in this endeavour have become more sophisticated and efficient, it is also true that the effort made to place one more brick in the house of knowledge is undoubtedly greater. Time effort is, in the course of History, undoubtedly attached with maths problem solving.

Spreading along Pregel River banks, Konigsberg city was the centre of a puzzling problem in the 18th century. For better mobility, seven bridges were built in the city and, since then, its inhabitants wondered about the possibility of crossing them all without recurrence. They tried to accomplish that goal but without success. Unable to find a solution, the mayor of Danzing asked Leonhard Euler (1707 – 1783), who lived in St. Petersburg, for help. The answer came swiftly.

> *Thus you see, most noble Sir, how this type of solution bears little relationship to Mathematics, and I do not understand why you expect a mathematician to produce it, rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on any mathematical principle. Because of this, I do not know why even questions which bear so little relationship to Mathematics are solved more quickly by mathematicians than by others* (Hopkins & Wilson, 2004).

Despite Leonhard Euler's words, he realized that his reasoning, presented to the St. Petersburg Academy on August 26th, 1735, and originally published in 1741 *(Figure 11)*, was a breakthrough in Maths, originating a new branch which, today, is known as the Graph Theory.



(Figure 11)

To explore the *Konigsberg Seven Bridges Problem* let's engage into George Pólya (1888 - 1985) four-step problem solving model.

What does the problem say?

It challenges the reader to travel along Konigsberg seven bridges without repetition, establishing a network between banks and two islands in the river bed.

Is there more available information regarding the statement of the problem?

Yes. The 18th century Konigsberg city map.

Are there restrictions concerning the place where the path should start?

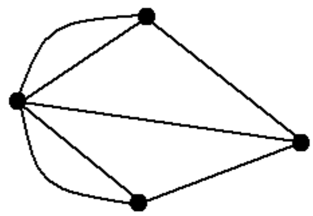
No. The travel can begin on the coast or in any one of the small islands.

After some attempts is the pathway revealed?

No. Under problem restrictions it appears that one bridge is always missing.

How can we represent the problem?

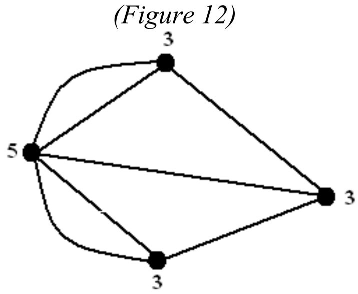
North, South and both islands are symbolised by points and each bridge by a line *(Figure 12)*.



(Figure 12)

Does the diagram exhibit data that have not yet been analysed?

Yes. It allows identifying, in a clear way, the number of bridges connected to each point *(Figure 13)*.



(*Figure 13)*

To arrive and depart, the place must be coupled by an even number of bridges. If all dots are connected to an even number of segments, then the tour can begin and end at the same place (circuit). If all dots are linked to an even number of segments, except two, the route (path) is also possible, but now it starts and ends at a place with an odd number of segments.

Is a circuit achievable?

No, to accomplish that goal all locations should be linked to an even number of bridges.

Is a path possible?

No, because to accomplish that goal all locations, with the exception of the departure and arrival, should be connected with an even number of bridges.

The problem has no solution! What happens if one bridge is withdrawn?

Regardless of the bridge subtracted it is always possible to complete a path.

The *Konigsberg Seven Bridges Problem* was quickly solved but in Maths that does not always happen. Pierre de Fermat, in his copy of the book *Arithmetic* from Diophantus, wrote, around 1637, in the margin of a page, next to the problem of the sum of squares, that $x^n + y^n = z^n$, with n greater than two and x, y, z, n positive integers, did not have a solution. Unfortunately, he could not write the proof there due to lack of space.

Later he did not revisit the subject again. After Pierre de Fermat's death in 1665, his son, Clément-Samuel Fermat, published in 1670 a new edition of *Arithmetic* with his father's notations. For more than three hundred years, professional and amateur mathematicians tried to demonstrate Pierre de Fermat's statement, all without success, despite some major progress made in the 19th and 20th century. Then, on 1993, Andrew Wiles announced at the Conference of Sir Isaac Newton Institute for Mathematical Sciences, in Cambridge, the most wanted proof to *Fermat's Last Theorem*, so called because all the remaining observations of the French mathematician had already been decided or demonstrated. Later, inaccuracies were found, but Andrew Wiles did not give up and returned to work on his proof, accomplished his goal together with his student Richard Taylor, then published on 1995 Annals of Mathematics.

> *Perhaps I could best describe my experience of doing Mathematics in terms of entering a dark mansion. Because, when one goes into the first room and it's dark, completely dark one stumbles around bumping into the furniture, gradually you learn where each piece of furniture is, and finally after six months or so you find the light switch, you turn it on, suddenly it's all illuminated, you can see exactly where you were* (Singh & Lynch, 1996).

For n equal to one there is an infinity number of solutions. Replacing n by two, it is still possible to solve the problem. Solutions (x, y, z), such as (*3*, *4*, *5*), are called Pythagorean Triple, numbers that fulfil the equation $x^2 + y^2 = z^2$. *Fermat's Last Theorem* is an expansion of this problem with n integer, greater than two.

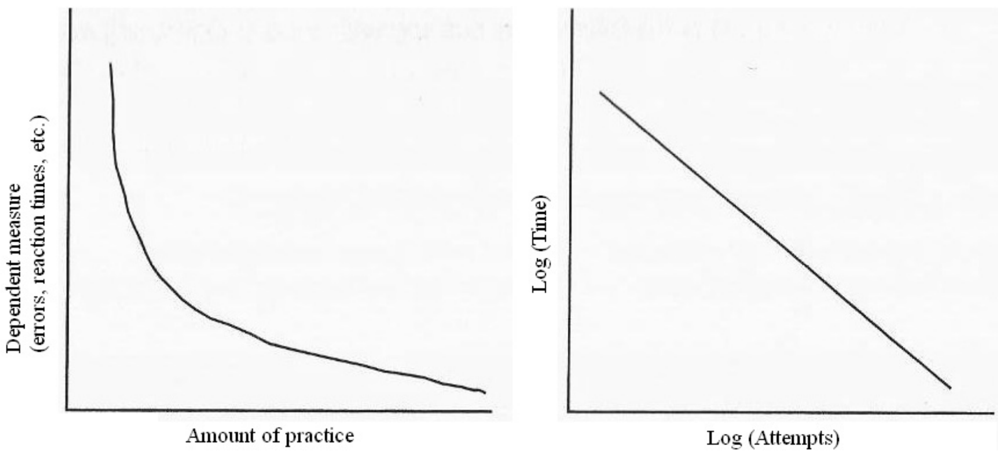
> *I never use a computer. I sometimes might scribble, I do doodles, I start trying to find patterns really, so I'm doing calculations which try to explain some little piece of Mathematics and I'm trying to fit it in with some previous broad conceptual understanding of some branch of Mathematics. Sometimes that'll involve going and looking up in a book to see how it's done there, sometimes it's a question of modifying things a bit, sometimes doing a little extra calculation, and sometimes you realize that nothing that's ever been done before is any use at all, and you just have to find something completely new and it's a mystery where it comes from* (Singh & Lynch, 1996).

Since introduced to *Fermat's Last Theorem* at the age of ten, Andrew Wiles was amazed with the statement and his open ended demonstration, *"I had this very rare privilege of being able to pursue in my adult life what had been my childhood dream"* (Singh & Lynch, 1996). We can argue about how to unlock hard problems. The master key is to study with method, acquire experience with time; only then is it possible to become an expertise in problem solving.

Today, the *Goldbach's Conjecture* is one of the greatest mathematical challenges. Until now no one has been capable to prove that all even numbers greater than two can be written as sums of two prime numbers. The quest began in 1742 with a letter from Christian Goldbach (1690 - 1764) to Leonhard Euler, and despite important steps forward, it remains open. Resilience is essential to overcome difficulties, as control, beliefs, resources and heuristics conspire fortuitously to establish connections between different issues.

Theoretical models put into practice give us a better understanding of the complex mechanisms engaged to produce learning.

Power Law of Learning: $T = cP^x$, *where T is time taken to perform a task, c is constant, P is the amount of practice, and x represents the slop of the line (Figure 14)* (Newell & Rosenbloom, 1981).



A hypothetical practice curve (left) and the same curve represented as a logarithmic scale (right) *(Figure 14)*

After this brief digression through the History of Mathematics concerning problems with different temporal horizons demanding specific knowledge and reasoning, complex psychological activities and resilience, we anticipate that structural changes will only come with Time. Although Problem Solving is shaped by metacognition processes that can be self-regulated, thus allowing autonomy and learning, it is also true that cognitive achievements are favoured by motivational aspects (Mayer, 1998). The conceptions, attitudes, and expectations of the students regarding Mathematics and Mathematics teaching are considered to be very significant for their school experience and achievement (Schoenfeld, 1985).

Are students hopelessly fond of school? How to rebuild hope in school with the frustration of being forced to learn *irrelevant* Maths?

Substantial research on Future Time Perspective (FTP) and academic achievement reported that students with high Grade Point Averages (GPA) were characterised by a long FTP (Davids & Sidman, 1962), (Epley & Ricks, 1963), (Goldrich, 1967), (Klineberg, 1967), (Lessing, 1968), (Teahan, 1958) and (Vincent & Tyler, 1965). However, Lessing (1972) found that a more extended FTP could, eventually, correspond to wish fulfilling fantasy.

Motivation can play an important role in instigating and sustaining an action such as studying. The strength of motivation is a function of the emotional force of the goals towards which the action is related (eventually aimed). The instrumental value of an action can be quantified as the difference between the subjective probability of reaching the goal when the action is performed, and the subjective probability of not reaching the goal when the action is not performed, de Volder (1980). Despite this straightforward arithmetic, behaviour is quite more intricate and actions that have instrumental value for reaching a certain anticipated goal may eventually have negative instrumental value (when the probability of reaching a goal is higher if a certain act is not performed than when it is performed) for reaching other goals (e.g., studying for long periods of time can compromise achievements in sports or studying a single school subject for long periods of time can compromise success in other school contents). Considering only motivational goals, comprehending that the valence of an objective decreases with increasing temporal distance to that goal (Mischel, 1981) is quite natural. Another correlation is: the lower the anticipated subjective value of a goal, the lower the strength of motivation for that goal.

> *For some people, the world is limited to all the forces they perceive in their immediately present situation, their biological urges, their social setting and that which others are doing or urging them to do, and the sensuous appeal of the stimulus itself. Those folks who usually limit their decision-making by referring only to the current circumstances are Present-oriented. Other people making a decision in the same setting downplay the present and search their memories for similar past situations; they recall what they did in the past and how these decisions turned out. These folks are Past-oriented. Finally, a third type of person makes up her or his mind entirely based on imagined future consequences – the cost and benefits – of an action. If anticipated costs outweigh anticipated benefits or gains, they won't go forward. They only go forward when they expect gains to predominate* (Zimbardo & Boyd, 2009, p. xiii).

Motivation is a critical factor in the school performance of the students, with strong correlations regarding future time perspectives. Hence, it is important to understand how the Portuguese school system embraces students who, after completing the Third Cycle, are not very fond of school and evaluate mathematical contents as irrelevant for their future.

In Portugal, Secondary Professional Courses grant a level 4 certificate in conformity with the European Qualifications Framework (EQF). In the author's view, the Maths programme for these courses is highly unsuitable. In fact, it is a simplification of the regular Secondary School curriculum and hence unattractive to students (ages ranging from fifteen to twenty one years old, and mainly dropouts from the regular courses) who seek the straightest way to start a career.

The same design is used for a very different range of courses, independently of the 100 or 300 hours of Maths lessons planned for the three years. Tackling the same issues with students who desire to initiate jobs as different as cook or environmental technician seems to be unsuitable work. Consequently, classes are noisy and badly behaved, a widely spread opinion among the teaching community. In the last decade these courses became an option to an increasing number of students. Classes of about a dozen students would be desirable so that the teacher could work individually with each one; today, classes of up to thirty learners are common in Secondary Education.

In the past, adolescents aged 15-19 contributed to the big dropout rate in Portugal, compared with other countries (Ferreira & de Lima, 2006). To solve this long term problem, in the 1990's, Professional Schools started to flourish, first with little feedback but recently they became an option to a significant portion of the school population. What mathematics is it suitable to teach? In general, the available textbooks are simple adaptations from the standard curriculum that does not correspond to the Mathematics these students will one day use, and consequently it does not capture their attention. We were reminded of the words of the mathematician and problem solver George Pólya (1887 – 1985), who stated that a teacher who only makes routine exercises jeopardises annihilating the student's interest and potential.

More time allocated to problem solving work in the maths classroom could contribute to diminish school failure rates, for which, in Portugal, Mathematics is considered the main cause, by already enriching the critical 9th grade maths curriculum with a problem oriented approach. In other words, the rationale is: if students had been challenged early enough by problems, they would have a less passive attitude towards Mathematics. The aim is to engage students into Mathematics; otherwise they will miss know-how which will be essential to further their studies and professional careers.

Children start official school at the age of six (even though before that age many attend kindergarten education), then reach a second level at the age of ten and a third level at the age of twelve. After nine years of education, *Lower Secondary* students engage in Secondary Education, a three-year period of scientific-humanistic courses. For some students it will be the end of the journey, twelve years of compulsory education *(Chart 4)*.

Compulsive Education (Chart 4)

<table>
<tr><th colspan="16">Age</th></tr>
<tr><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td><td>11</td><td>12</td><td>13</td><td>14</td><td>15</td><td>16</td><td>17</td><td>18</td></tr>
<tr><td colspan="3" rowspan="2">Optional Frequency</td><td colspan="13">Compulsive School</td></tr>
<tr><td colspan="10">Lower Secondary</td><td colspan="3" rowspan="2">Secondary Education</td></tr>
<tr><td colspan="3" rowspan="2">Kindergarten</td><td colspan="5">First Cycle</td><td colspan="2">Second Cycle</td><td colspan="3">Third Cycle</td></tr>
<tr><td colspan="5">1st to 4th grade (4 years study)</td><td colspan="2">5th and 6th grade (2 years study)</td><td colspan="3">7th to 9th grade (3 years study)</td><td colspan="3">10th to 12th grade (3 years study)</td></tr>
</table>

We will focus on the answers that the Portuguese Education System offers to teenagers who do not follow a traditional educational path. In order to understand the current framework, we must go back to 1986, when Portugal became member of the European Economic Community (EEC). Structural changes were implemented in all domains, including Education. At that time students attended nine years of compulsory schooling. In the 1990s a new concept of school, Professional Schools, started to flourish. At first, it was a model developed and promoted by private institutions to educate students after they had completed compulsory education. The objective was to fill the need of qualified technicians in a wide range of jobs according to society demands.

In their early years, Professional Schools did not attract students' interest mainly for two reasons: courses were outside the range of traditional education and expectations to outcompete students that only had compulsory education were low. It was no coincidence that the Portuguese Government engaged in a partnership with Portuguese-speaking African countries, mainly Cape Verde and São Tomé and Príncipe, to allow students from those countries to take this educational path in Portugal. In the first decade of the new millennium this window of opportunity allowed many private institutions to have students and consequently obtain European funds, first subsidised by *Programa de Desenvolvimento Educativo para Portugal* (PRODEP), and, since 2007, by *Programa Operacional Potencial Humano* (POPH). The Stability and Growth Programme for 2002-2005 - in the chapter dedicated to the development of public education - among several goals, ascertains the need

> *...to offer kindergarten school education to 75% of the kids, to start secondary education courses with emphasis on technological literacy and to start professional courses (in the next two years); accomplish curricular revision of secondary education and develop a professional curriculum for the 10th grade to suit market needs and students' interests and to avoid early school leavers* (2001).

Profound changes started to happen. To accomplish European Union standards, Portugal needed to reduce the dropout rate and increase the number of students in Professional Courses. In 2010s compulsory education was extended to twelve years. This regulation sped up the process that was already on the way. Today Professional Courses are quite popular and in a near future the government aims to have 50% of secondary students attending this educational path. Public Schools started offering Professional Courses at a large scale and private schools ended up with fewer students and consequently shorter funds.

To access Professional Courses, students need to succeed in the third school level (9th grade), but many do not achieve that goal. For those who, until the 6th grade, have failed two consecutive years in the same cycle or three years in different cycles, one or two years Lower Level Vocational Courses (1100 hours), according to each student's profile, were created. For those students of ages thirteen (or more) the modular framework is as follows *(Table 5)*:

Lower Level Vocational Courses modular framework (Table 5)

Formative Areas	Allocated Time $(60')$
General	
Portuguese Language	110
Mathematics	110
English	65
Physical Education	65
Complementary Subjects	
History / Geography	65
Natural Sciences / Physics	65
Second foreign language	50
Vocational	360
Simulated Practice (Work context)	210

After undergoing an evaluation process conducted by school psychologists, students can be advised to engage in this educational path. But this recommendation can be questioned. Should a student who is unsuccessful for two consecutive years in the first cycle or in the 5th grade but who has overcome their difficulties be led to this path after successfully completing the 6th grade? Shouldn't curricular programmes be consistent with the goals of each course? Do not these guidelines prevent many students who are more practically oriented to begin a path that is labelled with failure? Will parents put their thirteen-year-old children, who are willing to learn a trade after completing the 6th grade, in classrooms where the majority of the students are problematic fifteen-year-olds or even older? Will companies be interested in employing workers with such a profile? Lower Level Vocational Courses will allow students to complete three years of studies, from the 7th to the 9th grade, and replace similar programmes such as *CEF*[14] and *PCA*[15]. In the 10th grade those students could – if they succeed in Maths and Portuguese Language exams at the end of the 9th grade – restart the regular curricular path. Even though, after the conclusion of a Lower Level Vocational Course, the ordinary route is to start a Professional Course, regrettably, at the age of eighteen, many students leave school without finishing the 12th grade. In the 2014/2015 school year, in order to reduce the dropout rate, students aged sixteen (or more) may choose Secondary Vocational Courses (3000 hours) with a framework geared towards the society needs *(Table 6)*. In two years they can finish Secondary Education.

Curricular Framework for Secondary Vocational Courses (Table 6)

Formative Areas	Allocated Time $(60')$	
	1st Year	2nd Year
General		
Portuguese Language	150	150
English (conversation)	90	90
Physical Education	60	60
Complementary Subjects		
Applied Mathematics	90	90
Subjects offered by the School	60	60
Vocational	350	350
Formative Period of Training	700	700

The Department of Education and Science started in September 2012 a school experiment with thirteen Vocational Lower classes (285 students) spread throughout Portugal. Information has been gathered (209 students, 13 from the 2nd Cycle and 196 from the 3rd Cycle) and results analysed.

[14] *Cursos de Educação e Formação*.
[15] *Percursos Curriculares Alternativos.*

According to the *Vocational Evaluation Report*[16] (2013), the mean age of a 2nd Cycle student at the beginning of this path is 13.85 years *(Table 7)*. If we consider that the expected average age is 10 or 11 years old, we find a significant age variation, thus suggesting a prior school path with many adversities.

Vocational 2nd Cycle students' age (Table 7)

Age	Attendance	Proportion
13	6	46.2%
14	5	38.5%
15	1	7.7%
17	1	7.7%
Total	13	100%

Statistics regarding the 3rd Cycle show that the mean age of students at the beginning of these courses is 15.3 *(Table 8)*, when they are expected to begin this cycle at the age of 12. Again, we perceive a considerable age difference.

Vocational 3rd Cycle students' age (Table 8)

Age	Attendance	Proportion
13	10	5.1%
14	36	18.4%
15	65	33.2%
16	46	23.5%
17	28	14.3%
18	4	2.0%
19	1	0.5%
Invalid Data	6	3.1%
Total	196	100%

Concerning parents' schooling *(Table 9)*, no substantial gender differences were found. The majority of the parents have only compulsory schooling.

Parents' schooling (Table 9)

	Attendance (Father)	Proportion	Attendance (Mother)	Proportion
Did not attend school	1	0.6 %	0	0 %
1st Cycle (4th grade)	45	25.6 %	46	24.3 %
2nd Cycle (6th grade)	35	19.9 %	34	18.0 %
3rd Cycle (9th grade)	45	25.6 %	56	29.6 %
Secondary Education (12th grade)	37	21.0 %	44	23.3 %
Lower Tertiary Degree	9	5.1 %	5	2.6 %
Higher Tertiary Degree (Masters)	2	1.1 %	3	1.6 %
≥ PhD	2	1.1 %	1	0.5 %
TOTAL	176 (33 missing)	100 %	189 (20 missing)	100 %

[16] Committee members: Anabela Maria de Sousa Pereira; Cristina Santos Correia; Maria Isabel Ribeiro do Rosário Hormigo; Paulo Jorge de Castro Garcia Coelho Dias; Piedade Redondo Pereira; Ramiro Fernando Lopes Marques.

When questioned about the difficulties in their prior curricular paths, the students answered as follows: 42.3% did not like to attend school (46.1% were boys and 30.6% were girls), ($\chi^2 = 4.240, p < 0.05$); 19.1% did not like the teachers, with no significant differences between boys and girls, ($\chi^2 = 0.001, p > 0.05$); 10.5% did not like the school subject (10.6% were boys and 11.3% were girls), ($\chi^2 = 0.019, p > 0.05$); and 26.4% mentioned difficulties in accompanying the contents of the school subjects and low motivation because they do not feel accepted by their classmates. The main reason for choosing this educational path is the possibility to pursue their studies and only afterwards start a career *(Table 10)*. Despite a trajectory with years of failure, students understand the importance of acquiring further knowledge before starting a professional career.

What Would You Like to Learn in This Course?
*(Table 10) / *Multiple choise question*

	Frequency	Proportion*
Useful subjects to start a job	68	32.5%
Useful subjects to pursuit studies and then start a job	136	65.1%
Other Answers	13	6.2%

Answers are not associated to students' age ($\chi^2 = 3.822, p > 0.05$) or gender ($\chi^2 = 3.555, p > 0.05$). In the category *Other Answers*, 50% mentioned a willingness to study *"subjects to apply in daily life instead of subjects disconnected from reality"*.

When inquired about the possibility of return to the Regular Courses *(Table 11)*, the majority of the answers was "*No"*, due to a *"difficult, boring or not motivating"* educational path.

Pursuing Studies in Regular Courses
*(Table 11) / *10 are missing*

	Attendance	Proportion
Yes	40	20.1%
No	159	79.9%
TOTAL	199*	100%

We can argue about the advantages and disadvantages of these curricular paths but without them, students would have few alternatives and school would not meet those students' interests.

Now we will focus on Professional Courses, another alternative to complete Secondary Education. The Scientific Component of each Professional Course offers two or three subjects which allow for the completion of Secondary Educational level 4 in compliance with the European Qualifications Framework (EQF). Students develop knowledge, skills and aptitudes which enable them to acquire a set of competences focused on a branch of professional activities. For these students, developing competencies related to logical rules and symbols is not fundamental. Even though the teaching of computational tools is advisable, the essential mathematical learning must be at the level of the exploration of ideas regarding problem solving and applications. Still, empirical evidence shows that problem solving as only an instrument to teach and learn Mathematics is insufficient. Mathematical contents cannot be reduced to problem solving and it is not possible to become a mathematician only by practicing problem solving.

Research on chess expertise supports the idea that memorisation is the main factor to become a proficient player (De Groot, 1946). In fact, over time professional players learn to recognise thousands of game configurations and the best movements associated to each configuration (Simon & Gilmartin, 1973). Long term memory, a crucial factor in human cognition, is used to store big piles of connected complex data, crucial to work an inventory of problem solving, and not isolated, wide-ranging information. How to become a more proficient general problem solver is a question to be analysed further ahead.

As 2012/2013 statistics of Coimbra Public Secondary Schools show, classes of Professional Courses embrace a major part of the Secondary Education population *(Table 12)*.

Coimbra Public Secondary Schools (Table 12)

Secondary School	Classes of General Courses			Classes of Professional Courses		
	10th grade	11th grade	12th grade	1st Year	2nd Year	3rd Year
Escola Secundária Avelar Brotero, Coimbra	9	10	11	10	12	10
Escola Secundária D. Dinis, Coimbra	3	3	4	3	3	1
Escola Secundária D. Duarte, Coimbra	4	4	5	5	6	5
Escola Secundária Infanta D. Maria, Coimbra	9	9	9			
Escola Secundária Jaime Cortesão, Coimbra	2	2	2	3	3	5
Escola Secundária José Falcão, Coimbra	8	8	8	1	1	1
Escola Secundária Quinta das Flores, Coimbra	7	7	9	3	3	1

Nowadays, public schools offer a wide range of options regarding Professional Courses. However, *Escola Secundária Infanta D. Maria* does not follow this trend. Coincidently or not, the students from this school have consistently over time achieved, on average, the best public school results in Secondary National Exams. Mathematical significant knowledge acquisition is far from being the result of lucky chance.

Maths modular framework for Professional Courses respects the sequence A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10} and the respective syllabuses are established for all courses with 300 hours of lessons along the three years. The modular sequence A_2, B_2, A_3, A_7, A_6, A_9, A_{10} and respective syllabuses are fixed for all courses with 200 hours of lessons along the three years *(Table 13)*. The syllabus and modular sequence for courses with 100 hours of lessons is defined by the Mathematics department of each school according to the following criteria *(Table 14)*:

Maths Modular Courses Framework with 300 or 200 hours (Table 13)

Number	Designation	Allocated time (hours)
A_1	Geometry	36
A_2	Polynomial Functions	36
A_3	Statistics	27
A_4	Periodical Functions	36
A_5	Rational Functions	36
A_6	Variation Rate	27
A_7	Probability	21
A_8	Discrete Models	27
A_9	Growing Functions	27
A_{10}	Optimization	27
B_1	Periodical and Non-Periodical Functions	36
B_2	Computational Statistics	36
B_3	Functions Modelling	36
B_4	Linear Computation	30
B_5	Games and Mathematics	36
B_6	Geometrical Patterns	36

Maths Modular Courses Framework with 100 hours (Table 14)

Number	Designation	Allocated time (hours)	Precedence
A_1	Geometry	36	
A_2	Polynomial Functions	36	
A_3	Statistics	27	
A_4	Periodical Functions	36	
A_5	Rational Functions	36	A_2
A_6	Variation Rate	27	A_2 A_5 *or* A_2 B_1
A_7	Probability	21	
A_8	Discrete Models	27	
A_9	Growing Functions	27	A_2 A_5 *or* A_2 B_1
A_{10}	Optimization	27	A_2 A_5 A_6 *or* A_2 B_1 A_6
B_1	Periodical and Non Periodical Functions	36	A_2
B_2	Computational Statistics	36	A_3
B_3	Functions Modelling	36	
B_4	Linear Computation	30	
B_5	Games and Mathematics	36	A_3
B_6	Geometrical Patterns	36	A_1

Optional modules allow more suitable choices. The time allocated to each module can be adjusted if the total number of hours does not complete the global curricular plan.

This is a simplified version of the Mathematical curriculum for the general courses. Contents are apparently wide-ranging, but ignore Complements of Analytic Geometry, Complex Numbers and Combinatory. Concerning this last topic, a total absence of enumerative combinatory is observed, which should provide the framework to solve permutation, combination and partition problems. Pascal's triangle, a triangular array of the binomial coefficients was also withdrawn from the syllabus.

We understand that it is virtually impossible to teach this curriculum due to the lack of mathematical awareness shown by the majority of the students. Schoolbooks mainly present direct exercises, with plenty of illustrations such as drawings and pictures, but rarely offer more refined questions. To learn sophisticated Maths we need to understand and work with abstract contents. The main reason to teach this curriculum is a romantic view that students may choose to continue studies in Science Courses, at University, and must therefore pass the *Mathematics A* national exam as regular students. Another option to access University is to take the *MACS*[17] exam. However, the *MACS* curriculum, not so demanding as *Mathematics A*, is greatly different from the syllabus of Professional Courses.

In 2012/2013, statistics show a huge gap between internal and external students, mainly in *Mathematics A* and *Portuguese (Table 15)*. This trend is consistent in time and highlights the hardness of learning Mathematics. We agree with different mathematics syllabuses, which are especially designed for different school paths. In what concerns Professional Courses, we believe that it is possible to do more about the curriculum, schoolbooks and teacher guidance. If the results of the internal students who take the 1st Phase of the exam are satisfactory, with the exception of *Biology and Geology*, and *Physics and Chemistry A*, the performance of self-proposed students is far from acceptable.

As more students chose Professional Courses, editors began printing manuals for this syllabus. Frequently, these manuals are written by authors who have already published schoolbooks for the regular courses. Today we may find an assortment of titles in opposition to the early years, when teachers needed to elaborate their own worksheets, the only resources they had for the classroom. Variety brings the power to choose and provides more time to focus on lessons plans and students' needs. Despite those advantages, these manuals are not flawless: they are simplifications of general courses' manuals, and their pages are colourfully illustrated to become eye catching to the students, although often with photos and drawings which are useless for exercise practice and problem solving.

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[17] *Mathematics Applied to Social Sciences.*

Secondary Education National Final Exams (Table 15)

Exam	Results (score)								Students Evaluated			
	Totals			Internal Students (IS)					2013			2012
	1st Phase/2012	1st Phase/2013	Difference 2012-2013	1st Phase/2012	1st Phase/2013	Difference 2012-2013	Self proposed students (SP)	Score IS–SP 2013	Total	Internal	Self proposed (%)	Self proposed (%)
Biology and Geology	93	81	-12	98	84	-14	76	8	50933	29620	42	44
Drawing A	120	121	1	123	124	1	112	12	5307	3944	26	34
Economy A	101	100	-1	117	113	-4	88	25	11010	5073	54	64
Philosophy	78	91	13	89	102	13	72	30	8427	5445	35	31
Physics and Chemistry A	75	78	3	81	81	0	74	7	52591	30333	42	38
Geography A	103	94	-9	107	98	-9	82	16	19757	15059	24	26
Descriptive Geometry A	90	102	12	107	122	15	71	51	9113	5592	39	37
History A	110	99	-11	118	106	-12	80	26	15705	11462	27	30
Mathematics A	**87**	**82**	**-5**	**104**	**97**	**-7**	**54**	**43**	**47562**	**31413**	**34**	**31**
Mathematics Applied to Social Sciences	95	87	-8	106	99	-7	59	40	9343	6587	29	25
Portuguese	95	89	-6	104	98	-6	67	31	70807	50127	29	32

Students' scores between 0 and 200 points for each exam.
Subjects with a number of exams equal or superior to 2500.

	Negative years (score < 95)
	Annual difference inferior to 1 value (≤ score 14)

(GAVE, 2013)

Real life mathematical applications for specific professional areas may be difficult to produce. Nonetheless, improvements to schoolbooks can and should be made. One of the possibilities is to bring the curriculum closer to the *MACS* curriculum. To inspire authors and teachers in order to write effective classroom worksheets we suggest *Excursions in Modern Mathematics* (Tannenbaum, 2003), a book profusely illustrated with exercises and problems that survey mathematical daily life contents.

The Mathematics of Social Choice

- *The Mathematics of Voting;*
- *Weighted Voting Systems;*
- *Fair Division;*
- *The Mathematics of Apportionment;*

Management Science

- *Euler Circuits;*
- *The Travelling-Salesman Problem;*
- *The Mathematics of Networks;*
- *The Mathematics of Scheduling;*

Growth and Symmetry

- *Spiral Growth in Nature;*
- *The Mathematics of Population Growth;*
- *Symmetry;*
- *Fractal Geometry;*

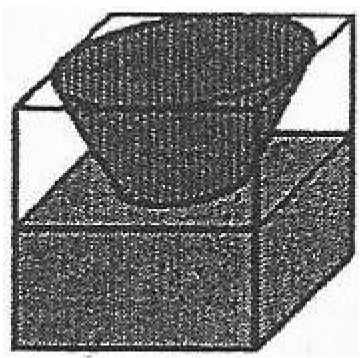
Statistics

- *Collecting Statistical Data;*
- *Descriptive Statistics;*
- *Chances, Probabilities and Odds;*
- *Normal Distributions.*

The voting theory is a good example of the role that Mathematics plays in society. In our opinion, this chapter of the *MACS* curriculum is a resourceful instrument for Professional Courses if the goal is to teach a selected set of methods instead of an endless list with similar patterns. Repeatedly tackling the same main subject is not the most advisable way to inspire students, to increase motivation or to improve cognitive aptitudes.

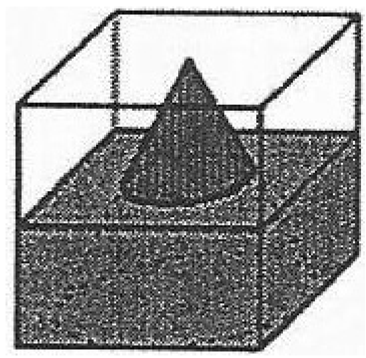
The selection of activities which promotes profitable knowledge acquisition is critical in the classroom. A practical purpose is always desirable to a theoretical purpose, be it either an exercise or a problem. The student must understand the question and assume its importance to start working on what is asked without further help. Questions with references to areas and volumes are well-known to students from an early age. Geometry, as a main issue in Mathematics and with numerous applications in daily life, allows practical explorations without an intricate theoretical framework. An example is presented below.

The image (Figure 15 and *Figure 16) shows two containers with a cubic shape made of a transparent material which can be filled with a fluid. Inside each cube, regardless of thickness, we find a massive cone. In the recipient on the left, the cone basis is inscribed on the upper cube face, and its vertex is in the centre of the lower face. In the recipient on the right, the cone basis is inscribed on the lower cube face, and its vertex is in the centre of the upper face. Each cube edge measures 1m.*



Container A

(Figure 15)



Container B

(Figure 16)

a) *Find the volume of the fluid in container A when it reaches a height of 50 cm.*

Students are familiarised with the procedures to determine the volume of a cube or a cone and, if they are not, the formulas can easily be found in the schoolbook or in lesson notes. First, we need to understand the problem and the figure which is attached to it. We must calculate the volume of water inside the cube when it is at a height of $0.5m$. How do we subtract the volume of a cone with $0.5m$ of height? It seems that data concerning the cone basis is missing. Now a critical point is reached. Students with low resilience levels due to years of school failures and a lackadaisical attitude give up more easily. The teacher should elicit opinions about how to obtain more information from the given data. All suggestions are welcome; even if they do not contribute to the solution of the problem, they allow for a constructive brainstorming. Sketches are a good idea and eventually one student will advance *"We can use the Thales Theorem"*. This connection is the missing link. Calculations determine $0.25m$ as the radius of the circumference of the basis of the cone. It is now possible to compute the volume of the cone ($\cong 0.0327m^3$). The solution ($\cong 0.4673m^3$) comes after subtracting the volume of the section of the cone from $0.5m^3$, *i.e.*, the volume of the parallelepiped with dimensions $1m \times 1m \times 0.5m$.

b) Find the volume of the fluid in container B when it reaches a height of 60 cm.

A slight change avoids routine and stimulates students' interest. Most of the hard work has already been done. From $0.6m^3$, the volume of a parallelepiped with dimensions $1m \times 1m \times 0.6m$, we must subtract one section of the cone. To obtain the volume of that section of the cone, first, we must find the volume of the section of the cone which is above the water line, a solid with a height of $0.4m$ and a basis with a $0.2m$ radius, a value which came from the application of the Thales Theorem. The volume of the section of the cone,

$$\frac{1}{3}(\pi \times 0.5^2 \times 1) - \frac{1}{3}(\pi \times 0.2^2 \times 0.4)$$

is now straightforward ($\cong 0.245m^3$). The fluid volume is approximately $0.355m^3$.

Attractive mathematical questions do not usually walk hand in hand with a simple, small thought which takes us from point A to point B. Routine exercises can accomplish that purpose but fail to feed students with fresh mathematical thoughts. Theoretical explanations are needed to go further, but only in the required amount to practise problem solving.

Most students' in Professional Courses believe that Mathematics is confined to calculation work. In order to discredit this widely spread conception, it is important to produce questions that appeal to students' critical opinion.

c) If f is a function that, to fluid height x (metres) in container A, corresponds the fluid volume (cubic metres) in container A, and g a function that, to fluid height x (metres) in container B, corresponds the fluid volume in container B, then

c_1) *Consider the sentence "Functions f and g have the same domain and the same counter domain". Justify this sentence and present the domain and counter domain of f and g.*

Regarding the functions' domain, it is easy to see that *x* can assume real measures from 0 to 1. In what concerns the counter domain, the fluid volume inside each container is a value superior or equal to 0 and inferior or equal to $1 - \frac{\pi}{12}$, obtained from the difference between the volume of the container ($1m^3$) and the volume of the cone $\left(\frac{1}{3}\left(\pi \times \left(\frac{1}{2}\right)^2 \times 1\right) m^3\right)$.

c_2) *Show that* $f(x) = x - \frac{\pi}{12}x^3$.

The volume inside container A is variable and depends on *x*, the height of the fluid inside the cube-shaped container. The importance of recovering previous techniques has already been underlined. For this question students should remember the previously used process. The fluid volume value is the difference between the volume of the parallelepiped with a height of *x* meters and the volume of the cone with a height of *x* meters. The application of the Thales Theorem is necessary to find a relationship between the radius of the basis of the cone and the water height, *i.e.*, the height of the cone. We calculate that the radius of the basis of the cone is the value of half the height of the water. Given the value of *x*, the fluid volume which is inside container A can be determined.

$$(1m \times 1m \times xm) - \left(\frac{1}{3}\left(\pi \times \left(\frac{1}{2}x\right)^2 \times x\right) m^3\right)$$

c_3) *Let* $k \in D_f'$, *a and b solutions to equations* $f(x) = k$ *and* $g(x) = k$.
Justify why $b \geq a$.

Concerning container A:

- the radius of a section of the cone at height *x* is $\frac{x}{2}$;
- as the volume of a cone with a radius *r* and a height *h* is $\frac{r^2\pi h}{3}$, the submerged piece of the cone at a level *x* has the volume $\frac{x\pi}{3} \times \left(\frac{x}{2}\right)^2 = \frac{x^3\pi}{12}$. Therefore, the water volume in container A, with the height *x*, is the value of the volume at the height *x* without the cone volume:

$$f(x) = x - \frac{x^3\pi}{12}, D_f' \in \left[0, \frac{12 - \pi}{12}\right].$$

$g(x)$ can be established through a similar process.

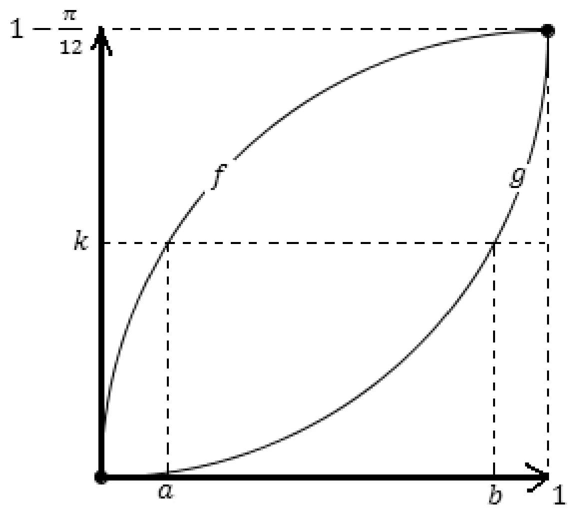
$$g(x) = x - \{volume\ of\ the\ truncated\ cone\ at\ a\ height\ x\ in\ container\ B\}^{18}$$

$$volume\ of\ the\ truncated\ cone\ at\ a\ height\ x\ in\ container\ B$$
$$= \left(\frac{1}{2}\right)^2 \frac{\pi}{3} - \left(\frac{1}{2}(1-x)\right)^2 \frac{\pi}{3}(1-x) \Rightarrow g(x) = x - \frac{\pi}{12}(1-(1-x)^3) = x - \frac{\pi}{12} + \frac{(1-x)^3\pi}{12}.$$

Now let us assume that: $f(a) = g(b)(= k)$, then

$$a - \frac{a^3\pi}{12} = b - \frac{\pi}{12} + \frac{(1-b)^3\pi}{12}.$$

As has been previously explained, $f(x)$ and $g(x)$ have the same counter domain $\left[0, 1-\frac{\pi}{12}\right]$ and evidently both functions are increasing: but, while f increases rapidly at first and then more slowly, the opposite happens to g, so if $f(a) = k = g(b)$, we can see that $a \leq b$ *(Figure 17)*.



(Figure 17)

Theorem: If f and g are two continuous growing functions, $f(0) = g(0)$; $f(1) = g(1)$, with $f(x) > g(x), \forall x \in]0,1[$, then $\forall\ k \in [f(0), f(1)]$, and if $f(a) = g(b)(= k)$, then $a < b$.

Prove: Supposing $f(a) = k$, then $g(a) < f(a)$. As g grows, $\forall x \leq a, g(x) \leq g(a)$, then $\forall x \leq a, g(x) < f(a)$. An hypothesis: $f(1) = g(1)$. As g continuous growing function, there is b such as $g(b) = f(a)$, *intermediate value theorem*[19]. Therefore, since $\forall x \leq a$, $g(x) < f(a), b > a$.

By definition, the problem solving activity requires much more than going from point A to point B. Still, if we go from point A to point Z by walking on all alphabet letters, problem solving practice is not granted. We could only be practising an acquainted algorithm.

[18] $\{volume\ of\ the\ whole\ cone\} - \{volume\ of\ a\ cone\ with\ height\ (1-x)\}$

[19] If a continuous function f with an interval $[a, b]$ as its domain takes values $f(a)$ and $f(b)$ at each end of the interval, then it also takes any value between $f(a)$ and $f(b)$ at some point within the interval.

Recent studies show that the practice of guided examples is a more efficient and effective way to learn how to solve problems than only practising problem solving without reference to guided examples (Paas & Van Gog, 2006). The same subject can be worked with different difficulty levels; complexity is positively correlated with the quality of learning. Curriculum and rigour are main vectors in Education. As compulsory education increased to twelve years, it is expectable / desirable that students from Professional Courses aspire to additional specialised education.

In consequence of the growing number of students that enrol in Professional Courses and, eventually, desire to pursue their studies, the Portuguese government promulgates a new tertiary educational path, Professional High School Courses, available in Polytechnic Schools. Those courses incorporate a general scientific component, a technical component and time allocated to work in firms or corporations. This syllabus also grants a level 5 in the European Qualifications Framework (EQF), as prescribed by the *European Credit Transfer and Accumulation System*, until now only granted by Technological Specialised Courses (CETs). The aim is to diversify educational options and consequently increase the number of citizens with higher qualifications so as to develop economic competitiveness. Professional High School Courses will gradually replace Technological Specialized Courses (CET's) in tertiary educational establishments, and CETs will only be available in non tertiary educational institutions.

Methodological considerations about Maths learning and teaching

Civilisation steps forward by solving theoretical, practical and multidisciplinary problems. Therefore, problem solvers are important for society: in fact many scientists and other influential people tell us that more important than mere competence is to be able to create novelty from it. They worry that our courses and students' evaluations rely too much on rod memory. Teaching problem solving by drawing on the experience of proficient problem solvers is perfectly natural. Often, mathematical problem solving is often a difficult task and requires hard work. Fortunately, we can look for help in treatises on the *art of invention*, and in books like *The Psychology of Invention in the Mathematical Field* by Jacques Hadamard (1865 – 1963).

The ideas of how to teach problem solving were crystallized in George Pólya's *How to Solve It* (2004). George Pólya's book does not promise miraculous solutions to all problems, but it contains excellent guidelines. Progress in Mathematics, such as in other fields, is achieved with work, perseverance, method and serendipity.

> *Solving problems is a practical skill like, let us say, swimming. We acquire any practical skill by imitation and practice. Trying to swim, you imitate what other people do with their hands and feet to keep their heads above water, and, finally, you learn to swim by practicing swimming. Trying to solve problems, you have to observe and to imitate what other people do when solving problems, and finally, you learn to do problems by doing them* (Pólya, 2004, pp. 4-5).

We wish to underline for the mathematical layperson that solving mathematical problems is harder than swimming. Anyway, mathematical activity can indeed be compared with sports (Schoenfeld, 1983).

The problem solving practitioner must be dynamic, but this is not sufficient; a teacher has the responsibility to optimise students' results. They can resemble a team coach and do more than just pull off the procedure. Under coach supervision the athlete performs training and, when necessary, the instructor corrects and specifies a more efficient method. Equally, a good teacher has to present/propose/induce efficient decisions that lead students to save time.

Problem solving achievement uses specific domain strategies besides weak methods or general heuristics (Anderson, 1987). In athletics, high jump is an extremely demanding technical sport. During the 1896 Athens Olympic Games, Ellery Harding Clark (1874 – 1949) won the competition with a $1.81m$ leap. In the 19^{th} century the athletes' had a diagonal approach to the bar, raising one leg first and afterwards the remaining body. New techniques emerged;

one of the most astonishing was performed in 1968 Mexico City Olympic Games by Richard *"Dick"* Fosbury. In a similar way an adequate and sophisticated method can solve a problem much more efficiently than standard procedures or the first approach that comes to mind. Without such efficiency it would be impossible for some individuals to solve all the six problems they are confronted with in the *International Mathematical Olympiads*[20].

We are herewith at the centre of our quest. What does it mean to teach problem solving well? It means to accelerate by guided instructions the acquisition of some of the higher order skills that can be potentially useful for attacking any mathematical problem. We are not saying that we will solve every problem through these methods, but proficient problem solvers almost always use them. Even if Alan Schoenfeld refers in his book *Mathematical Problem Solving* (1985) that a Putnam's[21] trainer, about Pólya's strategies, said *"It's worthless"*, we wish to note the background in which that statement was made: he trained some of the best problem solving students in the USA, individuals who already had years of training and dominated all corners of general heuristics. Our aim is to get very inert and mathematically passive students to work. We think that Pólya's suggestions are very useful as a first step for these students as they are for the high school teachers we wish to educate. But it is also clear that we should heed Schoenfeld's detailing of why heuristics fails if we leave students with little more than a guide. So teachers should be informed by teacher educators of the third chapter of Schoenfeld's *Mathematical Problem Solving*, for the heuristics that we teach must be specialised in subcategories which have to be taught by examples in order to become really effective.

We mentioned that fostering interaction between problem solving teachers would be helpful. This is of particular importance in our country where this activity has no tradition. Educators left alone would easily loose all enthusiasm. It is therefore important to create a problem solving community, because as William James (1842 – 1910) said *"The community stagnates without the impulse of the individual. The impulse dies without the sympathy of the community"* (Lee, Jor, & Lai, 2005, p. 3). We enlighten this concept, even though it is not completely clear and depends on the background of the solver, for those who are not familiar with the mathematical problem solving process.

> *... Problem solving becomes the process by which the subject extricates himself from his problem... Defined thus, problems may be thought of as occupying intermediate territory in a continuum which stretches from the puzzle at one extreme to the completely familiar and understandable situation at the other* (Brownell, 1942, p. 416).

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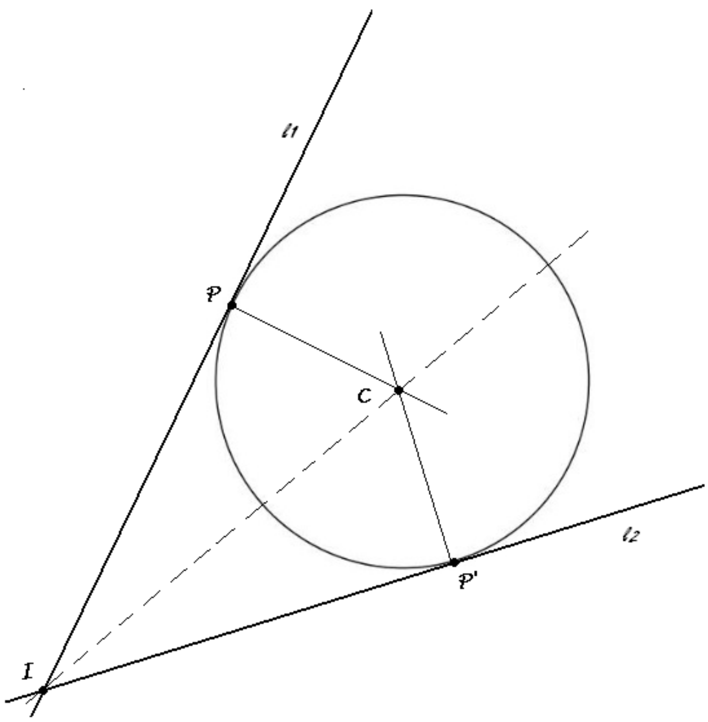
[20] World annual Championship Mathematics Competition for High School Students.
[21] Annual contest organised by The Mathematical Association of America for college students.

> *An exercise is a question that tests the student's mastery of a narrowly focused technique, usually one that was recently "covered". Exercises may be hard or easy, but they are never puzzling, for it is always immediately clear how to proceed. Getting the solution may involve hairy technical work, but the path towards solution is always apparent. In contrast, a problem is a question that cannot be answered immediately. Problems are often open-ended, paradoxical, and sometimes unsolvable, and require investigation before one can come close to a solution. Problems and problem solving are at the heart of mathematics* (Zeitz, 2007, p. x).

> *At the initial stage of problem analysis, the problem solver attempts to understand the problem by construing an initial problem representation. The quality, completeness, and coherence of this internal representation determine the efficiency and accuracy of further thinking. And these characteristics of the problem representation are determinate by the knowledge available to the problem solver and the way the knowledge is organized* (Glaser, 1984, p. 93).

Consider the following request. Given two intersecting straight lines l_1 and l_2, and a point P on l_1. Draw a circle through P that is tangent to l_1 and l_2 *(Figure 18)*. For somebody who does not know the precise notion of what a circle and a straight line are, this problem is, until the study of the concepts, out of reach. The same applies to mathematicians when they are engaged in themes that are apart from their area of expertise. Even after our subject, with illustrative figures, intuitively acquires the notions of circle and straight line, and is told what tangentially means and, with these intuitive ideas, is able to do a free hand sketch or can be instructed to draw the correct figure with the tools allowed, he will still be unable to justify the construction step by step. The subject will intuitively feel that the bisector of an angle contains the centres of the circles which touch l_1 and l_2 and the centre of the circle which passes tangentially to l_1 through P and lies on the perpendicular to l_1. If someone assures him or her that these intuitions are right, there is still a missing piece: if one draws a circle with centre C in the intersection of the bisector of l_1 and l_2 with the perpendicular to l_1 through P and radius CP, why should this circle be tangential to l_2? The reason is there a unique perpendicular to l_2 through C which yields a point $P' \in l_2$. The triangles $[IPC]$ and $[IP'C]$ have in I equal angles that are extended to P and P' as well. Furthermore, $\overline{IC}$ occurs in both as hypotenuses. We can invoke a theorem called SAA[22] which guarantees that the two triangles are congruent. Hence $\overline{CP}$=$\overline{CP'}$. The circle with centre C will pass through P' and tangentially to l_2 since the radius CP' is by construction perpendicular to l_2.



(Figure 18)

The most simple partial solution and justification for a straight-edge construction makes quite a number of mental acts that exceed routine tasks necessary. Hence, a complete justification relies on very precise definitions and specific theorems that must be known and presented in the correct order. From this point of view – in order to practise Mathematics with some fluency - one needs a strong mental personality; and this is only possible with resilience.

Heuristics, called *heurética* in Ancient Greece, is the study of the methods and rules of discovery and invention. Extended investigations on heuristics were scarce until Pólya’s time, but he refers the critical work done by Pappus of Alexandria (290 – 350) regarding Euclid’s *Elements*, Bernard Bolzano’s (1781 - 1848) interesting observations about the matter, and the most famous attempts of systematisation of Gottfried Leibniz (1646 – 1716) and René Descartes

[22] Side Angle Angle: If two angles of one triangle are congruent to two angles of another and a side of the first triangle that is not common to both angles is congruent to a side of the second triangle that is not common to both angles, then the triangles are congruent.

(1596 – 1650) which, however, had a wider scope than Mathematics. Heuristic procedures represent an opportunity to introduce fresh activities to the classroom. The goal is to stimulate students' creativity and critical appraisal about what is being learned, how and for what purpose.

A suggestion that comes naturally is to introduce problem solving into the curriculum. The question of where and how within the context of the current schooling system has a natural answer: in Mathematics. The density of cognitive problems and the clear cut manner by which they are solved - the clear answer to *"is S a solution to Q?"* - have no equal in other sciences. Also, Mathematics is not an expensive science, for it can be done with paper and pencil and by engaging brain activity, mechanisms incorporated in four categories of behaviour and knowledge: resources, heuristics, control and students' beliefs (Schoenfeld, 1985).

To exemplify this argumentation let us plunge into a problem solving activity.

What is the result of the addition of the first one hundred natural numbers? Most probably, Third Cycle classroom students' would become surprised with this question. Eventually, some students' would start adding up; others would remain motionless, thinking about the question or just indifferent to the challenge. In the 18th century, ten-year old Carl Friedrich Gauss (1777 - 1855) presents the correct solution: 5050. His argument combines simplicity and effectiveness. If 1 + 100 = 101, 2 + 99 = 101, 3 + 98 = 101, 4 + 97 = 101… 49 + 52 = 101, 50 + 51 = 101, then the total value results from 50 × 101. Gauss, without formal knowledge, used the symmetry property of arithmetic progressions. The sums of terms equally distant from the extremities are equal. How could Gauss have found that result? Possibly by specialising the problem to four or five terms and seeing the pattern referred. Indeed specialisation is an important technique in heuristics.

Educational Psychology is engaged in finding and understanding variables involved throughout problem solving activity. Task difficulty and students' knowledge play an important role in dynamics between the starting point (initial situation) and the end (problem conclusion).

Another example: Two hundred people are gathered in an international congress. One hundred and twenty speak Portuguese and one hundred speak Hebrew. How many conversations between two people are possible without a translator?

The following discussion is taken according to Pólya's problem solving guidelines, with *T* and *S* meaning teacher and student.

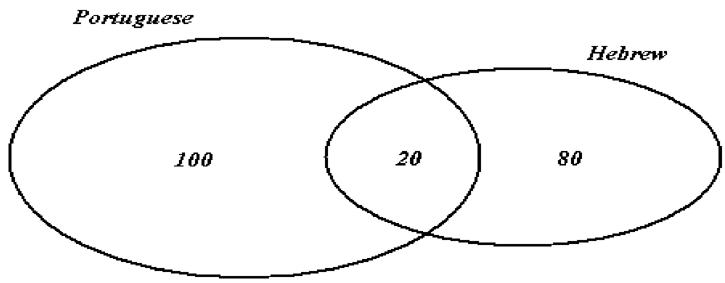
Let us first analyse the questions related to comprehension.

T: What's the unknown?

S: The number of conversations between two people that can be established. … But, I do not understand …how is it possible that one hundred and twenty people speak Portuguese and one hundred speak Hebrew if there are only two hundred participants? … Ah! There are twenty people who speak both Portuguese and Hebrew.

T: Good! Draw a figure!

S: (*Draws the following figure*):



T: And what will you do now?

S: This is somewhat complex.

T: So let's design a plan of how to do this. (*Comment: this is Polya's second step.*) Is there an easier problem that is related to this?

S: … Yes: How many conversations can be established between the one hundred people who only speak Portuguese?

T: Good! So let's solve this first. (*Comment: here the teacher lets the student deviate from thinking the plan to the end. As we shall see in a moment, this is less than optimal.*)

S: The first speaker can talk with 99 others. Let's say Portuguese 1 with Portuguese 2, Portuguese 3 …and so on.

T: Use better notation!

S: $P_1P_2; P_1P_3; P_1P_4; \dots; P_1P_{100}$, next $P_2P_3; P_2P_4; P_2P_5 \dots; P_2P_{100}$, then $P_3P_4; P_3P_5; \dots; P_3P_{100}$. Ah! There will be 99 + 98 + 97 + … + 3 + 2 +1 conversations. Now I know how to proceed. The sum of terms equally distant from the extremes is one hundred, which is identical to the sum of the terms at the extremes. After joining forty nine pairs of those numbers the central value must be added: 50. In this group 4950 conversations can be established (49×100 + 50 = 4950).

T: Good! Can we extend this reasoning?

S: Yes I can similarly find … wait a minute … the number between only Hebrew speakers. There will be 79 + 78 + 77 + … + 3 + 2 +1 conversations. This makes 39×80 + 40 = 3160 conversations. Similarly we find 19 + 18 + 17 + … + 3 + 2 +1= 190 conversations between the people who speak both languages. Now we sum and it is done.

T: Really? (*Comment: the mistake, here not too critical, would not have happened if the teacher insisted with the student to think the problem to the end. In more complicated situations plunging into a problem without thinking to the end can lead to a blind alley, thus causing great waste of time.*)

S: (*After a pause*) No. In fact the 20 Hebrews who speak 2 languages can also talk to the 100 people who only speak Portuguese and the 20 Portuguese who speak 2 languages can also enter into conversation with the 80 people who only speak Hebrew. This gives us 100×20 = 2000 pairs plus 80×20 = 1600 more Portuguese/Hebrew conversations. The sum is 3600+190+3160+4950 = 11900.

T: Great. You have solved the problem. But it's not a very elegant solution. So do you see a simpler way?

S: (*After a pause*) Well, we could put it as follows: In principle, according to my knowledge of Combinatory Analysis, there are $\binom{200}{2}$ conversations between 200 people, but the people from one of the nationalities who only speak one language cannot speak with the people from the other nationality who also only speak one language. This gives:

$$\binom{200}{2} - 100 \times 80 = \frac{200!}{2!\,198!} - 8000 = \frac{200 \times 199}{2!} - 8000 = 11900.$$

This is a good example of how a trainer can contribute with good questions or suggestions. How much time could our student have saved! For learning routine skills, organising and dividing a school textbook into curricular themes seems reasonable, but a student's manual should also contain problems that require the use of knowledge acquired months or even years before. Even though nowadays Portuguese school books have considerable graphic quality; they are less adept to convey the spirit of definition - theorem - proof style which is so typical of Mathematics, and one finds in them much more routine exercises than problems. So, this is not an approach that would challenge students' imagination.

The student's manual is one of the many instruments which teachers can use in the classroom. The tools necessary to operate with the problems should not simply be on the mental shelves; students should open those shelves of memory, manipulate and combine those tools, because otherwise they get out of practice. So a slight methodological change could generate substantial impact on students' mind, motivation and, consequently, acquisition of significant maths knowledge.

Let us now focus on another example. Second degree equations are studied in the 9th grade. Theoretical exercises are abundant in the student's book but lessons must go beyond algorithmic or routine applications and it is the teacher's responsibility to make it happen.

Solve the equation system: $\begin{cases} x^2 - xy - y^2 + 1 = 0 \\ x^3 - x^2y - xy^2 + x - y + 2 = 0 \end{cases}$

The student is surprised because he or she does not see which procedure to apply. Are the equations linked? Through this key question emerges one *Eureka* insight:

$$x^3 - x^2y - xy^2 + x - y + 2 = 0 \Leftrightarrow x(x^2 - xy - y^2 + 1) - y + 2 = 0.$$

The main difficulty has disappeared and gives place to straight calculations.

$$\begin{cases} x^2 - xy - y^2 + 1 = 0 \\ x(x^2 - xy - y^2 + 1) - y + 2 = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 - xy - y^2 + 1 = 0 \\ x \times 0 - y + 2 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x^2 - xy - y^2 + 1 = 0 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} x^2 - 2x - 3 = 0 \\ y = 2 \end{cases}.$$

The second degree equation $x^2 - 2x - 3 = 0$ is easily solved.

$$x^2 - 2x - 3 = 0 \Leftrightarrow x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times (-3)}}{2 \times 1} \Leftrightarrow$$

$$x = \frac{2 \pm \sqrt{4 + 12}}{2} \Leftrightarrow x = \frac{2 \pm 4}{2} \Leftrightarrow x = \frac{2 + 4}{2} \vee x = \frac{2 - 4}{2} \Leftrightarrow x = 3 \vee x = -1$$

The solutions for the equation system are$\{(-1; 2), (3; 2)\}$.

Indeed replacing *x* and *y* by those values in the initial system proves to be satisfactory.

In the words of Portuguese mathematician Pedro Nunez (1502 – 1578), Algebra is the *"... way to find the unknown quantity in every purpose of Arithmetic and Geometry and in all art of counting and measuring, as also in Cosmography, Astrology and mercantile purposes"* (Nunez, 1567, p. a ii).

Published in 1567, the *Libro de Algebra en Arithmetica y Geometria* is vividly written, explaining that number, *cosa* (x) and *censo* (x^2) are quantities that can be differently combined. Royal Cosmographer since 1529 and nominated Chief Royal Cosmographer in 1547, his navigation procedure techniques were often accepted with reluctance by captain sailors. In mathematics workshops and other educational events teacher educators deal with the same resistance as Pedro Nunez: often the school teachers do not want to assume the role of students. A qualified teacher is not necessarily the one with decades of school experience. If their methodology stays the same year after year, decade after decade, that is like endlessly repeating their first year as a school teacher.

It is quite possible that the small methodological change we propose generates a large positive effect. As a metaphor we recall Edward Lorenz's (1917 – 2008) weather forecast studies, in the 1960's, using variables like temperature, wind speed and atmospheric pressure. Before having started a second computer simulation, he changed parameter 0.506127 to 0.506, a small adjustment. He believed that the difference was irrelevant with the advantage of obtaining faster final computer results (in that time computers did not have today's calculation capacity). He thought the two simulations would generate similar outcomes, but in reality, this proved to be wrong. The idea that initial adjustments can have, in the long term, significant impact is known as the *Butterfly Effect*[23].

Since 1974 the educational system suffered deep changes due to economical, cultural, social and political transformations. The increase of compulsory education, more and better school infrastructures built to receive the growing number of students, access to new technologies and the dissemination of teaching degrees provided by public and private Universities have democratised access to Education in Portugal. Despite this development, in the last decades internal evaluations show students' poor performance in Maths exams. Concerning the OECD Programme for International Student Assessment (PISA)[24], progress has been made since 2000. The 2009 data show that Portugal presents standard results within the European Union (EU) as was written in the report *Progress towards the Common European objectives in Education and Training – Indicators and benchmarks 2010/2011*[25]. Regarding early dropouts from educational training aged 18-24, in spite of this evolution statistics are very negative: 42.6% in the year 2000 and 31.2% in the year 2009, when, in 2009, the EU average is 14.4%.

Bringing heuristics into play allows the teacher to approach contents with the demands of life in society. This behaviour inspires students' creativity and endorses reasoning development and, eventually, leads to the reduction of early dropouts. The teacher, as a knowledge supporter, must refine strategies, explain and guide, without forcing resolution methods, and allow the student to endeavour strategies under his supervision (Schoenfeld,

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[23] Sensitive dependence linked to problem initial conditions.

[24] Triennial study that evaluates reading level, scientific knowledge and mathematical ability of fifteen-year old students.

[25] Reading: 17.6%; Mathematics: 23.7%; Science: 16.5% (p. 183).

1983). Creative learning is related to problem solving apparently in opposition to the execution of algorithms or routines, but, in fact, proficiency also means to proceed in leaps which use bunches of memorised knowledge (Ausubel, 1960). *"What the teacher says in the classroom is not important, but what the students think is a thousand times more important"* (Pólya, 1981, p. 104).

Fifteen-year old students begin secondary school, a new three-year cycle (10^{th} to 12^{th} grade), which is a critical moment for their academic future and professional choices. It is here that our teacher training sets in: enriching mathematical food with nutritive supplements in order to improve instruction.

Today information travels at the speed of light. We can download gigabytes of data within seconds, but the essential, that is, comprehension, needs time to be acquired and requires effort to be assimilated by the brain. Technology has flooded the classrooms, but we must realise that computers and graphic calculators do help students' work. However, these tools do not replace pencil and rubber, just as multimedia boards do not substitute a white sheet of paper. Teaching and learning is not a hundred-metre race, it is a life-long marathon.

Problem solving is widely recognised as a higher order thinking skill; it is considered the opposite of what today's information processing machines can do, namely the execution of algorithms which they are fed with by programmers, *"Software does exactly what one tells it to do, and can make incorrect decisions trying to focus on a single important goal"* (Jones, 2008, p. 3). It is also a truism that we can hardly extend compulsory schooling beyond a 12^{th} year. Schools have to prepare the citizen of tomorrow to live in an ever more complex world.

PART II: EMPIRICAL RESEARCH

Research objectives

Since the major concern of this research study is to improve mathematical learning abilities based on problem solving activities, a path of research-based action with several methodological stages was followed. In general, this proposal aims to show that the teaching and learning process can, and should, be supported by the implementation of an intervention programme capable of promoting the construction of heuristics, thus leading to better performance in Mathematics.

In the Literature Review special attention was paid to models which explain the mathematical problem solving performance by taking into account motivational and metacognition variables, strategies, as well as empirical data displayed in international reports and the assessment of learning and performance tasks in the classroom. Hence, this study aims to assess: *1)* the possibility of engaging students in mathematical learning activities with a high focus level and interest, thus reducing negative beliefs about mathematics; *2)* the importance of presenting mathematical problems in a systematic and organised way, following an algorithmic method to process information, or monitoring a sustainable problem-solving heuristic scheme, which later would support self-regulated learning, and *3)* the significance of an organised problem solving intervention programme, once the need to intensively practice problem solving in specific domains is accepted. The goal would be to improve problem-solving performances and change negative beliefs to succeed in Mathematics. Each of these points could become a new focus of research. Results converge towards a better understanding of the relevance of learning Mathematics based on a problem solving approach, and focus on how to promote thinking and active attitudes regarding these issues among both teachers and students.

Educational goals for mathematical education are rooted in Psychology knowledge, mainly in studies on instruction and vocational education and training (VET), as well as in the awareness of individual development, particularly in learning contexts. Such incursions in both heuristics for mathematical problem solving and mathematical education resulted in oral presentations in thematic scientific meetings, where educational issues related to personal promotion, scholarship and professional choices were under discussion. As teachers and students are the main beneficiaries, we would like to propose a consistent intervention under the hypothesis that teaching and learning processes could be optimised with the implementation of a problem solving instructional programme. A sample[26] of 9th grade students and Third Cycle and

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[26] Data gathering authorisation (nº 0170100007) by *Monitorização de Inquéritos em Meio Escolar* (MIME) from *Direção-Geral de Inovação e de Desenvolvimento Curricular* (DGIDC).

Secondary Education teachers (from the 7th grade to the 12th grade) from Coimbra and surrounding villages was gathered to observe the prior assumptions.

The 2-hour session *The Man and the Infinite – A Mathematical Approach*, held on 30th November 2012 at *Centro Educativo dos Olivais* (CEO), in Coimbra, (its programme and report can be found on the following pages) provided the opportunity to test the hypothesis that selected mathematical activities can contribute to students' motivation to establish a more friendly relation with mathematical contents, even when they only master elementary mathematical concepts. Students' motivation should be observable in their engagement in the activities, manifestations of interest or active participation and attention, as well as in effective problem solving. This study was held in the scope of the project *Time and Man*[27].

A systematic and organised approach to the problem solving activity should be emphasised in the teaching of adequate strategies for learning specific curricular mathematical contents. The research plan incorporated encompassed an experimental component, inspired in *Kolb's model*[28], which was put into practice in the form of two Mathematics workshops for teachers held in the 2011/2012 and 2012/2013 school years at Nova Ágora's, educational facilities in Coimbra. The second edition, as it met previously identified needs, was an improved version of the first workshop. After handling the concrete experience of learning and learning to teach problem solving, as suggested by the model, the aim of our rationale is to achieve self-regulation and apply this method in new situations of pedagogical orientation.

The workshop *The use of Heuristics in Mathematics – Tools to problem solving* aimed to enhance the range of heuristic tools for problem solving used by the teachers in their classrooms, so that they would be able to better explore problem solving of both a theoretical and a practical nature which can be applied to life in society with *Basic*, *Secondary* and *Professional Course* students. The liaison between these areas is an asset for the Mathematics learning/teaching process. Heuristic procedures may be found in the demonstration of theorems, even though in a veiled way. Such procedures are an excellent opportunity to revisit George Pólya's *How to Solve It* and analyse his problem solving strategies. The curricular reforms of the last decades withdrew the binomial theorem - demonstration from compulsory education programmes and school manuals, and therefore from the classroom, thus curtailing the reasoning, consolidation and development of students' mathematical knowledge. The emphasis on exercise solving, even though useful for content consolidation, follows the rigidity of mostly

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[27] Educational project for youths (< 18 years old) in detention at *Centro Educativo dos Olivais* (CEO) (a National Museum Machado de Castro partnership with DGRS-MJ/CEO).

[28] Supported by the Experiential Learning Theory (ELT). The ELT is a model that combines two connected approaches towards acquisitive experience: Concrete Experience and Abstract Conceptualization, as well as two connected approaches towards transforming experience: Reflective Observation and Active Experimentation.

easily identifiable algorithms, thus promoting routine work and a less stimulating intellectual activity.

Transition to Secondary Education is a critical moment, for the choice of an educational path based on negative beliefs regarding Mathematics will probably restrain career choices. Solutions are needed to avoid an early divorce from Mathematics, so that more students successfully pursue their studies and forge scientific careers. The lack of a problem solving oriented approach in Mathematics programmes, and the difficulties felt by the students when tackling issues outside exercise solving are two good reasons for the development and implementation of such workshops.

As a natural result of the workshops, the author gathered an assorted set of topics in a problem solving manual for intensive practice, so as to improve performances and enhance positive beliefs which aim to contribute to students' life projects, both educational and professionally, in areas where mathematical contents are required. The syllabus aims to be an effective instrument to disseminate the practice of problem solving. Some questions are strictly focused on more advanced mathematics but the reader can find problems with lower difficulty level that are aimed for Third Cycle or Secondary Education students'. Heuristic procedures are presented and the reader is invited to make use of such tools.

Methodology

By embracing multiple dimensions of life in society, Mathematics explores the misleading dichotomy between abstract concepts and the real world. Connected to different areas of knowledge, Mathematics provides a particular perception of Infinity. This is an intricate concept, linked to the unattainable, which Humanity learned to understand. These ideas were the core of our presentation at *Centro Educativo dos Olivais*, where explanations about facts and the location of mathematical contents in a specific historical time were offered through artistic manifestations and the use of metaphors. In this session seven students from a Waiter Training vocational course, an alternative curricular path to Third Cycle Compulsory Education, were systematically challenged to participate, to think and verbalise their thoughts. An immediate feedback was given after each student's intervention.

A glimpse of Renaissance, one of the most sparkling periods in History and whose superior achievements immortalised the genius of their authors, can be caught at the National Museum Machado de Castro (MNMC), in Coimbra, where we can perceive *Santa Inês*, the masterpiece of the sculptor João de Ruão (1500 – 1580). This sculpture was at the heart of the 2-hour session supported in a PowerPoint presentation and complemented with selected activities.

1. GEOMETRY IN ART AND ARCHITECTURE
A brief historical approach.
1.1 Leonardo da Vinci (1452 – 1519)
Geometrical interpretation of the masterpieces:
Vitruvian Man (1490); *The Last Supper* (1495 – 1497); *Mona Lisa* (1503 – 1506).
1.2 Golden Rectangle and Golden Number
Geometrical construction of a Golden Rectangle.
Relation between the Golden Rectangle and the irrational number *phi* (ϕ) (infinite non-periodic decimal).

2. FROM DIGITS TO NUMBERS
A brief historical approach.
2.1 Leonardo Fibonacci (1170 – 1250)
Fibonacci's Rabbit Problem from *Liber Abaci* (1202).
The Fibonacci sequence. Relation between the Fibonacci sequence and the number ϕ, with illustrative examples (credit cards, the *Nautilus* spiral …).

3. INFINITY
Construction of a Möbius Strip.
Geometric analysis and interpretation of works of M. C. Escher (1898 – 1972).
Infinite sums: the paradox of Achilles and the Tortoise.
The meaning of the irrational number *pi* (π) (infinite non-periodic decimal).
Relation between the different sizes of A-format sheets of paper and the square root of 2 $(\sqrt{2})$.
Powers of Ten: a journey from the infinitely large to the infinitely small.

In a journey through Time, the students were invited to return to the Renaissance[29]. Painters and architects conveyed fresh perspectives to their works. The notion of geometric perspective was put in stone on cathedrals' vaults, as well as on 2-dimension paintings such as Filippo Brunelleschi's dome of the Basilica of San Lorenzo, in Florence, Raphael's *School of Athens* (1506-1510), Piero della Francesca's *The Resurrection* (1463) and *Brera Madonna* (1472-1474), or Masaccio's *The Tribute Money* (1420s). Art allows for the exploration of theoretical notions in the representation and visualisation of figures, particularly in 2- and 3-dimensional Geometry.

[29] Period of European History that goes from the late 13th century to the mid-17th century and which is characterised by a new interest in the culture of Classical Antiquity, both Greek and Roman.

Leonardo da Vinci's *Vitruvian Man* (1490), *The Last Supper* (1495 – 1497) and *Mona Lisa* (1505) were employed to show the liaison between Man and Geometry. The *Mona Lisa* was used to display a geometric figure that mathematicians define as the *Golden Rectangle*.

Geometry, a major area of Mathematics, is related to many other non-mathematical subjects. An example of this connection is the *Fibonacci's Rabbit Problem*[30], introduced with the purpose of studying sequences of numbers. Powerful tools to perform calculations, numbers are combinations of symbols invented by the Hindus and later disseminated by the Arabs. As the participants in the session acknowledged, the *Fibonacci's Rabbit Problem* has a surprising relation with the *Golden Number*[31], which can also be connected to the *Nautilus* shell spiral[32].

Arithmetic evolved from ten digits, and then magic happened, as it was observed by the group when engaged in a numerical activity based on Algebra principles.

- Think of a natural number between 10 and 99.
- Write it on a piece of paper. Keep your number in secret!
  (yx: y; tens digit, x; ones digit)
- Multiply the tens digit by 5.
  $(5y)$
- Add 3.
  $(5y + 3)$
- Multiply by 2.
  $2\times(5y + 3) = 10y + 6$
- Add the ones digit of the initial number.
  $10y + 6 + x$
- Tell the final result to the teacher. If calculations are accurately performed, the teacher will *guess* the initial number that you had written on the piece of paper!
  To identify the number, the teacher mentally subtracts six ones from the final result,
  $(10y + 6 + x) - 6 = 10y + x = yx$.

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[30] A man puts a pair of baby rabbits into an enclosed garden. Assuming each pair of rabbits bears a new pair every month, which, from the second month on, itself becomes productive, how many pairs of rabbits will there be in the garden at the beginning of each month?

[31] Real algebraic irrational constant whose value is $\frac{1+\sqrt{5}}{2}$.

[32] A marine cephalopod with a shell formed by chambers which work as floating devices. The cephalopod only inhabits the last chamber.

A Möbius Strip[33] structure gives students the sense of continuous motion, an inspiring piece for Maurits Cornelis Escher (1898-1972), who created masterly presentations of continuous motion in drawings such as *Reptiles* (1943) and *Drawing Hands* (1948).

Motion is an important issue in the paradox of *Achilles and the Tortoise*, imagined by Zeno of Elea (ca. 490 – 430 BC). Achilles, the Greek mythological warrior, competes with a tortoise. Confident of victory, Achilles, who runs ten times faster, grants the tortoise an advance of 100 metres. The race starts! What happens? When Achilles reaches the place where the tortoise started the race, it is now ten metres ahead. The athlete runs those ten metres, but now the tortoise is one metre ahead. That distance covered, the tortoise is now 10 centimetres ahead... The swift Achilles gets closer and closer but never reaches the slow tortoise. Is it indeed so? The solution to the problem is the result of infinite sums, whose outcome is the moment when Achilles achieves his desideratum.

The value of *pi*, an irrational number and an infinite non-periodic decimal, is the ratio of a circumference's perimeter to its diameter. More than repeating its approximate value to two decimal digits, students ought to understand its meaning when it is used in everyday contexts such as the computation of areas of circles and volumes of spheres.

Mathematics encourages learning in multiple areas of knowledge. *In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.* This relation, known as the Pythagoras' Theorem, allows calculating the diagonal of a square. Hence, if the side of the square measures one length unit, the diagonal will measure $\sqrt{2}$. This irrational number emerges in the sizes of A-format paper as the result of the division of length by width.

The voyage to Infinity ended with base 10 powers, in a journey from the infinitely large to the infinitely small called *Powers of Ten*[34]. The interest and active participation of the group is an indicator of the importance of presenting and explaining significant mathematical contents within problem solving activity.

The workshops for Third Cycle and Secondary Education teachers' aimed to produce the following outcomes: *1)* to develop teachers' problem solving skills beyond strict classroom contents. As a result teachers will feel more confident to use fresh approaches which will enhance performances and motivate both the students who already possess a solid mathematical knowledge and motivation, and those who do not have the prerequisites or the self-esteem required for immediate success. As Mathematics is mostly oriented to problem solving and therefore tackles many different areas of knowledge, the main goal of the workshop was to

[33] Topological space obtained from the gluing of the two ends of a strip after giving one of the ends a half-twist.

[34] Documentary film written and directed by Ray and Charles Eames in 1977.

familiarise the teachers with engaging mathematical problems so that they can introduce them in their classrooms with relevant contexts and contents; *2)* to propose new foundations based on problem solving for a more practical school subject, which would lead to a better transition to Secondary and Tertiary Education. Conversely to other countries, in Portugal, Tertiary Mathematics studies, namely those which provide teacher training, do not currently contemplate practical sessions for each theoretical issue (with separate assessment). However, this practice is essential for preparing well-trained professionals to enter the teaching career. The statements provided by the teachers regarding their students' performance in problem solving activities are vivid testimonials of the difficulties felt by the students and underline the importance of establishing a new paradigm in how to teach and learn mathematics.

WORKSHOP CONTENTS

1st moment - face to face on-site sessions *(12 $\frac{1}{2}$ hours)*

- *How to Solve It* by George Pólya, a brief introduction (foreword).
- Presentation and solving of problems by implementing the George Pólya's 4-step model, which may boost the teaching/learning process.
- Contextualised demonstration of theorems relevant for the teaching and learning of Mathematics.
- Enunciation of problems to be solved by the workshop participants, individually or in group.
- Reading and interpretation of mathematical texts.

Autonomous work *(15 hours)*
Implementation of problem solving activities in the classroom.

2nd moment - face to face on-site sessions *(10 hours)*

- Presentation of the activities developed by each teacher with their students.
- Presentation of new problems resulting from the experiences and difficulties felt by the teachers during previous autonomous work.
- Literature survey on the theme.
- Analysis of selected problems and solution methods extracted from the *International Mathematical Kangaroo* and the *Portuguese Mathematics Olympiad*.

Autonomous work *(10 hours)*
Problem solving implementation to assess students' classroom performance.

3rd moment - face to face on-site sessions *(2 $\frac{1}{2}$ hours)*

- Presentation of data regarding the performance of students in problem solving.
- Teachers' self- and hetero-assessment.
- Global assessment of the workshop.

The workshop had a total of 25 hours (theoretical explanations; group dynamics; debates; individual support concerning the implementation of problem solving in the classroom) and 25 hours of autonomous work (activity planning in the scope of problem solving; implementation of selected problems in the classroom; portfolio).

Summary of sessions in the classroom *(25 hours)*

- Session 1: (30-09-2011) Presentation of the contents of the workshop and the expectations of the participants. Group dynamics game inspired on John Nash's *Equilibrium Theory*. Psychological perspective of the analysis of the results of the decisions made in small groups and their repercussions.
- Session 2: (04-10-2011) Holistic view of Educational problem solving. Analysis of some classical problems of the History of Mathematics. Problem solving in the fields of Geometry, Algebra, Numbers Theory, Probabilities and Statistics…, by using different mathematical techniques.
- Session 3: (07-10-2011) Reading and interpretation of the first pages of *Livro de Álgebra* by Pedro Nunez. Evidence on the roles of the teacher and the student in the four stages of problem solving described by George Pólya in the book *How to Solve It*. Problem solving group activity and discussion regarding the heuristic procedures applied to attain the solution.
- Session 4: (12-10-2011) *International Mathematical Kangaroo*, organised in Portugal by the Department of Mathematics of the University of Coimbra. Problem solving group dynamics using George Pólya's methodology.
- Session 5: (14-10-2011) Diagnosis test based on questions extracted from the *International Mathematical Kangaroo* to be given to the students in the classrooms so as to assess their problem solving strategies. Guide for problem solving (2 pages) to be used in the classrooms as an instrument to help the teachers during this problem solving activity.
- Session 6: (04-11-2011) Presentation of the activities developed by the teachers in their classrooms. Discussion on the results and analysis of the main difficulties felt by the students when following Pólya's model for problem solving.

- Session 7: (11-11-2011) Presentation of the activities developed by the teachers in their classrooms. Discussion on the results and analysis of the main difficulties felt by the students when following Pólya's model for problem solving. Reading and interpretation of problems extracted from the *Brazilian Mathematics Olympiad*. Analysis and discussion of *Changing Education Paradigms*, a short animation film adapted from one of Sir Ken Robinson's lectures.
- Session 8: (18-11-2011) Presentation of the activities developed by the teachers in their classrooms. Discussion on the results and analysis of the main difficulties felt by the students when following Pólya's model for problem solving. Problem solving dynamics. Analysis of *A Valsa dos Brutos*, a short film on bullying and possible strategies to eradicate/minimise it in the school context.
- Session 9: (25-11-2011) PowerPoint presentation on the different Voting Methods. Group dynamics on George Pólya's methodology for problem solving. Portfolio recommendations.
- Session 10: (09-12-2011) Brief summary of the topics analysed in the workshop. Open discussion where the teachers expressed their opinions regarding the work developed in each session.

An article describing the activities developed in this workshop, which included eighteen teachers from eleven schools, was published (Rodrigues, 2012, pp. 53-55). As a result of the positive feedback of the workshop *The use of Heuristics in Mathematics – Tools to problem solving* 2011/2012, held in the 2012/2013 school year, a second workshop was organised with a new group of teachers. Hence, *The use of Heuristics in Mathematics – Tools to problem solving (II)* was conceived with the same design but also with a new set of fresh topics for our problem solving instructional programme (Rodrigues, 2014, pp. 64-67).

Summary of on-site sessions *(25 hours)*

- Session 1: (19-11-2012) Presentation of the contents of the workshop and the expectations of the participants. Introduction to George Pólya's model for problem solving. *Meno* from Plato, an illustrative recreation of the teaching and learning process. Group dynamics game inspired on John Nash's *Equilibrium Theory*. Psychological perspective of the analysis of the results of the decisions made in small groups and their repercussions.
- Session 2: (26-11-2012) Holistic view of Educational problem solving. The application of the heuristic procedures used to solve the problem *What is the result of the addition of the first one hundred natural numbers?* to other mathematical problems. The *Four Colour Theorem*. The *Goldbach's Conjecture*. Binary writing computation. The *Seven*

Bridges of Königsberg. Classical problems of the History of Mathematics: Zeno's paradoxes. The Ancient Egypt's method to determine an approximate value of *pi*. Reading and interpretation of the first pages of *Livro de Álgebra* by Pedro Nunez.

- Session 3: (03-12-2012) Al-Khwarizmi's Algebra: Geometry as an instrument to solve second degree equations. Evidence on the roles of the teacher and the student in the four stages of problem solving described by George Pólya in his book *How to Solve It*. Problem solving group activity and discussion regarding the heuristic procedures applied to attain the solution. Introduction to Voting Methods. John Banzhaf's power index and the United Nations Security Council voting system.
- Session 4: (10-12-2012) Probabilities. Problem solving dynamics under George Pólya's methodology for problem solving. Questions related to gambling. The *Monty Hall* problem.
- Session 5: (14-01-2013) Mathematics applied to Astronomy. The Eratosthenes' procedure to measure the Earth's circumference. Analysis of extracts form *Sidereus Nuncius* by Galileo Galilei, focussing on the method used to estimate the height of lunar mountains.
- Session 6: (21-01-2013) Introduction to Network Mathematics. Steiner's tree problem. Construction of minimal nets with the software *GeoGebra*. The Pythagoras' Theorem: *Chou Pei Suan Ching*, an Ancient Chinese mathematical text which includes a demonstration of the theorem.
- Session 7: (28-01-2013) Problem solving group dynamics on George Pólya's methodology for problem solving.
- Session 8: (04-02-2013) Relation between the Pythagoreans' five-pointed star and the *Golden Number*. Why do bees build honeycombs with a hexagonal configuration? *The Marshmallow Challenge*: group dynamics. *Flatland: A Romance of Many Dimensions*, analysis of some excerpts of the animation film.
- Session 9: (18-02-2013) Areas of lunes. Factors involved in the problem solving activity according to Alan Schoenfeld. Analysis of the written evaluation test (29-07-2008) on the topic *Mathematical Problem Solving* which allows students to enter the Master's Programmes of the Department of Mathematics of the University of Coimbra.
- Session 10: (25-02-2013) Brief summary of the topics analysed in the workshop. Open discussion where the teachers expressed their opinions regarding the work developed in each session. *Powers of Ten*, film analysis, a voyage to the infinitely large and the infinitely small. *Kiwi*, an animation film which illustrates how resilience can be a key factor in problem solving.

A set of guiding principles were kept in mind. From the Literature Review (previously presented) and from professional experience, these were extracted:

- The attitudes and behaviours revealed by the mathematicians in the course of problem solving activity are good examples of the intellectual skills required for teaching mathematics.
- Since problem solving is the core of mathematics, teachers should be skilled problem solvers.
- Mathematics is simultaneously a complex and straightforward system of coherent knowledge. It is impossible to successfully understand mathematics if we assume a compartmentalised conception of this science.
- Teachers should feel comfortable with mathematical language: *1)* being able to critically follow other people's reasoning; *2)* clearly explaining their analysis to others in different ways; *3)* formulating suitable enunciations for their thoughts; *4)* writing quality representations and explanations for others.
- Teachers do not think themselves as mathematicians in such a way that they can make significant discoveries; furthermore, as professionals, they also do not see themselves as significant elements within the education framework.
- It is difficult to establish teaching strategies when teachers are not familiarised with them.

These premises were observed during the on-site sessions.

Participants

In the 2-hour session *The Man and the Infinite – A Mathematical Approach*, held on 30th November 2012 at *Centro Educativo dos Olivais* (CEO), seven students from a Waiter Training vocational course, an alternative curricular path to Third Cycle Compulsory Education, were systematically challenged to participate, to think and verbalise their mathematical thoughts, where an immediate feedback is given after each intervention.

In *The use of Heuristics – Tools to Problem Solving* workshop we had a total of 34 participants, compulsory education Mathematics teachers (from the 7th to the 12th grade).

A sample[35] of 96 students' from the 9th grade, from the classes taught by the teachers who participated in *The use of Heuristics in Mathematics – Tools to Problem Solving* workshop was gathered to observe their attitudes towards Mathematical Problem Solving during the 2011/2012 school year.

[35] Data gathering authorisation (nº 0170100007) by *Monitorização de Inquéritos em Meio Escolar* (MIME) from *Direção-Geral de Inovação e de Desenvolvimento Curricular* (DGIDC).

A total of 339 students' from the classes of the teachers who participated in *The use of Heuristics in Mathematics – Tools to Problem Solving* 2011/2012 workshop solved assorted questions so that the quality of their answers and the eventual benefit of the use of heuristics could be evaluated.

Instruments

Maria da Luz Nogueira, teacher at *Centro Educativo dos Olivais* (CEO), summarises the session carried out on 30th November 2012 (10.30*h* – 12.30*h*) and the students' feedback with evaluation instruments *(Annex 1)* regarding Discipline (Attitude / Relation with pairs / Relation with teachers and other educational agents) and Performance (Interest / Adaptability / Creativity / Autonomy) to report students' interest in the session.

Prior to the practical work developed in the 2011/2012 and 2012/2013 workshops, research on mathematical problem solving, subjects and activities which would stimulate the interest and participation of the teachers enrolled in the training sessions was carried out. Pedagogical resources aiming different school levels and contents were compiled in a problem solving *Manual*. From the analysis of Pólya's methodology in the on-site sessions of the workshop emerged the opportunity to draw up a guide to help students solve problems (Almeida, 2004) *(Annex 2)*. This collective effort brought to light a 2-A_4-format page instrument whose purpose was to assist the students through problem solving activities on the subject of reading and understanding the statement of the problem, devising a plan, implementing the plan, and checking the solution.

A questionnaire about students attitudes towards Mathematics (Nicolaidou & Philippou, 2003) *(Annex 3)* was presented to 96 youngsters from the 9th grade *(Annex 4)*. Results and subsequent data analysis concerning Beliefs/Motivation, External Control, Mood, Resources, Heuristics and Knowledge Exhibition, mechanisms activated during the problem solving process, as anticipated by Schoenfeld, will be presented further on and the outcome show evidence of students' beliefs as an important variable in the teaching and learning practice.

In the course of the workshop a questionnaire for the teachers was implemented *(Annex 6)* supported on the Mathematics Teaching Efficacy Beliefs Instrument [36] (MTEBI) (Esterly, 2003) and on prior researcher (Anderson, 2000) *(Annex 5)*, and an assessment test *(Annex 7)* was given to the workshop participants, as well as exercises and problems *(Annex 8)* to be solved by their students so as to evaluate their performance in mathematical problem solving. Results, discussion and conclusions from statistic data will be presented further on.

[36] The MTEBI consists of 21 items, 13 items on the Personal Mathematics Teaching Efficacy (PMTE) subscale and eight items on the Mathematics Teaching Outcome Expectancy (MTOE) subscale.

Data analysis and results

Problem solving activity promotes reasoning and that triggers neurological impulses. What Alexander Pope (1688 – 1744) said about how Newton enlightened the world: *"Nature and nature's laws lay hidden in the night, and God said "Let Newton be!" and all was light."*, we say about Problem Solving: *"Let Heuristics be"* and the path to solution will be revealed!

The Man and The Infinite - A Mathematical Approach was a single 2-hour session, performed for a class of seven students in an atypical learning environment. Despite the positive reaction from the class to the proposed activities, the author is aware that to learn and practice more serious mathematical contents the revelation of a fragile glimpse of motivation by the participants is insufficient. The goal was to capture youngsters' attention and involvement with an appealing storyline focused on *Time and Man* with a mathematical approach. Only a systematic implementation of sessions could, eventually, validate the idea that selected activities can contribute to students' motivation to establish a more friendly relation with mathematics, even when they only master elementary mathematical concepts. Regardless of all constraints, this episode illustrates that it increasing students' motivation in order to practice maths might actually be possible.

The project Time and Man brought to Centro Educativo dos Olivais (CEO) the session The Man and The Infinite – A Mathematical Approach, conducted by Nuno Álvaro Ferreira Rodrigues, a Mathematics teacher and doctoral student at the Faculty of Psychology and Education Sciences of the University of Coimbra. The session was presented to the Waiter Training class on 30th November 2012 (10.30 h – 12.30 h) at CEO facilities.

The piece Santa Inês, a 16th century sculpture by João de Ruão, was the leitmotif that guided us through the Renaissance, a period of flourishing scientific development in Europe. After a thousand years of scant knowledge production, when compared to the extraordinary intellectual Ancient Greek thinking, the inventive capacity of human beings, to which mathematics was a main point of support, was reborn.

A PowerPoint presentation and worksheets with practical exercises were used as teaching resources and didactic materials.

The students' motivation and interest surpassed all expectations, with no exception. On a 1 to 5 scale, the students' informal evaluation ranged from $4\frac{1}{2}$ to 5. It was a moment of high pedagogical and training interest.

Maria da Luz Nogueira, Centro Educativo dos Olivais (CEO)

Concerning the 2011/2012 *The use of Heuristics in Mathematics – Tools to problem solving* workshop, teachers referred that the majority of the students were receptive and seemed to enjoy more doing Mathematics with problems than with the usual curriculum. Herewith we come to the limitations we encountered. We think that a much more ideal environment for making problem solving work would take place if the teachers themselves would firstly be exposed to these specific methodologies. The lecture and analysis of the first pages of Pedro Nunez's book *Libro de Algebra en Arithmetica y Geometria* regarding how to solve first and second degree equations were a success among teachers'. This illustrates that it is possible to teach and surprise skilled school teachers. However, the best place for this to happen would be the University. Otherwise, in very specialised courses on Educational Mathematics. Only then should they train their students, and only after that should we expect better school outcomes.

Teacher education is what educates teachers: it is what adds to the professional development of teachers. Participants were quite surprised that their expectations were met or even surpassed. Hence, for example, one participant says:

> *In my opinion, on the whole, this course was quite positive. My initial expectations were met or even surpassed. It served to deepen my mathematical and didactical knowledge, as well as that of the school curriculum, having in mind the construction of thought using specifically the Pólya method for solving problems. We carried out experiments aiming at developing the school curriculum in mathematics which contemplated not only the elaboration of problems, but also their analysis, and for this we could count on the help of the teacher educator.*

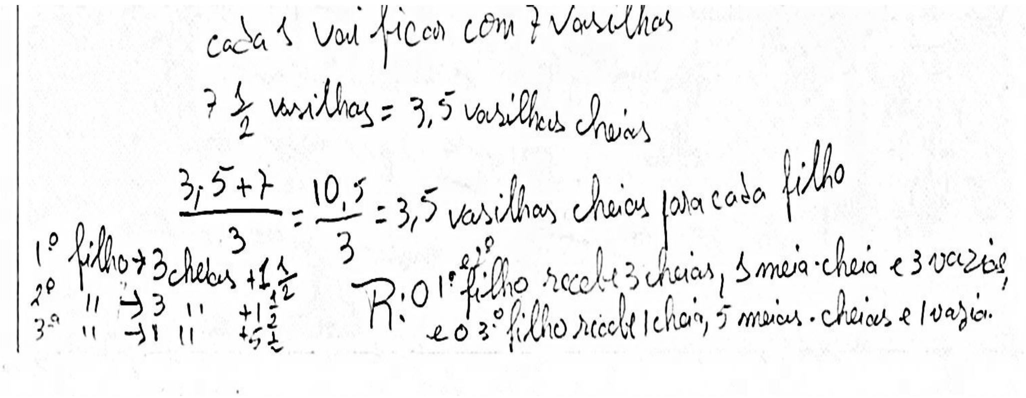
Let us now show how the participants imbued with the spirit of problem solving tried to transmit their excitation to their students; and let us also see some of the answers given by 9th grade students to the following question: How to divide 21 pots of honey, 7 full, 7 half-full and 7 empty, between three sons so that each one receives the same quantity of honey and the same number of pots without transferring any honey from one pot to the other? After a while, and using Pólya's guide, some of them said: *"this is equal to the problem of the camels"*[37], a problem that they had solved earlier. Such experiences consolidate their perceptions that what is learned can be transferred, and that abstraction can be useful *(Figure 19* and *Figure 20)*.

[37] Malba Tahan's book *O Homem Que Sabia Contar* (p. 14 – 16).

This student clarifies the given data: **1)** 7 pots of each type and three sons. **2)** Supposing (implicitly without loss of generality) that each pot has 1 litre, it is possible to calculate how much honey each son is entitled to. The answer is that there are 10.5 litres to distribute, and so each son should receive 3.5 litres of honey. **3)** The student draw a figure with 7 full pots (first row), 7 half-full pots (second row), 7 empty pots (third row), and correct borders. **4)** The student justifies the division.

(Figure 19)

Another solution:



(Figure 20)

The student: each son receives 7 pots. He notes that the seven half-full pots make 3.5 full pots, then adds these to the seven full pots and divides the amount by the three brothers: "$\frac{3.5+7}{3} = \frac{10.5}{3} = 3.5$ *full pots for each son"*. At the inferior left side corner he says: 1st son ⟶ 3 full + 1 $\frac{1}{2}$ (one half-full pot), 2nd son ⟶ 3 full + 1 $\frac{1}{2}$ (one half-full pot), 3rd son ⟶ 1 full + 5 $\frac{1}{2}$ (five half-full pots). Finally, he translates this into words: *"The first and the second sons receive 3 full pots, 1 half-full pot and 3 empty pots; and the third son receives 1 full pot, 5 half-full pots and 1 empty pot."*

> *"There is a very real danger that the type of Mathematics instruction we provide students is training them to be rigid in their thinking, not flexible and adaptable, is teaching them how to perform procedures but not when and under what conditions to perform them, and is showing them what to do but not why to do it"* (Lester, 1985, p. 43).

The relative enthusiasm showed by the students, when confronted with the problems, and the apparently greater motivation for Mathematics, is of course encouraging and important. Nevertheless, fluency in problem solving is not possible without knowledge, as we hope to have made clear, and chunks of knowledge are what we have committed to memory by hard work. Therefore, to avoid failure, courses which include problem solving must be well designed. The problems should not be predominantly pure brain teasers which cannot be connected to the Mathematics that the children know or will know. A pure brain teaser would be *e.g.* a chess problem, or a puzzle similar to the one where a string should be separated from a topologically complicated object. Of course such problems have their place in school but only sparingly, since connections to Geometry, Algebra, Probability, Functions, Analysis and other classic mathematical topics will not be easily established. Such problems would waste too much of the time that should be used for acquiring more fundamental knowledge such as the formula for the quadratic equation, the congruence theorems of Geometry, the basic formulae of Combinatory, the rules of differentiation… that is, the Mathematics that has proved useful in sciences and everyday life. So problem based learning promotes learning to learn, that is learning skills apart from learning facts, and learning to teach, as well as teaching to learn. Education is more than passive knowledge transmission. It requires ability to support student thinking into deep mathematical concepts, a milestone for those who embrace this vocation.

The use of Heuristics in Mathematics – Tools to problem solving was an opportunity to work mathematical questions, but also to listen to the teachers' concerns about Education. The topics selected for the workshop sessions captured the participants' interest and, in some situations, it was the first time that the teachers worked with several mathematical topics.

A theoretical framework shows evidence that positive durable mathematical beliefs can only be achieved within a long-term period. From the identification of what is a mathematical problem, through Pólya's four-step problem solving model and Schoenfeld's educational variables, we enlightened our hypothesis with two workshops (fifty hours each) for school teachers conducted in 2011/2012 and 2012/2013, complemented with students answers to questionnaires and selected mathematical questions.

There are no easy answers to complex questions. In mathematical instruction, difficulties are overcome with rational thinking, persistent work, and good teaching guidance, among many other success-enhancing factors. Serendipity can play an important role, especially when emphasised by a *Eureka* moment. But first the individual must always master a significant set of knowledge. Those who have studied advanced mathematics are well aware of the difference between an exercise and a problem, and know how to think in order to solve problems, even when they cannot reach the most desirable solution.

The last Curricular Revision, validated by Law No. 139/2012 of 5th July as well as by deliberation No. 5306/2012 of 18th April, aims to improve the quality of both teaching and learning with a rigorous instruction since Elementary School. To materialise such intentions, the Mathematics Curricular Goals, approved on 3rd August 2012, have been systematised. This document, written under the spirit of the 2007 Elementary Education Mathematics Programme, presents clearly enounced general objectives , divided into specific goals which, when accomplished, will lead to precise and quantifiable performances. Mathematics teaching should structure thought; it should engage the individual in real world analysis and society interpretation. Objectives which copy the essential performances that the students must reveal in each cycle of elementary instruction were established to fulfil such drive.

In the course of Third Cycle Elementary Education seven performances are required: *1)* Identify/Label: The student should correctly use designation, define concepts such as are presented to him or define such concepts in equivalent manner; *2)* Recognise: The student should unveil a coherent argumentation, eventually more informal than the teachers' reasoning. However, the student must be able to justify each particular step used in his argumentation; *3)* Recognise, given…: The student should be able to justify enouncements in straightforward cases with no need to present a generalisation; *4)* Know: The student should know the result, even without the requirement of justification or consistent proof; *5)* Evidence/Proof: The student must be able to present a mathematical proof as accurate as possible; *6)* Go further: This action is used in two distinct situations: *a)* To extend an already-known definition to a wider range. The student should define exactly the concept, or in an equivalent mode. *b)* To extend a property to a wider range. The student should be able to recognise such property, but eventually such perception can be restrained to particular cases. *7)* Justify: The student should be able to justify the enouncement in a simply manner, by using a property of his knowledge.

In 2015/2016, 10th grade students' will start a new Maths curriculum. Boolean logic with truth tables, the basics of computing, and Statistics will emerge with a fresh approach. Teachers must upgrade their knowledge to be prepared for changes and find the best way to communicate mathematical subjects in the classroom. Problem solving meetings, as those developed at Nova Ágora educational centre, are a good opportunity to enrich the mathematical culture of each teacher so as to become a better professional.

Problem solving is considered a powerful learning tool in Maths Education, both for the learner and the teacher. It might strengthen the motivation of the subjects involved through the challenges that they have to overcome, and it should foster the development of three main vectors: metacognition, knowledge and experience. Successful implementation depends on teacher educators capable to induce learning through effective teaching. Educators must be conscientious of their influence on students' attitude facing unexpected questions. To achieve this conscientiousness, teacher training should: *1)* be based on a model of mathematical-problem-solving activity which underlines psychological behaviour. A classical description was given by the Hungarian mathematician George Pólya; more recent work was done by Alan Schoenfeld. We suggest a balanced approach which uses applications of Mathematics but also pure problems. We must motivate but also wish to train formal methods and argumentation. So teacher training should also *2)* contain ingredients that will help to arouse students' interest and participation; it should *3)* provide possibilities of sharing experiences/resources/information between teachers, considering them the agents of dissemination of teaching tricks in the educational community; and to be more effective; *4)* teacher training should concentrate on problems which spread or need significant knowledge and provide opportunity for teachers to control the cognitive and metacognitive processes.

In each classroom, school and country, students score differently in Maths. PISA outcomes tell us that, on average, North European students score better than South European students. Why does that happen? What do they have that others don't? As was emphasised before, time plays an important role in the process of understanding and doing Maths. Why will I stay quietly closed surrounded by four walls with my Maths textbook as a companion if outside pleasure calls for me? If in the past I was not able to achieve success in problem solving activity, today I am certainly doomed to failure. National school educational system, the Maths curriculum and teachers' competences are external variables that play a decisive role in students' performance, but the cultural heritage of each country should also be considered as an important factor. Portuguese inhabitants are quite attached to the past and fatalism, and they still recall the glorious period of the Discoveries and wait for the return of Sebastian I, who disappeared, probably killed, in the battle of *Alcácer Quibir* in the 16th century.

The five dimensions of Time, Past-Negative, Past-Positive, Present-Hedonistic, Present-Fatalistic and Future-Oriented (Zimbardo & Boyd, 2009) shape people's choices. A Past-Negative perception combined with a Present-Hedonistic attitude jeopardises a Future-Oriented perspective related to Maths.

Nowadays adolescents have an almost infinite number of gadgets, mobile phones with Internet, videogame consoles and hundreds of TV channels at their disposal. All these instruments are attractive, colourful, entertaining, ready to be consumed, in opposition to students' awareness of Maths. A more appealing curriculum put into practice with vivid lessons may hold students' attention. However, in what concerns learning speed, it's impossible to go over all the stages of problem solving if we want to consolidate and apply knowledge far from routine exercises. Feedback can be quite different among adolescents; Future-Oriented teenagers engage, in principle, more from problem solving work than Present-Oriented teenagers, even though the latter understand the value of mathematical knowledge for their professional and social future. Perseverance, a main factor in problem solving activity, does not find much echo among Present-Hedonistic individuals.

In the course of the problem solving activity the outcome is only a fraction of the procedure and its significance depends on the question under analysis. Since the year 2005, 9th grade students have to perform a National Final Exam. The results of the first year where catastrophic! On average, more than seven out of ten students only obtained the classification of insufficient. On average, only one in one hundred students obtained the full score in the final question of the 2005 National Final Exam (1st Phase). The question is presented below.

Three identical spheres are put inside a cylindrical box (*Figure A*), as represented in the drawing (*Figure B*).

The height of the box is equal to the triple of the diameter of one of the spheres.

The radius of the basis of the cylinder is equal to the radius of one of the spheres.

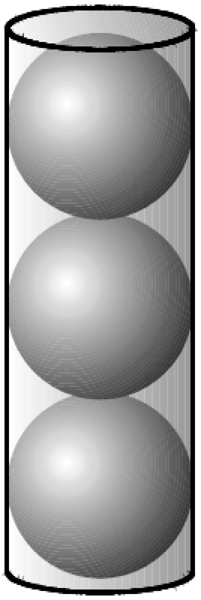


Figure A

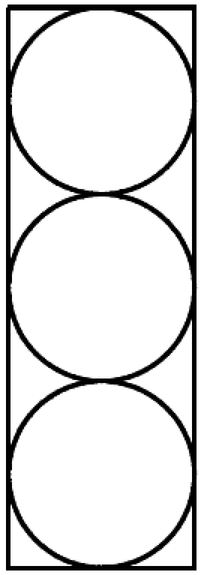


Figure B

Proof *The volume of the box not filled by the spheres is equal to half the volume of the three spheres*.

(Note: use *r* to define the radius of one of the spheres.)

The question above is a good example of how to practice problem solving, step by step, and not expect to immediately reach the proof. To give a complete answer we should perform a sequence of well-informed computation. The student has the formulas of the volume of the sphere and the volume of the cylinder in the form given with the exam.

$$V_{sphere} = \frac{4}{3}\pi r^3$$

$$V_{cylinder} = \pi r^2 \times height$$

The volume of the three spheres came straightforward: $4\pi r^3$.

Now the student should apply the abovementioned formulas. To succeed, the student must realise that the height of the cylinder is equal to the radius of the sphere multiplied by six.

$$V_{cylinder} = \pi r^2 \times 6r = 6\pi r^3$$

Subsequently, the volume of the empty space inside the cylinder can be defined by subtracting the volume of three spheres from the volume of the cylinder: $2\pi r^3$.

With a simple computation we verify that,

$$2\pi r^3 = \frac{1}{2}(4\pi r^3).$$

Q. E. D.

In order to reduce such low achievements in the Third Cycle National Final Exam, in 2006 the government implemented an action plan: educational courses to teachers and more hours allocated to Mathematics. Since then the results have balanced between mediocre and acceptable *(Table 16)*, and the main reason for such variation can be linked, primarily, with the level of difficulty of the exam.

The scenario concerning Secondary Education Maths A Final Exam is also far from good *(Table 17)*. The complexity of contents rises and subsequently requires much more intellectual accurateness from the student. Despite three more years of schooling and a more mature brain, results show that difficulties in answering mathematical questions did not disappear.

Concerning mathematical understanding, the slow progress showed by data helps us answer why the distance between what we know and what we do not know is always beyond what we can achieve with our learning speed.

(Table 16) - 9th Grade Maths National Final Exam (1st Phase) – Global Average Score (0 – 100), until the year 2011 contents were Statistics and Probability; Numbers and Operations; Algebra and Functions, Geometry.

Year	**Students**	**Total**	**Data Organization and Treatment**	**Numbers and Operations**	**Algebra**	**Geometry**
2008	90 064	55.3	80.1	59.9	52.8	43.0
2009	85 859	57.8	67.1	61.1	64.5	48.8
2010	84 437	51.1	45.4	57.9	55.5	46.7
2011	86 297	44.4	52.9	54.9	45.8	35.7
2012	88 228	54.4	55.7	65.8	52.8	50.4
2013	91 259	44.2	45.3	43.2	41.1	46.2
2014	92 082	52.8	50.8	44.1	55.3	72.9

Data Source: IAVE/MEC / JNE/MEC
Source: PORDATA

(Table 17) - Secondary Education Maths A National Final Exam (1^{st} Phase) – Global Average Score (0 – 100), until the year 2011 the 1^{st} Phase was not compulsory, about 80% of the students attended the 1^{st} Phase; it became compulsory in 2012.

Year	Students	Total	Probability and Combinatorics	Functions	Complex Numbers
2008	27 034	69.8	68.1	72.1	66.4
2009	26 684	58.4	64.0	53.9	60.6
2010	27 545	61.1	63.9	63.5	51.2
2011	28 132	52.9	47.5	56.3	50.4
2012	31 450	52.2	70.0	48.2	38.2
2013	31 620	48.6	58.8	41.6	51.7
2014	32 106	45.9	54.3	45.0	41.1

Data Source: IAVE/MEC / JNE/MEC
Source: PORDATA

Expertise only comes with time and dedication (Chi, Glaser, & Rees, 1982). If techniques can be hard to master, problem solving goes beyond them, because it requires deeper thinking. In Maths, simplicity and complexity sustain each other as two faces of the same coin.

Problem A: What is the value of 12 + 23 + 34 + 45 + 56 + 67 + 78 + 89?[38]

Problem B: What is the value of $x - y$, if $x = 1^2 + 2^2 + 3^2 + \ldots 2011^2$ and $y = 1\times 3 + 2\times 4 + 3\times 5 + 2010\times 2012$?[39]

Problems A and B regard methods to sum sequences of integers, but they are quite different. The first problem is straightforward and apparently without significance. We can label it an exercise. On the other hand, the second can be included in the domain of problem solving, as it connects knowledge, heuristic procedures, metacognition and beliefs (Schoenfeld, 2011).

[38] In a shortcut process, the student adds the eight parcels. Is he able to use a more efficient strategy, less susceptible to computational error and capable of generating the solution in the shortest time possible? To do so, students must have mathematical sensitivity, a quality that comes from practice. From the analysis of the problem emerges that pairs of numbers at the same distance from the extremes generate equal results. After the pattern is found, the solution becomes easy: 101 + 101 + 101 + 101= 4 × 101 = 404.

[39] What should we do after reading a question which is difficult to solve? As a marathon racer, problem solving is done step by step, the number of stages used and the time needed to accomplish the goal depend on the complexity of the subject and the skills of the problem solver.

$x = 1^2 + 2^2 + 3^2 + \cdots + 2011^2 = 1\times 1 + 2\times 2 + 3\times 3 + \cdots + 2011\times 2011.$

$y = 1\times 3 + 2\times 4 + \cdots + 2010\times 2012.$

$x - y = (1\times 1 + 2\times 2 + \cdots + 2011\times 2011) - (1\times 3 + 2\times 4 + 3\times 5 + \cdots + 2010\times 2012)$

$= (1\times 1 - 1\times 3) + (2\times 2 - 2\times 4) + \cdots + (2010\times 2010 - 2010\times 2012) + 2011\times 2011$

$= 1\times(1-3) + 2\times(2-4) + 3\times(3-5) + \cdots + 2010\times(2010-2012) + 2011\times 2011$

$= 1\times(-2) + 2\times(-2) + 3\times(-2) + \cdots + 2010\times(-2) + 2011\times 2011$

$= (-2)\times(1+2+3+\cdots+2010) + 2011\times 2011$

$= (-2)\times(2011\times 1005) + 2011\times 2011 = (2011)\times(-2010) + 2011\times 2011 = 2011\times(2011-2010) = 2011.$

We can question how consistently and successfully feasible it is to engage students in high-level mathematical tasks in the classroom, when high-level tasks are obviously more complex and extended in time than routine classroom activities, and are subsequently susceptible to several factors which could cause rejection in students' engagement if compared with less demanding thought processes (Doyle, 1988). Low motivation and negative beliefs work together to diminish students' engagement in the problem solving task.

In the next section we will present the results of a problem solving questionnaire – students' engagement, which supports the prior conceptions in order to answer our research questions.

Reporting a problem solving questionnaire experiment –students' engagement

> *We hypothesize that one of the reasons why some students get higher grades than others can be found in the fact that they are more highly motivated. Furthermore, we assume that they are more motivated because they possess a longer FTP. FTP is, however, not operationalized as the capacity to fantasize about future events. We conceptualize of FTP as consisting of two aspects. The dynamic aspect of FTP is formed by the disposition to ascribe high valence to goals in the distant future. The cognitive aspect of FTP is formed by the disposition to grasp the long-term consequences of actual behavior, as reflected in the concept of instrumental value of a behavioral act. This means that each category of behavior (studying, social contacts, and so on) has its own FTP, at least the cognitive aspect of it* (de Volder & Lens, 1982).

Learning and using what is learned is not restricted to school. Mathematical knowledge should be an integral part of an individual and an educational priority. Although it is true that the construction and consolidation of knowledge happens throughout life, adolescence is undoubtedly one of the most important and sensitive periods in terms of learning and the perfect time to develop further cognitive skills. It is also largely spent at school. In Portugal, the results obtained by students in the school national exams are, on average, below the expected. Unfortunately, the educational reforms implemented over the past decades have not solved this problem. How can we motivate students to devote time and energy to the study of Mathematics because it is relevant their future success? Which strategies catalyse good performances for lifelong learning? Heuristics! Our research started in 2011/2012 with a workshop devoted to mathematical problem solving supported by George Pólya's model for Third Cycle and Secondary Education teachers. Teachers were invited to apply this methodology to 9th grade students during the 2011/2012 school year. In parallel with the tracking of results by solving selected questions, students answered a questionnaire about their attitudes towards Maths and

Problem Solving at the beginning and at the end of the school year. We explore the promise of learning problem-based Mathematics to shorten the distance between what we do not know and our willingness to learn.

Throughout the last trimester of 2011 we conducted *The use of Heuristics in Mathematics – Tools to problem solving*, a fifty hour course for high school teachers at Nova Ágora Educational Centre, in Coimbra. In parallel to the workshop, participants were invited to collaborate in an exploratory study focused on 9th grade students' attitudes towards Mathematics and their performances concerning mathematical problem solving. Why the focus on them? Because 9th grade students are less than one year away from starting Secondary Education, a moment when they need to make curricular options with future consequences, both academic and professional. In the 2011/2012 school year more than one in four students from the 9th grade did not accomplish success in Maths. Behaviours are consequences of each one's motivations. In a socio-cognitive perspective, motivation is conceived as a lively process about individual experiences and perceptions with interactions between subject and context (Dweck & Leggett, 1988).

In general, a complex collection of factors, when mastered by the teacher, act more or less harmoniously in orchestrating classroom activity. These factors include the manner in which order is established in the classroom, the physical organisation of the space, the amount of time allotted to the various activities, the manner in which transition periods between tasks are done and the ways in which discipline interventions are handled (Doyle, 1986).

To identify what explains students' attitudes and their influence towards Mathematical Problem Solving, we implemented a questionnaire to be answered by students at the beginning and at the end of the school year *(Chart 5)*.

(Chart 5) – Items of the Attitudes towards Mathematical Problem Solving

Q_1. I like doing problem solving.
Q_2. Time spent doesn't matter when I'm trying to find the solution.
Q_3. When I reach a solution I always check my reasoning and calculations.
Q_4. My justifications are well organised so as to be easily understood.
Q_5. I like when my teacher sees me doing problem solving well.
Q_6. I enjoy solving problems on the board and correcting resolutions.
Q_7. Learning to solve problems can be useful in my daily life / professional future.
Q_8. I believe problem solving is a good mental exercise because I learn to think.
Q_9. When I'm engaged in doing problem solving I like to decide what to do.
Q_{10}. I feel embarrassed when I don't know how to solve a problem.
Q_{11}. Problem solving makes me anxious.
Q_{12}. I read the statement of the problem carefully.
Q_{13}. I should be more swift and effective in problem solving.
Q_{14}. I need help to decide what to do.
Q_{15}. Mistakes make me badly humoured.
Q_{16}. When I have difficulties I know that I can do something to improve.
Q_{17}. I analyse the problem before doing any calculations.
Q_{18}. I believe I do problem solving well.
Q_{19}. This is a funny activity.
Q_{20}. Problem solving is exhausting.
Q_{21}. If I realise the question is complex, I give up.
Q_{22}. Even when I check the solution I still don't know if it is right or wrong.

For each question the student had to choose one of four degrees of agreement about what he felt as true for him/herself for the respective statement: Very Much, More or Less, Sometimes and Not Really, scoring 3, 2, 1 and 0 points in Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, Q_9, Q_{12}, Q_{16}, Q_{17}, Q_{18} , Q_{19} and 0, 1, 2 and 3 points in the remaining items, consistently inversing the scores, as the most adequate attitudes for successful learning and problem solving were questioned. All data were organised with SPSS[40] tools.

> When students ascribe higher valence to goals in the distant future and higher instrumental value to studying hard for reaching goals in the distant future, they will be more persistent in their daily study and obtain better academic results. The causality of these relationships is not derived from the design, which is correlational, but from the cognitive theories of motivation in which the variables are embedded. We identify the motivational effects of a long FTP with ascribing high valence to goals in the distant future (dynamic aspect of FTP) and with ascribing high instrumental value to present (study) behavior for reaching goals in the distant future (cognitive aspect of FTP). No hypotheses were stated regarding the open present. Results were ambiguous and difficult to interpret (de Volder & Lens, 1982).

The study was conducted during the 2011/2012 school year, on a sample of 96 youngsters, 9th grade students, from the classes of the 18 Third Cycle and Secondary Education teachers who participated in *The use of Heuristics in Mathematics – Tools to problem solving* workshop. We intended to characterise problem-solving performances and attitudes. Fifty-two girl (54.2%) and forty-four boy students (45.8%) from seven classes in six public schools from urban areas in the district of Coimbra participated. The questionnaire regarding problem solving attitudes follows Schoenfeld's model, aiming at the assessment of heuristics, resources, control and beliefs.

Firstly, an Alpha Factor Analysis (AFA) was conducted with the 94 valid answers. This procedure attempts to create factors, which are linear combinations of the variables (the 22 questionnaire items), so as to estimate the latent variables or constructs that the instrument is measuring. The AFA method generates factors in such a way that the Cronbach's alpha (reliability) is maximised. In a global scale, concerning the first term of the school year, data reveals internal consistency, showing a Cronbach's alpha of 0.778.

An Exploratory Factor Analysis (EFA) was conducted to determinate the most appropriate dimensional structure for the data. An Exploratory factor analysis was performed using the principal components method (PCA) with the Statistical Program for Social Sciences (SPSS 20®).

[40] Statistical Program for Social Sciences.

Labels and items are related as follows *(Chart 6)*.

(Chart 6) - Correspondence between Labels and Items

Labels	Items
Beliefs/Motivation	$(Q_1, Q_2, Q_7, Q_8, Q_9, Q_{18}, Q_{19})$
External Control	(Q_{13}, Q_{14}, Q_{22})
Mood	(Q_{10}, Q_{11}, Q_{15})
Resources	(Q_{20}, Q_{21})
Heuristics	$(Q_3, Q_4, Q_{12}, Q_{16}, Q_{17})$
Knowledge Exhibition	(Q_5, Q_6)

The mean and standard deviation regarding each question is shown below *(Table 18)*.

(Table 18) – Mean and standard deviation of each question

	Mean	Std. Deviation
$Q_1 1^{st} Term$	1.62	.735
$Q_2 1^{st} Term$	2.05	.920
$Q_3 1^{st} Term$	2.21	.654
$Q_4 1^{st} Term$	2.32	.736
$Q_5 1^{st} Term$	1.85	.903
$Q_6 1^{st} Term$	2.59	.576
$Q_7 1^{st} Term$	2.10	.704
$Q_8 1^{st} Term$	1.88	1.096
$Q_9 1^{st} Term$	1.29	.650
$Q_{10} 1^{st} Term$	1.63	1.087
$Q_{11} 1^{st} Term$	.88	.746
$Q_{12} 1^{st} Term$	1.33	.694
$Q_{13} 1^{st} Term$	1.82	1.057
$Q_{14} 1^{st} Term$	2.33	.646
$Q_{15} 1^{st} Term$	2.14	.615
$Q_{16} 1^{st} Term$	1.50	.913
$Q_{17} 1^{st} Term$	1.74	.903
$Q_{18} 1^{st} Term$	1.95	.753
$Q_{19} 1^{st} Term$	2.34	.811
$Q_{20} 1^{st} Term$	1.48	.744
$Q_{21} 1^{st} Term$	1.27	.882
$Q_{22} 1^{st} Term$	1.37	.776

The item – item correlation matrix data are as follows *(Table 19)*.

(Table 19) / Item – Item correlation matrix

Inter-Item Correlation Matrix

	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	Q_9	Q_{10}	Q_{11}	Q_{12}	Q_{13}	Q_{14}	Q_{15}	Q_{16}	Q_{17}	Q_{18}	Q_{19}	Q_{20}	Q_{21}	Q_{22}
Q_1		.174	.216	.169	.286	.383	.383	-.163	-.082	.021	.094	.208	.048	.133	.118	.705	.175	.390	.330	.555	.275	-.030
Q_2			.374	.467	.242	.347	.208	-.015	-.116	-.023	-.289	.056	.143	.350	.215	.211	.081	.190	.293	.182	.075	-.209
Q_3				.371	.072	.380	.095	-.130	-.019	-.008	-.235	-.038	-.006	.443	.300	.198	-.034	.264	.329	.164	-.155	-.073
Q_4					.234	.214	.210	-.100	-.081	.150	-.068	.107	.006	.319	.210	.224	-.183	.264	.392	.307	-.082	-.022
Q_5						.211	.158	.080	.037	.031	-.090	-.024	.039	.104	.134	.352	.032	.130	.276	.155	.172	-.089
Q_6							.285	-.095	-.080	.025	-.215	.077	.140	.401	.346	.501	.084	.370	.421	.393	.135	-.180
Q_7								-.194	-.131	.117	.001	.199	-.208	.166	.093	.443	.073	.334	.300	.527	.132	.032
Q_8									.350	.531	.286	.065	.520	-.112	.040	-.263	.241	-.203	-.076	.056	.233	.090
Q_9										.229	.292	.122	.280	-.074	-.100	-.063	.255	-.056	-.188	-.021	.128	.148
Q_{10}											.330	.407	.380	-.022	.142	-.016	.121	.199	.182	.329	.228	.192
Q_{11}												.429	.177	-.142	-.105	.071	.195	-.050	-.111	.199	.358	.318
Q_{12}													.199	.115	.068	.229	.290	.199	.085	.337	.400	.309
Q_{13}														-.069	.088	.050	.199	.055	.060	.221	.248	-.074
Q_{14}															.534	.155	-.002	.235	.338	.160	.014	-.098
Q_{15}																.124	.161	.202	.250	.277	.070	-.222
Q_{16}																	.156	.446	.349	.530	.367	-.023
Q_{17}																		.011	-.100	.200	.478	-.047
Q_{18}																			.329	.507	.183	.145
Q_{19}																				.369	.113	-.152
Q_{20}																					.263	.023
Q_{21}																						.027
Q_{22}																						

Statistics allows identifying the multiple connections that underline the complexity of behaviours in relation to Maths Problem Solving. Answers to the first term questionnaire exposed a set of pairs, namely, Beliefs/Motivation – Resources, Beliefs/Motivation – Heuristics, Beliefs/Motivation – Mood, External Control – Mood, Heuristics – Resources. It is not by chance that Beliefs/Motivation is such a powerful instrument; it captures the idea of success *vs.* failure and perseverance *vs.* renunciation. It also supports theoretical understanding and the determination to show good work to others. Control skills are linked to disposition and knowledge, undoubtedly the lack of aptitude to deal with emotions can be the source of bad humour, as metacognition helps to organise, open and use what is inside each one's shelves of information. When I understand what to do I like to do it! The good use of heuristics seems to give confidence, allowing the student to share their reasoning with the teacher or colleagues.

Communalities (Table 20) tell us what proportion of each variable's variance is shared with the factors which have been created. The values in the initial column show how much variance each variable shared with all the other variables. Items Q_5: *I like when my teacher sees me doing problem solving well* and Q_9: *When I'm engaged in doing problem solving I like to decide what to do* have disturbingly low values here.

(Table 20) - Communalities

	Initial	Extraction
$Q_1 1^{st} Term$	.570	.561
$Q_2 1^{st} Term$	.463	.395
$Q_3 1^{st} Term$	.404	.467
$Q_4 1^{st} Term$	.451	.491
$Q_5 1^{st} Term$	.288	.240
$Q_6 1^{st} Term$	.487	.465
$Q_7 1^{st} Term$	.468	.387
$Q_8 1^{st} Term$	.644	.719
$Q_9 1^{st} Term$	.287	.314
$Q_{10} 1^{st} Term$	.569	.693
$Q_{11} 1^{st} Term$	.481	.499
$Q_{12} 1^{st} Term$	.492	.475
$Q_{13} 1^{st} Term$	.487	.436
$Q_{14} 1^{st} Term$	.486	.588
$Q_{15} 1^{st} Term$	.430	.453
$Q_{16} 1^{st} Term$	.689	.817
$Q_{17} 1^{st} Term$	.410	.522
$Q_{18} 1^{st} Term$	.466	.381
$Q_{19} 1^{st} Term$	.415	.460
$Q_{20} 1^{st} Term$	.643	.680
$Q_{21} 1^{st} Term$	.500	.497
$Q_{22} 1^{st} Term$	.329	.446

Extraction Method: Alpha Factoring.

We asked SPSS to create six factors. The *Communalities* in the extracted column tell us how much variance each variable has in common with the six factors that we kept. Again, items Q_5 and Q_9 retain disturbingly low values. If a variable does not share much variance with the other variables or with the retained factors, it is unlikely to be useful in defining a factor.

Initial results show that the first six factors explain, approximately, 63% of all variance. Factors have been labelled according to structure matrix data and Schoenfeld's theoretical model.

The *Total Variance Explained (Table 21)* shows the *Eigenvalues* for our factor analysis. SPSS started out by creating 22 factors, each a weighted linear combination of the 22 items. The initial eigenvalues tell us, for each of those 22 factors, how much of the variance in the 22 items was captured by that factor. A factor with an eigenvalue of 1 has captured as much variance as there is in one variable. Extraction Sums of Squared Loadings are interpreted in the same way of eigenvalues.

(Table 21) – Total Variance Explained outcome

	Initial Eigenvalues			Extraction Sums of Squared Loadings			Rotation Sums of Squared Loadings		
Factor	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %
1	4.856	22.073	22.073	4.361	19.824	19.824	3.391	15.414	15.414
2	3.269	14.857	36.930	2.802	12.734	32.559	2.320	10.543	25.958
3	2.002	9.102	46.031	1.522	6.916	39.475	2.013	9.148	35.105
4	1.537	6.988	53.020	1.071	4.868	44.343	1.398	6.354	41.459
5	1.232	5.602	58.621	.722	3.282	47.625	1.328	6.035	47.494
6	1.024	4.656	63.277	.509	2.313	49.938	.538	2.443	49.938

Extraction Method: Alpha Factoring.

The *Factor Matrix (Table 22)* presents the loadings, that is, the correlations between each variable and each factor. Items Q_{16} , Q_{20} , Q_6 , Q_{19} , Q_1 , Q_{18} , Q_{14} , Q_4 , Q_2 and Q_7 are significantly positively correlated with factor 1; items Q_{11} , Q_{10} , Q_8 , Q_{21} and Q_{12} are significantly positively correlated with factor 2; item Q_8 is significantly positively correlated with factor 3.

(Table 22) – Factor Matrix

	Factor					
	1	2	3	4	5	6
$Q_{16} 1^{st} Term$	.717	.141	-.370	-.323	-.116	.171
$Q_{20} 1^{st} Term$	.685	.327	-.195	.062	-.023	-.247
$Q_6 1^{st} Term$	.651	-.093	.110	-.112	.084	-.011
$Q_{19} 1^{st} Term$	.620	-.118	.049	.078	-.165	-.161
$Q_1 1^{st} Term$	.620	.142	-.302	-.236	-.077	.061
$Q_{18} 1^{st} Term$	.562	.068	-.194	.151	.016	-.006
$Q_{14} 1^{st} Term$	.520	-.219	.246	.211	.395	.094
$Q_4 1^{st} Term$	.518	-.163	.117	.337	-.231	.124
$Q_2 1^{st} Term$	.512	-.181	.277	-.022	-.071	.135
$Q_7 1^{st} Term$	.502	.018	-.354	.011	-.010	-.094
$Q_3 1^{st} Term$	.472	-.269	.229	.222	.115	.238
$Q_{15} 1^{st} Term$	.447	-.074	.321	.035	.337	-.173
$Q_5 1^{st} Term$	.358	.006	.072	-.179	-.254	.100
$Q_{11} 1^{st} Term$	-.107	.659	-.177	.140	.045	.026
$Q_{10} 1^{st} Term$	.191	.590	.199	.396	-.152	-.298
$Q_8 1^{st} Term$	-.135	.571	.557	.023	-.229	-.109
$Q_{21} 1^{st} Term$	.272	.562	-.043	-.312	.093	-.004
$Q_{12} 1^{st} Term$	.259	.546	-.164	.208	.197	.026
$Q_{17} 1^{st} Term$	.131	.479	.124	-.364	.340	.110
$Q_{13} 1^{st} Term$	.108	.452	.442	-.049	-.134	-.064
$Q_9 1^{st} Term$	-.125	.428	.204	.023	-.040	.266
$Q_{22} 1^{st} Term$	-.124	.321	-.323	.414	-.014	.228

Extraction Method: Alpha Factoring.

a. 6 factors extracted. 11 iterations required.

A $Varimax$ rotation minimises the complexity of the factors by composing the large loadings larger and the small loadings smaller within each factor. The Rotated Factor Matrix *(Table 23)* gives the loadings after the rotation. Note that items Q_{16} , Q_{20} , Q_1 , Q_7 , Q_{18} and Q_{19} , are positively correlated with factor 1; items Q_3 and Q_{15} are positively correlated with factor 2; items Q_{10} and Q_{13} are positively correlated with factor 3; items Q_{22} , Q_{11} and Q_{12} are positively correlated with factor 4; items Q_{17} and Q_{21} are positively correlated with factor 5.

(Table 23) – Rotated Factor Matrix

	Factor					
	1	2	3	4	5	6
$Q_{16} 1^{st} Term$	.832	.094	-.129	.003	.226	.219
$Q_{20} 1^{st} Term$	.715	.179	.206	.167	.078	-.246
$Q_1 1^{st} Term$	.712	.088	-.071	.011	.179	.097
$Q_7 1^{st} Term$	.574	.103	-.147	.098	-.023	-.122
$Q_{18} 1^{st} Term$	.512	.281	-.026	.181	-.038	-.070
$Q_{19} 1^{st} Term$	.506	.350	.112	-.146	-.206	-.073
$Q_5 1^{st} Term$	.349	.106	.124	-.176	-.008	.246
$Q_{14} 1^{st} Term$	.099	.743	-.093	-.003	.061	-.115
$Q_3 1^{st} Term$	.120	.645	-.067	-.016	-.118	.133
$Q_{15} 1^{st} Term$	.119	.550	.105	-.165	.155	-.273
$Q_2 1^{st} Term$	.260	.484	.088	-.218	-.056	.187
$Q_4 1^{st} Term$	.317	.463	.104	.058	-.371	.156
$Q_6 1^{st} Term$	.452	.461	.014	-.192	.103	-.012
$Q_8 1^{st} Term$	-.184	-.082	.810	.022	.134	.063
$Q_{10} 1^{st} Term$	.170	.051	.682	.354	-.076	-.257
$Q_{13} 1^{st} Term$	.021	.065	.626	-.024	.188	.060
$Q_9 1^{st} Term$	-.148	-.042	.337	.242	.226	.258
$Q_{22} 1^{st} Term$	-.006	-.130	-.013	.646	-.084	.071
$Q_{11} 1^{st} Term$	.060	-.260	.284	.528	.253	-.060
$Q_{12} 1^{st} Term$	.270	.081	.199	.515	.266	-.142
$Q_{17} 1^{st} Term$	.086	.042	.186	.049	.690	.020
$Q_{21} 1^{st} Term$	.364	-.085	.270	.105	.523	.004

Extraction Method: Alpha Factoring.
Rotation Method: Varimax with Kaiser Normalization.

a. Rotation converged in 12 iterations.

Months later, in the course of third term of the school year, the same students answered the same questionnaire. Statistics of the 84 valid answers are shown below. In a global scale, concerning the third term of the school year, data reveals internal consistency, showing a Cronbach's alpha of 0.813.

Mean and standard deviation regarding each question are presented below *(Table 24)*.

(Table 24) – Mean and standard deviation

	Mean	Std. Deviation
$Q_1 3^{rd} Term$	1.69	.760
$Q_2 3^{rd} Term$	1.93	.861
$Q_3 3^{rd} Term$	2.04	.685
$Q_4 3^{rd} Term$	2.26	.730
$Q_5 3^{rd} Term$	1.82	.867
$Q_6 3^{rd} Term$	2.46	.648
$Q_7 3^{rd} Term$	2.17	.848
$Q_8 3^{rd} Term$	1.71	1.059
$Q_9 3^{rd} Term$	1.21	.695
$Q_{10} 3^{rd} Term$	1.49	.988
$Q_{11} 3^{rd} Term$	.68	.563
$Q_{12} 3^{rd} Term$	1.11	.602
$Q_{13} 3^{rd} Term$	1.58	.972
$Q_{14} 3^{rd} Term$	2.18	.679
$Q_{15} 3^{rd} Term$	1.95	.599
$Q_{16} 3^{rd} Term$	1.38	.890
$Q_{17} 3^{rd} Term$	1.43	.854
$Q_{18} 3^{rd} Term$	1.98	.744
$Q_{19} 3^{rd} Term$	2.27	.750
$Q_{20} 3^{rd} Term$	1.46	.813
$Q_{21} 3^{rd} Term$	1.04	.828
$Q_{22} 3^{rd} Term$	1.31	.744

Surprisingly, students' scored lower in most of the items at the end of the school year. The occurrence was most perceptible in questions Q_{11}, Q_{12}, Q_{13}, Q_{17} and Q_{21}. In our twenty two-item questionnaire, five questions (Q_3, Q_4 , Q_{12}, Q_{16} and Q_{17}) were related to Heuristics. The scores of the students showed a significantly lower mean in the third term when compared with the first term.

In the third term of the school year students maintained lack of Control, Mood, Resources and Heuristics. Such data appears to emphasise how difficult it is to observe progress regarding Beliefs/Motivation and Knowledge Exhibition attitudes in the students who initially did not perform well in these categories. Eventually, progress concerning Control, Mood, Resources and Heuristics can be achieved more swiftly by those who, in the first term were already imbued with these qualities.

Item – Item correlation matrix data is showed below *(Table 25)*.

(Table 25) / Item – Item correlation matrix

	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	Q_9	Q_{10}	Q_{11}	Q_{12}	Q_{13}	Q_{14}	Q_{15}	Q_{16}	Q_{17}	Q_{18}	Q_{19}	Q_{20}	Q_{21}	Q_{22}
Q_1		.371	.207	.343	.391	.417	.586	-.036	.013	.123	-.038	.390	-.046	.435	.364	.657	.355	.583	.489	.684	.209	.129
Q_2			.331	.471	.467	.362	.363	-.274	-.216	-.029	-.122	.108	-.108	.475	.507	.256	.157	.373	.515	.392	.088	-.003
Q_3				.367	.234	.179	.280	-.069	-.118	-.062	-.064	.137	-.068	.297	.298	.254	.221	.262	.403	.165	.040	-.022
Q_4					.399	.376	.337	-.042	-.183	-.029	-.144	.155	-.082	.536	.359	.308	.166	.455	.395	.259	.104	-.084
Q_5						.278	.238	-.174	.004	-.052	-.070	.014	-.118	.239	.123	.386	.202	.236	.317	.324	.127	.087
Q_6							.449	.038	-.116	.018	-.246	-.006	-.014	.356	.182	.316	.224	.348	.528	.432	.171	.148
Q_7								.067	-.061	.146	-.164	.319	-.119	.282	.253	.473	.250	.503	.458	.446	.094	.070
Q_8									.248	.538	-.035	.162	.316	-.079	-.117	-.151	.283	-.039	-.143	.044	.039	.282
Q_9										.214	.024	.146	.116	-.107	-.293	-.017	.330	-.223	-.229	.014	.133	.592
Q_{10}											.091	.337	.440	-.006	.060	-.008	.363	.016	-.020	.254	.229	.087
Q_{11}												.210	.105	-.100	.097	-.041	.089	-.047	-.245	.040	.180	.010
Q_{12}													.263	.041	.148	.260	.332	.221	.068	.439	.331	.140
Q_{13}														-.233	-.055	-.121	.232	-.147	-.090	.065	.153	.130
Q_{14}															.524	.245	.240	.390	.494	.306	.010	-.087
Q_{15}																.192	.158	.484	.485	.367	.052	-.264
Q_{16}																	.258	.396	.293	.518	.177	.111
Q_{17}																		.168	.153	.421	.234	.376
Q_{18}																			.465	.437	.138	-.182
Q_{19}																				.362	.003	-.089
Q_{20}																					.208	.138
Q_{21}																						.119
Q_{22}																						

The answers to the questionnaire given to the students in the third term of the school year highlighted the pairs Beliefs/Motivation – Mood, Beliefs/Motivation – Heuristics, Beliefs/Motivation – Resources, Beliefs/Motivation – External Control, Heuristics – External Control, Heuristics – Resources, External Control – Mood, Beliefs/Motivation – Knowledge Exhibition.

Regardless of our small sample of 9th grade students, results corroborate Schoenfeld's Mathematical Problem Solving model, with strong connections between Beliefs/Motivation, Control, Resources and Heuristics.

Despite the teachers' vivid participation in the workshop and their willingness and even eagerness to use and promote heuristics in mathematical problem solving classroom activities, this sole initiative to improve students' attitudes is evidently clearly insufficient. Maths comprehension is an intricate process. Literature on Problem Solving highlights methodologies which can be used to improve performances; but still it is not one *Lapis Philosophorum*. The way students engage themselves in mathematical problem solving is the result of many components, a never ending journey into the fields of cognition.

Concerning communalities *(Table 26)*, items Q_3: *When I reach a solution I always check my reasoning and calculations*, Q_{11}: *Problem solving makes me anxious* and Q_{21}: *If I realise the question is complex, I give up* have disturbingly low values.

(Table 26) - Communalities

	Initial	Extraction
$Q_1 3^{rd} Term$	.734	.803
$Q_2 3^{rd} Term$	.593	.580
$Q_3 3^{rd} Term$	.303	.235
$Q_4 3^{rd} Term$	.509	.468
$Q_5 3^{rd} Term$	.427	.414
$Q_6 3^{rd} Term$	.542	.507
$Q_7 3^{rd} Term$	.535	.559
$Q_8 3^{rd} Term$	.506	.577
$Q_9 3^{rd} Term$	.495	.597
$Q_{10} 3^{rd} Term$	.519	.584
$Q_{11} 3^{rd} Term$	.247	.347
$Q_{12} 3^{rd} Term$	.477	.489
$Q_{13} 3^{rd} Term$	.362	.413
$Q_{14} 3^{rd} Term$	.563	.561
$Q_{15} 3^{rd} Term$	.562	.655
$Q_{16} 3^{rd} Term$	.533	.548
$Q_{17} 3^{rd} Term$	.453	.543
$Q_{18} 3^{rd} Term$	.541	.544
$Q_{19} 3^{rd} Term$	.588	.608
$Q_{20} 3^{rd} Term$	.621	.577
$Q_{21} 3^{rd} Term$	.243	.239
$Q_{22} 3^{rd} Term$	.553	.604

Extraction Method: Alpha Factoring.

(Table 27) – Total Variance Explained outcome

	Initial Eigenvalues			Extraction Sums of Squared Loadings			Rotation Sums of Squared Loadings		
Factor	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %
1	5.939	26.996	26.996	5.485	24.934	24.934	3.229	14.678	14.678
2	3.103	14.103	41.098	2.603	11.834	36.767	3.128	14.217	28.896
3	1.665	7.570	48.668	1.190	5.410	42.177	1.790	8.136	37.031
4	1.435	6.522	55.190	.967	4.395	46.572	1.617	7.350	44.381
5	1.186	5.392	60.582	.733	3.334	49.906	1.011	4.593	48.974
6	.988	4.492	65.074	.474	2.152	52.058	.678	3.084	52.058

Extraction Method: Alpha Factoring.

The results of the third term show that the first six factors explain approximately 65% of all variance *(Table 27)*.

(Table 28) – Factor Matrix

	Factor					
	1	2	3	4	5	6
$Q_1 3^{rd} Term$	.781	.194	.039	-.195	-.317	.125
$Q_{19} 3^{rd} Term$	.698	-.244	.062	.235	.007	-.044
$Q_{18} 3^{rd} Term$	.667	-.090	-.185	.028	-.210	.110
$Q_2 3^{rd} Term$	.666	-.209	-.016	-.009	.258	-.160
$Q_{20} 3^{rd} Term$	.663	.307	-.048	-.102	-.176	.000
$Q_7 3^{rd} Term$	.653	.072	.113	.073	-.317	.092
$Q_4 3^{rd} Term$	.630	-.154	-.005	.101	.183	-.068
$Q_{14} 3^{rd} Term$	.617	-.212	-.011	.151	.226	.248
$Q_{16} 3^{rd} Term$	.608	.085	.107	-.328	-.228	.016
$Q_6 3^{rd} Term$	.579	-.010	.271	.166	-.124	-.234
$Q_{15} 3^{rd} Term$	.573	-.170	-.449	.147	.175	.210
$Q_5 3^{rd} Term$	.501	-.065	.197	-.222	.167	-.209
$Q_3 3^{rd} Term$	.440	-.101	-.012	.042	.168	.038
$Q_{10} 3^{rd} Term$	.089	.639	-.175	.364	-.035	-.063
$Q_{17} 3^{rd} Term$	.408	.539	.110	.044	.236	.127
$Q_9 3^{rd} Term$	-.147	.537	.424	-.153	.181	.226
$Q_8 3^{rd} Term$	-.088	.532	.107	.503	-.066	.133
$Q_{12} 3^{rd} Term$	.330	.529	-.282	-.116	-.073	.044
$Q_{13} 3^{rd} Term$	-.097	.498	-.158	.239	.070	-.261
$Q_{21} 3^{rd} Term$	.210	.353	-.123	-.166	.036	-.163
$Q_{22} 3^{rd} Term$	.024	.500	.545	-.142	.168	.091
$Q_{11} 3^{rd} Term$	-.114	.251	-.399	-.296	.137	.074

Extraction Method: Alpha Factoring.

a. 6 factors extracted. 11 iterations required.

Note that items Q_1, Q_{19}, Q_{18}, Q_2, Q_{20}, Q_7, Q_4, Q_{14}, Q_{16}, Q_6, Q_{15} and Q_5 are significantly positively correlated with factor 1; items Q_{10} , Q_{17} , Q_9 , Q_8 and Q_{12} are significantly positively correlated with factor 2; item Q_{22} is significantly positively correlated with factor 3.

Most Portuguese students start developing negative behaviours concerning Mathematics from a very early age. The causes were previously discussed. When they arrive to the 9th grade, prejudice concerning this subject has developed and spread deep roots. Regardless of teachers' guidance into maths problem solving, that influence is limited because there is a curricular programme to accomplish and limited time to do it. On 10th May 2012, students did a national mid-term exam before the national final exam. The 9th grade school population scored, on average, the mediocre result of 31%. Few weeks later, on 21st June 2012, 93 435 students did the national exam, now with a mean score of 53%.

The Rotated Factor Matrix *(Table 29)* gives items Q_{14}, Q_{15}, Q_2, Q_{19} and Q_4 positively correlated with factor 1; items Q_1, Q_{16}, Q_{20}, Q_7 and Q_{18} positively correlated with factor 2; items Q_{10}, Q_8, Q_{13} positively correlated with factor 3; items Q_{22} and Q_9 positively correlated with factor 4; item Q_{11} positively correlated with factor 5.

(Table 29) – Rotated Factor Matrix

	Factor					
	1	2	3	4	5	6
$Q_{14}3^{rd}Term$	.705	.195	-.091	.016	-.071	-.110
$Q_{15}3^{rd}Term$	.678	.195	.048	-.288	.222	-.152
$Q_{2}3^{rd}Term$	.648	.207	-.085	-.068	-.055	.320
$Q_{19}3^{rd}Term$	.614	.330	-.007	-.142	-.306	.089
$Q_{4}3^{rd}Term$	.609	.225	-.005	-.055	-.107	.180
$Q_{3}3^{rd}Term$	.454	.150	-.035	.008	-.022	.070
$Q_{1}3^{rd}Term$	.305	.835	.014	.096	.023	.055
$Q_{16}3^{rd}Term$	.196	.664	-.142	.112	.031	.188
$Q_{20}3^{rd}Term$	.284	.651	.197	.087	.090	.134
$Q_{7}3^{rd}Term$	.299	.646	.075	.014	-.213	-.033
$Q_{18}3^{rd}Term$	.433	.554	-.001	-.215	-.005	-.052
$Q_{12}3^{rd}Term$	.077	.422	.378	.097	.389	.038
$Q_{10}3^{rd}Term$	-.008	.123	.739	.100	.092	-.061
$Q_{8}3^{rd}Term$	-.090	-.009	.626	.254	-.167	-.291
$Q_{13}3^{rd}Term$	-.117	-.088	.599	.041	.108	.141
$Q_{22}3^{rd}Term$	-.097	.089	.121	.745	-.064	.109
$Q_{9}3^{rd}Term$	-.189	-.009	.130	.732	.079	-.050
$Q_{17}3^{rd}Term$	.313	.250	.373	.471	.140	.045
$Q_{11}3^{rd}Term$	-.087	-.024	.072	-.002	.578	.009
$Q_{6}3^{rd}Term$	.324	.396	.110	.025	-.403	.265
$Q_{21}3^{rd}Term$	.028	.225	.229	.098	.251	.250
$Q_{5}3^{rd}Term$	.354	.250	-.157	.146	-.060	.420

Extraction Method: Alpha Factoring.
Rotation Method: Varimax with Kaiser Normalisation.

a. Rotation converged in 12 iterations.

Empirical data explain that positive durable mathematical beliefs can only be achieved within a long-term period. From the identification of what is a mathematical problem, through Pólya's four step problem solving model and Schoenfeld's educational variables, we enlightened our hypothesis with two workshops (fifty hours each) for school teachers conducted in 2011/2012 and 2012/2013, complemented with students answers to questionnaires and selected mathematical questions. The results regarding students' performances in the problem solving activities are as follows.

Students' problem-solving skills

Proposed answers for the exercise/problem test given to students

A set of three questions (A_1, B_1, C_1) with an increasing difficulty level was given to the students to be solved in 15 minutes. After a short break, but during the same lesson, another set of three questions (A_2, B_2, C_2), also with an increasing difficulty level, was given to the students to be solved in another 15 minutes. In the second set of questions some heuristic procedures were provided. This methodology was replicated in another lesson, first with questions D_1 and E_1, to be solved in 15 minutes, followed by questions D_2 and E_2, also to be solved in 15 minutes.

Proposed answers for problems A_1 and A_2

What is the value of 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10?

+
What is the value of 12 + 23 + 34 + 45 + 56 + 67 + 78 + 89?

The intuitive procedure leads to a sequential addition of addends. However, our goal is to assess students' understanding and how they select a more efficient strategy to find the solution to the problem, a solution which is both less exposed to computation error and less time consuming. For us to observe such behaviour, the students must possess the mathematical sensitivity which comes from systematic practice. From the analysis of the problem statement a pattern emerges. The value of the sum of numbers equally distant from the extremes is always the same. Students should replace the method of the addition of sequential addends by a more powerful one, by using a heuristic procedure to find the value of 12 + 23 + 34 + 45 + 56 + 67 + 78 + 89, thus writing 101 + 101 + 101 + 101, 4 × 101 = 404.

Proposed answers for questions B_1 and B_2

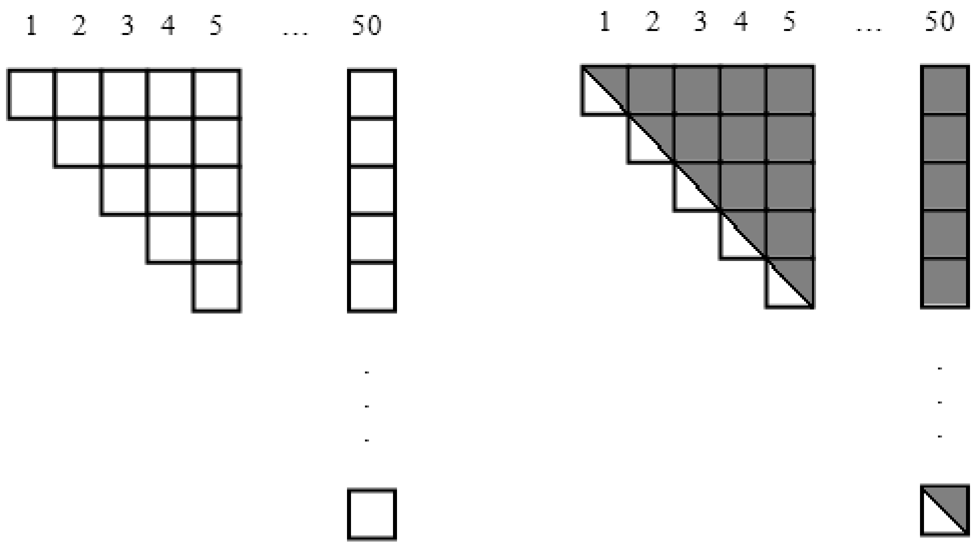
What is the result of the sum of the first fifty natural numbers?

What is the result of the sum of the first one hundred natural numbers?

Both problems are about additions where the sum of terms equally distant from the extremes is equal to the sum of the extremes; 1 + 50 = 2 + 49 = 3 + 48 = … = 25 + 26 = 51 and 1 + 100 = 2 + 99 = 3 + 98 =… = 50 + 51 = 101. To answer question B_1 we have twenty five pairs of numbers, and to answer question B_2 we have fifty pairs of numbers. Thus, the sum of the first fifty natural numbers can be achieved by computing 25 × 51 = 1275 and the sum of the first one hundred natural numbers can be reached by computing 50 × 101 = 5050.

A geometrical approach can also be considered, as we associate each natural number to the same amount of squares. The relation is not straightforward but as we dispose the squares in a sequential order a geometric figure comes into view, the triangle.

When the area of the grey triangle is added ($Area_{triangle} = \frac{50\times 50}{2} = 1250$) to the area of the 50 white triangles ($50 \times 0.5 = 25$), the result is 1275, which is the solution to question B_1. Identical reasoning can be applied to question B_2.



Proposed answers for questions C_1 and C_2

What is the value of $x - y$, if $x = 1^2 + 2^2 + 3^2 + \cdots + 105^2$ and $y = 1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + 104 \times 106$.

What is the value of $x - y$, if $x = 1^2 + 2^2 + 3^2 + \cdots + 2005^2$ and $y = 1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + 2004 \times 2006$.

Students question themselves about what they should do after reading the statement of a more difficult problem. Eventually they will remember the story of the marathon athlete, amateur or professional, who takes a step at a time in order to complete the race, and that the time required to complete the task is associated to the amount and quality of training hours, the technique used and many other variables such as physical well-being, weather conditions and the marathon route. The finish line is the end of the different stages undertaken since the starting point. The number of steps and the difficulty of each step depend on the statement of the problem and the level of knowledge of each individual.

After careful observation of the whole numbers which compose this problem, a skilful student will realise that both sums have the sequence 1, 2, 3, … , 104 and will certainly perform mental transformations such as:

$x = \sum_{i=1}^{105} i^2 = \sum_{i=1}^{105} i.i$; $y = \sum_{i=1}^{104} i.(i+2)$, then $x - y = \sum_{i=1}^{104} i\big(i-(i+2)\big) + 105^2$.

$$x = 1^2 + 2^2 + 3^2 + \cdots + 105^2 = 1\times 1 + 2\times 2 + 3\times 3 + \cdots + 105\times 105.$$

$$y = 1\times 3 + 2\times 4 + 3\times 5 + \cdots + 104\times 106.$$

$x - y =$

$= (1\times 1 + 2\times 2 + \cdots + 105\times 105) - (1\times 3 + 2\times 4 + \cdots + 104\times 106)$

$= (1\times 1 - 1\times 3) + (2\times 2 - 2\times 4) + \cdots + (104\times 104 - 104\times 106) + 105\times 105$

$= 1\times(1-3) + 2\times(2-4) + \cdots + 104\times(104-106) + 105\times 105$

$= (-2)\times(1+2+\cdots+104) + 105\times 105$

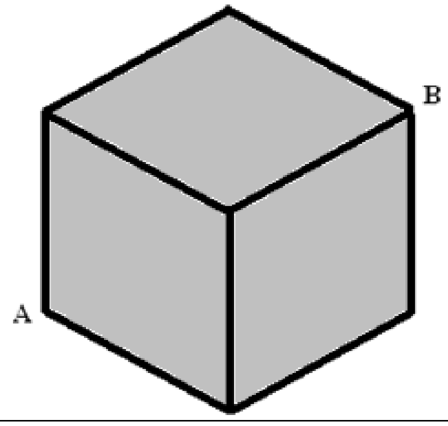
$= (-2)\times(105\times 52) + 105\times 105$

$= (105)\times(-104) + 105\times 105 = 105\times(105-104) = 105.$

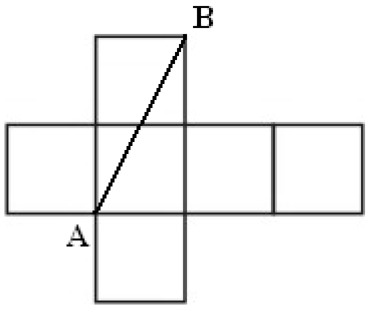
The same procedure can be applied in order to solve question C_2.

Proposed answer to question D_1

Consider a massive cube whose edge measures $1m$. What is the length of the trajectory which corresponds to the smallest distance between point A and point B?



The cube is drawn and the two points (A and B) are marked on it. As in the plane the shortest distance between two points is a straight line segment, then:

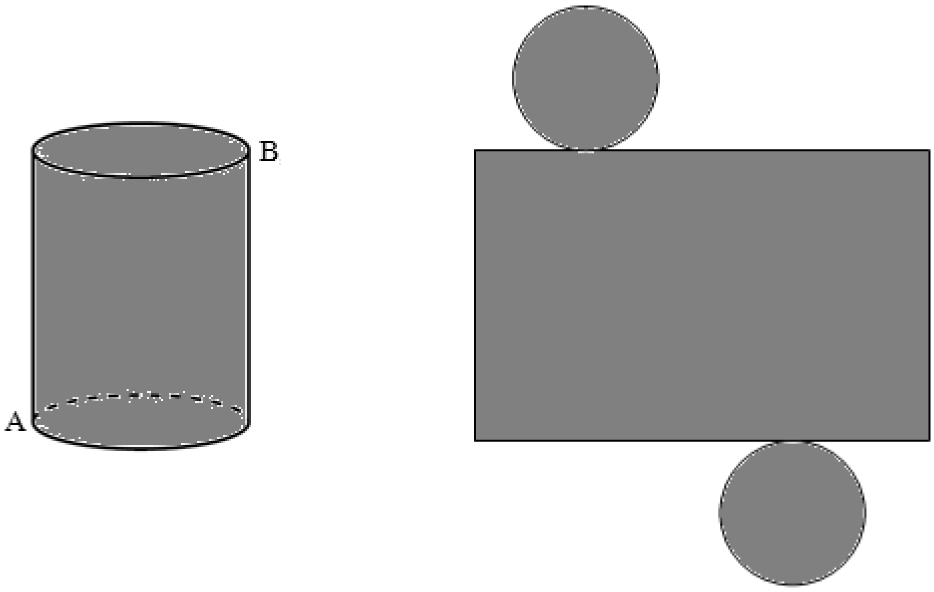


Making use of the Pythagoras theorem and considering x as the shortest distance between point A and point B, we have $x^2 = 2^2 + 1^2 \Leftrightarrow x^2 = 5 \Leftrightarrow x = \sqrt{5}$. Hence, the answer to the problem is, approximately, $2.24m$.

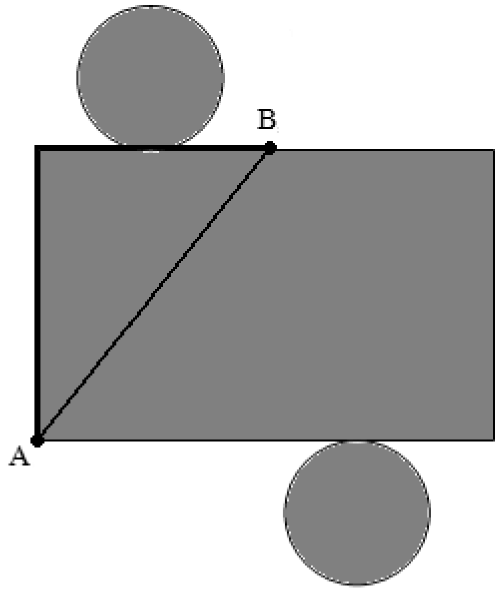
The question, which apparently has an easy solution, is prone to error. The first impulse is to add the length of three edges, which is the path from A to B. A more careful analysis leads to value $(\sqrt{2} + 1)m$, approximately, $2.41m$, which is obtained by adding the diagonal length of the face of the cube and the measure of one edge. Surprisingly for many students, this solution, more efficient than the previous one, still does not correspond to the shortest distance between point A and point B (as we have already shown).

Proposed answer to question D_2

Consider a massive cylinder with a 1-metre radius and 6-metres height. What is the length of the trajectory which corresponds to the smallest distance between point A and point B?



The cylinder should be drawn and the two points (A and B) marked on it. It is important that point B be marked in the middle of the line segment whose length is the perimeter of the circle (basis of the cylinder). In the plane the shortest distance between two points is a line segment, so:



After applying the Pythagoras' Theorem we obtain $\overline{AB}$, $x^2 = 6^2 + \pi^2 \Leftrightarrow x = \sqrt{36 + \pi^2} \cong 6.77m$.

Proposed answers to questions E_1 and E_2

When the first round of a group of the Champions League was complete, each team had played once against each of the other teams. The scores were as follows: A (7 points); B (4 points); C (3 points); D (3 points). As 3 points are awarded for each win and 1 point for each draw, what was the outcome of the game between teams A and D?

The goal is to find the outcome of the game between teams A and D from the information provided in the statement of the problem. A double entry table helps to compute the total of games played at the end of the first round, a total of six. This data, when combined with the scores of the team, enables us to estimate the number of wins, draws and defeats of each team. In the six games, the four teams, together, can score between 12 points (if the six games end in a draw) and 18 points (the 6 games were won by one of the teams). We know that at the end of the first round all the teams, together, scored a total of seventeen points, which means that five games were won by one team and one game ended in a draw. Regarding the performance of each team, we know that: team A – 2 wins, 1 draw and 0 defeats; team B – 1 win, 1 draw and 1 defeat; team C – 1 win and 2 defeats; and team D – 1 win and 2 defeats. Hence, the game between teams A and B ended in a draw, and the game between teams A and D was won by team A.

Teachers' testimonials about students' performance in the school year 2012/2013

Testimony of AFPMO:

In order to implement the tasks proposed in the training session in a 9th grade classroom, permission had to be obtained from the colleague who teaches the class with me (co-teaching).

The class is composed of 26 students, 15 boys and 11 girls. The average age of the class is 14.58 years. This class has 2 children with Special Educational Needs (SEN). The behaviour and the performance of the class may be classified as regular.

The 2 tasks proposed were accepted in quite a different way by the students. The first task $(A_1, B_1, C_1, A_2, B_2, C_2)$ *was carried out in a quiet set of lessons, whereas the second task* (D_1, E_1, D_2, E_2) *had to be completed in the course of a very busy week at the school (end of term activities) which I was unaware of as I am new in the school.*

Since I am not the class' main teacher (I only co-teach for 90 minutes/week) and the connection between me and the students is not very strong, the students thought "I don't have to make an effort because I'm not going to be evaluated for this". Nevertheless, in my opinion, the first task would always be more stimulating, since students dislike Geometry and reveal many difficulties in performing geometry problems.

As expected, there is a discrepancy between the results of the first and the second tasks. When the problems have numbers, the students try to make something of them. Otherwise, they give up easily. Even though the students' engagement in the second task was low, some students tried to solve the equations E_1 *and* E_2 *since they didn't include Geometry.*

In the first task, question A_2*, the majority of the students understood the suggested proceeding. However, many got stuck in the horizontal addition and could not get out of it. Some students did not understand what was suggested, which means that they are not used to using schemes and simplifications. Therefore, question* B_2 *was chaotic. If the addition of 50 numbers is difficult, the addition of 100 is even worse. That justifies the discrepancy of scores between questions* A_2 *and* B_2*, why so few could find a strategy to add so many numbers.*

Some students found a proceeding: "addition of numbers from 1 to 10 = 55; addition of numbers from 11 to 20 = 155; addition of numbers from 21 to 30 = 255; additions of numbers from 31 a 40 = 355; addition of numbers from 41 to 50 = 455. Total: 1275".

Regarding the third question of the first task, it was a total debacle, for only one student scored. In what concerns question D_2*, even with the proceedings to help solve the problems, the students revealed many difficulties to interpret the plan, which means that passing from three dimensions to two dimensions may be difficult.*

Testimony of AMPMC:

The problems were given to 9th grade students. In the first worksheet, among the 12 students, 4 had a positive mark (4 points), 2 scored 0 points, and the remaining students obtained a score inferior to 4 points. In general, these students show lack of prerequisites, lack of study habits and organisation, and they don't always complete their school tasks or do them carelessly. Hence, their performance is well below the desired level.

In the second worksheet, only 3 students obtained positive marks (5 and 6 points). However, no students had 0 points. Similarly to the first worksheet, from the 12 students, 6 obtained 3 points in the first question of this worksheet (it was the question with more correct answers – 8 students). Nonetheless, I must stress that in the first worksheet 3 students had 0 points, whereas in the second worksheet none of the students had 0 points.

In both worksheets, the students either did not answer the last question or answered it incorrectly.

I can conclude that there was a slight improvement in the results of the second worksheet.

In the third worksheet 6 students obtained 0 points, only 1 student had a positive mark and the remaining students obtained less than 3 points.

In the fourth worksheet, the results were even worse than the results of the third worksheet. Actually, they were the worst of the four worksheets; 9 students had 0 points and the remaining students had less than 3 points.

All the students scored 0 points in the first question of both the third and the fourth worksheets. None of the students drew the solids (cube and cylinder), which means that none of the students applied the adequate strategy to find the solution to the problem. The scores of these two worksheets come exclusively from the answers given to the second problem.

The results obtained in these problems were not surprising, for, as I mentioned before, these students usually reveal lack of attention/concentration in the classroom, lack of study habits and methods, an insufficient consolidation of prerequisites, difficulties in terms of calculation and logical-deductive reasoning. Most of the students also reveal great detachment from school activities. However, I must also highlight that the students thought that the problems were interesting but difficult and that they became aware that a lot of effort and dedication are required to succeed. If the student does not have enough willpower to insist, the goal of making the right choice when trying to solve a problem will hardly be achieved.

Testimony of AMMS:

The tests were applied to an 8th grade class composed of 16 students (10 boys and 6 girls). The average age of the class is 13.8 years. Five students are repeating the 8th grade and three are covered by Decree-Law 3/2008, thus having the support of the Special Educational Needs team.

Before the tests were applied, the students were informed that they would be dealing with questions related to problem solving and so they would have to follow a sequential process, the stages defined by George Pólya (1887 – 1985):

- *Understanding the problem;*
- *Devising a plan;*
- *Implementing the plan;*
- *Checking the solution.*

After the stages were explained, the tests were handed out. Both the first and the second parts of the first test were performed on 6th February; the first part of the second test was performed on the 6th March and the second part of the same test on the 7th March.

Analysis of the answers:

Problem A_1: most of the students answered this question correctly. However, only 5 students obtained the maximum score, fully justifying their reasoning and performing partial additions to achieve the final result.

Two of the students presented an answer close to the correct result, which shows a lack of commitment to solve the problem, considering that this is a question for the 2nd or 3rd grade.

Problem B_1: this was a more difficult question and only 4 students presented a solution to the problem.

Three of the students who had given a correct answer to the first question did not answer this question.

One of the students answered the question correctly, by adding the numbers to the same ones digit, and then computed the final result

1	2	3	4	5	6	7	8	9	10	
11	12	13	14	15	16	17	18	19	20	
21	22	23	24	25	26	27	28	29	30	
31	32	33	34	35	36	37	38	39	40	
41	42	43	44	45	46	47	48	49	50	
105	+ 110	+ 115	+ 120	+ 125	+ 130	+ 135	+ 140	+ 145	+ 150	= 1275

One of the best students of the class wrote:

$$55 + 55 \times 2 + 55 \times 3 + 55 \times 4 + 55 \times 5 = \cdots = 825$$

Even though a wrong strategy was followed, it is obvious that the student tried to relate what was asked in the statement with the result previously obtained. The mistake lies in considering that

$$20 = 2 \times 10, 30 = 3 \times 10, 40 = 4 \times 10 \text{ and } 50 = 5 \times 10$$

and not realising that these relations were not common to the remaining numbers, i.e., twofold, threefold, fourfold and fivefold.

When I was correcting the problems in the classroom, I tried to help the student correctly develop his strategy based on the completion of the table below (the 3 last tens).

11	=	*10*	+	*1*
12	=	*10*	+	*2*
13	=	*10*	+	*3*
...	...	...	...	...
20	=	*10*	+	*10*
11 + 12 + 13 + ... + 20	=	*100*	+	*55*

After having analysed and understood the table, the student was able to deduce what was missing:

$$1 + 2 + \cdots + 10 + 11 + \cdots + 20 + \cdots + 50$$
$$= 55 + (100 + 55) + (200 + 55) + (300 + 55) + (400 + 55) = \cdots = 1275$$

Finally, one of the students wrote: $55 + 55 + 55 + 55 + 55 = 275$.

In this case, the student did not follow all the stages of Pólya's method. The student skipped the last step and thus did not realise that there was an error in the answer.

Problem C_1*: This was an even more difficult question which, as expected, had the worst results.*

Some students only computed the value of the powers present in the given expressions. Even though the theme "Sequences and Regularities" had been tackled in class during the school year, and even in previous school years, the majority of the students did not realise that there were hidden terms, which made solving the problem impossible. Moreover, even though the students were familiar with the distributive property, I think that if this question had emerged right after the theme "Factor Decomposition" was taught, some students (not many) could have found the adequate strategy.

Problem A_2: Eleven students followed the strategy suggested by the diagram but only 8 reached the solution. One student obtained the correct result without resorting to the suggested strategy. He added the 2 first numbers and then successively added the following numbers.

Problem B_2: The students revealed many difficulties in this question. The only students who followed a possibly adequate strategy wrote:

1st Student: 1275 + 1275.

The student understood the problem but did not devise a correct solving plan and did not realise that the sum of the last 50 numbers had to be greater than the sum of the first 50. By not devising the correct plan and by not following the last step of Pólya's method, checking the solution, the student did not realise the error.

The 2nd student made the table presented on the right, where a hard-working and time-consuming strategy is clearly visible. If completed, this strategy could have led to the correct answer.

...	8	9	10
...	18	19	20
...	28	29	30
...	38	39	40
...	48	49	50
...	58	59	60
...	68	69	70
...	78	79	80
...	88	89	90
...	98	99	100

Problem C_2: Similarly to question C_1, the performance of the students was weak for they only calculated the value of some of the addends.

When the 2 parts of the test are compared, we can conclude that the hint provided by the diagram of question A_2 was clear to most of the students, thus allowing them to obtain the correct solution.

As I suspected, none of the students was capable of generalising the strategy for the calculation of the addition of the first 100 natural numbers. In my opinion, the low performance of the students in the third question can be justified with the same reasons used for the first part of the test. Furthermore, I believe that the fact that the 2 parts were given on the same day and that the students did not have access to the correction of the first part of the test prevented the students, or at least some of the students, from applying a problem solving strategy. Aiming to clarify this situation, the two parts of the second test were solved on consecutive days, and the correction of the first part of the test was done on the board before the second part was handed out.

Question D_1: For this question, 11 students measured the edges from A to B, thus answering 3 meters. One of the students wrote 2m. He drew the diagonal of the face which contains A and considered that it measured 1m and then added the edge until B. The student could not have determined the length of the diagonal for at that time they were not familiar with the Pythagoras' theorem. However, he could have applied what he had already learned regarding the relation between the angles and the sides of triangles, thus concluding that the diagonal must be greater than 1 m and smaller than 2m. Anyway, this student was the only one to fully understand the statement of the problem. The remaining students inferred that the route ought to follow the edges of the cube and therefore did not understand the statement, i.e., skipped the first step of Pólya's method.

Question E_1: The only student who gave a totally correct answer started by determining the number of wins, draws and defeats of each team. Then he assigned the scores obtained by the several games, and finally presented the solution to the problem. Five students adopted a good strategy. They tried to calculate the number of wins, draws and defeats of each team but were not able to conclude it. Six students gave the correct answer but did not present any solving strategies.

Only after the first part had been corrected was the second part of the test handed out. In the correction, I began by asking the students to read the statement of the first question again, so that they would fully understand the problem. After reading the problem, I asked them if the route from A to B had to be made through the edges. Most students answered correctly, showing that this time they had understood the problem. Since at this time of the school year the students are still not familiar with the Pythagoras' theorem, I used the programme of Dynamic Geometry, GeoGebra, to make them see what the shortest route from A to B is.

Question D_2: Even if the students understood the problem and devised a plan, they could not complete it since they cannot use the Pythagoras' theorem. Moreover, they found several difficulties in the approach to the problem. Considering that the students had already seen how to solve the problem of the cube (from the first part of the test) and that in this question the plan of the cylinder was part of the statement of the problem, the students were supposed to correctly mark the shortest distance between A and B on the figure. However, that did not happen. Only 3 of the 16 students represented points A and B correctly, as well as the line segment whose measure corresponds to the shortest distance between them. Another difficulty felt by the students, maybe because they do not deal with this type of questions on a daily basis, was the checking of the fact that one of the sides of the rectangle of the given plan matched the perimeter of the base (even though one of the students thought about it and even wrote the formula for calculating the perimeter).

Question E_2: Astonishingly, in this question the performance of the students got worse when compared to the same question on the first part of the test. Only one student improved their performance. The double-entry table was not very helpful as the majority of the students did not understand the problem. This type of hint is probably more adequate for 9th grade students, who are more used to working with double-entry tables.

For me, the element which contributed the most to such low results in this question was the fact that the students had not been able to solve the same question the day before. Even though this time there was a hint to help them, the students simply gave up.

Testimony of ACV:

When I asked the students if they would like to solve some problems, they immediately posed a lot of questions: "What is that for?", "I'm not doing this.", "Is it for assessment?", "Can we use a calculator?", "What if I miss the class?"... I answered those questions and tried to motivate them to participate in the challenge, which was not easy. I told them that the purpose of the problem solving sessions was not only to check their skills and identify their difficulties, but also a contribution for an assignment of a training course that I was attending whose goal was to improve their future performances.

I had already explained them that a problem is a question involving a new situation that you don't know how to solve, but with the knowledge that you have already acquired and using different strategies, you may be able to solve, and that there is no time limit to reach the solution (theoretically!). I was immediately questioned about the time allowed to complete each set of problems (15 minutes). I explained that the problems were relatively easy and had a solution which could easily be found within that time frame by the majority of the students. I also handed out a copy of Pólya's problem solving strategy, provided by the trainer, and which we analysed together.

Problem A_1: The students only performed the simple addition.

Students (%)	*Points*	*Result*
88.9 %	*2*	*Answered correctly*
11.1 %	*1*	*Only presented the result*

Problem B_1: Regarding A_1, the increase of the sequence of numbers to be added increased the level of difficulty of problem solving.

Students (%)	*Points*	*Result*
16.7 %	*3*	*Answered correctly*
22.2 %	*2*	*Presented a correct strategy but did not complete it*
11.1 %	*1*	*Only presented the result*
50 %	*0*	*Did not answer correctly*

Problem C_1: A more difficult problem. None of the students was able to find a correct strategy to solve it, which is natural due to the profile of these students.

Students (%)	Points	Result
55.6 %		Tried to answer
44.6 %		Did not answer

Problem A_2: With the suggested strategy, which was understood by the students, they easily reached the solution.

Students (%)	Points	Result
88.9 %	3	Answered correctly
5.5 %	2	Presented a correct strategy but did not complete it
5.5 %	1	Only presented the result

Problem B_2: Again, the increase of the series of numbers to be added increased the level of difficulty of problem solving.

Students (%)	Points	Result
27.8 %	3	Answered correctly
27.8 %	1	Only presented the result
33.3 %	0	Did not answer correctly
11.1 %	0	Did not answer

Problem C_2: A problem with a high level of difficulty for students with such a profile; results were terrible.

Students (%)	Points	Result
44.4 %	0	Tried to answer
55.6 %	0	Did not answer

Problem D_1: The students should have been able to correctly solve the problem. However, only 2 succeeded.

Students (%)	Points	Result
11.1 %	3	Answered correctly
44.4 %	2	Presented an adequate strategy but did not complete it
16.7 %	0	Did not present any strategies nor responded correctly
27.8 %	0	Did not solve the problem

Problem E_1:

Students (%)	Points	Result
22.2 %	3	Answered correctly
22.2 %	2	Presented an adequate strategy but did not complete it
16.7 %	1	Did not present a strategy but answered correctly
27.8 %	0	Did not present a strategy nor answered correctly
11.1 %	0	Did not solve the problem

Problem D_2: The students felt great difficulties to solve this problem. Even those who tried to correctly solve it could not easily mark the points A and B in the plan.

Students (%)	*Points*	*Result*
5.55 %	*3*	*Answered correctly*
16.7 %	*2*	*Presented an adequate strategy but did not complete it*
72.2 %	*0*	*Did not present any strategies nor responded correctly*
5.55 %	*0*	*Did not solve the problem*

Problem E_2: Problem similar to problem E_1 but with a given strategy. Possibly some of the students did not try to solve it for they did not think it would be necessary as they had already solved problem E_1.

Students (%)	*Points*	*Result*
5.55 %	*3*	*Answered correctly*
16.7 %	*2*	*Presented an adequate strategy but did not complete it*
72.2 %	*0*	*Did not present any strategies, nor responded correctly*
5.55 %	*0*	*Did not solve the problem*

I can say that these bad results are mainly caused by the lack of prerequisites, lack of attention/concentration, lack of motivation to solve problems, difficulties in interpreting the statements of the problems, difficulties in mental calculation, all allied to not knowing how to "be" in the classroom. I will try to improve these results by applying what I have learned in the heuristic workshop sessions. However, the students must also want to learn.

Testimony of CILMJR:

Analysis of the results of the problems:

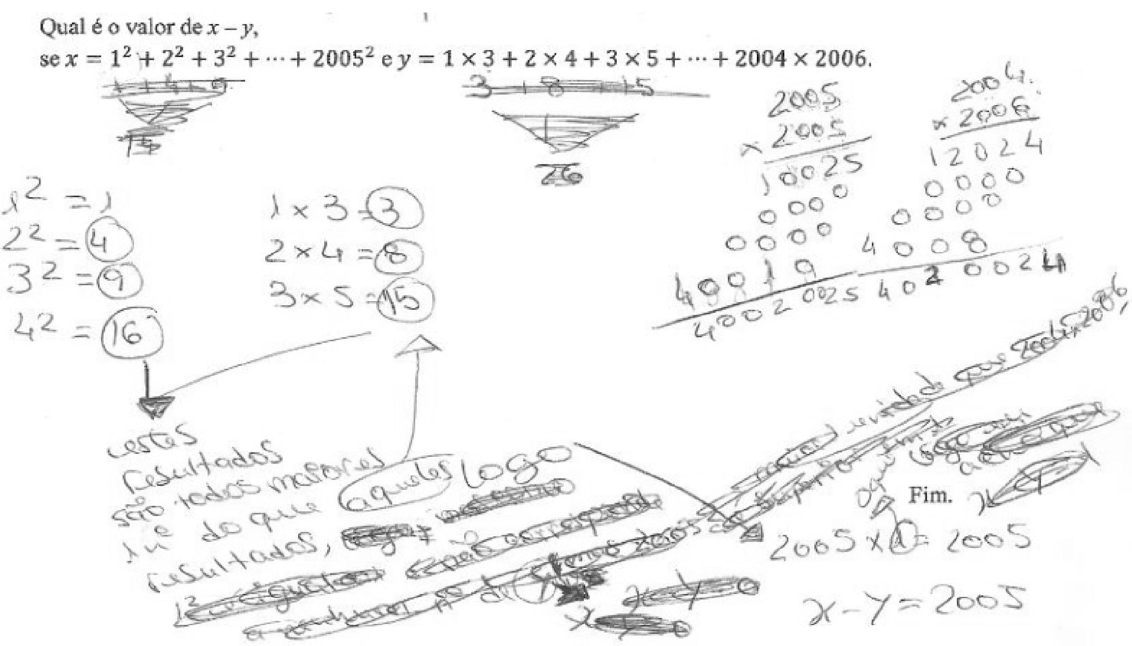
As can be seen in the table, the results of the tests were not satisfactory. The students did not solve some of the problems because they found them too difficult, the 15 minutes allowed to complete the test were not enough to solve the whole test, or they were not motivated to solve it. The students often base their work on an assumption, incorrect data or an inadequate conjecture. Several answers are correct but the students do not give any justifications or reasoning. This situation was more frequent in question A_1(the sum of the first 10 natural numbers) and questions E_1 and E_2 (the scores of the teams). Here, one of the students wrote after the result "I based my answer on my football knowledge". Still regarding questions E_1 and E_2, in general, the students did not understand the difference between these 2 questions which are part of different tests. They also did not find the table useful.

In question A_1, the sum of the first 10 natural numbers, almost all the students reached the result. Some only presented the final value and the ones who presented their reasoning mostly reached that result by associating 2 or 3 addends.

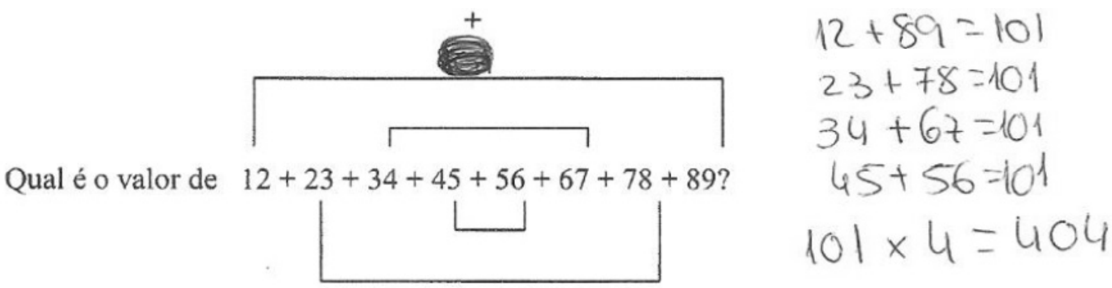
In question B_1, the students performed the additions addend by addend. Some didn't even use the result of the previous question, thus making very time-consuming calculations.

Not many students tried to solve question C_1 and only 2 students started calculations which might have led to the final answer. None of the students related the question to the special cases of multiplication.

In C_2, a similar problem of the second worksheet, one of these students presented a better explanation for his reasoning, even though his solution was not correct. Another student answered it correctly:



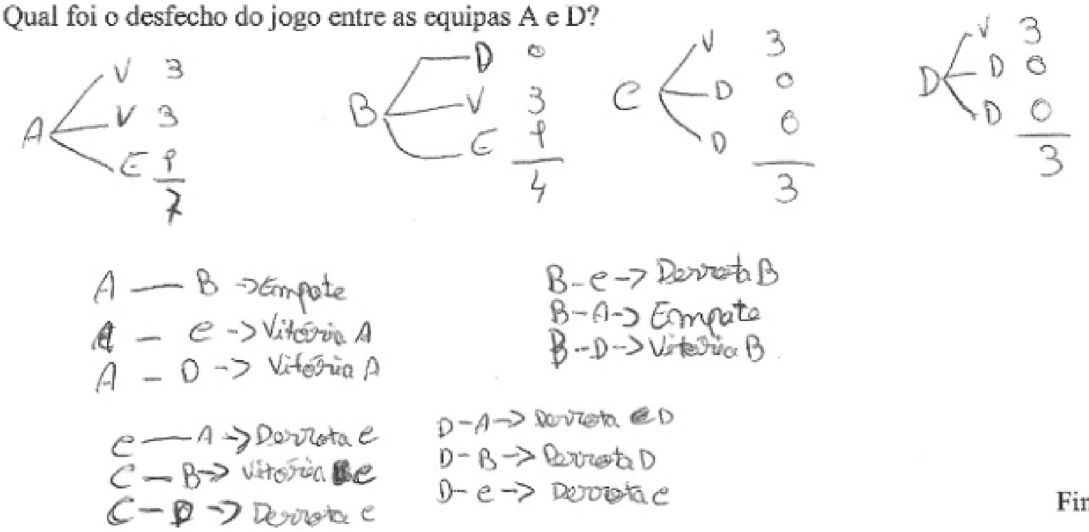
In the second worksheet, question A_2, the majority of the students understood the required reasoning. However, not all who used it to answer the first question realised that they could have also used it to answer the following question. In question B_2, there were students who used the results they had obtained in the previous worksheet followed by extensive and time-consuming calculations. Nevertheless, several students resorted to the previous reasoning and easily solved the question.



In the second class that solved the problems, none of the students gave a correct answer to the question of the cube. Some calculated the diagonal of the face and then added an edge, while others calculated the space diagonal line, which made it obvious that they had not understood the data of the problem. Only 2 students answered the problem of the cylinder correctly, and they presented a similar reasoning. Nonetheless, they only wrote their calculations and did not have the skill to communicate their ideas.

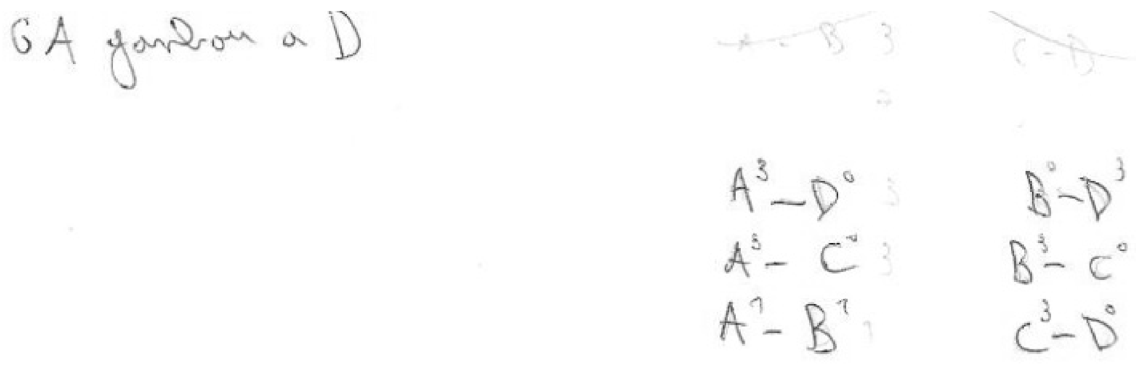
The problem of the scores of the football games was the one where more diagrams were made in order to reach a result, even though sometimes these diagrams were not correct. In the second worksheet many students answered in exactly the same way, because they didn't realise that the table would make reasoning easier.

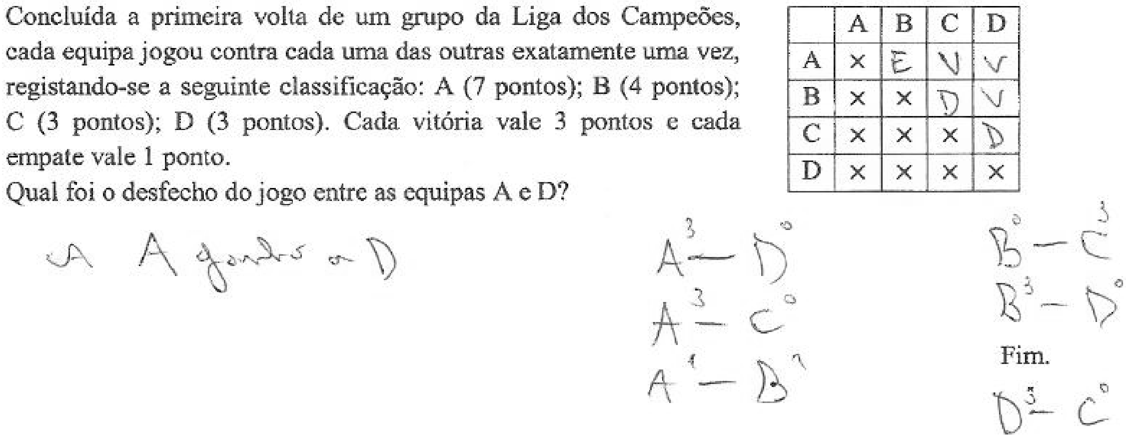
Concluída a primeira volta de um grupo da Liga dos Campeões, cada equipa jogou contra cada uma das outras exatamente uma vez, registando-se a seguinte classificação: A (7 pontos); B (4 pontos); C (3 pontos); D (3 pontos). Cada vitória vale 3 pontos e cada empate vale 1 ponto. Qual foi o desfecho do jogo entre as equipas A e D?



Fim.

The two following answers belong to the same student, who, in the second worksheet, used the table to complete the previous answer.





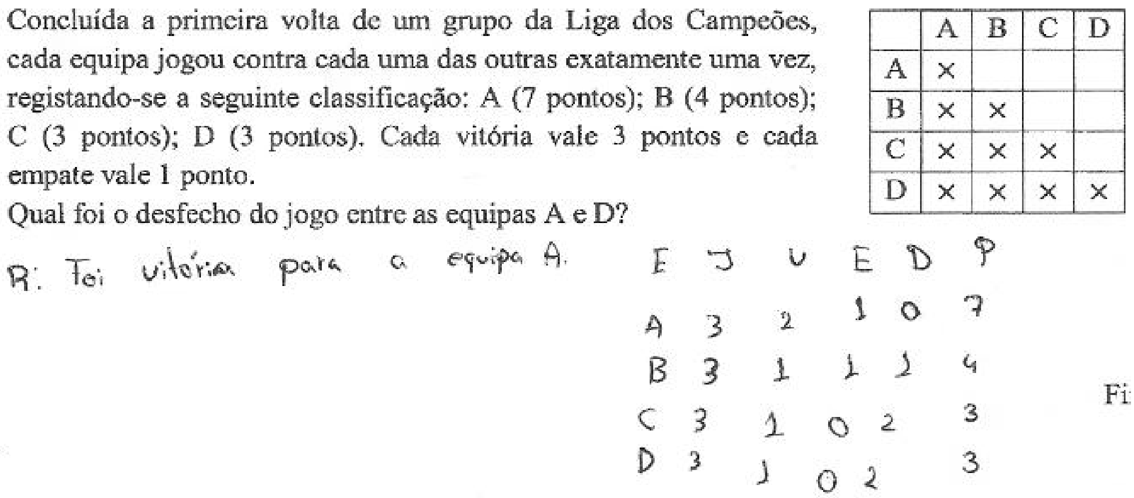
Concluída a primeira volta de um grupo da Liga dos Campeões, cada equipa jogou contra cada uma das outras exatamente uma vez, registando-se a seguinte classificação: A (7 pontos); B (4 pontos); C (3 pontos); D (3 pontos). Cada vitória vale 3 pontos e cada empate vale 1 ponto.
Qual foi o desfecho do jogo entre as equipas A e D?

	A	B	C	D
A	×	E	V	V
B	×	×	D	V
C	×	×	×	D
D	×	×	×	×

A A ganhou a D

A³ — D⁰
A³ — C⁰
A¹ — B¹
B⁰ — C³
B³ — D⁰
Fim.
D³ — C⁰

In the following example, the student made his own diagram and thus did not use the double entry table.



Concluída a primeira volta de um grupo da Liga dos Campeões, cada equipa jogou contra cada uma das outras exatamente uma vez, registando-se a seguinte classificação: A (7 pontos); B (4 pontos); C (3 pontos); D (3 pontos). Cada vitória vale 3 pontos e cada empate vale 1 ponto.
Qual foi o desfecho do jogo entre as equipas A e D?

	A	B	C	D
A	×			
B	×	×		
C	×	×	×	
D	×	×	×	×

R: Foi vitória para a equipa A.

	E	J	V	E	D	P
A	3	2	1	0	7
B	3	1	1	1	4
C	3	1	0	2	3
D	3	1	0	2	3

Fi

I think that this analysis allows seeing what type of answers are given by the students and assessing their difficulties in problem solving: analysing data, arguing, providing a better basis for their ideas and using natural and mathematical languages correctly.

I also think that there should have been more time to discuss the results after the application of the problems, and that students should have been given time to advocate their results in small group debates or with the whole class. This way the students would develop their oral skills and discuss and share the solutions found, as well as the mathematical processes and ideas used.

Testimony of LMCR:

WORKSHEET 1

Problem A_1*: The majority of the students answered correctly. Two students used a similar strategy to the one proposed in problem* A_2.

Problem B_1*: Five students applied satisfactory adequate strategies to reach the solution. Only 1 out of the 5 students proceeded in line with what is suggested in WORKSHEET 2. However, this student did not reach the correct result.*

Problem C_1*: Most students did not answer this question. The 3 students who did, did not give the right answer – they ignored the ellipsis mark and only considered some addends of the expressions of x and y. None of the students used the difference of squares.*

WORKSHEET 2

Problem A_2*: Most students gave the correct answer but only 8 followed the strategy suggested in the problem statement.*

Problem B_2*: None of the students answered correctly. Only 1 student applied a similar strategy to the one suggested but did not reach the correct result.*

Problem C_2*: The majority of the students did not answer this question. The 4 students who answered it make the same errors of problem* C_1.

WORKSHEET 3

Problem D_1*: None of the students gave the correct answer. Two students calculated the spacial diagonal of the cube. Even though the statement refers that it is a massive cube, maybe the presentation of the problem could be clearer. Something like: "An ant walks on the surface of a cube… What is the length of the shortest possible route for the ant to go from point A to point B?"*

Problem E_1*: None of the students gave a correct answer which emerged from an adequate strategy. Several students seem to not have understood what was asked (the outcome of 1 of the games) and provided the name of the team which won the first round of the tournament.*

WORKSHEET 4

Problem D_2*: None of the students answered correctly. Four students applied the Pythagoras' theorem to calculate the length of routes inside the cylinder. Again, maybe the statement of the problem might be altered to make it clearer: "An ant walks on the surface of a cylinder…"*

Problem E_2*: The students provided the same answers given to problem* E_1*. None of the students applied the strategy suggested in the table included in the statement.*

Note: In the problems of the cube and the cylinder the calculator may have been useful, as it would have allowed the students to compare the lengths of the several possible routes.

Testimony of MAGSA:

I chose a 10th grade class of a Professional Course to apply the worksheet. The class is composed of 18 boys and 10 girls. All the students have been retained in the course of their school path (from 1 to 5 times) and some come from Education and Training Courses. Three students benefit from special learning programmes with the special educational measures team such as 1) individual pedagogic support, 2) evaluation adjustments 2) personalised learning programmes 4) support technologies. Two other students are also referred for having benefited from special educational measures in previous years.

Taking into account the learning difficulties detected, the different school paths and the significant number of students with special educational needs allied to the excessive number of students in the class, a poor performance is expected. These are students who lack attention/concentration during lessons, have a poor level of acquired knowledge from previous lessons, learning difficulties, absence or lack of regular study and who frequently have inadequate attitudes in class. The difficulties revealed by the students to autonomously understand and interpret statements of problems, as well as the low skills to rigorously explain their reasoning using mathematical language is a matter of concern. Even though often reminded, some students still do not bring all the material necessary to participate in the activities of the lessons or copy in whole and/or with no mistakes what is registered during the lessons. They do their homework with no organisation whatsoever or they just copy the solutions, showing little or no effort to improve their performance.

Testimony of MGCCMFP:

General social characteristics of the students who compose this 9th grade class:

- *They usually respect the rules and norms of the classroom. However, there are occasional cases of students who intervene incorrectly with inadequate comments or disrespect their classmates, or do not abide the rules/norms which are included in the school rules procedure;*
- *Most of the students are motivated and engaged in the school activities;*
- *They usually complete the tasks proposed for the lessons or as homework.*

I know these students since the last school year, with the exception of 4 students, 3 of whom are retained students. In the previous year, they achieved quite good results in Mathematics (only 34% scored 2 as a final grade of the school year). Nevertheless, several students show great difficulty in interpreting and understanding statements, as well as difficulties in applying the various mathematical contents, in understanding the proceedings of mathematical communication. These factors condition problem solving and cause frustration *and lack of motivation, therefore drifting the students away from our goal – to capture their attention, thus providing different and more stimulating learning experiences which are more adequate to their learning rhythms, as well as promote their interest in the classroom and the development of their mathematical skills.*

Before the students started solving the problems, copies of the "Problem Solving Guide", based on George Pólya's ideas, were handed out, as suggested by the trainer. The Guide was analysed in detail, stage by stage. All the doubts were clarified and answers were given to every question. I suggested that the students not write on their Guides as it would be a consultation document. They should use it and follow the steps to solve all the problems, thus saving the copies. Then, the 2 first problems were handed out, according to the instructions of the trainer. The next two problems were carried out on another day. The corrections of the problems were not made immediately after the performance of the worksheets due to lack of time, since the lessons were only 45 minutes and took place at the end of March, the end of the term. I will make the correction of the problems in one of the first lessons of the third term or after the Mathematics national mid-term test (12th April).

I will now provide the various problem solving strategies used by the students and then proceed to the analysis and discussion of the detected errors on the devised plans which lead to incorrect solutions. I hope that the students are aware of their errors and are capable of overcoming them. At the same time, they acquired a wider range of problem solving strategies, thus enabling them to apply the same methods to similar problems after assessing each case.

Analysis of the results of problems A_1, B_1 and C_1

Question A_1 is very easy. Question B_1 is a little more difficult, but still it was easy enough for the students to find a solving strategy. Question C_1 was very difficult. Almost the whole class gave an incorrect answer or did not answer at all.

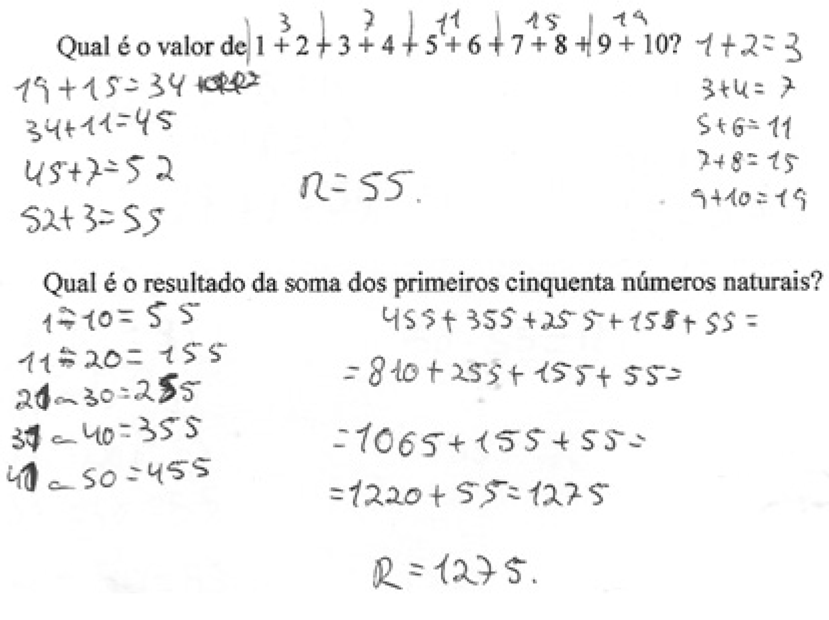
After the different stages of the Guide were followed, I observed that:

- *Almost all the students attentively read and understood questions A_1 and B_1.*
- *Point 2 – devising a plan – in question A_1 most of the students were able to devise a solving plan and implement it. In question B_1 approximately half of the students were able to devise a plan.*
- *In question A_1, 70.8% of the students implemented the plan and answered correctly. However, in question B_1, only 50% of the students obtained a correct answer, and 1 student was not able to devise a plan.*
- *Some of the students only presented the (correct) result to question A_1. They did not read the heading of the worksheet or they made mental calculations.*

A piece of evidence emerges from this analysis – the students may even be able to read and understand the statements but, most of the times, they do not know what to do with that information, i.e., they are not able to devise a plan to solve the problem, thus not being able to follow the different stages of the Guide. This became evident in question C_1, which almost none of the students even tried to solve (they considered it as very difficult). Nonetheless, I observed during the lesson that they read and understood it. Some decided that they were not capable of solving it because there were too many calculations involved and it was very time-consuming (which they considered they did not have). Others were not interested or thought it was not worth it for the solution they would find would certainly be wrong. One of the students wrote in the answer sheet "No answer is possible". Only 2 students tried to implement a plan but they 'shortened' the expressions of x and y, so their answers were not correct.

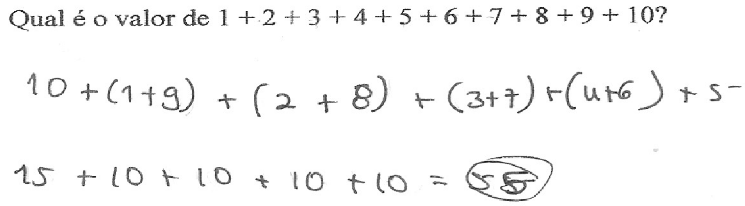
The average score obtained per student was 3.2 (in a total of 8 points) and only 9 students obtained a score equal to or greater than 4, which indicates an ***insufficient performance****.*

At the end of the test I heard some of the comments made by the students; "In question A_1 I made the sums mentally"; "Question A_1 was very easy."; "I didn't have enough time to answer question B_1 ! It was very long! It had many sums!"; "I'm not able to do sums without a calculator."; "The test was very difficult! I didn't answer question C_1 and B_1 is wrong!"; "Question C_1 was very difficult."


Qual é o valor de 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10?

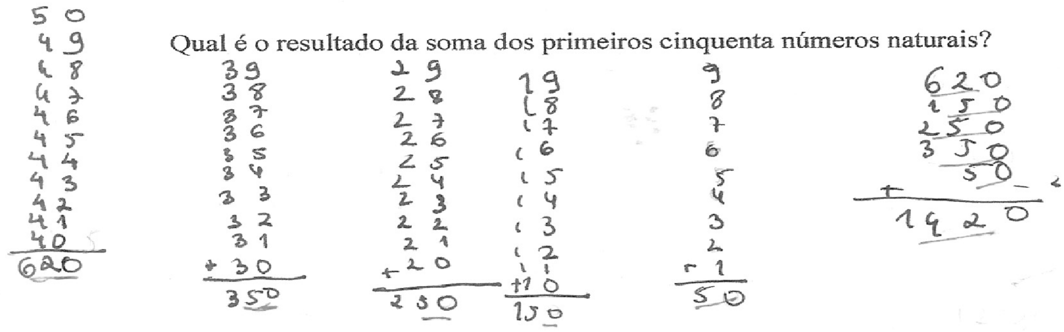
Qual é o resultado da soma dos primeiros cinquenta números naturais?

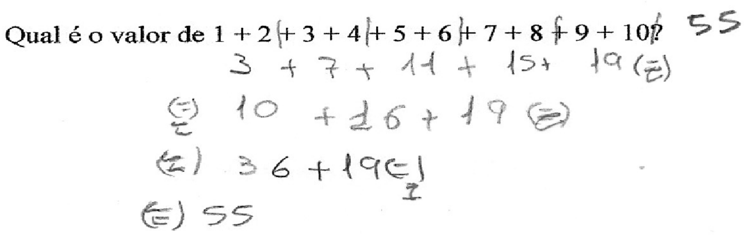
Next I will present some of the solutions for the stage "Implementing the plan".


Qual é o valor de 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10?

10 + (1+9) + (2 + 8) + (3+7) + (4+6) + 5

15 + 10 + 10 + 10 + 10 = 55


Qual é o resultado da soma dos primeiros cinquenta números naturais?

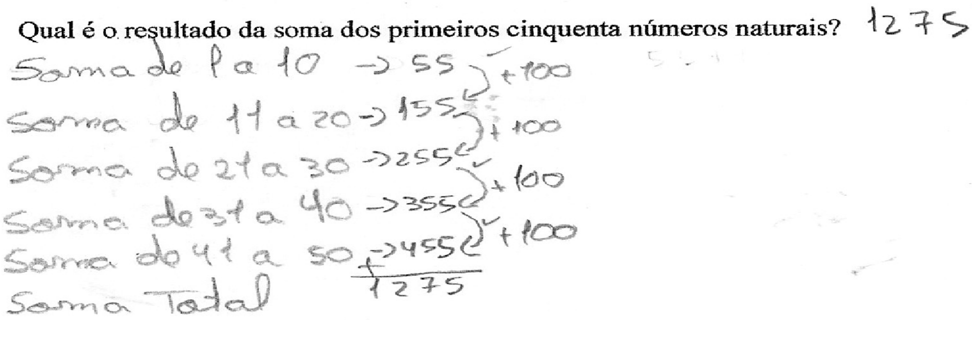

Qual é o valor de 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10? 55

3 + 7 + 11 + 15 + 19 (=)

(=) 10 + 26 + 19 (=)

(=) 36 + 19 (=)

(=) 55


Qual é o resultado da soma dos primeiros cinquenta números naturais? 1275

Soma de 1 a 10 → 55

Soma de 11 a 20 → 155

Soma de 21 a 30 → 255

Soma de 31 a 40 → 355

Soma de 41 a 50 → 455

Soma Total 1275

Analysis of the results of problems A_2, B_2 and C_2

Question A_2 is easy, but question B_2 already presents an intermediate level of difficulty. And question C_2 has an upper level of difficulty (almost none of the students answered this question).

After the several stages of the Guide were completed, I observed that:

- *Almost all the students read questions A_2 and B_2 attentively and understood them.*

- *In point 2 – devising a plan – in question A_2 approximately 2/3 of the students were able to devise and implement a plan. In question B_2, 1 out of 2 students was able to devise a plan.*
- *In question A_2, 70.8% of the students implemented the plan and obtained a correct answer. However, in question B_2, only 33.3% reached a correct answer and 4 were not able to conclude the devised plan.*
- *Some students only presented the (correct) answer to question B_2 but did not provide any plans.*

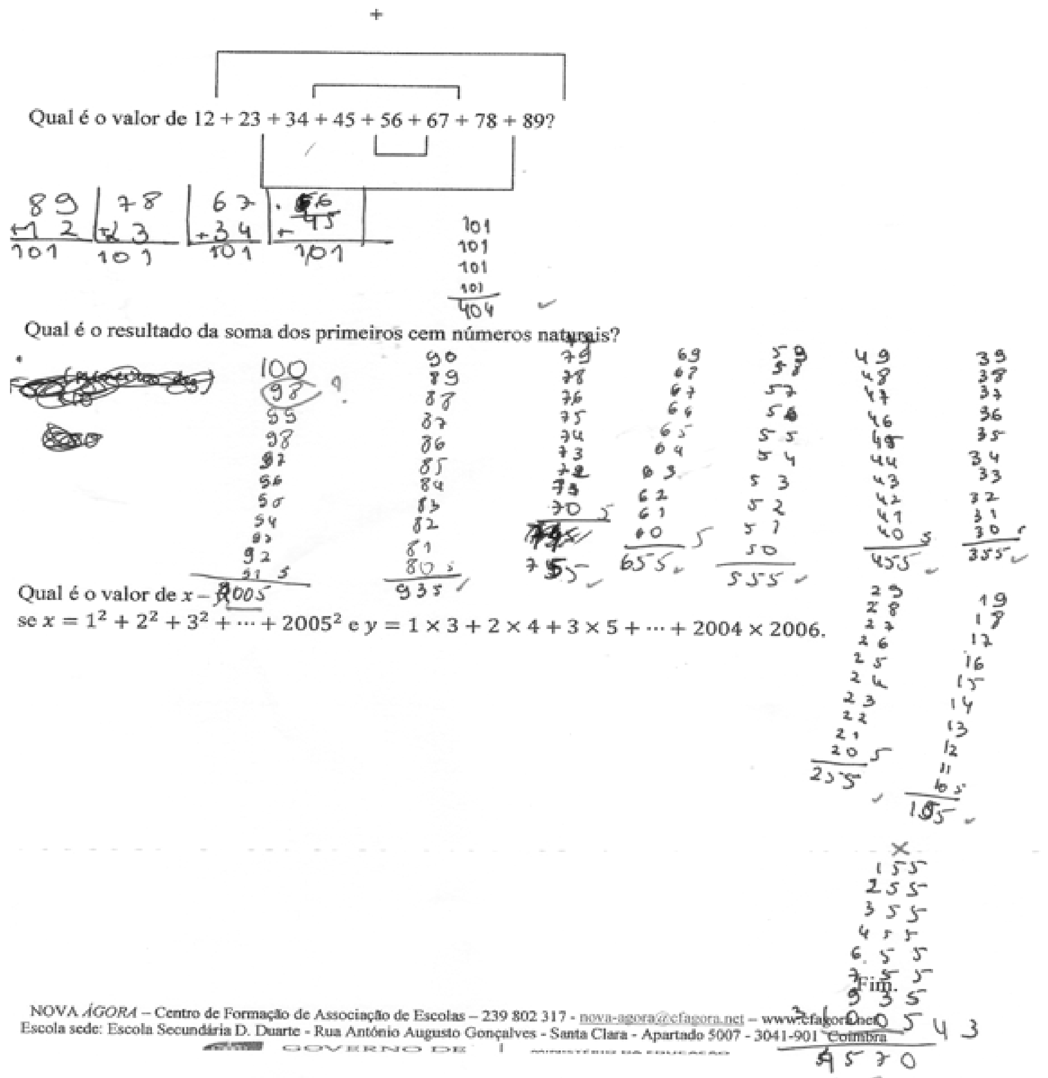
Similarly to problems A_1, B_1 and C_1, I observed that the students were capable of reading and understanding the statements. However, most of the times they could not devise a plan to solve the problems, thus not being able to follow the different stages of the Guide. This was perceptible in question C_2, which almost all of the students did not even try to solve.

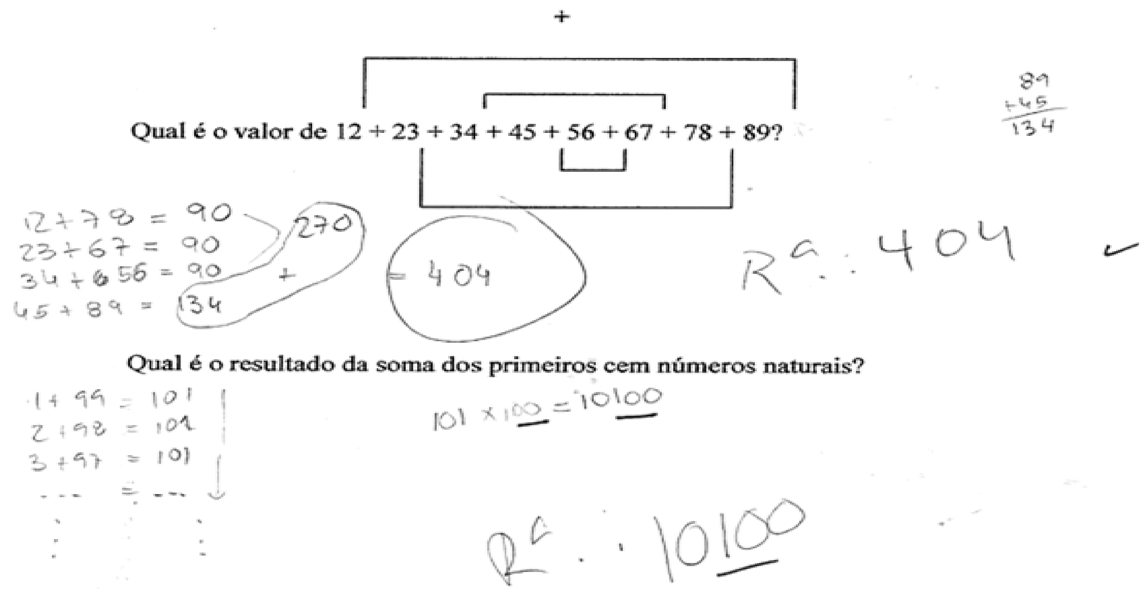
I also observed that in question A_2, the most common strategies were used, namely addition algorithms or the associative property of addition. However, many students worked with the addition of pairs of numbers equally distant from the extremes, which is constant, in line with the heuristics suggested in the statement, or even created other pairs of numbers, some of which had constant sums. In question B_2, several students applied the same heuristic proceeding of question A_2 (the sum of equally distant terms from the extremes equals the sum of the extremes, in this case 101), but made an error in the number of pairs, which should have been 50.

The average score obtained per student was 3.1 (in a total of 8 points) and only 11 students obtained scores equal to or greater than 4 (slightly greater than the scores obtained for problems A_1, B_1 and C_1), thus showing a poor performance. However, quality improved a little since 2 more students obtained scores equal to or greater than 4, especially if we consider that this test has a higher difficulty level than problems A_1, B_1 and C_1 . I am pleased to see that ***in such short time*** *my students tried to find strategies which are different from the most intuitive or common ones and apply heuristic proceedings in problem solving, sometimes the same in the same type of problems!* ***They are learning to think about how to solve problems****! I hope that they can improve their performance with more systematic practice.*

After the test was completed I heard some of the comments made by the students: "I only did question A_2. It was the one with less calculations!"; "In question A_2 I made the sum of all the numbers!"; "I didn't have time to complete B_2! It's very long! I got lost in the sums!"; "In a test with so many operations why couldn't we use a calculator?"; "I don't know how to answer question C_2!"; "This test is much more difficult than the last!".

Problem solving from stage "Implementing the plan".





Analysis of the results of problems D_1 and E_1

Question D_1 is included in the scope of Geometry and involves a massive cube. Given that the cube is in perspective and the two points are not in the same plane, one of the paths to solve the problem would be to plan it. In the plane, the shortest route between two points is a line segment and so the tool – the Pythagoras' theorem - can be used to compute the shortest distance between the 2 points. This question is quite difficult for besides having to find the shortest route between 2 points which do not belong to the same plane in the cube, the students still have to plan and translate the route found for the plane so that they can use the Pythagoras' theorem. Question E_1 has an intermediate level of difficulty but requires the study of all possibilities.

After the various stages of the Guide were followed, I observed that:

- *Almost all the students read questions D_1 and E_1 attentively and understood them.*
- *In point 2 – devising a plan, most of the students were not able to devise a plan for both questions and therefore could not follow the stages of the Guide.*
- *Some students only presented the solution to question E_1.*

Similarly to problems A_1, B_1, C_1,A_2, B_2 and C_2, I observed that the students could read and understand the statement but most of the times they could not devise a plan, thus not being able to follow the stages of the Guide. It is widely known that Geometry is one of the branches of Mathematics which students dislike the most. Nonetheless, if they systematically practise, they will be able to master more techniques and develop their problem solving skills.

*I observed that, in question D_1, some of the students considered the route along the edges of the cube. Others made a more attentive observation and considered the route that includes the diagonal of the face of the cube which starts in A and then extended it through the edge until B (a shorter route than the previous). Both these routes lead to a wrong solution for they do not correspond to the shortest distance between points A and B, and therefore to the shortest route. In question E_1, 10 students only presented the solution to the question, 12 gave an incorrect answer or did not answer, and 2 applied a strategy – 1 did not complete it and the other withdrew wrong conclusions. The average score per student was 0.7 (in a total of 6 points) and none of the students obtained a score equal to or greater than 3, which means that their performance was **very poor**.*

Analysis of the results of problems D_2 and E_2

Question D_2 is included in the scope of Geometry and involves a massive cylinder. To obtain the solution, the problem must be translated into a plane. In the plane, the shortest route between 2 points is a line segment and so a tool - the Pythagoras' theorem – can be used to calculate the shortest distance between the 2 points. This question is quite difficult for besides having to find the shortest route between 2 points, the students still have to mark them on the plane and know that B is the middle point of the line segment whose length is the perimeter of the circle so as to be able to apply the Pythagoras' theorem. Question E_2 has an intermediate level of difficulty but requires the study of all possibilities.

After the different stages of the Guide were followed, I observed that:

- *Almost all the students read questions D_2 and E_2 carefully and understood them.*
- *In point 2 – devising a plan – this did not happen in questions D_1 and E_1. Most of the students were not able to devise a plan for both questions and therefore were unable to follow the stages of Pólya's problem solving sequence.*
- *Some students only presented the solution to question E_2.*

Similarly to problems A_1, B_1, C_1,A_2, B_2, C_2,D_1 and E_1 I observed that the students could read the statements but that most of the times they were not able to devise a plan to solve the problems and therefore could not follow the stages of the Guide. Similarly to what happened in question D_1, none of the students was able to find a solution to question D_2. In question E_2 the performance of the students was similar to that observed for question E_1. Despite the heuristic aid, there was no improvement. The students ***will only be able to master other techniques if they develop their problem solving skills, if they practise systematically****. The average score per student was 0.7 (in a total of 6 points), the same of the worksheet with questions D_1 and E_1, and none of the students reached a score equal to or greater than 3, which indicates a* ***very poor*** *performance.*

I don't find these results surprising for the following reasons - the characteristics of the class: 1) several students reveal a huge lack of basic mathematical concepts (and other which are already signalled) and they are not motivated; 2) other students are not interested in solving problems because "they won't be assessed".

Even though their motivation is usually high and they usually like to solve problems, there was no "competitive factor" which would be validated by the score of the tests as an evaluation tool, for this is what usually motivates their performances; the moment when these tests were conducted (already in March – at the end of the second term): 1) for *planning/coordination reasons – with the other 9th grade classes - they could not be conducted before 2) at this time of the school year the students were already very tired for, besides common evaluation, they would have to prepare for national mid-term tests; taking into account that some of the questions made the students face new situations, there was not enough time to solve the problems according to the individual rhythms of the students; lack of practice: 1) prevents the students from understanding and therefore solving the problem or conditions the quality of the answer and the time spent in solving it. Other factors which should also be considered are the level of difficulty and level of knowledge involved in the activity; 2) the application of heuristic procedures to problem solving requires a lot of dedication, study and systematic practice both from the student and the teacher, for this is how the mathematical thinking and intellectual dexterity of the individuals in the teaching/learning process is developed; 3) it makes problem solving more difficult because you don't suddenly learn to solve problems. When these difficulties are overcome, then I should believe in better problem solving performances from my students!*

PROBLEM	*SOLUTION*
A_1	55
B_1	1275
C_1	105
A_2	404
B_2	5050
C_2	2005
D_1	$\sqrt{5} \approx 2{,}24\ m$
E_1	Team A won
D_2	$\sqrt{36+\pi^2} \approx 6{,}77m$
E_2	Team A won

Testimony of MFRN:

In question A_1, most of the students mentally calculated the sum of the 10 first natural numbers or presented the vertical operation and performed it.

In question B_1, some students applied the associative property as a strategy, thus grouping summands or, for example,

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$$

$$11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 = 155$$

$$21 + 22 + 23 + 24 + 25 + 26 + 27 + 28 + 29 + 30 = 255$$

$$31 + 32 + 33 + 34 + 35 + 36 + 37 + 38 + 39 + 40 = 355$$

$$41 + 42 + 43 + 44 + 45 + 46 + 47 + 48 + 49 + 50 = 455$$

and finally, 55 + 155 + 255 + 355 + 455 = 1275. Some were able to observe the regularity of the sums: 55; 155; 255; 355 and 455 (every one hundred), but the calculations were wrong.

None of the students was able to solve question C_1 .

In question A_2*, some students performed the sums as usual, by adding the numbers. Many of the students did not understand the problem and only performed the referred operations between each pair of numbers, obtaining the result of 101, and therefore did not finish solving the problem. Other students solved it as follows: (12 + 78) + (23 + 67) + (34 + 56) + (45 + 89) = 90 + 90 + 90 + 134 = 404.*

In question B_2 *, some students followed the strategy suggested in the previous question; 1 + 100 = 101; 2 + 99 = 101; ... but then performed the operation 100 × 101 and not 50 × 101, which means that they were not capable of identifying the number of groups. Still, others followed the strategy suggested in the previous question,*

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55

11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 = 155

21 + 22 + 23 + 24 + 25 + 26 + 27 + 28 + 29 + 30 = 255

31 + 32 + 33 + 34 + 35 + 36 + 37 + 38 + 39 + 40 = 355

41 + 42 + 43 + 44 + 45 + 46 + 47 + 48 + 49 + 50 = 455

51 + 52 + 53 + 54 + 55 + 56 + 57 + 58 + 59 + 60 = 555

61 + 62 + 63 + 64 + 65 + 66 + 67 + 68 + 69 + 70 = 655

71 + 72 + 73 + 74 + 75 + 76 + 77 + 78 + 79 + 80 = 755

81 + 82 + 83 + 84 + 85 + 86 + 87 + 88 + 89 + 90 = 855

91 + 92 + 93 + 94 + 95 + 96 + 97 + 98 +99 + 100 = 955

and finally, 55 + 155 + 255 + 355 + 455 + 555 + 655 + 755 + 855 + 955 = 5050.

None of the students was able to solve question C_2 .

In question D_1*, none of the students gave a correct answer. They used the Pythagoras' theorem in* $\mathbb{R}^3$, $\overline{AB} = \sqrt{1^2 + 1^2 + 1^2}$*, or answered* $1 + \sqrt{2}$*. None remembered to plan the cube.*

In question E_1*, one of the students presented the following reasoning:*

Teams	***A***	***B***	***C***	***D***	*Score*
A		*Dr*	*W*	*W*	*7*
B	*Dr*		*W*	*D*	*4*
C	*D*	*D*		*W*	*3*
D	*D*	*W*	*D*		*3*

(W: Win; Dr: Draw; D: Defeat)

Nevertheless, the answer was incorrect.

Another reasoning:

***A**: W, W, Dr* ***B**: W, Dr, D* ***C**: W, D, W* ***D**: W, D, D*

But the answer was also incorrect.

Some students forgot the Draws and only counted Wins and Defeats.

None of the students solved question D_2 *, even though some presented the answer:* $\overline{AB} = \sqrt{2^2 + 6^2}$.

Conclusion:

Most of the students of this class did not enjoy solving the problems. They did it with no enthusiasm and revealed difficulties in understanding some statements, namely the meaning of the ellipsis mark. These students had difficulty to plan the solids.

I believe that the time allowed to solving the problems, namely the groups of questions (D_1, E_1) *and* (D_2, E_2) *was insufficient. We should also have had enough time to discuss the assessment of all these problems in the classroom, which is almost impossible due to the length of the 9th grade programme of studies and the fact that all the contents have to be taught for the national mid-term test and then for the national exam. Mathematical problems for this type of students need to have previous motivation. Some classes should be dedicated to the solving of simple problems, namely in groups.*

Furthermore, we have to take into account that this class is composed of 19 students, of which 1 has special educational needs and 12 have Progress Monitoring Plans. These students lack essential prerequisites, work habits and methods, commitment and motivation, as well as attention/concentration. Some of the students do not bring the required material to the classroom and refuse to participate in the tasks/activities of the lessons, do not study on a daily basis and do not abide by what is proposed in their Progress Monitoring Plans. All this was reflected in the solving of the problems of these worksheets. These attitudes are exhibited not only in Mathematics, but in most of the other school subjects. The global performance of the class is considered insufficient as, at the moment, there are 12 students who risk retention. Their reduced capacity to solve problems, aggravated by the lack of concentration in the reading and interpreting of the statements should also be taken into account. Hence, these students reveal great difficulties to correctly understand what is required, which produces incorrect answers, with no critical sense regarding the results obtained. Parents are not really involved in the follow-up of their children in what concerns the monitoring of daily school tasks.

Testimony of MJNRL:

Regarding question A_1*, only 3 students presented miscalculations and they all just added the numbers without resorting to any other strategies. Only 7 students gave a correct answer to question* B_1*, but they also did not provide any different strategies. None of the student gave a complete answer to question* C_1*. Only 2 students outlined a possible route but they were not able to conclude it.*

In question A_2*, only 5 students did not present the right result, for even though they understood the suggestion, their calculations were incorrect. In question* B_2*, only 10 students applied the suggestion of the previous question. Of these, 3 could not determine the correct number of addends to be added. Besides these students, another 2 answered the question, but they only performed the sums. None of the students was able to solve question* C_2*, nor could they devise a correct strategy.*

When the test was over I was a little disappointed by the performance of my students, for I had great expectations for them. I realised that when they were faced with new situations they were not capable of outlining a strategy which would help them solve the problems.

When I pointed out what they should have done, the students were "angry" because they hadn't known which strategy to use. The positive side of this task is that at least some of the students will remember the strategy used if ever they have to face an identical situation again.

When the second set of questions was conducted some of the students were participating in a sports activity and therefore did not attend the lesson. Since there were no more spare lessons until the end of the term, only 18 out of the 28 students solved these problems.

In question D_1*, 12 students said that the length of the shortest route was* $\sqrt{2}+1$*. None of the students made the plan of the cube or gave a correct answer, In question* E_1*, only 6 students answered correctly and 3 understood the problem but did not realise that team D could not have 3 draws in 3 games. In question* D_2*, only 1 student found a strategy and presented the correct result. Six students began an adequate strategy but did not apply the Pythagoras' theorem correctly or miscalculated the perimeter of the circle and therefore did not reach the correct result. In question* E_2*, the suggestion did not help improve the results observed for* E_1*.*

This time I was not so disappointed for the students' performance in the first set of questions lead me to envisage that the same could happen again. I realised that, even though I have been telling them that whenever they have to solve a geometrical problem they should make a sketch of the figure in question, the students are not used to making the plans of the solids to help them solve the problems.

As this task was conducted at the end of the term I did not have the chance to explore the correction of the problems with the students and so I cannot describe their reactions.

To conclude, I can say that this path is still at the beginning. We need to work problem solving a lot more because if in this 9th grade class the results were not satisfactory, I cannot even imagine what the results would be in the others classes, where the students are academically weaker.

Testimony of MPJJS:

The work conducted in the scope of this training session was developed in a 9th grade class composed of 19 students (10 boys and 9 girls). The average age of the students is 15 years old. It is a homogeneous class where the students are motivated and committed.

The class has 2 students who have special educational needs and who, therefore, benefit from a special learning programme and assessment adjustments.

The worksheets were applied at 2 different moments, in 2 45-minute lessons. Two 15-minute worksheets were applied in each lesson. Before the worksheets were handed out the students were told to carefully read the statements of the problems, define a strategy and present all their calculations and reasoning. At the end of each lesson, a brief oral assessment of the activity was carried out. In general, the students enjoyed solving the problems. Defining the adequate strategy to solve the problems was considered as the most difficult part of the task.

Even though this class has a regular/good performance in Mathematics, the students revealed many difficulties in understanding/interpreting the statements and in mathematical communication, which interfered in problem solving.

Testimony of PCOA:

The class chosen to apply the problems (10th grade) has 17 students, of which 2 have special educational needs. Hence, in practice, only 15 students attend my lessons. These students reveal some difficulties to interpret the statements and apply their knowledge to new situations.

When the first set of problems was introduced and I explained what was required of them, the students showed concern for they related these worksheets with the problems of the Mathematical Olympiad. They did not see the proposal as a challenge, but rather with the concern of assessment. Only 4 students showed enthusiasm. In the course of the solving of the problems I observed unease in some of the students and lack of motivation in others. However, at the end they were all interested in knowing how they should have solved the problems and what the solutions were.

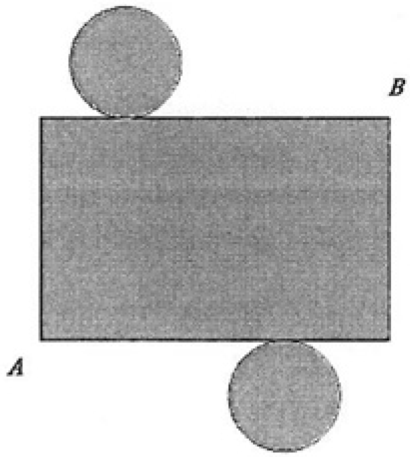
Regarding A_1*, 9 students presented the correct result but did not indicate the strategies used to solve the problem; 2 students presented adequate strategies but miscalculated the results; and 4 students presented adequate strategies and answered correctly. In question* B_1*, 10 students scored 0 points; 2 did not answer the question and 8 gave an incorrect answer;*

3 students used an adequate strategy but did not finish solving the problem or presented an incorrect result, and 2 students presented adequate strategies and presented the right result. In problem C_1*, 6 students did not answer and 9 started solving the problem but were not capable of finishing it.*

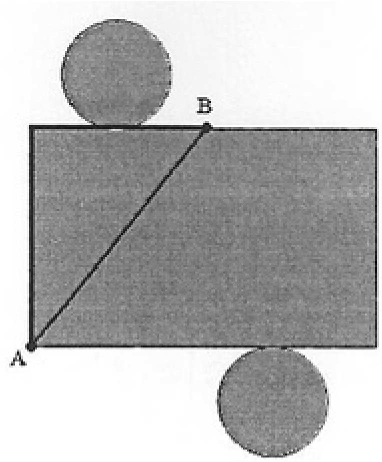
When the second worksheet was handed out, the reactions of the students were different, highly positive when they saw the similarities with the previous worksheet. Despite the similarities, 1 of the students did not understand the strategy proposed in the statement of problem A_2 *and answered as follows: 45 + 56 = 101, 34+67+101 = 202, 23+78+202 = 303, 12+89+303 = 404. Then the 4 results were added, 101+202+303+404=1010; 5 students answered correctly but did not present the strategies used; and 9 students presented the correct result and showed the strategies used to achieve it. In problem* B_2*, 7 students who had understood and applied the suggested strategy in the previous problem were not able to use it to answer this question; 7 students gave an incorrect answer, and only 2 used the strategy suggested in problem* B_1 *and presented the correct result. In what concerns problem* C_2*, the majority of the students only calculated the value of the powers and products. Some of the students tried to perform the subtraction x-y, but as they did not understand the meaning of the ellipsis mark in the expressions* $x = 1^2 + 2^2 + 3^2 + \cdots + 105^2$ *and* $y = 1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + 104 \times 106$*, they were not able to reach any conclusions. Only 1 of the students used a correct strategy but was not able to reach the final result.*

When worksheets 3 and 4 were handed out, the students became more interested and confident for the problems were similar to some that they had already solved in the classroom. Nevertheless, they achieved very poor results.

None of the students found the correct answer to problem D_1*. Nine out of the 15 students considered that the shortest distance to go from A to B was the diagonal of the cube, and 2 considered that it was the diagonal of the face of the cube added to the length of the edge; the remaining 4 students presented inadequate strategies. In problem* E_1*, 4 students did not present the correct result nor an adequate strategy. From the remaining 11, 7 reached the solution to the problem and 4 did not present the correct solution even though they used an adequate strategy. Regarding problem* D_2*, none of the students gave a correct answer. Only 2 students used the plan of the cylinder to mark point B: 1) marked point B so that* $[AB]$ *coincided with the diagonal of the rectangle and A was one of the extremes of that diagonal; 2) even though the student considered that the location of B corresponded to the middle point of the line segment whose length equals the perimeter of the circle, when the Pythagoras' theorem was applied the measures of the cathetus used were 6 m and 2π m, respectively, and* $x^2 = 6^2 + (2\pi)^2 \Leftrightarrow x = \sqrt{36 + 4\pi^2}$.

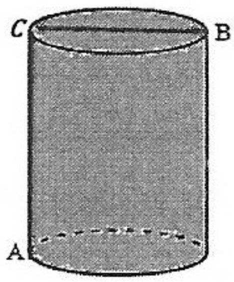


1)



2)

Nine students considered that, in the massive cylinder, the shortest route between A and B goes from A to C and then from C to B. Hence, the shortest distance between A and B was 8m.



Four students considered that the shortest distance would be given by:

$$x^2 = 6^2 + 2^2 \Leftrightarrow x^2 = 36 + 4 \Leftrightarrow x = \sqrt{40} \Leftrightarrow x = 2\sqrt{10}$$

The results of problem E_2 were similar to those obtained for question E_1.

The results were not satisfactory, which confirms the abovementioned gaps. Nevertheless, I highlight that the three best scores belong to the students who usually are more at ease to interpret statements, solve problems and make critical reflexion.

Testimony of PCMLEC:

I chose to apply the problems in a 11th grade class because many of the students reveal great difficulties in problem solving, even though the first module of year 10 starts with Problem Solving. This topic allows students to mobilise knowledge acquired in the 3rd Cycle, as well as improve the strategies they use. However, last year I did not teach this class, so I can only say that many of these students are not always committed enough for a subject like Mathematics, which is very demanding and requires regular work.

General social characteristics of the students:

- *A small group is hard-working but many students are not committed;*
- *Some students do not have work habits: they do not always do their homework or regularly study Mathematics;*
- *Many students reveal great difficulties to manage their study and can only study the contents of the subject under assessment on the following days (sometimes they are not concentrated in class because they are always thinking about that subject);*
- *These students are not very autonomous and when they face a difficulty they do not try to overcome it;*
- *Generally, they respect the rules/norms of the classroom.*

Application of the first worksheet: 25 students were present.

Application of the second worksheet: 21 students were present. Copies of the Guide to Problem Solving were handed out to the students. The Guide was briefly analysed. The stages for problem solving which had already been taught in the previous years (especially in 10^{th} grade) were reminded. The students were invited to follow the Guide to Problem Solving when solving the problems.

Problem A_1*:*

What is the value of 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10?

This was a basic problem and all the students gave a correct answer (55) except for 1 student who made an error in the calculations.

Problem B_1*:*

What is the result of the addition of the first fifty natural numbers?

Only 2 students were capable of calculating the correct result.

Eight students used adequate strategies but miscalculated the operations – our students depend a lot on the calculator!

The remaining 15 students gave a wrong answer: 5 × 55 = 275.

This answer is a good indicator of the way these students see Mathematics – answering before thinking.

Problem C_1*:*

What is the value of $x - y$, if $x = 1^2 + 2^2 + 3^2 + \cdots + 105^2$ and $y = 1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + 104 \times 106$.

No one answered this question correctly. Four students tried a strategy and if they had had more time, they might have succeeded.

Problem A_2:

+

What is the value of 12 + 23 + 34 + 45 + 56 + 67 + 78 + 89?

With the suggestion to associate the terms equally distant from the extremes it became easy, and 19 students presented the correct result. Of these, 5 did not follow the suggested strategy. Six students started an adequate strategy but miscalculated the result. Again, the fact that the students could not use the calculator conditioned the results.

Problem B_2:

What is the result of the addition of the first one hundred natural numbers?

Only 5 students were capable of finding the correct result.

Seven students started an adequate strategy but made an error in the calculations. Of the remaining 13 students who did not answer the question or gave a wrong answer, 3 followed the wrong strategy of the previous test and wrote 10 x 55 = 550 and 1 wrote 100 x 55 = 5500.

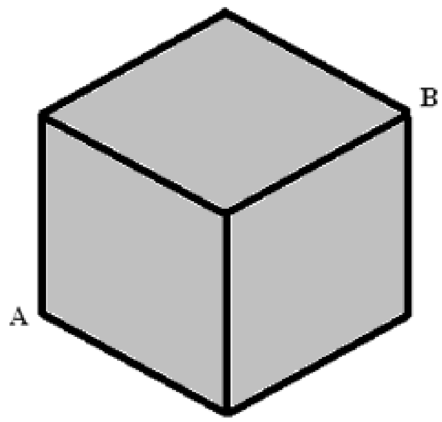
Problem C_2:

What is the value of $x - y$,
if $x = 1^2 + 2^2 + 3^2 + \cdots + 2005^2$ and $y = 1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + 2004 \times 2006$.

Only 1 student answered this question correctly. Given more time, the 7 students who started a strategy might have been able to reach the correct result.

Problem D_1:

Consider a massive cube whose edge measures $1m$. What is the length of the trajectory which corresponds to the smallest distance between point A and point B?



Only 6 students answered this question correctly. Of the remaining 15 students, 9 indicated a route through the surface of the cube but which was not the shortest; they chose a route whose length was 1+ $\sqrt{2}$*, the sum of the measures of the edge and the diagonal of the face of the square. One student ignored that it was a massive cube and considered that the route coincided with the* $\mathbb{R}^3$ *diagonal.*

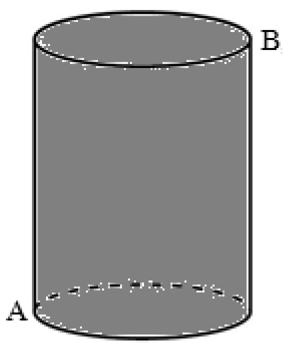
Problem E_1*:*

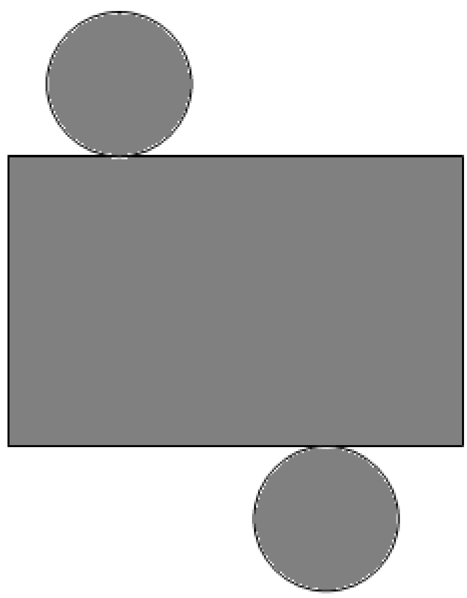
When the first round of a group of the Champions League was complete, each team had played once against each of the other teams. The scores were as follows: A (7 points); B (4 points); C (3 points); D (3 points). As 3 points are awarded for each win and 1 point for each draw, what was the outcome of the game between teams A and D?

Almost all the students realised that 6 games had been played between the four teams. Eleven students gave the correct answer. However, 3 of them obtained 3 points because they used a more or less adequate strategy and 8 students obtained 2 points because the strategy used presented some flaws.

Problem D_2*:*

Consider a massive cylinder with a 1-metre radius and 6-metres height. What is the length of the trajectory which corresponds to the smallest distance between point A and point B?





Even though the plan of the cylinder was given in the statement, only 2 students reached the right result and 1 of them used a number rounded off to the ones place as π. Four students started an adequate strategy but made an error in the calculations (2 considered that the root of a sum is the sum of the roots). Of the remaining 15 students, 5 indicated a route through the surface of the cylinder which was not the shortest one, 3 chose a route whose length is 8 = 6 + 2, the height of the cylinder and the diameter of the circle of the base, and 2 chose a route whose length is $6 + \pi$, the height of the cylinder and the semi-circumference. 4 students ignored the fact that the cylinder was massive and crossed right through it.

Problem E_2:

When the first round of a group of the Champions League was complete, each team had played once against each of the other teams. The scores were as follows: A (7 points); B (4 points); C (3 points); D (3 points). As 3 points are awarded for each win and 1 point for each draw, what was the outcome of the game between teams A and D?

	A	B	C	D
A	×			
B	×	×		
C	×	×	×	
D	×	×	×	×

This problem, even though equal to a problem of the previous worksheet, had an additional help (a table of the possible games). The students improved their performances. Fifteen students answered the question correctly. Nonetheless, 7 students obtained 3 points because they presented a more or less adequate strategy and 8 students obtained only 2 points because the strategy they used had some flaws.

In conclusion, the results obtained by these students are quite negative, which shows that, in general, the students do not know how to solve problems. Furthermore, they did not make an effort. The truth is that the programme of studies is long and there isn't enough time to explore the problems. The concepts are taught, planned according to the national programme of studies and many consolidation activities are developed in the classroom. However, hard individual work is required to acquire, reflect and explore contents. It is here that the work of most of the students fails. Nevertheless, it is also true that, at this level of instruction, most of the students think that only the activities that are going to be evaluated are worth their effort. When these problems were proposed, that was exactly what they asked.

I shall take more problems to the classroom because I consider that problem solving is an asset to motivate the students for Mathematics, help overcome difficulties to understand/interpret the statements, and positively contribute for decision making in the future.

Testimony of RMVF:

Characterisation of the class: 9th grade students who, in general, present poor competences in Mathematics.

Required material: a worksheet was handed out to the students at 2 different moments.

First moment of problem solving

Expected assessment: Taking into account the characteristics of both classes, the students were expected to easily solve the first questions but not the last ones, as these have a greater degree of abstraction.

Report of the experiment: the 2 sets were applied according to what was required (15 minutes for each set). The task was carried out on a Friday, in a 45-minute lesson.

Reflexion on the task: the students started by engaging in the activity. However, after a while they were no longer interested. When the second part was applied, many students had already given up.

Analysis of the results (Class A)

In the first set of problems, I observed that almost all the students answered question A_1 correctly, but the opposite happened in question C_1, where all the students scored 0 points.

In the second battery of problems the same trend was observed. The students obtained good results in question A_2 (14 students answered the question correctly), but when difficulties arose it became evident that they had started to give up. In problem B_2, all the students scored 0 points and 12 students did not even try to answer problem C_2.

Analysis of the results (Class B)

When the results of the first set of problems were analysed, I verified that almost all the students answered question A_1 correctly, but the opposite occurred with question C_1, where all the students obtained 0 points. When difficulties arose it became evident that they had started to give up. In problem B_2, 9 students scored 0 points and 14 students did not even try to answer problem C_2.

Second moment of problem solving

Expected assessment: Considering the specific characteristics of the class, the students were expected to easily solve the first questions but not the last ones, as these have a greater degree of abstraction. Poorer results than those obtained at the first moment of problem solving were expected.

Report of the experiment: the 2 sets were applied according to what was required (15 minutes for each set). The task was carried out on a Friday, in a 45-minute lesson.

Reflexion on the task: the students started by engaging in the activity. However, after a while they were no longer interested. When the second part was applied, many students had already given up.

Analysis of the results (Class A)

I observed that the students felt more difficulties than at the first moment of problem solving. A few gave up. The aid included in the statement turned out not to be an asset.

Analysis of the results (Class B)

The students were not interested in the activity. The aid did not contribute to an improvement of the results.

Testimony of RMSSV:

The problems were applied in a 9th class with 16 students (7 girls and 10 boys, 2 of whom have special educational needs). The average age of the students is 15 years. Only 3 students were retained once in the course of their school path. In terms of academic performance, this class is quite homogeneous. In Mathematics, there are a reasonable number of good and very good students (8), but also some students with insufficient performances (5). They are interested and committed students, even though a little restless.

From the table we can extract that, in general, the results were poor, since only 2 students scored more than half of the total of points. Several reasons justify this: some indifference (they easily give up), lack of concentration… In general, if they committed more, they would do better. Nonetheless, I consider that problems C_1 and C_2 are too difficult for the education level of these students, not only in terms of form (the presence of the ellipsis mark), but also in the amount of numbers.

In the second worksheet, I observed that the diagram provided in the statement of the problem helped the students find the adequate strategy, as there were a greater number of correct answers. And when the strategy was properly assimilated, it was correctly applied in the second exercise. However, most of the students did not do it.

The application of worksheets 3 and 4 took place on a second stage and the results were not better than the previous results.

At the end of each stage there was time for a brief oral debate on the experience. In general, the students referred that the time allowed was not enough, and that the main difficulties were felt in the definition of adequate strategies. Nevertheless, they enjoyed the activity and were enthusiastic regarding problem solving.

Testimony of TCRO:

I believe that this training session will lead to the alteration of some aspects of my pedagogic practice. I realised that I can go farther in terms of the interests of the students, and that this does not have to lead to conflict with the ideals of education claimed/designed by educators.

Despite my lay knowledge of the theme of the training session, its attendance broadened my horizons in mathematical problem solving, thus enriching and enhancing my teaching practice.

My participation became quite important since everything that I have learned is very enlightening and will lead to a better teaching/ learning practice, as well as to a better teacher/student, student/student and student/teacher interaction.

The good performance of the trainer should be highlighted, for the methodology and the strategies used were quite appealing and adequate to the group of trainees. I must also stress the personalised support and the patience of the trainer during the sessions, his openness to dialogue, his willingness to interactively clarify any doubts and his friendliness, which made the sessions pleasant and enriching.

Critical reflexion on the results obtained by the students

Even though the data available from the students' answers to mathematical questions is large enough, nonparametric tests were applied since we cannot assume that the outcome follows the normal distribution (the bell-shape distribution). Also, test application based on normality assumptions is more limited due to a lack of precise measurement. For instance, is a 2 points evaluation *Presented a correct strategy but did not complete it* twice as good as 1 point evaluation *Only presented the result*? Probably not! We have a rudimentary measure of scholastic accomplishments that only allows us to grade from poor performances to good performances. Moreover, it is not evident that equally spaced intervals on our scale can be significantly compared (*e.g.*, *Only presented the result* differs from *No answer/Did not answer correctly* in the same way that *Answered correctly* differs from *Presented a correct strategy but did not complete it*). Similarly, a rigid criteria to evaluate students' answers was not applied.

Central tendency measures are as follows (Table 30):

(Table 30) – Central tendency measures

	N	Median	Mode
Problem A_1	330	3.00	3
Problem B_1	322	.00	0
Problem C_1	292	.00	0
Problem A_2	328	3.00	3
Problem B_2	317	.00	0
Problem C_2	278	.00	0
Problem D_1	301	.00	0
Problem E_1	311	1.00	0
Problem D_2	305	.00	0
Problem E_2	310	.00	0

Score: Problem A_1(0: No answer/Did not answer correctly; 1: Only presented the result; 3: Answered correctly), subsequent problems (0: Did not answer correctly; 1: Only presented the result; 2: Presented a correct strategy but did not complete it; 3: Answered correctly).

Note: Initially we considered the scores of Problem A_1 as 0: No answer/ Did not answer correctly; 1: Only presented the result; 2: Answered correctly. The change had the purpose of a better assessment between questions A_1 and A_2.

Median and mode results concerning the first worksheet with questions A_1, B_1 and C_1 highlight a consistent increasing difficulty level. If the first question was straightforward, the second question needed the application of some elementary heuristics, and the third question required a more sophisticated procedure to attain the solution. Even though all questions are related with numbers and operations, they aim at very different sets of mental abilities and processes related to knowledge, namely, attention, long term memory and working memory, judgment and evaluation, reasoning and *computation*, comprehension, decision making, exercise and problem solving. In the subsequent worksheet students' receive a suggestion to be applied in question A_2. The purpose was to evaluate if individuals would incorporate such practice in order to solve question B_2. Results unveil both a similar median and mode for questions B_1 and B_2. Therefore, the students' difficulty in quickly assimilating and using a particular procedure, in a new situation, without an explicit instruction is noticeable. An equal median and mode were achieved for question C_2, and similar results were obtained for question C_1. This lower score illustrates how difficult it is to have a sophisticated insight in the absence of less demanding performances.

In another moment students' initially answered a worksheet with questions D_1 and E_1, the first question about Geometry and the second question a mathematical modelling problem. After a pause students' solved questions D_2 and E_2. Question D_2 was analogous to question D_1, eventually, a little more demanding, but it was presented with a hint, the cylinder plan, in order to facilitate a solution. Despite the suggestion provided, results revealed a consistently lower performance. Finally, question E_2 is the copy of question E_1 but with the support of a double entry chart. To our surprise, students' scored a lower median in question E_2. The display of heuristics without an adequate framework apparently does not benefit students' problem solving performance.

After this analysis, which separates the higher half from the lower half (median) of a data sample, and regarding the most frequent value of data (mode), we used the SPSS Statistics tools to introduce more detailed information about the students' performance in each problem ($Analysis \rightarrow Descriptive\ Statistics \rightarrow Frequencies{:}\ Statistics$).

(Table 31) - Problem A_1

		Frequency	Percentage	Valid Percentage	Cumulative Percentage
Valid	No answer/Did not answer correctly	64	18.9	19.4	19.4
	Only presented the result	30	8.8	9.1	28.5
	Answered correctly	236	69.6	71.5	100
	Total	330	97.3	100	
Missing	System	9	2.7		
Total		339	100		

Question A_1 only requires basic computational skills, but despite its lower difficulty level, still 19.4% of the students' did not answer or did not answer correctly. Many school teachers are well aware of this behaviour. Despite efforts to capture students' attention to subjects and engage them on a proficient aptitude towards mathematics, still a conduct of indifference persists. This result also encloses those students' whose work, despite their commitment to computation, contains one or more inaccuracies and therefore miss the correct answer.

(Table 32) - Problem B_1

		Frequency	Percentage	Valid Percentage	Cumulative Percentage
Valid	No answer/Did not answer correctly	184	54.3	57.1	57.1
	Only presented the result	18	5.3	5.6	62.7
	Presented a correct strategy but did not complete it	77	22.7	23.9	86.6
	Answered correctly	43	12.7	13.4	100
	Total	322	95.0	100	
Missing	System	17	5.0		
Total		339	100		

Question B_1 engages the use of a strategy to reach a solution. Without the expertise to draw a proper plan most students' do not answer or present a wrong result. Only 13.4% answered correctly. From question A_1 to question B_1 we perceive a large drop in performance.

(Table 33) - Problem C_1

		Frequency	Percentage	Valid Percentage	Cumulative Percentage
Valid	No answer/Did not answer correctly	270	79.6	92.5	92.5
	Only presented the result	3	.9	1.0	93.5
	Presented a correct strategy but did not complete it	19	5.6	6.5	100
	Total	292	86.1	100	
Missing	System	47	13.9		
Total		339	100		

In question C_1, none of the 292 students reached a complete answer. We categorise this problem as higher order difficulty level. Even though 13.4% of the students' achieved a full score in question B_1, they seemed to be unable to perform a more skilful reasoning, eventually due to time restrictions, because the worksheet containing questions A_1, B_1 and C_1 needed to be solved in 15 minutes.

(Table 34) - Problem A_2

		Frequency	Percentage	Valid Percentage	Cumulative Percentage
Valid	No answer/Did not answer correctly	35	10.3	10.7	10.7
	Only presented the result	18	5.3	5.5	16.2
	Presented a correct strategy but did not complete it	67	19.8	20.4	36.6
	Answered correctly	208	61.4	63.4	100
	Total	328	96.8	100	
Missing	System	11	3.2		
Total		339	100		

Question A_2 was quite similar to question A_1. Data reveal that the number of no answers or not corrected answers diminishes significantly from 19.4% to 10.7%. We can eventually attribute this positive factor to the diagram supplied with question A_1, which helps students' achieve a better performance.

(Table 35) - Problem B_2

		Frequency	Percentage	Valid Percentage	Cumulative Percentage
Valid	No answer/Did not answer correctly	198	58.4	62.5	62.5
	Only presented the result	15	4.4	4.7	67.2
	Presented a correct strategy but did not complete it	41	12.1	12.9	80.1
	Answered correctly	63	18.6	19.9	100
	Total	317	93.5	100	
Missing	System	22	6.5		
Total		339	100		

Question B_2 was quite similar to question B_1. The strategy suggested to solve question A_2 seemed to be neglected by the students when answering' question B_2. The number of no answers or not correct answers slightly increases, from 57.1% to 62.5%. Indeed, the connection is not obvious, but a proficient individual should be able to do such inference. Still, an increasing number of correct answers, from 13.4% to 19.9%, is encouraging. The gap between those who did not score and those who scored better is now bigger.

(Table 36) - Problem C_2

		Frequency	Percentage	Valid Percentage	Cumulative Percentage
Valid	No answer/Did not answer correctly	260	76.7	93.5	93.5
	Only presented the result	1	.3	.4	93.9
	Presented a correct strategy but did not complete it	14	4.1	5.0	98.9
	Answered correctly	3	.9	1.1	100
	Total	278	82.0	100	
Missing	System	61	18.0		
Total		339	100		

Question C_2 was quite similar to question C_1. Results are analogous. In the absence of a given strategy in such a high order task, only a minority of students' attain partial or full success.

(Table 37) - Problem D_1

		Frequency	Percentage	Valid Percentage	Cumulative Percentage
Valid	No answer/Did not answer correctly	233	68.7	77.4	77.4
	Only presented the result	36	10.6	12.0	89.4
	Presented a correct strategy but did not complete it	22	6.5	7.3	96.7
	Answered correctly	10	2.9	3.3	100
	Total	301	88.8	100	
Missing	System	38	11.2		
Total		339	100		

Question D_1 is related to Geometry. To answer correctly the students needed to plan the cube and locate point A and point B. Then, they should apply the Pythagoras' theorem. More than three out of four students' failed completely and did not find a solution.

(Table 38) - Problem E_1

		Frequency	Percentage	Valid Percentage	Cumulative Percentage
Valid	No answer/Did not answer correctly	135	39.8	43.4	43.4
	Only presented the result	56	16.5	18.0	61.4
	Presented a correct strategy but did not complete it	67	19.8	21.5	83.0
	Answered correctly	53	15.6	17.0	100
	Total	311	91.7	100	
Missing	System	28	8.3		
Total		339	100		

Question E_1 is a modelling mathematical problem. The students' performed much better in this question than in the Geometry question.

(Table 39) - Problem D_2

		Frequency	Percentage	Valid Percentage	Cumulative Percentage
Valid	No answer/Did not answer correctly	265	78.2	86.9	86.9
	Only presented the result	2	.6	.7	87.5
	Presented a correct strategy but did not complete it	31	9.1	10.2	97.7
	Answered correctly	7	2.1	2.3	100
	Total	305	90.0	100	
Missing	System	34	10.0		
Total		339	100		

Question D_2 is also related to Geometry. Despite the suggestion of the plan of the cylinder, the students' performance remained disappointing. The hint was neglected and only 2.3% answered correctly. Here we observe that a given suggestion, without additional support, does not necessarily imply enhanced answers.

(Table 40) - Problem E_2

		Frequency	Percentage	Valid Percentage	Cumulative Percentage
Valid	No answer/Did not answer correctly	163	48.1	52.6	52.6
	Only presented the result	64	18.9	20.6	73.2
	Presented a correct strategy but did not complete it	32	9.4	10.3	83.5
	Answered correctly	51	15.0	16.5	100
	Total	310	91.4	100	
Missing	System	29	8.6		
Total		339	100		

Question E_2 is a copy of question E_1. Even though a double entry chart was provided to help the students visualise the total of games played, performance did not increase. Eventually, the students' did not find motivation to solve the problem again. This justifies the increasing number of individuals who did not answer or did not answer correctly, from 43.4% to 52.6%. Again, motivation plays an important role in students' attitude towards problem solving activity.

After separately analysing each variable we looked for statistical dependence between two variables. For that purpose, the Spearman's rho, a non-parametric measure of statistical dependence, was extracted so as to measure the strength of the association between two ranked variables. The exact sampling distribution can be obtained without requiring knowledge of the joint probability distribution of X (the independent variable) and Y (the dependent variable). The procedure was performed with the SPSS statistics programme ($Analize \rightarrow Correlate \rightarrow Bivariate\ Correlations \rightarrow Spearman, Missing\ Values: Exclude\ Cases\ Pairwise$).
Despite some well-known disadvantages concerning the use of the Spearman's rho, Pearson' assumes a normally distributed data, so conclusions and predictions tend to be more reliable and easy to interpreted. Spearman's correlations *(Table 41)* do not require the assumption that the relationship between variables is linear and, therefore, is more appropriate for measurements obtained from ordinal scales (Pearson's r correlation is more appropriate for measurements obtained from continuous level data from an interval scale).

(Table 41) – Spearman's correlations

			Problem A1	Problem B1	Problem C1	Problem A2	Problem B2	Problem C2	Problem D1	Problem E1	Problem D2	Problem E2
Spearman's rho	Problem A1	Correlation Coefficient		.230**	-.005	.126*	.236**	.092	.259**	.109	.118*	.129*
		Sig. (2-tailed)		.000	.936	.023	.000	.124	.000	.058	.042	.025
		N		322	292	328	317	278	296	303	297	302
	Problem B1	Correlation Coefficient			.076	.106	.532**	.091	.128*	.171**	.127*	.218**
		Sig. (2-tailed)			.199	.058	.000	.132	.030	.003	.030	.000
		N			287	321	311	277	290	295	292	296
	Problem C1	Correlation Coefficient				.089	.045	.286**	.020	.150*	.034	.088
		Sig. (2-tailed)				.131	.450	.000	.744	.014	.583	.148
		N				291	286	268	270	270	267	270
	Problem A2	Correlation Coefficient					.106	.102	.043	.102	.170**	.083
		Sig. (2-tailed)					.059	.090	.465	.077	.003	.150
		N					317	277	295	302	297	301
	Problem B2	Correlation Coefficient						.138*	.167**	.185**	.239**	.209**
		Sig. (2-tailed)						.022	.004	.001	.000	.000
		N						276	289	293	290	292
	Problem C2	Correlation Coefficient							.046	.078	.259**	.193**
		Sig. (2-tailed)							.460	.212	.000	.002
		N							260	260	260	261
	Problem D1	Correlation Coefficient								.028	.302**	.094
		Sig. (2-tailed)								.638	.000	.108
		N								295	294	294
	Problem E1	Correlation Coefficient									.220**	.415**
		Sig. (2-tailed)									.000	.000
		N									300	304
	Problem D2	Correlation Coefficient										.304**
		Sig. (2-tailed)										.000
		N										300
	Problem E2	Correlation Coefficient										
		Sig. (2-tailed)										
		N										

**. Correlation is significant at the 0.01 level (2-tailed).

*. Correlation is significant at the 0.05 level (2-tailed).

Similarity, we found strong positive correlations between problems B_1 and B_2; C_1 and C_2; D_1 and D_2; E_1 and E_2. The absence of a strong significant correlation between problems A_1 and A_2 is justified by the differences in the evaluation of the answers for such questions, as was previously referred.

Question A_1, a straightforward number and operation exercise, is correlated with question B_1 and also with question B_2regarding the subject. The difference lies in the difficulty level. We observe a positive correlation with question D_1, a question on Geometry which requires some computation, even though not as intricate as question D_2. It is no coincidence that such strong positive correlation was not found between question A_1 and question D_2.

Regarding question B_2, we found a significant positive correlation with question D_2 and question E_2; as well as with question D_1 and question E_1, *i.e.* all the questions presented in the second moment of problem solving. This is justified by questions similarly difficult which we consider of average difficulty. Question B_1 shows a pattern similar the pattern of questions D_1, E_1, D_2 and E_2.

If the first two problems of the second worksheet, D_1 and E_1, do not show any correlation, the second problems, D_2 and E_2, have a significant positive correlation. As students' practise for a few minutes before they solve the new but identical questions, they eventually become more insightful of their capabilities / limitations, which consubstantiates a significant positive correlation.

Results show a significant positive correlation between questions C_2 and D_2. A possible explanation is that both questions have, respectively, a high and medium-high level of difficulty. By opposition, we do not find such correlation between questions C_1 and D_2.

Questions E_1 and D_2 present a significant positive correlation. Again, despite the different mathematical nature of these questions, they share similar a difficulty level.

In conclusion, we can state that as the level of difficulty of the questions increases, the quality of the answers decrease dramatically. In spite of some heuristic proposals to solve questions, in general, the students did not consider them as instruments to reach a solution. This behaviour underlines the resistances of the students against the use of strategies to help solving mathematical problems, eventually because they are not familiar with such techniques. A single intervention without a continuous problem solving intervention programme implemented along the school year is insufficient to produce better student performance.

Next we present the problem solving intervention *The use of Heuristics in Mathematics – Tools to problem solving* workshop for teachers carried out in the 2012/2013 school year.

Teachers' attitudes towards problem-solving

Problem solving proposal addressed to the teachers

1. Consider the multiplications:

$$12\,345\,679 \times 18 = 222\,222\,222$$
$$12\,345\,679 \times 27 = 333\,333\,333$$
$$12\,345\,679 \times 54 = 666\,666\,666$$

What number should be multiplied by 12 345 679 in order to obtain 999 999 999?

A solution to a problem can be attained by appealing to different procedures. The strategy followed may be a sign of the quality of the mathematical tools applied by the individual and of their dexterity in using them, as well as of the cognition performances activated throughout the whole process.

Solution a) $999\,999\,999 = 333\,333\,333 + 666\,666\,666 =$

$12\,345\,679 \times 27 + 12\,345\,679 \times 54 = 12\,345\,679 \times (27 + 54) = 12\,345\,679 \times 81$

Solution b) $999\,999\,999 = 333\,333\,333 \times 3 = 12\,345\,679 \times 27 \times 3 = 12\,345\,679 \times 81$

Solution c) Consider the existence of a pattern,

$12\,345\,679 \times 18 = 222\,222\,222$

$12\,345\,679 \times 27 = 12\,345\,679 \times (18 + 9)$
$= 12\,345\,679 \times 18 + 12\,345\,679 \times 9 = 222\,222\,222 + 111\,111\,111 = 333\,333\,333$

$12\,345\,679 \times 36 = 12\,345\,679 \times (27 + 9)$
$= 12\,345\,679 \times 27 + 12\,345\,679 \times 9 = 333\,333\,333 + 111\,111\,111 = 444\,444\,444$

$12\,345\,679 \times 45 = 12\,345\,679 \times (36 + 9)$
$= 12\,345\,679 \times 36 + 12\,345\,679 \times 9 = 444\,444\,444 + 111\,111\,111 = 555\,555\,555$

$12\,345\,679 \times 54 = 12\,345\,679 \times (45 + 9)$
$= 12\,345\,679 \times 45 + 12\,345\,679 \times 9 = 555\,555\,555 + 111\,111\,111 = 666\,666\,666$

$12\,345\,679 \times 63 = 12\,345\,679 \times (54 + 9)$
$= 12\,345\,679 \times 54 + 12\,345\,679 \times 9 = 666\,666\,666 + 111\,111\,111 = 777\,777\,777$

$12\,345\,679 \times 72 = 12\,345\,679 \times (63 + 9)$
$= 12\,345\,679 \times 63 + 12\,345\,679 \times 9 = 777\,777\,777 + 111\,111\,111 = 888\,888\,888$

$12\,345\,679 \times 81 = 12\,345\,679 \times (72 + 9)$
$= 12\,345\,679 \times 72 + 12\,345\,679 \times 9 = 888\,888\,888 + 111\,111\,111 = 999\,999\,999.$

2. If a digital clock marks from 0:00 to 23:59, how many times a day does it display all equal digits?

The solver must correctly understand the statement so as to prevent the inaccuracy which often results from routine procedures. The clock displays at least 3 digits. Daily, at 8 distinct moments, it displays all equal digits (it displays 3 equal digits 6 times a day and 4 equal digits twice a day).

Inventory without constraints	0:00	1:11	2:22	3:33	4:44	5:55	6:66	7:77	8:88	9:99	11:11	22:22
Inventory taking into account the constraints of the problem	0:00	1:11	2:22	3:33	4:44	5:55					11:11	22:22

3. A promotional campaign allows the change of 4 empty 1-litre bottles for a 1-litre bottle full of milk. How many litres of milk can a person who has 43 empty bottles obtain?

Intuition is important, but without reasoning it can, eventually, guide us to a wrong answer. Critical thinking on the devised plan and the solution attained is vital to optimize practice. A less clear-cut reading can lure the individual to answer 10 litres of milk, the result of 43 divided by 4 (the outcome of swapping 4 empty 1-litre bottles for a 1-litre bottle full of milk). However, this way of thinking does not take into account the reuse of the full bottles received, which can, and must, also be converted into more litres of milk.

Solution *a)* FB (Full Bottle); EB (Empty Bottle)

To fully benefit from this promotional campaign, the 43 empty bottles are grouped in sets of 4 bottles, and 3 empty bottles remain (3 EB). After the exchange, we get 10 full bottles (40 EB→ 10 FB). Then, we pour those 10 litres of milk into another container and we obtain another 10 empty bottles. Together with the previous 3 empty bottles, we have a total of thirteen empty bottles (10 EB + 3 EB). With this procedure (see chart below), a total of 14 litres of milk is obtained.

	40 EB→ 10 FB		12 EB → 3 FB		4 EB → 1 FB
43 EB (40 EB + 3 EB)		13 EB (10 EB + 3 EB)		4 EB	
	3 EB		1 EB		

Solution *b)* The exchange of 4 empty bottles for a full bottle results in 1 litre of milk and the exclusion of 3 empty bottles. From the 4 bottles delivered we must deduct 1, the received bottle of milk. As a result of the entire division of 43 by 3, the value 14 is obtained, which is the solution to the problem.

Solution *c)* Initially we have 43 empty bottles. If we give 4 empty bottles, it means that we receive a full bottle in return. The number of empty bottles is then reduced to 40 (39 from the initial lot plus the 1 received in return for the 4 empty bottles). Repeat the procedure and at the end you will have a total of 14 bottles of milk.

Bottles	43	40	37	34	31	28	25	22	19	16	13	10	7	4
Given	4	4	4	4	4	4	4	4	4	4	4	4	4	4
Received	1	1	1	1	1	1	1	1	1	1	1	1	1	1

4. In an empty box there were several balls of different colours: 5 blue balls, 4 yellow, 3 red, 2 white and 1 black. Renato took three balls from the box. Knowing that the balls taken were neither blue, nor yellow, nor black, can we say that:

A) the three balls have the same colour? B) the three balls are red?

C) 1 ball is red and 2 are white? D) 1 ball is white and 2 are red?

E) at least 1 of the balls is red?

After reading the statement of the problem, the solver concludes that the extracted balls belong to the set of 3 red balls and 2 white balls. Let's consider each option:

A) All balls have the same colour.

False, for even though there is the possibility that the 3 extracted balls are red, that is not a certain event because there are also white balls inside the box.

B) All balls are red.

False, as previously explained.

C) 1 ball is red and 2 are white.

False, for even though that may occur, it is not a certain event. For example, it is possible that the 3 balls extracted from the box are red.

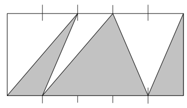
D) 1 ball is white and 2 are red.

False, for even though that may occur, it is not a certain event. For example, it is possible that the 3 balls extracted from the box are red.

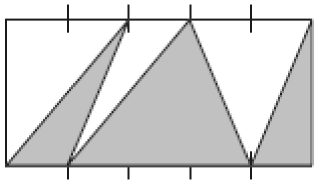
E) At least 1 ball is red.

True. If 3 balls are taken from the box and the colours of the balls in the box are restricted to red and white (3 red balls and 2 white balls), at least 1 of the balls is guaranteed to be red.

5. Considering that the area of the rectangle is 12, what is the area of the part in grey?



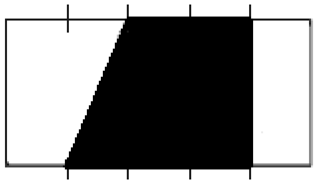
Solution *a)* We can see that the rectangle measures 5 units of length. This information combined with the value of the area allows us to compute the value of its width.



$$5 \times w = 12 \Leftrightarrow w = 2.4$$

Compute the areas of each triangle and, after the values are added, the solution is attained $\left(\frac{1\times 2.4}{2} + \frac{3\times 2.4}{2} + \frac{1\times 2.4}{2} = 6\right)$.

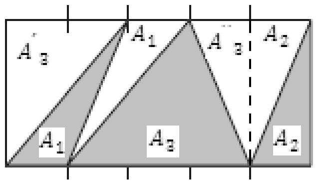
Solution *b)* Move the grey sections in order to obtain a parallelogram (in black).



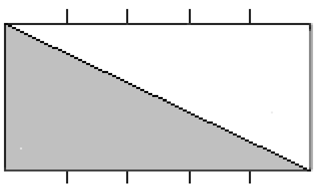
$$Area\ of\ the\ Parallelogram\ = \frac{Larger\ Base + smaller\ base}{2} \times height$$

$$Area\ of\ the\ Parallelogram\ = \frac{3+2}{2} \times 2.4 = 2.5 \times 2.4 = 6$$

Solution *c)* Compare sections. The grey area equals half of the area of the rectangle. Hence, correspondence may be established between regions A_1 and A_2, coloured in grey and white. This procedure can be extended to region A_3, with regions A_3' and A_3''.



Solution *d)* Triangles with an equal base and an equal height have identical areas. Move the vertices of the grey triangles to the left. Notice how the areas remain the same. Two geometrically equal triangles are formed, one of which corresponds to the grey area. Hence, the solution to the problem is half of the area of the rectangle.



6. The number 10 can be written as the sum of two prime numbers in two different ways: 10 = 5 + 5 and 10 = 7 + 3. How many ways are there to express the number 25 as the sum of two prime numbers?

We may say that 2 + 23 = 25, but how can we prove that this is the only solution? We must take into account that all prime numbers are odd, with the exception of prime number 2, and that to obtain an odd number by adding two prime numbers, the addition must include both an even and an odd number.

Even Number + Even Number = Even Number

Even Number + Odd Number = Odd Number

Odd Number + Even Number = Odd Number

Odd Number + Odd Number = Even Number

7. A given 2-digit number *N* is the square of a natural number. If the order of the digits is inverted, an odd number is obtained. The difference between the two numbers is the cube of a natural number. Can we say that the sum of the digits of *N* is:

A) 7? B) 10? C) 13? D) 9? E) 11?

Solution *a)* Identify all 2-digit numbers *N* which are the square of a natural number: 16, 25, 36, 49, 64, 81. Then, reverse the order of the digits in order to obtain an odd number. The only candidates to the solution are numbers 61 and 63. Next, the difference between the two numbers must be the cube of a natural number. Since $61 - 16 = 45$ and $63 - 36 = 27 = 3^3$, the number *N* which complies with the statement of the problem is the number 36, and the sum of its digits equals 9.

N – 2-digit numbers which are the square of a natural number	**16**	**25**	**36**	**49**	**64**	**81**
Reverse the order of the digits to obtain an odd number	**61**	52	**63**	94	46	18
The difference between the two numbers is the cube of a natural number	**61 – 16 =** 45		**63 – 36 = 27 =** 3^3			
Add the digits of *N*			**3 + 6 = 9**			

Solution *b)* Consider $N = n_1n_2$. For n_2n_1 to be an odd number, n_1 can assume the values 1, 3, 5, 7 or 9. Hence, the tens digit of *N* can assume the values 1, 3, 5, 7 or 9. Being *N*, a 2-digit number, the square of a natural number, among the values 16, 25, 36, 49, 64 or 81, it can only be 16 or 36. If *N* = 16, 61 - 16 = 45, if *N* = 36, 63 - 36 = 27 = 3^3, therefore the sum of the digits of *N* is 9.

8. If natural numbers are put in columns, as set out in the table, under which letter will the number 2,000 be?

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
1		2		3		4		5
	9		8		7		6	
10		11		12		13		14
	18		17		16		15	
19		20		21		...		...

Solution *a)* Perform the entire division of 2000 by 9, obtaining a quotient of 222 and a remainder of 2. The number 2000 is the 223rd term of an arithmetical progression whose first term is 2 and whose common difference between successive terms is 9. The answer is letter *C*.

Solution *b)* The general term of each sequence is the value of the first term added to the expression for the multiples of 9. Let us now analyse each option.

Sequence *A*: 1, 10, 19…
Will the value 2000 appear under the letter A? $1 + 9n = 2000$, $n \notin \mathbb{N}_0$

Sequence *B*: 9, 18, 27 …
Will the value 2000 appear under the letter B? $9 + 9n = 2000$, $n \notin \mathbb{N}_0$

Sequence *C*: 2, 11, 20 …
Will the value 2000 appear under the letter C? $2 + 9n = 2000 \Leftrightarrow n = 222$

Solution *c)* For each letter *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H* and *I*, the general expressions for each sequence are, respectively, $9n - 8$, $9n$, $9n - 7$, $9n - 1$, $9n - 6$, $9n - 2$, $9n - 5$, $9n - 3$ and $9n - 4$, $n \in \mathbb{N}$. If each expression is equal to the value 2000, we obtain *A*: $9n = 2008$; *B*: $9n = 2000$; *C*: $9n = 2007$; *D*: $9n = 2001$; *E*: $9n = 2006$; *F*: $9n = 2002$; *G*: $9n = 2005$; *H*: $9n = 2003$ and *I*: $9n = 2004$. Concerning the second member of the equations, the only multiple of 9 is number 2007, since $2 + 0 + 0 + 7 = 9$, which is a multiple of 9. Therefore, the number 2000 belongs to column C.

9. The emir Abdel Azir had more than 39 children, including several twins, triplets and quadruplets. In fact, the historian Ahmed Aab says in one of his writings that all the children of the emir were born as twins, except for 39; all were triplets, except for 39; all were quadruplets, except for 39. How many children did the emir Abdel Azir have?

Solution *a)* Let us consider x as the number of twins, y the number of triplets, z the number of quadruplets, and T the total number of children.

$$x = T - 39 \Rightarrow T = x + 39 \qquad y = T - 39 \Rightarrow T = y + 39 \qquad z = T - 39 \Rightarrow T = z + 39$$

The relations show that $x = y = z$, *i.e.*, the number of twins, triplets and quadruplets is the same. However, the number of twins must be divisible by 2, the number of triplets must be divisible by 3 and the number of quadruplets must be divisible by 4. When the least common multiple between 2, 3 and 4 is determined, lcm (2, 3, 4) = 12, we conclude that there are at least six groups of twins, four groups of triplets and three groups of quadruplets, in a total of 51 children, *i.e.*, $x = 12, T = 12 + 39 = 51$.

Solution *b)* Consider F as the set of children. Each child is either an only child or part of a group of twins, triplets or quadruplets, thus satisfying only one of these conditions. Thus, $F = U \uplus D \uplus T \uplus Q$.

$$U = \{only\ children\},$$
$$D = \{children\ born\ as\ twins\ \},$$
$$T = \{children\ born\ as\ triplets\},$$
$$Q = \{children\ born\ as\ quadruplets\}.$$

We know that *"all except 39 were twins"*, which means that if we deduct thirty nine from F, we obtain the number of twins. The equation $F = U \uplus D \uplus T \uplus Q$ allows to write $|D| = |F| - |U| - |T| - |Q|$, because we have a disjoint union, *i.e.*, $|F| = |U| + |D| + |T| + |Q|$. We know that $|D| = |F| - 39$. Similarly, we can say that $|T| = |F| - 39$ and $|Q| = |F| - 39$. Since each child who is not an only child is part of a group of two, a group of three or a group of four, then:

$$d := elements\ of\ twins;$$
$$t := elements\ of\ triplets;$$
$$q := elements\ of\ quadruplets.$$

We obtain $|D| = 2.d$; $|T| = 3.t$; $|Q| = 4.q$. Hence, as $|F| - 39 = 2.d = 3.t = 4.q$, then $|F| - 39$ is divisible by 2; $|F| - 39$ is divided by 3; $|F| - 39$ is divided by 4, subsequently, $|F| - 39 = k.lcm(2,3,4) = 12k$, then $|F| = 39 + 12k$. We have $k > 0$, because, according to the statement of the problem, the emir had more than 39 children. The possible solutions are $\{51, 63, 75, \dots\}$.

10. Four friends go visit a museum and one of them decides to enter without paying for the ticket. A controller shows up and wants to know which of them entered without paying.

– It wasn't me, says Benjamin. – It was Carlos, says Mario.

– It was Peter, says Carlos. – Mario is wrong, says Peter.

Only one of them is lying. Who did not pay for their ticket to the museum?

A) Mario. B) Peter. C) Benjamin. D) Carlos.

E) It is not possible to find a solution for the problem, for the statement lacks information.

Solution *a)* Let us designate Benjamin, Carlos, Mario and Peter as ***B***, ***C***, ***M*** and ***P***, respectively. We will fill in a double entry table with the information *Y* (Yes) and *N* (No), in compliance with the boys' statements on who entered the museum without paying for his ticket. With this plan we devise contradictions in Mario and Peter's statements concerning Carlos. Which of the two is lying? Let us assume that Mario spoke the truth. In this case, Carlos and Peter entered the museum without paying! It does not respect the statement of the problem. So, Mario is lying. Peter was the one who entered the museum without paying for his ticket.

	B	***C***	***M***	***P***
B	*N*			
C				*Y*
M		*Y*		
P		*N*		

Solution *b)*

The liar is…	then…
Benjamin	According to Benjamin and Carlos' statements, Benjamin and Peter entered the museum without paying. This does not respect the statement of the problem.
Peter	Mario is right. So, it must be Carlos who entered without paying. But if what Carlos says is true (he says that Peter entered without paying); then both Carlos and Peter entered without paying. This does not respect the statement of the problem.
Carlos	Peter did not enter without paying. Benjamin states that he paid for his ticket. According to Mario, Carlos entered without paying. But from Peter's statement, we know that it was not Carlos who entered without paying. Mario and Peter's statements are contradictory!
Mario	It was not Carlos who entered without paying. Benjamin says that he paid for his ticket. According to Carlos, Peter entered without paying. From Peter's statement, we know that Carlos paid for his ticket to the museum.

The only option is for Mario to be lying. The boy who entered without paying was Peter, just as Carlos said.

Solution *c)* Let us identify Benjamin, Carlos, Mario and Peter as B, C, M, P, respectively.

ϕ_X : $X's\ statement$ $\qquad\qquad$ X := X did not pay

The boys' statements can be written as follows:

$\phi_B: \neg B \qquad \phi_C: P \qquad \phi_M: C \qquad \phi_P: \neg\phi_M = \neg C$

"Only one of them is lying" means exactly that one statement ϕ_X has a false value: $\mathcal{V}(\phi_X) = 0$. As C and $\neg C$ have different logical values, then ϕ_M or ϕ_P are false. So, *M* or *P* are lying. If *P* lied, then $\mathcal{V}(\phi_C) = \mathcal{V}(\phi_M) = 1$, which means that C and P are right. But according to *C*, *P* did not pay, and according to *M*, *C* did not pay. But now two boys would have entered the museum without paying. This contradicts the statement of the problem! We conclude that *M* must be lying. Therefore, $\neg C$ and $\neg B$. Since *C* was telling the truth, *P* (Peter) was the boy who did not pay for his ticket.

11. How many whole positive numbers under 1,000,000 are there whose cubes end in 1?

What do positive whole numbers inferior to 1,000,000 whose cubes finish in 1 have in common? All these numbers have 1 as the ones digit (justification below).

…1 × …1 × …1 = …1; …2 × …2 × …2 = …8; …3 × …3 × …3 = …7;
…4 × …4 × …4 = …4; …5 × …5 × …5 = …5; …6 × …6 × …6 = …6;
…7 × …7 × …7 = …3; …8 × …8 × …8 = …2; …9 × …9 × …9 = …9.

How many numbers have this characteristic?

1	101	…	901
11	111	…	911
21	121	…	921
31	131	…	931
41	141	…	941
51	151	…	951
61	161	…	961
71	171	…	971
81	181	…	981
91	191	…	991
10	**10**	**…**	**10**

In each 1,000 numbers, 100 finish with the digit 1. In the first 1,000,000 natural numbers we find 100,000 numbers with this characteristic.

12. The 61 candidates approved for a competition had different marks. They were divided into 2 classes, according to the marks obtained: the first 31 were put in class A and the remaining 30 in class B. The means of the two classes were calculated. However, later it was decided that the last candidate to be put in class A should move to class B. Hence:

A) The mean of class A increased, but the mean of class B decreased.

B) The mean of class A decreased, but the mean of class B increased.

C) The means of both classes increased.

D) The means of both classes decreased.

E) The means of both classes can increase or decrease, depending on the marks of the candidates.

Solution *a)* Transferring the less qualified candidate from class A to class B increases the averages of both class A and class B. The less-qualified candidate of the set of the best-qualified candidates contributes to a lower average of the set. However, when that individual is transferred to class B, thus becoming the best of the group, it will raise the average of that group.

Solution *b)* Class A is composed of 31 students whose marks are identified by $x_1, x_2, \dots, x_{30}, x_{31}$.

$$\overline{x_A} = \frac{x_1 + \cdots + x_{31}}{31}$$

Class B is composed of 30 students whose marks are identified by $x_{32}, x_{33}, \dots, x_{60}, x_{61}$.

$$\overline{x_B} = \frac{x_{32} + \cdots + x_{61}}{30}$$

Transfer the less-qualified element of class A to class B.

$$\overline{x_{A'}} = \frac{x_1 + \cdots + x_{30}}{30} \qquad \overline{x_{B'}} = \frac{x_{31} + \cdots + x_{61}}{31}$$

Obviously $\overline{x_A} < \overline{x_{A'}}$.

$$\overline{x_A} = \frac{x_1 + \cdots + x_{31}}{31} = \frac{x_1 + \cdots + x_{30}}{31} + \frac{x_{31}}{31} = \frac{30\overline{x_{A'}}}{31} + \frac{x_{31}}{31} < \frac{30\overline{x_{A'}}}{31} + \frac{\overline{x_{A'}}}{31} = \overline{x_{A'}}$$

It is also obvious that $\overline{x_{B'}} > \overline{x_B}$.

$$\overline{x_{B'}} = \frac{x_{31} + \cdots + x_{61}}{31} = \frac{x_{31}}{31} + \frac{x_{32} + \cdots + x_{61}}{31} = \frac{x_{31}}{31} + \frac{30\overline{x_B}}{31} > \frac{\overline{x_B}}{31} + \frac{30\overline{x_B}}{31} = \overline{x_B}$$

We explain that the averages of both classes have improved, $\overline{x_{A'}} > \overline{x_A}$, $\overline{x_{B'}} > \overline{x_B}$.

The answers were rated from 1 to 6 *(Table 42)* with the following criterion *(Chart 7)*:

Classification evaluation (Chart 7)

6: <u>Exemplary answer.</u> The answer is complete and includes a clear and accurate explanation of the techniques used to solve the problem. It includes accurate diagrams (where appropriate), identifies important information, shows full understanding of ideas and mathematical processes used in the solution, and clearly communicates this knowledge.

5: <u>Competent answer.</u> The answer is fairly complete and includes a reasonably clear explanation of the ideas and processes used. Solid supporting arguments are presented, but some aspect may not be as clearly or completely explained as possible.

4: <u>Satisfactory with minor flaws.</u> The problem is completed satisfactorily, but explanation is lacking in clarity or supporting evidence. The underlying mathematical principles are generally understood, but the diagram or description is inappropriate or unclear.

3: <u>Nearly satisfactory, but contains serious flaws.</u> The answer is incomplete. The problem is either incomplete or major portions have been omitted. Major computational errors may exist, or a misuse of formulas or terms may be present. The answer generally does not show full understanding of the mathematical concepts involved.

2: <u>Begins problem but fails to complete solution.</u> The answer is incomplete and shows little or no understanding of the mathematical processes involved. Diagram or explanation is unclear.

1: <u>Fails to begin effectively.</u> The problem is not effectively represented. Parts of the problem may be copied, but no solution was attempted. Pertinent information was not identified.

(Meier, 1992)

Problem solving evaluation results (Table 42)

	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	Q_9	Q_{10}	Q_{11}	Q_{12}	Total
AFPMO	4	6	6	6	6	5	6	3	4	2	5	5	58
AMPMC	5	5	6	6	6	5	6	5	2	1	5	DA	52
AMMS	2	2	5	2	6	5	5	4	2	DA	5	6	44
ACV	4	6	5	6	6	5	6	6	5	4	1	5	59
CILMJR	4	5	4	6	6	5	6	5	2	1	6	5	55
LMCR	DA	5	4	6	6	5	3	5	4	6	6	DA	50
MAGSA	5	5	4	6	6	5	5	5	2	1	5	DA	49
MGCCMFP	5	5	6	6	6	6	6	DA	DA	DA	DA	5	45
MFRN	5	5	6	6	6	5	6	5	2	1	6	DA	53
MJNRL	4	6	5	6	5	5	6	5	5	5	5	6	63
MPJJS	4	5	5	6	5	5	6	2	DA	DA	5	5	48
PCOA	DA	5	5	6	5	5	6	DA	DA	3	2	6	43
PCMLEC	5	5	6	6	6	6	6	5	2	5	6	5	63
RMVF	4	6	5	6	6	5	5	5	4	2	DA	4	52
RMSSV	4	6	5	4	3	5	5	4	2	5	6	5	59
TCRO	4	6	5	6	5	5	6	5	4	5	5	4	60
Total	**59**	**83**	**82**	**90**	**89**	**82**	**89**	**64**	**40**	**41**	**68**	**61**	

DA: Did not Answer

The answers provided by the 16 teachers who attended the workshop *The use of Heuristics in Mathematics – Tools to problem solving (2012/ 2013)* reveal: *1)* low-quality answers in questions Q_9 and Q_{10} which are related to Algebra and Logic, due to difficulties felt in organising and using the information provided by the statement of the problem; *2)* 16 out of 192 answers were classified as DA (Did not Answer), which is a significant number in view of the qualifications of the participants. We also observed that several teachers did not answer questions Q_{12} (4 DA), Q_9 (3 DA), Q_{10} (3 DA), Q_1 (2 DA), Q_8 (2 DA) and Q_{11} (2 DA), which can be justified by the limited time allowed to complete the test (120 *min.*) and / or a great level of difficulty.

Theoretical knowledge was required to answer the test, as well as the use of heuristic procedures to organise given information, devise and implement a plan, and find a solution.

The room where the written test took place was equipped with tables for two people and individual chairs which were disposed in a panoramic ⊓, without partition walls. There were no restrictions regarding seats, so the participants chose where they would like to sit. The workshop instructor had a privileged view over all the participants. When confronted with the written test, in general, the initial reaction was of annoyance and dissatisfaction, a behaviour driven by the embarrassment of knowing that a formal evaluation would take place. Since all the participants had tertiary education in Mathematics or Engineering, my option was to use an informal approach in order to relieve the stress felt by the group. Throughout the two hours sporadic dialogues between the participants and glimpses to neighbours' work were noticed. Teachers expect, but do not approve, these behaviours in their classrooms, when their students take a written test. In the role of student, these 16 teachers involuntarily showed behaviours crystallised from their adolescence. The fear of not being successful compels the individuals to act in such a way. In the role of teachers, such behaviour would be object of criticism.

Critical reflexion on the teachers' perceived practices

We investigated teachers' opinions and perceived practices, as well as the promotion of the learning, especially in what concerns problem solving, from a questionnaire answered by 30 teachers, 7 male and 23 female, with mean age of 46.67 years old, participants in *The use of Heuristics in Mathematics – Tools to problem solving* 2011/2012 and 2012/2013 workshops. Next we present the data gathered from the 65 questions included in the questionnaire and a brief analysis under the scope of descriptive statistics. Questions 1, 2, 4, 5, 7, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 24, 25, 26, 30, 32, 35, 36, 38, 39 and 40 options were: *Strongly Agree* (5), *Agree* (4), *Indifferent / Not Sure* (3), *Disagree* (2), and *Strongly Disagree* (1); about questions 41, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, and 65 options were: *Almost Always* (4), *Frequently* (3), *Sometimes* (2), and *Almost Never* (1).

Due the phrasing of questions 3, 6, 8, 12, 21, 22, 23, 27, 28, 29, 31, 33, 34, and 37 results were considered as follows: *Strongly Agree* (1), *Agree* (2), *Indifferent / Not Sure* (3), *Disagree* (4), and *Strongly Disagree* (5); questions 42 and 43 should be considered as follows: *Almost Always* (1), *Frequently* (2), *Sometimes* (3), and *Almost Never* (4).

1. When a student achieves a better performance than usual in Mathematics, it is because the teacher made an extra effort. Median value 3 *(Table 43)*.

(Table 43)

Question 1	Frequency	Percent
Strongly Disagree	1	3.3
Disagree	8	26.7
Indifferent / Not Sure	11	36.7
Agree	10	33.3

A better achievement from the student can't be strictly labeled with a teacher's more proactive role in the classroom. The teacher plays an important role in the teaching and learning process but there are many others variables that endorse success.

2. I am always looking for better ways to teach Mathematics. Median value 5 *(Table 44)*.

(Table 44)

Question 2	Frequency	Percent
Agree	4	13.3
Strongly Agree	26	86.7

Teachers reveal the desire to find better strategies to apply in the classroom and assume a proactive attitude to achieve such purpose.

3. Even if I try really hard, I will not be as efficient as the best teachers I know. Median value 4 *(Table 45)*.

(Table 45)

Question 3	Frequency	Percent
Agree	1	3.3
Indifferent / Not Sure	3	10.0
Disagree	14	46.7
Strongly Disagree	12	40.0

Teachers disagree on the statement, so they believe that becoming more relentless on their effort in improving instruction will guide them to become more efficient in their vocation.

4. When students' marks improve in Mathematics, it is often due to a more effective teaching approach. Median value 4 *(Table 46)*.

(Table 46)

Question 4	Frequency	Percent
Agree	30	100.0

If we compare this score with the score of question 1 we realize that teaching method is significantly more relevant for students' success than the vividly lesson performed by the teacher also called *extra effort*.

5. I know how to efficiently teach Mathematical concepts. Median value 4 *(Table 47)*.

(Table 47)

Question 5	Frequency	Percent
Strongly Disagree	1	3.3
Indifferent / Not Sure	2	6.7
Agree	26	86.7
Strongly Agree	1	3.3

Teachers trust in their theoretical knowledge and acquired practice to efficiently teach Mathematical concepts. Such perception is important in the teaching and learning process because the teacher should be confident and master the subject matter that teaches in the classroom.

6. I am not very efficient in what concerns the monitoring of learning activities. Median value 4 *(Table 48)*.

(Table 48)

Question 6	Frequency	Percent
Indifferent / Not Sure	5	16.7
Disagree	23	76.7
Strongly Disagree	2	6.7

Teachers disagree on the statement, so they consider that they are able to supervise the learning activities that take place in the classroom. Of course it can be difficult for a teacher to confess, even in a confidential questionnaire, that he/she is less able to monitor such activities.

7. Bad results in Mathematics are often due to inadequate teaching. Median value 3 *(Table 49)*.

(Table 49)

Question 7	Frequency	Percent
Strongly Disagree	2	6.7
Disagree	11	36.7
Indifferent / Not Sure	3	10.0
Agree	11	36.7
Strongly Agree	3	10.0

Teachers are not sure that an inadequate teaching can be the main cause of students' poorer results. Indeed, gifted students perform well independently of the teaching method, and the subject matter is independent of the method used to present it.

8. In general, the teaching of Mathematics is not adequate. Median value 4 *(Table 50)*.

(Table 50)

Question 8	Frequency	Percent
Agree	7	23.3
Indifferent / Not Sure	3	10.0
Disagree	16	53.3
Strongly Disagree	4	13.3

Teachers disagree on the statement, so they endorse the curriculum and the classroom practices advocated in the math program for Third Cycle and Secondary Education.

9. The lack of prerequisites in the learning of Mathematics can be surpassed with adequate teaching. Median value 4 *(Table 51)*.

(Table 51)

Question 9	Frequency	Percent
Strongly Disagree	2	6.7
Disagree	3	10.0
Indifferent / Not Sure	6	20.0
Agree	15	50.0
Strongly Agree	4	13.3

Teachers agree that with suitable classroom approaches it is feasible helping the student to acquire basic knowledge that is critical to more complex learning.

10. When a student with a poor performance suddenly has better results in Mathematics, it is usually due to extra-attention of the teacher. Median value 4 *(Table 52)*.

(Table 52)

Question 10	Frequency	Percent
Disagree	2	6.7
Indifferent / Not Sure	10	33.3
Agree	18	60.0

Considering only lower performance students, teachers agree that a suddenly better achievement can be consequence of the teacher's surplus effort to explain the subject matter regarding that particular student. If we compare question 10 with question 1, we can consider that teachers believe their extra effort to teach subject matter produces outcomes more straightforward with lower performance students than with average students.

11. I have a sufficient command of the concepts to be an efficient Mathematics teacher. Median value 4 *(Table 53)*.

(Table 53)

Question 11	Frequency	Percent
Indifferent / Not Sure	1	3.3
Agree	16	53.3
Strongly Agree	13	43.3

As previous revealed on question 5 results, teachers believe that they are skillful in mathematics to carry on their profession with efficiency.

12. Usually, the teacher is responsible for the performance of the students. Median value 2 *(Table 54)*.

(Table 54)

Question 12	Frequency	Percent
Strongly Agree	3	10.0
Agree	16	53.3
Indifferent / Not Sure	6	20.0
Disagree	5	16.7

Generically, regardless of the distinctiveness of each student, teachers are aware of the responsibility they have in the performance of their students.

13. The performance of the students in Mathematics is directly related to the efficacy of the teaching. Median value 3 *(Table 55)*.

(Table 55)

Question 13	Frequency	Percent
Disagree	7	23.3
Indifferent / Not Sure	13	43.3
Agree	10	33.3

We found correspondence with question 7. The teacher is a key element in the teaching and learning process but it is not the only one, so students' performance also depends on other factors.

14. If parents comment that their child is revealing more interest in the Mathematics taught at school, it is probably due to the performance of the teacher. Median value 4 *(Table 56)*.

(Table 56)

Question 14	Frequency	Percent
Indifferent / Not Sure	10	33.3
Agree	19	63.3
Strongly Agree	1	3.3

Teachers agree with this statement. They are aware of their role to motivate students to gain interest about mathematical subject matter.

15. I experience difficulties in applying manageable materials to explain the applicability of mathematics to my students. Median value 2 *(Table 57)*.

(Table 57)

Question 15	Frequency	Percent
Strongly Disagree	7	23.3
Disagree	14	46.7
Indifferent / Not Sure	6	20.0
Agree	2	6.7
Strongly Agree	1	3.3

Teachers disagree with this statement. As previous revealed on questions 2, 5 and 11, teachers consider themselves capable to master mathematical concepts, both theoretical and practical, with a chalk on the black board or with manageable materials.

16. I can usually answer the questions posed by my students. Median value 5 *(Table 58)*.

(Table 58)

Question 16	Frequency	Percent
Strongly Disagree	1	3.3
Indifferent / Not Sure	1	3.3
Agree	8	26.7
Strongly Agree	20	66.7

Answers underline what was previous analysed mainly concerning questions 5 and 11.

17. I question myself whether I have the necessary skills to teach Mathematics. Median value 2 *(Table 59)*.

(Table 59)

Question 17	Frequency	Percent
Strongly Disagree	14	46.7
Disagree	9	30.0
Indifferent / Not Sure	2	6.7
Agree	5	16.7

Answers to this question corroborate the teachers' consciousness about their high-quality skills. Due to teachers' perception regarding mastering the subject matter, they don't question about their competence to teach Mathematics.

18. If I had the chance, I would invite different pedagogical entities of the education system to assess my way of teaching Mathematics. Median value 3 *(Table 60)*.

(Table 60)

Question 18	Frequency	Percent
Strongly Disagree	1	3.3
Disagree	9	30.0
Indifferent / Not Sure	10	33.3
Agree	8	26.7
Strongly Agree	2	6.7

Teachers' answers reveal restlessness about exposing their teaching practice to outsiders.

19. When a student reveals difficulties to understand a mathematical concept, I usually help them understand that concept. Median value 5 *(Table 61)*.

(Table 61)

Question 19	Frequency	Percent
Agree	9	30.0
Strongly Agree	21	70.0

Teachers are engaged in doing the best they can to help students to learn Mathematics. Such fortitude may or may not produce outcomes in the student's understanding of the mathematical concept that is explained by the teacher.

20. I promote a question-friendly environment in my classroom. Median value 4 *(Table 62)*.

(Table 62)

Question 20	Frequency	Percent
Indifferent / Not Sure	2	6.7
Agree	18	60.0
Strongly Agree	10	33.3

Teachers agree with this statement. They promote friendly classroom environment and welcome with satisfaction the students' questions.

21. I do not know how to lure students into studying Mathematics. Median value 4 *(Table 63)*.

(Table 63)

Question 21	Frequency	Percent
Agree	5	16.7
Indifferent / Not Sure	1	3.3
Disagree	20	66.7
Strongly Disagree	4	13.3

The answers to this statement corroborate previous ones, namely, answers to questions 2, 3 and 5. Teachers consider themselves as capable to stimulate students' interest regarding Mathematics.

22. I am very stressed out when I am going to teach Mathematics to a class of problematic / low-performance students. Median value 4 *(Table 64)*.

(Table 64)

Question 22	Frequency	Percent
Agree	2	6.7
Indifferent / Not Sure	8	26.7
Disagree	16	53.3
Strongly Disagree	4	13.3

Teachers disagree with this statement. They don't recognize psychological burnout or, eventually, don't want to express that state of mind when teaching to problematic or low-performance students.

23. I like Mathematics but I dislike teaching it. Median value 5 *(Table 65)*.

(Table 65)

Question 23	Frequency	Percent
Strongly Agree	1	3.3
Disagree	2	6.7
Strongly Disagree	27	90.0

Teachers strongly disagree with this statement. Again, we have a consistent trend about the satisfaction not also in enjoying Mathematics but also in teaching this subject matter.

24. Mathematics is an interesting subject that I like to teach. Median value 5 *(Table 66)*.

(Table 66)

Question 24	Frequency	Percent
Agree	4	13.3
Strongly Agree	26	86.7

Once more is evident the teachers' satisfaction about Mathematics and the teaching practice of Mathematics.

25. Mathematics is a fun and fascinating subject. Median value 5 *(Table 67)*.

(Table 67)

Question 25	Frequency	Percent
Indifferent / Not Sure	1	3.3
Agree	5	16.7
Strongly Agree	24	80.0

Teachers' opinion corroborates the answers given to question 24.

26. When I teach Mathematics, I feel secure. I find it stimulating. Median value 5 *(Table 68)*.

(Table 68)

Question 26	Frequency	Percent
Agree	14	46.7
Strongly Agree	16	53.3

Teachers' answers express confidence in their activity with the students in the classroom, and without unveiling psychological burnout.

27. Some concepts are difficult to teach if I do not check the manual. Median value 4 *(Table 69)*.

(Table 69)

Question 27	Frequency	Percent
Strongly Agree	2	6.7
Agree	2	6.7
Indifferent / Not Sure	3	10.0
Disagree	17	56.7
Strongly Disagree	6	20.0

Teachers disagree with this statement. The opinion is consistent with the answers to questions 11 and 26.

28. I feel insecure when I have to solve an unknown problem. Median value 3 *(Table 70)*.

(Table 70)

Question 28	Frequency	Percent
Agree	7	23.3
Indifferent / Not Sure	10	33.3
Disagree	11	36.7
Strongly Disagree	2	6.7

Teachers acknowledge some uneasiness when they have to solve an unknown problem. Outside their comfort area they express lack of confidence, such behavior reveals that teachers are more comfortable doing well-known exercises and problems that already master procedure to achieve solution than exploring new mathematical problems.

29. Teaching Mathematics to students who do not learn easily makes me restless, unhappy, angry and impatient. Median value 2.5 *(Table 71)*.

(Table 71)

Question 29	Frequency	Percent
Strongly Agree	1	3.3
Agree	14	46.7
Indifferent / Not Sure	3	10.0
Disagree	10	33.3
Strongly Disagree	2	6.7

In slight opposition to answers to question 22, teachers unveil discomfort when teaching students that have learning difficulties.

30. I feel good when I am teaching Mathematics. Median value 5 *(Table 72)*.

(Table 72)

Question 30	Frequency	Percent
Agree	7	23.3
Strongly Agree	23	76.7

Teachers' opinion corroborates the answers given to questions 24, 25, and 26.

31. If I do not comply with my lesson plans, it is as if I am lost in a labyrinth. Median value 4 *(Table 73)*.

(Table 73)

Question 31	Frequency	Percent
Strongly Agree	1	3.3
Indifferent / Not Sure	6	20.0
Disagree	16	53.3
Strongly Disagree	7	23.3

Teachers disagree with this statement because they are aware of the importance to be flexible about the schedule of each lesson. When necessary the teacher needs to present and work with the students subject matter already taught in order to consolidate information that is essential to understand the present theme.

32. Mathematics is something I fully appreciate. Median value 5 *(Table 74)*.

(Table 74)

Question 32	Frequency	Percent
Agree	6	20.0
Strongly Agree	24	80.0

Teachers' opinion corroborates the answers given to questions 24, 25, 26 and 30.

33. I feel anguished when a student says "I don't understand". Median value 4 *(Table 75).*

(Table 75)

Question 33	Frequency	Percent
Strongly Agree	1	3.3
Agree	4	13.3
Indifferent / Not Sure	3	10.0
Disagree	20	66.7
Strongly Disagree	2	6.7

Teachers disagree with this statement. Motivated teachers consider this student's reaction as a challenge and not as a stress factor.

34. I face Mathematics with a feeling of indecision which results from the fear of not being able to teach efficiently. Median value 5 *(Table 76).*

(Table 76)

Question 34	Frequency	Percent
Indifferent / Not Sure	1	3.3
Disagree	12	40.0
Strongly Disagree	17	56.7

Teachers strongly disagree with this statement as already noticeable in answers to questions 3, 5, 6, 11, 16, 17, 26, and 32.

35. I enjoy Mathematics, independently of my students' learning. Median value 4.5 *(Table 77).*

(Table 77)

Question 35	Frequency	Percent
Strongly Disagree	1	3.3
Indifferent / Not Sure	4	13.3
Agree	10	33.3
Strongly Agree	15	50.0

Teachers' opinion corroborates the answers given to questions 23, 24, 25, 26 and 30.

36. I feel pleasure when I teach Mathematics. Median value 5 *(Table 78).*

(Table 78)

Question 36	Frequency	Percent
Disagree	1	3.3
Agree	6	20.0
Strongly Agree	23	76.7

Teachers' opinion supports the answers given to questions 23, 24, 25, 26, 30 and 35.

37. Thinking about having to correct Mathematics exercises make me nervous. Median value 4 *(Table 79)*.

(Table 79)

Question 37	Frequency	Percent
Agree	4	13.3
Indifferent / Not Sure	2	6.7
Disagree	12	40.0
Strongly Disagree	12	40.0

Teachers disagree with this statement. Correcting mathematical exercises is undoubtedly less stressful when compared with problem solving exploratory activity.

38. I remember that I did not like Mathematics in Basic Education. Median value 1 *(Table 80)*.

(Table 80)

Question 38	Frequency	Percent
Strongly Disagree	20	66.7
Disagree	4	13.3
Indifferent / Not Sure	1	3.3
Agree	4	13.3
Strongly Agree	1	3.3

Teachers strongly disagree with this statement; this means that since early age they enjoy practicing mathematics at school. Such behavior endorses individuals to pursue studies oriented towards Mathematics without negative beliefs about this subject matter.

39. I prefer teaching Mathematics to doing any other school task. Median value 5 *(Table 81)*.

(Table 81)

Question 39	Frequency	Percent
Agree	6	20.0
Strongly Agree	24	80.0

Teachers strongly agree with this statement. School responsibilities often include bureaucratic work, namely school meetings with the need to fill paperwork and writing reports. Teachers prefer to teach mathematics rather than doing such kind of work.

40. I feel at ease when I am teaching Mathematics because I like studying Mathematics. Median value 5 *(Table 82)*.

(Table 82)

Question 40	Frequency	Percent
Indifferent / Not Sure	2	6.7
Agree	7	23.3
Strongly Agree	21	70.0

Teachers' opinion corroborates the answers given to questions 24, 25, 26, 30 and 32.

41. I enjoy Mathematics because it has many practical applications. Median value 3 *(Table 83)*.

(Table 83)

Question 41	Frequency	Percent
Almost Never	1	3.3
Sometimes	6	20.0
Frequently	11	36.7
Almost Always	12	40.0

Teachers frequently agree with this statement. The appliance to concrete situations is not the only reason teachers enjoy Mathematics. Theoretical subject matter under the scope of Mathematics is also a light motive to take pleasure in doing Mathematics.

42. I am a mathematician. Nevertheless, I know that I am not a good Mathematics teacher. Median value 4 *(Table 84)*.

(Table 84)

Question 42	Frequency	Percent
Sometimes	6	20.0
Almost Never	24	80.0

Teachers consider that they have not only a solid theoretical mathematical background but they are also proficient in the classroom.

43. It is impossible to learn Mathematics if students do not like it. Median value 3 *(Table 85)*.

(Table 85)

Question 43	Frequency	Percent
Frequently	7	23.3
Sometimes	16	53.3
Almost Never	7	23.3

"Sometimes" was the median evaluation for this statement. Despite not liking Mathematics, teachers believe, eventually, students can still learn the subject matter.

44. I make sure that my students develop individual work in the classroom. Median value 3 *(Table 86)*.

(Table 86)

Question 44	Frequency	Percent
Sometimes	4	13.3
Frequently	16	53.3
Almost Always	10	33.3

Teachers frequently agree with this statement. For a most advantageous teaching and learning process, students should be able to execute autonomous reasoning and not be confined to listen the teacher and copy subject mater from the black board.

45. I explain in detail what my students have to do to solve the problems. Median value 3 *(Table 87)*.

(Table 87)

Question 45	Frequency	Percent
Almost Never	1	3.3
Sometimes	6	20.0
Frequently	15	50.0
Almost Always	8	26.7

Teachers frequently agree with this statement. To solve mathematical problems students should be conscious of what is asked, and be oriented by the teacher that promotes a propitious problem solving environment.

46. At the end of a problem solving class, I establish a debate in the classroom so that students share their solutions and strategies. Median value 3 *(Table 88)*.

(Table 88)

Question 46	Frequency	Percent
Sometimes	3	10.0
Frequently	13	43.3
Almost Always	14	46.7

Teachers' attitude corresponds to what was expressed in question 44.

47. My students can use calculators. Median value 4 *(Table 89)*.

(Table 89)

Question 47	Frequency	Percent
Almost Never	1	3.3
Sometimes	7	23.3
Frequently	3	10.0
Almost Always	19	63.3

"Almost always" is the median opinion that matches with the recommendations established in the prescribe curriculum for Third Cycle and Secondary Education. Ultimately, the excessive use of calculators to perform simple computation should be eradicated.

48. I encourage my students to cooperatively work in small groups. Median value 3 *(Table 90)*.

(Table 90)

Question 48	Frequency	Percent
Almost Never	1	3.3
Sometimes	9	30.0
Frequently	12	40.0
Almost Always	8	26.7

Teachers frequently agree with this statement. It is necessary to promote cooperative work among students, in the classroom. Teachers are aware of the importance of such methodology but, eventually, they don't have enough time available in the course of school year to implement such practice.

49. I present open and imaginative problems in the classroom and provide a minimum of indications as to how to solve them. Median value 2 *(Table 91)*.

(Table 91)

Question 49	Frequency	Percent
Almost Never	5	16.7
Sometimes	18	60.0
Frequently	7	23.3

"Sometimes" is the teachers' median opinion. Problem solving activity in the classroom is often relegated due to time constraints to teach the complete syllabus and train students to final exams.

50. I encourage my students to take notes of their own problem solving procedures and methods. Median value 3.5 *(Table 92)*.

(Table 92)

Question 50	Frequency	Percent
Almost Never	2	6.7
Sometimes	4	13.3
Frequently	9	30.0
Almost Always	15	50.0

Teachers recognize the importance of problem solving activity and give confidence to their students to list their own problem solving strategies.

51. I encourage my students to present their mathematical problems. Median value 2 *(Table 93)*.

(Table 93)

Question 51	Frequency	Percent
Almost Never	8	26.7
Sometimes	9	30.0
Frequently	11	36.7
Almost Always	2	6.7

"Sometimes" is the teachers' median opinion. Teachers' opinion suggests that, although problem solving is an important activity to promote success in Mathematics, it is not a standard procedure to encourage students to suggest mathematical problem to be analysed in the classroom.

52. I regularly provide a set of problems so that my students can choose one they would like to solve. Median value 1 *(Table 94)*.

(Table 94)

Question 52	Frequency	Percent
Almost Never	17	56.7
Sometimes	12	40.0
Frequently	1	3.3

"Almost never" is the teachers' median opinion. Regardless of the significance of problem solving, teachers prefer being in complete control of what occurs in the classroom rather than being exposed to uncertainty that could come from a not complete previous home study problem.

53. I let my students solve the same problem in one or more lessons. Median value 1 *(Table 95)*.

(Table 95)

Question 53	Frequency	Percent
Almost Never	25	83.3
Sometimes	4	13.3
Frequently	1	3.3

"Almost never" is the teachers' median opinion as expected due to time constraints to complete the school year syllabus.

54. I use problems to show the students that there is knowledge, skills and procedures that they need to master. Median value 3 *(Table 96)*.

(Table 96)

Question 54	Frequency	Percent
Sometimes	5	16.7
Frequently	18	60.0
Almost Always	7	23.3

Teachers frequently agree with this statement, so they consider problem solving activity a useful methodology to endorse mathematical understanding.

55. I give my student problems which may be applied in other contexts. Median value 3 *(Table 97)*.

(Table 97)

Question 55	Frequency	Percent
Sometimes	4	13.3
Frequently	17	56.7
Almost Always	9	30.0

Teachers' opinion corroborates the answer given to question 54.

56. I hand out consistent materials for students who require them. Median value 3 *(Table 98)*.

(Table 98)

Question 56	Frequency	Percent
Almost Never	3	10.0
Sometimes	3	10.0
Frequently	17	56.7
Almost Always	7	23.3

Teachers frequently provide extra worksheets with exercises and problems that complement the school handbook.

57. I provide the class with a problem solving model. Median value 3 *(Table 99)*.

(Table 99)

Question 57	Frequency	Percent
Sometimes	10	33.3
Frequently	13	43.3
Almost Always	7	23.3

Teachers answered that they frequently make available a problem solving model, as prescribed by the Hungarian mathematician George Pólya.

58. I discuss useful strategies for problem solving (organising lists, drawing diagrams ...). Median value 4 *(Table 100)*.

(Table 100)

Question 58	Frequency	Percent
Sometimes	1	3.3
Frequently	12	40.0
Almost Always	17	56.7

Almost always teachers make use of heuristic procedures to engage students in problem solving activity.

59. I discuss problem solving procedures (i.e., devising a plan, following the plan, checking calculations ...). Median value 4 *(Table 101)*.

(Table 101)

Question 59	Frequency	Percent
Frequently	14	46.7
Almost Always	16	53.3

More than provide a problem solving guide, teachers almost always implement such procedures in the course of problem solving activity.

60. I use problems from the school context or which are related to the students' experiences. Median value 3 *(Table 102)*.

(Table 102)

Question 60	Frequency	Percent
Sometimes	3	10.0
Frequently	17	56.7
Almost Always	10	33.3

Teachers' opinion about their practice is that frequently they choose suitable problems to present in the classroom. Such procedure means that they spend more time when previously plan the lesson.

61. I provide open problems so that students may autonomously explore mathematical situations. Median value 2 *(Table 103)*.

(Table 103)

Question 61	Frequency	Percent
Sometimes	19	63.3
Frequently	10	33.3
Almost Always	1	3.3

"Sometimes" is the median opinion. Eventually, teachers don't offer more open problems to be worked autonomously by the students because in many classrooms students don't have the ability to perform such activity by themselves.

62. I provide exercises which enable the students to practice their skills. Median value 3 *(Table 104)*.

(Table 104)

Question 62	Frequency	Percent
Sometimes	1	3.3
Frequently	15	50.0
Almost Always	14	46.7

Teachers frequently provide exercises to be solved by their students. Aside with theoretical explanations, teachers understand the importance of mathematical practice to better assimilate subject matter. Such practice should combine exercises and problem solving activity.

63. I introduce new problems but also use problems which the students already know. Median value 2 *(Table 105)*.

(Table 105)

Question 63	Frequency	Percent
Almost Never	1	3.3
Sometimes	17	56.7
Frequently	11	36.7
Almost Always	1	3.3

"Sometimes" is the median opinion that corroborates the teachers' answers to question 49.

64. I promote problem solving in the classroom to "relax". Median value 2 *(Table 106)*.

(Table 106)

Question 64	Frequency	Percent
Almost Never	10	33.3
Sometimes	16	53.3
Frequently	4	13.3

Teachers sporadically use problem solving with didactic purpose. Such use of problem solving is also important and can help students to avoid negative beliefs about mathematics.

65. I believe that the act of learning the contents is more important than the way they are worked by the students. Median value 2 *(Table 107)*.

(Table 107)

Question 65	Frequency	Percent
Almost Never	7	23.3
Sometimes	16	53.3
Frequently	5	16.7
Almost Always	2	6.7

"Sometimes" is the median opinion. This is a sensitive point in the teaching and learning process. If learning is undoubtedly crucial, the *modus operandi* to achieve such goal should not be neglected.

From the answers given to the questionnaire we came to several conclusions: *i)* teachers are a critical element in the teaching and learning process. Teachers are aware of the responsibility in the performance of their students but they are not the only key element since students also participate in the procedure, and there are also other variables such as teaching methodologies that affect the course of action. It is interesting to underline that teachers believe that an extra effort to teach mathematics produces outcomes more straightforward with lower performance students than with average students (Questions 1, 4, 7, 10, 12, 13 and 43); *ii)* teachers consider themselves good professionals with a solid theoretical mathematical knowledge but unveil some discomfort if they have to solve an unknown mathematical problem (Questions 3, 5, 6, 11, 15, 16, 17, 27, 28, 31, 40 and 42); *iii)* teachers have a proactive attitude looking for better ways to improve teaching and with that engaging students in mathematical activities (Questions 2, 19, 20, 21, 44, 47, 48, 56 and 62); *iv)* teachers enjoy mathematical subject matter, the practice of the profession but they unveil some discomfort when teaching students that have learning difficulties (Questions 22, 23, 24, 25, 26, 29, 30, 32, 33, 34, 35, 36, 37, 38, 39 and 41); *v)* teachers are aware of the importance of promoting problem solving activities in the classroom and providing an instructional model with the appliance of heuristic procedures but time constraints don't allow to implement such methodology with the desirable regularity (Question 45, 46, 49, 50, 51, 52, 53, 54, 55, 57, 58, 59, 60, 61, 63 and 64); *vi)* teachers are restlessness about exposing their practice to outsiders. Considering problem solving a more dynamic activity than theoretical subject matter presentation and the practice of exercise solving, such methodology recommends, eventually, two teachers in the classroom, but the evidence taken from the answer doesn't make easy the cooperative work of two teachers in the classroom (Question 18); *vii)* teachers majorly endorse the curriculum and the classroom practices advocated in the math program for Third Cycle and Secondary Education but in our sample of 30 professionals 7 consider that in general the teaching of mathematics is not adequate. Furthermore, with adequate teaching it is possible to help students to overcome the lack of prerequisites. Also, it is not clear for the teachers that the act of learning is more important than the way subject matter is worked by the students (Question 8, 9 and 65).

CONCLUSION

Heuristics in Problem Solving for the Teaching and Learning of Mathematics aspires to be an open research study on mathematical problem solving in an educational perspective supported by essential psychological processes to build significant learning.

Empirical research was carried out in two workshops projects promoted during the 2011/2012 and 2012/2013 school years at Nova Ágora, an Education Training Centre in Coimbra, with a high positive feedback from the teacher participants. The application of problem solving heuristics by those teachers in their classrooms was a goal which was only partially achieved due to the reduced number of lessons available to perform more problem solving activities. Teachers have an extended mathematics programme to cater until the end of the school year. Credit must be given to the teachers, who recognised the importance of problem solving and were receptive to its implementation in the classroom, as well as to the assessment of their students' performances. This effort provided the possibility to monitor teaching and, simultaneously, the dynamic evaluation of their students' procedures, thus allowing for the identification of deficient stages in problem solving and content learning.

Our quest was oriented towards four main questions. Firstly, why do so many students have a negative conception of Mathematics? Clues can be identified through the analysis of the data of the 96 9th grade students who answered the questionnaire *Attitudes towards Mathematics.* Even though our sample was limited, the data gathered corroborates Alan Schoenfeld's model regarding the mechanisms activated in the course of problem solving activity. Variables such as *Beliefs/Motivation*, *External Control*, *Mood*, *Resources*, *Heuristics*, *Knowledge Exhibition* and correlations between them were found. We also perceived a poorer attitude towards Mathematics from 9th grade students during one of the school years encompassed in this study. As Secondary Education entrance approaches, many students reveal more disbelief regarding Mathematics. In our study, we have also devoted attention to alternative educational curricular paths such as Vocational and Professional Courses, currently favoured by an increasing number of students. In many cases, such options are made due to lack of success in Mathematics during the early ages. Statistics show the Portuguese annual contest *Canguru Matemático sem Fronteiras*, Cadet category (9th grade students), reveals a decreasing trend of participation in recent school years. The history of results of the Maths Third Cycle national exam is also far from satisfactory. A better average in a particular school year would be difficult to justify as the result of structural educational measure implementation or due to a less demanding exam. Also, as the history of national exams is gathered, the teachers have access to a bigger archive of typified questions to practice with their students, and students to practice by themselves, which will eventually help achieve better results.

We were also interested in understanding the cognitive processes which are activated during problem solving tasks. In the Literature Review we revisited different models and theories to attain mathematical knowledge processes as devised by Norman (1980). Mathematical learning is about complex processes that imply structures to receive, filter, organise, modulate and retain data from the external environment. As mathematicians use a formal logic language which is closer to the scope of Artificial Intelligence, we should not forget the human factor. Cognition is vital to humans as software is vital to hardware. Schoenfeld (1992) explains how knowledge, heuristic strategies, belief systems and metacognition interact as four main vectors of individual problem solving activity. Moreover, problem solving strikes as a powerful instructional methodology which promotes a profitable interaction between teacher and student if the adequate learning environment is settled. We remind the stages identified by Schoenfeld (2007) during problem solving activity (read, analyse, explore, plan, implement, verify) and how different individual procedures are, as individuals master different levels of mathematical knowledge and mathematical training. Students need knowledge (declarative and procedural) (Anderson, 1981) but they also need to use it with personal benefit so as to further knowledge acquisition. The way knowledge is acquired, stored in long term memory or short term working memory, used and transformed is a challenge, as it is to better understand how such intricate mechanisms are put in motion. *Meno*, a character from Plato's dialogue, already lead us to understand that mathematical activity is perfect for this search.

Subsequently, we analyse how teachers can support the learning of Mathematics by promoting problem solving activity. In the course of the workshops we realised that the teachers were receptive to our proposals. We tried to present a wide range of problem solving activities which, on the one hand, and taking into account the scientific professional enrichment of each participant, we considered interesting, and, on the other hand, had concrete applicability in classrooms of the 7th to 12th grades, with a special incidence in the 9th grade. With such a wide goal, it was impossible for us to present a complete and organised problem solving curricular proposal for a particular school year. Our intention is that some of the proposed mathematical contents can be used with added value by readers from different academic backgrounds and in diversified areas of interests. Explanations, comments and suggestions are made with the purpose to enlighten ideas and sustain the readers' autonomous comprehension. We are aware that the systematic practice of problem solving contributes to improve students' performance but always under a theoretical framework, as Karpicke's (2009) study on reading and active recovery shows. To be effective, a problem solving programme must be extended in time. This statement is corroborated by our students' results during our flash problem solving intervention, which suggests that heuristic procedures did not improve performances. That was not surprising, given that solid mathematical learning only comes after a long period of systematic work. Shortcuts cannot be followed in mathematics learning. A good instructional programme based on the application of problem solving heuristics captures the interest of both teachers and their students. As an example, let us remember the significance of the value of *pi*: almost all 9th grade students know the approximate value of *pi*, but only a minority recognises its significance.

Time is a precious resource and, at school, teachers struggle to teach the full programme and therefore favour classroom exercise solving. That option is understandable, as the difference between concepts, problem solving and exercise solving, is divided by a very thin line. What we call problem solving activity, if repeated in time and concerning the same contents, becomes a routine procedure. Furthermore, a diversified problem solving activity provides students with an enlarged number of tools so that they can pick those which are most suitable to solve the next proposed problem. Even though, according to the teachers' testimonials, the menace of problem solving turning into exercise solving is ever present, the problem solving activity plays an important role in student motivation, allowing working curricular concepts in a more challenging environment. The lack of time to implement those activities in the classroom during the school year and the fragile background of students' theoretical knowledge are two of the main aspects which make this practical approach difficult to implement. With compulsory education extended to the 12th grade, the educational system needs to meet the aspirations of 21st century students, both for those who want to pursue tertiary education, and those who want to start a professional career right away. Those students who apparently do not share any of these interests, namely the ones contemplated in our first intervention, at *Centro Educativo dos Olivais*.

Heuristics in Problem Solving for the Teaching and Learning of Mathematics aims to research and stimulate students' perseverance, a decisive factor in the learning process. Problem solving is undoubtedly a thorny activity and many of the tasks that are presented in our *Manual* underline the need to be resilient. Students can have fun from doing Mathematics but problem solving is a serious game that requires thoroughness.

The *Manual* was written with the aim to develop the use of problem solving heuristics, focusing on the needs of teachers, as mediators to access contents and mathematical problem solving procedures for students who are starting Third Cycle Education. The subjects presented are to be worked by students and teachers, by teachers with students in the classroom, and also in workshops for teachers like *The use of Heuristics in Mathematics – Tools to problem solving*. Mathematical milestones such as Geometry and Algebra, but also several other branches of Mathematics, are vividly displayed and explained under the spirit of George Pólya's problem solving model. As we cannot evoke all the contents of this study here, we underline the lecture on the first pages of Pedro Nunez's *Livro de Algebra*. Algebra, Geometry and History of Mathematics were brought together in a single activity with a framework where knowledge input and resolution strategies focused on understanding and representation refresh the information process so as to attain conscious thinking organisation and identification, as well as skills which enable the choosing of specific strategies, which facilitate auto-regulated learning with impact on motivation and engagement of the solvers in learning environments.

In conclusion, within the potential of a mathematical formula, the problem solving activity has no use if we don't understand the variables, but it may be powerful if we know its significance and use such kinetics to a better practice.

Annex: A PROBLEM SOLVING MANUAL – context and proposal

In Third Cycle of Elementary Education contents are commonly distributed in five areas: *Numbers and Operations*, *Geometry and Measurement*, *Functions*, *Sequences and Series*, *Algebra* and *Statistical Treatment of Data*. The problem solving *Manual* was written with the purpose to cover such mathematical contents but it also goes through Secondary Education themes, from *Geometry* to *Algebra*, as well as *Probabilities* and *Statistics*. Contents from the syllabus of Mathematics Applied to Social Sciences such as *Graph Theory* problems are also encompassed in the *Manual*. The reader can also find some more advanced issues which require upper mathematical knowledge. Each topic presented in the *Manual* can be worked separately from the remaining issues but ultimately the reader will find connections between the different themes. Some problems may require a previous theoretical introduction. This instruction manual is not a close syllabus; the advised reader can and should search for additional information. Technology is more that ever present in the teaching and learning of mathematics, but the *Manual* goes back to basics and mainly requires the use of paper and pencil. We hope the reader be induced to look into additional literature.

PREFACE; PÓLYA'S PROBLEM SOLVING MODEL: Polya's problem solving techniques; Schematic models; Problem solving in the maths curriculum; Schoenfeld: Control, Resources, Heuristics, Beliefs; Conceptual Framework; *Meno*; Hibert, Gödel, Kant, Brouwer; Problem solving activity inspired in *Meno's* dialogue; THE UNIT AS A SYNERGISTIC CATALYTIC ELEMENT: Prisoner's dilemma; John Nash's equilibrium game; WHEN ADDING IS MORE THAN A BASIC OPERATION: Gauss's problem and arithmetic series; International congress dialogue problem; Magic squares; Wheat and chessboard problem; Four colours problem; Adding and multiplying with subtraction operations; Partial Latin square; A TOOL TO MEASURE DISTANCE AND VOLUME: THALES' THEOREM: Thales of Miletus's triangle; Abraham de Graaf; Thales' theorem; Truncated square pyramid volume; Thales' problem solving activities; NARRATIVES WITH A MATHEMATICAL PLOT: Kitchen tap problem; Twenty one students and their friends; Share 80cl of water; The herd of camels; BINARY WRITING AND THE SEXAGESIMAL SYSTEM: Division flaw; Zero; Googol; Binary operations; Roman numerals; Egyptian hieroglyphs; Rhind mathematical papyrus; Sexagesimal (base 60) notation; Sieve of Eratosthenes; Perfect numbers; FIBONACCI – EMBASSADOR OF THE HINDU-ARABIC SYSTEM IN EUROPE: Fibonacci's water fountain problem; Logic; Special cases of binomial multiplication; Fibonacci's water fountain problem (text interpretation); Fibonacci's rabbit problem; Lucas, Zeckendorf, and Cassini's theorems; Golden ratio; FROM THEORY TO PRACTICE: A CONTINUOUS EXERCISE: Ellery Harding Clark's high jump; Schoenfeld; Zeno's paradox, Achilles and the tortoise;

Solving rate-time-distance problems; Königsberg bridge; Graph theory; Travelling salesman problem; SMALL CHANGES WHICH CAUSE BIG ALTERATIONS: Cylinder packing problem; Ancient Egyptians' value for *pi*; Lorenz and the butterfly effect; Fahrenheit and Celsius's scales; Babylonian value for *pi*; Philippe Starck; Quadratic equations; Triangular numbers; Party guests puzzle; Exponential equation in quadratic form; Diophantus' age problem; Solving Algebra problems; Pedro Nunez's quadratic equations; Al Khwarizmi and his geometric solutions for quadratic equations; Cardano's formula; A problem about Functions, Geometry and Algebra; THE USE OF HEURISTIC PROCEDURES IN PROBLEM SOLVING: Palindromic number; Positive integers average problem; One circuit - Two buses; Five lines intersect in one point; A geometrical problem; Exterior angle to a regular n-gon; Divisors of a natural number; Distance problem; Bricks with different colours; Two-digit number problem; A problem with numbers; Algebraic Geometry; Equations; Darts game; Rational inequalities; Different-colour cars problem; Sixty-four tennis players; System of polynomial equations; Jogging couple problem; An equation; Rectangular fence optimisation problem; Maximising the volume of a rectangular prism volume; Numbers and operations problem; Dissecting triangles; Car number plates; Archimedes' principle; Irrationality of the square root of 2; ISO standard; Diophantus; MATHEMATICS APPLIED TO LIFE IN SOCIETY: Voting methods; Banzhaf's power index; D'Hondt system exercise; Throwing dice; Microbe growing conditions; Mendel's laws; Hazard (dice games); Monty Hall's problem; Throwing and adding the value of three dice; Neper; Euler's continuous compounding; Hugo Steinhaus's fair division; Lone divider's method; Sealed bids' method; Fair division problem; *EPPUR SI MUOVE!*: Size of the Earth; Days of the week; Distances to the Moon and the Sun; *Sidereus Nuncius*; Kepler's laws of planetary motion; TWO POINTS / ONE LINE / INFINITY OF POINTS: Buckminsterfullerene; Minimum distance problem; Pompeiu's theorem; Fermat-Torricelli's point; Minimal network; PITHAGORAS' THEOREM: The Chinese Proof of the Pythagorean theorem; Geometrical problems; Pentagram and the Golden Ratio; Hexagonal honeycomb pattern; Vertices, edges and faces; Flatland and Alice's Adventures in Wonderland; Long-term memory; Lune (mathematics); PROBLEM SOLVING / ORGANISING AND FITTING THE PIECES TOGETHER: Lateral thinking and Logic Puzzles.

An effective evaluation of this *Manual* cannot be carried out with a sporadic intervention. Despite the positive feedback given by the teachers who participated in the workshops, they only practised a limited set of problem solving questions in the classroom due time constraints. Curricular programmes are extensive considering the amount of hours allocated to Maths, so there is no time to develop this kind of practice in the classroom. Allocating one week of school time (45 / 50 minutes lessons) to implement problem solving activities in the course of a school year (35 weeks), is the best way to evaluate the efficiency of a problem solving syllabus. Unfortunately this kind of study could not be implemented.

Preface

Pólya's problem solving model

The unit as a synergistic catalytic element

When adding is more than a basic operation

A tool to measure distance and volume: Thales' Theorem

Narratives with a mathematical plot

Binary writing and the sexagesimal system

Fibonacci – Ambassador of the Hindu-Arabic system in Europe

From theory to practice: a continuous exercise

Small changes which cause big alterations

The use of heuristic procedures in problem solving

Mathematics applied to life in society

Eppur si muove!

Two points / One line / Infinity of points

Pithagoras' Theorem

Problem solving / Organising and fitting the pieces together

Preface

Learning is an intrinsic capacity of humans which is expected to be for life. A generic concept embodied in individual or group activity, learning has a myriad of variables which determine its success. Hence, the learning process is intrinsically connected to the teaching practice. In Portugal, in the last decades, several changes in contents and methodological practices were introduced by the Ministry of Education and Science with the aim to improve the curriculum and, ultimately, enhance the quality of Education. Outcomes assessed internally at the end of every school year, as well as national exams and international test evaluations, although quantitative, allow for varying interpretations.

This instruction manual aims to contribute for the dissemination of problem solving practice through the use of heuristic procedures. The Maths Curriculum endorses problem solving activity, but still classroom implementation has been hindered by the difficulties felt by the teachers to fully teach the complete programme during the school year. It is reasonable to claim that a systematic practice of problem solving may be a catalyst for academic choice, as Mathematics is a core subject for further studies and adds professional value, thus allowing the student to aspire to full personal, intellectual and professional fulfilment. It is intended for math teachers who are already aware of the importance of problem solving, as well as for all participants in the teaching/learning process, be they educators or students.

This manual encompasses different study areas, difficulty levels and education levels. And even though chapters are internally consistent so as to be independent work units, a complete reading is recommended. Many of the questions present in this manual were didactically organised to illustrate possible applications in the classroom. The proposed problems, presented in a text box, are an invitation to the implementation of heuristic procedures, a varied and yet incomplete pathway. A mathematical text is not a simple collection of neatly distributed characters, its composition comprises arguments which, when correctly assimilated by the reader, become powerful agents of reasoning.

Pólya's problem solving model

Mathematics stimulates learning and knowledge. When studying Mathematics, individuals question themselves with the aim of answering theoretical, practical and multidisciplinary problems in a dynamic and frequently arduous process. George Pólya (1887 - 1985) wrote several books about problem solving methods; *How to Solve It*, *Mathematics and Plausible Reasoning (Volume I: Induction and Analogy in Mathematics*, and *Volume II: Patterns of Plausible Inference*), and *Mathematical Discovery: On Understanding, Learning, and Teaching Problem Solving* (volumes 1 and 2). In *How to Solve It* (1945), he systematically presents four phases which may contribute to the success of this activity.

The first phase would be the reading and understanding of the problem. The identification of the unknown, relevant data and condition is most important, for nobody can answer a question before it has been understood. If the problem includes a figure, it must be duly interpreted, relating it with what is intended in the problem statement.

When conditions to proceed are created, Pólya suggests devising a plan, *i.e.*, doing the inventory of the data and relating them to the intended answer. If the desired relation is not established, recalling a similar problem is suggested, in order to assess the possibility of replicating the solving method. Eventually, an auxiliary element must be introduced so as to make its application possible. This methodology aims to find an idea which may be implemented. All questions are valid, but the Hungarian mathematician stated that one must be aware of the difficulty of having a good idea when little is known about the subject, and that that a good idea is quite improbable when nothing is known about the subject.

The third phase is carrying out the plan, checking each argument so that everything is perfectly identified and flawless.

Finally, a review of the already accomplished work and the achieved solution must be carried out in order to consolidate knowledge and perfect skills. Failure to interpret, calculate and choose the appropriate tools is a threat which should not be ignored. Promoting problem solving through one or more alternative processes contributes to a better and quicker handling of mathematical tools, thus providing more confidence in their use.

Each of the four phases is essential. A brilliant idea which provides a swift solution may sometimes occur, but the process of bypassing the different phases should not be deliberate. Calculations and schemes are useless if the student does not fully understand the problem. Only after the general structure is assimilated and a plan is devised should the first procedures be implemented. George Pólya's book is not meant to be a prescription of how to obtain miraculously solutions for problems, but it provides excellent guidance. Progress in Mathematics, as in other areas of knowledge, comes from persistence, serendipity and methodical work, with the help of heuristics.

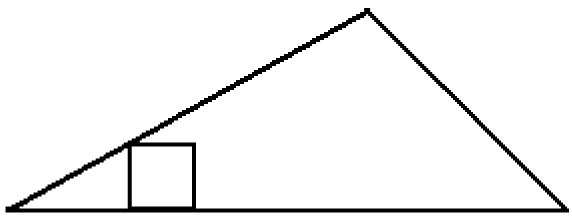
Regarding modern heuristics, Pólya refers that

> *It seeks to understand the problem solving process, particularly useful mental operations, which are typical of this process. It has various sources of information, none of which should be overlooked. A conscientious study of Heuristics must consider both its logical and psychological background. Experience in problem solving and experience in the observation of this activity by others should make the foundation of Heuristics. In this study, no problems should be neglected. We should, however, look for common aspects in dealing with all kinds of problems: general aspects should be considered, regardless of the specific subject of the problem. The study of Heuristics has "practical" objectives: a better knowledge of the typical mental operations applied to problem solving may have beneficial influence on the teaching process, particularly the teaching of Mathematics* (Pólya, 1995, p. 87).

Pólya mentions the following problem: Find a procedure to inscribe a square in a given triangle. Two vertices of the square should be on the base of the triangle and the other two vertices on the other two sides of the triangle, one on each.

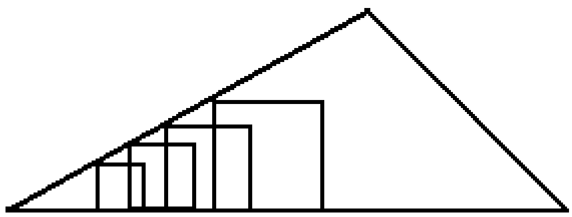
What is the goal? Inscribe a square in a given triangle. Two vertices of the square should be on the base of the triangle and the other two vertices on the other two sides of the triangle, one on each.

How should the problem be visualised? Inscribe a square in a given triangle.

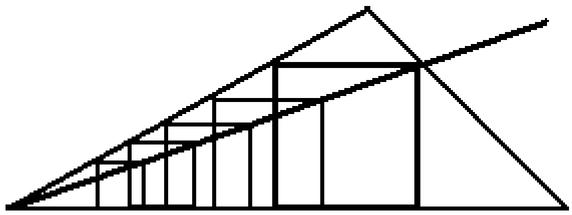


Does the square in this figure satisfy the conditions of the problem? No. Two vertices of the square are on the base of the triangle, but in what the other two vertices are concerned, only one satisfies the statement of the problem.

Is it possible to draw other squares in which three of the vertices are on the perimeter of the triangle? Yes.

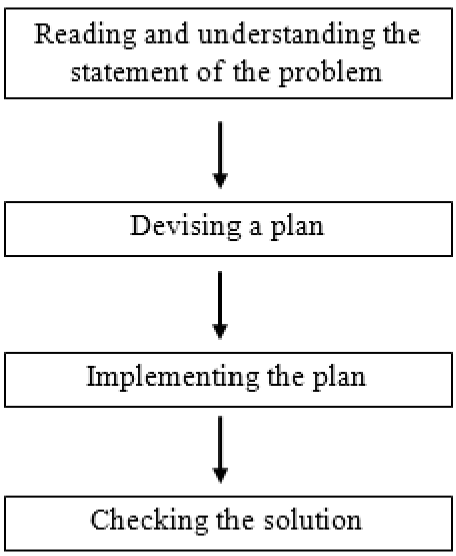


Which seems to be the geometrical locus of the fourth vertex? Intuitively we can speculate and answer as follows: the point of intersection of the straight line which contains the vertices of the previously drawn squares and the side of the triangle which is missing.



Check each phase of the problem.

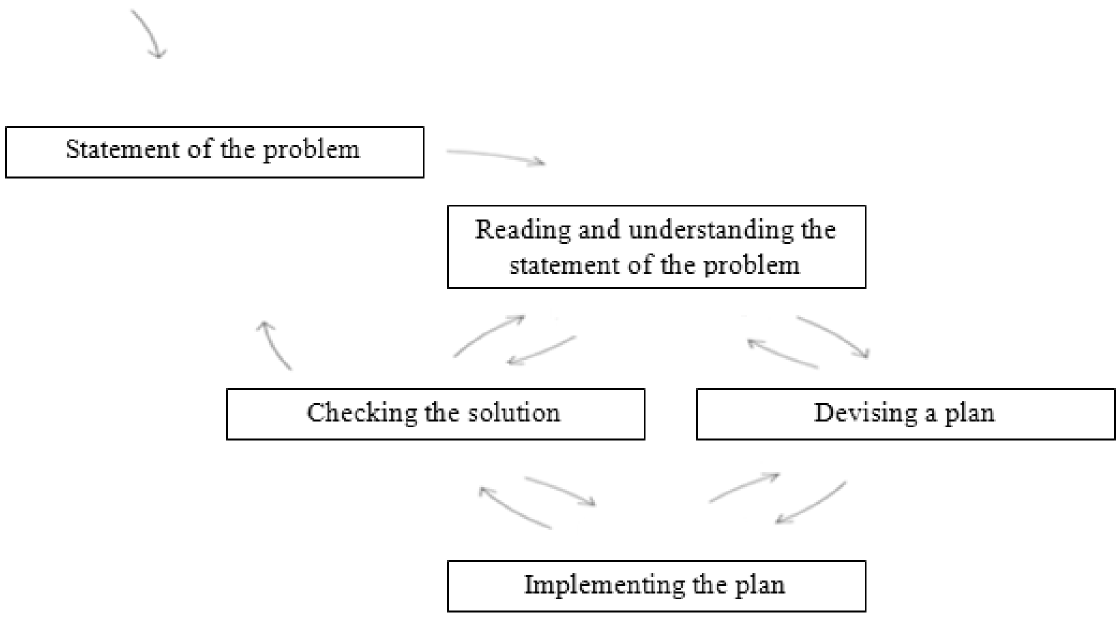
The *Ishango Bone*, discovered in the Belgian Congo, currently the Democratic Republic of the Congo, which dates to the Upper Palaeolithic period, is the most ancient evidence of the human need to count, interpret and explain phenomena. Questioning and searching for answers using simple or complex methods, individually or collectively, carried out in a short period of time or in years of work, is human nature. The importance of Cognitive Psychology was driven by the researchers' interest in problem solving, by the intrinsic potential for the understanding of thought and learning mechanisms.



(Figure 21)

As George Pólya had already stated, problem solving allows for the exploration of innumerable subjects through a multitude of heuristic procedures, and with unquestionable benefits for the teaching and learning processes. The model includes reading and understanding the statement of the problem, devising a plan, implementing and checking the solution *(Figure 21)*. The purpose of teaching the student to think about how to solve mathematical problems should not be distorted by a design which points to a teacher who teaches what to think or what to do. The scheme should not suggest that solving a problem is a linear process composed of a sequence of steps, thus implying that problem solving is carried out through memory, practice and routine.

This linear method is not consistent with mathematical activities associated with problem solving. Nonetheless, it may represent the way an experienced subject expresses the reasoning followed to reach a solution after it has been achieved. Similarly, the work developed by any mathematician who attempts to solve a problem which has as yet not been solved should be highlighted. This is a bumpy process, with breakthroughs and setbacks, which may be illustrated by *Fermat's Last Theorem*. In general, the construction of a convincing argument or even a proof is a complex and difficult process. However, the result comes out as simple and elegant, with no hints of the dynamic processes which occurred during the path to the discovery of the solution. Hence, the best suitable theoretical model should include these dynamics and complexity, *i.e.*, when students face a problem, a continuous reasoning is triggered in order to understand what is required. After one or more failed attempts to devise a plan, students may feel the need to go back and revise their interpretation of the statement of the question. After building a strategy to solve the problem, its implementation may reveal itself unfeasible or lead to a dead end. Eventually, the individual may opt for the drawing up of another plan or to go back to the starting point to revise the initial interpretation *(Figure 22)*. The possibility of analysing a similar problem should be considered, thus showing perseverance in the solving process.



(Figure 22)

The teacher is asked to promote knowledge, to question and guide the students, but without enforcing problem solving models. Students should be given time to implement their own strategies. Thus, the teacher should play the role of supervisor or coach. More important than solving many exercises by routinely applying a given algorithm is the promotion of creative learning. Significant learning and memorisation are connected by a *continuum* path inside the cognitive structure of the student.

In Portugal, during the last decades, the education system suffered major evolution. These changes were both influenced by ephemeral procedures taken from the different reference models provided by the theories of Education Psychology, and the consequence of political, economic, cultural and social transformation.

The extension of compulsory education, the development of infrastructures to host an increasing number of students, the introduction of new technologies, as well as teacher training provided by the Universities, among other measures, inducted expectations in the quality of Education and the development and consolidation of students' skills. However, internal assessments, quantifiable through tools such as assessment tests and national exams in Secondary Education, reveal disappointing outcomes. The curriculum for Basic Education stresses the relevance of problem solving activity. Also, the item regarding methodological indications based on the transversal skills to be developed states that such skills are

> *...crucial for mathematical knowledge construction, consolidation and mobilisation within the different themes, in connection with reasoning and communication. To have the capacity to solve mathematical problems means being able to successfully perform activities such as understanding the problem, with the identification of the unknown and conditions; selecting and applying the best strategies and resources, with the exploration of mathematical relationships to overcome difficulties; and checking solutions and revising processes*[41] (Ponte, et al., 2007, p. 62).

Mathematics requires imagination and accuracy in a harmonious combination. However, for many students, formality is the characteristic which stands out when discussing this school subject. Undeniably, Mathematics' symbolic language requires time to be appropriately interpreted and used. Therefore, the apparent complexity of this instrument is likely to create barriers to communication for the less fluent. Based on a universal language, thus confirming its efficiency and simplicity, the mathematical dialogue excludes those who are not capable of decoding the message.

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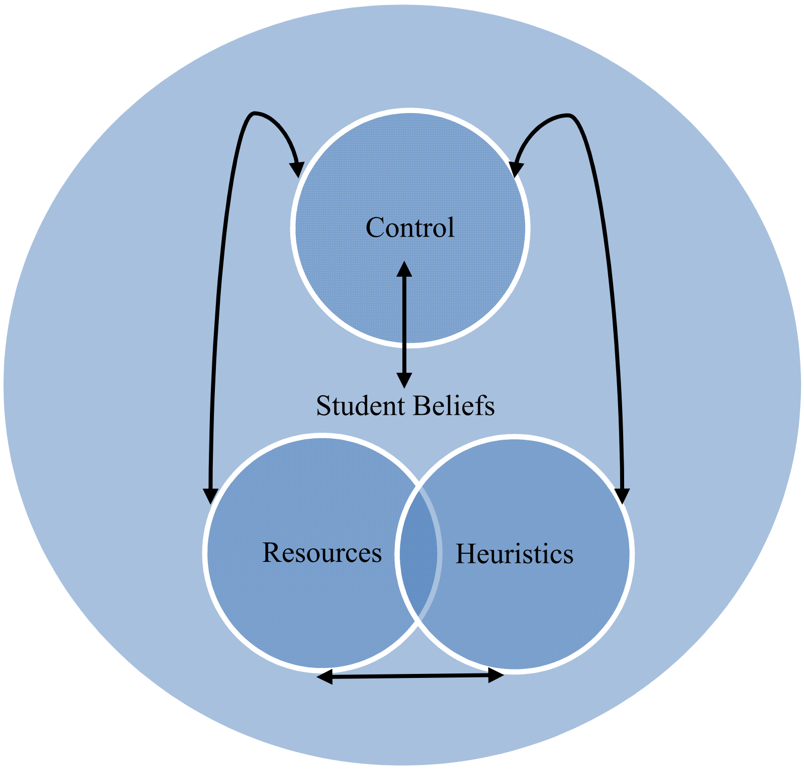
[41] Author's translation: *Resolver problemas é fundamental para a construção, consolidação e mobilização de conhecimentos matemáticos dos diversos temas, em conexão com o raciocínio e a comunicação. Possuir a capacidade de resolver problemas matemáticos significa ser capaz de realizar com sucesso atividades como compreender o problema, identificando a incógnita e as condições; selecionar as estratégias e os recursos apropriados e aplicá-los, explorando conexões matemáticas para superar dificuldades; e verificar soluções e rever processos.*

Hence, entering Secondary Education becomes a milestone in students' life. It is a transition which determines the students' academic path and professional future. A choice which, when taken, automatically closes a plethora of options, which are put aside due to fears of failure, particularly when Mathematics is involved. The promotion of successful learning depends on the quality of the curriculum created by the Portuguese Ministry of Education and science, on its effective implementation by the teachers and on how it is apprehended by the students. Nevertheless, other equally important variables may be involved, thus influencing the attitude, commitment and performance of the students in and out of the classroom, such as family participation in the school life of their children, the number of students per class or the heterogeneity of students per class.

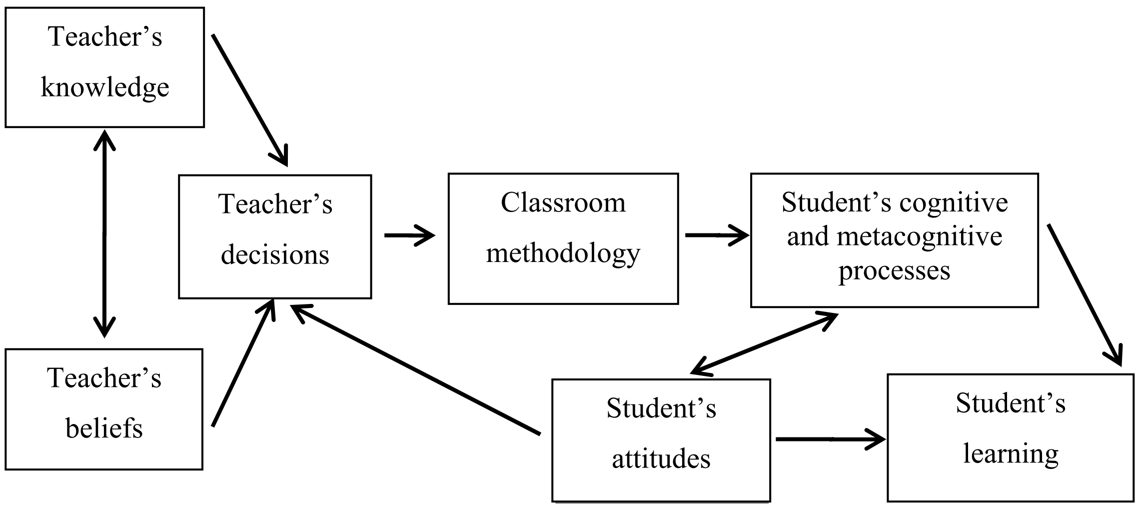
Schoenfeld (1985) classified the mechanisms activated during the problem solving process in four categories of behaviour and knowledge: resources, heuristics, control and beliefs of the subject *(Figure 23)*.



(Figure 23)

Memory resources which may be applied in problem solving are activated based on the knowledge held by the subject. Strategy mobilisation with the aim to overcome the difficulties involved in the complexity of the question favours the implementation of heuristics; casual procedures that lack formal accuracy can, eventually, be useful when skilfully applied. The outcome of the task is influenced by the way resources and heuristics are managed throughout the different steps of problem solving. Control also affects the conceptual systems of the student as it interferes in the cognitive processes involved in problem solving. The relationship which the students establish with Mathematics, visible in the way they make questions, in the level of expertise of the techniques used and in the commitment and time put in solving the proposed task, is the result of conceptual systems, *i.e.*, "*the context where resources, heuristics and control interconnect*"[42] (Borralho, 1993, p. 37). The variables involved (teacher, student, context, tasks, emotions...) and the difficulties that the cognitive sciences face in order to decode the phenomena which occur during problem solving prevent the generalisation of the results obtained from research, whether they are qualitative or quantitative, for they are limited to the sample and the period of time during which the study is carried out.

The convergence of the willingness to solve, to commit to solving and to think the solution in a classroom context is connected with multiple variables. Carpenter (1990) proposes a model which gives a central role to the thought developed both by students and teachers *(Figure 24)*. The methodology implemented in the classroom is based on the decisions of the teacher, and the impact of instruction on the students' attitudes and learning are measured by their cognition. Teachers' decisions are built upon scientific knowledge and beliefs, as well on the perception of the learning of the students through the direct observation of their attitudes.



(Figure 24)

Solving a problem is analogous to organising a jigsaw puzzle. The number of pieces is correlated with the degree of difficulty but it is not the only factor that comes into play. Each piece's colour and shape, experience in carrying out this kind of task, and the motivation and resilience of the subject all together have a decisive contribution to the progress and outcome of the activity. It may happen that the placement of a piece is the result of mere chance or the result of trial and error which requires the spending of an enormous amount of time and may

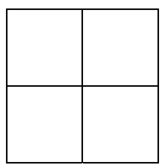
[42] Author's translation: *"contexto no qual os recursos, as heurísticas e o controlo se relacionam".*

compromise the completion of the puzzle. Conversely, the subject may apply a work methodology which allows optimising performance. This requires the domain of as much theoretical knowledge and as many heuristic procedures as possible.

The study of heuristics goes back to Antiquity. Through *Meno*, Plato guides us to a dialogue of Socrates with his disciple. When questioned about the meaning of virtue and whether it can be taught, the philosopher rejects that possibility, advocating that knowledge comes from recollections emerging from the soul. Socrates requires the presence of one of Meno's slaves so as to prove that the essence of Mathematics arises from the action of the soul, by encouraging him to build a square which doubles the area of a previously drawn square. The questions make the individual face his scientific prejudice, thus allowing for the distinction between right and wrong, which are recollections of what had never been taught to him. Both questions and answers emerge from a casual dialogue between teacher and learner, and the questions that any learner will ask themselves during the problem solving process may follow a thread similar to that of Plato’s.

We transcribe the full dialogue which the reader may wish to skip.

Socrates: Tell me, boy, do you know that a figure like this is a square? And you know that a square figure has these four lines equal?



Boy: I do.

Socrates: A square may be of any size?

Boy: Certainly.

Socrates: And if one side of the figure be of two feet, and the other side be of two feet, how much will the whole be? Let me explain: if in one direction the space was of two feet, and in other direction of one foot, the whole would be of two feet taken once?

Boy: Yes.

Socrates: But since this side is also of two feet, there are twice two feet?

Boy: There are.

Socrates: And how many are twice two feet? Count and tell me.

Boy: Four, Socrates.

Socrates: And might there not be another square twice as large as this, and having like this the lines equal? And of how many feet will that be?

Boy: Yes. Of eight feet.

Socrates: And now try and tell me the length of the line which forms the side of that double square: this is two feet- what will that be?

Boy: Clearly, Socrates. It will be double.

Socrates: Do you observe, Meno, that I am not teaching the boy anything, but only asking him questions; and now he fancies that he knows how long a line is necessary in order to produce a figure of eight square feet; does he not?

Meno: Yes.

Socrates: And does he really know?

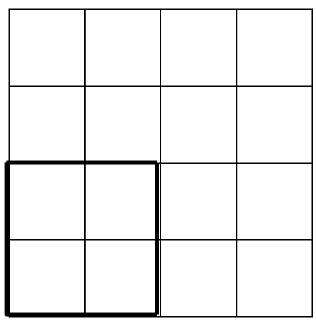
Meno: Certainly not.

Socrates: Observe him while he recalls the steps in regular order.

(To the boy): Tell me, boy, do you assert that a double space comes from a double line? Remember that I am not speaking of an oblong, but of a figure equal every way, and twice the size of this-that is to say of eight feet; and I want to know whether you still say that a double square comes from double line?

Boy: Yes.

Socrates: But does not this line become doubled if we add another such line here? And four such lines will make a space containing eight feet? And are there not these four divisions in the figure, each of which is equal to the figure of four feet? And is not that four times four?



Boy: Yes.

Socrates: And four times is not double?

Boy: No, indeed.

Socrates: But how much?

Boy: Four times as much.

Socrates: Therefore the double line, boy, has given a space, not twice, but four times as much. Four times four are sixteen, are they not? And the space of four feet is made from this half line? And is not a space of eight feet twice the size of this, and half the size of the other?

Boy: Certainly.

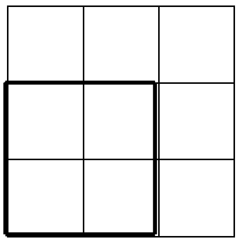
Socrates: Such a space, then, will be made out of a line greater than this one, and less than that one? I like to hear you say what you think. And now tell me, is not this a line of two feet and that of four? Then the line which forms the space of eight feet ought to be more than this line of two feet, and less than the other of four feet?

Boy: It ought.

Socrates: Try and see if you can tell me how much it will be.

Boy: Three feet.

Socrates: Then if we add a half to this line of two, that will be the line of three. Here are two and there is one; and on the other side, here are two also and there is one: and that makes the figure of which you speak?



But if there are three feet this way and three feet that way, the whole space will be three times three feet?

Boy: That is evident.

Socrates: And how much are three times three feet?

Boy: Nine.

Socrates: And how much is the double of four?

Boy: Eight.

Socrates: Then the figure of eight is not made out of three?

Boy: No.

Socrates: But from what line?-tell me exactly; and if you would rather not reckon, try and show me the line.

Boy: Indeed, Socrates, I do not know.

Socrates (to Memo): Do you see, Meno, what advances he has made in his power of recollection? He did not know at first, and he does not know now, what is the side of a figure of eight feet: but then he thought that he knew, and answered confidently as if he knew, and had no difficulty; now he has a difficulty, and neither knows nor fancies that he knows.

Meno: True.

Socrates: Is he not better off in knowing his ignorance?

Meno: I think that he is.

Socrates: We have certainly, as would seem, assisted him in some degree to the discovery of the truth; and now he will wish to remedy his ignorance, but then he would have been ready to tell all the world again and again that the double space should have a double side. But do you suppose that he would ever have inquired into or learned what he fancied that he knew, though he was really ignorant of it, until he had fallen into perplexity under the idea that he did not know, and had desired to know?

Meno: I think not, Socrates. He would not have inquired.

Socrates: However such torpor was good for him?

Meno: I think so.

Socrates: Mark now the farther development. I shall only ask him, and not teach him, and he shall share the enquiry with me: and you watch and see if you find me telling or explaining anything to him, instead of eliciting his opinion.
(To the boy): Tell me, boy, is not this a square of four feet which I have drawn? And now I add another square equal to the former one? And a third, which is equal to either of them? Suppose that we fill up the vacant corner? Here, then, there are four equal spaces?

Boy: Yes.

Socrates: And how many times larger is this space than this other?

Boy: Four times.

Socrates: But it ought to have been twice only, as you will remember.

Boy: True.

Socrates. And does not this line, reaching from corner to corner, bisect each of these spaces?

Boy: Yes.

Socrates: And are there not here four equal lines which contain this space?

Boy: There are.

Socrates: Look and see how much this space is.

Boy: I do not understand.

Socrates: Has not each interior line cut off half of the four spaces?

Boy: Yes.

Socrates: And how many spaces are there in this section?

Boy: Four.

Socrates: And how many in this?

Boy: Two.

Socrates: And four is how many times two?

Boy: Twice.

Socrates: And this space is of how many feet?

Boy: Of eight feet.

Socrates: And from what line do you get this figure?

Boy: From this.

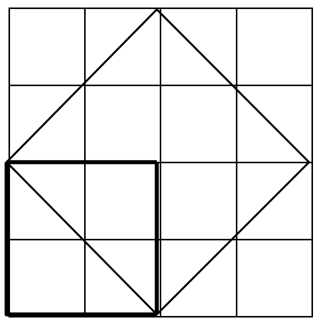
Socrates: That is, from the line which extends from corner to corner of the figure of four feet?

Boy: Yes.

Socrates: And that is the line which the learned call the diagonal. And if this is the proper name, then you, Meno's slave, are prepared to affirm that the double space is the square of the diagonal?

Boy: Certainly, Socrates.

Socrates: What do you say of him, Meno? Were not all these answers given out of his own head?



Meno: Yes, they were all his own.

Socrates: And yet, as we were just now saying, he did not know?

Meno: True.

Socrates: But still he had in him those notions of his, had he not?

Meno: Yes.

Socrates: Then he who does not know may still have true notions of that which he does not know?

Meno: He has.

Socrates: And at present these notions have just been stirred up in him, as in a dream; but if he were frequently asked the same questions, in different forms, he would know as well as anyone at last?

Meno: I dare say.

Socrates: Without any one teaching him he will recover his knowledge for himself, if he is only asked questions?

Meno: Yes.

Socrates: And this spontaneous recovery of knowledge in him is recollection?

Meno: True.

Socrates: And this knowledge which he now has must he not either have acquired or always possessed?

Meno: Yes.

Socrates: But if he always possessed this knowledge he would always have known; or if he has acquired the knowledge he could not have acquired it in this life, unless he has been taught geometry; for he may be made to do the same with all geometry and every other branch of knowledge. Now, has someone ever taught him all this? You must know about him, if, as you say, he was born and bred in your house.

Meno: And I am certain that no one ever did teach him.

Socrates: And yet he has the knowledge?

Meno: The fact, Socrates, is undeniable.

In the course of the 19th century, the conceptualisation of Mathematics was influenced by the increasing abstraction of the concepts, which are closely linked to a model of perfection, as the revisiting of *Platonism*.

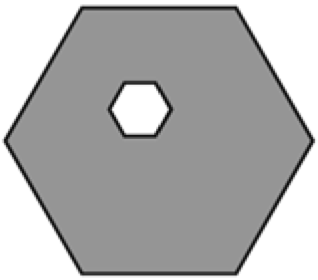
David Hilbert (1862 – 1943) fostered *Formalism* with the intent to reduce Mathematics to the application of mechanical procedures. Nonetheless, this intent was contested, among others, by Kurt Gödel (1906 – 1978), the author of the *Incompleteness Theorem*[43].

[43] The whole axiomatic system on the arithmetic of whole numbers includes not decidable propositions, *i.e.*, propositions which can be neither proved nor disproved by *mechanical* methods.

Immanuel Kant (1724 – 1804), who describes Science as a result of the interaction between rationalism (universal and immutable) and empiricism (experimental and uncertain), allowed for the expansion of *Intuitionism*, whose principles were established by Luitzen Brouwer (1881 – 1966). According to this doctrine, the individual is an active participant in the creative process. In essence, the practice of mathematics enhances the competence to distinguish between truth of proof and doubt of error.

(Subject: Geometry - 8th grade students / 9th grade students)

Figure shows two regular hexagons.



We know that:

- The length of the side of the exterior hexagon is five times bigger than the length of the side of the interior hexagon;
- The area of the interior hexagon is $23cm^2$.

Compute the area, in square centimetres (cm^2), of the shaded part of the exterior hexagon. Justify your answer.

From *Meno* we recall that if we double the side of a given square, the area becomes four times larger. If we consider *s* the length of the side of the initial square, the area becomes $s \times s = s^2$. A double side square, 2*s*, encompasses the area $2s \times 2s = 4s^2$, *i.e.*, four times bigger. This insight can be extended to other regular figures. The student can imagine the interior hexagon as a pavement of six triangles, each with the area $\frac{\sqrt{3}}{4}l^2$ (by applying the Pythagoras' Theorem), a total of $\frac{6\sqrt{3}}{4}l^2$. With an identical procedure the area of the exterior hexagon is computed, $\frac{150\sqrt{3}}{4}l^2$ thus proving that it is 25 times bigger than the previous hexagon.

Now, the student calculates the area of the exterior hexagon, $25 \times 23cm^2 = 575cm^2$, and after subtracting the area of the interior hexagon, $23cm^2$, obtains the area of the shaded part of the exterior hexagon, $552cm^2$.

The unit as a synergistic catalytic element

Problem solving presents the opportunity to foster individual skills and promote student interaction in small groups. Educational Psychology pays particular attention to this area of research, which is directly linked to Mathematics.

For example, two individuals who are suspect of having committed a crime are detained for questioning. The police officer questions them simultaneously but in separate interrogation rooms. If they do not confess the crime, the evidence collected will lead to a one-year conviction of actual jail time. The inspector who is in charge of the investigation offers freedom to whomever assumes his participation in the crime, but only if a single confession is obtained. The individuals, who are being kept in separate interrogation rooms and cannot communicate with one another, face a dilemma. What to do? Not to confess the crime in the expectation that the partner does the same? The evidence will convict each suspect to a one-year incarceration, but in the case of denouncement they risk being convicted to a six-year sentence in prison while their partner is set free because he collaborated with the police. If the two individuals confess the crime, the police will have proof against both, and even though there are mitigating circumstances due to the confession, each suspect will be sentenced to a three-year imprisonment *(Table 108)*.

The prisoner's dilemma, a model of cooperation and conflict, proposed by Albert Tucker (1905 – 1995) in the 1950s, shows how complex it is to predict human behaviour when facing a critical situation.

(Table 108) – Prisoner's dilemma

	SUSPECT B **CONFESSES THE CRIME**	**SUSPECT B** **DOES NOT CONFESS THE CRIME**
SUSPECT A **CONFESSES THE CRIME**	SUSPECT A: 3 SUSPECT B: 3	SUSPECT A: 0 SUSPECT B: 6
SUSPECT A **DOES NOT CONFESS THE CRIME**	SUSPECT A: 6 SUSPECT B: 0	SUSPECT A: 1 SUSPECT B: 1

> *He truly believed "less is more", that in simplicity lies elegance. I remember my father once saying that his favorite form of communication was the telegram because it forces clear and full expression using few words. (…) He showed an unflagging commitment to the importance of teaching and learning* (pp. 1143-1147).

John Nash (1928 – 2015) was the pioneer of the Equilibrium Theory[44]. Such concepts, which may be explored in the classroom, in a problem solving context, highlight the importance of cooperation to achieve the best solution. Four teams are given two cards each, one with the letter X and the other with the letter Y, and a table with the possible combinations of both letters and the scores according to the letter shown *(Table 109)*. For instance, if in the first turn all teams choose to show the card with the X, each group gets ten negative points. Objectively, the goal is to *obtain as many points as possible* after ten turns, *i.e.*, ten times playing the game.

(Table 109) – Score permutation

X = -10		Y = -20		X = 10		Y = -10		Y = 10
X = -10		Y = -20		X = 10		Y = -10		Y = 10
X = -10		X = 20		X = 10		Y = -10		Y = 10
X = -10		X = 20		Y = -30		X = 30		Y = 10

The players of each team can talk with each other for a period of time previously established by the teacher. After that time, the representatives of the teams simultaneously present their letter. If required, before the third turn communication between all the players is again allowed for a pre-established period of time so that a strategy may be set out. The scores obtained in that turn are multiplied by three. Equal opportunities are provided before the fifth turn, when scores are multiplied by five, and the tenth turn, when scores are multiplied by ten *(Table 110)*.

(Table 110) – Evaluation board

TEAMS / GAME	1	2	3	4	5	6	7	8	9	10	FINAL SCORE
SCORE	× 1	× 1	× 3	× 1	× 5	× 1	× 1	× 1	× 1	× 10	
A											
SUB-TOTAL											
B											
SUB-TOTAL											
C											
SUB-TOTAL											
D											
SUB-TOTAL											

[44] In situations where each key player is not aware of the behaviour of other players, there is always equilibrium. *"Von Neumann and Morgenstern have developed a very fruitful theory of two-person zero-sum games in their book Theory of Games and Economic Behavior. This book also contains a theory of n-person games of a type which we would call cooperative. This theory is based on an analysis of the interrelationships of the various coalitions which can be formed by the players of the game.*
Our theory, in contradiction, is based on the absence of coalitions in that it is assumed that each participant acts independently, without collaboration or communication with any of the others.
The notion of an equilibrium point is the basic ingredient in our theory. This notion yields a generalization of the concept of the solution of a two-person zero-sum game. It turns out that the set of equilibrium points of a two-person zero-sum game is simply the set of all pairs of opposing "good strategies" (Nash, 1950, p. 5).

The goal is to achieve the highest possible score at the end of the ten turns. To do it, all the teams ought to always show the letter Y, thus allowing each team to add 10 points, in a total of 40 points, or more in the case of turns three, five and ten. It is, however, quite possible that one team, or eventually more, decide to show the letter X to gain on two fronts: add more points and penalise their opponents.

Hence, in the following turns, the teams who had previously opted for the letter Y stop doing so. As all teams show the letter X, scores quickly drop. When communication is possible, some or all team members, who are aware of what is happening, ask for the letter Y to be once again shown. But the suspicion which has progressively been installed compromises collective confidence. At this moment, the probability that all teams show the letter Y is greatly reduced.

The team who achieves the best score cannot truly be declared the winner since winning is only possible if all teams reach 250 points: the highest possible score at the end of the ten turns!

Individual or group strategy actually influences the final score. In what concerns mathematical problem solving, the selection of effective procedures may represent the difference between failure and success.

When adding is more than a basic operation

In the classroom: almost complete silence. Some students, confident in their calculation skills, arduously write the first hundred natural numbers in order to add them. Other students remain absorbed, perhaps engrossed in mental strategy or, as J. G. Buttner, the teacher, suspected, indifferent to the challenge.

Multiple factors distinguish those who show clarity of reasoning from those who struggle in the dim or get lost in the dark. Shortly after the question was raised, Carl Friedrich Gauss (1777 - 1855), a ten-year-old student, surprises the teacher by answering it. The process used to reach the solution combines simplicity with efficiency. If,

$$1 + 100 = 101,\ 2 + 99 = 101,\ 3 + 98 = 101,\ 4 + 97 = 101 ...\ 49 + 52 = 101,\ 50 + 51 = 101$$

so, the 50 pairs of numbers which were established, $50 \times 101 = 5050$. Gauss unconsciously applies the property of symmetry of *arithmetic progressions*[45]. In an arithmetic progression, the

[45] A sequence (u_n) is an arithmetic progression where a real number r is $u_{n+1} - u_n = r$, for any n belonging to the set of natural numbers. The difference between each term of the sequence and its previous term is constant.

The ratio of the arithmetic progression is r.

The general term of an arithmetic progression is $u_n = u_1 + (n-1) \times r$.

The sum of n first terms is $S_n = \frac{u_1 + u_n}{2} \times n$.

sum of terms equally distant from the extremes is equal to the sum of the extremes. The mastery of a wide set of rules allows for better skills in problem solving.

Mathematics comprises the art of calculation, but the problem solver should avoid redundant calculation. The word originates from the Latin *calculus* and means counting with pebbles. The invention of the concept of number aimed at counting objects. To keep track of the herd, the shepherd used pebbles. Every morning, for each animal which left the barn, a pebble was introduced into a bag. In the evening, when bringing back the herd, the shepherd did the opposite, withdrawing a pebble from the bag for each animal which entered the barn. Having concluded the procedure, the presence of one or more pebbles inside the bag meant the loss of livestock.

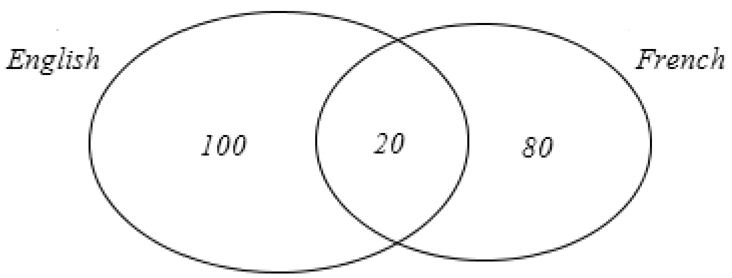
All basic operations can be applied in a variety of contexts. Problems can be classified according to several criteria: *1)* prerequisites necessary for solving the problem; *2)* goal of the problem; *3)* complexity of the problem; *4)* quantity and quality of the information provided by the statement; *5)* similarity with previously solved problems; *6)* workload quantified as the number of steps leading to the solution (Robertson, 2001).

Different questions often share a common solving matrix.

(Subject: Numbers and Operations – Third Cycle students)

Two hundred people attend an international conference of Mathematics. One hundred and twenty participants speak English and one hundred can make themselves understood in French. How many dialogues can be established without the presence of an interpreter?

According to the conditions of the problem, the aim is to determine the number of possible dialogues between two people. However, the statement suggests an apparent contradiction. How is it possible that one hundred and twenty participants can communicate in English and one hundred in French if only two hundred people participate in the conference? The difference between the number of people who attend the conference (200) and the sum of the people who speak English and French (220) is justified by the fact that 20 people speak both English and French. Consider the following diagram:



Which people can communicate among themselves without the help of an interpreter?

Everyone who has a language in common, thus constituting the speaking pairs: English - English; English – English & French; English & French – English & French; English & French – French; French - French.

How many different dialogues can be established among the one hundred people who exclusively communicate in English?

The first element (C_1) of the group of only English speakers establishes a dialogue with 99 participants:

$$C_1C_2; C_1C_3; C_1C_4; \dots; C_1C_{100}$$

The second element of the group(C_2) adds 98 dialogues to those previously stated.

$$C_2C_3; C_2C_4; C_2C_5; \dots; C_2C_{100}$$

It should be noted that the dialogue C_2C_1 is not included as it coincides with C_1C_2.

The third element of the group adds 97 conversations:

$$C_3C_4; C_3C_5; C_3C_6; \dots; C_3C_{100}$$

The dialogues C_3C_1 and C_3C_2 are not considered, as they coincide with C_1C_3 and C_2C_3.

Following this reasoning, 99 + 98 + 97 + … + 3 + 2 +1 dialogues take place.

What is the result of the sum 99 + 98 + 97 + … + 3 + 2 +1?

The sum of the numbers equally distant from the extremes of the sequence is 100. 49 pairs of numbers are established, to which the central value of the initial sequence, 50, ought to be added. Hence, 4950 dialogues can be counted.

Let us determine the number of dialogues which can be established among the one hundred people who exclusively speak English and the 20 who can communicate in both languages. Each element of the group who only speaks English establishes 20 dialogues with the people who speak both languages, summing up 2000 dialogues (100×20 = 2000).

Now, let us determine the number of different dialogues which may be established among the people who speak both English and French. Let us explore the situation in the classroom:

The first element of the group communicates with 19 participants:

$$C_1C_2; C_1C_3; C_1C_4; \dots; C_1C_{20}$$

The second element of the group adds 18 dialogues to the ones above mentioned:

$$C_2C_3; C_2C_4; C_2C_5; \dots; C_2C_{20}$$

It should be noted that the dialogue C_2C_1 is not considered for it coincides with C_1C_2.

Following this reasoning, 19 + 18 + 17 + … + 3 + 2 +1 dialogues are established. The sum of the numbers equally distant from the extremes of the sequence is 20. 9 pairs are established, to which the central value of the initial sequence, 10, ought to be added. Hence, 190 dialogues can be counted.

Let us now determine the number of different dialogues which can be established among the 20 people who can communicate in both languages with the 80 who only speak French. Each element of the group which only communicates in French establishes 20 dialogues with those who speak both languages, in a total of 1600 dialogues (80×20=1600).

Finally, let us determine the number of different dialogues which may be established among the 80 people who exclusively speak French.

The first element of the group communicates with 79 participants:

$$C_1C_2; C_1C_3; C_1C_4; \dots; C_1C_{80}$$

The second element of the group adds 78 dialogues to the previously mentioned:

$$C_2C_3; C_2C_4; C_2C_5; \dots; C_2C_{80}$$

It should be noted that the dialogue C_2C_1 is not included for it coincides with C_1C_2.

Following this reasoning, 79 + 78 + 77 + … + 3 + 2 +1 dialogues were established.

The sum of the numbers equally distant from the extremes of the sequence is 80. 39 pairs are established, to which the central value of the initial sequence, 40, ought to be added. Thus, 3160 dialogues can be counted.

According to the conditions of the statement of the problem, 4950 + 2000 + 190 + 1600 + 3160 = 11900 dialogues can take place. There is, however, a simpler procedure to achieve the solution. Using Combinatorial Analysis, $\binom{200}{2}$ dialogues are possible between two people. But as participants from different nationalities who only speak one language cannot communicate among themselves, we have to subtract these potential dialogues. This gives,

$$\binom{200}{2} - 100 \times 80 = \frac{200!}{2!\,198!} - 8000 = \frac{200 \times 199}{2!} - 8000 = 11900.$$

The History of Mathematics allows for multiple applications regarding sequential sums such as the ones presented before. Now, let us look at the properties of *magic squares*[46].

[46] Quadratic table of different natural numbers whose sum of the values of each row, column and diagonal is always constant. Its origin goes back to Ancient China.

4	9	2
3	5	7
8	1	6

15	10	3	6
4	5	16	9
14	11	2	7
1	8	13	12

Consider a magic square with n^2 grid squares (n rows by n columns) and the process to calculate the *magic constant*, the value of the sum of any of its main horizontal, vertical or diagonal lines. This is a useful value for it contributes to the construction of magic squares.

First, let us look at a less difficult question related with the problem. Which natural numbers should be part of the magic square? 1, 2, 3, 4, …, n, …, n^2.

Is it possible to calculate the sum of the first n^2 natural numbers? $S_n = \frac{1+n^2}{2} \times n^2$.

Taking into account that the magic square has n rows by n columns and that for each row and column the sum of all the values is constant, in order to attain the intended expression we must divide $\frac{1+n^2}{2} \times n^2$ by n so as to obtain the magic constant. That is, $\frac{\frac{1+n^2}{2} \times n^2}{n} = \frac{n}{2}(1+n^2)$.

The application of heuristics in calculation problems, like the above mentioned, is not limited to the presentation of an aesthetically harmonious solution. The increasing complexity of life in society requires greater skills and swiftness in acquire and selecting information. Let us consider chess. The sovereign of India, amazed by the strategic thinking which is intrinsic to the game, required that the inventor be brought to him to receive a reward. When questioned about what he desired, the inventor asked for a grain of wheat to be put on the first grid square of the chessboard, two on the second grid square, four on the third, eight on the fourth and continue doubling the number of grains until the sixty-four grid squares of the chessboard were filled out. The king considered the request as modest, and was therefore surprised to find that the stocks of wheat of his kingdom did not have enough grains to meet the reward. The quotient between the numbers of grains of wheat put on adjacent grid squares is constant and equals two. The total of grains is the sum of the first 64 terms of a geometric progression with a common ratio of two.

$$S_{64} = 1 \times \frac{1-2^{64}}{1-2} = \frac{1-2^{64}}{-1} = -(1-2^{64}) = 2^{64} - 1.$$

Let us assess how large this number is. It is known that for each natural number n there is only one k, so that $10^k \leq n < 10^{k+1}$. In compliance with logarithm rules, $k \leq \log n < k+1$, so $k = \lfloor \log n \rfloor$, where $(k+1)$ is the number of digits of n in a decimal representation.

For the number $2^{64} - 1$, $k + 1 = \lfloor log\ (2^{64} - 1) \rfloor + 1$, where $\lfloor log\ (2^{64} - 1) \rfloor$ represents the larger whole number less than or equal to $log\ (2^{64} - 1)$. The number of grains to be given has 20 decimal digits. One also knows that $log_c a = log_c b \times log_b a$, for $b > 1$. So $log_{10} a = log_{10} 2 \times log_2 a$ and $log_{10}(2^{64} - 1) = log_{10} 2 \times log_2 (2^{64} - 1) = log_{10} 2 \times 64$.

The construction of high-performance calculating machines enabled the exploration of problems which had seemed insurmountable due to the high number of arithmetical operations or the astronomical numbers involved such as the chess game problem. How to justify the minimal number of colours required to colour a flat map so that adjacent regions are not represented with the same colour (where regions with a single point in common were not considered as adjacent) can also be included in this type of problem. Common sense suggests that this number may be correlated to the number of regions to be coloured. Nevertheless, the chessboard, divided in 64 grid squares, is perfectly defined with only two colours. Francis Guthrie (1831 – 1899) suspected that four colours would be enough to colour any flat map (1852), independently of the number of regions in which it would be divided. He, however, could not prove it. The demonstration of the *Four Colour Theorem* was achieved in 1976 by Kenneth Appel, Wolfgang Haken and… an IBM 360 computer. Later on corroborated by more efficient algorithms, this demonstration does not dispense the computing capacity of the machine to examine the thousands of cases to which the problem was reduced. The importance of diminishing the tasks to be performed by simplification without loss of generality emerged from the *Four Colour Theorem*.

To illustrate the benefits of mathematical simplification, basic operations of addition, subtraction, multiplication and division in $\mathbb{C}$ can be carried out using only inverse operations, *i.e.*, by the composition of these binary and unary operations, respectively. Given that $x + y = x - (0 - y)$, therefore, the addition can be made under the stated conditions. To extend the concept to multiplication, it should be noted that the function $x \longmapsto x^2$ may be exclusively defined by the operations of subtraction and division. In fact,

$$\frac{1}{\frac{1}{x-1} - \frac{1}{x}} - (0 - x) = \frac{1}{\frac{x - (x-1)}{x(x-1)}} + x = \frac{x(x-1)}{1} + x = x^2 - x + x = x^2.$$

So we can see that constructing the functions $(x, y) \longmapsto x + y$ and $x \longmapsto x^2$ according to the statement is indeed possible. Hence, constructing the function m defined by $(x, y) \longmapsto 4xy = (x + y)^2 - (x - y)^2$, *i.e.*, multiplication, is also feasible, for:

$$m\left(m(x, y), \frac{1}{16}\right) = 4 \times m(x, y) \times \frac{1}{16} = 4 \times 4xy \times \frac{1}{16} = xy.$$

Finally, regarding division,

$$\frac{x}{y} = x \times \frac{1}{y}.$$

Although apparently artificial, this reasoning proves that it is possible to calculate complex functions using basic tools. The next problem presents a variation of the magic square.

(Subject: Numbers and Operations – Third Cycle students)

Consider the table (7×7) partially filled with the natural numbers 1, 2, 3, 4, 5, 6 and 7 so that neither row nor column has two equal natural numbers. Complete the table to form a *Latin Square*[47] or justify that, eventually in this case, it is not possible.

4	1		6			3
			7	3	5	
						6
		1		2	6	
6	3				1	
	6	4		5	2	
5		3		1		2

In order to systematise the completion of the table, consider that the natural number 6 is the most frequent. As each number must be written seven times, only number 6 is missing from two grid squares whose reading is *(row, column)*. It should be noted that number 6 is absent from rows two and seven and from columns three and five. Thus, it should occupy two of the positions (2,3), (2,5), (7,3), (7,5). Since the grid squares (2,5), (7,3), (7,5) are already filled out, completing the table according to the restrictions of the statement is impossible.

A tool to measure distance and volume: Thales' Theorem

Resilience is a decisive variable in terms of success, an attribute which the solver is able to develop and enhance if the proposed activity is understood and valued. Converse to exercise solving, which only requires the application of more or less obvious procedures, problem solving demands the activation and organisation of complex cognitive mechanisms with consistent effects but which only become visible in the long term.

Mathematics is a science of never ending challenges. Christian Goldbach (1690 – 1764), suggests that *Every even integer greater than 2 can be expressed as the sum of two primes*. Apparently simple, it has not yet been proven. Georg Cantor (1845 – 1918) validated the relation for a significant set of even numbers, and later on research between numbers and sums of two primes was expanded. Nevertheless, this conjecture is as yet unproven.

Thales of Miletus (640 BC – 550 BC) created an effective method to calculate inaccessible heights, namely the height of the Pyramids of Egyptian pharaohs. He placed a stick with a known height perpendicularly to the ground so that the tip of the shadow of the pyramid to be measured coincided with the shadow projected by the stick *(Figure 25)*. *Two triangles are in the Thales' position when there is an angle which defines sides shared by both triangles and whose vertex is opposed to two parallel sides, each one belonging to one of the triangles.* As the side of such triangles are proportional, the unknown of the problem can be determined.

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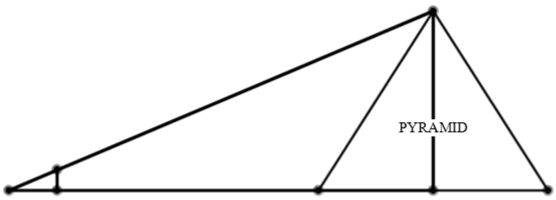
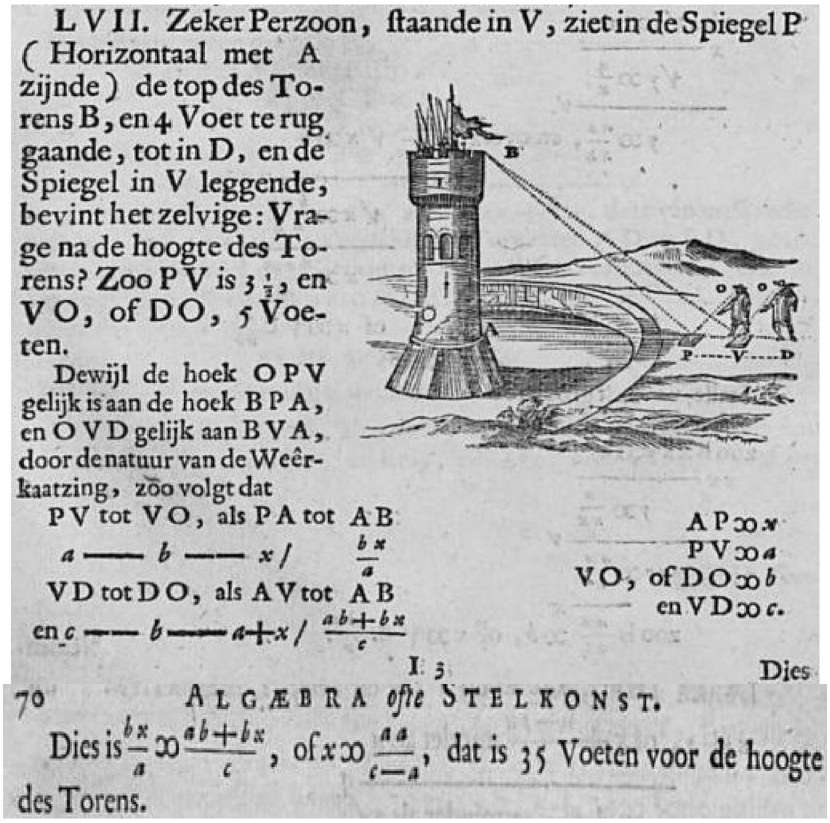
47 Quadratic table ($n \times n$) whose rows and columns comprise all natural numbers 1, 2, … n.



(Figure 25)

Books are sources of knowledge and learning is more than an easy reading exercise. It requires critical analysis. Theoretical concepts should be supported by illustrating examples. Let us see an application of the Thales' Theorem *(Figure 26)* by Abraham de Graaf (1672, p. 69). The language in which the problem is written does not necessarily constitute an unsurpassable hurdle to solve. Mathematics is based on a set of universal symbols and rules which contribute to the understanding of the message. Effective strategies to promote learning are required. Hypothetically, the greater the attention paid to the cognitive development of the student the more useful the History of Mathematics is in fostering effective didactic suggestions for a better pedagogical practice.



LVII. Zeker Perzoon, ſtaande in V, ziet in de Spiegel P (Horizontaal met A zijnde) de top des Torens B, en 4 Voet te rug gaande, tot in D, en de Spiegel in V leggende, bevint het zelvige: Vrage na de hoogte des Torens? Zoo PV is $3\frac{1}{2}$, en VO, of DO, 5 Voeten.

Dewijl de hoek OPV gelijk is aan de hoek BPA, en OVD gelijk aan BVA, door de natuur van de Weêrkaatzing, zoo volgt dat

PV tot VO, als PA tot AB

a — b — x / $\frac{bx}{a}$

VD tot DO, als AV tot AB

en c — b — $a+x$ / $\frac{ab+bx}{c}$

AP ∞ x
PV ∞ a
VO, of DO ∞ b
en VD ∞ c.

I 3 Dies

70 ALGÆBRA ofte STELKONST.

Dies is $\frac{bx}{a}$ ∞ $\frac{ab+bx}{c}$, of x ∞ $\frac{aa}{c-a}$, dat is 35 Voeten voor de hoogte des Torens.

(Figure 26)

The goal is to calculate the height of the tower. To visualise and organise the information provided, a sketch of a figure indicating the known values and the unknown(s) is necessary. We did this *(Figure 27)*. We see: $\overline{AP} = x$; $\overline{PV} = a = 3\frac{1}{2}$; $\overline{VO} = \overline{DO_1} = b = 5$ and $\overline{VD} = c = 4$. The solving method is based on the similarity of triangles, implied by the fact that sunbeams are reflected in a mirror, where the incidence angle is equal to the reflection angle.

Considering,

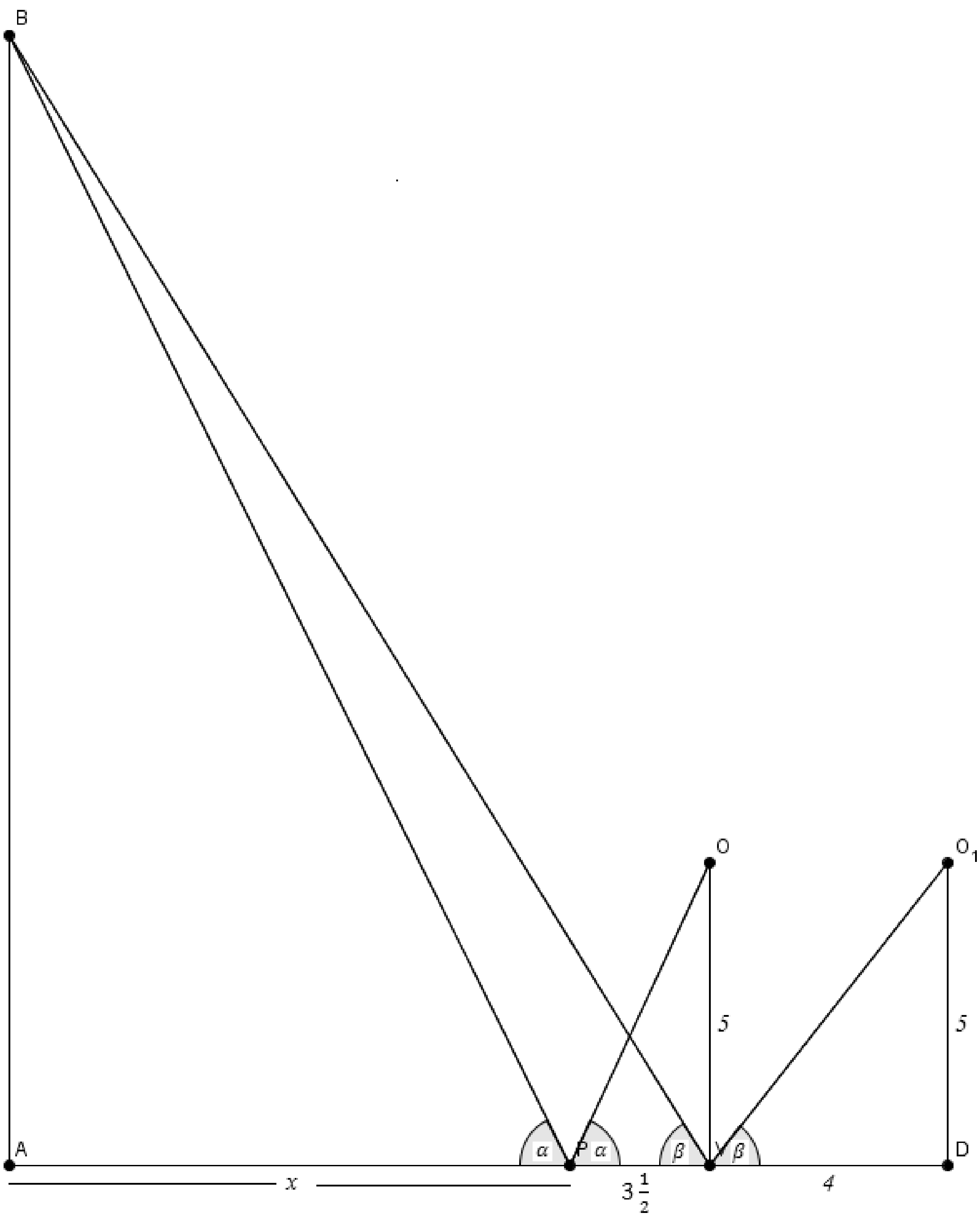
$$\frac{\overline{PV}}{\overline{VO}} = \frac{\overline{PA}}{\overline{AB}} \Leftrightarrow \frac{a}{b} = \frac{x}{\overline{AB}} \quad \text{and} \quad \frac{\overline{VD}}{\overline{DO_1}} = \frac{\overline{AV}}{\overline{AB}} \Leftrightarrow \frac{c}{b} = \frac{a+x}{\overline{AB}}$$

we have,

$$\overline{AB} = \frac{bx}{a} \quad \text{and} \quad \overline{AB} = \frac{ab+bx}{c}.$$

It follows that

$$\frac{bx}{a} = \frac{ab+bx}{c}.$$



(Figure 27)

If substitutions are done according to the values indicated in the statement, then

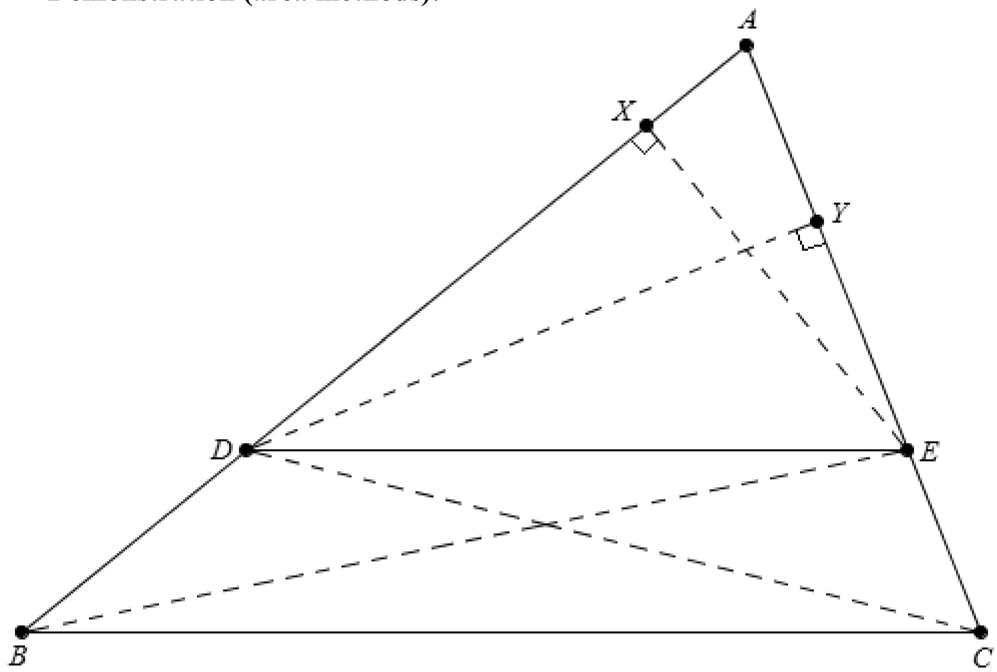
$$\frac{5x}{3\frac{1}{2}} = \frac{3\frac{1}{2} \times 5 + 5x}{4} \Leftrightarrow x = 24.5.$$

As the distance between the tower and the nearest observer is known, the height of the tower can be calculated through one of the equalities. Hence,

$$\overline{AB} = \frac{5 \times 24.5}{3\frac{1}{2}} = 35.$$

The proportional segments theorem (Thales' Theorem): *When two transversal lines intersect a set of parallel lines, then the lengths of the segments of the transversal lines that have been cut are proportional (Figure 28).*

Demonstration (area methods):



(Figure 28)

Considering triangle ΔABC and a point D on $\overline{AB}$, draw a line *r* parallel to $\overline{BC}$, being $\{D\} = r \cap AB$. The goal is to prove that

$$\frac{|AD|}{|DB|} = \frac{|AE|}{|EC|}.$$

The area of triangle ΔADE may be determined through two different processes. The area is given by

$$\frac{|AD| \times |XE|}{2}$$ as well as $$\frac{|AE| \times |YD|}{2},$$

and so $|AD| \times |XE| = |AE| \times |YD|$ *(1)*.

Triangles ΔEDB and ΔDEC have equal areas (the same base, $\overline{DE}$, and the same height). Therefore,

$$\frac{|DB| \times |XE|}{2} = \frac{|CE| \times |YD|}{2},$$

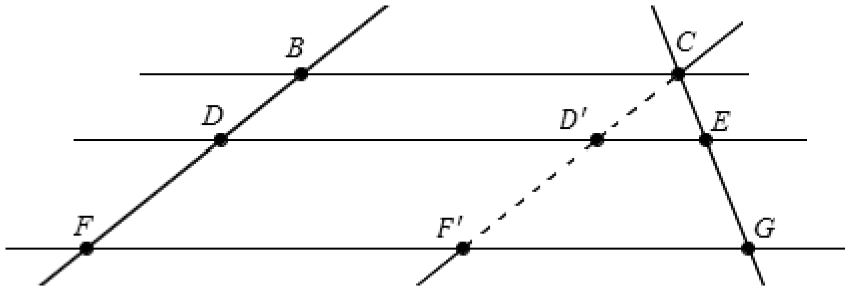
and so $|DB| \times |XE| = |CE| \times |YD|$ *(2)*.

Hence, *(1)* and *(2)* yield,

$$\frac{|AD|}{|DB|} = \frac{|AE|}{|CE|}.$$

Q. E. D.

The case below *(Figure 29)* is more general than the previous, but can be related to it.



(Figure 29)

Let us consider,

$$\frac{|CD'|}{|D'F'|} = \frac{|CE|}{|EG|}$$

as,

$|BD| = |CD'|$ and $|DF| = |D'F'|$

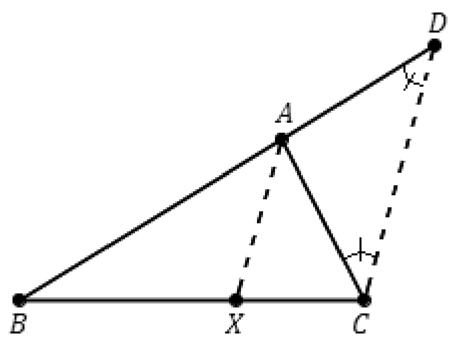
it follows that,

$$\frac{|CE|}{|EG|} = \frac{|CD'|}{|D'F'|} = \frac{|BD|}{|DF|}.$$

Q. E. D.

If ΔABC is a triangle, *X* the intersection point of the bisector of the angle in A with the segment BC *(Figure 30)*. Show that

$$\frac{|BX|}{|XC|} = \frac{|AB|}{|AC|}.$$



(Figure 30)

Considering triangle ΔABC, the bisector of the angle $B\hat{A}C$ and point D were marked so that $|AD| = |AC|$ and points B, A, D are collinear. The triangle ΔCAD is isosceles because $|AD| = |AC|$. Therefore,

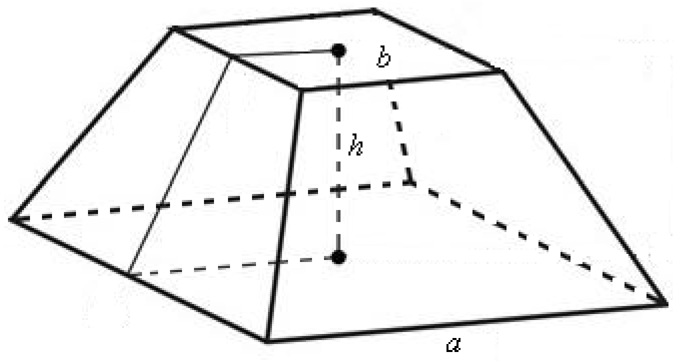
$$A\hat{D}C = \frac{1}{2}\left(180^{\circ} - D\hat{A}C\right) = \frac{1}{2}B\hat{A}C = B\hat{A}X.$$

The equality shows that *AX* and *DC* are parallel line segments, being $\Delta BAX \sim \Delta BDC$. According to the statement, we can infer from the Thales's Theorem that,

$$\frac{|BX|}{|XC|} = \frac{|BA|}{|AD|} = \frac{|AB|}{|AC|}.$$

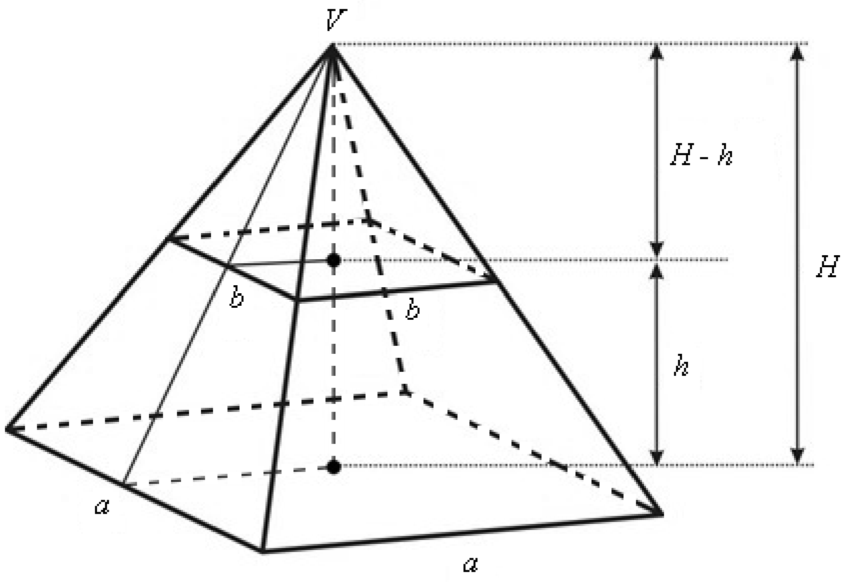
Q. E. D.

Deduce the formula for the volume of a truncated square pyramid *(Figure 31)*, $V = \frac{1}{3} \times h \times (a^2 + ab + b^2)$, where *a* and *b* are the sides of the square shown, and *h* the height.



(Figure 31)

Draw a regular square pyramid whose base edge is *a* and height is *H*, which is cut by a plane parallel to the base at a distance of *h* units of length *(Figure 32)*.



(Figure 32)

If V_1 is the volume of the pyramid of height H and V_2 the volume of the pyramid of height $H - h$, the formula which allows the calculation of the volume of the pyramid is

$$V = \frac{1}{3} \times Base\ Area \times Height$$

Considering,

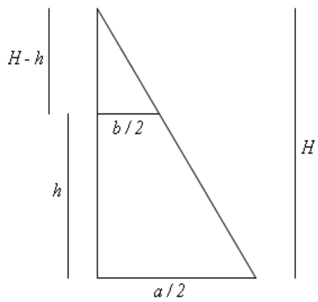
$$V_1 = \frac{1}{3} \times a^2 \times H,$$

$$V_2 = \frac{1}{3} \times b^2 \times (H - h),$$

$$V_{truncated\ pyramid} = V_1 - V_2 = \tfrac{1}{3} \times a^2 \times H - \tfrac{1}{3} \times b^2 \times (H - h)\ \textit{(I)}$$

$$= \frac{1}{3} \times a^2 \times H - \frac{1}{3} \times b^2 \times H + \frac{1}{3} \times b^2 \times h = \frac{1}{3}[(a^2 - b^2) \times H + b^2 \times h].$$

Let us divide the pyramid by a vertical plan containing the vertex and the perpendicular bisector of one of the lateral faces *(Figure 33)*. The figure shows two similar triangles.



(Figure 33)

Using triangle similarity we have the equivalences:

$$\frac{\frac{a}{2}}{\frac{b}{2}} = \frac{H}{H - h} \Leftrightarrow \frac{a}{b} = \frac{H}{H - h}$$

$$\Leftrightarrow a \times (H - h) = b \times H \Leftrightarrow H = \tfrac{a \times h}{a - b}.\ \textit{(II)}$$

Using the relation *(II)* in *(I)*, we get:

$$V_{truncated\ pyramid} = \frac{1}{3}\left[(a^2 - b^2) \times \frac{a \times h}{a - b} + b^2 \times h\right]$$

$$= \frac{1}{3}[(a + b) \times a \times h + b^2 \times h]$$

$$= \frac{1}{3} \times h \times (a^2 + ab + b^2).$$

Q. E. D.

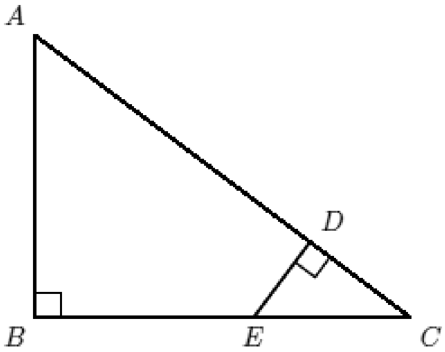
Instead of a formal structure, the statement of the problem may take the form of a story, thus aiming to engage the reader in the plot. The goal is to present a situation as close to life in society as possible, therefore avoiding routine statements with little more than abstract meaning. Next we present an example related to the same subject.

(Subject: Geometry – Third Cycle students)

The drawing *(Figure 34)* shows rectangle triangles $[ABC]$ and $[EDC]$.

We know that:

- point E is on $[BC]$
- point D is on $[AC]$
- $\overline{AD} = 11c$
- $\overline{DC} = 4cm$
- $\overline{EC} = 5cm$



(Figure 34)

a) Triangles $[ABC]$ and $[EDC]$ are similar. Justify.

b) Compute $\overline{BC}$. Present the answer in centimetres.

a) Angles ACB and DCE are congruent because they are coincident. As both triangles have a square angle, we can assert that triangles have two pairs of congruent angles, which is enough to justify similarity (Angle Angle criterion).

b) Due to the similarity of the triangles, we can state that the ratio between correspondent sides is equal, *i.e.*,

$$\frac{\overline{BC}}{\overline{DC}} = \frac{\overline{AC}}{\overline{EC}}.$$

($[AC]$ and $[EC]$ are opposite sides to the square angle in each triangle, thus, are correspondent; $[BC]$ and $[DC]$ are adjacent sides to the square angle and acute angle in C, thus, are correspondent sides).

Since $\overline{AC} = \overline{AD} + \overline{DC} = 11 + 4 = 15$, then,

$$\frac{\overline{BC}}{\overline{DC}} = \frac{\overline{AC}}{\overline{EC}} \Leftrightarrow \frac{\overline{BC}}{4} = \frac{15}{5} \Leftrightarrow \overline{BC} = 3 \times 4 \Leftrightarrow \overline{BC} = 12.$$

(Subject: Geometry – Third Cycle students)

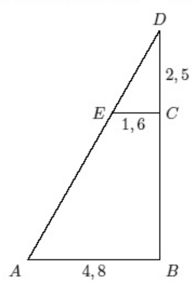
To ensure fire forest prevention and detection authorities build observation towers. To evaluate the height of the platform of the tower you can imagine two similar square triangles.
The figure shows a scheme (it is not a scale drawing) of such square triangles *(Figure 35)*.
We know:

- $\overline{DC} = 2.5m$
- $\overline{EC} = 1.6m$
- $\overline{AB} = 4.8m$

How much is $[CB]$ length?

9th grade National Intermediate Exam (11th May 2010)
8th grade National Intermediate Exam - Adapted (30th April 2008)





(Figure 35)

Since triangles $[ABD]$ and $[ECD]$ are similar (because they have a common acute angle and square angles *ECD* and *ABD*), we can establish that the ratio between correspondent sides is equal, *i.e.*,

$$\frac{\overline{BD}}{\overline{DC}} = \frac{\overline{AB}}{\overline{EC}}.$$

Then,

$$\frac{\overline{BD}}{2.5} = \frac{4.8}{1.6} \Leftrightarrow \overline{BD} = \frac{4.8 \times 2.5}{1.6} \Leftrightarrow \overline{BD} = 7.5.$$

Finally, as $\overline{BD} = \overline{BC} + \overline{DC} \Leftrightarrow \overline{BC} = \overline{BD} - \overline{DC}$,

$$\overline{BC} = 7.5 - 2.5 \Leftrightarrow \overline{BC} = 5.$$

A didactic view on mathematics, also illustrated in the previous problem, finds opposition in mathematicians throughout History, because

> *"[...] real mathematics has nothing to do with applications, nor with the calculating procedures that you learn at school. It studies abstract intellectual constructs which, at least while the mathematician is occupied with them, do not in any way touch the physical, sensible world. [...] Mathematicians [...] find the same enjoyment in their studies that chess players find in chess. In fact, the psychological make-up of the true mathematician is closer to that of the poet or the musical composer, in other words of someone concerned with the creation of Beauty and the search for Harmony and Perfection. He is the polar opposite of the practical man, the engineer, the politician or ... indeed, the businessman"*[48] (Doxiadis A. , 2001, pp. 28-29).

This dichotomy, highlighted by previously presented problems solved through the Thales' theorem, reinforces the need to wisely select the activities to be implemented in the classroom.

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[48] Author's translation: *" [...] a verdadeira matemática não tem nada a ver com aplicações, nem com os processos de cálculo que aprendes na escola. Estuda idealizações intelectuais abstratas que, pelo menos enquanto o matemático está ocupado com elas, não tocam de forma nenhuma no mundo físico e sensível. [...] Os matemáticos [...] sentem nos seus estudos o mesmo prazer que os jogadores de xadrez encontram no xadrez. Na verdade, a estrutura psicológica do verdadeiro matemático está mais próxima da do poeta ou do compositor musical, noutras palavras, de alguém preocupado com a criação da Beleza e a procura da Harmonia e da Perfeição. Ele é o oposto do homem prático, o engenheiro, o político ou o homem de negócios."*

Narratives with a mathematical plot

(Subject: Geometry – Third Cycle students)

The kitchen tap is dripping at a rate of six drops per minute. In order to not waste water, a container with scale is put in the sink. The kitchen watch marks $10h$ $26m$ $40s$. Meanwhile, unavoidable activities make the owner leave the house. He returns home at $16h$ and verifies that the container left under the tap has precisely $20cl$ of water. He smiles with contentment because even though the leaking problem was not fixed, he now knows how to calculate the volume of each drop of water and the approximate value of its radius.

The statement refers the collection time (20,000 seconds), the rhythm of the dripping (six drops per minute) and the volume of water in the container ($20cl = 0.2dm^3$). To calculate the volume of a drop of water, the collected volume is divided by the number of drops which fell inside the container. Knowing that six drops fall in one minute, one drop falls every ten seconds to obtain a total of 2,000 drops of water. The volume of a drop is therefore $\frac{0.2dm^3}{2{,}000}$, *i.e.*, $0.0001dm^3$. Considering that a drop is roughly spherical, it is possible to assess its radius ($V_{sphere} = \frac{4}{3}\pi r^3$).

$$\frac{4}{3}\pi r^3 = 0.0001dm^3 \Leftrightarrow 4\pi r^3 = 0.0003dm^3 \Leftrightarrow r^3 \cong 0.000023873dm^3 \Leftrightarrow r \cong 0.0288dm.$$

The radius of a water drop is $0.0288dm = 2.88mm$.

Other problems will be introduced next.

(Subject: Numbers and Operations – Second Cycle students)

There are 21 students in a class (boys and girls). Each girl may or may not have friends in the class. No two girls in the class have the same number of friends. What is the maximum number of girls in that class?

Taking the restrictions into account, the student must set up a plan *(Chart 8)*.

(Chart 8) – Problem solving strategy

Girls	Boys
$Girl_1$	
$Girl_2$	Boy_1
$Girl_3$	Boy_2; Boy_3
$Girl_4$	Boy_4; Boy_5; Boy_6
$Girl_5$	Boy_7; Boy_8; Boy_9; Boy_{10}
$Girl_6$	Boy_{11}; Boy_{12}; Boy_{13}; Boy_{14}; Boy_{15}

When reaching 21 elements, the algorithm is interrupted and the solution is presented. Surprisingly, the teacher asks for a reassessment of the answer. The student is startled and probably disappointed for he/she is certain beyond all doubt of the reasoning used. A more accurate analysis will eventually contribute for the so-called *Eureka* moment, which will enable the identification of the error and arrive at the correct answer. Even though no two girls can have the same number of friends, each boy may simultaneously be friends with more than one girl *(Chart 9)*.

(Chart 9) – Problem solving strategy

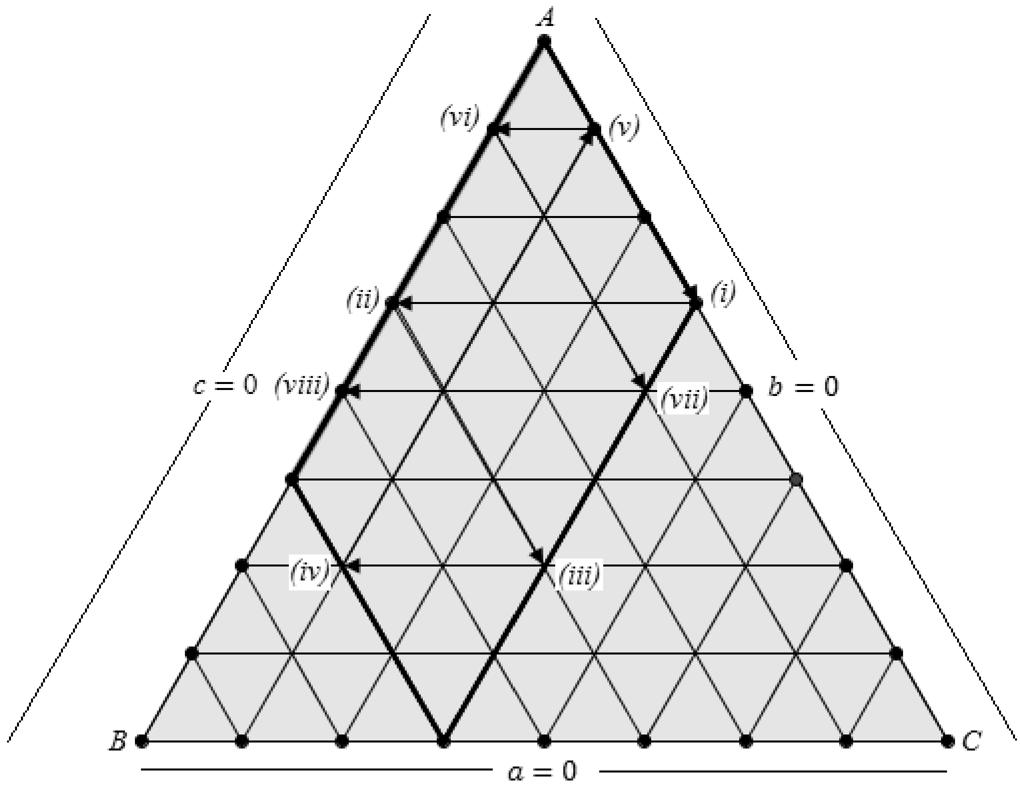
Girls	Boys
$Girl_1$	
$Girl_2$	Boy_1
$Girl_3$	Boy_1; Boy_2
$Girl_4$	Boy_1; Boy_2; Boy_3
$Girl_5$	Boy_1; Boy_2; Boy_3; Boy_4
$Girl_6$	Boy_1; Boy_2; Boy_3; Boy_4; Boy_5
$Girl_7$	Boy_1; Boy_2; Boy_3; Boy_4; Boy_5; Boy_6
$Girl_8$	Boy_1; Boy_2; Boy_3; Boy_4; Boy_5; Boy_6; Boy_7
$Girl_9$	Boy_1; Boy_2; Boy_3; Boy_4; Boy_5; Boy_6; Boy_7; Boy_8
$Girl_{10}$	Boy_1; Boy_2; Boy_3; Boy_4; Boy_5; Boy_6; Boy_7; Boy_8; Boy_8
$Girl_{11}$	Boy_1; Boy_2; Boy_3; Boy_4; Boy_5; Boy_6; Boy_7; Boy_8; Boy_8; Boy_{10}

The class may have a maximum of 11 girls.

(Subject: Applied Mathematics – Third Cycle students)

Two friends intend to equally share the content of a bottle which contains 80*cl* of water. For that, they have two non-graduated glasses, one with the capacity for 50*cl* and the other for 30*cl*. Is it possible to achieve a solution? If your answer is yes, how?

In order to circumvent the trial and error method, which may become tedious and eventually lead to the belief that the task may be impossible, procedure systematisation is required. The concept of barycentric coordinates should be applied to solve this type of problems. Suppose that the height of an equilateral triangle ΔABC is 80 and P is a point of coordinates (a, b, c) located at a distance a of the side BC, b of the side AC and c of the side AB. The sum of these distances is constant. So, $a + b + c = 80$. Consider a as the quantity of water, in *cl*, in the bottle, and b and c as the quantities of water in each non-graduated glass. It is known that $b \leq 50$ and $c \leq 30$. The possible stages are delimited in bold by the borderline of the parallelogram *(Figure 36)*.



(Figure 36)

The aim is to attain point (40, 40, 0), as the 30cl glass does not have the capacity to be used as a final container. The transference operations correspond to segments which connect points located at the border of the parallelogram. The initial situation is identified in ΔABC by point A whose coordinates are (80, 0, 0). A possible solution is *(i)* to pour 30cl from bottle *a* to glass *c*, (50, 0, 30); *(ii)* to pour 30cl from glass *c* to glass *b*, (50, 30, 0); *(iii)* to pour 30cl from bottle *a* to glass *c*, (20, 30, 30); *(iv)* to pour 20cl from glass *c* to glass *b*, (20, 50, 10); *(v)* to pour 50cl from glass *b* to bottle a, (70, 0, 10); *(vi)* to pour 10cl from glass *c* to glass *b*, (70, 10, 0); *(vii)* to pour 30cl from bottle *a* to glass *c*, (40, 10, 30); *(viii)* to pour 30cl from glass *c* to glass *b*, (40, 40, 0).

(Subject: Numbers and Operations – Second Cycle students / Third Cycle students)

We had been travelling for a few hours without stopping when occurred an episode worth retelling, wherein my companion Beremiz put to use his talents.

Close to an old abandoned inn, we saw three men arguing heatedly besides a herd of camels. Amid the shouts and insults the men gestured wildly in fierce debate and we could hear their angry cries.

"It cannot be!", "That is robbery!", "But I do not agree!"

The intelligent Beremiz asked them why they were quarrelling.

"We are brothers" the oldest explained, "and we received 35 camels as our inheritance.

According to the wishes of my father half of them belong to me, one-third to my brother Hamed Namir, and one-ninth to Harim Namir, the youngest. Nevertheless we do not know how to make the division, and whatever one of us suggests, the other two don't agree.

"Very simple" said the Man Who Counted. "I promise to make the division fairly but let me add to the inheritance of thirty-five camels this splendid beast that brought us here at such an opportune moment".

Turning to the eldest of the brothers, he spoke thus:"You would have half of 35 – that is 17.5. Now, you will receive half of 36 – that is 18. You have nothing to complain about because you gain by this division".

Turning to the second heir, he continued "And you, Hamed, you would have received one-third of 35 – that is, 11 and some. Now you will receive one-third of 36 - that is 12. You cannot protest as you too gain by this division".

Finally, he spoke to the youngest, "And you, young Harim Namir, according to your father's last wishes, you were to receive one-ninth of 35, or three camels and part of another. Nevertheless, I will give one-ninth of 36, or four. You have benefited substantially and should be grateful to me for it. And he concluded with the greatest confidence "By this advantageous division, which has benefited everyone, 18 camels belong to the oldest, 12 to the next, and 4 to the youngest, which comes out to: 18+12+4=34 camels. Of the 36 camels, therefore, there are two extra. One, as you know, belongs to my friend from Baghdad. The other rightly belongs to me for having resolved the complicated problem of the inheritance to everyone's satisfaction".

"Stranger, you are a most intelligent man," explained the oldest of the three brothers "and we accept your solution with the confidence that it was achieved with justice and equity".

The clever Beremiz, the Man Who Counted, took possession of one of the final animals in the herd and, handing me the reins of my own animal, said, "Now, dear friend, you can continue the journey on your camel, comfortable and content. I have one of my own to carry me."

And we travelled on towards Baghdad.

(Tahan, 2001)

The goal is to divide an inheritance of 35 camels by three brothers, receiving each $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{9}$ of the herd, without killing any of the camels.

Such division is not feasible because the values do not correspond to whole numbers.

$\frac{35}{2} = 17.5$ $\quad \frac{35}{3} \cong 11.67$ $\quad \frac{35}{9} \cong 3.89$

When an animal is added, a total of 36 camels is achieved, and this number is a multiple of 2, 3 and 9.

$\frac{36}{2} = 18$ $\quad \frac{36}{3} = 12$ $\quad \frac{36}{9} = 4$

Doubts regarding the fact that all heirs benefited from the division when only 34 animals were used cast suspicions on the division of the whole inheritance. Let us see if the sum of all parts equals the whole, *i.e.* the unit.

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{9}=\frac{9}{18}+\frac{6}{18}+\frac{2}{18}=\frac{17}{18}$$

The heirs received $\frac{17}{18}$ of the inheritance, thus meaning that $\frac{1}{18}$ of the herd was not integrated in the division. Let us consider that $\frac{1}{18}$ of 35 is less than 2, a fact unnoticed by the heirs since the result of the division cannot be expressed in whole numbers, namely in the number of animals to be received. However, when an animal is added, the part of the inheritance which is not distributed is two camels, thus enabling Beremiz Samir to solve the problem to the satisfaction of the heirs and, at the same time, profit from the situation.

Binary writing and the sexagesimal system

Among the four basic operations, the division is the one requiring a greater knowledge when performing mathematical algorithms. The student eventually ends up disrespecting division basic principles.

Identify the error(s) which is part of the following reasoning:
Knowing that $a = b$.
Multiply both members by a, whose result is $a^2 = ab$.
Add $a^2 - 2ab$ to both members: $a^2 + a^2 - 2ab = ab + a^2 - 2ab$.
When simplifying, you obtain: $2(a^2 - ab) = a^2 - ab$.
Finally, by dividing both members by $a^2 - ab$, $\frac{2(a^2-ab)}{a^2-ab} = \frac{a^2-ab}{a^2-ab}$, the result is 2 = 1.

The development of the equality $a = b$ leads to an absurdity, even though, apparently, the procedures were correct and it looked as if mathematical rules were respected. The error arises in the last step, when both members of the equation were divided by $a^2 - ab$. Assuming that $a = b$, the division by $a^2 - ab$ determines a quotient whose values for both the numerator and the denominator are zero, which makes simplification impossible.

The Hindu-Arabic numeral system was introduced in Europe at the beginning of the 13th century. Among its ten digits, the zero was the one most suspicious to a great extent due to difficulties in using it. After all, what was the use of a symbol which, by itself, means nothing?

Disbeliefs regarding its value eventually subsisted but it gradually assumed a prominent position in the progress of the mathematical sciences.

The googol, by which the number 10^{100} is known, inspired two computer science students, Larry Page and Sergey Brin, to call *Google* to the search engine that they created for the *Internet*. Used by millions of people all over the world, an increasing amount of information is made available by *Google*, thus making justice to the number to which it was named after.

Zero is closely connected to computer sciences. Any natural number may be represented in binary writing by using only the digits 0 and 1, logic 1 for true and logic 0 for false. In computer language, these symbols correspond to the situation turned off (0) and turned on (1), constituting a minimum element of information called *bit* (*b*inary dig*it*). In its full version, each digit or letter is represented by a string of eight bits, with an increasing place value when reading is done from right to left.

Consider an algorithmic procedure based on successive divisions by 2 used to convert decimal notation to binary notation. For example: in binary notation, 60 is interpreted as 111100 $(60 = 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0)$.

60	2				
0	30	2			
	0	15	2		
		1	7	2	
			1	3	2
				1	1

The value 10101 converted to decimal numeration is 21 $(1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 21)$. The algorithm which allows for the transformation of binary notation into decimal notation uses powers of 2, always bearing in mind that the states of turned on/turned off located immediately on the left of another state correspond to the double, in units *(Table 111)*.

(Table 111) – Decimal notation vs. Binary notation

DECIMAL NOTATION	BINARY NOTATION
0	0 (0×2^0)
1	1 (1×2^0)
2	10 $(1 \times 2^1 + 0 \times 2^0)$
3	11 $(1 \times 2^1 + 1 \times 2^0)$
4	100 $(1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0)$
5	101 $(1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0)$
6	110 $(1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0)$
7	111 $(1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0)$
8	1000 $(1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 0 \times 2^0)$
9	1001 $(1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0)$
10	1010 $(1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0)$
11	1011 $(1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0)$
12	1100 $(1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0)$
…	…

Consider the procedures which enable the performance of basic arithmetic operations in the binary system; addition, subtraction, multiplication and division:

ADDITION – Possible cases when summing by columns:

a)			*b)*			*c)*			*d)*			*e)*		
	0			0			1			1			1	
+	0		+	1		+	0		+	1			1	
=	0		=	1		=	1		=	10		+	1	
												=	11	

In situations *a)*, *b)* and *c)* there were no carries.
In situation *d)* carry occurred. The result of 1 + 1 is 0, with the carry of 1.
In situation *e)* the sum of 1+ 1 is 0, which summed to 1 is 1, with the carry of 1.

SUBTRACTION – Possible cases when subtracting by columns:

a)			*b)*			*c)*			*d)*					
	0			0			1			1				
-	0		-	1		-	0		-	1				
=	0		=	1		=	1		=	0				

In situation *b)* the result is 1, but carry for the next column occurs, which should be added to the subtrahend.

Example: $10110_2 - 1001_2$.

	1	0	1	1	0									
	1			***1***										
-		1	0	0	1									
=	0	1	1	0	1									

MULTIPLICATION – Possible cases:

a)			*b)*			*c)*			*d)*					
	0			0			1			1				
×	0		×	1		×	0		×	1				
=	0		=	0		=	0		=	1				

Example: $10110_2 \times 11_2$.

			1	0	1	1	0							
					×	1	1							
			1	0	1	1	0							
	1	***1***	***1***	***1***										
+		1	0	1	1	0								
=	1	0	0	0	0	1	0							

DIVISION

Let us start with the dividend. Select, among the most significant bits of the dividend, the same number of bits of the divisor. If the number is greater than or equal to the divisor, write 1 in the quotient; if that is not the case write 0 and attempt a new division by bringing down the next digit of the dividend. When the division is possible, write 1 in the quotient and subtract the value of the divisor from the part of the dividend that is used in the operation. Bring down the next digit of the dividend to the result of the subtraction and repeat the procedure until all the digits of the dividend are used. Similarly to decimal division, zeros can be added to the right of the radix point of the dividend when the remainder is not 0 so as to continue using the algorithm.

Example: $110100_2 \div 100_2$.

	1	1	0	1	0	0	1	0	0					
-	1	0	0				1	1	0	1				
	0	1	0	1										
	-	1	0	0										
		0	0	1	0	0								
			-	1	0	0								
				0	0	0								

Just like other numbers, fractional numbers may also be represented in binary writing,

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0,	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}

Example: $111.01_2 \times 11_2$.

			1	1	1.	0	1							
					×	1	1							
			1	1	1	0	1							
	1	***1***	***1***											
+		1	1	1	0	1								
=	1	0	1	0	1.	1	1							

Example: $1011.11_2 \div 11.1_2$.

Move the radix point to make the divisor a whole number, *i.e.* $10111{,}1_2 \div 111_2$. As, in this case, the three most significant *bits* of the dividend are inferior to the three most significant *bits* of the divisor, the following digit of the dividend must be brought down. Proceed according to the above-mentioned division algorithm. Once all the digits of the dividend are used, the radix point should be correctly placed in the quotient. If the remainder of the division is not zero, the radix point must be placed in the position corresponding to its initial position in the dividend.

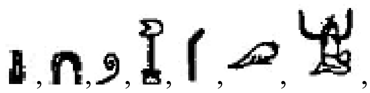
	1	0	1	1	1.	1	0	1	1	1				
-	***1***	1	1	1				1	1.	0	1			
	0	1	0	0	1									
		1	***1***											
	-		1	1	1									
		0	0	1	0	1	0							
				1	***1***	***1***								
			-		1	1	1							
				0.	0	1	1							

$$1011.11_2 = 11.1_2 \times 11.01_2 + 0.011_2$$

In Europe, the Hindu-Arabic numeral system was gradually introduced by Leonardo Fibonacci (1170 – 1250) and other mathematicians. As a young boy, Fibonacci accompanied his father in business trips to the North of Africa. There, he contacted with different cultures and discovered the digits; a novelty for the Europeans, who nowadays write with Roman numerals. As a result of that knowledge, Fibonacci writes the *Liber Abaci* (1202), a manual for teaching and learning the rules of elementary arithmetic operations.

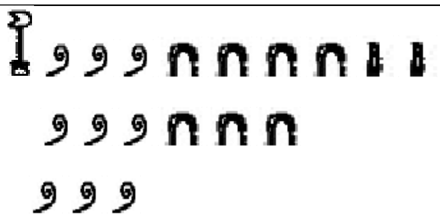
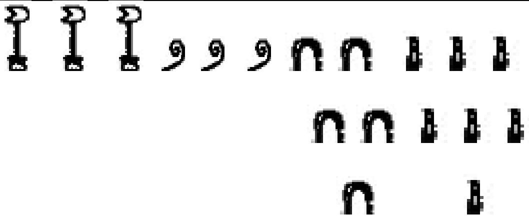
Associating symbols to represent quantity was not exclusive of the Hindu civilisation. The Roman Empire used seven letters of the alphabet (I, V, X, L, C, D, M), to which the values 1, 5, 10, 50, 100, 500 and 1000 were respectively attributed. Characters could not be repeated in succession more than 3 times. A character with an inferior value placed on the left of another with a greater value meant a subtraction. A character with a lesser value placed on the right of a character with a greater value meant an addition. Examples: XL = 40 and MCMLXII = 1962.

The interpretation of Egyptian hieroglyphs gained a strong impetus in 1799, when the Rosetta Stone was discovered during a Napoleonic expedition to Egypt. Its text was written in three languages, Hieroglyph, Demotic (a vernacular form adopted by the poorer people of Egypt) and Ancient Greek. The numeric system found in the Rosetta Stone comprised seven symbols *(Figure 37)*,



(Figure 37)

with the purpose of respectively represent 1, 10, 100, 1000, 10000, 100000 and 1000000.

ONE THOUSAND NINE HUNDRED AND SEVENTY-TWO	
+	
ONE THOUSAND THREE HUNDRED AND EIGHTY-FIVE	
=	
THREE THOUSAND THREE HUNDRED AND FIFTY-SEVEN	

By using successive duplications, the Egyptians performed multiplications quickly and efficiently. For example, to determine the result of twenty-two times eighteen, the scribe would start by choosing the multiplicand. Opting for eighteen meant that he would have to add it twenty-two times.

NUMBER OF SUMMANDS	RESULT
1	18
2	**36**
\ 10	**\ 360**

Partial multipliers whose sum is twenty-two were placed in the column on the left and the corresponding results, which would subsequently be added, were placed in the column on the right.

$$22 \times 18 = 20 \times 18 + 2 \times 18 = 360 + 36 = 396$$

The exact division, the inverse operation of multiplication, was performed through a similar process. Dividing *a* by *b* means knowing *how to operate on b in order to obtain a*, *i.e.*, *how often should b be multiplied in order to obtain a*.

Problem 69 of the Rhind Papyrus requires you to operate on 80 in order to obtain 1120. How often should 80 be multiplied in order to obtain 1120? As the dividend and the divisor are known, the goal is to determine the quotient and the remainder. The quotient of the division is 14 and the remainder is 0.

NUMBER OF SUMMANDS	RESULT
1	80
\ 10	**\ 800**
2	160
4	**320**

What are the quotient and the remainder of the division of 1276 by 68?

NUMBER OF SUMMANDS	RESULT
1	68
\ **10**	\ **680**
2	136
4	272
8	**544**

The scribe operated on 68 in order to obtain 1276 or a number that is inferior to that and whose difference to 1276 is lesser than 68 units. Therefore, 1276 – (680 + 544) = 52. The quotient is 18 and the remainder of the division is 52.

Let us now study the sexagesimal system, a system used since Antiquity.

Consider the decomposition of 60 and 10 in prime factors.

$60 = 2^2 \times 3 \times 5$ $\qquad$ $10 = 2 \times 5$

In base 10, fractions whose denominators are of type $2^\alpha \times 5^\beta$ with $\alpha, \beta \in \mathbb{N}_0$ are reducible to decimal fractions and therefore representable by decimals. All the other fractions are periodic infinite repeating decimals.

Examples:

FRACTIONS EQUIVALENT TO FINITE DECIMALS	FRACTIONS EQUIVALENT TO INFINITE DECIMALS
$\frac{4}{5} = \frac{4 \times 2}{5 \times 2} = \frac{8}{10} = 0.8$	$\frac{1}{15} = 0.0(6)$
$\frac{7}{8} = \frac{7}{2^3} = \frac{7 \times 5^3}{2^3 \times 5^3} = \frac{875}{1000} = 0.875$	$\frac{1}{3} = 0.(3)$
$\frac{72}{50} = \frac{2^3 \times 3^2}{2 \times 5^2} = \frac{2^4 \times 3^2}{2^2 \times 5^2} = \frac{144}{100} = 1.44$	$\frac{11}{12} = 0.91(6)$

In base 60, all the fractions whose denominators are of type $2^\alpha \times 3^\beta \times 5^\gamma$ with $\alpha, \beta, \gamma \in \mathbb{N}_0$ are reducible to sexagesimal fractions and therefore representable by finite decimals. Hence, some fractions which in the decimal system correspond to infinite repeating decimals have finite sexagesimal expansions. In the 1920s, Otto Neugebauer suggested that the whole part and the fractional part of a sexagesimal number should be separated by a semicolon *";"* and that the units concerning different orders should be separated by a comma ",".

Examples:

$$3{,}18{,}57 = 3 \times 60^2 + 18 \times 60 + 57$$

$$5{,}39; 2{,}41 = 5 \times 60 + 39 + \frac{2}{60} + \frac{41}{60^2}$$

When compared to the decimal system, the sexagesimal positional numeral system has a clear advantage. Some fractions which in the decimal system correspond to infinite repeating decimals also have a finite decimal configuration in the sexagesimal method[49].

Examples:

$$\frac{1}{15}=\frac{1}{3\times 5}=\frac{2^2}{2^2\times 3\times 5}=\frac{4}{60}=0;4$$

$$\frac{1}{3}=\frac{2^2\times 5}{2^2\times 3\times 5}=\frac{20}{60}=0;20$$

$$\frac{11}{12}=\frac{11}{2^2\times 3}=\frac{11\times 5}{2^2\times 3\times 5}=\frac{55}{2^2\times 3\times 5}=\frac{55}{60}=0;55$$

Neugebauer called the numbers whose inverses are finite sexagesimal expansions *regular numbers*, a category where 3, 6 and 15 are included.

Operations in base 60 are performed in a similar way as operations in base 10. The numbers of the same orders are added. When the result is greater than 60, it is divided by 60, the remainder is placed in the corresponding order and the obtained quotient is added to the numbers of the next higher order.

2; 31,6 + 50; 58, 59 + 40; 7 = ?

Fractional part:

6 + 59 = 65.

The sum is greater than 60. Divide by 60. Quotient 1. Remainder 5.

31 + 58 + 7 + 1 = 97.

The sum is greater than 60. Divide by 60. Quotient 1. Remainder 37.

Whole part:

2 + 50 + 40 + 1 = 93.

The sum is greater than 60. Divide by 60. Quotient 1. Remainder 33.

0 + 1 = 1.

Result: 1,33; 37,5

A similar process is used for multiplication.

(3; 45,21) × 8 = ?

Fractional part:

21 × 8 = 168.

---

[49] All base 10 fractions with finite decimals, and others under certain conditions, are finite decimals in the sexagesimal system.

The value obtained is greater than 60. Divide by 60. Quotient 2. Remainder 48.

$45 \times 8 + 2 = 362$.

The value obtained is greater than 60. Divide by 60. Quotient 6. Remainder 2.

Whole part:

$3 \times 8 + 6 = 30$.

Result: 30; 2,48.

Division may be turned into a multiplication.

In the case of $\frac{a}{b}$ it can be written as $a \times \frac{1}{b}$, where $\frac{1}{b}$ is the reciprocal of *b*.

Using the Babylonian method, what is the result of the division of 22 by 9?

$$\frac{22}{9} = \frac{2 \times 11}{3^2} = \frac{2^5 \times 11 \times 5^2}{(2^2 \times 3 \times 5)^2} =$$

$$= \frac{8800}{60^2} = \frac{2 \times 60^2 + 1600}{60^2} = \frac{2 \times 60^2 + 26 \times 60 + 40}{60^2} = 2 + \frac{26}{60} + \frac{40}{60^2} = 2;26{,}40.$$

In Geometry, the degree as a unit to measure an angle is a commonly used notation. A degree is divided in 60 minutes $(1^\circ = 60')$ and each minute in 60 seconds $(1' = 60'')$. The 2nd Cycle curriculum (5th and 6th grade) requires the operationalization of this content at the level of the addition and subtraction of angles. The resolution of this type of exercises allows an approach to the sexagesimal system with the use of algorithmic procedures.

Examples:

a) $38^\circ\,41'\,19'' + 07^\circ\,12'\,26''$

	38°		$41'$		$19''$
+	07°		$12'$		$26''$
	45°		$53'$		$45''$

b) $112^\circ\,39'\,48'' + 26^\circ\,02'\,34''$

	112°		$39'$		$48''$
+	26°		$02'$		$34''$
	138°		$41'$		$82''$
		+	$01'$	−	60°
	138°		$42'$		$22''$

c) $74^\circ\,00'\,52'' - 39^\circ\,55'\,17''$

	74°		$00'$		$52''$
-	39°		$55'$		$17''$

	73°		$60'$		$52''$
-	39°		$55'$		$17''$
	34°		$05'$		$35''$

Next we present two more algorithmic procedures, the first to attain prime numbers and the second to reach perfect numbers.

Eratosthenes (276 BC – 194 BC) invented a simple method to identify all *prime numbers*[50] up to n: write all natural numbers from 2 to n in ascending order; eliminate all the multiples of 2; the first number greater than 2 is 3, so eliminate all the multiples of 3 except 3; the first number greater than 3 is 5, so eliminate all the multiples of 5 except 5; the first number greater than 5 is 7, so eliminate all the multiples of 7 except 7. Repeat the process until elimination is no longer possible. The remaining numbers are prime numbers.

Nicomachus of Gerasa (~ 60 – 120) created a process to systematically determine *perfect numbers*[51]: *1)* write the numbers of the succession $U_n = 2^n$, $n \in \mathbb{N}_0$; *2)* add the two first values – if the result is a prime number, multiply that number by the last added term. The result of this operation is a perfect number; if it is not, add the next value of the list and examine the result again, successively, up to infinity. Take the algorithm which allows to identifying whether a number is a perfect number.

$U_n = 2^n$, $n \in \mathbb{N}_0$ 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096…

1 + 2 = 3 (3 PRIME NUMBER) 3 ×2 = 6 6 PERFECT NUMBER / DIVISORS OF 6 {1, 2, 3, 6}
1 + 2 + 4 = 7 (7 PRIME NUMBER) 7 × 4 = 28 28 PERFECT NUMBER / DIVISORS OF 28 {1, 2, 4, 7, 14, 28}
1 + 2 + 4 + 8 = 15
1 + 2 + 4 + 8 + 16 = 31 (31 PRIME NUMBER) 31 × 16 = 496 496 PERFECT NUMBER / DIVISORS OF 496 {1, 2, 4, 8, 16, 31, 62, 124, 248, 496}
1 + 2 + 4 + 8 + 16 + 32 = 63
1 + 2 + 4 + 8 + 16 + 32 + 64 = 127 (127 PRIME NUMBER) 127 × 64 = 8128 8128 PERFECT NUMBER / DIVISORS OF 8128 {1, 2, 4, 8, 16, 32, 64, 127, 254, 508, 1016, 2032, 4064, 8128}
…

[50] A prime number is a natural number greater than 1 whose only positive divisors are 1 and itself.

[51] The sum of its divisors equals two times its value.

Fibonacci – Ambassador of the Hindu-Arabic system in Europe

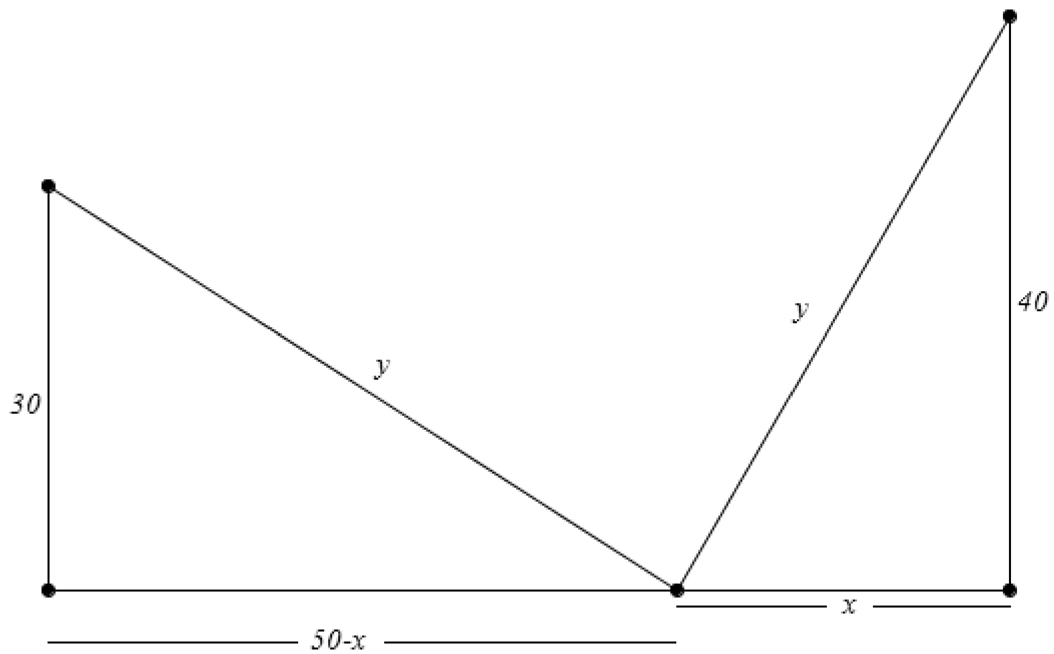
(Subject: Geometry / Algebra – Third Cycle students)

Two towers are 50 cubits apart: one tower is 30 cubits high and the other 40 cubits high. There is a bird at the top of each tower. Between the two towers there is a fountain where the birds drink. Both birds fly at the same rate and reach the fountain at the same time.
Starting at the same time and flying at the same rate, the birds reach the fountain between the bases of the towers at the same moment. How far is the fountain from each tower?

Liber Abaci

The goal is to determine how far the fountain is from each tower. In order to better visualize and organize the available information, a diagram with the towers, the birds' flight and the point where they reach the fountain is useful *(Figure 38)*.

Consider x as the distance between the highest tower and the fountain. Taking the distance covered by each bird, and knowing that both birds fly at the same rate and reach the fountain at the same time, we can conclude that they both travel the same distance. Let us then represent this value by the letter y.



(Figure 38)

The diagram shows two right triangles, thus suggest the application of the Pythagoras' theorem. The result of its application to both the right and the left triangle is, respectively, $y^2 = 30^2 + (50 - x)^2$ and $y^2 = 40^2 + x^2$. When combining both equalities, we obtain:

$$30^2 + (50 - x)^2 = 40^2 + x^2 \Leftrightarrow 900 + 2500 - 100x + x^2 = 1600 + x^2 \Leftrightarrow$$

$$\Leftrightarrow 3400 - 1600 = 100x \Leftrightarrow x = 18.$$

The fountain is located at a distance of 18 cubits from the highest tower. The solution to the problem of the birds' flight allows us to obtain additional information. It is now possible to determine the distance covered by each bird. For that purpose, let us find the solution for *y* by substituting $x = 18: y^2 = 40^2 + 18^2 = 1600 + 324 = 1924$. Each bird flies a distance of 1924 cubits.

In this context, as in similar circumstances, the notation $A \Leftrightarrow B \Leftrightarrow C \Leftrightarrow D$ was used, which is usually used by high school teachers. It should be noted that this is an ambiguous abbreviation for what should be written $(A \Leftrightarrow B) \wedge (B \Leftrightarrow C) \wedge (C \Leftrightarrow D)$ or even $A \wedge (A \Leftrightarrow B) \wedge (B \Leftrightarrow C) \wedge (C \Leftrightarrow D)$. This formula is true if and only if *A*, *B*, *C*, *D* are all true. Consider the truth tables *(Table 112 / Table 113)* where 0 means false and 1 means true.

(Table 112)

A	B	$\sim A$	$\sim B$	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
0	0	1	1	0	0	1	1
0	1	1	0	0	1	1	0
1	0	0	1	0	1	0	0
1	1	0	0	1	1	1	1

(Table 113)

A		B		C		D		$(A \Leftrightarrow B) \Leftrightarrow C$		$A \Leftrightarrow (B \Leftrightarrow C)$		$(A \Leftrightarrow B \Leftrightarrow C) \Leftrightarrow D$		$A \wedge (A \Leftrightarrow B \Leftrightarrow C \Leftrightarrow D)$
0		0		0		0		0		0		1		0
0		0		0		1		0		0		0		0
0		0		1		0		1		1		0		0
0		0		1		1		1		1		1		0
0		1		0		0		1		1		0		0
0		1		0		1		1		1		1		0
0		1		1		0		0		0		1		0
0		1		1		1		0		0		0		0
1		0		0		0		1		1		0		0
1		0		0		1		1		1		1		1
1		0		1		0		0		0		1		1
1		0		1		1		0		0		0		0
1		1		0		0		0		0		1		1
1		1		0		1		0		0		0		0
1		1		1		0		1		1		0		0
1		1		1		1		1		1		1		1

From the table of truth we can observe that $A \wedge (A \Leftrightarrow B \Leftrightarrow C \Leftrightarrow D)$ is true, for example, if A, C are true, but B, D are false. We notice that $(A \Leftrightarrow B) \Leftrightarrow C$ is equivalent to $A \Leftrightarrow (B \Leftrightarrow C)$, that is, $\Leftrightarrow$ is an associative connective and we can dispense with parenthesis.

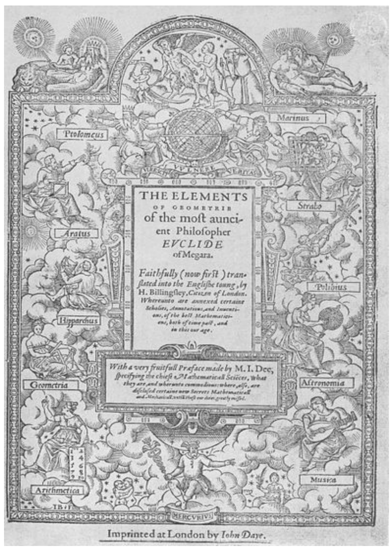
The next truth table *(Table 114)* shows that the second or third column is what the high school teacher wishes to express.

(Table 114)

$(A \Leftrightarrow B) \wedge (B \Leftrightarrow C)$		$(A \Leftrightarrow B) \wedge (B \Leftrightarrow C) \wedge (C \Leftrightarrow D)$		$A \wedge (A \Leftrightarrow B) \wedge (B \Leftrightarrow C) \wedge (C \Leftrightarrow D)$
1		1		0
1		0		0
0		0		0
0		0		0
0		0		0
0		0		0
0		0		0
0		0		0
0		0		0
0		0		0
0		0		0
0		0		0
0		0		0
0		0		0
1		0		0
1		1		1

In the problem of the birds' flight, among other procedures, the development of the square of a difference was used. The reasoning contributes to an effective teaching/learning process. Once again, the connection between contents can be explored with resources of History of Mathematics.

Elements, a treatise consisting of 13 books written and organised by Euclid, includes definitions, axioms, theorems and demonstrations. The rigour and methodology of this work dated from 300 BC are a source of inspiration for whoever wants to learn Geometry, but also Arithmetic and Algebra *(Figure 39)*. In fact, the arrangement *theorem* followed by *demonstration* used by modern manuals was already present in the work of the Greek mathematician.



(Figure 39) - The Frontispiece of Sir Henry Billingsley's first English version of Euclid's Elements (1570)

For the classroom we propose the following:

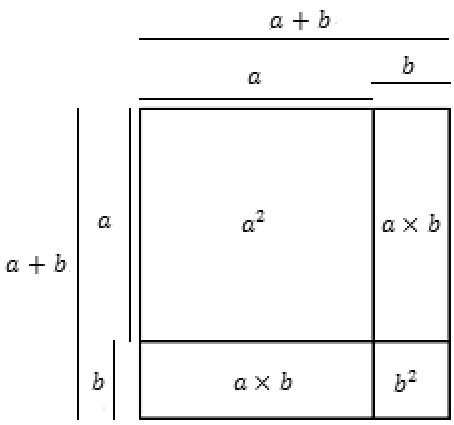
(Subject: Geometry / Algebra – 8th grade students)

Demonstrate the equality: $(a+b)^2 = a^2 + 2 \times a \times b + b^2$.

Euclid writes:

If a straight line is randomly cut, the square of the whole equals the squares of the segments plus two times the rectangle contained by the segments.

Proposition IV – Book II



Development of a square of a sum:
$(a+b)^2 = a^2 + 2 \times a \times b + b^2$

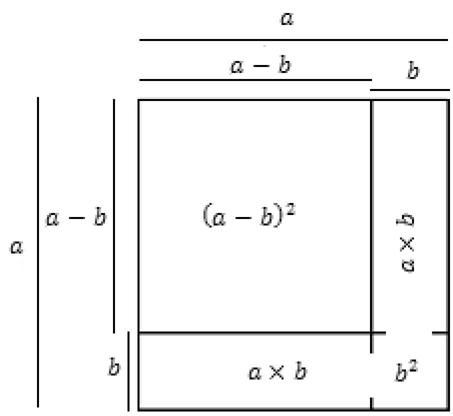
(Subject: Geometry / Algebra – 8th grade students)

Demonstrate the equality: $(a-b)^2 = a^2 - 2 \times a \times b + b^2$.

The goal is to prove the development of the square of a difference.

$$(a-b)^2 = a^2 - 2 \times a \times b + b^2$$

Let us consider the introduced alterations to the figure, which are associated to the development of the square of a sum. Take a square whose side measures a. Subtract b from the line segment which measures a, hence creating two squares with areas $(a-b)^2$ and b^2, and two geometrically equal rectangles with areas $a \times b$.



The area of the square whose side measures (a - b) is equal to the difference between the initial square a^2 and two times the area of the rectangles. However, the area of the smaller square b^2 must be added, as it had previously been wrongfully subtracted twice. Therefore, $(a-b)^2 = a^2 - 2 \times a \times b + b^2$, as we wanted to demonstrate.

Let us now observe the method used by Fibonacci for the flight of the two birds.

If the highest tower is at a distance of 10 cubits from the fountain, 10 multiplied by 10 is 100, which when added to the measure of the highest tower multiplied by itself, 1600, is 1700.

The remaining distance should then be multiplied by itself, and added to 900, which is 2500. The difference of the sums is 800 (2500 – 1700). Thus, the fountain is located more than 10 cubits away from the highest tower. Let us suppose that this tower is 5 cubits farther, in a total of 15 cubits, which when multiplied by itself, is 225. This last value added to the squared length of the highest tower is 1825. The distance from the fountain to the smaller tower multiplied by itself and to which the length of the smaller tower is added is 2125. The difference between the two sums is 300 (2125 – 1825).

When 5 cubits were added, the distance was reduced by 500. By multiplying 5 by 300 and dividing the result by 500, the result is 3. When this value is added to the actual 15 cubits, we obtain a total of 18 cubits, which is the distance from the fountain to the highest tower.

Interpreting and translating the process described by Fibonacci to mathematical language is an activity which can be performed in the classroom, individually or in small groups. The analysis of the solving strategy highlights the application of the Pythagoras' theorem.

$$10^2 + 40^2 = 100 + 1600 = 1700$$

$$(50 - 10)^2 + 30^2 = 40^2 + 30^2 = 1600 + 900 = 2500$$

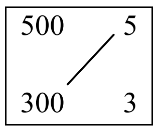
Fibonacci wrote that *the difference of the sums is 800 (2500 – 1700).*

$$15^2 + 40^2 = 225 + 1600 = 1825$$

$$35^2 + 30^2 = 1225 + 900 = 2125$$

Fibonacci wrote that *the two sums differ by 300 (2125 – 1700).*

Fibonacci used a diagram (on the right) to perform the operations; $(5 \times 300): 500 = 3$ and 3 + 15 = 18.

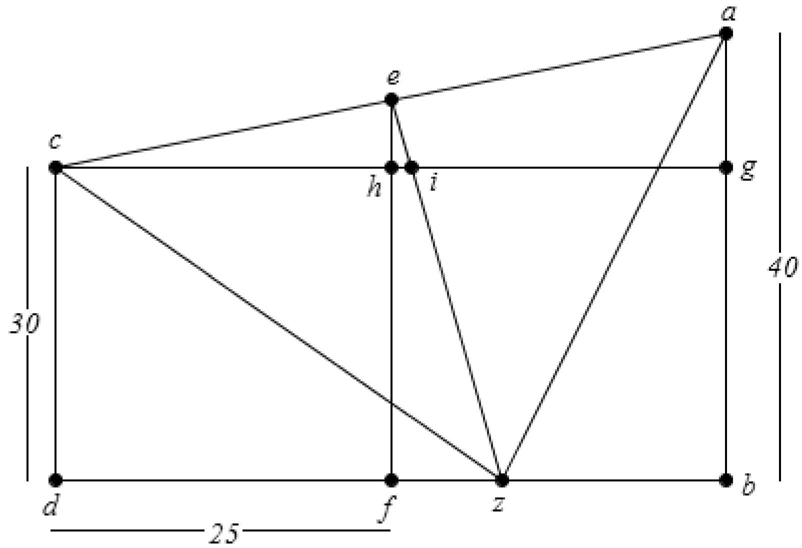


5 cubits added reduce 500,
3 (more cubits) added reduce more 300.

We should note that the correctness of Fibonacci's strategy is closely connected to the fact that the problem turns out to solve in *x* and thus to be linear in *x*.

Exploring different methodologies enables you to work with a wider variety of mathematical tools. Fibonacci used the *Method of the False Position*, a problem solving strategy that goes back to Ancient Egypt. When comparing the two solving processes, it is clear that the conversion of the statement of the problem to an algebraic expression significantly simplifies the procedures to attain the solution.

Fibonacci also considered a strategy based in geometric principles *(Figure 40)*.

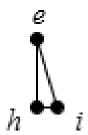


(Figure 40)

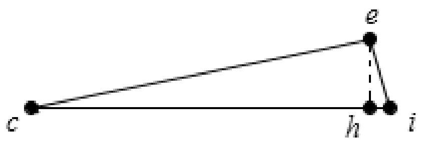
Explain why *acz* (*z* is the location of the fountain) is an isosceles triangle (*az* = *cz*), with *ae* = *ec*.

40 and 30 are 70; the half is 35, at the line ef. The lines df and fb measure 25, the difference between 35 and the smaller tower is 5, which when multiplied by 35 is 175, which divided by half of the distance between the towers, 25, is 7 (line fz). Hence, dz is 32 and the remaining, 18, is the measure of line zb.

Prove that *ae* = *ec*. Consider the point *g*, which results from the intersection of the line *ab* with the parallel line to *db* which passes through *c*. As the triangles *acg* and *ech* are similar, we can say that the line *eh* measures 5. By applying the Pythagoras' theorem to both right triangles we estimate that the line *ac* measures $\sqrt{2600}$ and, similarly, the line *ec* measures $\frac{1}{2}\sqrt{2600} = \sqrt{650}$, the same as the line *ae*.



$$\overline{ei}^2 = \overline{hi}^2 + 5^2$$

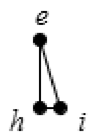


$$\overline{ci}^2 = \overline{\imath}^2 + \overline{ce}^2 \Leftrightarrow \left(\overline{ch} + \overline{hi}\right)^2 = \overline{ei}^2 + \overline{ce}^2$$

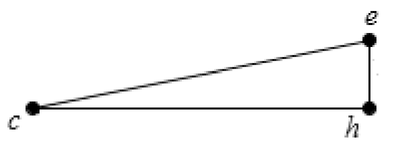
$$\Leftrightarrow \left(25 + \overline{hi}\right)^2 = \overline{ei}^2 + \overline{ce}^2$$

$$\Leftrightarrow 625 + 50\overline{hi} + \overline{hi}^2 = \overline{hi}^2 + 5^2 + 650$$

$$\Leftrightarrow 50\overline{hi} = 50 \Leftrightarrow \overline{hi} = 1$$



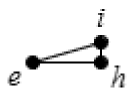
$$\overline{ei}^2 = \overline{hi}^2 + 5^2 \Leftrightarrow \overline{ei} = \sqrt{26}$$



$$\overline{ce} = \sqrt{650}$$

$$\overline{ch} = 25$$

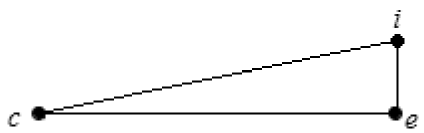
$$\overline{eh} = 5$$



$$\overline{ei} = \sqrt{26}$$

$$\overline{eh} = 5$$

$$\overline{hi} = 1$$



$$\overline{ci} = 26$$

$$\overline{ce} = \sqrt{650}$$

$$\overline{ei} = \sqrt{26}$$

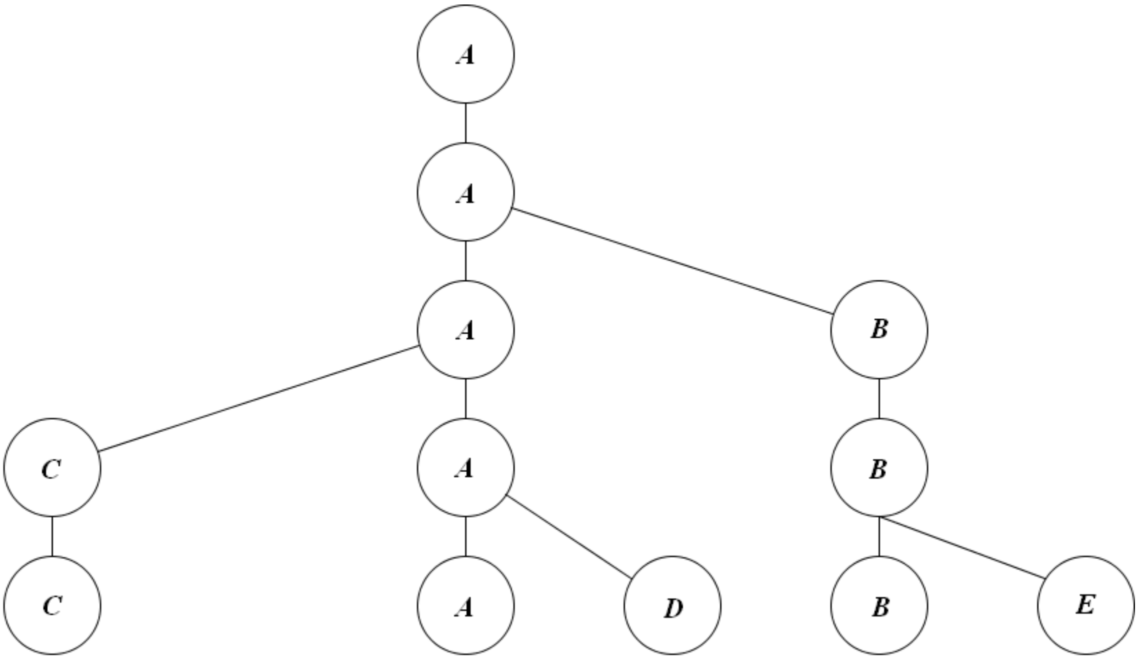
Consider the point *i* which results from the intersection of line *cg* with *ez*. Considering that triangles *cei*, *ehi* and *che* are similar enables you to conclude that triangles *che* and *efz* are similar. By triangle similarity we obtain $\frac{5}{25} = \frac{zf}{35}$. Therefore, the line *zf* measures 7. The distance from point *d* to point *z* is 32 (25 + 7) and the distance from point *b* to point *z* is 18 (25 – 7).

Problem solving activity and discussion of the problem are didactic strategies which should be implemented in the classroom as often as possible.

Let us see another of Fibonacci's famous problems.

A man puts a pair of baby rabbits into an enclosed field. Assuming that each pair of rabbits generates a new pair every month, which, from the second month on, produces a new pair, how many pairs of rabbits will there be in the field at the beginning of each month?

Let us build a diagram for the beginning of each month where each pair of rabbits is represented by a circumference identified by a capital letter *(Figure 41)*.



(Figure 41)

Pair A, the one put *into the enclosed field on 1st January, conceives a new pair, identified as B, which is born on 1st March. At that moment, there are two pairs of rabbits in the field. The 1st and the 2nd terms of the sequence are 1. Each of the following terms is the result of the addition of the two terms which precede it, thus originating a numeric sequence which identifies the number of rabbit pairs on the 1st day of each month*:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …

$$F = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & \cdots \end{pmatrix}$$

An explanation for the above claims is as fallows.

Definition: The Fibonacci sequence is a numerical sequence which may be recursively written: $F_n = F_{n-1} + F_{n-2}$, $n \in \mathbb{N}$, for $n \geq 3$, with the initial conditions $F_1 = F_2 = 1$.

Being $F_k := \#\, pairs\ on\ the\ 1st\ day\ of\ month\ k, k \geq 3$, therefore,

$$\begin{aligned} F_k &= \#\, pairs\ on\ the\ 1st\ day\ of\ month\ (k-1) + \#\, pairs\ born\ on\ the\ 1st\ day\ of\ month\ \ k \\ &= \#\, pairs\ on\ the\ 1st\ day\ of\ month\ (k-1) + \#\, pairs\ on\ the\ 1st\ day\ of\ month (k-2) \\ &= F_{k-1} + F_{k-2}. \end{aligned}$$

The first known additional record regarding this recursive relation was written by Albert Girard (1595 – 1632). Since then, results obtained from the Fibonacci sequence have been many. In 1876, François Anatole Lucas (1842 – 1891) proved that the sum of the first *n* terms of the sequence is equal to the order *n+2* term minus 1.

Theorem: The sum of the first *n* terms of the Fibonacci sequence is equal to the order term *n+2* minus one:

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1.$$

Demonstration:

As the Fibonacci sequence, $F_n = F_{n-1} + F_{n-2}$, *i.e.*, $F_{n-2} = F_n - F_{n-1}$, $n \geq 3$.

Being $k = n - 2$, the previous relation can be written as $F_k = F_{k+2} - F_{k+1}, k \geq 1$.

Hence, the sum of the first *n* terms of the Fibonacci sequence is equal to:

$$\sum_{k=1}^{n} F_k = \sum_{k=1}^{n} (F_{k+2} - F_{k+1}) = \sum_{k=1}^{n} (-F_{k+1} + F_{k+2}) =$$

$$= (-F_2 + F_3) + (-F_3 + F_4) + (-F_4 + F_5) + \cdots + (-F_n + F_{n+1}) + (-F_{n+1} + F_{n+2}) =$$

$$= -F_2 + (F_3 - F_3) + (F_4 - F_4) + \ldots + (F_{n+1} - F_{n+1}) + F_{n+2}$$

$$= F_{n+2} - F_2 = F_{n+2} - 1.$$

Q. E. D.

The astronomer Giovanni Domenico Cassini (1625 – 1712) demonstrated that the square of any given term differs from the product of the consecutive terms by 1 or -1.

Theorem: For each $n \geq 2, {F_n}^2 = F_{n-1}F_{n+1} - (-1)^n$

Demonstration:

Proof by induction on *n*.

For $n = 2$, we have: ${F_2}^2 = 1$ e $F_1F_3 - (-1)^2 = 2 - 1 = 1$.
Therefore, the formula is valid for $n = 2$.

Let us now suppose that the formula is valid for a whole positive number, *k*:

$$F_{k-1}F_{k+1} - F_k^2 = (-1)^k.$$

Let us prove that it is also valid for $k + 1$, instead of *k*.

$$F_kF_{k+2} - F_{k+1}^2 = (F_{k+1} - F_{k-1})(F_k + F_{k+1}) - F_{k+1}^2 =$$

$$= F_kF_{k+1} + F_{k+1}^2 - F_{k-1}F_k - F_{k-1}F_{k+1} - F_{k+1}^2 = F_kF_{k+1} - F_{k-1}F_k - F_{k-1}F_{k+1}.$$

By induction hypothesis, this latter expression is

$$F_kF_{k+1} - F_{k-1}F_k - F_k^2 - (-1)^k = F_kF_{k+1} - F_k(F_{k-1} + F_k) + (-1)^1(-1)^k =$$

$$= F_kF_{k+1} - F_kF_{k+1} + (-1)^{k+1} = (-1)^{k+1}.$$

Q. E. D.

Theorem (Zeckendorf, 1972): Every natural number has a unique representation as a finite sum of distinct, non-consecutive Fibonacci numbers.

Demonstration:

Proof of existence will be carried out by induction on a natural number *n*.

Note that $1 = F_1$, $2 = F_3$ and $3 = F_1 + F_3$.

Supposing that the theorem is true for every natural number inferior to F_n, each natural number 1, 2, 3, 4 … $F_n - 1$, may be represented as a finite sum of distinct, non-consecutive Fibonacci numbers of the set $\{F_1, F_2, F_3, \dots, F_{n-1}\}$.

Let us prove that the result is true for all natural numbers inferior to F_{n+1}.

Let *k* be a natural number between F_n and F_{n+1}: $F_n \leq k < F_{n+1}$.

If $k > F_n$, there is $r > 0$ such that $k = F_n + r$, which means that, $r = k - F_n < F_{n+1} - F_n = F_{n-1}$, which, in turn, is equivalent to writing that $r < F_{n-1}$.

By the induction hypothesis *r* may be represented as the sum of distinct non-consecutive Fibonacci numbers belonging to the set $\{F_1, F_2, F_3, \dots, F_{n-2}\}$. Hence, $k = F_n + r$ may be represented as the sum of distinct, non-consecutive Fibonacci numbers of the set $\{F_1, F_2, F_3, \dots, F_n\}$.

Uniqueness of decomposition remains to be proven, and for that the following lemma is necessary:

Lemma: The sum of the elements of any non-empty set *S* of distinct, non-consecutive Fibonacci numbers whose largest member is F_j is strictly less than F_{j+1}.

Demonstration (on the cardinality of S):

Being S $=\{F_j\}$, for any $j \in \mathbb{N}$, then *Sum S* = $F_j < F_{j+1}$.

Suppose that the lemma is true for S \ $\{F_j\}$, whose largest member is F_i, with $i < j - 1$. Then, *Sum S* = *Sum* (*S* \ $\{F_j\}$) + $F_j < F_{i+1} + F_j \leq F_{j-1} + F_j = F_{j+1}$.

Q. E. D.

In order to prove the uniqueness of decomposition of a natural number provided by the Zeckendorf's Theorem, let us consider two sets, *S* and *T*, of distinct, non-consecutive elements of the Fibonacci sequence whose sum has the same value. When common elements to *S* and *T* are eliminated, sets S' and T' are obtained.

We want to demonstrate that S' and T' are empty sets, *i.e.*, that *S* is equal to *T*.

Let us first assume that they are both non-empty sets, $S' \neq \emptyset \wedge T' \neq \emptyset$.

Assume F_s is the largest element of set S' and F_t the largest element of set T'. Suppose without loss of generality that $F_s < F_t$. According to the previously mentioned lemma, the sum of the elements of S' is strictly lesser than F_{s+1}. Being F_{s+1} the Fibonacci number immediately following F_s, then F_{s+1} is lesser or equal to F_t. Hence, the sum of the elements of set S' is inferior to F_t.

As F_t belongs to T', the sum of the elements of T' is greater or equal to F_t. It so happens that the sum of the elements of S' and of T' is not equal, and the same happens to the sum of the elements of *S* and of *T*. The assumption that S' and T' are both non-empty sets is wrong.

Let us now suppose that S' is empty and T' is not empty. Then *S* is a sub-set of *T* and, therefore, the sum of the elements of *S* and the sum of the elements of *T* cannot be equal.

The same thing can be said for $S' \neq \emptyset$ and $T' = \emptyset$.

If S' and T' are both empty sets, then *S* is equal to *T*, for *S* and *T* have the same elements.

Hence, every natural number may be uniquely represented as a finite sum of distinct, non-consecutive Fibonacci numbers.

Q. E. D.

François Anatole Lucas (1842 – 1891) proves that by transforming Pascal's Triangle into a right triangle, the sum of the numbers in the diagonals coincides with the terms of the Fibonacci sequence.

n		$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	
0	F_1	1							
1	F_2	1	1						
2	F_3	1	2	1					
3	F_4	1	3	3	**1**				
4	F_5	1	4	**6**	4	1			
5	F_6	1	**5**	10	10	5	1		
6	F_7	**1**	6	15	20	15	6	1	
7	F_8	1	7	21	35	35	21	7	1
…	…				...				

The previous table reads as follows: F_n is the sum of the elements of the diagonal whose origin is in the left lower corner, beginning on the line where F_n is located. For instance, the sum of the elements of the diagonal which are in bold is 13, *i.e.*, F_7.

For example,

$$21 = F_8 = \binom{7}{0} + \binom{6}{1} + \binom{5}{2} + \binom{4}{3} = 1 + 6 + 10 + 4 = 21. \textit{ (1)}$$

$$13 = F_7 = \binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3} = 1 + 5 + 6 + 1 = 13. \textit{ (2)}$$

Seeing that, by Pascal's Triangle property,

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

we have,

$$\binom{6}{0} + \binom{6}{1} = \binom{7}{1}, \binom{5}{1} + \binom{5}{2} = \binom{6}{2} \text{ and } \binom{4}{2} + \binom{4}{3} = \binom{5}{3}. \textit{ (3)}$$

Then, using ***(1)*** and ***(2),*** and through the substitutions taken from ***(3)*** we obtain

$$F_9 = F_8 + F_7 = \binom{7}{0} + \binom{7}{1} + \binom{6}{2} + \binom{5}{3} + \binom{3}{3} = \binom{8}{0} + \binom{7}{1} + \binom{6}{2} + \binom{5}{3} + \binom{4}{4}.$$

Theorem (Lucas):

For every natural $n \geq 0$, if $j = \left\lfloor \frac{n}{2} \right\rfloor$, then,

$$F_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \ldots + \binom{n-j}{j} = \sum_{j=0}^{n} \binom{n-j}{j}.$$

Demonstration:

Let us use induction on *n* to establish proof.

For *n* = 0 we have,

$$F_{0+1} = \binom{0}{0} = 1,$$

$$\sum_{j=0}^{0} \binom{0-j}{j} = \binom{0}{0} = 1,$$

which means that the result is true.

Let us now suppose that

$$F_{n+1} = \sum_{j=0}^{n} \binom{n-j}{j}$$

It is proved for F_{n-1} and F_n, instead of F_{n+1}. We then have

$$F_{n+1} = F_n + F_{n-1} = \sum_{j=0}^{n-1} \binom{(n-1)-j}{j} + \sum_{j=0}^{n-2} \binom{(n-2)-j}{j}$$

$$= \binom{n-1}{0} + \sum_{j=1}^{n-1} \binom{n-1-j}{j} + \sum_{j=0}^{n-2} \binom{n-2-j}{j} = 1 + \sum_{j=1}^{n-1} \binom{n-1-j}{j} + \sum_{j=1}^{n-1} \binom{n-1-j}{j-1}$$

$$\left(\begin{array}{c} Note\ that\ \binom{n-2-j}{j} = \binom{(n-1)-(j+1)}{(j+1)-1} \\ and\ if\ j\ is\ between\ 0\ and\ n-2, then\ (j+1)\ is\ between\ 1\ a\ n-1 \end{array} \right)$$

$$= 1 + \sum_{j=1}^{n-1} \left(\binom{n-1-j}{j} + \binom{n-1-j}{j-1} \right) = 1 + \sum_{j=1}^{n-1} \binom{n-j}{j} = \sum_{j=0}^{n} \binom{n-j}{j}.$$

Q. E. D.

The Fibonacci sequence enables you to explore and relate apparently distinct contents, thus becoming an adequate didactic tool to implement a holistic approach to problem solving in Mathematics.

Theorem: If a line segment is divided in two unequal parts such that the ratio of the whole to the longer part is equal to the ratio of the longer part to the shorter part, then this ratio is the golden ratio. Its value is commonly represented by ϕ (*Phi*[52]).

Demonstration: According to the concept of the golden section, *x* should satisfy the relations pointed out by the following chain of equivalences where, without loss of generality, the value 1 is attributed to the larger part:

$$\frac{1+x}{1} = \frac{1}{x} \Leftrightarrow (1+x)x = 1 \Leftrightarrow x^2 + x - 1 = 0 \Leftrightarrow x = \frac{-1+\sqrt{5}}{2} \vee x = \frac{-1-\sqrt{5}}{2}.$$

The negative value is ignored, for *x* represents a length. Therefore, $\phi = \frac{1+x}{1} = \frac{1+\sqrt{5}}{2}$.

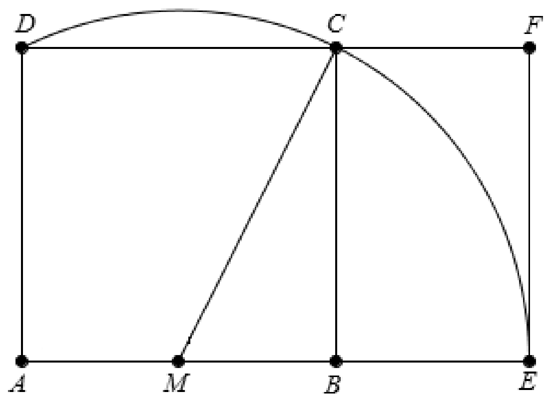
Q. E. D.

[52] *Phi* is an irrational number, the g*olden number*, named after the sculptor *Phideas*.

A rectangle is a golden rectangle when the ratio of the length of the longer side to the shorter side is equal to ϕ. In order to build a figure which matches this property, a square [ABCD] with a side length of 1 unit should be drawn. Consider the middle point M of side AB. Then build a circumference whose centre is M and which contains $[AB]$. Point E is the intersection point of the arc of the circumference with the line which contains $[AB]$. Finally draw a line $[EF]$ vertical to the line which contains $[AB]$ with a length of 1 unit.

(Subject: Geometry – 8th grade students / 9th grade students)

Proposition: The construction $[ADFE]$ is a golden rectangle *(Figure 42)*.



(Figure 42)

Demonstration:

In this geometric construction we can see that:

$\overline{AB} = 1$; $\overline{AD} = 1$; $\overline{BE} = x$ (the lesser part of the division). $\overline{MB} = \frac{1}{2}$; $\overline{BC} = 1$.

$[MBC]$ is a right triangle. When applying the Pythagoras' theorem, we get: $\overline{MC} = \frac{\sqrt{5}}{2}$.

Hence, $\overline{AE} = \overline{AM} + \overline{ME} = \frac{1}{2} + \frac{\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}$, which means that, $\frac{\overline{AE}}{\overline{AD}} = \frac{1+\sqrt{5}}{2}$.

So the sides of the rectangle $[ADFE]$ are in the golden ratio.

Q. E. D.

In order to fulfil its purpose, the teaching/learning process of Mathematics requires competence and perseverance from the participants. These are two of the main variables which contribute to educational attainment, a goal which can only be achieved after an extended period of time. The understanding and assimilation of subtle relations do not occur in an immediate manner, but rather gradually in a learning-friendly environment.

Let us see another example of the subtle relations which are part of Mathematics. The sequence defined by the ratio of each term of the Fibonacci sequence to the preceding term converges to a limit. The value of this limit is an irrational number, curiously enough, the *golden number*.

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \dots, \frac{F_{n+1}}{F_n}, \dots$$

From theory to practice: a continuous exercise

Success in problem solving depends on the selection of adequate strategies which may belong to specific domains of knowledge. High jump, for example, is technically very demanding. How to jump as high as possible without knocking the bar to the ground? In 1896, during the Athens Olympic Games, the North-American Ellery Harding Clark (1874 – 1949) was, among all the athletes, the one who achieved the best result with a 1.81*m* jump. The method used in the early days, known as the scissors jump, consisted in a diagonal approach to the bar, starting the jump with the elevation of one leg and then of the rest of the body. During the 20th century new techniques emerged, but they all had the same goal: to jump as high as possible without knocking the bar to the ground. Nowadays, the athletes use the *flop* or the *Fosbury Flop*, a technique made popular by Dick Fosbury, who amazed the world during the Olympic Games in Mexico, in 1968, when he jumped over the bar on his back, with his face turned to the sky, clearing 2.24*m* and winning the competition. Other athletes gradually adopted the innovation. When combined with increasingly refined mental and physical training, this technique challenges the Law of Gravitation, as the lemma *Citius, Altius, Fortius*[53].

Schoenfeld (1983) compares the mathematical activity to the practice of sport, where the players involve themselves mentally and physically and do not just sit back passively watching the course of events. The analogy can be extended to the teacher, who should play the role of coach so as to enhance students' performances. The coach does not only verbalise a technique or show how a procedure should be performed. Similarly to the teacher in the classroom and in addition to more or less detailed theoretical explanations, other stages are implemented with the aim to maximise the potential of each individual. The player is told to practise what has been transmitted under the observation of the coach. When deemed appropriate the coach interrupts the procedure to correct faulty performances or to point to a more effective exercise. As well as the teacher in the classroom, the purpose of the coach is to foster intelligent decision making which may lead to success. Let us invoke Achilles, a warrior of the Greek mythology, in a footrace with a tortoise. Confident of victory because he is ten times faster than the tortoise, Achilles allows the tortoise a head start of one hundred cubits. Zeno of Elea (495 BC – 430 BC) suggests that when the warrior reaches the starting point of the tortoise, the tortoise would be ten cubits farther, a distance thereafter shortened to one cubit, a tenth of the cubit and so on. Therefore, Achilles can never reach the tortoise. Nevertheless, sports' practice, and mathematical reasoning, demonstrates that the paradox of Achilles and the Tortoise is, in fact, a deceptive theoretical reasoning.

[53] *Faster, higher, stronger.*

Consider the formula to sum all the terms of a geometric progression with initial term u_1 and ratio r with $|r| < 1$, $S = \frac{u_1}{1-r}$. The athlete continuously covers the distances 100; 10; 1; 0.1; ... The sequence is a geometric progression with ratio 0.1.

$$S_{athlete} = \frac{100}{1-0{,}1} = \frac{100}{0{,}9} = 111.(1).$$

The tortoise continuously covers the distances 10; 1; 0.1; 0.01; ... The sequence is a geometric progression with ratio 0.1.

$$s_{tortoise} = \frac{10}{1-0.1} = \frac{10}{0.9} = 11.(1).$$

The athlete reaches the tortoise after running 111. (1) cubits. Even though the solver may not have theoretical knowledge regarding geometric progressions, the problem can still be solved by using less sophisticated techniques.

(Subject: Algebra – Third Cycle students)

Achilles, a Greek hero of Homer's *Iliad*, competes in the racing track with a tortoise. The athlete, who runs ten times faster than the tortoise, allows the tortoise a head start of 100 cubits. What distance does the tortoise cover until it is reached by Achilles?

The goal is to calculate the distance travelled by the tortoise until it is reached by Achilles, knowing that Achilles is ten times faster and starts the race with a disadvantage of 100 cubits. While translating the statement of the problem to mathematical language, a relation can be established between the distances covered by both Achilles and the tortoise: $d_{Achilles} = d_{tortoise} + 100$. A relation between Achilles' speed and the speed of the tortoise can also be established: $v_{Achilles} = 10 \times v_{tortoise}$.

Now let us search for a relation between the distance covered by Achilles and the distance covered by the tortoise. Considering that $speed = \frac{distance\ covered}{time}$ and that the time necessary for Achilles to reach the tortoise is equal for both, then $d_{Achilles} = 10 \times d_{tortoise}$.

Given the previous relations, we have:

$$d_{Achilles} = 10 \times d_{tortoise};\ d_{Achilles} = d_{tortoise} + 100,$$

$$10 \times d_{tortoise} = d_{tortoise} + 100 \Leftrightarrow 9 \times d_{tortoise} = 100 \Leftrightarrow$$

$$\Leftrightarrow d_{tortoise} = \frac{100}{9} \Leftrightarrow d_{tortoise} = 11.(1).$$

The tortoise covers a little over 11 cubits until it is reached by Achilles.

(Subject: Algebra – Third Cycle students)

A two-hour walk consisted of a route with a flat part followed by a climbing part, and then the return through the same trail (first the descending part and then the flat part). The speed was 4 *km/h* in the flat part, 3 *km/h* during the climb and 6 *km/h* in the descent. What was the total distance covered?

A formula relates two or more variables. Identifying the most adequate of the formula(s) is a significant step to find a solution to a given problem. Information regarding the walk and the speed in each part of the route (flat, climb and descent) are provided in the statement of the problem. The goal is to determine the total distance covered during the walk.

The total time taken in the walk is known (2 hours).

According to the statement of the problem, when the relation is translated to mathematical language *f*, c, d, f represent the distances covered during the route: flat, climb, descent, flat, respectively.

Consider the formula which relates Time, Speed and Distance:

$$time\ (h) = \frac{distance\ covered\ (km)}{Speed(km/h)}.$$

The solution to the problem is obtained by the equality $2 = \frac{f}{4} + \frac{c}{3} + \frac{d}{6} + \frac{f}{4}$. When this relation is simplified, we have $6f + 4c + 2d = 24 \Leftrightarrow 3f + 2c + d = 12$. According to the statement, the distances c and d are equal. Hence, $3f + 2c + d = 12 \Leftrightarrow 3f + 2c + c = 12 \Leftrightarrow 3f + 3c = 12 \Leftrightarrow f + c = 4$.

Knowing that the two parts of the path are done via the same route and therefore have the same length, the total distance is $2 \times (f + c)$, *i.e.*, $8km$.

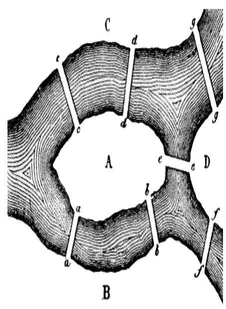
Problems with distances and routes are recurrent in the teaching of Mathematics because they permit the exploration of practical situations which are easily identifiable by the students and simultaneously have the potential to promote solving strategies. The *Problem of the Bridges of Königsberg* is included in this group of questions. The inhabitants of *Königsberg* wondered whether it might be possible to establish a route which would cross all the seven bridges of their city once and only once. They tried several possibilities but could not find a way to do it, thus considering it an impossible task. In 1736, Leonard Euler (1707 – 1783) proved that the inhabitants' suspicions were right.

We engage on a branch of Mathematics called the Graph Theory; the following definitions and facts are critical to understand the *Problem of the Bridges of Königsberg*.

- A graph is a pair (V, A), where V represents a set of points (vertices) and A represents a set of lines (edges) which connect some of the pairs of points of set V.
- If one or more vertices of a graph have an odd degree, then an Euler circuit cannot be established. An Euler circuit is a circuit that uses every edge of a graph exactly once.
- If the graph is connected (if any two of its vertices are connected by a walk) and all the vertices have an even degree, then there is at least one Euler circuit (a circuit which visits every edge of a connected graph once and only once, and starts and ends on the same vertex).
- If the graph has more than two odd-degree vertices, then an Euler path cannot be established. An Euler path uses all the edges of a connected graph once and only once.
- If a connected graph has exactly two odd-degree vertices, then at least one Euler path is possible. Whatever the walk, it starts on an odd-degree vertex and ends on the other odd-degree vertex.
- The degree of a vertex equals the number of edges incident with it.
- The sum of the degrees of all the vertices equals twice the number of edges.
- The number of odd-degree vertices ought to be even.

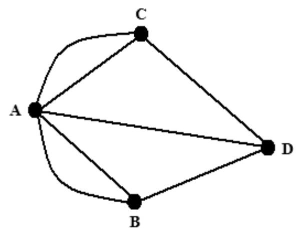
Let us now go back to the *Problem of the Bridges of Königsberg*. There are no restrictions regarding the starting point. Several routes were tried but none fulfilled the requirements of the problem. The hypothesis that there is no solution for this problem becomes therefore more consistent. What would the justification for that possibility be? A graph was sketched from an 18th century map. Only odd-degree vertices were found. Hence, an Euler path cannot be established, so crossing only once all the bridges of Königsberg is unfeasible.

The city of Königsberg prospered on both sides of River Pregel, a watercourse with two small islands *(Figure 43)*. To ensure greater mobility, seven bridges were built. The population wondered whether it would be possible to walk through all the different bridges of the city without repeating any. In the 18th century, the local authorities invited the best mathematicians of the Kingdom of Prussia (in 1945 Königsberg was integrated in Russia with the name of Kaliningrad) to study the enigma and find the solution. In 1736 Leonard Euler cleared the mystery of the S*even Bridges of Königsberg*. Study the problem and unravel the secret!



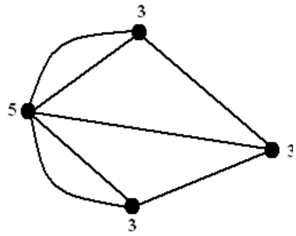
(Figure 43)

The *Problem of the Bridges of Königsberg* may be represented in a diagram where the north bank, south bank and islands are points and each of the bridges a line segment *(Figure 44)*.



(Figure 44)

The diagram helps to identify the number of bridges which are connected to the north bank, south bank and islands *(Figure 45)*.



(Figure 45)

In order to be able to come and go, the place has to be connected to an even number of bridges. If all the points are connected to an even number of line segments, the walk can start and end at the same spot (circuit). If all points but two are connected to an even number of line segments, the route is feasible (path) if it starts and ends at the two points with an odd number of connections.

Three bridges converge to the north bank, south bank and east island, and five bridges converge to the west island. A circuit is not possible for it would require that all the points be associated to an even number of bridges. A path is also not possible for all the points except the start and the finish would have to be connected to an even number of bridges.

We can then conclude that the problem does not have a solution. However, if any one of the bridges is taken out, the problem would be solvable. The justification for this case would be based on the previous reasoning.

Draw each figure without lifting your pencil from the paper and without tracing any line more than once.

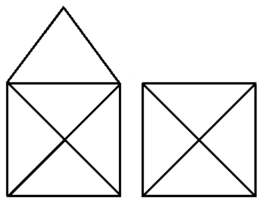


Figure A *Figure B*

In the classroom: Students, the goal is to draw figures A and B without lifting the pencil from the paper and without repeating the route.

Figure A: The task can be successfully carried out by using the method of "*trial and error*".

Figure B: Can the figure be drawn if the conditions of the problem are respected?

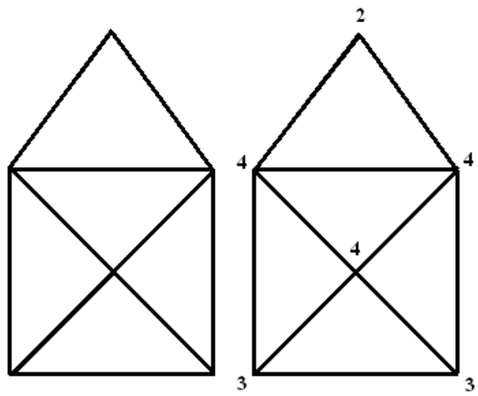
What happens when you try to draw the figures?

We have a starting point and a finish point, which may or may not be coincident.

Consider that the starting point coincides with the finish point. What would be the implications for the problem? If during the drawing process the pencil does not return to the point, it will only be visited twice. Otherwise the arrival and the departure of the pencil will increase the number of times the point is visited by two units. This reasoning can be extended to the remaining points of the figure. All points have an even value.

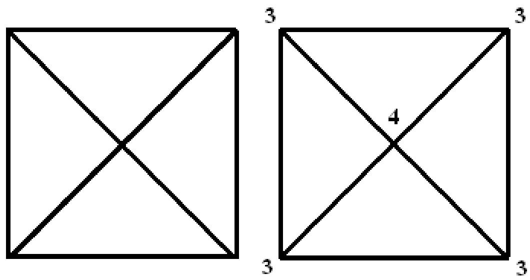
Let us now consider that the finish point is not coincident with the starting point. What would the implications to the problem be? Independently of the number of times that the pencil will return to the starting point, this point will be associated with an odd value (one, if the pencil does not go back, plus the two units for each time the pencil goes back). The same reasoning may be applied to the finish point. Regarding the remaining points, the value is always even due to the arrival and departure of the pencil.

Two *vertices* have odd degrees and the remaining have even degrees. The construction starts in an odd vertex and ends in the other odd vertex.



(Figure A)

One *vertex* has an even degree and the others have odd degrees.



(Figure B)

Figure A can be drawn according to the restrictions of the problem, but the same does not happen with *Figure B*.

In Mathematics, problems with similar statements do not necessarily require identical solving processes. The belief that a recipe may be repeatedly applied to apparently similar questions prevents the students from imagining solving strategies, thus being stuck in exercise solving.

Let us now look at the *Problem of the Travelling Salesman*. A salesman ought to visit different cities once and only once and as economically as possible. Then he has to return to the starting point. The aim is to set the best route, known as the Hamilton Circuit as a tribute to William Rowan Hamilton (1805 – 1865).

A solving methodology, the *Brute Force Algorithm*, is a systematic analysis of a problem by abiding to the following stages: *1)* list all possible Hamilton circuits; *2)* find the weight of each circuit; *3)* choose the one(s) with the *smallest weight*, *i.e.*, smallest cost. The application of this methodology using only paper and pen is compromised by the increase of the number of cities to be visited. The amount of calculations associated to the brute force algorithm may be astronomical, thus making this an unfeasible task, even with a computer. The impossibility of achieving the optimal answer in a suitable time range requires the use of other algorithms. Even if these may not produce the optimal solution, they offer a good solution in a reasonable time.

The *Closest Neighbour Algorithm* emerges as a solving strategy which may produce the desired answer according to the stages: *1)* identify the vertex which corresponds to the starting point; *2)* follow the *closest neighbour* vertex whose connection/edge is the least valuable. If more than one is possible, choose randomly; *3)* follow the circuit, a vertex at a time, always towards the *closest neighbour* vertex of the set of vertices which have not been visited yet. If more than one is possible, choose randomly; *4)* when the last vertex is reached return to the starting point.

(Subject: Numbers and Operations – 7th grade students)

A travelling salesman who lives in Coimbra has to visit, professionally, the cities of Aveiro, Lisbon and Oporto, and then return to his hometown. What should he do?

	Coimbra	Aveiro	Lisbon	Oporto
Coimbra		58	196	117
Aveiro			244	68
Lisbon				313
Oporto				

Distances between the cities (in kilometres)

The goal is to leave Coimbra, visit once and only once the cities of Aveiro, Lisbon and Oporto, and then return to the city of origin, by covering the minimum possible distance. Let us identify the routes that the salesman may choose to complete the travel *(Table 115)*.

(Table 115) - Brute Force Algorithm

ROUTE	DISTANCE COVERED (*kms*)
COIMBRA-AVEIRO-LISBON-OPORTO-COIMBRA	58 + 244 + 313 + 117 = 732
COIMBRA-AVEIRO-OPORTO-LISBON-COIMBRA	58 + 68 + 313 + 196 = 635
COIMBRA-LISBON-AVEIRO-OPORTO-COIMBRA	**196 + 244 + 68 + 117 = 625**
COIMBRA-LISBON-OPORTO-AVEIRO-COIMBRA	196 + 313 + 68 + 58 = 635
COIMBRA-OPORTO-LISBON-AVEIRO-COIMBRA	117 + 313 + 244 + 58 = 732
COIMBRA-OPORTO-AVEIRO-LISBON-COIMBRA	**117 + 68 + 244 + 196 = 625**

The route COIMBRA – LISBON – AVEIRO – OPORTO – COIMBRA and the respective mirror-route COIMBRA – OPORTO – AVEIRO – LISBOA – COIMBRA are the best choice.

Let us look at the solution provided by the closest neighbour algorithm. The salesman is in Coimbra and goes to Aveiro, the nearest city, located at 58*kms*. Then he has two options: Lisbon, at a distance of 244*kms*, and Oporto, which is only 68*kms* away. He decides for Oporto. After finishing his business in this city, he must go to Lisbon and then he finally returns to Coimbra. In total, he covers 635*kms*. The choice for an apparently effective strategy does not necessarily mean that it is the best solution. In this case, the small number of cities allows the use of the *Brute Force Algorithm* for there are only six circuits to be considered.

For a total of *n* cities, starting the journey in a specific location and returning to the same place at the end, the number of circuits would be $(n - 1)!$[54]

(Subject: Numbers and Operations – 12th grade students)
A travelling salesman ought to visit the cities A, B, C, D, E, F, G, H, I and J only once. For professional reasons, he ought to visit A before E, and E before I. He also has to visit B before C. How many routes are possible?

We know that there are 10! = 3,628,800 possibilities to order the letters A, B, C, D, E, F G, H, I, J. Taking into account the restrictions of the problem, there are 12 (3! 2!) possible ways to sequentially combine the cities A, E and I with the cities B and C in the ten-city route. The sequence A - E - I / B – C is the only one to respect the conditions of the statement of the problem. We have the following twelve possibilities:

A - E - I / B - C	A - E - I / C - B
A - I - E / B - C	A - I - E / C - B
E - A - I / B - C	E - A - I / C - B
E - I - A / B - C	E - I - A / C - B
I - A - E / B - C	I - A - E / C - B
I - E - A / B - C	I - E - A / C - B

Hence, from the 3,628,800 possible unrestricted routes $\left(10! = {}^{10}A_3 \times {}^{7}A_2 \times 5!\right)$, which correspond to the staging of three cities (A, E, I), two cities (B, C) and the remaining five cities, only $\frac{1}{12}$ of this number actually represents a possible route.

[54] $n! = 1 \times 2 \times 3 \times \ldots \times (n - 1) \times n$ (where *n* is a positive whole number).

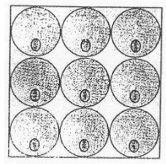
Let us highlight the capacity of Mathematics to establish liaisons with different areas of knowledge. Even though a problem may be labelled as belonging to a theoretical or a practical domain, it is the merging of the two which makes them theoretical-practical problems. Why would students want to know the approximate value of *pi* if they do not understand its meaning? Convenient problem solving which emphasises the applicability of *pi* (π) contributes to an illustration of the practical nature of Mathematics. This perception, however, is often absent from the classroom where the teacher asks their students to open the manual on page *x* and solve exercise *y* on the board. Students must not be confined to a passive role where they only copy the contents written on the board by the teacher. In the classroom, a methodology which promotes a more proactive behaviour from students must be implemented and communication optimised.

Small changes which cause big alterations

Tinned food is commonly put in cylindrical tin cans, but the same does not happen whit other goods. Packaging is done under criteria which take into account the comfort of the client, among other factors.

(Subject: Geometry – Third Cycle students)

Nine cylindrical packages were stored in a box as represented in the picture. Find the relation between the occupied value and the total volume of the box?



The goal is to determine the ratio between the volume of the packages and the total volume of the box. The question is illustrated with a picture, a square whose side measures 6 × the radius of the package, with 9 geometrically equal circles inside a square. The reader is now questioned about the area occupied by the 9 circles and the area of the square.

Without loss of generality, this problem about volumes was converted into a problem about areas, for the ratio that we want to determine does not depend on the height of the box. Let us apply the calculation formula for the area of the circle: we multiply the obtained result by 9. The area of the square may now be calculated. Then the value resulting from the division of the area of the 9 circles by the area of the square is calculated. By multiplying the value of the ratio between the area of the packages and the area of the square by 100, the value of the occupied volume in percentage is obtained. The radius of the cylindrical package does not condition the result, as the value of the ratio is a constant. The solution, the ratio between the volume occupied by the packages and the volume of the box, $\frac{9\pi r^2}{(6r)^2} = \frac{\pi}{4}\,(\cong 0.785)$, shows the importance of the number *pi* in a practical situation. Here *r* is the radius.

The teacher will possibly be questioned about the calculation process of *pi*. If that does not happen, the teacher could take the initiative of questioning the students and present a procedure that goes back to Ancient Egypt.

The River Nile provided the ideal conditions for the advent of agriculture, thus settling nomadic peoples. The Egyptian civilisation erected sumptuous buildings, invented an elaborate communication system, the hieroglyphic writing, established a complex system of social hierarchy and provided eternal rest for their most famous dead. Aware of the importance of Mathematics, the Egyptians applied it to everyday problem solving. Egyptian hieroglyphs were carved on stone or painted in ceramics or papyrus. The latter was produced from the fibres of rushes which grew by the banks of the Nile.

The scribe Ahmes copied a document, originally written between 2000BC and 1800BC. Also known as the Rhind Papyrus, this manuscript is a tribute to the lawyer and antiquities dealer Alexander Henry Rhind (1833 – 1863), who bought it in Egypt in the middle of the 19th century. The document contains problems which aimed at teaching to solve practical problems, namely the calculation of the area of a circular field with a diameter *d* of 12 units of length.

$Area_{circle} = \left(d - \frac{1}{9}d\right)^2$ was used to determine this value. When replacing *d* by the value provided by the problem statement, we obtain $\left(12 - \frac{12}{9}\right)^2 = \left(\frac{96}{9}\right)^2 = \frac{9216}{81} \cong 113.78$.

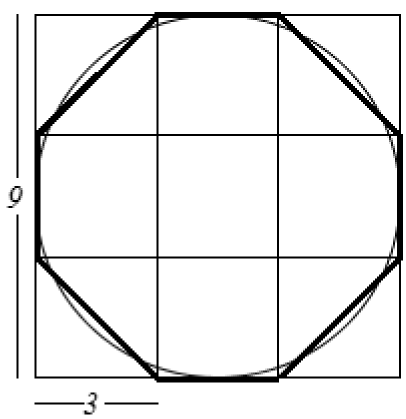
At a cursory glance, the Egyptian formula seems to be significantly different from the one that is taught today in our classrooms: $Area_{circle} = \pi r^2$. Let us take a detailed look at the process used in Ancient Egypt.

$$Area_{circle} = \left(d - \frac{1}{9}d\right)^2 = \left(\frac{8d}{9}\right)^2 = \frac{64}{81}d^2 = \frac{64}{81}(2r)^2 = \frac{256}{81}r^2 \cong 3.16r^2$$

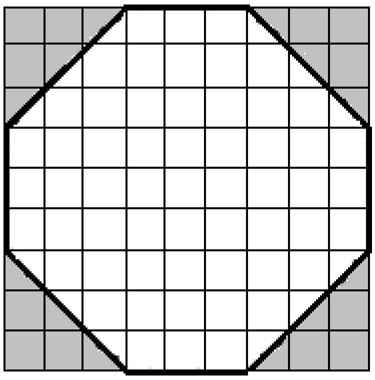
The formula uses a approximate value to the value of *pi*, an irrational number which comes from the division of the perimeter of a circumference by its diameter. The Ahmes papyrus explains that:

The scribe tries to build a square with an area which is equal to the area of the circle. The figure suggests that the area of the circle is approximately equal to the area of the octagon.

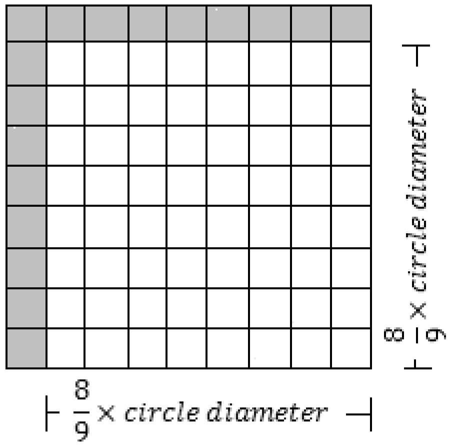
We see that the area of the octagon (*Figure 46* and *Figure 47*) corresponds to $\frac{5}{9} + 4 \times \frac{1}{2\times 9} = \frac{7}{9}$ of the area of the square, *i.e.*, $\frac{7}{9} \times 9^2 = \frac{7}{9} \times 81 = 63$ (Katz, 1998, pp. 20-21).



(Figure 46)



(Figure 47)



(Figure 48)

The area of the square whose side measures $\frac{8}{9}$ of the original square is 64 *(Figure 48)*. This value is very close to 63, thus corresponding to approximately the area of the octagon and, simultaneously, the area of the circle. This formula has a minor error which does not compromise the result when it is applied to the majority of daily life problems.

Theoretical mathematics accuracy is always a virtue. The student may be asked about the expression $\sqrt{2}^2$ and required to compare the expected value with the result produced by the scientific calculator, by introducing $\sqrt{2}$ and squaring the obtained value. Many students show amazement at the result on the display for it is not 2, which is the expected value. The rounding of a number may have an innocuous effect or, conversely, trigger compromising effects, as Edward Lorenz (1917 – 2008) discovered in 1961, when he was working with weather forecast models which implied the use of mathematical formulas and values for temperature, wind and air masses (climatic variables). When repeating a simulation using a computer, Lorenz interrupted the process to round a value, replacing 0.506127 by 0.506. The difference was minimal and had the advantage of producing results with smaller time intervals, because at the beginning of the 1960s, computers did not possess the same calculation capacity that they have today. Results were stunning. The two simulations should have generated similar results, but reality proved to be quite different. When comparing the values of the lists based on the processing of data, an increasing difference was observed in the outcomes. By that time, slide rules, instruments whose accuracy was limited to one, two or, at the most, three decimal places, were common, which led Lorenz to introduce the value 0.506 instead of 0.506127 without anticipating the consequences of such rounding. In 1979 Edward Lorenz developed the idea that an apparently insignificant action may have enormous repercussions, a concept known as the *Butterfly Effect* that illustrates the foundations of the chaos theory.

(Subject: Numbers and Operations – 8th grade students)

Demonstrate 0.(9) = 1.

Mathematical notation tell us that 0.(9) is a decimal number with an infinite number of digits, with period 9. Let us write $x = 0.999$ ...

We can also write $10x = 9.999$ ... Then, $10x - x = 9 \Leftrightarrow 9x = 9 \Leftrightarrow x = 1$.

Q. E. D.

Improving the teaching and learning of Mathematics is an objective whose success depends not only on the implementation of structural measures but also, and equally important, on good practices in the classroom. Understanding and being a proactive agent in this micro cosmos contributes to triggering the *Butterfly Effect*. Efficient and continuous pedagogical actions are key contributors to the future education of students. Students may be asked to carry out two experiments: enter the value $r = 0.99$ in the calculator and successively apply the function $q(x) = x^2$, *i.e.*, repeatedly calculate $q(r)$, $q(q(r))$, $q(q(q(r)))$ … Then, enter the value $r = 1.01$ in the calculator and repeat the procedure. The behaviour of the values resulting from the experiments is an example of the phenomenon observed by Edward Lorenz.

Regarding temperature forecast, a reference to the Celsius scale as defined by Anders Celsius (1701 – 1744) is suggested for 3rd Cycle students. The Celsius scale is the standard measure for atmospheric temperature, even though some countries, namely the United States of America, use the Fahrenheit unit, a scale proposed by Gabriel Fahrenheit (1686 – 1736). The goal is to establish a conversion table between the two temperature units, by only knowing that 0º*C* (temperature at which water passes from liquid to solid state) corresponds to 32º*F* and that 100º*C* (temperature at which water passes from liquid to gas state) corresponds to 212º*F*. The drawing of a Cartesian referential is advised, where the horizontal axis is identified by the letter *C* (Celsius) and the vertical axis by the letter *F* (Fahrenheit), and where the points P_1 ز (0, 32) and P_2 ز (100, 212) are marked.

Two points allow to define a line whose reduced equation is $y = mx + b$, being *m* the value of the slope of the line, and *b* the value of the ordinate at its origin. Consider that $\frac{y_2 - y_1}{x_2 - x_1}$ is the value of the slope of the line, built from P_1 ز (x_1, y_1) and P_2 ز (x_2, y_2), $m = \frac{212-32}{100-0} = \frac{9}{5}$.

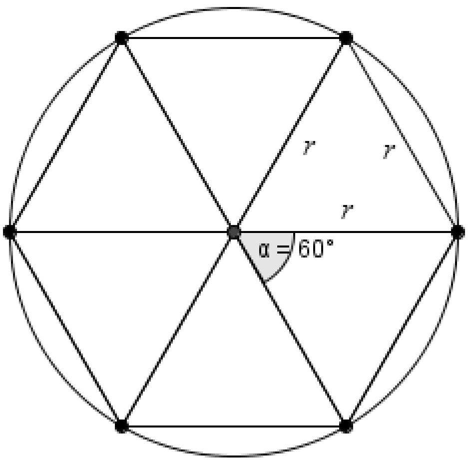
In order to identify the ordinate at its origin, let us use one of the known points, P_2 , for example, $212 = \frac{9}{5} \times 100 + b \Leftrightarrow 212 = \frac{900}{5} + b \Leftrightarrow 212 = 180 + b \Leftrightarrow 32 = b$. In this problem, point P_1 , which has abscissa 0, immediately reveals the value of *b*. Thus, the reduced equation of the line which passes through points P_1 and P_2 is: $y = \frac{9}{5}x + 32$. If the variables *x* and *y* are replaced by *C* and *F*, respectively, we get $F = \frac{9}{5}C + 32$.

(Subject: Geometry – 8th grade students)

In Antiquity, the Babylonians used the formula $A = \frac{1}{12}P^2$ to calculate the area of a circle, where *P* represents the perimeter of the circle. For practice purposes, this means assigning the value 3 to the number π, for $P = 2\pi r$ and $\pi r^2 = \frac{1}{12}(2\pi r)^2 \Leftrightarrow \pi = 3$.

In the 20th century, archaeologists found a text inscribed in clay tablets with a relation between the perimeter P_6 of a regular hexagon and the perimeter *P* of the circumference which circumscribes it: $P_6 = 0;57{,}36 \times P$. What is the value assigned to π?

The goal is to determine the value of π. Consider the relation between perimeter P_6 of a regular hexagon and perimeter *P* of the circumference circumscribed to that hexagon ($P_6 = 0;57{,}36 \times P$, in sexagesimal notation). The geometrical construction *(Figure 49)* allows visualising that the hexagon is composed of six equilateral triangles. The letter *r* represents the measure of the side of each equilateral triangle. The perimeter of the hexagon is 6*r*. The perimeter of the circumference is $2\pi r$.



(Figure 49)

Hence,

$$P_6 = 0;57{,}36 \times P \Leftrightarrow$$

$$\Leftrightarrow 6 \times r = 0;57{,}36 \times 2 \times \pi \times r \Leftrightarrow \frac{6 \times r}{0;57{,}36 \times 2 \times r} = \pi \Leftrightarrow$$

$$\Leftrightarrow \frac{3}{0;57{,}36} = \pi \Leftrightarrow \frac{3}{\frac{57}{60} + \frac{36}{60^2}} = \pi \Leftrightarrow \frac{10800}{3456} = \pi \Leftrightarrow \pi = 3.125.$$

Dexterity in the use of mathematical tools is acquired after intense methodical training, because a tool is only effective when its function is understood and the techniques are mastered. In 1990, Philippe Starch created a 14*cm* diameter and 25*cm* height object made of cast aluminium *(Figure 50)* which became a designer piece.

What was it made for?

(Figure 50)

Assimilating the functioning of a tool requires a higher level of dexterity when compared to simply handling the tool. Students should be encouraged to question and question themselves about the contents taught. Stating the solving formula for *quadratic equations*[55] without exploring the deduction process seems to be a rather simplistic approach. Learning by heart and applying a rule to routine exercises contributes to strengthening the belief that Mathematics is like a recipe book which can be used but does not need to be understood.

Apparently, the following chain of correspondences demonstrates the general formula for second degree equations:

$$ax^2 + bx + c = 0 \Leftrightarrow ax^2 + bx = -c \Leftrightarrow \frac{ax^2 + bx}{a} = -\frac{c}{a} \Leftrightarrow x^2 + \frac{b}{a}x = -\frac{c}{a} \Leftrightarrow$$

$$\Leftrightarrow x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} \Leftrightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \Leftrightarrow$$

$$\Leftrightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \Leftrightarrow x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \Leftrightarrow$$

$$\Leftrightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This deduction deserves a critical analysis to stimulate the students' critical sense. Should we always accept this reasoning? The first equivalence is universally valid, but the second is only true if $a \neq 0$; assuming this, the next question regards the equivalence that induces the root. Roots of negative numbers do not make sense to 9th grade students'. Therefore, we must assume $b^2 - 4ac \geq 0$. As the square of an expression $-\sqrt{\dots}$ (with meaning) is the same as the square of $\sqrt{\dots}$, in this equivalence we must introduce two signals. What comes after is straighforward. Thus, we can enunciate the final result.

Theorem: Let *a*, *b*, *c* be real numbers whose value is known. If $a \neq 0$ and $\Delta = b^2 - 4ac \geq 0$, then there are two real numbers x_- and x_+ which solve the quadratic equation $ax^2 + bx + c = 0$. These two real numbers are identical only if $\Delta = 0$ and are given by the formulas $x_- = \frac{1}{2a}\left(-b - \sqrt{\Delta}\right)$ and $x_+ = \frac{1}{2a}\left(-b + \sqrt{\Delta}\right)$.

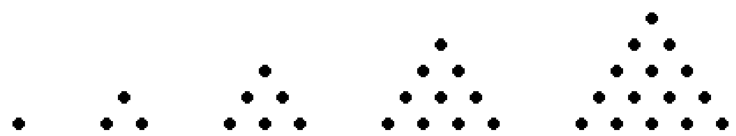
[55] The general form for this type of equation is: $ax^2 + bx + c = 0$, where x is a variable, and *a*, *b* and *c* are constants, and where $a \neq 0$ (otherwise it would be a linear equation). The constants *a*, *b* and *c* are, respectively, the quadratic coefficient, the linear coefficient and the constant coefficient. The variable *x* represents the value to be determined, *i.e.*, the unknown. The word quadratic comes from the Latin *quadratus*, which means square.

The teacher may have to answer the question "What happens when $\Delta < 0$?" from a more curious student. The answer may be: *"We saw that the implications $\Rightarrow$ yield that all potential solution x satisfie*

$$\left(x+\frac{b}{2a}\right)^2=\frac{\Delta}{4a^2}.$$

So, if $\Delta < 0$, and as the square of real numbers is not negative, there is not a real number *x* that satisfies the equation. The step from real numbers to complex numbers was one of the greatest moments in the History of Mathematics. Complex numbers are a 12th grade contents. With them quadratic equations with $\Delta < 0$ become solvable.

Connecting *triangular numbers*[56], numbers a_n in dot sequences *(Figure 51)* that define equilateral triangles of size length *n – 1*, with the number of different handshakes in a group of *n* people may seem an improbable association. However, it constitutes an opportunity to complement the everyday application of the solving formula for quadratic equations.



(Figure 51)

It is evident that, if we add a person to a group of *n – 1* people, this new individual must greet *n – 1* people. We end up with $a_n = 1 + 2 + \cdots + (n-1)$. It is also obvious that if we extend a tiled equilateral triangle of side *n – 2* with an extra layer of points on the right side, in order to get a triangle of side *n – 1*, we will have to add *n – 1* points.

The prior information can be organise for better use *(Table 116)*.

(Table 116) – Triangular numbers / Number of different handshakes in a group of n people

Triangular numbers	Number of different handshakes in a group of *n* people ($n \geq 2$)
1	1 ($n = 2$)
3	3 = 1 + 2 ($n = 3$)
6	6 = 1 + 2 + 3 ($n = 4$)
10	10 = 1 + 2 + 3 + 4 ($n = 5$)
15	15 = 1 + 2 + 3 + 4 + 5 ($n = 6$)
…	…

[56] A triangular number is a natural number which may be represented as an equilateral triangle.

Consider the number of different handshakes, a_n, according to the variable n ($n \geq 2$).

$$a_n = \frac{n \times (n-1)}{2}, n \geq 2.$$

The statement will be proven by induction on n,

$$P(n): \sum_{k=1}^{n-1} k = \binom{n}{2}.$$

The equality is true for $n = 2$.

Supposing that *P(n)* is true, we will prove that *P(n+1)* is also true.

Hence,

$$\sum_{k=1}^{n} k = \sum_{k=1}^{n-1} k + n = \binom{n}{2} + n = \frac{n(n-1)}{2} + \frac{2n}{2} = \frac{n^2+n}{2} = \frac{(n+1)\,n}{2} = \binom{n+1}{2}.$$

In general, for every whole number $r \geq 0$, consider the statements

$$P(n): \sum_{k=1}^{n-1} \binom{k}{r} = \binom{n}{r+1}$$

and

$$P(n+1): \sum_{k=1}^{n} \binom{k}{r} = \binom{n+1}{+1}.$$

Then *P(n)* implies *P(n+1)*. The proof comes from the development

$$\sum_{k=1}^{n} \binom{k}{r} = \sum_{k=1}^{n-1} \binom{k}{r} + \binom{n}{r} = \binom{n}{r+1} + \binom{n}{r} = \binom{n+1}{r+1}.$$

The formula enables you to calculate expressions of type

$$\sum_{k=1}^{n} k^r, r \in \mathbb{Z}^+.$$

For example, we may compute

$$\sum_{k=1}^{n-1} k^2.$$

Knowing that,

$$\sum_{k=1}^{n-1} \binom{k}{2} = \sum_{k=1}^{n-1} \frac{k(k-1)}{2} = \binom{n}{3}, i.e, \sum_{k=1}^{n-1} (k^2 - k) = \sum_{k=1}^{n-1} k^2 - \sum_{k=1}^{n-1} k = 2\binom{n}{3},$$

we get

$$\sum_{k=1}^{n-1} k^2 = 2\binom{n}{3} + \binom{n}{2} = \frac{(n-1) \times n \times (2n-1)}{6}.$$

Now we introduce an example which implies the use of the general formula of second degree equations.

(Subject: Probability and Combinatorics – 12th grade students)

At a party, the son of the hosts wants to know how many guests are in the house. Concluding that the number is too great to be counted without making any mistakes, he chooses a more effective strategy. He knows that the guests salute each other with a toast. From the twinkling of crystal glasses, he counted 66 salutations. By applying the knowledge acquired in his mathematics lessons, the boy intends to find the number of guests that are at the party.

The answer to the problem is the solution to the equation $\binom{n}{2} = 66$, *i.e.*,

$$\frac{n \times (n-1)}{2} = 66 \iff n^2 - n - 132 = 0.$$

After solving this quadratic equation, the only valid solution is *n* = 12.

Algebra allows to codify problem statements, but to use it with effectiveness the individual must also understand what is asked and pinpoint problem restrictions. Algebra is also applied when we solve a straight equation.

(Subject: Algebra – 9th grade students / Secondary Education students)

Solve the equation: $12 \times 3^x - 3^{2x} = 27$.

The difficulty lies in knowing how to transform the equation so that computation is possible. Intuitively, $27 = 3^3$emerges, but $12 \times 3^x - 3^{2x} = 3^3$ is still difficult to solve. Taking into account exponentiation properties, we have,

$$12 \times 3^x - (3^x)^2 = 27.$$

Now the presentation of the equation suggests a change of variable, *i.e.*, $y = 3^x$.

Therefore, we have,

$$12 \times y - y^2 = 27 \iff -y^2 + 12 \times y - 27 = 0.$$

The solution set is $\{3; 9\}$. In conclusion, $3^x = 3 \vee 3^x = 9$.
Thus the solution set for the equation $12 \times 3^x - 3^{2x} = 27$ is $\{1; 2\}$.

In Mathematics, the discovery of solutions is not the only goal of problem solving. Understanding and modelling cognitive mechanisms which are activated during the process are a key component for learning and storing knowledge. The teacher ought to understand those mechanisms, for therein lies a major part of the efficacy of their teaching.

We invite the reader to go further in the Algebra domain, but first let us introduce some easy questions. After that we will merge in the writing of Pedro Nunez.

(Subject: Algebra – 9th grade students / Secondary Education students)

'Here lies Diophantus,' the wonder behold.
Through art algebraic, the stone tells how old:
'God gave him his boyhood one-sixth of his life,
One twelfth more as youth while whiskers grew rife;
And then yet one-seventh ere marriage begun;
In five years there came a bouncing new son.
Alas, the dear child of master and sage
After attaining half the measure of his father's life chill fate took him. After consoling his fate by the science of numbers for four years, he ended his life.'

Let us assign the letter x to the unknown, *i.e.*, Diophantus' age when he died. Information from the epitaph enables us to write the equation:

$$\frac{1}{6}x+\frac{1}{12}x+\frac{1}{7}x+5+\frac{1}{2}x+4=x.$$

Adding fractions with different denominators requires a common denominator, thus, making it necessary to calculate the *least common multiple*[57].

$2 = 2$ $\quad$ $6 = 2\times3$ $\quad$ $7 = 7$ $\quad$ $12 = 2^2\times 3$

$$lcm\,(2, 6, 7, 12) = 2^2\times3\times7=84$$

$$\frac{14}{84}x+\frac{7}{84}x+\frac{12}{84}x+\frac{420}{84}+\frac{42}{84}x+\frac{336}{84}=\frac{84}{84}x$$

The problem comes down to the solving of a linear equation: $756=9x \Leftrightarrow x=84.$

Diophantus lived 84 years.

Subject: Numbers and Operations / Third Cycle students)

a and *b* are 2 natural numbers.
We know that :

- if *a* is multiplied by *b* the product is 882.
- the greatest common divisor (*gcd*) for *a* and *b* is 7.

From the following options, choose the possible values for *a* and *b*.
(A) 7 and 119 (B) 14 and 63 (C) 21 and 42 (D) 18 and 49

57 The least common multiple of two or more integers is equal to the product of the highest power of each of their common and non-common prime factors.

This question can be hard to solve for a 9th grader if the student does not remember or does not complete the equivalence,

$$lcm\ (a,b) \times gcd(a,b) = a \times b.$$

The least common multiple of two numbers is the product of the highest power of each of their common and non-common prime factors.

Now let us think about what is missing in this product when compared with $a \times b$ prime number decomposition. The absent prime factors are those that are common to a and b, *i.e.*, the greatest common divisor of a and b.

By using the previous equality and the problem data $gcd(a,b) = 7$, then $lcm(a,b) = 126$.

Let us analyse option (A) by doing the decomposition of each number in prime factors: $a = 7$ and $b = 7 \times 17$, so $gcd(7{,}119) = 7$ and $lcm(7{,}119)$ =119. Wrong option!

Let us analyse option (B): $a = 2 \times 7$ and $b = 3^2 \times 7$, so $gcd(14{,}63) = 7$ and $lcd(14{,}63) = 3^2 \times 2 \times 7 = 126$. Right option!

Even though it is now useless, let us confirm our option by doing the computations for the remaining options.

Let us analyse option (C): $a = 3 \times 7$ and $b = 2 \times 3 \times 7$, so $gcd(21{,}42) = 3 \times 7 = 21$. Wrong option!

Let us analyse option (D): $a = 2 \times 3 \times 3$ and $b = 7 \times 7$, so $gcd(18{,}49) = 1$. Wrong option!

(Subject: Algebra – 7th grade students)

Grandmother Maria told her grandchildren: "If I bake 2 pies for each, I will still have enough dough for 3 more pies. However, I cannot bake 3 pies for each, for the dough won't be enough for the last 2 pies." How many grandchildren does Grandmother Maria have?

The quantity of pies baked by Grandmother Maria is not known, but information is given which relates that value to the number of grandchildren. Let us then identify the variable x as the number of grandchildren and the variable y as the number of pies which can be baked with the available dough.

If I bake 2 pies for each, I will still have enough dough for 3 more pies: $2x + 3 = y$.

However, I cannot bake 3 pies for each, for the dough won't be enough for the last 2 pies: $3x - 2 = y$.

The answer is obtained by solving the equation $2x + 3 = 3x - 2$.

Grandmother Maria has 5 grandchildren.

(Subject: Algebra – Secondary Education students)

The last digit of a 3-digit number is 2. If the ones digit is moved to the beginning of the number, the initial number is reduced by 36 units. What is the sum of the digits of the initial number?

Consider a 3-digit number whose hundreds and tens digits are unknown but which has a 2 in the ones place value. When the ones digit is moved to the hundreds place value, the number has less 36 units than the initial number. The goal is to calculate the sum of the three digits of the initial number.

The hundreds and the tens digits are unknown. Let us define the hundreds digit as a and the tens digit as b. Then the number represented as $ab2$ has the value $100a + 10b + 2$. When the ones digit is moved to the hundreds place value, the initial number is reduced by 36 units. In terms of representation, this means $ab2 - 2ab = 36$, and we have,

$$ab2 - 2ab = 36 \Leftrightarrow$$
$$\Leftrightarrow (100a + 10b + 2) - (200 + 10a + b) = 36$$
$$\Leftrightarrow 90a + 9b = 234 \Leftrightarrow 9\,(10a + b) = 234 \Leftrightarrow 10a + b = 26.$$

Hence, since $a, b \in \{0, 1, \ldots, 9\}$, $a = 2$, $b = 6$. The initial number is 262. The sum of its digits is 10. The difference between 262 and 226 equals 36, according to the conditions stated in the problem.

Equation solving is an opportunity to use the potential of Algebra in what concerns problem solving. Pedro Nunez (1502 - 1578), on the first pages of *Livro de Álgebra* (1567) gives special attention to the solving of quadratic equations. This book, which took him over three decades to write, was published in Castilian.The mathematician defines Algebra as "*easy and brief calculations to find an unknown quantity, for any purpose in Arithmetic and Geometry, and in any other art which uses calculations and measures, such as Cosmography, Astrology, Architecture and Trade*". His demonstrative discourse uses the *número* (art of expressing any quantity which is composed of units) the *cosa* (x in modern notation) and the *censo* (the square of x, x^2 in modern notation); quantities which, when combined in the equality, originate the following situations:

Simple combinations:

1) Censos equals cosas.

2) Censos equals número.

3) Cosas equals número.

Compound combinations:

4) Censo and cosas equals número.

5) Cosas and número equals censo.

6) Censo and número equals cosas.

Each of these six combinations has its own rule. Let us make a contemporary reading of the text written by Pedro Nunez. The reader must remember that Pedro Nunez is unaware of negative numbers and zero (0), as well as modern symbol notation.

FIRST RULE: Censos equals cosas.

When Pedro Nunez writes *censos equals cosas*, he means $ax^2 = bx$, in modern notation.

Divide the número of cosas by the número of censos.

The result is the value of cosa , $\frac{b}{a} = x$.

Let us consider the example, 4 censos equals 20 cosas: $4x^2 = 20x$.

Divide 20 by 4. The result is 5.

Experience says so because $20 \times 5 = 100$ *and* $5^2 = 25$, $4 \times 25 = 100$.

Since the language of Pedro Nunez has been identified, as of now we will prune his discourse.

SECOND RULE: Censos equals número: $ax^2 = c$.

Divide número by censos. Calculate the root.

The result is the value of cosa, $\sqrt{\frac{c}{a}} = x$.

Let us consider the example: $7x^2 = 63$.

Divide 63 *by* 7*. The result is* 9 *and the value of its root is* 3.

The value of cosa is 3*, of censo is* 9 *and* 7 *censos are* 63*, for* $7 \times 9 = 63$.

THIRD RULE: Cosas equals número: $bx = c$.

Divide número by cosas.

The result of the division is the value of cosa, $\frac{c}{b} = x$.

Let us look at the following example: $10x = 25$.

Divide 25 *by* 10. *The result is* $2\frac{1}{2} = \frac{2\times2+1}{2} = \frac{5}{2}$.

The value of cosa is $2\frac{1}{2}$, *for* $10 \times 2\frac{1}{2} = 10 \times \frac{5}{2} = \frac{50}{2} = 25$.

FOURTH RULE: Censo and cosas equals número: $x^2 + bx = c$.

Multiply half of cosas by itself, $\left(\frac{b}{2}\right) \times \left(\frac{b}{2}\right) = \left(\frac{b}{2}\right)^2$.

Add the value of número, $\left(\frac{b}{2}\right)^2 + c$.

Calculate the square root of, $\sqrt{\left(\frac{b}{2}\right)^2 + c}$.

Subtract half of the value of cosas to the value of the square root, $\sqrt{\left(\frac{b}{2}\right)^2 + c} - \frac{b}{2}$.

Let us consider the example: $x^2 + 10x = 56$.

Multiply half of cosas by itself. The result is $\left(\frac{10}{2}\right)^2 = 25$.

Add the value of número, $25 + 56 = 81$. *Calculate the square root,* $\sqrt{81} = 9$.

Subtract half of the cosas and the result is the value of cosa, $9 - \frac{10}{2} = 9 - 5 = 4$.

The value of censo is 16, *which added to* 40, *the value of* 10 *cosas*, *makes* 56.

FIFTH RULE: Cosas and número equals censo: $bx + c = x^2$.

Multiply half of cosas by itself $\left(\frac{b}{2}\right) \times \left(\frac{b}{2}\right) = \left(\frac{b}{2}\right)^2$.

Add the value of número $\left(\frac{b}{2}\right)^2 + c$. *Calculate the square root* $\sqrt{\left(\frac{b}{2}\right)^2 + c}$.

Add half of the value of cosas to the value of the square root, $\sqrt{\left(\frac{b}{2}\right)^2 + c} + \frac{b}{2}$.

Let us consider the example: $6x + 40 = x^2$.

Multiply half of cosas by itself. The result is $\left(\frac{6}{2}\right)^2 = 9$.

Add the value of número, $9 + 40 = 49$. *Calculate the square root,* $\sqrt{49} = 7$.

Add half of cosas and the result is the value of cosa, $7 + \frac{6}{2} = 7 + 3 = 10$.

This is because 6 *cosas are* 60*, which when added to* 40 *makes* 100*, a value which is equal to the censo of* 10.

SIXTH RULE: Censo and número equals cosas: $x^2 + c = bx$.

Multiply half of cosas by itself, $\left(\frac{b}{2}\right) \times \left(\frac{b}{2}\right) = \left(\frac{b}{2}\right)^2$.

Subtract the value of número, $\left(\frac{b}{2}\right)^2 - c$. *Calculate the square root,* $\sqrt{\left(\frac{b}{2}\right)^2 - c}$.

Add or subtract the previous value to half of cosas, $\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c}$.

Let us consider the example: $x^2 + 24 = 10x$.

Multiply half of cosas by itself. The result is $\left(\frac{10}{2}\right)^2 = 25$.

Subtract the value of número, $25 - 24 = 1$. *Calculate the square root,* $\sqrt{1} = 1$.

Add or subtract the previously found value to half of cosas, 5 ± 1. *Hence, the value of cosa may be* 6 *or* 4. *Experience sais so because if* 6 *is the value of cosa, then the censo of* 6 *added to* 24 *makes* 60*, which is equal to* 10 *cosas. Similarly, given that* 4 *is the value of cosa, the censo of* 4 *added to* 24 *corresponds to* 40*, which is equal to* 10 *cosas.*

PARTE PRIMERA DESTA OBRA.

Cap. primero del fin de la Algebra, y de sus Conjugaciones y Reglas.

EN esta Arte de Algebra el fin que se pretende, es manifestar la quantidad ignota. El medio de que vsamos para alcançar este fin, es ygualdad. Las principales quãtidades a q̃ por discursos demõstratiuos procuramos esta ygualdad, dandoles o quitandoles quanto cõuiene, como quien pone en balança, son tres: Numero, Cosa, Censo.

Numero en esta Arte se dize qualquiera quantidad, quando la entendemos compuesta de vnidades, o sea numero entero, o sea quebrado, o sea Raiz, aun q̃ sea sorda. Como quien dixesse: 8. 9. 10. $\frac{1}{2}$. $\frac{1}{3}$. $\frac{1}{4}$. $8\frac{1}{2}$. $8\frac{1}{3}$. $8\frac{1}{4}$. Raiz de .8. Raiz de .9. Raiz de .10. Raiz de $\frac{1}{2}$. Raiz de $\frac{1}{3}$. Raiz de $\frac{1}{4}$. Raiz de $.8\frac{1}{2}$. Raiz de $.8\frac{1}{3}$. Raiz de $.8\frac{1}{4}$. Cosa se dize la Raiz de qualquier quadrado, y Censo llamamos el quadrado que sale de aquella Raiz.

Estas tres quantidades se pueden conjugar en la ygualdad, que el Arte siempre procura por seis modos. Porque son tres conjugaciones simples, y tres compuestas.

Cõjugaciones simples:
1. Censos yguales a Cosas.
2. Censos yguales a Numero.
3. Cosas yguales a Numero.

Cõjugaciones compuestas:
4. Censo y cosas yguales a numero.
5. Cosas y numero yguales a censo.
6. Censo y numero yguales a cosas.

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A cada vna destas seis conjugaciones responde su regla: de suerte que son seis reglas, tres para las simples, y tres para las compuestas.

Primera Regla: Quando Censos fueren yguales a Cosas, partiremos el numero delas cosas por el numero de los censos, y lo q̃ viniere en la particion sera el valor de la cosa. Exemplo: Pongamos que siendo nos propuesto algun problema para resoluer, y procurando la ygualdad q̃ conuiene, hallamos q̃.4.censos son yguales a.20.cosas. Partiremos por tanto.20.por.4. y vernan.5. por valor dela cosa. E la experiencia assi lo dize: Porq̃.20.multiplicados por.5.q̃ es el valor de la cosa, hazen.100. y porq̃ siendo la cosa.5.el censo es.25.valdran por tanto.100. los.4.censos.

Segunda: Quando censos fueren yguales a numero. Partiremos el numero por los censos, y la raiz de lo que viniere en la particion sera el valor de la cosa. Exemplo; Pongamos que.7.censos son yguales al numero.63. Partimos 63. por.7. y vernan.9. cuya raiz que es.3. sera el valor de la cosa. Y assi es por que siendo el valor de la cosa.3. el censo sera.9. y siete censos valdran.63. porque.7.vezes.9. son.63.

Tercera: Quando las cosas fueren yguales a numero, partiremos el numero por las cosas, y lo que viniere en la particion, sera el valor de la cosa. Exemplo: Pongamos que.10.cosas son yguales a.25. partiremos.25.por.10. y lo q̃ viene que es.2½. sera el valor de la cosa, porque.10.vezes.2½. son los mesmos.25.

Quarta Regla, q̃ es la primera delas cõpuestas: Quando vn censo y las cosas fueren yguales a nu-

numero, multiplicaremos la mitad del numero de las cosas en sy mesma, criando quadrado, y a este quadrado juntaremos el numero ꝓpuesto, y de toda la sũma tomaremos la raiz. De laqual raiz quitaremos la mitad del numero de las cosas, y quedara manifesto el valor de la cosa. Exempla: Pongamos q̃ vn censo y diez cosas son yguales al numero .56. y q̃remos saber el valor de la cosa. Multiplicaremos en si .5. q̃ es la mitad del numero de las cosas, y haremos .25. los quales juntaremos con .56. y haremos .81. La raiz destes .81. es .9. y destes .9. quitaremos .5. que es la mitad del numero de las cosas, y quedaran .4. por valor de la cosa. E ansi es, porq̃ siendo .4. el valor de la cosa, sera el censo .16. los quales juntãdo con .40. q̃ es el valor de .10. cosas, haremos .56. q̃ pusimos ser yguales a .1. censo y .10. cosas.

Quinta Regla, que es la segunda delas compuestas: Quando cosas y numero fueren yguales a vn censo, multiplicaremos en si la mitad del numero de las cosas criando quadrado, y con este quadrado juntaremos el numero, como antes hezimos. Y de toda esta Sũma tomaremos la raix, con la qual juntaremos la mitad del numero delas cosas, y sera la Sũma el valor dela cosa. Exemplo: Pongamos q̃ .6. cosas con el numero 40. son yguales a vn censo. Multiplicaremos en si .3. q̃ es la mitad del numero de las cosas, y haremos .9. estes .9. con .40. hazen .49. cuya raiz q̃ es .7. juntaremos con los .3. y haremos .10. que sera el valor de la cosa. E la experiencia assi lo muestra, porque .6. cosas valen .60. los quales con .40. hazen .100. que es el censo de diez.

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Sexta Regla, q̃ es la tercera de las compueſtas: Quando vn cenſo y el numero fueren yguales a las coſas, multiplicaremos en ſi la mitad del numero de las coſas criando quadrado, del qual quitaremos el numero propueſto, y de lo que quedare tomaremos la Raiz. La qual juntando con la mitad del numero de las coſas, o quitandola ſi quiſieremos, nos dara el valor de la coſa. Exemplo: Pongamos que vn cenſo con el numero .24. ſon yguales a .10. coſas, y queremos ſaber quãto ſea el valor de la coſa. La mitad del numero de las coſas es .5. que multiplicados en ſi, hazen .25, de los quales quitando .24. queda vno, cuya raiz que es .1. juntaremos con .5, y haremos .6. que ſera el valor de la coſa. E podremos tambien ſi nos pluguiere quitar .1. de .5. y quedaran .4. q̃ otro ſi puede ſer valor de la coſa: mas en reſpeto de otro cenſo, y con entrambos los valores de la coſa, quadra el exemplo. E ſi acaeciere q̃ el numero propueſto ſea ygual al quadrado de la mitad del numero de las coſas, en tal caſo eſſa mitad del numero de las coſas ſera el valor de la coſa. Exemplo: 1. cenſo con .9. ſean yguales a .6. coſas. Digo que el valor de la coſa ſera .3. Porque el quadrado de la mitad del numero de las coſas es .9. del qual quitando el numero q̃ otroſi es .9. queda cifra, la qual o quitada de .3. o añadida a los .3. ſiempre reſultan .3.

The roots of Algebra as a branch of Mathematics cannot be found in Europe. They lie in the East, in Babylonia, and go back to the 9th century, to the Islamic civilization, where the concept of equation was born, as well as their classification. It was also there that it was associated with geometric evidence. In Islam, the first great book addressing Algebra, the *Compendious Book on Calculation by al-jabr and al-muqabala*, was written by Abū ‘Abd Allāh Muhammad ibn Mūsā al-Khwārizmī (780 – 850). The expression *al-jabr* has originated the word *algebra*, which means restoration, *i.e.*, removing the negative units from the equation, whether they be numbers, roots or squares, by adding the same quantity to both sides. For example, the equality $x^2 - 2 = x$ is transformed, by *al-jabr*, in $x^2 = x + 2$. The expression *al-muqabala*, which means to compare, is used to balance the equality so that like terms are on the same side of the equation. For example, the equation $x^2 + 14 = x + 5$ is transformed, by *al-muqabala*, in $x^2 + 9 = x$.

The book was dedicated to the Caliph al-Mamum who *“... has encouraged to compose a short work on calculating by al-jabr and al-muqabala, confining it to what is easiest and more useful in arithmetic, such as men constantly require in cases of inheritance, legacies, partition, law-suits and trade, and in all their dealings with one another, or where the measuring of lands, the digging of canals, geometrical computation, and other objects of various sorts and kinds are concerned...* (al-Khwarizmi)*”.*

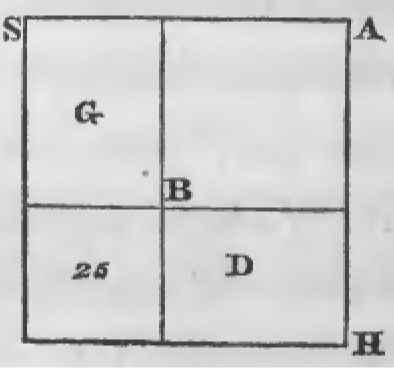
Al-Khwārizmī writes:

*Demonstration of the Case: “a Square and ten Roots are equal to thirty-nine Dirhems.”**

* Geometrical illustration of the case, $x^2 + 10x = 39$

We proceed from the quadrate A B, which represents the square. It is our next business to add to it the ten roots of the same. We halve for this purpose the ten, so that it becomes five, and construct two quadrangles on two sides of the quadrate A B, namely, G and D, the length of each of them being five, as the moiety of the ten roots, whilst the breadth of each is equal to a side of the quadrate A B. Then a quadrate remains opposite the corner of the quadrate A B. This is equal to five multiplied by five: this five being half of the number of the roots which we have added to each of the two sides of the first quadrate. Thus we know that

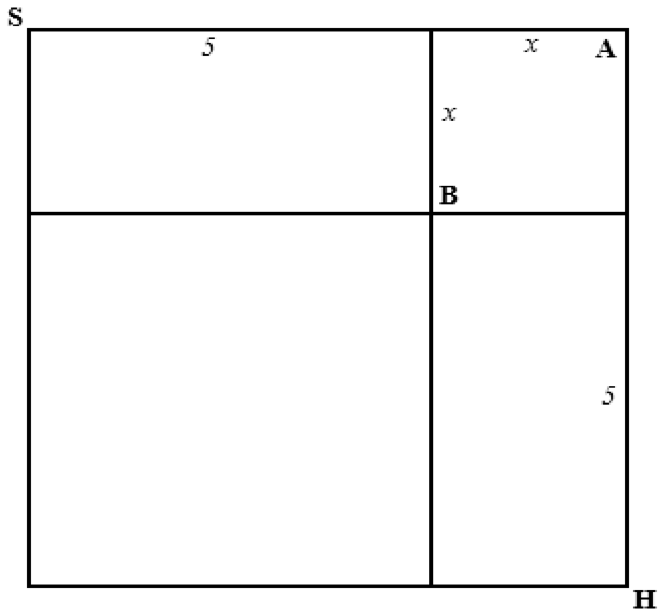
the first quadrate, which is the square, and the two quadrangles on its sides, which are the ten roots, make together thirty-nine. In order to complete the great quadrate, there wants only a square of five multiplied by five, or twenty-five. This we add to thirty-nine, in order to complete the great square S H. The sum is sixty-four. We extract its root, eight, which is one of the sides of the great quadrangle. By subtracting from this the same quantity which we have before added, namely five, we obtain three as the remainder. This is the side of the quadrangle A B, which represents the square; it is the root of this square, and the square itself is nine. This is the figure:—



A square and ten roots are equal to 39 dirhams. What must the square be such that when it is combined with ten of its own roots, it will amount to a total of 39?

Compendious Book on Calculating by al-jabr and al-muqabala

After analysing the text, we can conclude that, in symbol language, the problem matches the equation $x^2 + 10x = 39$. In order to determine solutions to equations of type $x^2 + bx = c$, al-Khwārizmī used Geometry *(Figure 52)*.



(Figure 52)

Two consecutive sides of a square AB with a side x originate two rectangles which, together, make 10 roots. Together, square AB and the two rectangles, make 39. A square 5 multiplied by 5, i.e. 25, is added to the largest square (39) so as to complete the square SH. The sum is 64. The root is extracted in order to obtain the measure of the side of square SH. When the previously added quantity, 5, is subtracted from this value, the side of square AB is obtained. The square is, in itself, 9.

Note that only after the value of *x* is known can this figure be completed. Nevertheless, whatever the measure of *x*, the kind of figure which al-Khwārizmī advocates is known to exist and this knowledge is enough to support his reasoning. Symbolically, according to al-Khwārizmī, the procedure is in line with the justification:

$$x^2 + 10x = 39$$

$$x^2 + 10x + (5 \times 5) = 39 + 25 = 64$$

$$\sqrt{x^2 + 10x + 25} = \sqrt{64}$$

$$x + 5 = 8$$

$$x = 8 - 5 = 3$$

*Demonstration of the Case: "three Roots and four of Simple Numbers are equal to a Square."**

* Geometrical illustration of the 3d case, $x^2 = 3x + 4$

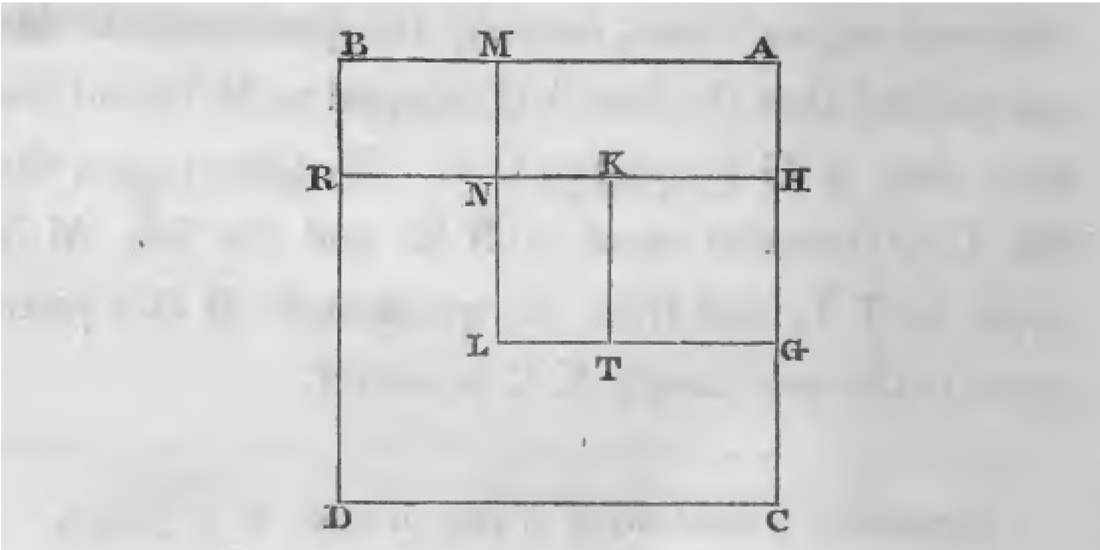
Let the square be represented by a quadrangle, the sides of which are unknown to us, though they are equal among themselves, as also the angles. This is the quadrate A D, which comprises the three roots and the four of numbers mentioned in this instance. In every quadrate one of its sides, multiplied by a unit, is its root. We now cut off the quadrangle H D from the quadrate A D, and take one of its sides H C for three, which is the number of the roots. The same is equal to R D.

It follows, then, that the quadrangle H B represents the four of numbers which are added to the roots. Now we halve the side C H, which is equal to three roots, at the point G; from this division we construct the square H T, which is the product of half the roots (or one and a half) multiplied by themselves, that is to say, two and a quarter. We add then to the line G T a piece equal to the line A H, namely, the piece T L; accordingly the line G L becomes equal to A G, and the line K N equal to T L. Thus a new quadrangle, with equal sides and angles, arises, namely, the quadrangle G M; and we find that the line A G is equal to M L, and the same line A G is equal to G L. By these means the line C G remains equal to N R, and the line M N equal to T L, and from the quadrangle H B a piece equal to the quadrangle K L is cut off.

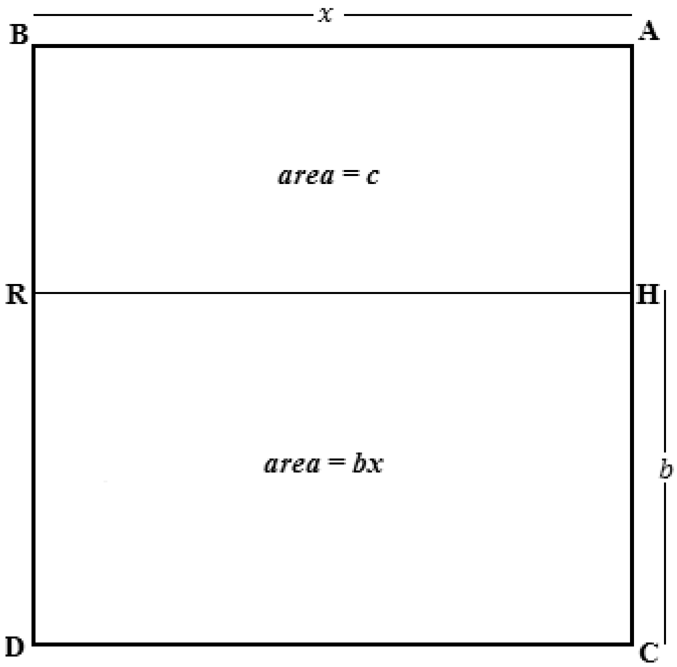
> But we know that the quadrangle A R represents the four of numbers which are added to the three roots. The quadrangle A N and the quadrangle K L are together equal to the quadrangle A R, which represents the four of numbers.

Al-Khwārizmī's writing is devoid of symbols. His algorithms do not contemplate the analysis of negative or zero roots of equations. However, they include irrational roots, which he identifies as *asamm*, *surdus* in Latin. Even though, at the time, Euclid's *Elements* had already been translated to Arabic, al-Khwārizmī's geometric fundaments do not have the accuracy which is characteristic of the Greek mathematician.

> We have seen, also, that the quadrangle G M comprises the product of the moiety of the roots, or of one and a half, multiplied by itself; that is to say two and a quarter, together with the four of numbers, which are represented by the quadrangles A N and K L. There remains now from the side of the great original quadrate A D, which represents the whole square, only the moiety of the roots, that is to say, one and a half, namely, the line G C. If we add this to the line A G, which is the root of the quadrate G M, being equal to two and a half; then this, together with C G, or the moiety of the three roots, namely, one and a half, makes four, which is the line A C, or the root to a square, which is represented by the quadrate A D. Here follows the figure. This it was which we were desirous to explain.



Al-Khwārizmī explains the rule for equations $bx + c = x^2$. *Draw a square whose side measure is x. Divide the square and label the parts as bx and the quantity as c. Then draw a square whose side measure is* $x - \frac{b}{2}$. *Consider that its area is* $\left(\frac{b}{2}\right)^2 + c$. $[See\ the\ dislocated\ piece]$. *Being x the measure of the side of the initial square, it is equal to the sum of* $\left(\frac{b}{2}\right)$ *with the root of the quantity inscribed in the smallest square, i.e.,* $x = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + c}$.



(Figure 53)

Al-Khwārizmī starts with the square AD *(Figure 53)* whose side is *x* and divides it with a line HR in such a way that the areas of the pieces are *c* and *bx*, thus meaning that HC = *b*. The figure will then represent the equation $x^2 = bx + c$.

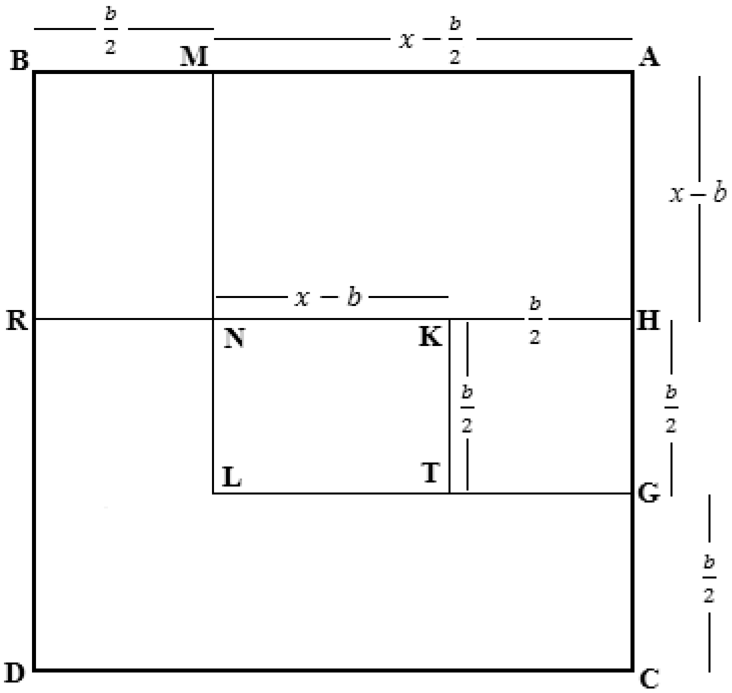
Next, he divides HC in half, thus obtaining G, so as to construct a square HT on HG and a square GM on AG. These constructions introduce points L, T, K and N *(Figure 54)*.

The indicated measures show that

$$|NK| = \left(x - \frac{b}{2}\right) - \left(\frac{b}{2}\right) = x - b = |MN|,$$

$$|NL| = \frac{b}{2} = |MB|.$$

Therefore, NKTL and MNRB are congruent quadrangles.



(Figure 54)

Hence:

$$area\left({}_{L}\square^{A}\right) = area\left({}_{N}\square^{A}\right) + area\left({}_{L}\square^{H}\right)$$

$$= area\left({}_{N}\square^{A}\right) + area\left({}_{L}\square^{K}\right) + area\left({}_{T}\square^{H}\right)$$

$$= area\left({}_{R}\square^{M}\right) + area\left({}_{N}\square^{A}\right) + area\left({}_{T}\square^{H}\right)$$

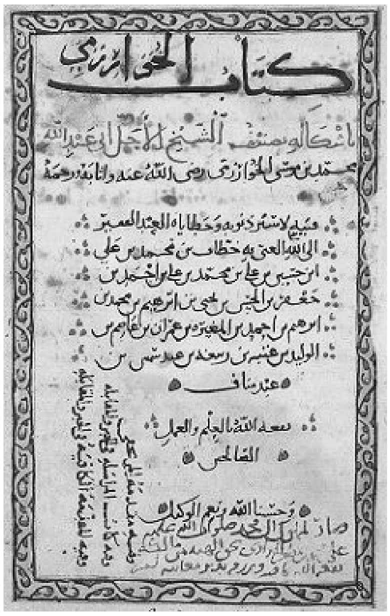
$$= area\left({}_{R}\square^{A}\right) + area\left({}_{T}\square^{H}\right)$$

$$= c + \left(\frac{b}{2}\right)^2.$$

This means that,

$$\left(x - \frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2; \text{ and so, } x = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + c}\ .$$

Al-Khwārizmī's approach to Algebra is different from the one presented by the Babylonian texts dated from 1800 BC to 1600 BC. The legacy of Babylonian mathematicians, printed in clay tablets, encompasses the study of fractions, the property of right triangles which will later on be named as the Pythagoras' theorem, and quadratic and cubic equations. The display of the contents is also different from Diophantus' *Arithmetic*. For al-Khwārizmī, solving concrete problems is not a main concern. His focus is to comprehensively present procedures based on general principles which, when combined, allow to solve different types of equations *(Figure 55)*.



(Figure 55)
Algebra (page of the book)

*Demonstration of the Case: " a Square and twenty-one Dirhems are equal to ten Roots."**

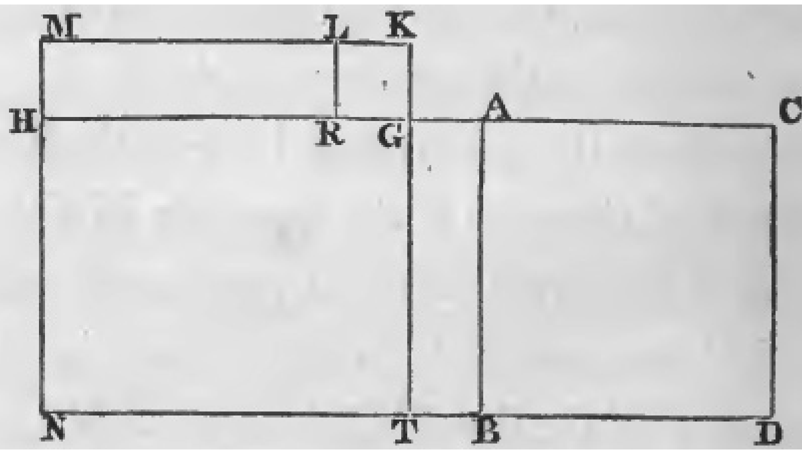
We represent the square by a quadrate A D, the length of whose side we do not know. To this we join a parallelogram, the breadth of which is equal to one of the sides of the quadrate A D, such as the side H N. This paralellogram is H B. The length of the two

* Geometrical illustration of the case, $x^2 + 21 = 10x$

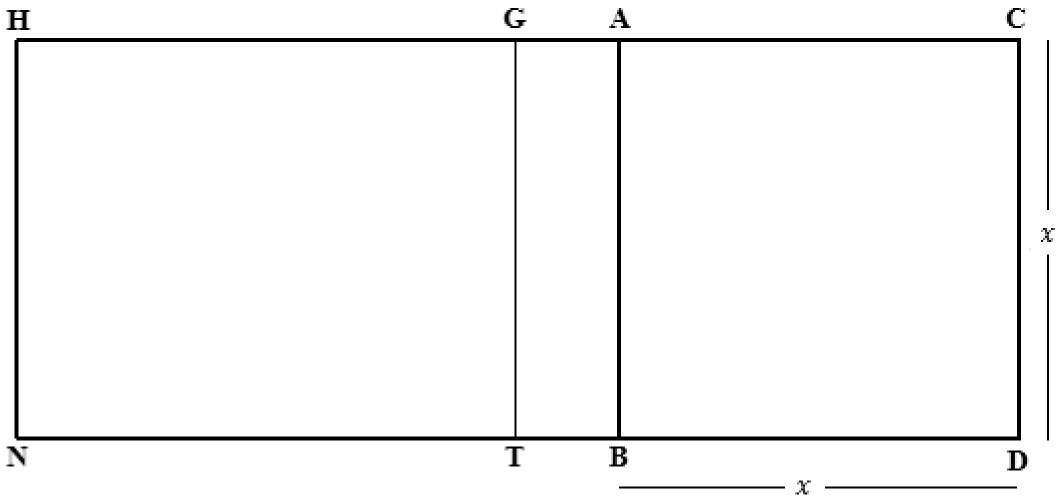
figures together is equal to the line H C. We know that its length is ten of numbers; for every quadrate has equal sides and angles, and one of its sides multiplied by a unit is the root of the quadrate, or multiplied by two it is twice the root of the same. As it is stated, therefore, that a square and twenty-one of numbers are equal to ten roots, we may conclude that the length of the line H C is equal to ten of numbers, since the line C D represents the root of the square. We now divide the line C H into two equal parts at the point G: the line G C is then equal to H G. It is also evident that the line G T is equal to the line C D. At present we add to the line G T, in the same direction, a piece equal to the difference between C G and G T, in order to complete the square. Then the line T K becomes

equal to K M, and we have a new quadrate of equal sides and angles, namely, the quadrate M T. We know that the line T K is five; this is consequently the length also of the other sides: the quadrate itself is twenty-five, this being the product of the multiplication of half the number of the roots by themselves, for five times five is twenty-five. We have perceived that the quadrangle H B represents the twenty-one of numbers which were added to the quadrate. We have then cut off a piece from the quadrangle H B by the line K T (which is one of the sides of the quadrate M T), so that only the part T A remains. At present we take from the line K M the piece K L, which is equal to G K; it then appears that the line T G is equal to M L; more-

over, the line K L, which has been cut off from K M, is equal to K G; consequently, the quadrangle MR is equal to T A. Thus it is evident that the quadrangle H T, augmented by the quadrangle M R, is equal to the quadrangle H B, which represents the twenty-one. The whole quadrate M T was found to be equal to twenty-five. If we now subtract from this quadrate, M T, the quadrangles H T and M R, which are equal to twenty-one, there remains a small quadrate K R, which represents the difference between twenty-five and twenty-one. This is four; and its root, represented by the line R G, which is equal to G A, is two. If you subtract this number two from the line C G, which is the moiety of the roots, then the remainder is the line A C; that is to say, three, which is the root of the original square. But if you add the number two to the line C G, which is the moiety of the number of the roots, then the sum is seven, represented by the line C R, which is the root to a larger square. However, if you add twenty-one to this square, then the sum will likewise be equal to ten roots of the same square. Here is the figure :—

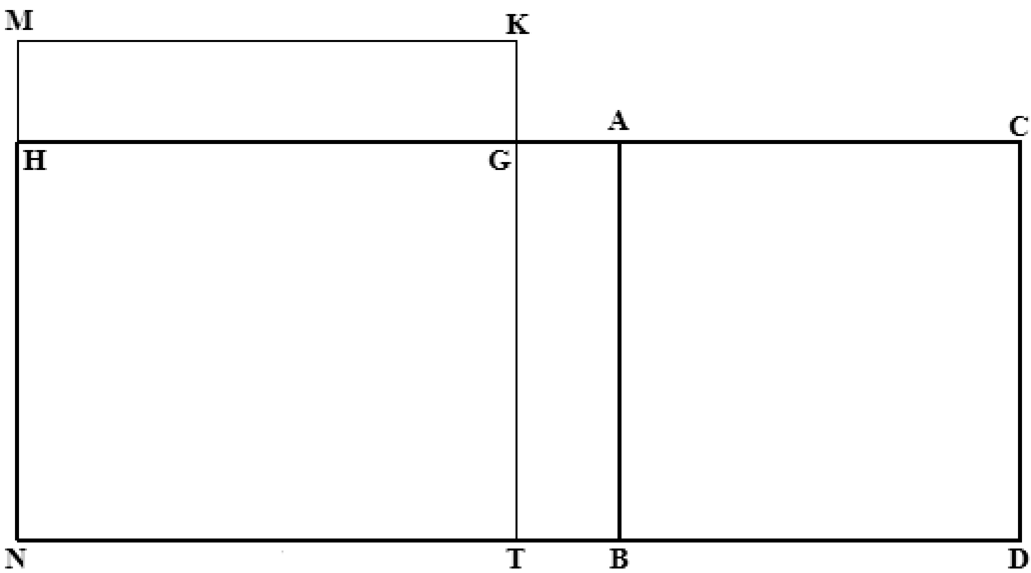


Consider al-Khwārizmī's study for equations $x^2 + c = bx$, with b, $c > 0$ (and $x > 0$).



(Figure 56)

Al-Khwārizmī imagines a square AD with side x *(Figure 56)*. Since $b > x$, on the left of B there is a point N, making $|ND| = b$, and, therefore, a point H which defines a rectangle HD with an area $\left({}^{H}\square_{D}\right) = bx$. Hence, from the equation, $\left({}^{H}\square_{B}\right) = c$. Being G the middle point of line segment HC (G may fall on line segment AC), the quadrilateral HT, which has the format $\frac{b}{2} \times x$, may be obtained *(Figure 57)*. Hence, $|TB| = \left|\frac{b}{2} - x\right|$.

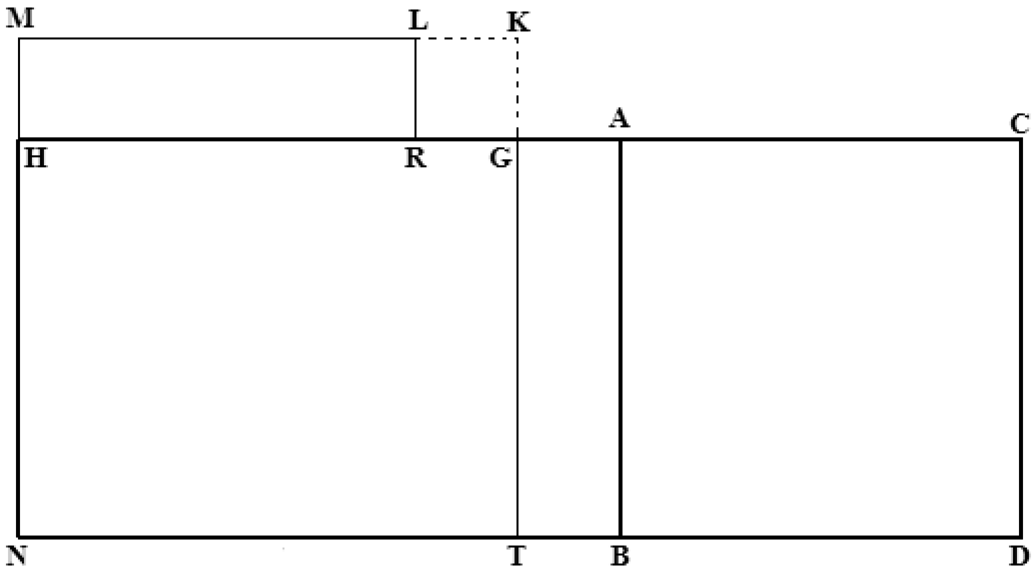


(Figure 57)

If $\frac{b}{2} - x$ is added to the line segment TG, a line segment TK with a length of $x + \left(\frac{b}{2} - x\right) = \frac{b}{2}$ and therefore a square KN may be obtained.

Quadrilateral KH has the form $\left(\frac{b}{2} - x\right) \times \frac{b}{2}$, and quadrilateral BG $\left(\frac{b}{2} - x\right) \times x$.

A point R is introduced in such a way that $|HR| = x$. Quadrilateral LH, which has the form $\left(\frac{b}{2} - x\right) \times x$ and is therefore congruent with quadrilateral BG, is obtained *(Figure 58)*. Quadrilateral KR is a square whose measures are $\left(\frac{b}{2} - x\right) \times \left(\frac{b}{2} - x\right)$.



(Figure 58)

Hence,

$$area\left({}^{M}\square_{T}\right) = \left(\frac{b}{2}\right)^2$$

$$area\left({}^{M}\square_{T}\right) - area\left({}^{L}\square_{G}\right) = area\left({}^{H}\square_{B}\right) = c$$

$$\Rightarrow \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2} - x\right)^2 = c$$

$$\Rightarrow \left(\frac{b}{2}\right)^2 - c = \left(\frac{b}{2} - x\right)^2$$

$$\sqrt{\left(\frac{b}{2}\right)^2 - c} = \frac{b}{2} - x$$

$$x = \frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - c}.$$

For $x^2 + 21 = 10x$, we have $x = 5 - \sqrt{25 - 21} = 5 - \sqrt{4} = 3$.

The appliances of Algebra reach cubic and high degree equations. The Italian mathematician Nicolo Tartaglia (1500 – 1557), who could solve cubic equations $x^3 + cx = d$, being *c* and *d* positive numbers, teach the method to Girolamo Cardano (1501 – 1576), that swore not reveal the solving formula.

$$x = \sqrt[3]{\sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3} + \frac{d}{2}} - \sqrt[3]{\sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3} - \frac{d}{2}}$$

However, in 1545, Cardano ended up publishing the solving formula in *Ars Magna*.

Let us look at the application of the formula to the equation $x^3 + 6x = 20$.

$$x = \sqrt[3]{\sqrt{\left(\frac{20}{2}\right)^2 + \left(\frac{6}{3}\right)^3} + \frac{20}{2}} - \sqrt[3]{\sqrt{\left(\frac{20}{2}\right)^2 + \left(\frac{6}{3}\right)^3} - \frac{20}{2}}$$

$$\Leftrightarrow x = \sqrt[3]{\sqrt{10^2 + 2^3} + 10} - \sqrt[3]{\sqrt{10^2 + 2^3} - 10}$$

$$\Leftrightarrow x = \sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10} \Leftrightarrow x = 2.$$

$x = \sqrt[3]{\sqrt{\left(\frac{20}{2}\right)^2 + \left(\frac{6}{3}\right)^3} + \frac{20}{2}} - \sqrt[3]{\sqrt{\left(\frac{20}{2}\right)^2 + \left(\frac{6}{3}\right)^3} - \frac{20}{2}}$ gives (with properly defined 3rd roots) the numerical value of one of the solutions to the equation.

Scipione del Ferro (1465 – 1526), a predecessor of Tartaglia and Cardano, who studied equations of type $x^3 + cx = d$, is supposed to have found the solving method, which he taught to some of his students.

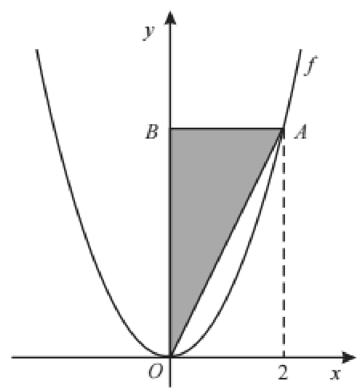
In 1535, Antonio Maria Fiore confronted Nicolo Tartaglia with several problems which were reducible to the form $x^3 + cx = d$. Tartaglia solved them and then questioned Fiore about equations of type $x^3 + bx^2 = d$ but Fiore was unable to solve them.

All cubic equations of the form $x^3 + bx^2 + cx + d = 0$ (in modern notation) are reducible by a simple change of variables $x = y - \frac{b}{3}$ to an equation in *y* where the quadratic term is missing. The expansion of Algebra brought new questions. The equation $x^3 - 15x = 4$ has a solution, 4, which is relatively easy to prove. When the solving formula for cubic equations is applied, the result obtained is $x = \sqrt[3]{\sqrt{-121} + 2} - \sqrt[3]{\sqrt{-121} - 2}$. Square roots of negative numbers, until then ignored by mathematicians because they did not have any geometric significance, gradually started to reveal their secrets. The set of complex numbers, $\mathbb{C}$, filled the gaps of the set of real numbers, $\mathbb{R}$.

Cubic equations, like the previous ones, are contents of Tertiary Education. However there are a myriad of exercises and problems about Algebra that can be explored with Third Cycle students'. For a given question to be classified as a problem, the level of knowledge of the solver must be taken into consideration. Still, it is possible to say that a problem differs from an exercise to the extent that after the reading of the statement the procedures to solve it are not evident.

(Subject: Functions / Geometry / Algebra – 9th grade students)

The figure shows, in the Cartesian coordinate system, a fraction of a quadratic function graphic and triangle $[OAB]$.



We know that:

- Point O is the origin of such referential.
- Point A belongs to f with an abscissa equal to 2.
- Point B belongs to the ordinate axis.
- Triangle $[OAB]$ is square at B with an area of 12.
- Function f is defined by $f(x) = ax^2$, where a is a positive number.

Solve the equation $f(x) = 5x - 2$.

First we need to find the value of a.

As point A belongs to f with abscissa 2, $f(2) = a2^2 \Leftrightarrow f(2) = 4a$. Knowing that triangle $[OAB]$ is square at B with an area of 12, this data allows us to find the value of a.

$$Area_{[OAB]} = \frac{4a \times 2}{2}$$

From there we can extract the value of a.

$$\frac{4a \times 2}{2} = 12 \Leftrightarrow a = 3$$

Now we should apply the second-degree equation formula to solve $3x^2 = 5x - 2$.

$$3x^2 = 5x - 2 \Leftrightarrow 3x^2 - 5x + 2 = 0 \Leftrightarrow x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 3 \times 2}}{2 \times 3} \Leftrightarrow x = \frac{5 \pm 1}{6}$$

Solutions are $x = \frac{2}{3} \vee x = 1$.

This question obviously appeals to a more complex degree of intellectual activity than simple second degree equation resolution.

Problem solving requires productive thinking, as opposed to exercise solving, which is associated with reproductive thinking. If reproductive thinking is limited to the mechanical use of previously acquired knowledge to solve a task, productive thinking requires that the solver deeply understands the structure of the problem. The sketching of diagrams helps to better organise and relate the information provided in the problem statement. However, this is not the only heuristic procedure which can be applied. Several problem solving proposals with reference to school level implementation will be presented, discussed and solved.

The use of heuristic procedures in problem solving

> Mathematical activity is human activity. Certain aspects of this activity – as of any human activity – can be studied by psychology, others by history. Heuristic is not primarily interested in these aspects. But mathematical activity produces mathematics. Mathematics, this product of human activity, 'alienates itself from the human activity which has been producing it. It becomes a living, growing organism, that acquires a certain autonomy from the activity which has produced it; it develops its own autonomous laws of growth, its own dialect. The genuine creative mathematician is just a personification, an incarnation, however, is rarely perfect. The activity of human mathematicians as it appears in history is only a fumbling realisation of the wonderful dialectic of mathematical ideas. But any mathematician, if he has talent, spark, genius, communicates with, feels the sweep of, and obeys this dialect of ideas (Lakatos I. , 1976, p. 146) .

(Subject: Numbers and Operations – 7th grade students)

A palindromic number is a number which remains the same when its digits are reversed. For example, 13931 is a palindromic number. What is the difference between the largest 6-digit palindromic number and the smallest 5-digit palindromic number?

Identify the largest 6-digit palindromic number: 999999.

Identify the smallest 5-digit palindromic number: 10001.

The difference between the largest 6-digit palindromic number and the smallest 5-digit palindromic number is 989998.

The fact that 6-digit palindromes are easy to identify may eventually lead to an erroneous reasoning by considering 11111 the smallest 5-digit palindrome.

(Subject: Numbers and Operations – 7th grade students)

The mean of 10 distinct whole, positive numbers is 10.
What is the highest possible value which can be assigned to one of those numbers?

The mean of 10 different whole, positive numbers is 10.

The goal is to identify the highest possible value which can be assigned to one of those numbers. Let us assign the designation *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *j* to each of the 10 different whole, positive numbers. Even though the value of each number is not known, their sum can be calculated.

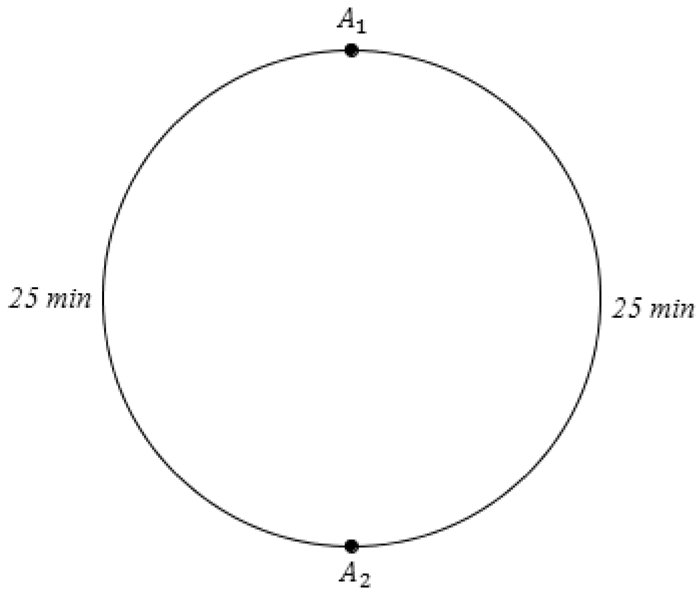
$$10 = \frac{a+b+c+d+e+f+g+h+i+j}{10}$$

$$\Leftrightarrow a+b+c+d+e+f+g+h+i+j = 100.$$

To determine the highest possible value which can be assigned to one of these numbers, let us assign to the remaining numbers the smallest possible value, according to the conditions of the problem: $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 5$, $f = 6$, $g = 7$, $h = 8$, $i = 9$. The sum of these is 45. Thus, $j = 55$.

(Subject: Applied Mathematics – Third Cycle students)

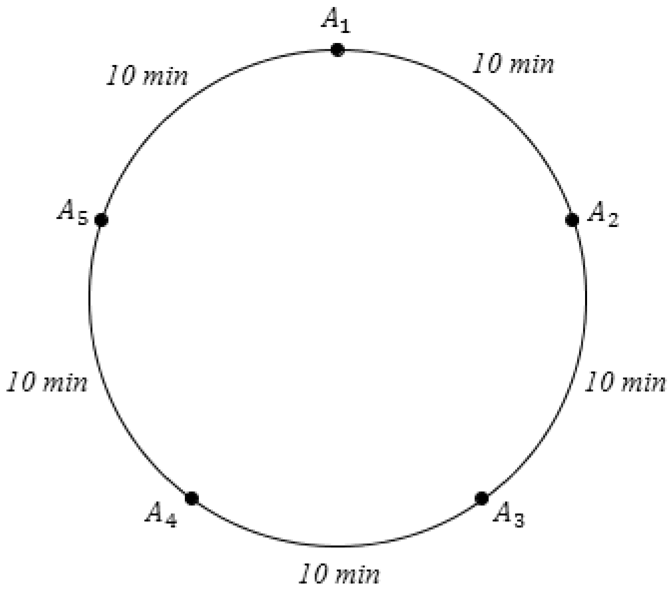
Two busses make the same route with a 25-minute interval. How many busses should be added to the route in order to reduce the interval of time between consecutive busses by 60%?



(Figure 59)

If 2 busses make the same route with a 25-minute interval, then each bus takes 50 minutes to complete the route *(Figure 59)*.

The goal is to calculate the number of additional busses which are necessary to shorten that interval of time by 60%, a percentage which corresponds to a 15-minute reduction (25*min* × 0.60). The interval between busses will be 10 minutes (25 minutes – 15 minutes). The information is presented bellow (Figure 60).

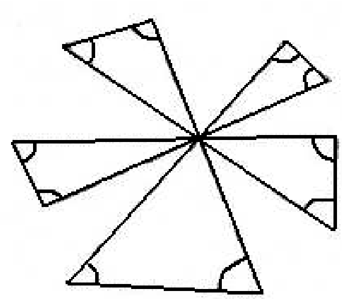


(Figure 60)

Other 3 busses will be necessary.

(Subject: Geometry – 7th grade students)

Five straight lines intersect in a common point. Compute the sum of the sizes of the 10 angles perceptible in the figure?

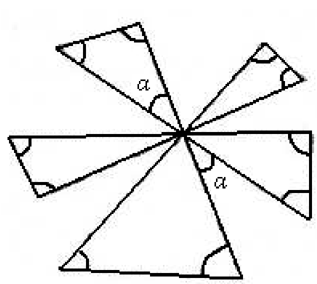


The statement does not provide any values regarding the size of the angles. Apparently, information is missing. In problem solving, success depends on the capacity to recall knowledge which may be used in a new context. Which property(ies) of the size of angles can be useful?

The sum of the internal angles of a triangle is 180°.

The sum of all the angles at the centre is 360°.

Opposite angles have the same size.

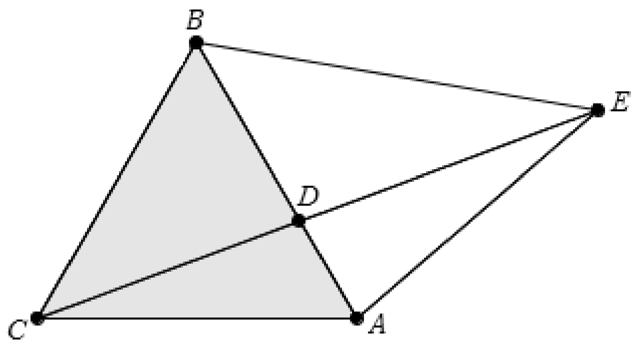


The sum of the sizes of the 5 angles at the centre which belong to the triangles of the figure equals the sum of the sizes of the remaining angles at the centre, *i.e.*, together, their size is 180°. The size of the sum of the 10 angles marked in the 5 triangles equals five times the sum of the sizes of the internal angles of a triangle subtracted to the five angles at the centre. Hence, the sum of the sizes of the 10 angles in the figure is $720^{\circ}\ (5 \times 180^{\circ} - 180^{\circ})$.

(Subject:
Geometry – Third Cycle students)

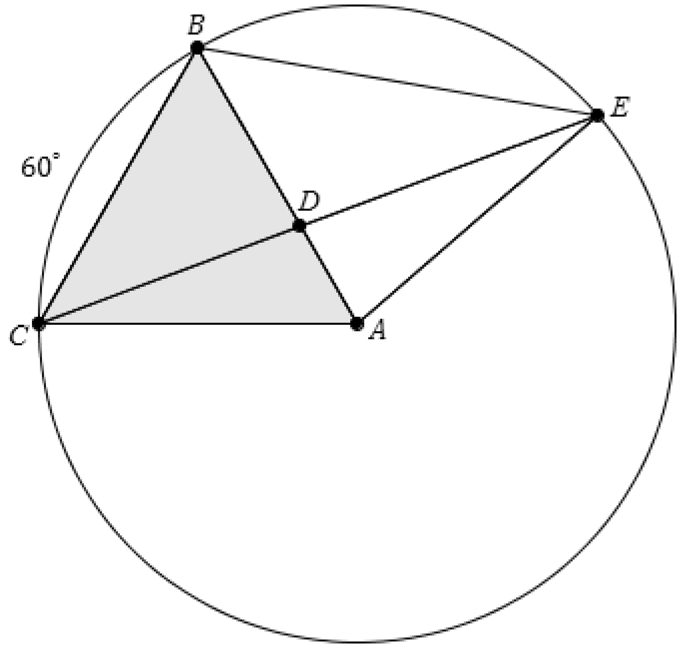
$[ABC]$ is an equilateral triangle and D is a point between A and B. Consider point E, different from C, on straight line CD, so that $\overline{AE} = \overline{AC}$.

Compute $C\hat{E}B$.



Two solving processes are provided. The first is based on geometric principles and the second has a stronger algebraic component.

Solution *a)* $[ABC]$ is an equilateral triangle, $\overline{AB} = \overline{BC} = \overline{CA}$, being, $C\hat{A}B = A\hat{B}C = B\hat{C}A = 60^{\circ}$. The statement of the problem shows that $\overline{AE} = \overline{AC}$. Construct a circumference which includes points *B*, *C* and *E*, and whose centre is point *A*. The arc of the circumference BC has a size of 60°, which is equal to the size of angle $C\hat{A}B$ at the centre. Considering that an angle inscribed in a circumference has half the size of the corresponding arc, the size of angle $C\hat{E}B$ is $\frac{1}{2} \times 60^{\circ} = 30^{\circ}$ *(Figure 61).*



(Figure 61)

Solution *b)* $[ABC]$ is an equilateral triangle, $\overline{AB} = \overline{BC} = \overline{CA}$. Consider that $\overline{AE} = \overline{AC}$. The size of angle $C\hat{E}B$ is x. Consider triangle $[ACE]$ *(Figure 62)* where $\overline{AC} = \overline{AE}$. Hence, we have, $A\hat{C}E = A\hat{E}C = y$. Knowing that $C\hat{A}B = 60^\circ$ and that the sum of the sizes of the internal angles of a triangle is 180°, we have,

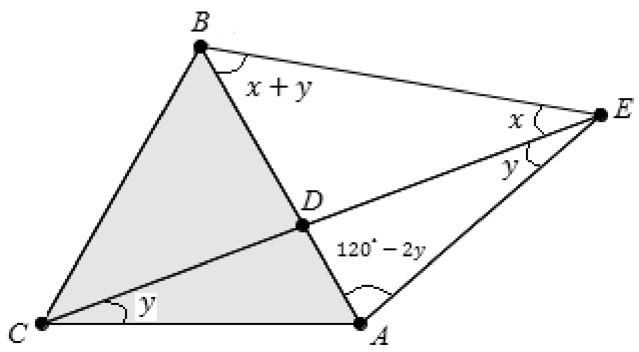
$$\left(60^\circ + B\hat{A}E\right) + y + y = 180^\circ$$

i.e.,

$$B\hat{A}E = 180^\circ - \left(60^\circ + y + y\right) = 120^\circ - 2y.$$

Observe triangle $[ABE]$. The statement implies that $\overline{AB} = \overline{AE}$. Hence, $A\hat{B}E = A\hat{E}B = x + y$. Therefore,

$$\left(120^\circ - 2y\right) + (x + y) + (x + y) = 180^\circ \Leftrightarrow x = 30^\circ.$$

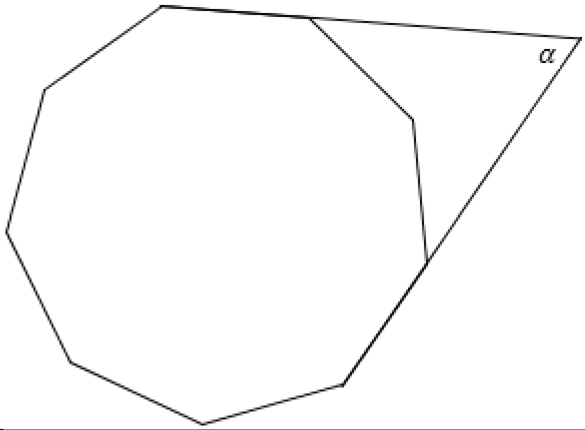


(Figure 62)

(Subject:
Geometry – Third Cycle students)

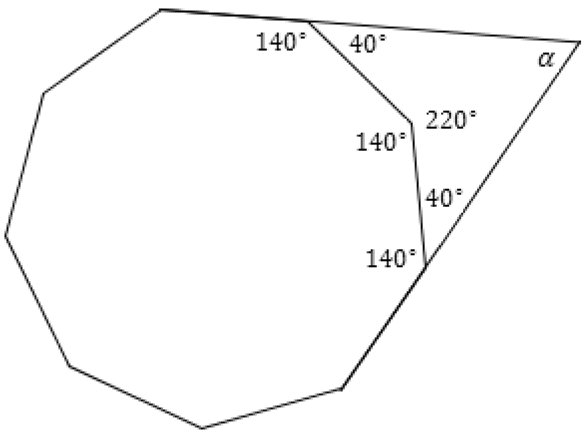
The figure represents a regular nonagon (a 9-sided polygon).

What is the measure of the size of angle α?



When one does not know what to do, one must collect relevant information to solve the problem. Recall that the size of internal angles of *n* regular polygons is $\frac{180°(n-2)}{n}$. The size of each internal angle of the nonagon is 140°. Let us relate this information to the size of angle α so as to extract further information *(Figure 63)*. The sum of the internal angles of a quadrilateral is 360º.

The size of angle α is 360º - (40º + 220º + 40º) = 60º.



(Figure 63)

(Subject: Algebra – 9th grade students)

A natural number *n* has two divisors. Natural number *n + 1* has three divisors.
Identify the number of divisors of natural number *n +2*.

Since *n* has only two divisors, *n* must be a prime number. If *n + 1* has three divisors, they are $1 < d < n + 1$. As *d* is strictly between 1 and *n + 1*, the divisor $\frac{n+1}{d}$ of *n + 1* is also strictly between 1 and *n + 1,* from where we can extract $\frac{n+1}{d} = d$; *n + 1* = d^2 ; $n = d^2 - 1 = (d-1).(d+1)$. Let us recall that *n* is prime, which means that *d – 1* = 1; *d* = 2. Considering this information, we have $n + 1 = 4; n = 3$. Hence, $n + 2 = 5$. As 5 is a prime number, the answer would be: *n + 2* has two divisors (1 and 5).

(Subject: Algebra – 9th grade students)

Helen and Peter are going on a hike between two cities. At the start they are informed that the route will take 2 hours and 55 minutes. They leave at midday and after walking for one hour they rest for 15 minutes. There they are informed that they are 1 hour and 15 minutes away from their destination. If they continue their walk at the same speed and do not make any more stops, at what time will they arrive at their destination?

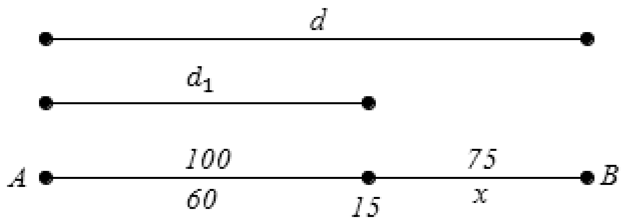
How to relate speed, distance covered and time spent on the journey?

$$speed = \frac{distance\ covered}{time}$$

If d is the value of the whole distance covered and 2 hours and 55 minutes the estimated time for the hike, then $speed_{estimated} = \frac{d}{175}$.

Does the estimated speed for the hike coincide with Helen and Peter's speed? No, because one hour after their departure they were told that they were only 1 hour and 15 minutes away from their destination. Therefore, if they keep their speed, they will complete their hike in less time than the time estimated by the organisation.

We can make a schematic representation *(Figure 64)*.



(Figure 64)

Consider the need to distinguish between information regarding the estimated speed for the hike and the speed at which Helen and Peter are going. The diagram shows that 60 minutes after starting the hike a 15-minute pause was made, and that the time Helena and Pedro took to complete the hike is not known.

$$speed_{HP} = \frac{d}{60+x} \qquad speed_{HP} = \frac{d_1}{60} \Leftrightarrow d_1 = 60 speed_{HP}$$

$$speed_{\ estimated} = \frac{d}{175} \qquad speed_{\ estimated} = \frac{d_1}{100} \Leftrightarrow d_1 = 100 speed_{\ estimated}$$

Hence, $60 speed_{HP} = 100 speed_{\ estimated} \Leftrightarrow speed_{HP} = \frac{5}{3} speed_{\ estimated}$.

We can extract that,

$$\frac{d}{60+x} = \frac{5}{3} \times \frac{d}{175} \Leftrightarrow 525d = 300d + 5dx \Leftrightarrow 45 = x.$$

Helen and Peter took (60*min.* + *15min.* + 45*min.*) to complete the route, *i.e.*, 2 hours. They started at 12.00h and will finish at 14.00h.

> **(Subject: Algebra – 9th grade students)**
>
> There are 50 white, blue and red bricks in a box, and each brick has only one colour. The number of white brick is 11 times greater than the number of blue bricks. There are less red bricks than white, but there are more red bricks than blue. How many less red bricks are there than white bricks?

Assume that w is the number of white bricks, b the number of blue bricks and r the number of red bricks. There are 50 white, blue and red bricks and each brick has only one colour: $w + b + r = 50$.

The number of white bricks is 11 times greater than blue bricks: $w = 11b$.

There are less red bricks than white: $r < w$.

There are more red bricks than blue: $r > b$.

As $w + b + r = 50$, $w = 11b$, $11b + b + r = 50 \Leftrightarrow 12b + r = 50 \Longrightarrow b \in \{0, 1, 2, 3, 4\}$.

Supposing $b = 0$. We have $r = 50 > w$. Impossible.

Supposing $b = 1$. We have $12 + r = 50 \Leftrightarrow r = 38 > 11 = w$. Impossible.

Supposing $b = 2$. We have $24 + r = 50 \Leftrightarrow r = 26 > 22 = w$. Impossible.

Supposing $b = 3$. We have $36 + r = 50 \Leftrightarrow r = 14 < 33 = w$.

Supposing $b = 4$. We have $48 + r = 50 \Leftrightarrow r = 2$. Then $b > r$. Impossible.

Hence, the only possible solution shows that there are 3 blue bricks, 33 white bricks and 14 red bricks. The difference between white bricks and red bricks is 19.

> **(Subject: Algebra – 9th grade students)**
>
> For how many 2-digit numbers will a number greater than its triple be obtained when its digits are reversed?

The 2-digit number is represented as ab ($ab = 10a + b$) in such a way that $a \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. After the reversal we obtain ba ($ba = 10b + a$). According to the statement of the problem, $ba > 3(ab)$, so that $b \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, since $b = 0$ is evidently impossible. Hence,

$$10b + a > 3\,(10a + b) \Leftrightarrow 10b + a > 30a + 3b \Leftrightarrow 7b > 29a \Leftrightarrow \frac{7}{29} > \frac{a}{b} \Leftrightarrow \frac{a}{b} < \frac{7}{29} < \frac{1}{4}.$$

If $a = 1$, b can assume the values 5, 6, 7, 8 or 9.

If $a = 2$, b can only be 9.

If $a \geq 3$, no values of b will make the inequality $\frac{a}{b} < \frac{1}{4}$ true.

The problem has six possible solutions: 15, 16, 17, 18, 19 and 29.

(Subject: Numbers and Operations – Third Cycle students)

A number *N* has 3 digits. The product of the digits of *N* is 126 and the sum of the last 2 digits of *N* is 11. Calculate the possible digits of the hundreds of *N*.

N is a 3-digit number: *x, y* and *z* are the digits of the hundreds, tens and ones.

$N = x\,y\,z$ The product of the digits is 126, *i.e.*, $x \,.\, y \,.\, z = 126$.

The sum of the last 2 digits is 11, *i.e.*, $y + z = 11$.

Consider that (y, z) or (z, y) are part of the set $\{(2,9), (3,8), (4,7), (5,6)\}$. The choice $(y, z) = (3,8)$ implies that $x \,.\, y \,.\, z = x \,.\, 3 \,.\, 8 = 126$, which is impossible because 126 cannot be divided by 8. Similar reasoning for other choices is summarised in the following table.

Addition is commutative. Consider all situations *(Table 117)*.

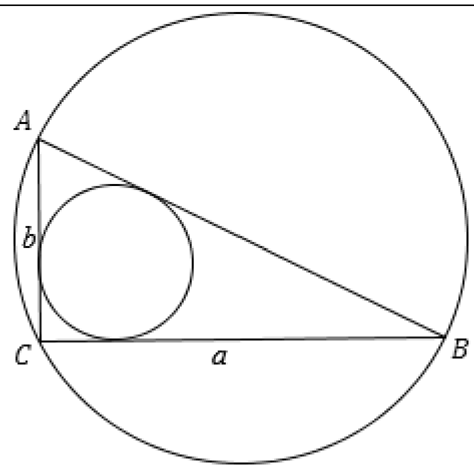
(Table 117)

$y + z$	$x \times y \times z = 126$ $126 = 2 \times 3^2 \times 7$
$2 + 9$	$x \times 2 \times 9 = 126 \Leftrightarrow x = 7$
$3 + 8$	$x \times 3 \times 8 = 126$ Impossible because 126 is not divisible by 8
$4 + 7$	$x \times 4 \times 7 = 126$ Impossible because 126 is not divisible by 4
$5 + 6$	$x \times 5 \times 6 = 126$ Impossible because 126 is not divisible by 5

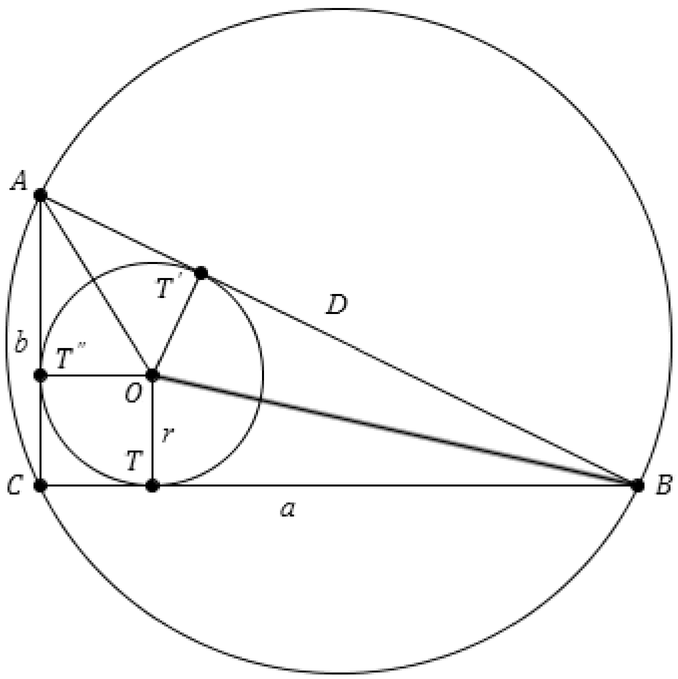
The figure of the hundreds place value is 7. $N = 7 \times 2 \times 9 = 7 \times 9 \times 2 = 126$.

(Subject: Geometry – 9th grade students)

The measures of the lengths of the cathetes of the right triangle in the figure are *a* and *b*. If *d* is the diameter of the incircle and *D* the diameter of the circumcircle of that triangle, what is the value of *d* + *D*?



Let us choose an adequate notation to identify variables and points *(Figure 65)*. The rightness of the triangle $[ABC]$ suggests the application of the Pythagoras' theorem. Using the notation of the statement of the problem and considering $\overline{BC} = a$ and $\overline{AC} = b$, we have $D = \sqrt{a^2 + b^2}$.



(Figure 65)

The radius of the incircle is represented by r.

$[OBT]$ and $[OBT']$are congruent triangles and $\overline{BT} = \overline{BT'} = a - r$.

$[OAT']$ and $[OAT'']$are congruent triangles and $\overline{AT'} = \overline{AT''} = b - r$.

$D = \overline{AB} = \overline{AT'} + \overline{T'B} = (b - r) + (a - r) = a + b - 2r = a + b - d$.

In conclusion, $d + D = a + b$.

(Subject: Functions – 8th grade students)

f is a function defined for all x real numbers, that satisfies the conditions:

$$\begin{cases} f(3) = 2 \\ f(x+3) = f(x) \times f(3) \end{cases}$$

What is the value of $f(-3)$?

Assign x the value -3.

$f(0) = f(-3 + 3) = f(-3) \times f(3) \Leftrightarrow f(0) = f(-3) \times f(3)$

Is it possible to determine the value of $f(0)$?

$f(0 + 3) = f(0) \times f(3) \Leftrightarrow 2 = f(0) \times 2 \Leftrightarrow f(0) = 1$

Hence $f(0) = f(-3) \times f(3) \Leftrightarrow 1 = f(-3) \times 2 \Leftrightarrow f(-3) = \frac{1}{2}$.

Verification: $f(-3 + 3) = f(-3) \times f(3) = \frac{1}{2} \times 2 = 1 = f(0)$.

(Subject: Algebra – 8th grade students)

In an archery game, the shooting target has 2 rings (A and B) each with different scores. Two archers shoot three arrows each. Knowing that the first archer obtained 17 points and the second archer 22 points, how many points are assigned to an arrow which hits ring A?

The three arrows are shot. Shooting ring A gives *x* points, and shooting ring B gives *y* points. Since neither 17 nor 22 are divisible by 3, one of the archers must have hit ring A with 2 arrows and ring B with 1 arrow; and the other archer must have hit ring A with 1 arrow and ring B with 2 arrows. The previous reasoning allows you to write the following system of equations:

$$\begin{cases} y + 2x = 17 \\ x + 2y = 22 \end{cases} \Leftrightarrow \begin{cases} y + 2(22 - 2y) = 17 \\ x = 22 - 2y \end{cases} \Leftrightarrow \begin{cases} y + 44 - 4y = 17 \\ x = 22 - 2y \end{cases} \Leftrightarrow \begin{cases} 27 = 3y \\ x = 22 - 2y \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 9 \\ x = 22 - 2 \times 9 \end{cases} \Leftrightarrow \begin{cases} y = 9 \\ x = 4 \end{cases}.$$

Each arrow which hits ring A is given a 4-point score. If an arrow hits ring B, the archer gets a 9-point score. The first archer shot 3 arrows, 1 hit B and 2 hit A: 1 × 9 + 2 × 4 = 17. The second archer shot 3 arrows, 1 hit A and 2 hit B: 1 × 4 + 2 × 9 = 22.

(Subject: Numbers and Operations – 9th grade students)

If *p* and *q* are whole, positive numbers which satisfy the inequality $\frac{7}{10} < \frac{p}{q} < \frac{11}{15}$, determine, from the given options, the least possible value for *q*.

a) 6 b) 7 c) 25 d) 30 e) 60

Consider *p* and *q* as whole, positive numbers: $\frac{7}{10} < \frac{p}{q} < \frac{11}{15}$.

From the statement of the problem, we can write that $10p > 7q$ and $11q > 15p$, which can be represented as a system of 2 inequalities with 2 unknowns $\begin{cases} 10p - 7q > 0 \\ 11q - 15p > 0 \end{cases}$.

Consider the case $q = 6$.

$$\begin{cases} 10p - 7 \times 6 > 0 \\ 11 \times 6 - 15p > 0 \end{cases} \Leftrightarrow \begin{cases} 10p > 42 \\ 66 > 15p \end{cases} \Leftrightarrow \begin{cases} p > 4.2 \\ p < 4.4 \end{cases}$$

It is impossible because *p* is a whole, positive number!

Consider the case $q = 7$.

$$\begin{cases} 10p - 7 \times 7 > 0 \\ 11 \times 7 - 15p > 0 \end{cases} \Leftrightarrow \begin{cases} 10p > 49 \\ -15p > -77 \end{cases} \Leftrightarrow \begin{cases} p > 4.9 \\ p < 5.1 \end{cases}$$

Therefore $p = 5$ and $q = 7$.

When substitutions are made, the relations: $\frac{7}{10} < \frac{5}{7} < \frac{11}{15}$ are confirmed.

(Subject: Applied Mathematics – Third Cycle students)

A yellow car, a green car, a blue car and a black car are in a line. The car which is immediately before the blue car is smaller than the one which is immediately after the blue car; the green car is the smallest; the green car is after the blue car; and the yellow car is after the black car. Which car is the first in the line?

Note: The first car in the line is the one which comes before all the other cars.

Solution *a)*

The car which is before the blue car is smaller than the car which is after the blue car.

The green car is the smallest.

The green car is after the blue car.

The yellow car is after the black car.

Which car is the first in the line?

The car which is before the blue car is smaller than the car which is after the blue car.

The green car is after the blue car.

What are the possible situations for this to happen? We know that there are cars before and after the blue car, so it has 2nd or 3rd place; and there is a green car behind it *(Chart 10)*.

(Chart 10)

1st Place	2nd Place	3rd Place	4th Place
	Blue	Green	
	Blue		Green
		Blue	Green

Knowing that the green car is the smallest, there is only one possibility *(Chart 11)*.

(Chart 11)

1st Place	2nd Place	3rd Place	4th Place
	Blue		Green

The yellow car is after the black car *(Chart 12)*.

(Chart 12)

1st Place	2nd Place	3rd Place	4th Place
Black	Blue	Yellow	Green

The first car of the line is black.

Solution *b)*

The yellow car is after the black car; therefore it cannot be the first.

The blue car has a car in front of it; therefore it cannot be the first.

The green car is after the blue car; therefore it cannot be the first.

Hence, the black car must be the first in the line.

(Subject: Applied Mathematics – Third Cycle students)

A tennis tournament is played by 64 players. In the first round 32 matches are played and whoever loses is eliminated. In the second round 16 matches are played, and similarly to the previous round, the players who lose are eliminated, and so on. If a better ranked player always wins against a lower ranked player, what is the maximum number of matches played by the tenth ranked player in the tournament?

Let us identify the tenth ranked player in the tournament as X.

Consider the extreme situation in which the 9 best players are paired (1 is left out). Then, only 5 better-ranked players than X go to the next round. At the end of round 1, 32 players are still in the tournament.

Now consider the extreme situation in which the 5 best players are paired (1 is left out). Only 3 better-ranked players than X go to the next round. At the end of round 2, 16 players are still in the tournament.

Then consider the extreme situation in which the 3 best players are paired (1 is left out). Only 2 better-ranked players than X go to the next round. At the end of round 3, 8 players are still in the tournament.

Consider the extreme situation in which the 2 best players are paired. Only 1 better-ranked player than X goes to the next round. At the end of round 4, 4 players are still in the tournament.

Consider the extreme situation in which the only better-ranked player than X does not play against him. At the end of round 5, 2 players are still in the tournament. In the next round, the best-ranked player in the tournament plays against X and wins the match. Hence, X plays a maximum of 6 matches.

(Subject: Algebra – 9th grade students)

Find the pairs (x, y) of real numbers (if there are any) which satisfy the system of equations:

$$\begin{cases} x^2 - xy - y^2 + 1 = 0 \\ x^3 - x^2y - xy^2 + x - y + 2 = 0 \end{cases}.$$

Let us relate the two equalities in such a way that the problem is simplified. The reasoning is that $x^2 - xy - y^2 + 1$ is implicit in $x^3 - x^2y - xy^2 + x - y + 2$.

$$x^3 - x^2y - xy^2 + x - y + 2 = 0 \Leftrightarrow x(x^2 - xy - y^2 + 1) - y + 2 = 0$$

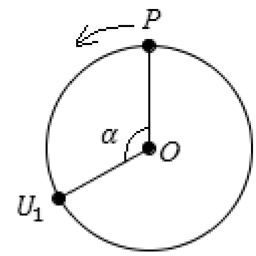
Hence, $x \times 0 - y + 2 = 0$, *i.e.*, $y = 2$. Consider the first equation and look for the solutions to the quadratic equation $x^2 - 2x - 3 = 0$. The solutions are $x = 3 \vee x = -1$. The ordered pairs (-1, 2) and (3, 2), among which the solutions should be, satisfy the system of equations.

(Subject: Applied Mathematics – Third Cycle students)

A couple was prescribed regular physical exercise by their doctor. Every morning they go jogging around their neighbourhood. They start together and finish together, and they jog at a constant speed in the same direction. During their run, the wife overtakes the husband twice while the husband completes only one lap. If the wife would run in the opposite direction, how many times would she pass her husband?

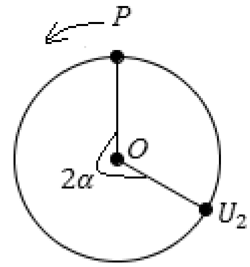
Solution *a)* Without loss of generality, let us suppose that the path is circular. Hypothetically, husband and wife start at the same time and at the same point *P*. The initial advantage that the wife has over her husband is not considered as overtaking.

The first overtaking takes place at point U_1(while the wife is in her second lap). The angle $P\hat{O}U_1$ is defined by α *(Figure 66)*.



(Figure 66)

As the speeds are constant, the second overtaking U_2 defines angle 2α *(Figure 67)*.



(Figure 67)

They next meet at the starting point. Therefore, $\alpha = \frac{360^0}{3} = 120^0$. Given that, during the same time frame, the husband completes one lap around the neighbourhood while the wife completes four laps, the wife is four times faster than the husband.

If the wife was running in the opposite direction when they meet for the first time, the distance covered by both would equal a complete lap to their neighbourhood. Knowing that the wife is four times faster than the husband, the route is divided in 5 segments of equal length. From these segments, four are covered by the wife and one is covered by the husband. The size of each angle at the centre is 72^0.

When they meet a second time, the angle at the centre is $2 \times 72^0 = 144^0$.

When they meet a third time, the angle at the centre is $3 \times 72^0 = 216^0$.

When they meet a fourth time, the angle at the centre is $4 \times 72^0 = 288^0$.

The final meeting occurs at the starting point, $5 \times 72^0 = 360^0$, a position which does not correspond to an overtaking. During their jogging, the wife passes her husband four times.

Remarks:

The speeds of the husband and the wife are, respectively, v_h and v_w. Hence, $v_h < v_w$, which is a necessary condition for an overtaking to occur. In order to have exactly 2 overtakes, these would take place in U_1 and U_2. If overtake U_1 takes place at the moment t_1, we can say that

$$v_w \times t_1 = v_h \times t_1 + \frac{2\pi}{3}$$

where, without loss of generality, $2\pi = perimeter\ of\ the\ circuit$. Hence,

$$(v_w - v_h) \times t_1 = \frac{2\pi}{3}.$$

(Subject: Algebra – 9th grade students)

Find in $\mathbb{R}$, the solution(s) to the equation $\sqrt{x+10} - \sqrt{2x+3} = \sqrt{1-3x}$.

Idea: Both members of the equation may be elevated to the square.

$$(x+10) - 2\sqrt{(x+10)(2x+3)} + (2x+3) = 1 - 3x$$

$$\Leftrightarrow 6x + 12 = 2\sqrt{(x+10)(2x+3)} \Leftrightarrow 3x + 6 = \sqrt{(x+10)(2x+3)}$$

$$\Rightarrow (3x+6)^2 = (x+10)(2x+3)$$

$$\left(\begin{array}{c} Note\ that\ the\ last\ equation\ might\ have\ solutions\ with\ 3x+6<0. \\ (3x+6)^2 = (x+10)(2x+3)\ may\ eventually\ lead\ to\ x : 3x+6<0. \\ Nevertheless, such\ are\ not\ possible\ for\ 3x + 6 = \sqrt{(x+10)(2x+3)} \end{array}\right)$$

$$\Leftrightarrow 9x^2 + 36x + 36 = 2x^2 + 3x + 20x + 30 \Leftrightarrow 7x^2 + 13x + 6 = 0 \Leftrightarrow x = -\frac{6}{7} \vee x = -1.$$

Confirmation is indeed necessary, as due to the implication ($\Rightarrow$) which emerges by the development of the equation, we only know that the solutions are, eventually, $-\frac{6}{7}$ or -1 .

If x assumes the value $-\frac{6}{7}$, then

$$\sqrt{-\frac{6}{7}+10}-\sqrt{2\times\left(-\frac{6}{7}\right)+3}=\sqrt{1-3\times\left(-\frac{6}{7}\right)}$$

$$\Leftrightarrow\sqrt{\frac{64}{7}}-\sqrt{\frac{9}{7}}=\sqrt{\frac{25}{7}}\,.(Both\ members\ of\ this\ equation\ are\ positive)$$

If both members are elevated to the square, then

$$\frac{64}{7}-2\times\sqrt{\frac{64}{7}}\times\sqrt{\frac{9}{7}}+\frac{9}{7}=\frac{25}{7}$$

$$\Leftrightarrow\frac{64}{7}-2\times\sqrt{\frac{576}{49}}+\frac{9}{7}=\frac{25}{7}\Leftrightarrow\frac{64}{7}-\frac{48}{7}+\frac{9}{7}=\frac{25}{7}\Leftrightarrow\frac{25}{7}=\frac{25}{7}.$$

Through their positiveness, these equivalences confirm the validity of $-\frac{6}{7}$ as a solution. If x is assigned the value -1, then

$$\sqrt{(-1)+10}-\sqrt{2\times(-1)+3}=\sqrt{1-3\times(-1)}\Leftrightarrow\sqrt{9}-\sqrt{1}=\sqrt{4}\Leftrightarrow 2=2.$$

(Subject:
Algebra – Second Cycle students / Third Cycle students / Secondary Education students)

With 100 m of wire mesh and 4 poles, we intend to build a fence surrounding a rectangle with the largest possible area. What would the dimensions of the fence be?

This problem may be presented to students from different school grades. Second Cycle students would explore and compute the area of rectangles. The experimental results obtained *(Table 118)* help to empirically infer the best option.

(Table 118)

Length (m)	Width (m)	Area (m^2)
40	10	400
35	15	525
30	20	600
25	25	625

In the Third Cycle (7th to 9th grade), when studying the parabola as a graphic representation of quadratic equations, namely vertex coordinates, students are able to determine the area of the field in terms of the length (x). The width of the field is given as $\frac{1}{2}(100-2x)=50-x$, thus the area A_f, in terms of x, is given by $A_t(x)=50x-x^2$, that is, it is a quadratic function $x\longmapsto ax^2+bx+c$ with $a=-1$, $b=50$, $c=0$. This function is a parabola, *i.e.*, a function of type ax^2+bx+c with $a\neq 0$, whose concavity is facing down ($a<0$). Its vertex coordinates $V\circlearrowleft\ (h;k)$ are obtained from writing it as $a(x-h)^2+k$.

Now, to find them in our case, we note

$$ax^2 + bx + c = (ax^2 + bx) + c = a\left(x^2 + \frac{b}{a}x\right) + c$$

$$\left(Note\ that\ \left(x + \frac{b}{2a}\right)^2 = x^2 + 2 \times x \times \frac{b}{2a} + \left(\frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right)$$

$$= a\left(\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c = a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 - \frac{b^2}{4a} + c.$$

Hence,

$$V \circlearrowleft \left(-\frac{b}{2a}; -\frac{b^2 - 4ac}{4a}\right),$$

and so in our case

$$V \circlearrowleft \left(-\frac{50}{2 \times (-1)}; -\frac{50^2 - 4 \times (-1) \times 0}{4 \times (-1)}\right),$$

$$V \circlearrowleft (25; 625).$$

In Secondary Education, if students' know the rules of derivation, they can write

$$A_f'(x) = 50 - 2x,$$

and then determine the zeros of the derivative so as to complete the table of signals *(Table 119)*.

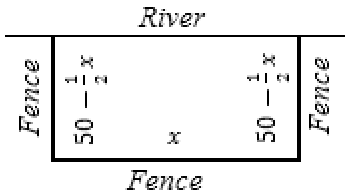
$$A_f'(x) = 0 \Leftrightarrow 50 - 2x = 0 \Leftrightarrow x = 25.$$

(Table 119)

x	0	25	50
$A_f'(x)$	+	0	-
$A_f(x)$	↗	*Max.*	↘

In order to maximise the area with the available mathematical resources, the field should be in the shape of a square with a 25*m* side, a geometrical figure with an area of $625m^2$.

Additionally, students' should be presented with variations to the problem, namely, to determine the measures of the field if the fence is to be placed near a river, where access to the water line is a necessary condition *(Figure 68)*.



(Figure 68)

In this situation we can compute some results *(Table 120)*.

(Table 120)

Length (m)	Width (m)	Area (m^2)
80	10	800
60	20	1200
50	25	1250

In terms of the length *x*, the area A_f is given by

$$A_f(x) = x\left(50 - \frac{1}{2}x\right) = 50x - \frac{1}{2}x^2.$$

The vertex of the parabola has, hence, the following coordinates,

$$V \circlearrowleft \left(-\frac{50}{2 \times \left(-\frac{1}{2}\right)}; -\frac{50^2 - 4 \times \left(-\frac{1}{2}\right) \times 0}{4 \times \left(-\frac{1}{2}\right)}\right), \quad \textit{i.e.},$$

$$V \circlearrowleft (50; 1250).$$

By using the rules of derivation,

$$A_f'(x) = 50 - x.$$

After computing the zeros of the derivative function, let us complete the table of signals *(Table 121)*.

$$A_f'(x) = 0 \Leftrightarrow 50 - x = 0 \Leftrightarrow x = 50.$$

(Table 121)

x	0	50	100
$A_f'(x)$	+	0	-
$A_f(x)$	↗	*Max.*	↘

To maximise the area using the available mathematical resources, the field ought to have the shape of a rectangle with a 50*m* length and a 25*m* width, a geometrical figure with an area of $1250m^2$.

(Subject: Algebra – Secondary Education students)

What should the measures of a closed square-based prism of capacity 2*l* be, so that the material necessary for its construction can be minimised?

We know that the volume of a rectangular parallelepiped is

$$V_{par} = Length \times Width \times Height\ .$$

If x is the measure of the side of the base (square) and h the height of the parallelepiped in *dm*,

$$(2l = 2dm^3)$$

then,

$$x \times x \times h = 2 \Leftrightarrow h = \frac{2}{x^2}.$$

The goal is to optimize the use of resources, *i.e.*, to minimize the surface area of the solid, which is formed by 4 lateral faces and 2 bases.

$$A_{par} = 4 \times (x \times h) + 2 \times (x \times x).$$

If the previously established relation between the unknowns *x* and *h* is applied,

$$A_{par}(x) = 4 \times \left(x \times \frac{2}{x^2}\right) + 2 \times (x \times x) = \frac{8}{x} + 2 \times x^2.$$

If the rules of derivation are applied,

$$A'_{par}(x) = \frac{(8)' \times x - 8 \times (x)'}{x^2} + 2 \times 2 \times x^{2-1} = -\frac{8}{x^2} + 4x.$$

After computing the zeros of the derivative function, we complete the table of signals *(Table 122)*.

$$-\frac{8}{x^2} + 4x = 0 \Leftrightarrow\ x = \sqrt[3]{2} \wedge x \neq 0$$

(Table 122)

x	0	$\sqrt[3]{2}$	$+\infty$
$A'_{par}(x)$	-	0	+
$A_{par}(x)$	↘	*min.*	↗

To minimise the area, the side of the base of the parallelepiped should measure $\sqrt[3]{2}dm$ and its height should measure $\sqrt[3]{2}\,dm$ $\left(h = \frac{2}{x^2} = \sqrt[3]{2}\right)$. Therefore, the solid should be a cube.

(Subject: Numbers and Operations – 12th grade students)

In a given set of 10 distinct 2-digit positive numbers it is always possible to choose 2 disjoint subsets whose elements have the same sum.

Let us try to answer less demanding questions related to the problem. Let us determine the number of subsets which can be formed from a set with *n* elements. We identify all the subsets of $A = \{\ \ \}$, $B = \{a\}$, $C = \{a, b\}$, $D = \{a, b, c\}$, $E = \{a, b, c, d\}$, in order to determine a general expression *(Chart 13)*.

(Chart 13)

$A = \{\ \ \}$	$B = \{a\}$	$C = \{a, b\}$	$D = \{a, b, c\}$	$E = \{a, b, c, d\}$
$\emptyset$	$\emptyset, \{a\}$	$\emptyset, \{a\}, \{b\}, \{a, b\}$	$\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$	$\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}$

Apparently the sequence 1, 2, 4, 8, 16… comes into view. Now we can speculate that a set of *n* elements has 2^n different subsets. One possible proof requires that the solver have knowledge of Combinatory Analysis, particularly of the properties of Pascal's Triangle and Newton's binomial formula.

The total of subsets is

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-2} + \binom{n}{n-1} + \binom{n}{n} = \sum_{i=0}^{n} \binom{n}{i}$$

being $\binom{n}{i}$ the number of subsets with *i* elements of a set of *n* elements.

Considering Newton's binomial,

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

if $x = y = 1$,

$$(1 + 1)^n = \sum_{i=0}^{n} \binom{n}{i}$$

i.e.,

$$2^n = \sum_{i=0}^{n} \binom{n}{i}.$$

Let us look at another proof.

Theorem: A set of *n* elements has 2^n different subsets.

Proof: Without loss of generality, the given set is $\Omega = \{1,2,\dots,n\}$. To each subset S of Ω corresponds its characteristic vector defined by $\chi_S(i) = \begin{cases} 1 \; if \; i \in S \\ 0 \; if \; i \notin S \end{cases}$.

χ_S can be adequately represented by an *n*-uple of 0*s* and 1*s* by assuming in the position i the value $\chi_S(i)$. For example: if $S = \{3,7,8\}$, then $\chi_S = (0,0,1,0,0,0,1,1,0,\dots,0)$. As choice is possible for each of the positions, there are 2^n different characteristic vectors; which means 2^n different subsets.

We now come back to our initial problem. There are $2^{10} = 1024$ subsets of a set of 10 different 2-digit whole, positive numbers. If Ω is the set of 10 2-digit whole, positive numbers, then

$$\sum_{\omega\in\Omega} \omega \leq 90+91+92+93+94+95+96+97+98+99 = 5\times(90+99) = 945.$$

Let us define an application *s*, $2^\Omega \ni X \xrightarrow{s} \sum_{x\in X} x \in \mathbb{Z}$, whose domain has $2^{10} = 1024$ elements and which has in its elements $s(X) \in \{0,1,2,\dots,945\}$. Note that $s(\emptyset) = 0$.

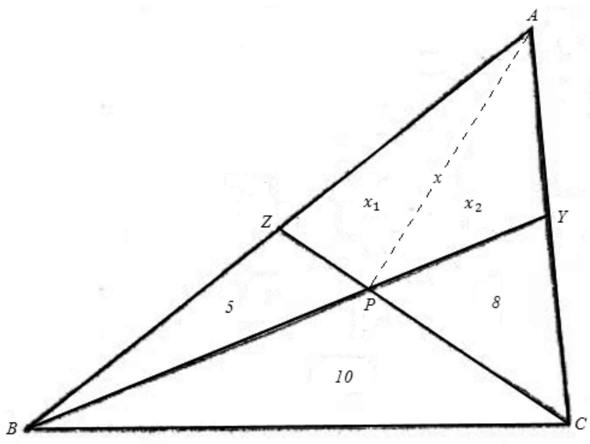
The function cannot be injective for the domain has more elements than the codomain. Hence, there are $X_1 \neq X_2$ in so that $s(X_1) = s(X_2)$. By definition of *s*, we have

$$s\big(X_1 \setminus (X_1 \cap X_2)\big) = s(X_1) - s(X_1 \cap X_2) = s(X_2) - s(X_1 \cap X_2) = s\big(X_2 \setminus (X_1 \cap X_2)\big)$$

The sets $X_i' = X_i \setminus (X_1 \cap X_2), i = 1,2$ are as required.

(Subject: Geometry – Secondary Education students)

In the figure, the numbers 5, 10 and 8 inscribed in the triangles stand for areas. Calculate the area *x* of the quadrangular region $[AYPZ]$.



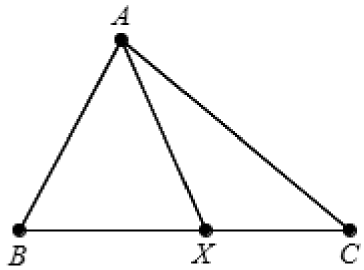
Let us use the following observation in order to solve the problem:

Lemma: For a given triangle ΔABC and a *cevian*[58] $\overline{AX}$ we have,

$$\frac{area(\Delta ABX)}{area(\Delta ACX)} = \frac{\overline{BX}}{\overline{XC}}.$$

Proof: If $h := height\ \Delta ABC$ in relation to straight line BC, then,

$$\frac{area(\Delta ABX)}{area(\Delta ACX)} = \frac{\frac{\overline{BX} \times h}{2}}{\frac{\overline{XC} \times h}{2}} = \frac{\overline{BX}}{\overline{XC}}.$$



(Figure 69)

We have proved that a cevian divides a triangle in two triangles whose ratio of the areas equals the ratio in which the cevian divides the side opposite to its vertex *(Figure 69)*.

Now let us go back to the problem. Complete the figure with notable points and the auxiliary segment $\overline{AP}$. Consider $x_1 = area\ (\Delta APZ)$ and $x_2 = area\ (\Delta APY)$. By applying the lemma to ΔBCZ with cevian $\overline{BP}$ and to ΔZCA with cevian $\overline{AP}$, we have, respectively,

$$\frac{5}{10} = \frac{\overline{ZP}}{\overline{PC}} \qquad \text{and} \qquad \frac{x_1}{x_2 + 8} = \frac{\overline{ZP}}{\overline{PC}}$$

i.e.,

$$\frac{x_1}{x_2 + 8} = \frac{5}{10}.$$

Similarly, we have triangle ΔBYA with cevian $\overline{PA}$ and triangle ΔYBC with cevian $\overline{PC}$,

$$\frac{x_1 + 5}{x_2} = \frac{\overline{BP}}{\overline{PY}} = \frac{10}{8}.$$

Hence, we obtain a two-equation system with two unknowns. When we solve this system by standard methods we obtain $x_1 = 10$ and $x_2 = 12$.

$$\begin{cases} \dfrac{x_1}{x_2 + 8} = \dfrac{5}{10} \\ \dfrac{x_1 + 5}{x_2} = \dfrac{10}{8} = \dfrac{5}{4} \end{cases}.$$

The area *x* of the square region is 22 ($x_1 + x_2$ = 10 + 12 = 22).

[58] Line segment which connects a vertex of a triangle to its corresponding opposite side and/or to its prolongation. Median, altitude and bisector are all examples of cevians. Cevians are named after the Italian mathematician Giovanni Ceva, who provided the conditions for three concurrent cevians.

Mathematics can always surprise us with its applications to real life needs, as we have seen before. Now let us look at a curiosity related to Portuguese car number plates.

In Portugal, car number plates are currently composed as a sequence of *Digit Digit - Letter Letter – Digit Digit* (*D D – L L – D D*)[59]. Occasionally or as a consequence of methodical study, surprising patterns emerge in Mathematics. For example, consider the number plate *42 – L L – 85*. The positive difference between 42 and 85 is 43. If the order of each number is inverted, we obtain 24 and 58, whose positive difference is 34, but if we invert the position of the digits we obtain 43. This pattern holds also for other number plates. However, there are exceptions. For example, the number plate *37 – L L – 42* has the positive difference of 05. But if the order of each number is inverted, the positive difference is 49 (73 – 24). The analysis of exceptions leads to a new pattern which may be intentionally identified as an alternate pattern. Hence, if the digits of 49 are inverted and 05 is added to 94, the total is always 99 (05 + 94).

The intuition one gets by looking of such examples may be corroborated by a mathematical explanation which justifies why these numerical patterns emerge.

Two 2-digit whole, positive numbers *ab* and $a'b'$ written in base 10 have a positive difference of *ef*. Similarly, let us consider the positive difference between *ba* and $b'a'$ as *gh*. We claim that then, $ef = hg$ or $ef + hg = 99$.

Remarks:

With accuracy, xy should read $10x + y$, where x and y are digits.

Without loss of generality, let us suppose that $ab < a'b'$. Therefore, the positive difference between ab and $a'b'$ is $10(a' - a)$ + $(b' - b)$, where $a < a'$ or $a = a'$ and $b < b'$.

Two cases should be considered:

- $b \leq b'$.

The positive difference between ba and $b'a'$ may be written as $10(b' - b) + (a' - a)$, and since $9 \geq b' \geq b \geq 0$, and by hypothesis, $9 \geq a' \geq a \geq 0$, we obtain $b' - b = g$ and $a' - a = h$. But we know $b' - b = f$ and $a' - a = e$ since $a'b' - ab = ef$. Therefore, $ef = hg$.

[59] The sequence *Letter Letter – Digit Digit – Digit Digit* ended in 1992, and was then replaced by the sequence *Digit Digit – Digit Digit – Letter Letter*. This sequence was exhausted in 2005, when it was replaced by the sequence *Digit Digit – Letter Letter – Digit Digit* currently in force.

- $b > b'$.

In this case, necessarily, $a' > a$ and the positive difference between ab and $a'b'$ may be written as $10(a' - a - 1) + (10 + b' - b)$, so that there are whole, non-negative numbers in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ in the tens and ones place values.

Hence, $a' - a - 1 = e$, and $10 + b' - b = f$. On the other hand, the positive difference between ba and $b'a'$ is $10(b - b') + (a - a') = 10(b - b' - 1) + (10 + a - a')$, so that $b - b' - 1 = g$ and $10 + a - a' = h$.

Therefore,

$$ef + hg = 10(a' - a - 1) + (10 + b' - b) + 10(10 + a - a') + (b - b' - 1)$$
$$= 10(a' - a - 1 + 10 + a - a') + (10 + b' - b + b - b' - 1)$$
$$= 10 \times 9 + 9 = 99.$$

The different strategies used to solve the preceding battery of mathematical problems highlight greater or smaller difficulties in the development of the procedures which lead to the solution. These strategies depend on the quality and quantity of knowledge of the individual on the subject, as well as on the mental schemata and its operationalisation.

Understanding the mechanisms which trigger the *insight* is an intricate process. Even so, the History of Mathematics is also written with *insight* episodes, the most famous is perhaps the one featuring the Hellenic Mathematician Archimedes (287 BC – 212 BC). When entering a container full of water to bathe, Archimedes realised that part of the liquid had spilled and was on the floor. Any other person would have shown displeasure at the situation, but for Archimedes, one of the brightest mathematicians of Ancient Greece, this episode triggered the discovery of the *Buoyancy Principle*[60]. Ecstatic with his discovery, Archimedes ran to the street shouting out loud "*Eureka!*". Archimedes had reason to be happy for he had just discovered the answer to a problem posed by the King of Syracuse. The King had ordered a gold crown but suspected that the goldsmith had mixed the gold of the crown with silver, a less expensive metal. The weight of the crown was correct but the King was still suspicious of the honesty of the craftsman. How to seek the truth? Archimedes put the King's crown and a crown made of pure gold with the same weight in two similar containers full of water. He observed that the volume of the liquid which had spilled from the first container was greater than the volume of the liquid which had spilled from the second container. The fraud was discovered, for the density of silver is inferior to that of gold.

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[60] All bodies immersed in a fluid experience an opposing upward force whose magnitude equals the weight of the volume of the dislocated fluid.

Narratives about mathematical episodes which describe the emotion of discovery may be a powerful motivational agent, especially for students who believe in their failure when performing mathematical activities. During his life, Pythagoras disseminated a doctrine based on Science, Ethics and Religion. The Pythagoreans saw the concept of whole number as the basis for everything. However, a theorem, which would end up being known by the name of its founder, brought down that belief. Terrified, the Pythagoreans discovered that the properties of natural numbers and *rational numbers*[61], obtained by the quotient of natural numbers, were not sufficient to define all quantities. The diagonal of a square whose side measures 1 unit of length became a Nemesis, unsettling Pythagoras to the point of forbidding the dissemination of the discovery. He was not successful.

The irrationality of $\sqrt{2}$ means that it cannot be written as $\frac{a}{b}$, a quotient of natural numbers. Let us recall that the square of an even number is also an even number, and that the square of an odd number is also an odd number. Consider a fraction whose value is $\sqrt{2}$ written in the reduced form $\frac{a}{b}$ (quotient of natural numbers).

$$\frac{a}{b} = \sqrt{2} \Leftrightarrow \left(\frac{a}{b}\right)^2 = 2 \Leftrightarrow a^2 = 2b^2$$

The value of a must be an even number because a^2 is also an even number.

The fraction $\frac{a}{b}$ is in its reduced form. Therefore, the value of the denominator must be an odd number. Otherwise, the numerator and the denominator would be divisible by 2. This is not possible because the fraction $\frac{a}{b}$ is irreducible.

The value of a is an even number. The value of b is an odd number.

This means that a equals 2 times a natural number, $a = 2n$.

After the substitution, $a^2 = 2b^2 \Leftrightarrow (2n)^2 = 2b^2 \Leftrightarrow 4n^2 = 2b^2 \Leftrightarrow 2n^2 = b^2$. The last equality shows that b^2 is an even number, thus implying that b is an even number. Contradiction!

As an alternative reasoning, consider

$$\frac{a}{b} = \sqrt{2} \Leftrightarrow \left(\frac{a}{b}\right)^2 = 2 \Leftrightarrow a^2 = 2b^2$$

[61] The Greeks did not actually know the designation *rational numbers*. However, they worked with *commensurable numbers*. Two line segments are commensurable when there is a third line segment which is common to each of the given line segments.

$$\begin{pmatrix} Note\ that\ by\ the\ unique\ prime\ factorisation\ theorem\ we\ have \\ a^2 = p_1^{\alpha_1}\, p_2^{\alpha_2} \ldots p_k^{\alpha_k}\ with\ p_1 < p_2 < \cdots < p_k primes; \alpha_1, \alpha_2, \ldots, \alpha_k \geq 1 \\ b^2 = p_1^{\beta_1}\, p_2^{\beta_2} \ldots p_k^{\beta_k}\ with\ p_1 < p_2 < \cdots < p_k\ primes; \beta_1, \beta_2, \ldots, \beta_k \geq 1 \end{pmatrix}$$

by the unique-prime-factorisation theorem applied to a^2 and $2b^2$, all primes on the left appear exactly as many times on the right. However, on the left, each prime number appears an even number of times (*i.e.*, in the form p^{α}, where α is an even number), whereas 2 on the right appears an odd number of times.

If questioned about the importance of the number $\sqrt{2}$ and why it is relevant to the understanding of the previous demonstration, the teacher may produce an activity which includes the applicability of the number $\sqrt{2}$. Hence, let us observe the relation between A-format sheets of paper and theoretical Mathematics. The A_4 format defined by the ISO 216 standard measures 297*mm* by 210 *mm* and its area corresponds to $\frac{1}{16}\, m^2$. Students are asked to fill a board *(Table 123)* with the length and width measures of the different sizes of A-format sheets of paper, thus encouraging them to discover the connection between the irrational $\sqrt{2}$, and the result of the division of the length and the width of A-format sheets of paper.

(Table 123)

SHEET OF PAPER	LENGTH / WIDTH (*mm*)
A_0	
A_1	
A_2	
A_3	
A_4	
A_5	

In school levels which precede Tertiary Education, the teacher should present and encourage activities which insert theoretical contents in a given context to motivate students and enhance their persistence so that they achieve better performances. In the History of Mathematics we find such good examples of the effort that is required to solve a problem.

In the 17th century, while studying problems related to the Pythagoras' theorem, Pierre de Fermat (1601-1665) wrote that the equation $x^n + y^n = z^n$, where *n* is greater than 2 and *x*, *y*, *z* and *n* represent whole positive numbers, did not have a solution. However, if n = 1, there is an infinite number of solutions. When n is replaced by 2, solutions are also possible. For example, $3^2 + 4^2 = 5^2$. The mathematician wrote in the margin of a page of his copy of Diophantus' *Arithmetica* that he could prove it but clearly did not have the space to do it. Years later, a new edition of Diophantus' *Arithmetica* completed with Pierre de Fermat's margin notes was published. Still, the 17th century ended and no one was able to present the long-awaited demonstration. In the following centuries progress was made to enhance the procedures which would allow to prove that the equation $x^n + y^n = z^n$, where n is greater than 2 and x, y, z and n represent whole positive numbers, was unsolvable. At last, in 1993, Andrew Wiles proved that there is no solution. Later, after a minor correction, Andrew Wiles' demonstration was validated by the scientific community. Nevertheless, Fermat's Last Theorem still inspires passion among mathematicians, for the demonstration presented by Andrew Wiles uses mathematical tools which did not exist in the 17th century, and therefore could not be known to Pierre de Fermat.

In mathematics, problem solving encompasses theoretical, practical and theoretical-practical issues with different levels of difficulty which require different stages of specific knowledge. Therefore familiar subjects may become an opportunity for students to get more involved in mathematical contents. Well-known situations provide a prolific source of questions to work in the classroom. Such kind of contents is linked to *Mathematics Applied to Social Sciences*, a school subject for students who opted for Humanity Courses in Secondary Education. A problem solving programme should explore such resources. The next topics include the Theory of Voting, Probabilities and Fair Division.

Mathematics applied to life in society

In France, the institution of the Republic (1789) revolutionised the political system. The study of voting methods receives major contributions from Marie Jean Antoine Nicolas Caritat (1743 - 1794), Marquis de Condorcet and Jean de Charles Borda (1733 - 1799).

In what concerns the right to vote in a Democracy, all citizens are equal. That ideal is implemented through the principle *one person – one vote*. But should this system be used in events which involve, for example, organisations or countries? Should all the participants in a voting process have the same power and influence in the outcome? An alternative approach advocates the principle *one voter – x votes*, where voters may not have the right to the same number of votes. The question arises as to what the relation between the number of votes assigned to a given voter and the weight of those votes in the results of the vote is. Three elements should be taken into account: the voters, the votes assigned to each voter and the quota, represented by the letter q (the minimum of votes required in order to pass a motion). Let us only consider votes where the voter is limited to a choice between *Yes* and *No*. It is universally known that the quota should be greater than half of all votes. If $E_1, E_2, \ldots, E_n$ represent the electors and $w_1, w_2, \ldots, w_n$ their respective weights (*weight* or votes), then, the inequality for q should be

$$\frac{w_1 + w_2 + \cdots + w_n}{2} < q \leq w_1 + w_2 + \cdots + w_n.$$

In the following examples, the notation $[q: w_1, w_2, \dots, w_n]$ is used, *i.e.*, the value of the quota and the number of votes assigned to each elector in decreasing order.

$[4: 3, 3, 2,1]$ The value of the quota, 4, is less than half of the total number of votes, 9. If electors E_1 and E_4 vote *Yes*, and E_2 and E_3 vote *No*, then the rival groups will reach the value of the quota. The result is anarchy since both groups reach / exceed the quota.

$[3: 1, 1, 1, 1, 1]$ In this voting system all voters have the same number of votes. For the motion to be approved, 3 of the 5 voters ought to have voted *Yes*. Here, the context is *one person – one vote*, where a simple majority is enough to pass the motion.

$[15: 5, 4, 3, 2, 1]$ In this situation, 5 voters have a total of 15 votes. The value of the quota is 15. For the motion to be approved, unanimity is required.

$[4: 5, 2, 1]$ Dictatorial voting system. Voter E_1 controls enough votes for the approval of the motion. All the other voters, irrespective of the number of votes assigned to each, do not have any influence in the final results. They are therefore called stooge voters.

$[7: 4, 3, 2, 1]$ Even though voter E_1 is not a dictator, he/she has the power to prevent the approval of a motion. All the other voters, together, do not have enough votes to reach the value of the quota. In this context, E_1 has veto power.

$[9: 8, 8, 1]$ What is the real power of each voter in this voting system? Voters E_1 and E_2 hold a large number of votes. However, for a deliberation to be approved, at least the votes of 2 voters are required. Hence, E_1, E_2 and E_3 have the same influence in the outcome.

John Banzhaf quantified the influence of each voter in a *one voter – x votes* system according to the algorithm: *1)* draw up a list of all possible coalitions; *2)* identify the winning coalitions; *3)* identify the voters necessary to win the election for each winning coalition (critical voter); *4)* count the total number of times that *E* is a critical voter (represent that value by B_E); *5)* sum all critical voters in each winning coalition (represent this value by *T*). The Banzhaf power index for each voter *E* is obtained from the division of B_E by *T*. Let us calculate the Banzhaf power index for a company managed by three partners, identified by the letters A, B and C, who have respectively 3, 2 and 1 votes. Any decisions require a majority of votes (a minimum of 4 in a total of 6), a condition defined by $[4: 3, 2, 1]$. To determine the decision power of each element, we note the following:

1) There are 7 possible coalitions, {A}, {B}, {C}, {A, B}, {A, C}, {B, C}, {A, B, C};

2) The winning coalitions are: {A, B}, {A, C}, {A, B, C};

3)

WINNING COALITIONS	CRÍTICAL VOTERS
{A, B}	A and B
{A, C}	A and C
{A, B, C}	A

4) A is 3 times critical; B is 1 time critical; C is 1 time critical.

5) In winning coalitions, sum the number of critical voters.

This example has 5: 2 + 2 + 1 = *T*.

The Banzhaf power index for each partner is:

Partner A: $\frac{B_A}{T} = \frac{3}{5} = 60\%$ Partner B: $\frac{B_B}{T} = \frac{1}{5} = 20\%$ Partner C: $\frac{B_C}{T} = \frac{1}{5} = 20\%$

Even though partners B and C have a different number of votes, they hold the same influence in the outcome.

(Subject: Applied Mathematics – Secondary Education students)

For a motion to be approved by the United Nations Security Council, an organisation which comprises 15 member-countries, 5 permanent members (United Kingdom, China, France, Russia and United States of America) and 10 non-permanent members elected for 2-year terms, on rotation; the consent of the 5 permanent members and of at least 4 non-permanent members is required.

What is the power relation between the members of the United Nations Security Council?

Let us suppose that each non-permanent member has the right to 1 vote and each permanent member has the right to *x* votes. This is a *1 voter – x votes* system. In compliance with the rules of the United Nations Security Council for motion approval, the quota *q* ought to be greater than the number of votes of all the non-permanent members (10) and of 4 permanent members (4*x*) and inferior or equal to the number of votes of all the permanent members (5*x*) and 4 non-permanent members (4) in order to tackle inequalities.

$$\begin{cases} 5x + 4 \geq q \\ 4x + 10 < q \end{cases}, x \in \mathbb{N}.$$

Let us observe the hypothetical number of x votes of the permanent members. Considering that $4x + 10 < q \leq 5x + 4$, necessarily $x > 6$. Let us assign the value 7 to x. Hence, the value of the quota q is 39, because $4 \times 7 + 10 = 38 < q \leq 39 = 5 \times 7 + 4$ admits no other integer for q.

The voting system of the United Nations Security Council can be modelled as follows: $[39: 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$. If the value of the quota is 39, then each permanent member has 7 votes and each non-permanent member 1 vote. The only winning coalitions are the following: 5 permanent members and 4 non-permanent members; 5 permanent members and 5 non-permanent members, 5 permanent members and 6 non-permanent members, 5 permanent members and 7 non-permanent members, 5 permanent members and 8 non-permanent members, 5 permanent members and 9 non-permanent members, and 5 permanent members and 10 non-permanent members. Thus we have,

$$\binom{5}{5}\binom{10}{4} + \binom{5}{5}\binom{10}{5} + \binom{5}{5}\binom{10}{6} + \binom{5}{5}\binom{10}{7} + \binom{5}{5}\binom{10}{8} + \binom{5}{5}\binom{10}{9} + \binom{5}{5}\binom{10}{10}$$

$$= \sum_{k \geq 4} \binom{10}{k} = 2^{10} - \sum_{k \leq 3} \binom{10}{k} = 1024 - 1 - 10 - \binom{10}{2} - \binom{10}{3} = 1024 - 176 = 848$$

winning coalitions. Every permanent member is critical, *i.e.*, indispensable for the 848 winning coalitions. Non-permanent member is critical in a small number of the winning coalitions, namely in those consisting of 5 permanent members and exactly 4 non-permanent members. This makes $\binom{5}{5}\binom{10}{4} = 210$ of the winning coalitions. The total number of critical voters in all possible situations is $848 \times 5 + 210 \times 4 = 4240 + 840 = 5080$. A non-permanent member becomes a critical voter only in the subclass of $\binom{9}{3} = \frac{9 \times 8 \times 7}{3!} = 84$ of the 210 winning coalitions.

JOHN BANZHAF'S POWER INDEX
Permanent member: $\frac{848}{5080} \cong 16.7\%$
Non-permanent member: $\frac{84}{5080} \cong 1.65\%$

According to John Banzhaf's index, the decision power of a permanent member, when compared to a non-permanent member, is substantially higher. Voting is about choices, but the outcome can eventually be conditioned by the method applied.

> *There is a set of conceivable actions which an individual could take, each of which leads to certain consequences. The individual has in mind an ordering of all possible consequences of actions, saying, for each pair of consequences, either that he prefers one or that he is indifferent between them; these relations of preference and indifference have the property (known as transitivity) that if consequence A is preferred to consequence B and B to C, then A is preferred to C, and similarly with indifference* (Arrow, 1951) .

D'Hondt's (1841 – 1901) method is commonly used to convert votes into mandates. The proliferation of voting systems such as Hamilton's apportionment method, Webster's method, Huntington-Hill's method, Jean-Charles Borda's counting method and Condorcet's method illustrates the importance assumed by voting in political science or economics.

The D'Hondt's procedure consists of: *1)* counting the number of votes obtained by each candidate/list in the constituency; *2)* dividing the number of obtained votes by each candidate/list by, successively, 1, 2, 3, 4, 5, etc..., then aligning the quotients into descending order according to quantity in a series of terms equal to the number of mandates for constituency; *3)* successively distributing the mandates by the candidates/list according to the highest terms of the series established by the previous rule; *4)* in case only one mandate remains to be distributed and the following terms of the series are equal and from different lists, the mandate will be given to the list which obtained an inferior number of votes.

Let us apply the Hondt's method to the results of the 2013 local elections in the municipality of Coimbra, where the City Council has 11 members *(Table 124)*.

(Table 124)

Voters	Votes	Blank	Invalid	CDS-PP	PAN	PCTP-MRPP	PS	PCP-PEV	PSD.PPM.MPT	CPC-IV
129,060	63,744	3,404	1,828	2,691	945	559	22,632	7,079	18,730	5,876

Source: (CNE)

(Table 125)

DIVISORS	CDS-PP	PAN	PCTP-MRPP	PS	PCP-PEV	PSD.PPM.MPT	CPC-IV
1	2,691	945	559	**22,632**	**7,079**	**18,730**	**5,876**
2	1,345.5	472.5	279.5	**11,316**	3,539.5	**9,365**	2,938
3	897	315	186.(3)	**7,544**	2,359.(6)	**6,243.(3)**	1,958.(6)
4	672.75	236.25	139.75	**5,658**	1,769.75	**4,682.5**	1,469
5	538.2	189	111.8	**4,526.4**	1,415.8	3,746	1,175.2
6	448.5	157.5	93.1(6)	3,772	1,179.8(3)	3,121.(6)	979.(3)

Partido Socialista (PS) obtained 5 mandates, the coalition formed by *Partido Social Democrata* (PSD), *Partido Popular Monárquico* (PPM) and *Movimento Partido da Terra* (MPT) obtained 4 mandates and the movement *Cidadãos por Coimbra* (CPC – IV) obtained 1 mandate *(Table 125)*. Proportionally, the Hondt's algorithm provides greater representativeness to the parties with greater electoral expression. The premise is that the Hondt's method endorses political forces to form pre-electoral coalitions in order to attain more mandates.

The Social Sciences are not confined to the study of voting methods. The use of the dice as a game object goes back to Ancient Times. Today, dice are used in many board games. The cube has its faces engraved with natural numbers from 1 to 6, and the sum of the values on the opposite faces is 7. But there are dice with different geometrical shapes and numbers.

Associated to dice games, an interest was developed to calculate the probability of victory for each player. Girolano Cardano (1501 - 1576) was the first to consistently study this subject. In 1526, Cardano wrote *Liber de Ludo Aleae*, a small manual on gambling. Others followed, among whom were Pierre de Fermat (1601 - 1665), Blaise Pascal (1623 - 1662) and Christiaan Huygens (1629 – 1695). The treatise *Théorie Analytique des Probabilités* (1812) written by Pierre Laplace (1749 – 1827) was a milestone concerning Probabilities. The treatise includes a fundamental law which is extremely useful for solving problems involving uncertainty, on the assumption that the elementary events studied have equal possibility of happening: *The probability of an event is equal to the quotient between the number of cases favourable to the event and the number of possible cases*.

(Subject: Probability and Statistics – 9th grade students)

When simultaneously throwing 2 dice, both numbered from 1 to 6, and summing the values, in what results should a player bet in order to have the greatest probability of winning?

The goal is to determine which value resulting from the sum of the numbers obtained by the throw of the two dice has the greatest probability of occurring. The range of results may be determined with the construction of a double-entry chart *(Table 126)*.

(Table 126)

+	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Consider Ω as the sample space;

$$\Omega = \begin{Bmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6), \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6), \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6), \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6), \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6), \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6), \end{Bmatrix}$$

If X is the real random variable *Sum of the values obtained by the throw of two dice,* defined in the sample space Ω, probabilities are inserted in the following chart *(Table 126)*. For example, the value 4 is obtained from the pairs (1,3), (2,2), (3,1) so $P(X = 4) = \frac{3}{36} = \frac{1}{12}$.

Table 126

x_i	2	3	4	5	6	7	8	9	10	11	12
$p\ (X = x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

The most probable value to occur (7) is symmetrically distributed.

Now, let us analyse a probability problem in the scope of Biology.

(Subject: Probability and Statistics – 12th grade students)

Consider a microbe which may or may not reproduced itself before it dies, being p the microbe's probability to die. If it does not die, it will originate two other genetically equal microbes. Demonstrate that the probability for the microbe to create a colony of infinite duration is

$$\frac{1-2p}{1-p}.$$

Let us identify the death of the microbe by the symbol †. The statement of the problem refers that the microbe's probability of survival is *(1 - p)*. For a colony of infinite duration to be created, the microbe does not die and replica A originates a colony of infinite duration or replica B originates a colony of infinite duration.

The fact that a microorganism forms a colony of infinite duration is identified by the symbol C_∞. Considering that A and B are replicas of the initial microbe, then $p_A(C_\infty) = p_B(C_\infty) = p_m(C_\infty)$ is the microbe's probability of producing a colony C_∞.

Hence,

$$p_m(C_\infty) = p_m(\sim †) \times \big(p_A(C_\infty) \vee p_B(C_\infty)\big)$$

$$= (1-p) \times \big(p_A(C_\infty) + p_B(C_\infty) - p_A(C_\infty) \times p_B(C_\infty)\big)$$

$$= (1-p) \times \big(p_m(C_\infty) + p_m(C_\infty) - p_m(C_\infty) \times p_m(C_\infty)\big)$$

$$= (1-p) \times \big(2p_m(C_\infty) - {p_m}^2(C_\infty)\big).$$

Consider

$$p_m(C_\infty) = (1-p) \times \big(2p_m(C_\infty) - {p_m}^2(C_\infty)\big).$$

When dividing both members by $p_m(C_\infty)$, then

$$1 = (1 - p) \times \big(2 - p_m(C_\infty)\big).$$

From the equation we can see that

$$\frac{1}{1-p} = 2 - p_m(C_\infty) \Leftrightarrow p_m(C_\infty) = \frac{1-2p}{1-p}.$$

Let us explore the application of probabilistic concepts to Biology a little further.

Three cards which are indistinguishable to touch are put in ballot box A. One of the cards has the letter *a* written in it and the other 2 the letter *b*. The probability of randomly taking the card with the letter *a* out of the ballot box is $\frac{1}{3}$. The probability of taking out a card with the letter b is $\frac{2}{3}$. As we will see, information should be organised in polynomic form as $\frac{1}{3}a + \frac{2}{3}b$. Now let us consider ballot box B, which contains 1 card with the letter *a*, 2 cards with the letter *b* and 3 cards with the letter *c*. For this case, the polynomial is $\frac{1}{6}a + \frac{2}{6}b + \frac{3}{6}c$. The situation of extracting 1 card out of each ballot box, two independent events, has as a probability the product of the probabilities of each event. Hence, $\left(\frac{1}{3}a + \frac{2}{3}b\right) \times \left(\frac{1}{6}a + \frac{2}{6}b + \frac{3}{6}c\right) = \frac{1}{18}a^2 + \frac{2}{9}ab + \frac{2}{9}b^2 + \frac{1}{6}ac + \frac{1}{3}bc$.

Due to the previous description, the rules of multiplication of polynomials reflect certain probabilistic considerations. This polynomial indicates that the probability to extract an *x* from a box and a *y* from another box is $\frac{2}{9}$. These considerations can be applied in school. Let us see how Gregor Mendel (1822 – 1884) applied probabilistic concepts to his work on hereditary transmission in plants. His option for the pea plant was a happy one, for this species reproduces itself in compliance with hereditary laws, which were the focus of his experiments.

Experiment One: to cross-breed purebred yellow pea seeds with purebred green pea seeds. Results: only yellow pea seeds.

Experiment Two: to cross-breed the yellow pea seeds produced in experiment one. Results: yellow pea seeds and green pea seeds were obtained in a ratio of 3: 1.

At the time, this was a surprising outcome for the scientific community, for only the yellow phenotype was expected. Mendel concluded that the phenotype was determined not by one but by two hereditary factors, the genotype. Hence, the cross-breeding of purebred yellow pea seeds (*YY*) with purebred green pea seeds (*GG*) would result in yellow pea seeds.

In mathematical language, the first experiment is represented as

$$\left(\frac{1}{2}Y+\frac{1}{2}Y\right)\times\left(\frac{1}{2}G+\frac{1}{2}G\right)=\frac{1}{4}YG+\frac{1}{4}YG+\frac{1}{4}YG+\frac{1}{4}YG=\frac{1}{4}(YG+YG+YG+YG)=YG$$

and the second experiment as

$$\left(\frac{1}{2}Y+\frac{1}{2}G\right)\times\left(\frac{1}{2}Y+\frac{1}{2}G\right)=\frac{1}{4}YY+\frac{1}{2}YG+\frac{1}{4}GG.$$

Corroborated by empirical data, the Mendel model corresponded to the expectable according to probabilistic laws. In what concerns the manifestation of colour, the yellow is *dominant* and the green is *recessive*. In these experiments the dominant allele was identified by a capital letter, and the recessive allele by a lowercase letter.

Hence, for experiments one and two, we have respectively

$$\left(\frac{1}{2}Y+\frac{1}{2}Y\right)\times\left(\frac{1}{2}y+\frac{1}{2}y\right)=\frac{1}{4}Yy+\frac{1}{4}Yy+\frac{1}{4}Yy+\frac{1}{4}Yy=\frac{1}{4}(Yy+Yy+Yy+Yy)=Yy,$$

expressed by the

Law of Dominance: From the cross-breeding of two purebred specimens, *i.e.*, homozigotics which have a different pair of alleles, the next generation hybrids (F^1) are all equal.

and

$$\left(\frac{1}{2}Y+\frac{1}{2}y\right)\times\left(\frac{1}{2}Y+\frac{1}{2}y\right)=\frac{1}{4}YY+\frac{1}{2}Yy+\frac{1}{4}yy,$$

expressed by the

Law of Segregation: From the cross-breeding of two specimens of the F^1 generation, in the case of dominant heredity, phenotypes occur in the statistical ratio 3: 1 and genotypes occur in statistical ratios 1: 2: 1. In the case of intermediate heredity (no dominant alleles), both phenotypes and genotypes occur in the statistical ratio 1: 2: 1.

Let us look at the expected ratios in cross-breeding *1) Yy* × *yy* and *2) YY* × *Yy*.

1) Yy × *yy*

$$\left(\frac{1}{2}Y+\frac{1}{2}y\right)\times\left(\frac{1}{2}y+\frac{1}{2}y\right)=\frac{1}{2}Yy+\frac{1}{2}yy$$

Phenotypes occur in the ratio 1: 1 and genotypes *Yy* and *yy* in ratios 1: 1.

2) YY × Yy

$$\left(\frac{1}{2}Y+\frac{1}{2}Y\right)\times\left(\frac{1}{2}Y+\frac{1}{2}y\right)=\frac{1}{2}YY+\frac{1}{2}Yy$$

Hence, only the yellow colour occurs while genotypes *YY* and *Yy* have an equal incidence.

Mendel researched a hypothetical dependence between pea colour and pod shape, round or wrinkled, of the pea plant. Hence, he crossbred round pod, yellow pea seeds (*YY RR*) with wrinkled pod, green pea seeds (*yy rr*), but the round pod was dominant and the wrinkled recessive. This corresponds to the computation,

$$\left[\left(\frac{1}{2}Y+\frac{1}{2}Y\right)\times\left(\frac{1}{2}R+\frac{1}{2}R\right)\right]\times\left[\left(\frac{1}{2}y+\frac{1}{2}y\right)\times\left(\frac{1}{2}r+\frac{1}{2}r\right)\right]$$
$$=\frac{1}{16}[(Y+Y)\times(R+R)\times(y+y)\times(r+r)].$$

The expanded polynomial *Yy Rr* shows that all F^1 generation pea plants have round pod, yellow pea seeds. If we now crossbreed plants from the F^1 generation we get

$$\left[\left(\frac{1}{2}Y+\frac{1}{2}y\right)\times\left(\frac{1}{2}R+\frac{1}{2}r\right)\right]\times\left[\left(\frac{1}{2}Y+\frac{1}{2}y\right)\times\left(\frac{1}{2}R+\frac{1}{2}r\right)\right]$$
$$=\frac{1}{16}\left[\left((Y+y)\times(R+r)\right)\times(Y+y)\times(R+r)\right]$$
$$=\frac{1}{16}[(YR+Yr+yR+yr)\times(YR+Yr+yR+yr)]$$
$$=\frac{1}{16}(Y^2R^2+Y^2Rr+YyR^2+YyRr+Y^2rR+Y^2r^2+YyrR+Yyr^2$$
$$+yYR^2+yYRr+y^2R^2+y^2Rr+yYrR+yYr^2+y^2rR+y^2r^2).$$

Considering the phenotypes, the letter $\bar{Y}$ was introduced for *YY* and *Yy*, the letter $\bar{R}$ for *RR* and *Rr*, the letter $\bar{y}$ for y^2, and the letter $\bar{r}$ for r^2.

$$\frac{1}{16}(\bar{Y}\bar{R}+\bar{Y}\bar{R}+\bar{Y}\bar{R}+\bar{Y}\bar{R}+\bar{Y}\bar{R}+\bar{Y}\bar{r}+\bar{Y}\bar{R}+\bar{Y}\bar{r}+\bar{Y}\bar{R}+\bar{Y}\bar{R}+\bar{y}\bar{R}+\bar{y}\bar{R}+\bar{Y}\bar{R}+\bar{Y}\bar{r}+\bar{y}\bar{R}$$
$$+\bar{y}\bar{r})=\frac{1}{16}(9\bar{Y}\bar{R}+3\bar{Y}\bar{r}+3\bar{y}\bar{R}+\bar{y}\bar{r})$$

Hence, in probabilistic terms, the polynomic calculates the expected number of pea plants with the following characteristics: round pod, yellow pea (9); wrinkled pod, yellow pea (3); round pod, green pea (3); wrinkled pod, green pea (1) for each 16 F^2 generation pea plants. Mendel's results show that pea colour and pod shape are independent results.

For example:

$$P(round\ pod, yellow\ pea\ seed) = P(yellow\ pea\ seed\ ) \times P(round\ pod\ seed)$$

$$\frac{9}{16} = \frac{12}{16} \times \frac{12}{16}$$

Actually, the polynomic has 9 terms which comprise the dominant characteristics (Y and R), 12 terms with Y and 12 terms with R. Similarly,

$$P(wrinkled\ pod, yellow\ pea\ seed) = P(yellow\ pea\ seed) \times P(wrinkled\ pod\ seed)$$

$$\frac{3}{16} = \frac{12}{16} \times \frac{4}{16}$$

$$P(round\ pod, green\ pea\ seed) = P(green\ pea\ seed\ ) \times P(round\ pod\ seed)$$

$$\frac{3}{16} = \frac{4}{16} \times \frac{12}{16}$$

$$P(wrinkled\ pod, green\ pea\ seed) = P(green\ pea\ seed\ ) \times P(wrinkled\ pod\ seed)$$

$$\frac{1}{16} = \frac{4}{16} \times \frac{4}{16}$$

Law of Independent Distribution of Characters: From the cross-breeding of hybrids, alleles of different phenotypes are independently transmitted.

Mendel's theory of heredity was a breakthrough even though it was object of criticism by many of his colleagues. A sensitive subject was the transmission of rare characteristics in humans, such as colour vision deficiency and left-handedness, which should have been extinguished after a few generations.

In autonomous research, Godfrey Harold Hardy and Weinberg revealed that regardless of the distribution of the frequencies of the genotypes in two-allele-systems, *in a panmictic population*[62], *from the first filial generation onwards the frequencies of the three possible genotypes are constant*. Let us now analyse the Hardy-Weinberg principle in detail. Consider a population with genotypes *YY*, *Yy*, *yy*, whose frequencies are *a*, *b*, *c*, respectively. Regarding the possibility of finding each of these genotypes in the filial generation, each parent transmits 1 of their alleles to their descendant, we may determine the respective probabilities via the computation below. Note that $\left(a + \frac{1}{2}b\right)$, for example, is the relative frequency with which allele Y exists in the parental generation.

[62] A panmictic population is a population where the probability of the crossing of two genotypes equals the product of the probabilities of finding these genotypes in that population.

$$\left[a\left(\frac{1}{2}Y+\frac{1}{2}Y\right)+b\left(\frac{1}{2}Y+\frac{1}{2}y\right)+c\left(\frac{1}{2}y+\frac{1}{2}y\right)\right]^2=\left[\left(a+\frac{1}{2}b\right)Y+\left(c+\frac{1}{2}b\right)y\right]^2$$

$$=\left(a+\frac{1}{2}b\right)^2 YY+2\times\left(a+\frac{1}{2}b\right)\times\left(c+\frac{1}{2}b\right)Yy+\left(c+\frac{1}{2}b\right)^2 yy$$

$$=a'YY+b'Yy+c'yy.$$

where a', b', c' are, respectively, the coefficients of the monotypes YY, Yy and yy, to which correspond the same probabilities of finding genotypes YY, Yy and yy in the filial generation.

An operational relation which characterises the evolution of the frequencies of the alleles from a generation to the next can thus be established:

$$(a,b,c)\overset{\phi}{\rightarrow}(a',b',c')$$

$$:=\left[\left(a+\frac{1}{2}b\right)^2,2\times\left(a+\frac{1}{2}b\right)\times\left(c+\frac{1}{2}b\right),\left(c+\frac{1}{2}b\right)^2\right]$$

In what concerns hereditary transmission from F^1 generation to F^2 generation,

$$(a'',b'',c'')=\phi(a',b',c').$$

This means that,

$$a''=\left(a'+\frac{1}{2}b'\right)^2=\left[\left(a+\frac{1}{2}b\right)^2+\frac{1}{2}\times 2\times\left(a+\frac{1}{2}b\right)\times\left(c+\frac{1}{2}b\right)\right]^2$$

$$=\left(a+\frac{1}{2}b\right)^2\times\left[\left(a+\frac{1}{2}b\right)+\left(c+\frac{1}{2}b\right)\right]^2$$

$$=\left(a+\frac{1}{2}b\right)^2\times(a+b+c)^2=\left(a+\frac{1}{2}b\right)^2=a'.$$

Note: a,b,c are probabilities whose sum is 1.

Similarly, $b''=b'$ and $c''=c'$. Then we can conclude that $\phi\circ\phi=\phi$, *i.e.*, ϕ is independent, thus corroborating the Hardy-Weinberg Law.

Let us now look at another group of problems about probabilities. In the 17th century Antoine Gombaud (1607 – 1684), a famous gambler and expert in calculations also known as Chevalier de Méré, a title adopted by himself as he did not belong to nobility, questioned mathematician Blaise Pascal about a problem which fascinated gamblers since the Middle Ages.

Two players who have the same probability of winning start gambling. At a certain moment they are interrupted and cannot conclude the game. According to the score of the game at the moment of the interruption, how should the money from the bet be divided?

(Subject: Probability and Statistics – 12th grade students)

Two players bet 32 silver coins each. The game is fair, *i.e.*, for each round, the probability of winning is equal for both. The winner of each round scores 1 point, the looser does not score. The rules say that the first player to achieve 3 points is declared the winner. However, something unexpected happens and the game is interrupted when player A scores 2 points and player B 1 point. How to fairly divide the 64 coins of the bet?

This problem aroused the interest of Blaise Pascal, who wrote a letter on the subject to Pierre de Fermat. What would have happened had the game not been interrupted ought to be studied in order to fairly divide the money. Surely, in this situation, the meaning of a fair division is a matter of opinion. Nevertheless, it is reasonable to think that the money should be divided according to A or B's relative probabilities of having won the game.

Let us make a register *(Chart 14)* to systematise what could have happened if the game had not been interrupted.

(Chart 14)

	Winner	Winner	Winner
4th Round	A	B	B
5th Round		A	B
Final winner	A	A	B

According to the information gathered from the chart, player A would have the right to $\frac{2}{3}$ of the total amount of money, and player B would have the right to the remaining money. Apparently, all possible situations have been identified but the chart above does not reflect that in round 4 players A and B have the same probability of winning. A corrected diagram would read as follows *(Chart 15)*.

(Chart 15)

	Winner	Winner	Winner	Winner
4th Round	A	A	B	B
5th Round			A	B
Final winner	A	A	A	B

Hence, player A would have the right to $\frac{3}{4}$ of the amount of money, *i.e.*, 48 coins, and player B would have the right to the 16 remaining coins.

$$Coins_{player\ A} = \frac{1}{2} \times 64 + \frac{1}{2} \times 32 = 32 + 16 = 48$$

Player A has the right to 48 coins and player B to the remaining 16 coins.

The initial error arose from the study of the strictly necessary results to find the winner of the game. However, all possible results must be considered.

What would the possible scores of each player be after the 4th round is completed? If player A wins, he/she scores 3 points, while player B still scores only 1 point. If player B wins the round, then both players will score 2 points. If the game had gone on without being interrupted, player A would have had a 50% chance of winning the 4th round, which would have given him/her the 64 coins. Otherwise, they would both have scored 2 points. In this case, there would have been no need to divide the money for both players would have kept their initial 32 coins.

How to perform the division of the money if at the moment of the interruption player A has 2 points and player B has 0 points? Let us check the possible situations *(Chart 16)*.

(Chart 16)

	Winner	Winner	Winner	Winner	Winner	Winner	Winner	Winner
3rd Round	A	A	A	A	B	B	B	B
4th Round					A	A	B	B
5th Round							A	B
Final winner	A	A	A	A	A	A	A	B

Player A has the right to $\frac{7}{8}$ of the money, *i.e.*, 56 coins, and player B to the remaining value, 8 coins.

How to divide the money if at the moment of the interruption player A had 1 point and player B 0 points? Player A would have a 50% chance to win the 2nd round, which would give him/her 2 points, while player B still scored 0. If player A lost the 2nd round, both players would score 1 point and the money would not have to be divided, as both players would keep their initial 32 coins.

$$Coins_{player\ A} = \frac{1}{2} \times 56 + \frac{1}{2} \times 32 = 28 + 16 = 44$$

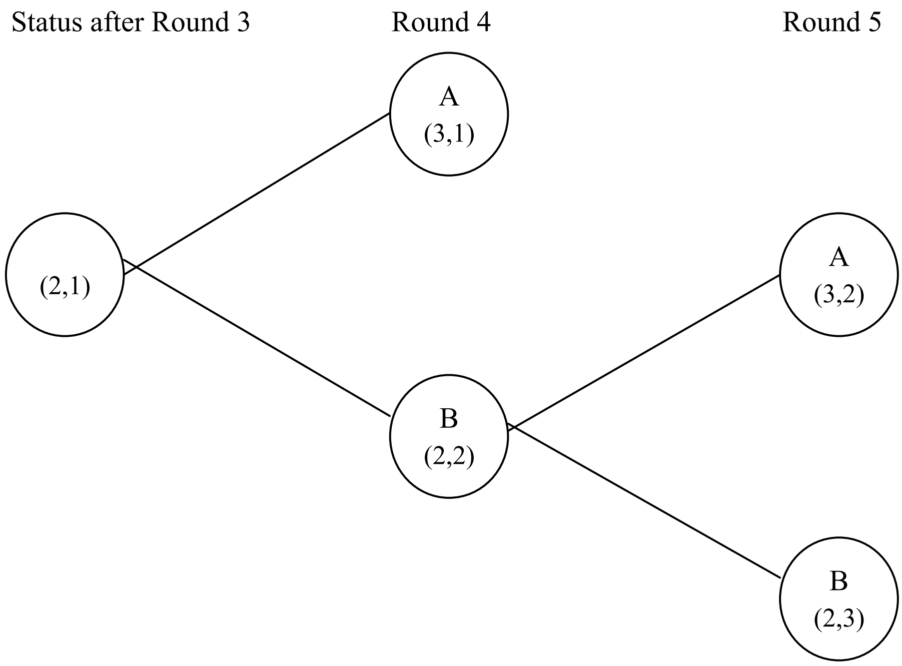
Player A has the right to 44 coins and player B to the remaining 20 coins.

Discussion on Chevalier de Mere's problem.

We want to know the probability of A (or B) winning the game after A has scored 2 points and B 1 point. We are in round 4. If A wins, then A scores 3 points and B remains with only 1 point, and the game is over.

If A loses in round 4, then A and B both score 2 points, and the game will move to round 5. Here there are 2 possibilities: whether A wins or B wins, and the final winner is the player who wins this last round.

The following diagram *(Figure 70)* shows the possible developments of the game after round 3. The diagram represents 2 scenarios favourable to player A and 1 scenario favourable to player B.

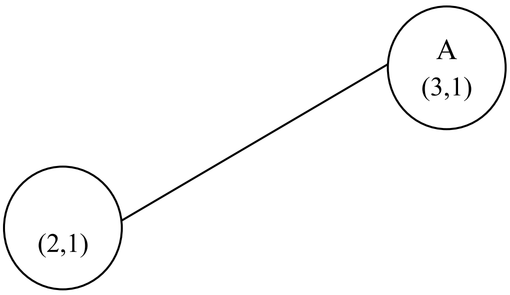


(Figure 70)

We may simulate a dialogue between 2 characters, P_1 and P_2, regarding this problem.

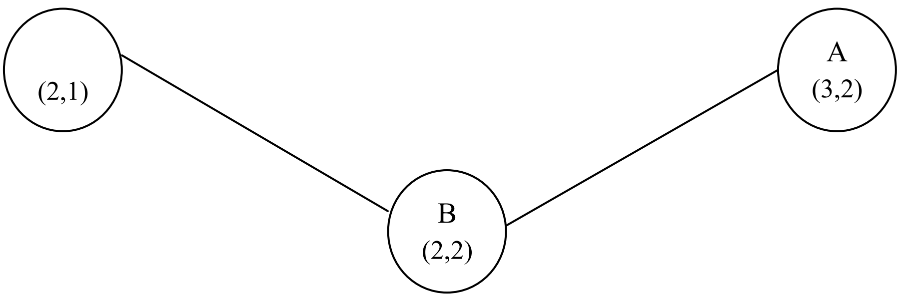
P_1: If after the 3rd round A has 2 points, A has a $\frac{2}{3}$ chance to win. I mean, in 99 games in which, after the 3rd round, A has 2 points, and B only 1 point, we see that A wins approximately 66 games and B 33. Right?

P_2: I doubt it. You're assuming that, from the 3rd round, the branches of the diagram, that is, the 3 paths to the end, will follow the same frequency: 33 times each. But that's not true! Half of the times another path will be followed *(Figure 71)*,



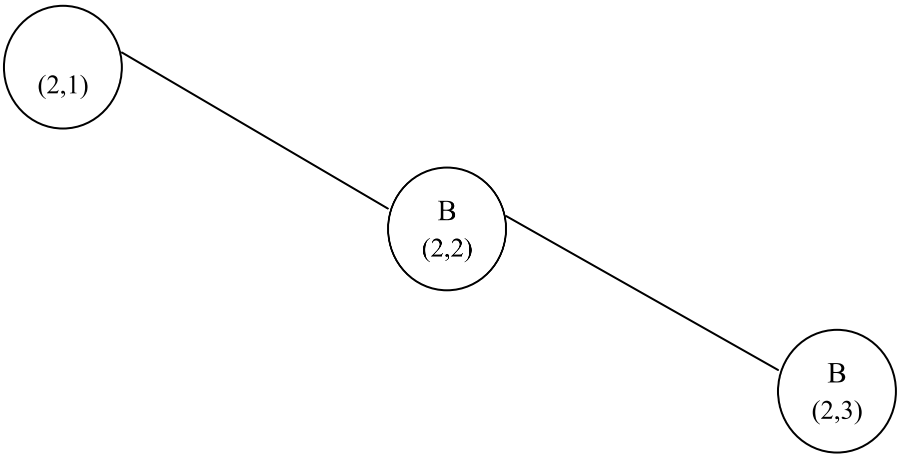
(Figure 71)

and half of the times one of the paths *(Figure 72)*,



(Figure 72)

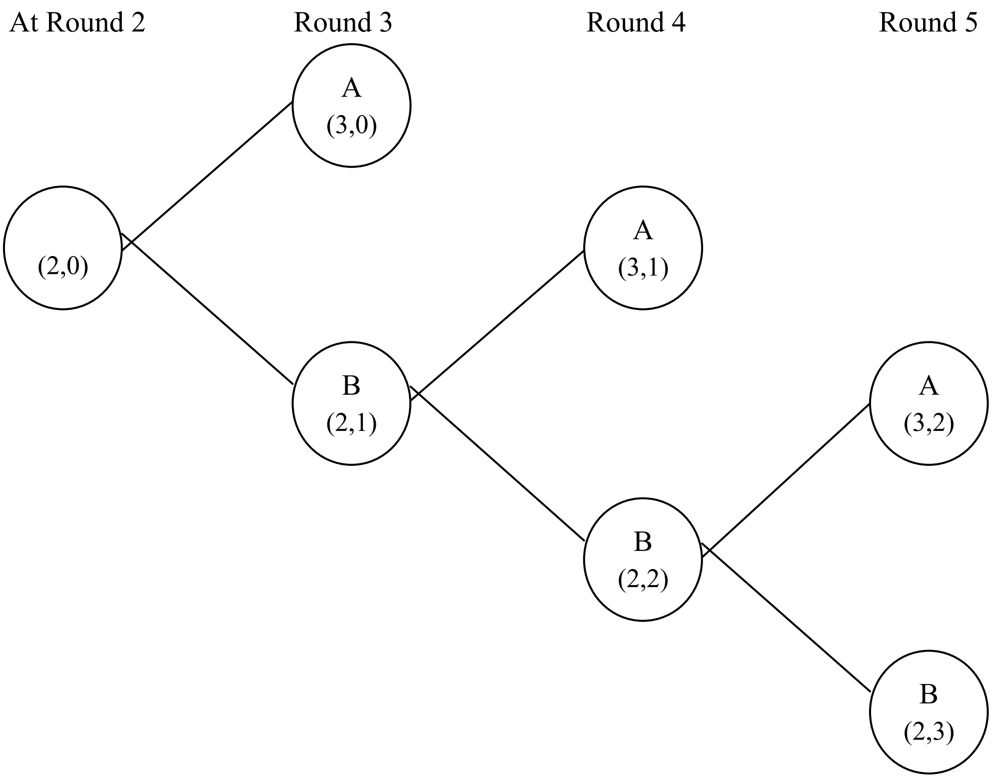
or *(Figure 73)*



(Figure 73)

In the second case, all paths are equally probable. Therefore, the truth is that A will win $\frac{99}{2}+\frac{99}{4}$ of the times, i.e., has a $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$ chance of winning.

If 2 rounds had been played and both were favourable to player A, then *(Figure 74)*,



(Figure 74)

In this case, A's probability of winning is $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{4+2+1}{8}=\frac{7}{8}$.

Draw a diagram of when only round 1 has been played and the winner was player A.

(Subject: Probability and Statistics – 12th grade students)

The Chevalier de Méré used a simple but effective strategy. By making bets with a possibility of success slightly greater than 50%, he achieved, after a considerable number of games, substantial gains. The plan required some initial investment because the first rounds could eventually benefit his opponents. However, as the number of games increased, gain was mathematically assured. In order to diversify his bets, this inveterate player proposed to achieve a *double 6* at least once, in 24 throws of two dice. He believed that the advantage was on his side. However, to his astonishment and incredulity, the games invariably ended in an accumulation of losses. His reasoning was as follows: *When only 1 dice is thrown, the hypothesis of obtaining the face 6 is 1/6, and as 3×1/6 = 50% and 4×1/6≅67%, he would have to throw the dice 4 times to have more than a 50% chance of obtaining the face 6 at least once. Hence, when a pair of dice is thrown, there is a total of 36 chances, i.e., 6 times more chances than with the throw of only 1 dice. Therefore, the 2 dice ought to be thrown 6×4 = 24 times in order to have more than a 50% chance of obtaining a double 6 at least once.*

After many games, how to justify substantial gains in the first situation and significant losses in the second? Identify and correct the errors committed by the Chevalier de Méré.

The goal is to calculate the probability of obtaining the face 6 at least once, in 4 throws of a balanced dice.

The following notation is used to determine the number of cases favourable to obtaining the face 6 once: * obtaining the face 6; ✖ obtaining a face other than 6. The sequences with exactly one 6 are of type *✖✖✖, ✖*✖✖, ✖✖*✖, ✖✖✖*. Obviously, there are 1×5×5×5 chances for a throw of type *✖✖✖. The same reasoning may be extended to the remaining cases. Hence, the total is 1×5×5×5 + 5×1×5×5 + 5×5×1×5 + 5×5×5×1, *i.e.*, 500 favourable cases.

The number of cases favourable to obtaining the face 6 twice is **✖✖, ✖**✖, ✖✖**, *✖*✖, *✖✖*, ✖*✖*,*i.e.*, 1×1×5×5 + 5×1×1×5 + 5×5×1×1 + 1×5×1×5 + 1×5×5×1 + 5×1×5×1= 150.

If a more elegant procedure is used, then,

$$\binom{4}{2} \times 5^2 = 6 \times 25 = 150.$$

Note: $\binom{4}{2}$ is the number of chances of distributing 2 stars ** in 4 different positions.

The number of cases favourable to obtaining 3 times the face 6 is determined by types ***✖, **✖*, *✖**, ✖***. Its number is

$$\binom{4}{3} \times 5 = 4 \times 5 = 20.$$

The number of cases favourable to attaining 4 times the face 6, ****, is 1×1×1×1 = 1.

When a balanced dice is thrown 4 times in a row, the number of possible cases is 6×6×6×6 = 1296.

If the Laplace Law is applied, the probability of event A: *Obtaining the face 6 at least once in 4 consecutive throws of a balanced dice is*

$$P(A) = \frac{500+150+20+1}{1296} = \frac{671}{1296} \cong 0.5177 = 51.77\%.$$

Considering that the calculation of the probability of the complement of event A is easier, the solution to the problem may be achieved by an alternative process. $\bar{A}$: *Never obtaining the face 6 in 4 consecutive throws of a balanced dice.*

The notation * obtaining the face 6; × obtaining a face other than 6; is used to determine the number of cases favourable to the occurrence of event $\bar{A}$. Hence, ××××, *i.e.*, 5×5×5×5 = 625.

$$P(\bar{A}) = \frac{625}{1296}$$

Knowing that $P(A) = 1 - P(\bar{A})$, then

$$P(A) = 1 - \frac{625}{1296} = \frac{1296 - 625}{1296} = \frac{671}{1296}.$$

What was the Chevalier de Mere's error? The throw of a balanced dice 4 times in a row cannot be described as a sum of isolated events, but rather as an indivisible situation.

Consider the second proposal of the Chevalier de Méré: calculate the probability of obtaining a *double 6* at least once by simultaneously throwing 2 balanced dice 24 consecutive times.

The number of cases favourable to that is significant but difficult to pinpoint in a sequence of 24 throws. In order to circumvent this problem, the probability of the occurrence of the opposite may even be calculated.

B stands for the event: *Obtaining a double 6 at least once in the simultaneous throw of 2 balanced dice 24 times in a row.*

$\bar{B}$: *"Never obtaining a double 6 in the simultaneous throw of 2 balanced dice 24 times in a row."*

How many possible cases of a *double 6* are there in the simultaneous throw of 2 balanced dice, and how many of those are actually a *double 6*?

There are 36 possible cases and only 1 is a *double 6 (Table 127)*.

(Table 127)

	1	**2**	**3**	**4**	**5**	**6**
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Determine the probability of never obtaining a *double 6* by simultaneously throwing 2 balanced dice 24 times in a row.

$$P(\bar{B}) = \underbrace{\frac{35}{36} \times \frac{35}{36} \times \ldots \times \frac{35}{36}}_{24\ vezes} = \left(\frac{35}{36}\right)^{24}$$

$$P(B) = 1 - P(\bar{B}) = 1 - \left(\frac{35}{36}\right)^{24} \cong 0.4914 = 49.14\%$$

So, the bet does not favour the Chevalier de Méré, thus explaining the accumulation of losses. Probabilistic concepts have multiple applications, also to recreational games or cases involving highly valuable assets such as the TV show *Let´s Make a Deal*, hosted by Monty Hall, which ran in the United States of America in the 1970s. In this show, guests had to choose 1 of 3 doors, 2 of which offered prizes of limited value. Only 1 door led to the most desired prize, a top model car! After the guest had made their choice, the host, with previous knowledge of which door would lead to the car, would open 1 of the doors which had not been chosen and show a minor prize. At that moment, the guest was given the chance to alter the initial choice and opt for the other door which was still closed.

According to Laplace's Law, the probability of finding the car behind the door which had been initially chosen by the guest is $\frac{1}{3}$, whereas the opposite event would have a probability of $\frac{2}{3}$. When 1 of the doors which did not lead to the car was opened, what should the guest do? Consider the following deceptive reasoning: now that only 2 doors are closed, the probability of the car being behind the initially chosen door raises to $\frac{1}{2}$. The host contributed to the possibility of winning the car! Since both closed doors had the same probability of hiding the prize, the alteration of the initial choice would not be advantageous.

Truth differs from intuition. When opening a door which does not have the big prize, the host does not change the probability of the car being behind the previously chosen door. However, he does introduce new data. The probability of the car being behind the door that he had just opened is 0, as this is an impossible event. Hence, the probability of the car being behind the other closed door is now $\frac{2}{3}$. Even though the location of the prize is still unknown to the guest, the logic option would be to alter the initial choice, which would actually double the probability of winning. A schematic of the Monty Hall problem is presented *(Chart 17)*.

(Chart 17)

Number of the door which leads to the car						
1		$G_1\ H_2\ D_3$		$\boldsymbol{G_2\ H_3\ D_1}$		$\boldsymbol{G_3\ H_2 D_1}$
1		$G_1\ H_3\ D_2$				
2		$\boldsymbol{G_1\ H_3\ D_2}$		$G_2\ H_1 D_3$		$\boldsymbol{G_3\ H_1 D_2}$
2				$G_2\ H_3 D_1$		
3		$\boldsymbol{G_1\ H_2\ D_3}$		$\boldsymbol{G_2\ H_1 D_3}$		$G_3\ H_1 D_2$
3						$G_3\ H_2 D_1$

G: Guest's initial choice.
H: The host opens a door where there is no car.
D: Door opened after the guest's change of option.
$(For\ each\ pair\ of\ possibilities\ not\ written\ in\ bold,\ only\ one\ is\ chosen)$

The probability of the car being behind door number 1 is $\frac{1}{3}$. If the guest initially chooses this door (and the probability of that happening is $\frac{1}{3}$) and then alters his choice, the probability of winning the prize is $\frac{1}{3} \times \left(\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times 1\right) = \frac{2}{9}$. The result in which the car is behind any of the other closed doors is analogous. Thus, the probability of winning the car with the indicated strategy is $3 \times \frac{2}{9} = \frac{2}{3}$.

If 3 balanced dice (numbered from 1 to 6) are rolled and the values obtained are summed, which is easier to obtain, 9 or 10?

In the 16th century, gamblers still believed that the probability was the same. According to their reasoning, each value could be obtained by 6 different decompositions *(Table 128)*. However, after a great number of games and after summing the values of the three dice, they verified that the value 10 was the most frequent, which was something that they could not explain.

(Table 128)

9 Points	10 Points
1 + 2 + 6	1 + 3 + 6
1 + 3 + 5	1 + 4 + 5
1 + 4 + 4	2 + 2 + 6
2 + 2 + 5	2 + 3 + 5
2 + 3 + 4	2 + 4 + 4
3 + 3 + 3	3 + 3 + 4

The list bellow *(Table 129)* will explain the outcome without further comments.

(Table 129)

9 Points (25 favourable cases)	10 Points (27 favourable cases)
1 + 2 + 6, 1 + 6 + 2, 2 + 1 + 6, 2 + 6 + 1, 6 + 1 + 2, 6 + 2 + 1	1 + 3 + 6, 1 + 6 + 3, 3 + 1 + 6, 3 + 6 + 1, 6 + 1 + 3, 6 + 3 + 1
1 + 3 + 5, 1 + 5 + 3, 3 + 1 + 5, 3 + 5 + 1, 5 + 1 + 3, 5 + 3 + 1	1 + 4 + 5, 1 + 5 + 4, 4 + 1 + 5, 4 + 5 + 1, 5 + 1 + 4, 5 + 4 + 1
1 + 4 + 4, 4 + 1 + 4, 4 + 4 + 1	2 + 2 + 6, 2 + 6 + 2, 6 + 2 + 2
2 + 2 + 5, 2 + 5 + 2, 5 + 2 + 2	2 + 3 + 5, 2 + 5 + 3, 3 + 2 + 5, 3 + 5 + 2, 5 + 2 + 3, 5 + 3 + 2
2 + 3 + 4, 2 + 4 + 3, 3 + 2 + 4, 3 + 4 + 2, 4 + 2 + 3, 4 + 3 + 2	2 + 4 + 4, 4 + 2 + 4, 4 + 4 + 2
3 + 3 + 3	3 + 3 + 4, 3 + 4 + 3, 4 + 3 + 3

When three dice numbered from 1 to 6 are rolled, there are 216 possible cases. From these, 25 are favourable to obtaining 9 points and 27 are favourable to obtaining 10 points. The probability of obtaining 9 and 10 points is, respectively, $\frac{25}{216}$ and $\frac{27}{216}$.

An intervention programme on problem solving in the scope of Mathematics should promote significant learning, without neglecting learning by memorisation. Since significant learning is crucial for the full intellectual development of the students, the need to explore different solving processes should be highlighted. The rigour of theoretical Mathematics, the knowledge and understanding of long-lasting theorems and properties are not a synonym of monotony. They are necessary conditions for the intellectual exercise which both stimulates and consolidates the development of skills.

Similarly to the number π, in Mathematics, an approximation to the Neper number *e* is an example of how different strategies converge to a desired outcome. Let us consider Isaac Newton's (1642 – 1727) and Euler's (1707 – 1783) infinite series.

Newton obtained an infinite series from the binomial sequence $\left(1+\frac{1}{n}\right)^n$, $n \to \infty$:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

> *Euler first undertook work on infinite series around 1730, and by that time, John Wallis, Isaac Newton, Gottfried Leibniz, Brook Taylor, and Colin Maclaurin had given series calculation of the constants* e *and* π *and the use of infinite series to represent functions in order to integrate those that could not be treated in closed form. Hence it is understandable that Euler should have tackled the subject. Like his predecessors, Euler's work lacks rigor, is often ad hoc, and contains blunders, but despite this, his calculations reveal an uncanny ability to judge when his methods might lead to correct results* (Kline, 1983, pp. 307-314).

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \cdots}}}}}$$

Another example of the application of Mathematics to the Social Sciences is the study of fair division methods. Let us recall the Judgement of Solomon: two women claimed to be the mother of a new-born. After a long quarrel, the case was brought to King Solomon, who, when listening to the women, considered both arguments and deliberated that the infant be split in two and one half delivered to each of the women. As cruel as the decision may have seemed, King

Solomon knew that the true mother would never accept it, preferring to see her child alive in the arms of the impostor. A maternal cry revealed the identity of the mother and put an end to such monstrous sentence.

In the 1940s, Hugo Steinhaus (1887 – 1972) a Polish mathematician and educator developed fair division methods, rules which allow asset division so that a fair share is obtained by all participants, with no need of an outsider. Compliance with the procedures and mutual respect complete a fair and balanced outcome for all participants.

The simplest method involves 2 people. One person divides and the other chooses. The person who divides the assets may be chosen by coin-tossing. This method should only be used if the value of the object to be divided is not reduced if divided in 2. A china porcelain object, for instance, cannot be divided by this method. In the point of view of the person who divides the assets, these are divided in 2 equally valuable shares. Otherwise the other person will choose the best part. In asset division, subjective factors such as the taste of the person who cuts and of the person who chooses the slices of a cake half part strawberry and half part chocolate may, eventually, be taken into account.

Solutions for fair division problems must be accepted by all participants. Mathematics offers a precious contribution, whether to divide a pack of candy by a group of children or to help a family divide the assets of an inheritance. In order to understand the meaning of receiving a fair share, consider *V* as the total value to be distributed. Each of the *N* elements receives a fair share whose value must be greater or equal to $\left(\frac{1}{N}\right)$ of *V*.

Remarks:

Each element should consider their share as fair. The opinion of the remaining elements regarding that share (fair or unfair) is irrelevant for the division process.

Each element ought to accept a share that he/she considers as fair, even though, in their point of view, another share may be more likable. Hence, in a division involving, for example, 4 people, each individual has the right to a share that, in their opinion, corresponds to at least $\frac{1}{4} = 0.25 = 25\%$ of the total value to be divided. Another parcel, which is believed to be more valuable, may be allocated to another person. But according to their judgement, he/she has received at least 25% of the total value to be divided by the 4 people.

The method of the lone divider

Four members of a family (Anna, Brüno, Charles and Diana) wish to divide a portion of land appraised at 120,000 €. One of the participants, who is randomly chosen, divides the property in 4 parcels. Let us suppose that it is Diana. Individually, the other members (Anna, Brüno and Charles) write the value that they assign to each of the parcels on a piece of paper and put it in closed envelopes. Then the envelopes are opened. Let us imagine that the proposals were as follows *(Table 130)*.

(Table 130)

	Parcel 1	Parcel 2	Parcel 3	Parcel 4
Anna	10,000 €	10,000 €	30,000 €	70,000 €
Brüno	30,000 €	10,000 €	45,000 €	35,000 €
Charles	15,000 €	5,000 €	80,000 €	20,000 €

In order for the division to be fair, each member of the family ought to receive a parcel which, according to their own judgement, equals at least 30,000 €, which is the result of the division of 120,000 € by 4. Let us see which parcels are considered as acceptable by each individual (in bold *Table 131*).

(Table 131)

	Parcel 1	Parcel 2	Parcel 3	Parcel 4
Anna	10,000 €	10,000 €	**30,000 €**	**70,000 €**
Brüno	**30,000 €**	10,000 €	**45,000 €**	**35,000 €**
Charles	15,000 €	5,000 €	**80,000 €**	20,000 €

Let us find whether it is possible to fairly divide that portion of land. Charles will only be satisfied if he gets parcel 3. Anna would accept both parcels 3 or 4. However, parcel 3 ought to be given to Charles. Finally, Brüno would like parcels 1, 3 or 4. As parcels 3 and 4 have already been assigned, Brüno will receive parcel 1. And since Diana was the one to divide the piece of land and, according to her own judgement, did it fairly, she will accept the remaining parcel *(Chart 18)*. In this case, it is parcel 2.

(Chart 18)

	Parcel 1	Parcel 2	Parcel 3	Parcel 4
Anna				✓
Brüno	✓			
Charles			✓	
Diana		✓		

The method of sealed bids

Four brothers and sisters (Anthony, Beatrice, Catherine and Daniel) have to divide an inheritance which is composed of three assets. The sentimental value of the inheritance makes them want to keep it in the family, thus selling for dividing the money afterwards is not an option. As the number of assets is different from the number of heirs, this problem is a source of great tension among family members. In order to solve the problem, each member of the family writes the value which they are willing to offer for the house, the boat and the car on a piece of paper. Next, each piece of paper is put in an envelope identified with the name of the heir and then the envelopes are sealed. The brothers and sisters are not aware of the value of each other's bids. Then, the 4 sealed envelopes are opened *(Table 132)*.

(Table 132)

	Anthony	Beatrice	Catherine	Daniel
House	180,000€	200,000€	165,000€	150,000€
Boat	80,000€	40,000€	90,000€	60,000€
Car	20,000€	25,000€	20,000€	30,000€

After the sum of the bids is performed, the fair share that each heir will receive is calculated. This value is reached by dividing the sum of the bids by 4 (the total number of heirs). Each item will belong to the heir which made the highest bid. Hence, Beatrice will have the house, Catherine the boat, and Daniel the car. The next step is to calculate the difference of value between the item received and the fair share which should duly belong to each heir. Anthony and Daniel are owed money and Catherine and Beatrice ought to give them a certain amount of money. In this situation, when what one owes and what is owed to one is calculated, the absolute value is 55,000€. The division of the total surplus by the four heirs enables the performance of the sum of 13,750€, a value which can be labelled as a bonus *(Table 133)*.

(Table 133) - The share results from the sum of the addends:
Item received + Difference + Division of the surplus

	Anthony	Beatrice	Catherine	Daniel
House	180,000€	200,000€	165,000€	150 000€
Boat	80,000€	40,000€	90,000€	60,000€
Car	20,000€	25,000€	20,000€	30,000€
Sum of the bids	280,000€	265,000€	275,000€	240,000€
Fair share (FS)	70,000€	66,250€	68,750€	60,000€
Item received (AR)		House	Boat	Car
Difference (PJ – OA)	70,000€	- 133,750€	- 21,250€	30,000€
Total surplus (55 000€)				
Division of the surplus	13,750€	13,750€	13,750€	13,750€
Final disposition of items	+ 83,750€	House - 120,000€	Boat - 7,500€	Car + 43,750€

The rationale for the method is as follows: Anthony (the heir who received nothing) should get what according to his calculations should be his share. The same is true for the others: Beatrice can say that her fair share should be 66,250€ (the sum of her bids divided by 4). But she received a house for 200,000€. So she should give 133,750€ to be split between the other participants.

(Subject: Numbers and Operations – 12th grade students)

Once upon a time, an elderly magnate, owner of extensive property (hotels, airline companies, oil wells, yachts…), decided to distribute his assets by his offspring. Worried with the division procedures, he called them to talk about the subject. Aware of his offspring's different interests, the patriarch listed his *2n* indivisible assets.

- I ask each of you to order my assets according to your wishes. Give 2n points to your favourite asset, 2n-1points to your second favourite and so on. When you are finished, give me your list.

His children went to different rooms to draw up their lists and then came back to their father. After comparing the list the magnate said:

- As I had anticipated, aware as I am of your different interests, none of my assets has the same position on your lists, which facilitates a fair distribution!

If Ahmed receives the first asset on the list, Musa may also receive the first item on his list. Then both items are crossed off both lists until all the inheritance is distributed. It is therefore evident that you are both going to receive items whose sum of the points is equal or superior to half the total sum of the points of the inheritance.

In general, is it true that, by using the process chosen by the magnate, each son will receive assets corresponding to a score equal or superior to half of the total sum of the points of the inheritance?

Even though the basis for this reasoning is apparently credible, it is actually not true.

Consider the example *(Table 134)*:

(Table 134)

Assets		*A*	*B*	*C*	*D*	*E*	*F*
Ahmed		1	2	3	4	5	6
Musa		3	1	2	6	4	5

The distribution of the assets would be: *Ahmed F*; *Musa D*; *Ahmed E*; *Musa A*; *Ahmed C*; *Musa B*. Hence, the total sum of points of the assets received by Musa is

$$6+3+1=10<\frac{1}{2}\sum_{i=1}^{6} i=\frac{21}{2}.$$

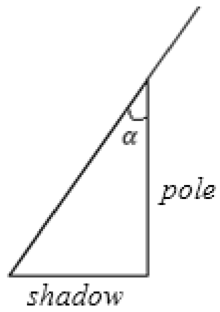
A close door becomes a challenge. The rational, in mathematical problem solving, is to identify the key which moves the mechanism of the lock instead of using trial and error methods, which is barren when there are too many possible solutions. Simple mechanisms are easily unblocked, while more complex ones require more refined techniques and more sophisticated solving processes. However, as we will see, problems of the size of planets and satellites sometimes have surprisingly simple solutions.

It is therefore obvious that, for a given question, the level of sophistication used by the solver will depend on their specific knowledge on the subject and their skill to use eventually useful techniques to find the solution.

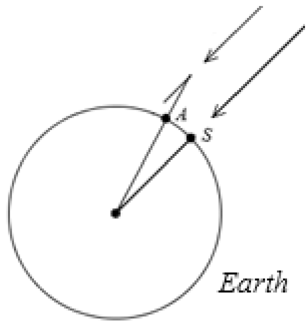
Eppur si muove![63]

In the summer solstice, the inhabitants of Siene, a town located on the Tropic of Cancer, observed that a pole in vertical position did not project its shadow on the ground. The sun was exactly perpendicular. In Alexandria, on the same day and at the same hour, sun beams reached the ground in an angle of $\frac{1}{50}$ of the circumference of the meridian, *i.e.*, 7.2º (*Figure 75* and *Figure 76*).

The distance between the two cities is of approximately 5,000 stages (*stadion*, in Greek). As the Greek measurement unit was not standardised, it is estimated that it corresponds to approximately 157.5*m*. If a proportion is used, the perimeter of the *Earth*[64] would be 39,375,000*m*, *i.e.*, 39,375*kms.* This is an approximate value, for the distance between these two cities is not 5,000 stages and they are not on the same meridian.



(Figure 75)



(Figure 76)

Even though conditioned by Ptolemy's model, which placed the Earth at the centre of the Universe and the other celestial bodies around it, ancient Greek mathematicians performed impressive calculations. With basic measuring tools and their knowledge of Geometry, they determined distances and measurement relations regarding the Moon and the Sun. Fascinated by the orbs, ancients named the days of the week after celestial bodies in different languages *(Chart 19)*.

---

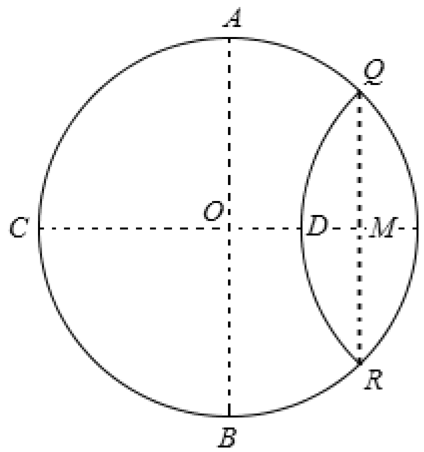
[63] Phrase uttered by Galileo Galilei when leaving the court room of the Roman Inquisition after being convicted for advocating the Heliocentric Theory.

[64] Presently, the perimeter of the Earth is estimated to have an approximate value of 40,075 *km*.

(Chart 19)

Latin	Spanish	French
Solis dies	*Domingo*	*Dimanche*
Lunae dies	*Lúnes*	*Lundi*
Martis dies	*Martes*	*Mardi*
Mercurie dies	*Miercoles*	*Mercredi*
Jovis dies	*Juéves*	*Jeudi*
Veneris dies	*Viernes*	*Vendredi*
Saturni dies	*Sábado*	*Samedi*

A solar eclipse can be an excellent opportunity to deduce that the Sun is at a greater distance from the Earth than the Moon. Another phenomenon, a lunar eclipse, allows estimating the ratio between the radius of the Earth and the radius of the Moon *(Figure 77)*. The image shows the moon with a shadow coming from Earth (arc QDR together with arc QR).



(Figure 77)

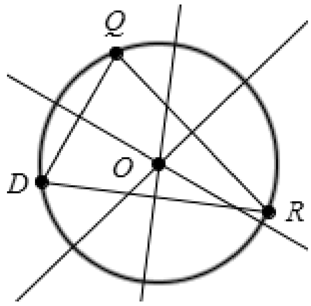
The relative distances $|AB|$, $|QR|$ and $|CD|$ are obtained with measuring tools. Consider *M* as the middle point of $[QR]$. $\overline{QM} = \frac{1}{2}\overline{QR}$. By the Pythagoras' theorem, we have:

$$\overline{OM}^2 + \overline{MQ}^2 = \overline{OQ}^2 = radius_{Moon}^2 = \left(\frac{1}{2}\overline{AB}\right)^2.$$

As $\overline{MQ}$ and $\overline{OQ}$ are known, $\overline{OM}$ can be extracted (note that *M* is not visible).

Since $\overline{DM} = \overline{CM} - \overline{CD} = \overline{CO} + \overline{OM} - \overline{CD}$, using trigonometric relations, we can infer that $tan\left(\frac{1}{2}Q\hat{D}R\right) = \frac{\overline{MQ}}{\overline{DM}}$, and calculate the size of the angle $Q\hat{D}R$.

The measure of the radius of the Earth corresponds to the circumradius of triangle $\Delta\, QDR$. Given triangle $\Delta\, QDR$, its bisections (straight lines whose points are equidistant from the limits of one of the sides of the triangle) are known to intersect, on a point O, the circumcentre, which is equidistant from vertices *Q*, *D* and *R*. Hence, $\overline{OQ} = \overline{OD} = \overline{OR}$. It is, therefore, possible to construct a circumference whose centre is O and whose radius is called circumradius, and which includes vertices D, Q and R *(Figure 78)*.

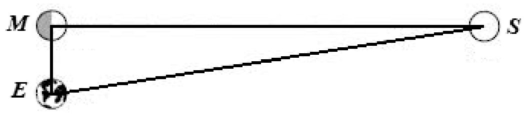


(Figure 78)

To determine the radius of the Earth, let us remember the *Sine theorem*. There is proportionality between the length of each side of the triangle and the sine of the opposite angle, whose value is the double of the circumradius, *i.e.*, $2r = \frac{\overline{QR}}{sin(Q\hat{D}R)} = \frac{\overline{QD}}{sin(Q\hat{R}D)} = \frac{\overline{DR}}{sin(D\hat{Q}R)}$. Hence, $radius_{Earth} = \frac{1}{2}\frac{\overline{QR}}{sin(Q\hat{D}R)}$.

From the formulas for $radius_{Moon} = \frac{1}{2}\overline{AB}$ and $radius_{Earth} = \frac{1}{2}\frac{\overline{QR}}{sin(Q\hat{D}R)}$, Greek mathematicians may have determined $\frac{radius_{Earth}}{radius_{Moon}} = 3$. The ratio between the radius of the Earth and the radius of the Moon is approximately 3.67, being the error explained by the deficient accuracy of the instruments of the time.

Aristarchus of Samos (310 BC – 250 BC) determined the distance between the Earth and the Sun by previously observing the Moon in the first-quarter phase and the moment when the sun is at the horizon. He measured the angular separation between the Sun and the Moon, and obtained the value 87º, one of the angles of the right triangle Earth-Moon-Sun, whose right-angle vertex is the Moon. With this procedure, Aristarchus determined the distance between the Earth and the Sun, $\overline{ES}$, which was approximately 20 times greater than the distance between the Earth and the Moon, $\overline{EM}$ *(Figure 79)*.



$$\cos 87^{\circ} = \frac{\overline{EM}}{\overline{ES}}$$

(Figure 79)

Nowadays, with precision instruments, the angular separation between the Sun and the Moon is known to be approximately, 89.8º, which means that the distance between the Earth and the Sun is much greater than that determined by Aristarchus of Samos.

> *Again, it is a most beautiful and delightful sight to behold the body of the Moon, which is distant from us nearly sixty semi-diameters of the Earth, as near as if it was at a distance of only two of the same measures; so that the diameter of this same Moon appears about thirty times larger, its surface about nine hundred times, and its solid mass nearly 27,000 times larger than when it is viewed only with the naked eye; and consequently any one may know with the certainty that is due to the use of our senses, that the Moon certainly does not possess a smooth and polished surface, but one rough and uneven, and, just like the face of the Earth itself, is everywhere full of vast protuberances, deep chasms, and sinuosities*[65] (Galilei, 2010, pp. 151-152).

Galileo Galilei (1564 – 1642), Italian astronomer, physicist, engineer, philosopher and mathematician, describes the topography of the Moon and identifies some satellites of Jupiter in the *Sidereus Nuncius*, a book published in 1610. These discoveries arise from astronomical observation with the use of a telescope, an instrument Galileo[66] himself conceived, built and perfected.

Galileo Galilei advocated the heliocentric theory proposed by Aristarchus of Samos and later fostered by Nicolaus Copernicus (1473 – 1543). This theory collided with the geocentric theory in force at the time which said that the Earth was the centre of the Universe, followed by the Moon, Mercury, Venus, Mars, Jupiter and Saturn. In 1633, Galileo was tried by the Inquisition and, under threat to his life, was forced to renounce his theory of the revolving Earth. However, intimidation was not enough to diminish his determination, which was based on astronomic observations. When he was leaving the room where the interrogation had taken place, and convinced of his arguments he murmured *Eppur si muove* (*but it moves*).

The *father of modern science* (Finocchiaro, 2007) describes with geometric argumentation why the Moon is not a flat surface *(Figure 80)*. The procedure used by Galileo is a bright example of the application heuristics.

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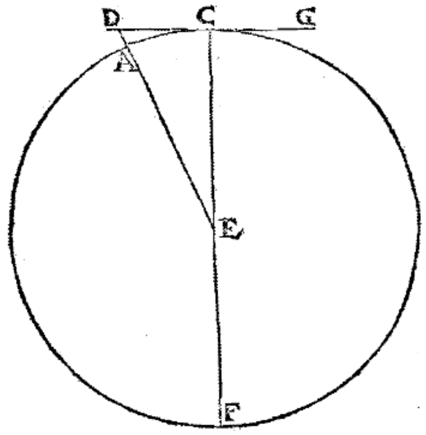
[65] Author's translation: *É magnífico, e muito agradável ao olhar, poder observar o corpo lunar, que está afastado de nós cerca de sessenta raios terrestres, como se não estivesse mais distante do que duas dessas unidades; a tal ponto que o diâmetro dessa mesma Lua parece quase trinta vezes, a sua superfície novecentas vezes e o seu volume quase vinte e sete mil vezes maiores do que quando são vistos simplesmente à vista desarmada. Daí, consequentemente, que qualquer pessoa compreenda, com a certeza dos sentidos, que a Lua não é de maneira nenhuma revestida de uma superfície lisa e perfeitamente polida, mas sim de uma superfície acidentada e desigual, e que, como a própria face da Terra, está coberta em todas as partes por enormes protuberâncias, depressões profundas, e sinuosidades.*

[66] Galileo used approximate values. The italic mile or roman mile (ca. 1478m) was widely used in Europe.

As I often observed in various positions of the Moon with reference to the Sun, that some summits within the portion of the Moon in shadow appeared illuminated, although at some distance from the boundary of the light (the terminator), by comparing their distance with the complete diameter of the Moon, I learnt that it sometimes exceeded the one-twentieth (1/20th) part of the diameter. Suppose the distance to be exactly 1/20 th part of the diameter, and let the diagram represent the Moon's orb, of which CAF is a great circle, E its center, and CF a diameter, which consequently bears to the diameter of the Earth the ratio 2:7; and since the diameter of the Earth, according to the most exact observations, contains 7000 Italian miles, CF will be 2000, and CE 1000, and the 1/20th part of the whole, CF, 100 miles. Also let CF be a diameter of the great circle which divides the bright part of the Moon from the dark part (for, owing to the very great distance of the Sun from the Moon this circle does not differ sensibly from a great one), and let the distance of A from the point C be 1/20th part of that diameter; let the radius EA be drawn, and let it be produced to cut the tangent line GCD which represents the ray that illumines the summit, in the point D. Then the arc CA or the straight line CD will be 100 of such units, as CE contains 1000. The sum of the squares of DC, CE is therefore 1,010,000, and the square of DE is equal to this; therefore the whole ED will be more than 1004; and AD will be more than 4 of such units, as CE contained 1000. Therefore the height of AD in the Moon, which represents a summit reaching up the Sun's rays GCD, and separated from the extremity C by the distance CD, is more than 4 Italian miles; but in the Earth there are no mountains which reach to the perpendicular height of even one mile. We are therefore left to conclude that it is clear that the prominences of the Moon are loftier than those of the Earth (Galilei, 2004, pp. 26-28).



(Figure 80)

The calculation of distances combines the use of scientific instruments with the application of mathematical concepts. When the available tools do not allow achieving satisfactory answers, new, more efficient instruments must be conceived. In Euclidean Geometry, the shortest difference between two points is determined by a line segment. In Spherical Geometry, the shortest distance between two points is an *orthodromic distance*[67].

Astronomers systematised the laws which rule the Universe. For two decades, Tycho Brahe (1546 - 1601) observed planets and stars. His records helped Johannes Kepler (1571 – 1630) to enunciate *Kepler's Laws*[68] which are essential to understand planetary movement.

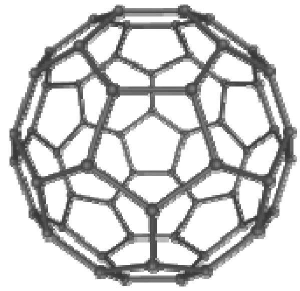
There is a very fine line between appearance and reality, thus making problem solving difficult and compromising the judgement of the conclusions which may arise.

Two points / One line / Infinity of points

A truncated icosahedron is a geometric solid with 32 faces, of which 20 are regular hexagons and 12 are regular pentagons. Its origin goes back to Ancient Greece. In the 20th century, it inspired the design of a football, the *Buckyball*, named after Richard Buckminster Fuller (1895-1983) the inventor of the *geodesic dome*[69]. In 1985, Harold W. Kroto, Robert F. Curi and Richard E. Smalley produced fullerenes in the laboratory, namely a molecule composed of 60 atoms of carbon connected as a truncated icosahedron *(Figure 81)*.

In Engineering the design of effective networks to connect points in the plane is critical. Topography can condition the execution of a structure, so Mathematics helps optimise resources.

Truncated Icosahedron



(Figure 81)

[67] Section of the maximum circle defined by two points.

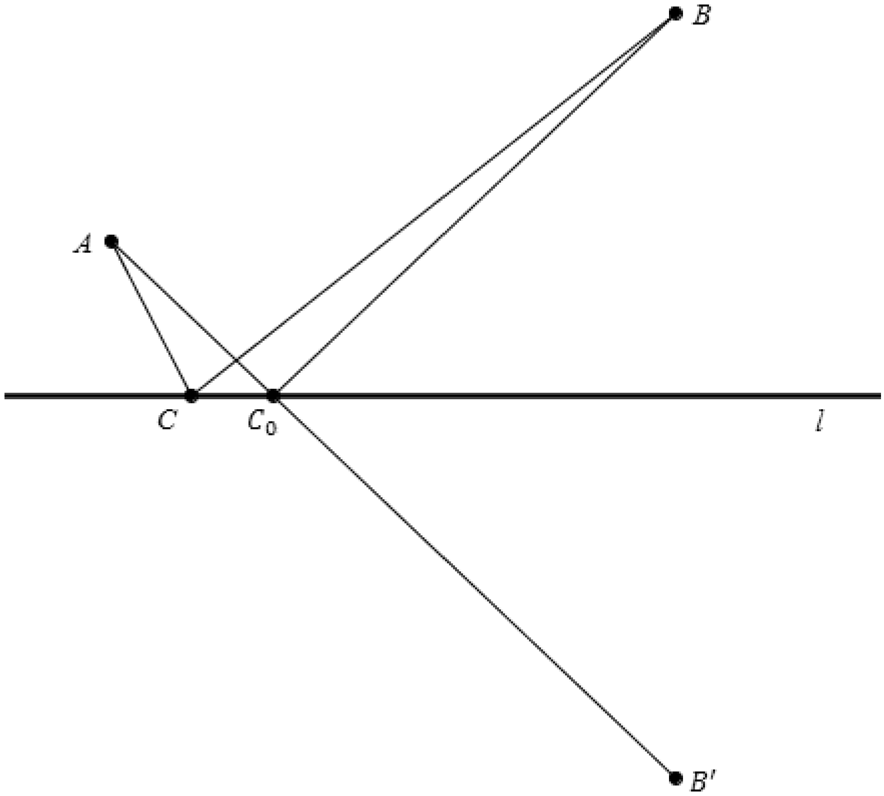
[68] Law of Elliptical Orbits: planets move in elliptical orbits around the Sun, where the Sun is one of the focal points of the ellipse. The distance from a planet to the Sun varies in the course of its orbit. Law of Areas: the line segment which connects a planet to the Sun sweeps out equal areas in equal times. A planet's orbital speed is not constant. When a planet is closer to the sun, it moves faster. Law of Harmonies: the square of the orbital period *(T)* of a planet is proportional to the cube of the semi-major axis of its orbit. The semi-major axis of an orbit indicates the mean distance from a planet to the Sun $\left(\frac{T^2}{S^3} = k,\ k\ constant\right)$.

[69] Highly resistant structure comprised by equilateral triangles and isosceles triangles.

(Subject: Geometry – Secondary Education students)

Two cities *A* and *B* are located on the same side of the motorway. The goal is to build a petrol station (*C*) in the motorway and a road from *A* to *B* with a connection to *C*. Where should the petrol station be built so that the road has the minimum possible length?

The statement suggests a sketch *(Figure 82)* where each city is marked by a point.



(Figure 82)

Suppose that the route of the motorway (*l*) is linear. Find the place to build the petrol station taking into account that the sum of the distances $\overline{AC}+\overline{CB}$ should be as short as possible.

On a plane, the shortest distance between two points is represented by a line segment. However, in this situation, that is not possible for the road must connect A and B to C. If B' is the reflection of B on straight line l, then C_0 is the intersection point of straight line l with AB'. Note that by definition of B', for every point $C \in l$, $\overline{CB'}=\overline{CB}$. By the observation of the figure,

$$\overline{AC}+\overline{CB}=\overline{AC}+\overline{CB'} \geq \overline{AB'}=\overline{AC_0}+\overline{C_0B'}=\overline{AC_0}+\overline{C_0B},$$

where the inequality comes from the triangular inequality and the equality from the fact that for $C=C_0$, the $\Delta ACB'$ degenerates into a straight line. Hence, the petrol station should be built on point C_0.

In what concerns the building process of the shortest connection between three points, consider the following fact composed of two items:

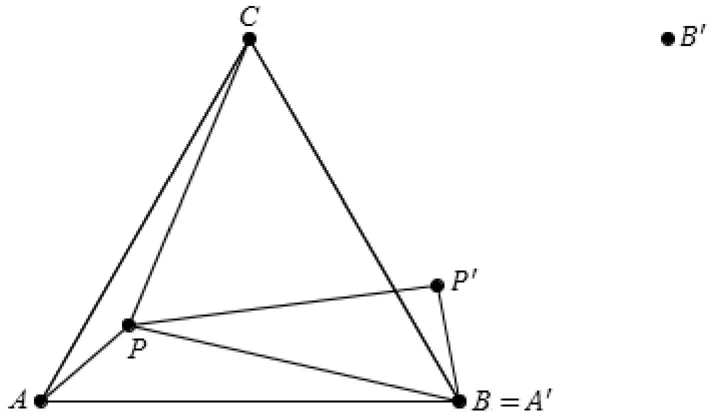
1) **Dimitrie Pompeiu's Theorem:** *Given an equilateral triangle ΔABC in the plane and any point P in the plane of ΔABC , $\overline{PA}$, $\overline{PB}$ and $\overline{PC}$ form the sides of a triangle, i.e. the sum of any two of these quantities is greater than the third (Figure 83).*

Demonstration: Consider a rotation of 60º around point *C*. Point *A* is transformed in point A', which is coincident with *B*, point *B* in point B' and point *P* in point P'.

Given $\overline{CP} = \overline{CP'}$ and $P\hat{C}P' = 60^\circ$, then $\Delta PCP'$ is equilateral, where $\overline{PP'} = \overline{PC}$.

Rotation maintains the distances between any two points, thus $\overline{PA} = \overline{P'A'} = \overline{P'B}$.

In conclusion, the measures of the sides of $\Delta PBP'$ occur as lengths $\overline{PB}$, $\overline{PA}$ and $\overline{PC}$, which proves that the statement is true.



(Figure 83)

Q. E. D.

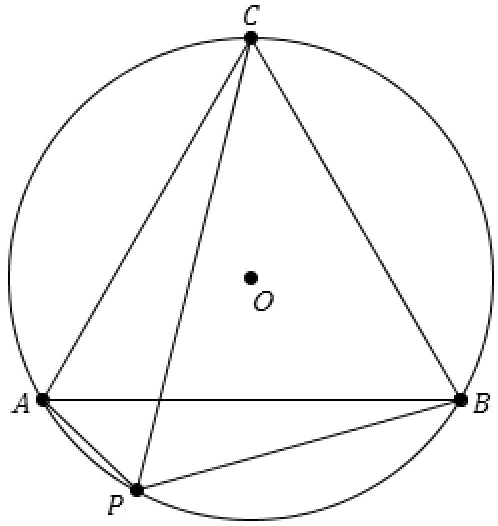
2) Supplement: If P belongs to arc $\widehat{AB}$ of the circumference circunscribed to ΔABC , which does not contain C, then $\overline{PA} + \overline{PB} = \overline{PC}$ (Figure 84).

The equality comes from Ptolemy's Theorem: *If a quadrilateral is inscribable in a circle then the product of the measures of its diagonals is equal to the sum of the products of the measures of the pairs of opposite sides.*

$$\overline{PC}.\overline{AB} = \overline{PA}.\overline{BC} + \overline{PB}.\overline{AC}$$

If ΔABC is equilateral, $\overline{AB} = \overline{BC} = \overline{AC} = a$, then,

$$\overline{PC}.a = \overline{PA}.a + \overline{PB}.a, \text{ i.e., } \overline{PC} = \overline{PA} + \overline{PB}.$$



(Figure 84)

Problems with recognised applicability are an opportunity for the teacher to illustrate its importance and enhance students' motivation levels. These problems frequently require a previous approach to theoretical concepts, a condition which can become a critical factor in what concerns the success of the implementation of problem solving as a formula to improve the teaching/learning process.

(Subject: Geometry – Secondary Education students)
Build a connection system of the shortest possible length which connects three points A, B and C.

A hasty interpretation may suggest that the two shortest line segments which are the sides of ΔABC are the solution. However, this is not the best answer.

A, B and C are points which correspond to cities whose connection system, a road network, should be as short as possible so as to minimise building costs (suppose that the building costs per length unit is constant). Build random connections inside or on the borders of triangle ΔABC. Is it possible to locate a point P in order that the connection $\overline{PA} + \overline{PB} + \overline{PC}$ is the shortest possible?

Being the three points, A, B and *C in the plane, locate in that same plane a point P so that the sum of the distances from P to A, B and C is minimal.*

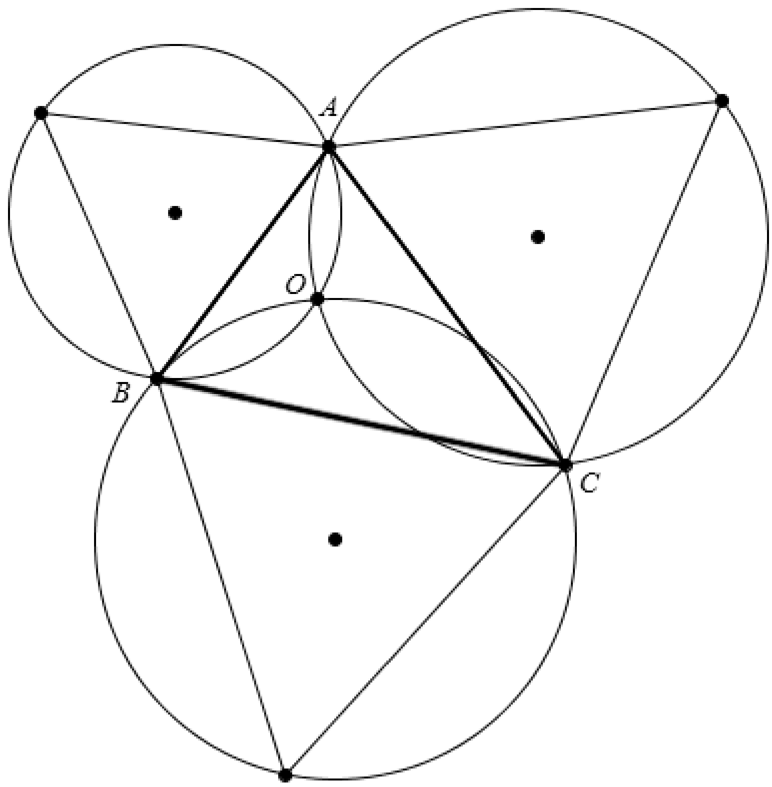
This problem, suggested by Fermat, was solved by Evangelista Torricelli (1608 – 1647). If the greatest angle of triangle ΔABC is inferior to 120º, then point P, also known as the Torricelli point, can be located. If the greatest angle of triangle ΔABC is equal or greater than 120º, the minimal network is composed of the two smaller sides of triangle ΔABC.

First, construct equilateral triangles exterior to the sides of ΔABC. Then, draw the circumferences which circumscribe each of the equilateral triangles. The point of intersection of the three circumferences is the Torricelli point.

Knowing and diversifying the use of mathematical tools gives the solver scientific leverage which, when allied to reasoning, method and imagination, lead to optimal performance. In Mathematics, the teaching/learning process encompasses the recapitulation of contents with an increasing level of difficulty over the different school cycles. The main goal is that basic concepts are well stored in the long-term memory so as to create a basis for the practice of Mathematics.

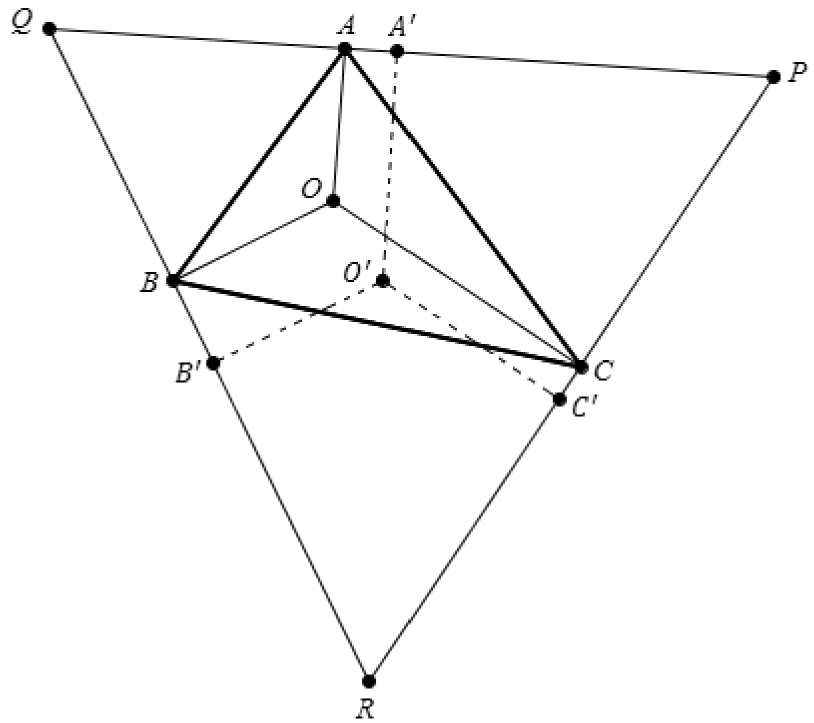
Theorem: If A, B and C are points of a triangle whose internal angles are inferior to 120º, then point O, which minimises the distances to A, B and C (*i.e.*, $|OA| + |OB| + |OC| = min$), is the only point in the plane where $A\hat{O}B$, $A\hat{O}C$, $B\hat{O}C$ are all equal to 120º.

Demonstration: O is the point obtained by the Torricelli method which minimises the sum of the distances to A, B and C *(Figure 85)*.



(Figure 85)

Proof:



(Figure 86)

When straight lines $[OA]$, $[OB]$, $[OC]$ are drawn, another triangle is obtained, ΔPQR *(Figure 86)*. Quadrilateral *AOCP* has angles $\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{2}, C\hat{P}A$, whose sum necessarily equals

$$2\pi = \frac{2\pi}{3} + \frac{\pi}{2} + \frac{\pi}{2} + C\hat{P}A.$$

Hence $C\hat{P}A = \frac{\pi}{3}$. Also, it follows that $A\hat{Q}B = \frac{\pi}{3}$, $B\hat{R}C = \frac{\pi}{3}$. Thus ΔPQR is equilateral.

In an equilateral triangle, the sum of line segments' perpendicular to the sides from any interior point to the sides is constant.

If O' is another point belonging to the interior of triangle ΔABC and A', B' and C' are the projections of O' on the sides *PQ*, *QR*, *RP*, then, according to what has been previously stated and by evident Geometry,

$$O'A \geq O'A'$$
$$O'B \geq O'B'$$
$$O'C \geq O'C'$$

$$(here\ at\ least\ once > for\ otherwise\ we\ would\ have\ O = O')$$

therefore,

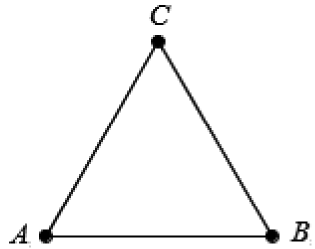
$$O'A + O'B + O'C > O'A' + O'B' + O'C'.$$

In the scope of the problem, consider *O* as the desired point.

Q. E. D.

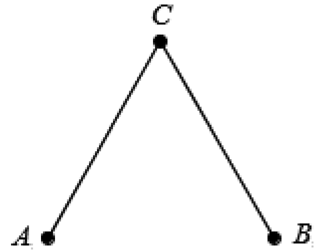
(Subject: Geometry – Secondary Education students)

Three cities A, B and C form an equilateral triangle whose side measures $50km$ *(Figure 87)*. A telecommunications company wants to connect the cities by optic fibre cable at the lowest possible cost. The region, located in a vast plain, implies certain restrictions in the trajectory of the installation of the equipment, even though the cost per *km* of the optic fibre cable is constant. Draw the network with the shortest possible length, justifying efficiency gains when compared to other possible solutions.



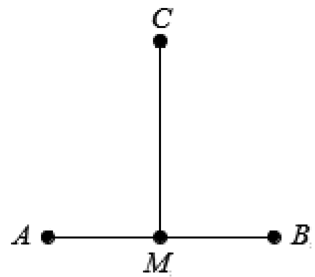
(Figure 87)

An immediate answer would suggest that the network comprises two line segments, a solution which would require a length of 100*km* of optic fibre cable *(Figure 88)*.



(Figure 88)

Check for the possibility of establishing a network with an inferior length. The equilateral triangle induced by the location of the cities may be decomposed into two right triangles. If M is the middle of $[AB]$, then $\overline{MC}$ and $\overline{AB}$ are a more efficient connection whose measure is approximately $93.3km$, for as $\overline{MC}$ is the cathetus of the right triangle ΔAMC and therefore has a length inferior to that of the hypotenuse $\overline{AC}$ *(Figure 89)*.



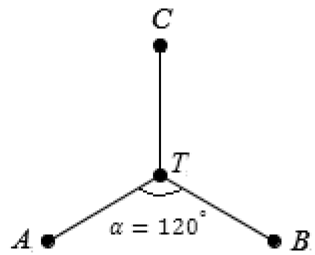
(Figure 89)

Point *T*, the Torricelli point for ΔABC,

$$A\hat{T}B = B\hat{T}C = C\hat{T}A = 120^\circ$$

assures the construction of a minimal network, whose measure is approximately 86.6 *km (Figure 90)*. This value results from an elementary trigonometric relation $\left(sin60^\circ = \frac{25}{AT}\right)$.

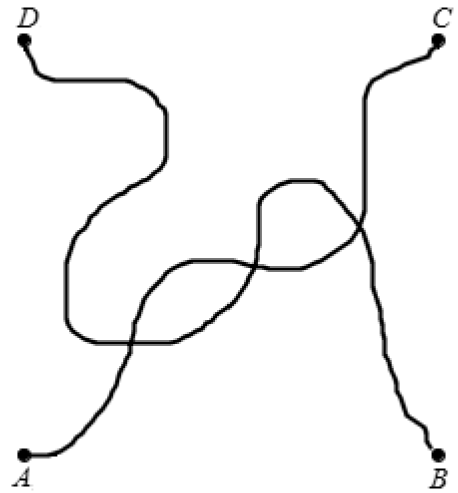
The cost reduction is of approximately 13. 4%.



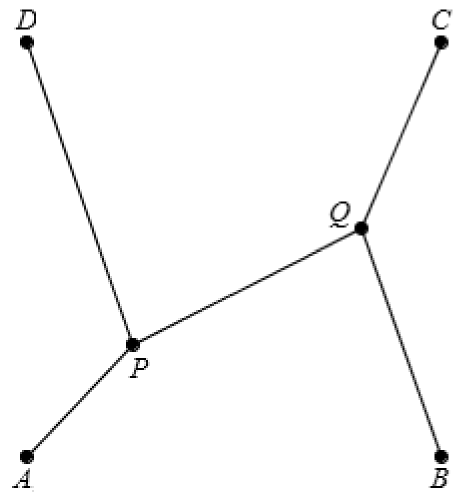
(Figure 90)

A general objective would be to outline the minimal system which connects an arbitrary number of points. However no general method is known. We will be satisfied to solve the problem with 4 points.

Consider 4 points A, B, C and D, which are vertices of a square. The network, also known as mathematical tree, has its vertices in these points and also two Steiner points. To establish proof, create an arbitrary system of paths which connect A, B, C and D. Assume that the connections from A to C and B to D are inside or on the border of the fictitious square, and intersect at some points *(Figure 91)*.



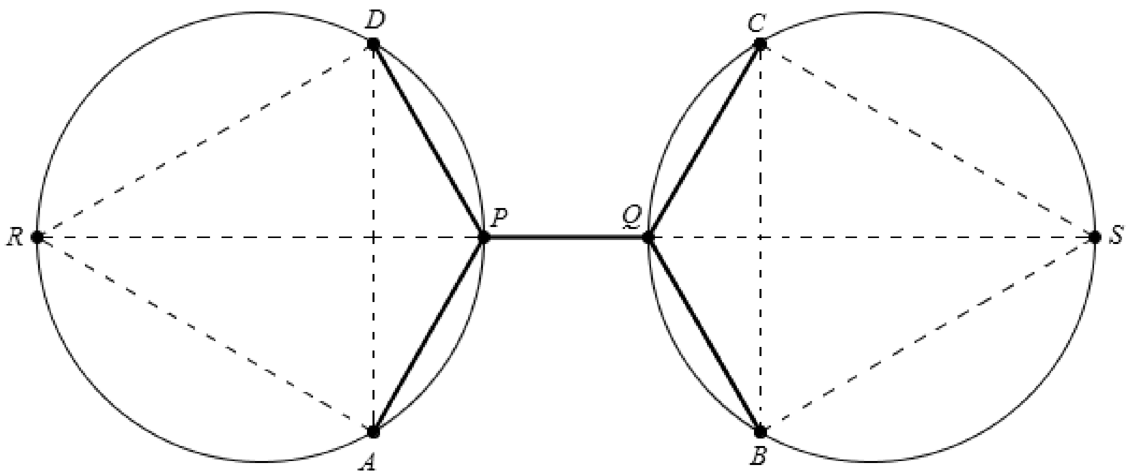
(Figure 91)



(Figure 92)

Points P and Q are, respectively, the first and the last of the connection from A to C which intersect the connection from D to B. Hence, the tree formed by $[AP]$, $[DP]$, $[PQ]$, $[BQ]$ and $[CQ]$ obviously has a length which is inferior to that of the initial structure *(Figure 92)*.

Determine the configuration of the minimal tree by using the Dimitrie Pompeiu's Theorem. Construct the equilateral triangles ΔADR and ΔBCS which are exterior to the square. Hence, $\overline{PA} + \overline{PD} \geq \overline{PR}$ and $\overline{QB} + \overline{QC} \geq \overline{QS}$, where $\overline{PA} + \overline{PD} + \overline{PQ} + \overline{QB} + \overline{QC} \geq \overline{PR} + \overline{PQ} + \overline{QS} \geq \overline{RS}$ *(1)*. The problem requires the minimisation of the length. According to the previous inequality *(1)*, this value is always greater than or equal to $\overline{RS}$ *(Figure 93)*.



(Figure 93)

In order to obtain the minimal tree, P and Q are, respectively, the intersection points of $[RS]$ with the circumcircles of ΔADR and ΔBCS. The length of the minimal network may be equal to $\overline{RS}$.

$$\overline{PA} + \overline{PD} = \overline{RP} \qquad\qquad \overline{QB} + \overline{QC} = \overline{QS}$$

$$\overline{PA} + \overline{PD} + \overline{PQ} + \overline{QB} + \overline{QC} = \overline{RP} + \overline{PQ} + \overline{QS} = \overline{RS}$$

The size of the angles formed by the line segments which meet at each Steiner point is 120º. The minimal trees for 3 and 4 points may be extended to an arbitrary number of points, in line with the following properties: *1)* all original points are connected to 1, 2 or 3 points; *2)* all Steiner points are connected to 3 points; *3)* connections between any 2 vertices form an angle of at least 120º. If the connection point is a Steiner point, the size of the angle is 120º; *4)* for *n* points, the Steiner minimal tree has a maximum of *n – 2* points.

Generally, the construction of the Steiner minimal tree is a complex problem whose solution for an increasing number of initial points requires advanced algorithms supported by computational instruments.

Let us go back to the difference between the concepts of equivalence and implication, and introduce solutions to systems of equations. The usual methods to find the solution to a system of equations,

$$\begin{cases} F_1(x_1, x_2, \dots, x_n) = 0 \\ F_2(x_1, x_2, \dots, x_n) = 0 \\ \qquad\quad \dots \\ F_n(x_1, x_2, \dots, x_n) = 0 \end{cases}$$

perform successive transformations of the form ε_1, then ε_2, then ε_3, … *(i)* where ε_3 says something like $x_1 \in S_1$, $x_2 \in S_2 = S_2(x_1)$, $x_3 \in S_3(x_1, x_2)$, …, where $S_1, S_2, S_3, \dots$ are sets of possible values, for S_j may depend on the already chosen values $x_1, x_2, x_3, \dots, x_{j-1}$.

It should be noted that *(i)* encompasses a series of implications of type *if then*, where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots$ are usually intermediate systems. These implications are not always reversible without additional conditions, thus implying that transformation outcome ought to be carefully read. If there is a solution, then the solution necessarily satisfies $x_1 \in S_1$, $x_2 \in S_2, \dots$.

The necessary conditions found are not always sufficient, *i.e.*, the $x_1, x_2, \dots$ found may eventually not solve the system. From all the possible solutions, the ones which, when the unknowns are replaced, satisfy the system are the ones to be chosen.

Suppose that the goal is to know *x*. If something such as $a(x) = \sqrt{b(x)}$ is deduced, then it may be an indication that *x* does not exist in ℝ. For example:

$$-1 - x = \sqrt{8x} \; (1)$$

for the root will be interpreted only for number $x \geq 0$, and then $\sqrt{8x} \geq 0$. The left side of the equation will assume a negative value! Nevertheless, calculations may proceed by writing

$$(-1 - x)^2 = 8x$$
$$1 + 2x + x^2 = 8x$$
$$1 - 6x + x^2 = 0$$
$$x_{1,2} = \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm \sqrt{8} \; (2)$$

When these perfectly defined real numbers are replaced in *(1)*, they become an absurdity. Result *(2)* should then be interpreted: there is a solution to *(1)*, then

$$x_{1,2} \in \{3 \pm \sqrt{8}\}.$$

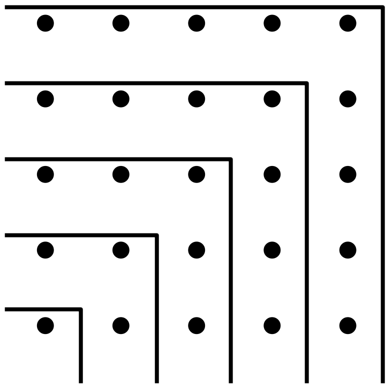
Hence, the notation ⟺ to replace ⟹ must be carefully used.

Mathematical problem solving requires specific knowledge, decision-making skills and ability to relate and apply information. A myriad of concepts, definitions and theorems makes choosing the right strategy, as well as applying it correctly, a difficult procedure. The Pythagoras' theorem is introduced to 8th grade students. However, if in the 9th grade students are asked about its significance, they will all most probably repeat the famous statement but not all will know what it means. That is not by chance. The Pythagoras' theorem is a milestone within the learning process of Mathematics. If the student wishes to proceed to Secondary Education, then their choice ought to be free of failure beliefs and wrong ideas regarding Mathematics. Mathematics is not a recipe book.

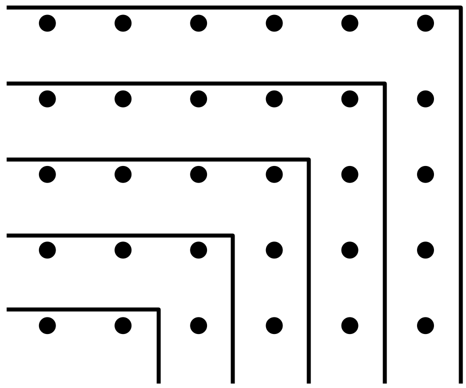
> *The Pythagorean Theorem provides the link between geometry and algebra required for the study of rational right triangles. For, as we have seen, if ΔABC is a triangle with sides of length a, b, and c, then $x = a$, $y = b$, and $z = c$ is a solution of the algebraic equation $x^2 + y^2 = z^2$ if and only if ΔABC is a right triangle with legs of length a and b, and hypotenuse of length c. The algebraic equation $x^2 + y^2 = z^2$ is often called the Pythagorean equation* (Sally & Sally, 2007, p. 63).

Pithagoras' Theorem

The Pythagoreans were in the habit of representing arithmetical numbers as geometrical forms through which they arrived at some interesting insights. Regarding the following figures, Aristotle said: First *(Figure 94)* gnomons were assembled around the One. Each band of points is odd. Then *(Figure 95)* gnomons were assembled around the Dyad. Each band of points is even. We can understand why the Pythagoreans identified the Odd with the Square and the Even with the Oblong.



(Figure 94) Square Number

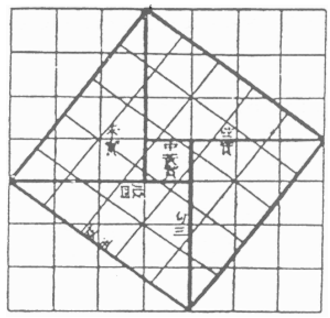


(Figure 95) Oblong Number

The principles of Limit and the Unlimited are also noticeable in these illustrations. The Square is a stable form and the Oblong is infinitely variable: *with each successive gnomon, the shape and its corresponding lateral to horizontal ratio changes each time, for it is the nature of the Unlimited to be eternally variable and multifarious.* (Guthrie, 1987, p. 24). The relation between the sides of a right triangle was established by different civilisations. In Ancient China, the equality was introduced in the *Chou Pei Suan Ching* (*The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven***)**, an anonymous collection written between 500 BC and 200 BC, whose most ancient known copy dates from 1213.

> *Zhou Kung addressed Shang Gao, saying: There are no steps by which one may ascent the heavens, and the earth is not measurable with a footrule. I should like to ask you what was the origin of these numbers? Shang Gao replied: Let us cut a rectangle (diagonally), and make the width 3 (units) wide, and the length 4 (units) long. The diagonal between the two corners will then be 5 (units) long. Now after drawing a square on this diagonal, circumscribe it by half-rectangles like that which has been left outside, so as to form a (square) plate. Thus the (four) outer half-rectangles of width 3, length 4, and diagonal 5, together make two rectangles (of area 24); then (when this is subtracted from the square plate of area 49) the remainder is of area 25. This (process) is called 'piling up the rectangles' (chi chu).* (Joseph, 1991, p. 180).

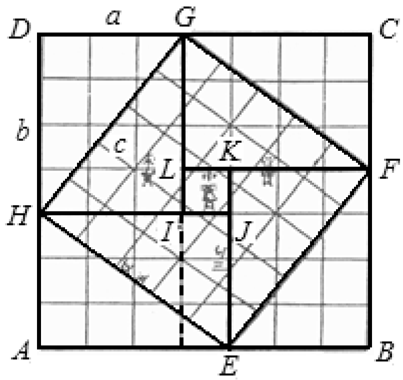
Exercise 1: The illustration *(Figure 96)* was taken from *The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven* where it illustrates a theorem on right angles described by Shang Gao. With the help of the figure, explain the meaning of Shang Gao's text.



(Figure 96)

The figure shows a grid of seven vertical grid squares by seven horizontal grid squares, and a square whose side corresponds to the diagonal of a rectangle with 3 units by 4 units. If the square is circumscribed, 4 geometrically equal right triangles can be identified.

Exercise 2: Identify the points which correspond to vertices of squares or rectangles, as well as the measures of the cathetus and the hypotenuse of one of the right triangles exterior to the square, using, for example, the letters a, b and c *(Figure 97)*. Demonstrate that, for this case, $c^2 = a^2 + b^2$.



(Figure 97)

The area of the square whose side measures *c* units of length is c^2.

Decompose the square whose side measures *c* units of length in 4 geometrically equal right triangles and a square. The measures of the cathetus of each right triangle are *a* and *b*. The measure of the side of the square $[IJKL]$ is $(b - a)$.

After determining the area of the figures,

$$4 \times \frac{a \times b}{2} + (b - a)^2 = 2 \times a \times b + b^2 - 2 \times a \times b + a^2 = a^2 + b^2$$

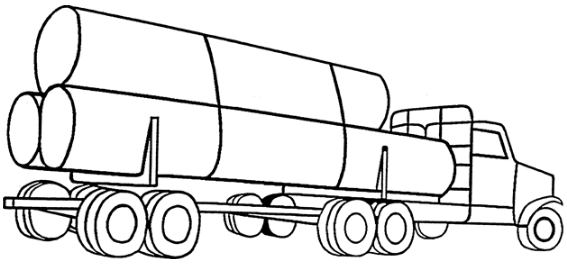
then,

$$c^2 = a^2 + b^2.$$

The grid with seven vertical grid squares by seven horizontal grid squares has 49 units of area. If the 4 right triangles are subtracted, which together have 24 units of area, a square with 25 units of area is obtained.

(Subject: Geometry – 9th grade students)

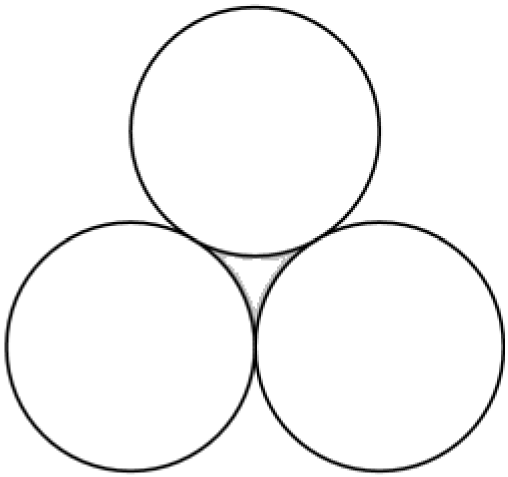
A truck carries three equal logs of wood as presented *(Figure 98)*.
Logs have a cylindrical form with a 1-metre diameter.
Compute the cargo height.
Present your result in centimetres and estimate your computations in units.



(Figure 98)

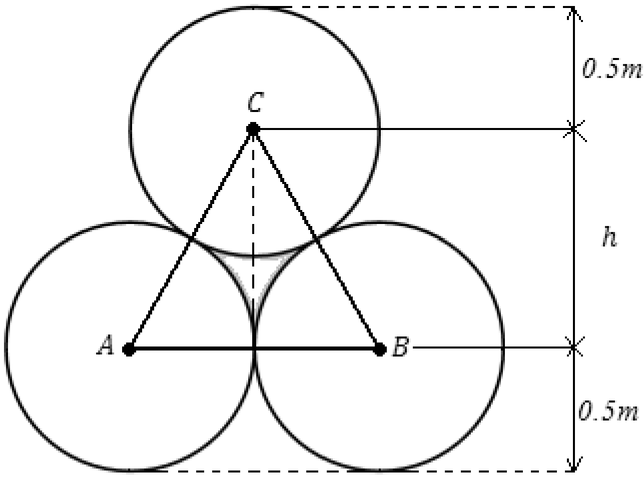
Students' impulsive answer is 2 metres. If each log has a 1-metre diameter, the reasoning is that one log on top of the remaining reach a 2-metre height *(Figure 99)*. Unfortunately, impetuousness is mathematics usually does not lead to high-quality results.

We could ask the students to draw a figure with the three logs in position.



(Figure 99)

If even then they sustain the previous answer, they should be questioned about the positions of the upper log and of the other two logs. By now the students should have realised that their answer is wrong, for the lower point of the upper log is less than 1 metre high. So the correct answer should be inferior to 2*m*. Now they should devise a plan to reach the solution but they do not know how to proceed. If this problem is presented by the teacher shortly after a Pythagoras' Theorem discussion, the students will possibly deduce the need to use it. If not, the occurrence of such association will not come that easily. However, there still remains a long way to achieve a solution, because the triangle rectangle needed for the application of the Pythagoras' theorem remains *invisible*. If within a given time the students do not figure out such diagram, a suggestion from the teacher is welcome *(Figure 100)*.

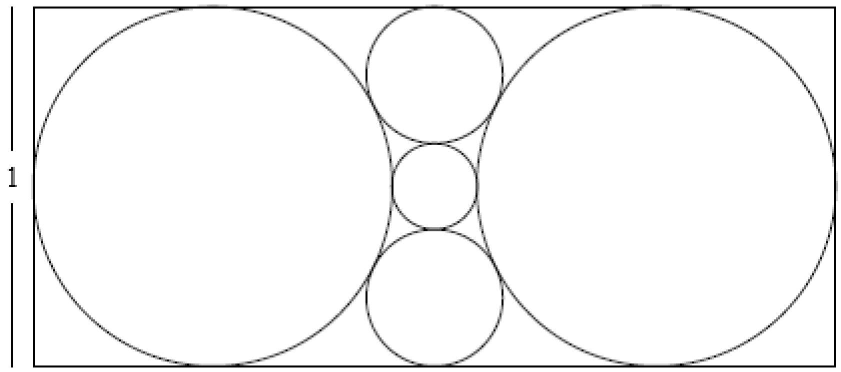


(Figure 100)

Now the question becomes a straightforward exercise. ΔABC is equilateral with the side length of $1m$ and the unknown height h. By using the Pythagoras' theorem, $1^2 = h^2 + 0{,}5^2$, we extract that $h = \sqrt{0{,}75}$. Then, the cargo height is of approximately $187cm$.

(Subject: Geometry – 9th grade students)

What is the length of the largest side of the rectangle *(Figure 101)*?

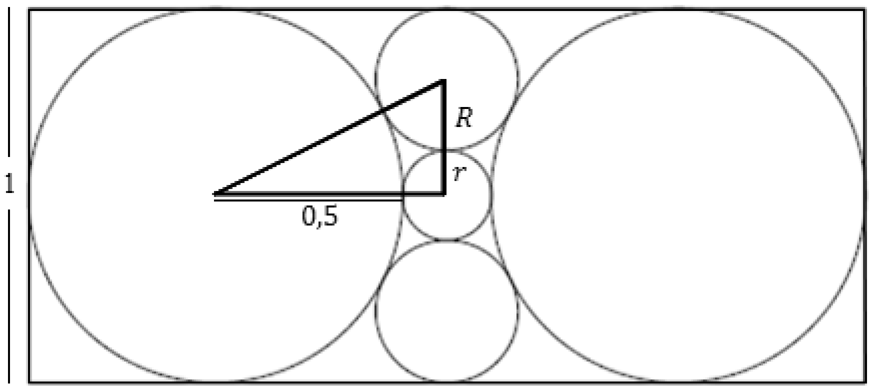


(Figure 101)

Use adequate notation. The figure shows circumferences with different radii. The measure of the smallest side of the rectangle corresponds to one unit of length, a value which coincides with the diameter of the circumferences which have the largest radius. The measure of the radius of the remaining circumferences is not known. Each of these quantities is identified by *R* and *r (Figure 102)*.

Let us devise a plan. The solver possesses a set of mathematical tools whose variety depends on the quality and quantity of their mathematical knowledge. Hence, the solver may be aware of part or all the required techniques to solve the problem. The difficulty does not necessarily lie in the application of the solving method(s) but rather in their identification. Eventually, students randomly state procedures, among which the Pythagoras' theorem.

Nevertheless, verbalising the Pythagoras' theorem does not mean knowing how to use it. Being able to state and use the procedure is crucial, but it may not be enough to reach the solution. The diagram identifies a right triangle and emphasises the need to apply the Pythagoras' theorem.



(Figure 102)

With the application of the Pythagoras' theorem,

$$(0{,}5 + R)^2 = (0{,}5 + r)^2 + (r + R)^2$$

$$0{,}25 + R + R^2 = 0{,}25 + r + r^2 + r^2 + 2rR + R^2 \Leftrightarrow R = r + 2r^2 + 2rR.$$

The solver obtains a relation between the measures of the radii of the circumferences. However, he/she also realises that the equality is not sufficient to solve the problem. The value of *R* and *r* must still be quantified. The goal is to calculate the length of the rectangle. The figure shows that that measure corresponds to 2 times the diameter of the circumference with the largest radius plus the value of the diameter of the smallest circumference $(2 \times 1 + 2r)$.

As the width of the rectangle corresponds to a unit of length (a piece of information which allowed the identification of the radius of the largest circumference), the width of the rectangle is equal to 2 times the diameter of the circumference with the radius *R*, to which the value of the diameter of the circumference with the radius *r* should be added $(4R + 2r = 1)$.

$$\begin{cases} R = r + 2r^2 + 2rR \\ 4R + 2r = 1 \end{cases}$$

Multiply the first equation by 4 and use the equality $4R = 1 - 2r$ extracted from the second equation.

$$\begin{cases} 4R = 4(r + 2r^2 + 2rR) \\ 4R = 1 - 2r \end{cases}$$

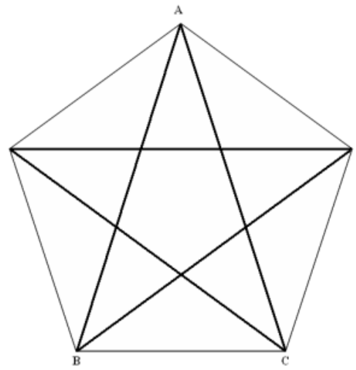
Hence,

$$1 - 2r = 4r + 8r^2 + 2r(1 - 2r).$$

Therefore, the valid solution for the problem is $r = \frac{\sqrt{5}}{2} - 1$.

It is now possible to calculate $2 \times 1 + 2r$. The rectangle measures $\sqrt{5}$ units of length.

In Ancient Greece, the 5-pointed star inscribed in a pentagon and filled with irrational numbers had an aura of mysticism for the Pythagoreans *(Figure 103)*.



(Figure 103)

A diagonal point resulting from the intersection of 2 diagonals divides a diagonal in 2 line segments. As we shall see, the ratio between the total diagonal and the largest line segment equals the ratio between the largest line segment and the smallest segment. The value of this ratio coincides with the golden number. When 1 point of the star is connected to the 2 opposite points, an isosceles triangle with 2 angles of 72º and another of 36º is obtained. The ratio between the lengths of the largest side and the smallest side coincides with the golden number.

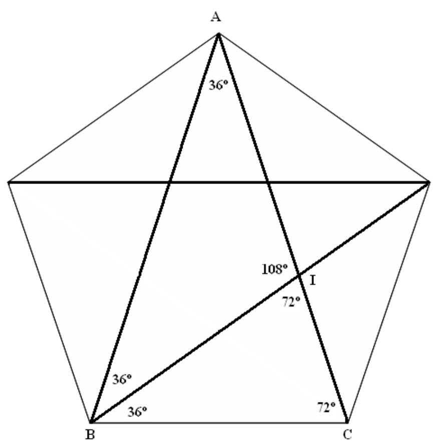
We remember that the formula which relates the size of the internal angles of a regular polygon with the number *n* of sides is: $\alpha = \frac{180^\circ(n-2)}{n}$ and also that the size of an angle inscribed in a circumference is equal to half of the size of the corresponding arc. So, the size of each internal angle of a regular pentagon is 108º and $B\hat{A}C$ =36º.

If the Law of Sines is applied, then *in any triangle, the sides are proportional to the sines of the angles opposite to them and the constant of proportionality is the diameter of the circumscribed circumference,* hence,

$$\frac{sin\ 72^\circ}{\overline{AB}} = \frac{sin\ 36^\circ}{\overline{BC}}.$$

The ratio between the length of the largest side of the triangle and the length of the smallest side unites through the golden number, Geometry with the Fibonacci sequence.

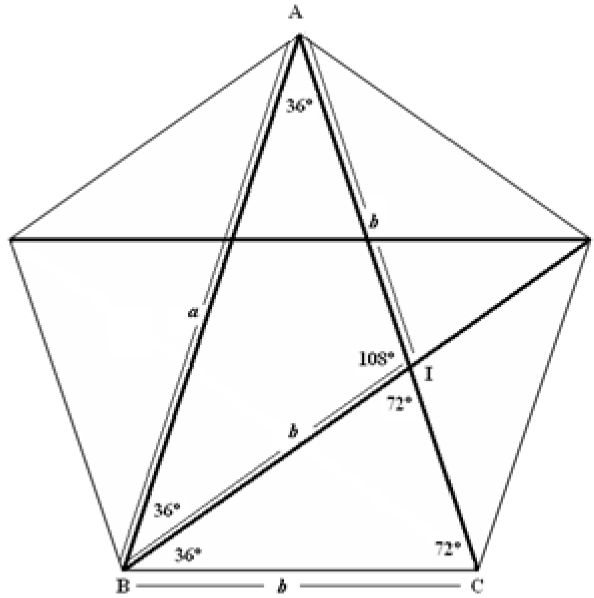
The Pythagorean star holds more surprises. Point *I*, which results from the intersection of 2 diagonals, divides 1 diagonal in 2 line segments. The ratio between the total and the largest line segment is equal to the ratio between the largest segment and the smallest segment *(Figure 104)*. The value of this ratio is coincident with the golden number.



(Figure 104)

To prove it, consider the isosceles triangle $[ABC]$ with $\hat{B} = \hat{C} = 72^{\circ}$ and $\hat{A} = 36^{\circ}$.

If $\dot{B}I$ is the bisector of angle $A\hat{B}C$, then ΔAIB and ΔCBI are isosceles triangles. Hence, $\Delta CBI \sim \Delta BAC$, for the corresponding internal angles are respectively congruent *(Figure 105)*.



(Figure 105)

Regarding ΔCBI, then $\overline{BC} = \overline{BI} = b$.

Point I is in $[AC]$. Then, $\overline{AC} = \overline{AI} + \overline{IC}$. Consider $a = b + \overline{IC}$, *i.e.*, $\overline{IC} = a - b$.

From the similarity of triangles $[BAC]$ and $[CBI]$, we have:

$$\frac{\overline{AC}}{\overline{BI}} = \frac{\overline{CB}}{\overline{CI}} \Leftrightarrow \frac{a}{b} = \frac{b}{a-b} \Leftrightarrow b^2 = a^2 - ab \Leftrightarrow \frac{b^2}{a^2} = \frac{a^2-ab}{a^2} \Leftrightarrow \left(\frac{b}{a}\right)^2 = 1 - \frac{b}{a}.$$

When we have $\frac{b}{a} = m$, then $m^2 = 1 - m$, which is equivalent to $m^2 + m - 1 = 0$.

The solution $m = \frac{-1+\sqrt{5}}{2}$ allows to proof that point I divides $[AC]$ in the golden ratio.

$$\frac{a}{b} = \left(\frac{b}{a}\right)^{-1} = \frac{2}{-1+\sqrt{5}} = \frac{2(-1-\sqrt{5})}{(-1+\sqrt{5})(-1-\sqrt{5})} = \frac{-2 \times (1+\sqrt{5})}{-2 \times 2} = \frac{1+\sqrt{5}}{2}.$$

(Subject: Geometry – 9th grade students)

Why do bees build their honeycombs with the shape of regular hexagons?

Bees build their honeycombs to store honey by using the minimum material possible. The goal is to maximise the volume with a given amount of material. Without loss of generality, consider the problem in $\mathbb{R}^2$. Find the regular polygons which can be used to pave a flat surface.

- Equilateral triangles are an option. Each internal angle measures 60º. When 6 equilateral triangles are put together, the sum of the size of the angles around the common point is 360º.

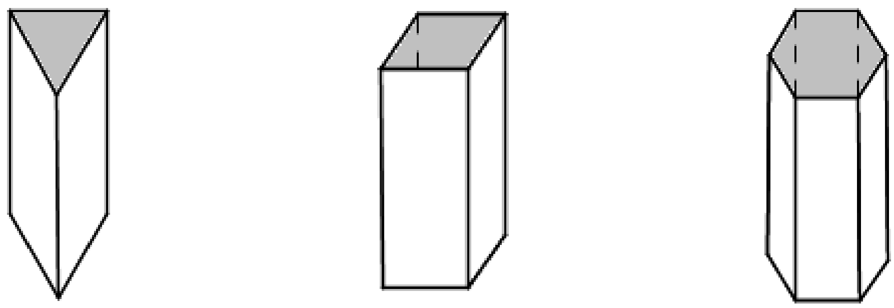
- Squares are a solution to the problem. Each internal angle measure 90º. When 4 squares are put together, the sum of the size of the angles around the common point is 360º.

Regular pentagons cannot be used to pave the surface. The size of each internal angle of a pentagon is 108º. When 3 regular pentagons are put together, the sum of the size of the angles around the common point is 324º, a value which is inferior to 360º. When 4 regular pentagons are put together, the sum of the size of the angles around the common point is 432º, a value which is greater than 360º.

- Can regular hexagons be used to pave? The size of each internal angle of a regular hexagon is 120º. When 3 regular hexagons are put together, the sum of the size of the internal angles around the common point is 360º.

Hence, equilateral triangles and squares or regular hexagons can be used to pave the surface. The goal of the problem is to determine the geometric figure which is more suitable. Each figure corresponds to the base of a prism whose volume must be maximised. If the three solids have the same height, which of the prisms, whose faces have the same area, has the greatest volume.

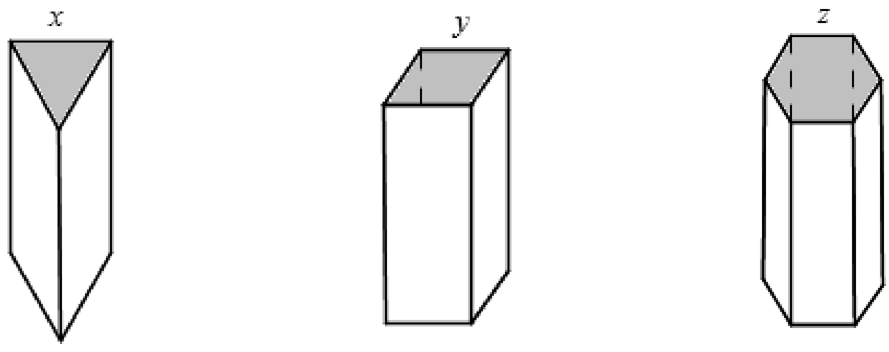
Build the prisms with A_4-format sheets of paper by folding the largest side in three, four and six equal parts, respectively *(Figure 106)*.



(Figure 106)

The volume of each prism is equal to the area of its base multiplied by its height (constant). Hence, the volume depends exclusively on the area of the base.

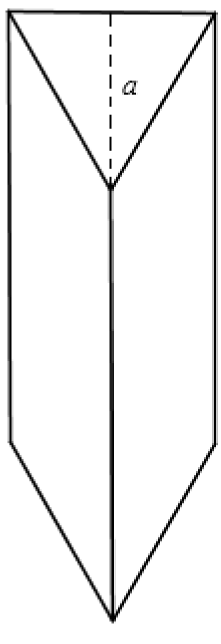
Let us see how areas and edges of the base of prisms relate *(Figure 107)*. Consider x, y and z, which are respectively the measures of the bases of the triangular, quadrangular and hexagonal prism. Supposing that the bases of the prisms have the same perimeter (p), the length of the sheet of paper, then $p = 3x, p = 4y$ and $p = 6z$.



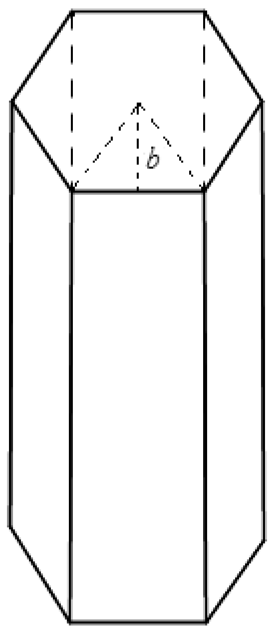
(Figure 107)

Let us create an equality: $3x = 4y$ and $3x = 6z$, *i.e.*, $y = \frac{3}{4}x$ and $z = \frac{1}{2}x$.

The areas of the equilateral triangle, the square and the regular hexagon may be determined in function of x (the edge of the base of the triangular prism). By using the Pythagoras' theorem to determine the height (*a)* of the equilateral triangle *(Figure 108)*, the base of the triangular prism, and the height *(b)* of any of the six isosceles triangles in which form a regular hexagon *(Figure 109)*, the base of the hexagonal prism.



(Figure 108)



(Figure 109)

$$x^2 = a^2 + \left(\frac{x}{2}\right)^2 \Leftrightarrow x^2 - \frac{1}{4}x^2 = a^2 \Leftrightarrow \frac{3}{4}x^2 = a^2 \Leftrightarrow a = \frac{\sqrt{3}}{2}x$$

Therefore,

$$Area\ of\ the\ equilateral\ triangle = \frac{Base \times Height}{2} = \frac{x \times \frac{\sqrt{3}}{2}x}{2} = \frac{\sqrt{3}}{4}x^2;$$

whereas for the square,

$$Area\ of\ the\ square = Side\ \times Side = \frac{3}{4}x \times \frac{3}{4}x = \frac{9}{16}x^2;$$

and finally, for the hexagon,

$$\left(\frac{1}{2}x\right)^2 = b^2 + \left(\frac{1}{4}x\right)^2 \Leftrightarrow \frac{1}{4}x^2 - \frac{1}{16}x^2 = b^2 \Leftrightarrow \frac{3}{16}x^2 = b^2 \Leftrightarrow b = \frac{\sqrt{3}}{4}x$$

$$Area\ of\ the\ regular\ hexagon = 6 \times \frac{\frac{1}{2}x \times \frac{\sqrt{3}}{4}x}{2} = \frac{3\sqrt{3}}{8}x^2.$$

Considering the coefficients of x^2 which measure areas in terms of x, we have:

$$\frac{\sqrt{3}}{4} < \frac{9}{16} < \frac{3\sqrt{3}}{8}.$$

The inequalities show that, of the three polygons, the regular hexagon is the one to have a greater area given its perimeter, thus having the greater volume. The bees' goal is to provide stability to viscous mass (honey). Hence, they introduce a partition in small cells; the easiest way to do it is with triangles, squares or hexagons. Calculations demonstrate that hexagonal paving is the most economical. In fact, bees close their hexagonal prisms in a curious way, a behaviour which was the basis for Réaumur's problem (Dörrie, 1965, p. 366).

A flat surface can also be paved with combinations of different geometrical figures. It is now up to the reader to find the possible combinations.

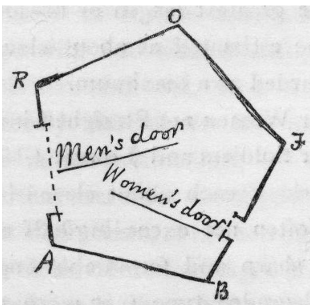
(Subject: Geometry – 7th grade students)

How many edges does a prism with 2014 faces have?

This solid is formed by an upper face and a bottom face (individual surfaces), which are parallel and congruent, connected by edges (which join 1 vertex to another vertex). The prism has 2012 lateral faces, trapeziums or parallelograms. Whatever the prism, the number of lateral faces coincides with the number of lateral edges. Hence, the prism has 2012 lateral edges, to which the 2012 edges which define the bottom face and the 2012 edges which define the upper face must be added. Hence, the solid has 6036 edges. The answer substantiates Euler's polyhedron formula. Any convex polyhedron surface has Euler's characteristic *Faces + Vertices = Edges + 2*.

Regular polyhedron solids are exclusively limited by flat faces, regular polygons, which are connected in an equal number to vertices. In any regular polyhedron, at least 3 faces are required for each vertex, and the sum of the sizes of the angles which surround it must be inferior to 360º.

The plane/space duality/dichotomy, $\mathbb{R}^2$ and $\mathbb{R}^3$, suggested in the previous paragraph is highlighted in *Flatland: A Romance of Many Dimensions (Figure 110)*, by Edwin A. Abbott, a fiction novel which carries the reader to a two-dimensional world as a satire to British Victorian society. In *Flatland*, women are represented by line segments and men by regular polygons with as many sides as greater their position in society is. The narrator, a humble Square, is visited by the Sphere, a being of a three-dimensional world. The solid makes great effort to explain the existence and the rules of Space, but the challenge is complex. Aware of the difficulty of the task, the Sphere takes the Square to $\mathbb{R}^3$, where he has the opportunity to experience what, until that moment he had only known by words. From Space, they hear the leaders of *Flatland* conspire to silence those who dare to disseminate information on the existence of the three-dimensional world. Consequently, depending on cast or social position, many witnesses are killed or imprisoned. Imbued with the existence of $\mathbb{R}^3$, the Square suggests the possibility of the existence of new dimensions. But unable to visualise such scenario, the Sphere angrily sends the Square back to *Flatland*, where he is not welcome. The Square's imagination makes him dream once more of the Sphere. Now the Sphere takes him to *Pointland*, whose only inhabitant does not know anything beyond its own singularity, and, unable to conceive other scenarios, interprets their attempts to communicate as the result of his own thoughts. He unsuccessfully tries to explain the existence of a second dimension to the king, who is not capable of seeing beyond an infinite line. The narrator draws a parallel between the ignorance of the kings of *Pointland* and *Flatland* with his own initial ignorance regarding the existence of superior dimensions. On his return to *Flatland*, he is incarcerated for disseminating subversive ideas.



(Figure 110) - Illustration of a simple house in Flatland

Knowledge comes from curiosity, imagination, purpose and use of stored knowledge. In Alice in Wonderland, written in 1865, Charles Lutwidge Dodgson (1832 – 1898), with the pseudonym Lewis Carroll, introduces a young lady called Alice who, lead by her curiosity, is carried to a world of fantasy. The presence of mathematical episodes throughout the book is a reflex of the author's education, a mathematician at Christ Church College, Oxford, and can be used by teachers to illustrate significant concepts and relate them with creative writing. The 19th century was a turning point in mathematics with the emergence of new and controversial concepts, such as imaginary numbers. Dodgson was a conservative mathematician and

> *he valued the ancient Greek textbook Euclid's Elements as the epitome of mathematical thinking. Broadly speaking, it covered the geometry of circles, quadrilaterals, parallel lines and some basic trigonometry. But what's really striking about Elements is its rigorous reasoning: it starts with a few incontrovertible truths, or axioms, and builds up complex arguments through simple, logical steps. Each proposition is stated, proved and finally signed off with QED.*
>
> *For centuries, this approach had been seen as the pinnacle of mathematical and logical reasoning. Yet to Dodgson's dismay, contemporary mathematicians weren't always as rigorous as Euclid. He dismissed their writing as "semi-colloquial" and even "semi-logical". Worse still for Dodgson, this new mathematics departed from the physical reality that had grounded Euclid's works.*
>
> *By now, scholars had started routinely using seemingly nonsensical concepts such as imaginary numbers – the square root of a negative number – which don't represent physical quantities in the same way that whole numbers or fractions do. No Victorian embraced these new concepts wholeheartedly, and all struggled to find a philosophical framework that would accommodate them. But they gave mathematicians a freedom to explore new ideas, and some were prepared to go along with these strange concepts as long as they were manipulated using a consistent framework of operations. To Dodgson, though, the new mathematics was absurd, and while he accepted it might be interesting to an advanced mathematician, he believed it would be impossible to teach to an undergraduate* (Bayley, 2009).

One of the theories which was strongly criticised by Dodgson was the *Principle of Continuity* advocated by Jean-Victor Poncelet (1788 – 1867), a French engineer and mathematician, who said as follows *"Let a figure be conceived to undergo a certain continuous variation, and let some general property concerning it be granted as true, so long as the variation is confined within certain limits; then the same property will belong to all the successive states of the figure."*

Let's consider the case of 2 intersecting circles. After solving their equations we find that they intersect at two points. The *Principle of Continuity* endorses that any continuous transformation of these circles, moving their centres away one from the other, for example, will conserve their intersection at 2 points. When the centres are far enough, the solution will involve an imaginary number that collides with mathematical dogmas.

If we consider two circumferences centred on $(0,0)$ and $(1,0)$ with equations $(x -$ $02+y-02=12$ and $x-12+y-02=12$, these circumferences intersect at 2 points, $12,32$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Under the *Principle of Continuity* they are expected to continue to intersect in 2 points even if moved apart and no longer in contact. Hence, if we move the circumference with centre $(1,0)$ to $(3,0)$, now equations $(x-0)^2 + (y-0)^2 = 1^2$ and $(x-3)^2 + (y-0)^2 = 1^2$, these circumferences still intersect in 2 points, $\left(\frac{3}{2}, \frac{\sqrt{5}}{2} i\right)$ and $\left(\frac{3}{2}, -\frac{\sqrt{5}}{2} i\right)$, where $i^2 = -1$.

The study of excerpts taken from creative writing may be an excellent starting point for problem solving activities. Articles published in scientific journals, namely the *Gazeta de Matemática*[70], a repository of knowledge for teachers and students, are excellent didactic tools. However, we highlight that problem solving alone does not fill students' gaps, and that its implementation ought to abide by a well-defined methodology, so that

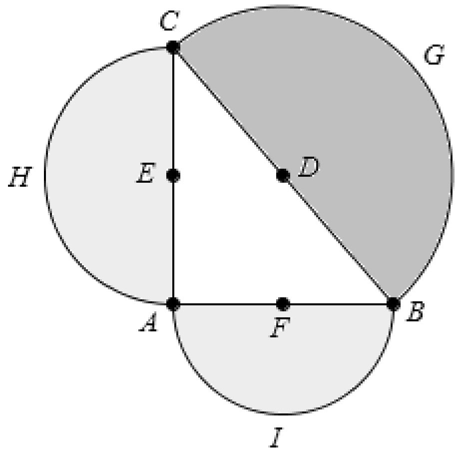
> *…long-term memory is no longer seen as a passive repository of discrete, isolated fragments of information that permit us to repeat what we have learned. Nor is it seen only as a component of human cognitive architecture that has merely peripheral influence on complex cognitive processes such as thinking and problem solving. Rather, long-term memory is now viewed as the central, dominant structure of human cognition. Everything we see, hear, and think about is critically dependent on and influenced by our long-term memory.*
>
> *Expert problem solvers derive their skill by drawing on the extensive experience stored in their long-term memory (...) they then quickly select and apply the best procedures for solving problems in the form of concepts and procedures, known as mental schemes. (...) In short, our long-term memory incorporates a massive knowledge base that is central to all our cognitively based activities. Hence, research suggests that novice mathematicians can be taught to be efficient problem solvers by providing them with a great storage of specific knowledge* (Sweller, Clark, & Kirschner, 2012, pp. 26-29).

[70] Published 3 times a year, it disseminates mathematical culture in Portugal. Created in 1939 by a group of mathematicians which included Bento de Jesus Caraça (1901-1948), it is currently published by Sociedade Portuguesa de Matemática (SPM).

Let us present proposals for students to work on the areas of figures such as the lune[71].

(Subject: Geometry – 9th grade students)

Prove that in a right triangle, the area of a semicircle constructed on the length of the hypotenuse is equal to the areas of the semicircles constructed on the lengths of the cathetus.



(Figure 111)

Solution: the diagram *(Figure 111)* shows:

Area of semicircle Ø BC: $A_{BGCD} = \frac{\pi \overline{BC}^2}{8}$;

$$\left(= \frac{1}{2}\pi \overline{BD}^2 = \frac{1}{2}\pi \left(\frac{\overline{BC}}{2}\right)^2 \right)$$

Area of semicircle Ø AC: $A_{AHCE} = \frac{\pi \overline{AC}^2}{8}$;

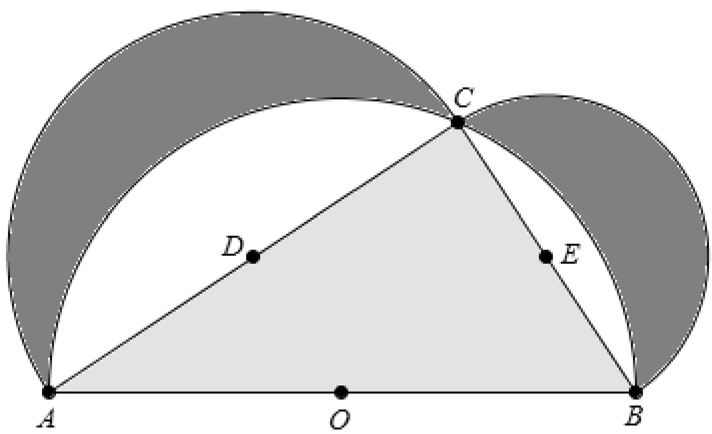
Area of semicircle Ø AB: $A_{AIBF} = \frac{\pi \overline{AB}^2}{8}$;

$$\frac{\pi \overline{AC}^2}{8} + \frac{\pi \overline{AB}^2}{8} = \frac{\pi}{8}(\overline{AC}^2 + \overline{AB}^2) = \frac{\pi \overline{BC}^2}{8}$$

$$A_{AHCE} + A_{AIBF} = A_{BGCD}$$

(Subject: Geometry – 9th grade students)

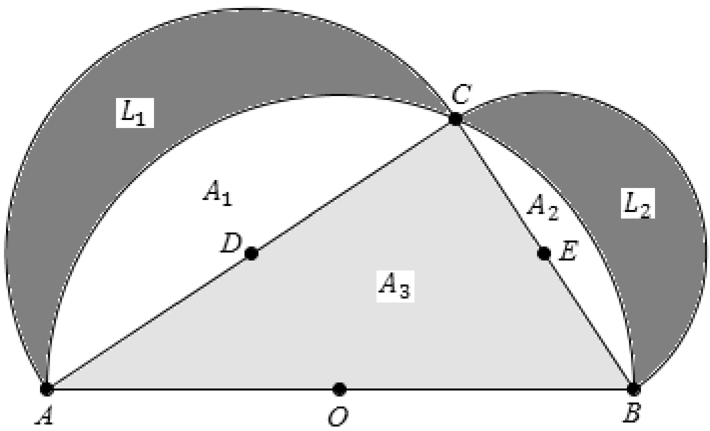
Prove that the sum of the areas of the lunes (grey regions) is equal to the area of ΔABC *(Figure 112)*.



(Figure 112)

[71] Concave-convex area bounded by two circular arcs.

Solution: From the semi-circumference ∅ AB, draw ΔABC, which is right in *C*. Draw a semi- circumference ∅ AC whose centre D is the middle point of $[AC]$. Identify the lunes L_1 and L_2, the circular segments of the areas A_1 and A_2 and the right triangle with an area A_3, inscribed in the semi-circumference ∅ AB. The goal is to prove that $L_1 + L_2 = A_3$ *(Figure 113)*. How to group the values of the areas so that complementary information can be extracted? $L_1 + A_1$ corresponds to semicircle ∅ AC, $L_2 + A_2$ corresponds to semicircle ∅ BC and $A_1 + A_2 + A_3$ corresponds to semicircle ∅ AB. Hence, $L_1 + L_2 = (L_1 + A_1) + (L_2 + A_2) - [(A_1 + A_2 + A_3) - A_3]$.



(Figure 113)

Therefore,

$$L_1 + L_2 = \frac{1}{2}\pi\left(\frac{1}{2}\overline{AC}\right)^2 + \frac{1}{2}\pi\left(\frac{1}{2}\overline{BC}\right)^2 - \left[\frac{1}{2}\pi\left(\frac{1}{2}\overline{AB}\right)^2 - \frac{\overline{BC}\times\overline{AC}}{2}\right]$$

$$= \frac{1}{8}\pi\overline{AC}^2 + \frac{1}{8}\pi\overline{BC}^2 - \frac{1}{8}\pi\overline{AB}^2 + \frac{\overline{BC}\times\overline{AC}}{2}$$

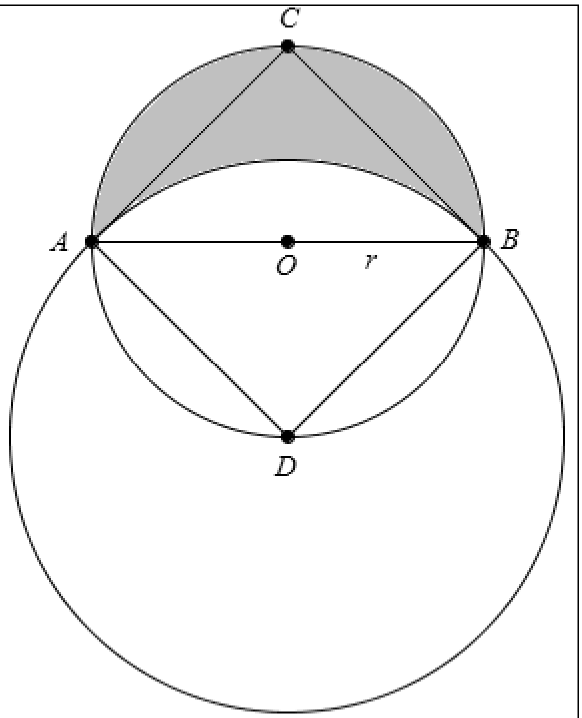
$$= \frac{1}{8}\pi(\overline{AC}^2 + \overline{BC}^2 - \overline{AB}^2) + \frac{\overline{BC}\times\overline{AC}}{2}.$$

The Pythagoras' theorem, applied to the right triangle ΔABC allows to establish the relation $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$. We conclude that the sum of the areas of the lunes is equal to the area of ΔABC inscribed in the semi-circumference ∅ AB.

$$L_1 + L_2 = \frac{\overline{BC}\times\overline{AC}}{2}$$

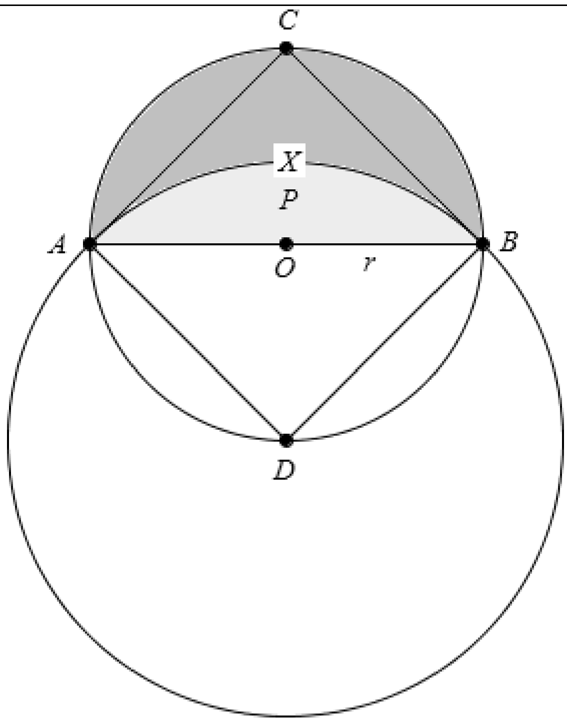
(Subject:
Geometry – 9th grade students)

Consider a circumference whose centre is point O and radius OB circumscribed to the square *ABCD (Figure 114)*. The lune (grey region) is delimited by the arc of the circumference whose centre is O and radius r, and by the arc of a circumference whose centre is D and radius DB. What is the area of the lune?



(Figure 114)

Is it possible to determine the areas of other regions? A circle whose centre is point D and radius DB has an area of $2\pi r^2$. The area of ΔABD is r^2. Consider section *P (Figure 115)*. The area of the lune corresponds to the difference between the area of a semicircle with the centre in point O and radius r, whose value is $\frac{\pi r^2}{2}$, and the area of section P.



(Figure 115)

Hence,

$$Area\ Circular\ Sector$$

$$AB = \frac{1}{4}\left(\pi\left(\sqrt{2}r\right)^2\right) = \frac{\pi r^2}{2};$$

$$Area\ \Delta ABD = r^2;$$

$$Area\ P = area\left(\widehat{AXB}\right) - area(ABD)$$

$$= \frac{\pi \times \left(\sqrt{2}r\right)^2}{4} - r^2$$

$$= \frac{\pi r^2}{2} - r^2 = \frac{r^2(\pi - 2)}{2}$$

$$Area\ Lune = \frac{1}{2}\pi r^2 - \frac{r^2(\pi-2)}{2} = r^2.$$

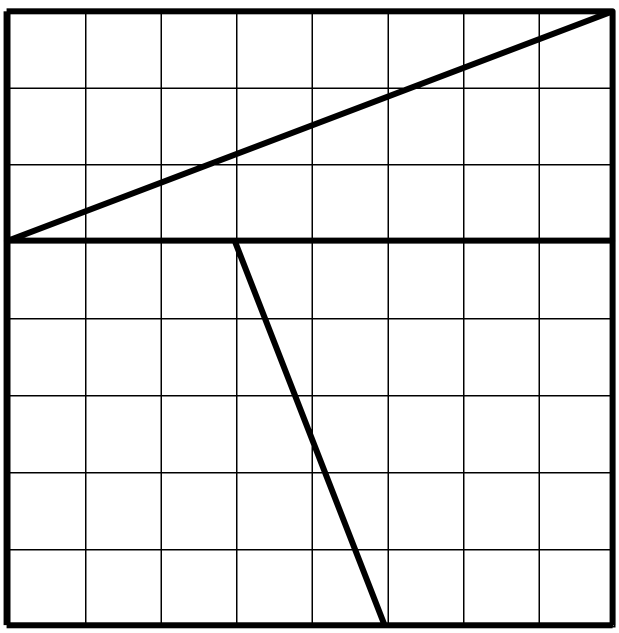
To successfully solve a problem, the pieces of a puzzle must all fit together. Its complexity level may be assessed by the number of pieces which compose the puzzle. However, this is not the only factor of the equation. Each unit's structure has a principal influence in the difficulty level of the problem, for if the number of elements is reduced and the differences between them are subtle, the pieces of the puzzle are more difficult to fit. When the pieces are different and there is no place for mistakes, the quantity is not prohibitive to achieve success.

Repeated practice and exercise solving enhance the efficacy of the answer. The film character performed by Charles Spencer Chaplin (1889 – 1977) in the *Modern Times* shows that the effect of repetition may be counterproductive for the individual. Routine, eventually, helps to consolidate basic procedures but does not develop the use of heuristic mathematical tools. We advocate that problem solving practice cannot be dissociated from the teaching/learning process of Mathematics and that it should be systematically implemented in the classroom.

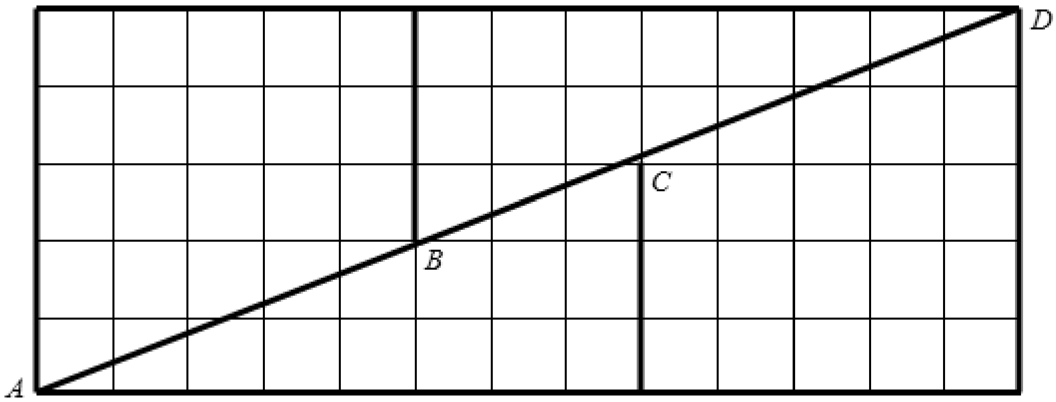
Problem solving / Organising and fitting the pieces together

The exploration of theoretical contents by the teacher and its assimilation by the students are crucial for both the practice of solving exercises and the practice of solving problems, since often the line between them is blurred. Didactic procedures used for the transmission of theoretical knowledge are also of no small importance.

Consider a square whose measures are 8×8, and whose area is 64 *(Figure 116)*. This square is cut into four pieces, two triangles and two trapeziums. These pieces can be reorganised to form a rectangle with approximately $13 \times 5 = 65$ units of area *(Figure 117)*.

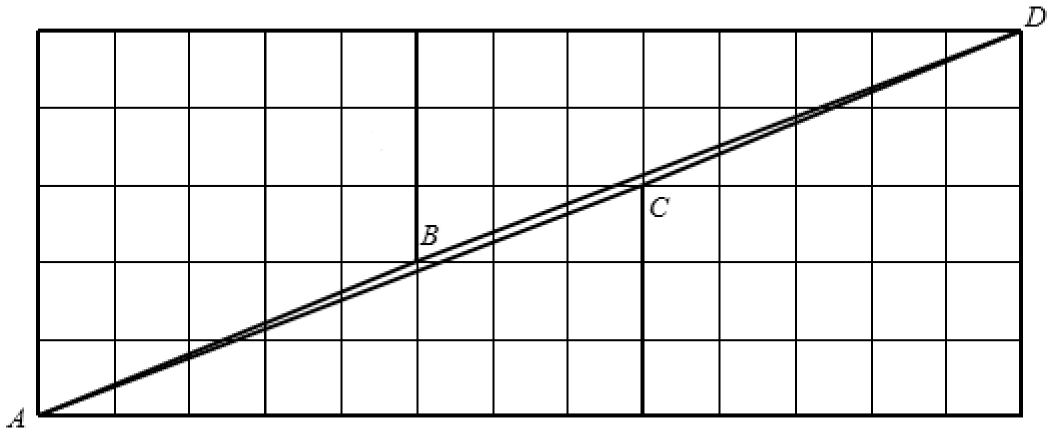


(Figure 116)



(Figure 117)

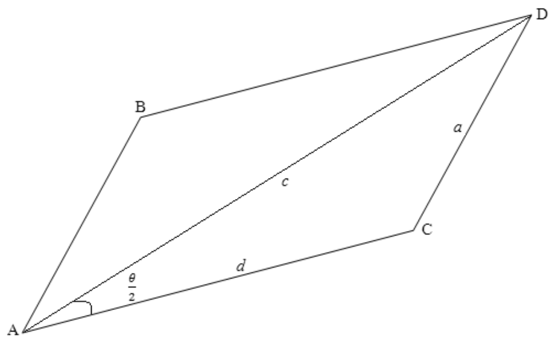
To understand the paradox of the gain of a unit of area, consider the diagonal ABCD of the rectangle. Points A, B, C and D are actually the vertices of a parallelogram *(Figure 118)*.



(Figure 118)

$\overline{AB} = \overline{CD} = \sqrt{29}$ $\qquad$ $\overline{AC} = \overline{BD} = \sqrt{73}$ $\qquad$ $\overline{AD} = \sqrt{194}$

The goal is to prove that the area of the parallelogram is equal to the difference between the area of the rectangle and the area of the square, *i.e.*, 1. We draw a caricature of the parallelogram exaggerating its features, and introducing *a*, *c*, *d* as shown *(Figure 119)*.



(Figure 119)

If the Law of the Cosines is applied to ΔACD, then:

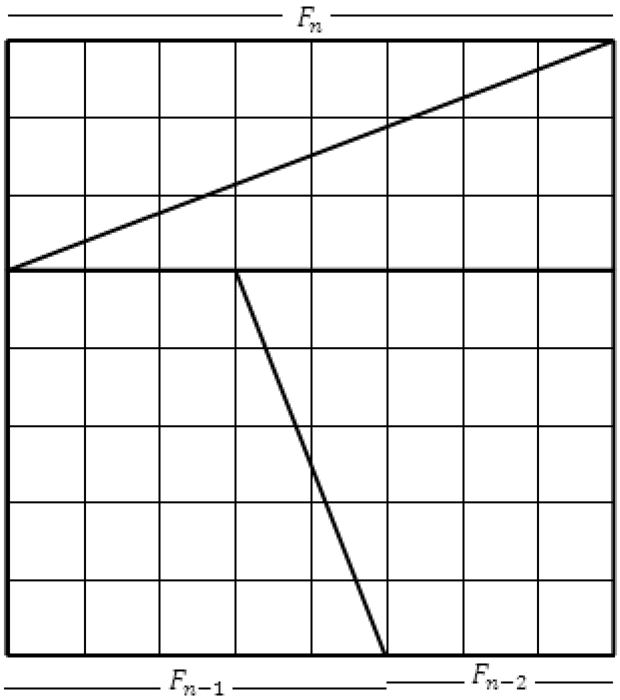
$$a^2 = d^2 + c^2 - 2cd\, cos\left(\frac{\theta}{2}\right),$$

$$cos\left(\frac{\theta}{2}\right) = \frac{d^2 + c^2 - a^2}{2cd} = \frac{(8^2 + 3^2) + (13^2 + 5^2) - (5^2 + 2^2)}{2\sqrt{(8^2 + 3^2)}\sqrt{(13^2 + 5^2)}} = \frac{73 + 194 - 29}{2\sqrt{73}\sqrt{194}} \approx \frac{119}{120}.$$

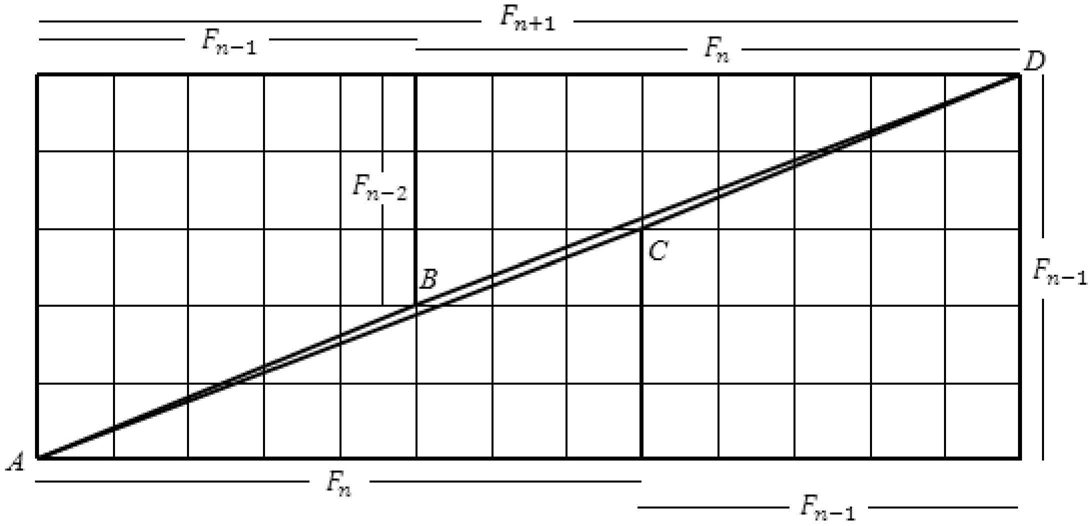
$$\theta \approx 1°.$$

The small size of θ makes the parallelogram invisible to the naked eye.

The solution to the problem is based on a previously mentioned theorem stated by Giovanni Cassini on the Fibonacci sequence (*Figure 120* and *Figure 121*): the square of any term is different from the product of the adjacent terms by 1 or -1. Consider the measures of the pieces under the perspective of the Fibonacci sequence (F_n for n = 6).



(Figure 120)



(Figure 121)

Why is there a parallelogram? The pieces (triangles rectangles) are obviously congruent, just as the two trapeziums below the *horizontal cut* of the square. When rearranging the pieces, $|AC| = |BD|$ and $|AB| = |CD|$ are obtained. Hence, the quadrilateral ACDB has two pairs of opposite sides of equal length. That can only happen to a parallelogram. The *hypotenuse* of *triangles* of cathetus $(F_n + F_{n-1})$ and F_{n-1} alternately overlap or leave a quasi-linear void along the diagonal according to if *n* is an odd numbers (for in this case ${F_n}^2 - F_{n-1}F_{n+1} = -(-1)^n > 0$) or an even numbers (for in this case ${F_n}^2 - F_{n-1}F_{n+1} = -(-1)^n < 0$).

In the first case, ${F_n}^2 > F_{n-1}F_{n+1}$, which means that *the square does not fit inside the rectangle*. Hence, the excess of area is the parallelogram. In the second case, ${F_n}^2 < F_{n-1}F_{n+1}$, which means that *the square does not completely fill the rectangle*, thus creating a void.

The parallelogram is already *invisible* in case $n = 6$. The size of the angle in *A* is minimal. Let us now determine the height of the parallelogram. The area of the parallelogram is equal to the product of the length of its base by the value of height *h*. Hence, 1 = $\sqrt{{F_n}^2 + {F_{n-2}}^2}$ × *h*, *i.e.*, the height of the parallelogram is:

$$\therefore\ h = \frac{1}{\sqrt{{F_n}^2+{F_{n-2}}^2}}.$$

The relation reveals that the greater the measure of the side of the initial square, the smaller the height of the parallelogram, and, consequently, the more difficult to visualise.

The paradox presented in this final section illustrates how difficult it can be to solve mathematical problems. The solver is required to establish connections which are not obvious. This digression into problem solving activities aims to be a proposal for an intervention programme for teachers so as to suggest topics and activities for the classroom. Nonetheless, these pages do not exclude those who as yet are not so fluent in the mathematical language, namely students. Systematic practice is not a magic formula which solves learning problems in Mathematics. The difficulties inherent to the assimilation of mathematical concepts are amplified by the complexity of cognitive mechanisms. These cognitive mechanisms are activated during the reasoning process so as to prevent a cursory linear interpretation which would always be reductive. All those who study Mathematics have already questioned themselves, after reading a statement and understanding the solving process for a question whose solution they have not found yet, why they have not come up with the same idea. The phrase *Why didn't I remember this?* is frequently used for this kind of situations, thus highlighting the importance of long-term memory.

The web page of the Portuguese Society of Mathematics (*Sociedade Portuguesa de Matemática*) (SPM), *www.spm.pt*, offers an archive of exercises from the *International Mathematical Kangaroo* and the *Portuguese Mathematics Olympiads*. As these are competitions with different characteristics, they allow the practice of problem solving by students of different levels. The *International Mathematical Kangaroo* is an annual multiple-choice-question test with an increasing level of difficulty, but still accessible to average students, when compared to the *Mathematics Olympiads*.

The promotion of proficient pedagogic practices comes from Ancient Greece. The teaching and learning of Mathematics by recurring to heuristics for problem solving is nowadays an issue as relevant as it was in the past, which demonstrates its importance as an educational methodology.

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APPENDIX

Appendix 1 – AFPMO classroom results

	Problem		A_1	B_1	C_1		A_2	B_2	C_2		D_1	E_1		D_2	E_2
Student															
A			2	0	0		2	0			0	0			0
B			2	0			0	0				0			0
C			2	0			3	0			0	0		0	0
D			2	0			3	0				3			1
E			2	0	0		3	0				0			
F			2	0	0		2	0			0	0			1
G			1	0	0		2	0							
H			2	3			3	2				0			0
I			1	0	0		3	0			0	2		0	0
J			2		0		3					0			0
K			2	2			3		0		0	0		0	3
L			2	2			3	0			0	3		0	3
M			2		0		3	3			0	3		0	3
N			2				3	0			0	0		0	
O			2	0			3	0			0	0		0	0
P			2	2	0		3	0	0		0	1		0	1
Q			2	2			2	3							
R			2	2	2		3	3	0			0		0	1
S			2	0			3	2			0	3		2	3
T			2	2			2	0			0	0		0	1
U			2	2			3	0	0		0	1		0	0
V			2		0		3	0			0	0		0	
W			1	0	0		3	0			0	1		2	1
X			2	0			0					0		0	0
Y			2	0	0		3	0			0	0		0	0
Z			2				2	0			0	1		0	0

Appendix 2 – AMPMC classroom results

	Problem		A_1	B_1	C_1		A_2	B_2	C_2		D_1	E_1		D_2	E_2
Student															
A			2	0	0		3	0	0		0	2		0	0
B			2	2	0		3	0	0		0	2		0	1
C			0	0	0		3	0	0		0	2		0	2
D			1	0	0		3	0	0		0	2		0	0
E			2	2	0		2	2	0		0	3		0	1
F			0	3	0		2	3	0		0	0		0	0
G			2	2	0		3	3	0		0	2		0	0
H			2	2	0		3	2	0		0	0		0	0
I			0	0	0		3	0	0		0	0		0	0
J			2	0	0		2	0	0		0	0		0	0
K			2	0	0		3	0	0		0	0		0	0
L			2	0	0		1	0	0		0	0		0	0

Appendix 3 – AMMS classroom results

	Problem		A_1	B_1	C_1		A_2	B_2	C_2		D_1	E_1		D_2	E_2
Student															
A			1	3	0		3	0	0		0	2		2	2
B			1	0	0		2	0	0		0	1		0	0
C			1	0	0		1	0	0		0	1		0	0
D			0	0	0		3	0	0		0	1		0	0
E			2	0	0		3	0	0		0	3		0	2
F			1	0	0		2	0	0		0	0		0	0
G			1	0	0		3	0	0		0	1		0	1
H			1	0	0		2	0	0		0	2		0	0
I			1	0	2		2	0	0		0	2		0	0
J			2	2	0		3	0	0		0	2		0	2
K			2	0	0		0	0	0		0	1		0	1
L			2	3	0		2	0	0		0	0		0	0
M			2	0	0		2	0	0		0	2		0	1
N			1	0	0		3	0	0		0	1		0	0
O			1	0	0		3	0	0		0	0		0	2
P			1	2	0		3	0	0		0	0		0	0

Appendix 4 – ACV classroom results

	Problem		A_1	B_1	C_1		A_2	B_2	C_2		D_1	E_1		D_2	E_2
Student															
A			2	2	0		3	1	0		2	0		0	0
B			2	0	0		3	1	0		0	2		0	0
C			2	1	0		3	0	0		0	1		0	0
D			2	3	0		3	1	0		3				
E			2	0	0		3	0	0		2	3		0	3
F			2	0	0		3	3	0		0	0		0	1
G			2	0	0		3	0	0		0	1		0	0
H			1	2	0		3	0	0		2	0		0	1
I			2	0	0		3	0	0		0	0		0	0
J			2	2	0		3	3	0		0	2		0	1
K			1	0	0		3	0	0		2	0		0	0
L			2	1	0		2	3	0		0	2		0	0
M			2	2	0		3	0	0			0		0	0
N			2	0	0		3	1	0		2				
O											2	3		2	1
P											2	2		2	0
Q			2	3	0		3	3	0		2	3		2	3
R			2	3	0		3	3	0		0	3		0	3
S			2	0	0		1	0	0		3	1		0	0
T			2	0	0		3	1	0			0		3	0

Appendix 5 – CILMJR classroom results

	Problem		A_1	B_1	C_1		A_2	B_2	C_2		D_1	E_1		D_2	E_2
Student															
A			2	3	0		3	2	0		0	3		0	2
B			2	2	0		2	3	0		0	3		0	0
C			2	0	0		3	2	0		0	3		2	1
D			0	0	0		3	0	0		0	1		0	1
E			2	1	0		3	3	0		0	0		2	1
F			2	3	0		3	3	0		0	3		0	0
G			0	0	0		3	0	0		0	0		0	1
H			2	0	0		0	3	0		0	0		0	0
I			2	2	0		3	0	0		0	0		0	0
J			2	0	0		0	0	0		0	2		0	2
K			2	0	0		3	3	0		0	0		0	1
L			2	0	0		3	3	0		0	3		2	3
M			2	3	0		3	0	0		0	3		0	3
N			0	0	2		3	0	2		0	0		0	0
O			2	0	0		3	0	0		0	2		0	0
P			2	3	0		0	3	0		0	1		0	0
Q			2	0	0		2	0	0		0	0		0	3
R			1	2	2		3	3	2		0	3		3	3
S												3		3	3
T												2		0	1
U			2	2	0		3	3	3		0	1		0	0
V												0		0	0
W			2	3	0		3	3	0		0	3		0	3
X			2	2	0		3	3	0		0	1		0	0
Y			0	2	0		3	2	0		0	1		0	1
Z			2	0	0		3	0	0		0	1		0	1

Appendix 6 – LMCR classroom results

	Problem		A_1	B_1	C_1		A_2	B_2	C_2		D_1	E_1		D_2	E_2
Student															
A			0	2	0		2	0	0		0	0		0	0
B			2	2	0		3	0	0		0	2		0	2
C			2	0	0		3	0	0		0	0		0	1
D			0	0	0		2	0	0		0	0		0	0
E			2	0	0		3	0	0		0	0		2	2
F			2	2	0		3	2	0		0	0		0	0
G			2	1	0		3	0	0		0	1		0	1
H			2	2	0		3	0	0		0	0		0	0
I			2	3	0		3	0	0		0	0		0	0
J			2	0	0		2	0	0		0	0		0	0
K			2	0	0		0	0	0		0	1		0	1
L			2	0	0		2	0	0		0	0		2	2

Appendix 7 – MAGSA classroom results

	Problem		A_1	B_1	C_1		A_2	B_2	C_2		D_1	E_1		D_2	E_2
Student															
A			0	0	0		3	0	0		0	0		0	0
B			2	0	0		3	0	0		0	2		0	0
C			2	0	0		3	0	0		0	2		0	0
D			0	1	0		3	0	0		0	3		0	1
E			0	1	0		3	0	0		0	0		0	1
F			0	0	0		3	0	0		0	2		0	0
G			0	0	0		3	0	0		2	0		0	0
H			0	0	0		3	0	0		0	1		0	1
I			0	0	0		3	0	0		0	0		0	0
J			0	0	0		3	0	0		0	2		0	0
K			0	0	0		3	0	0		0	0		0	0
L			0	0	0		3	0	0		0	0		0	0
M			2	0	0		1	0	0		0	2		0	0
N			2	0	0		3	0	0		0	2		0	0
O			2	0	0		3	0	0		0	3		0	3
P			0	0	0		1	0	0		0	0		0	0
Q			0	0	0		3	0	0		0	0		0	0
R											0	2		0	0
S			2	3	0		3	0	0		0	0		0	0
T			0	0	0		3	0	0		0	0		0	0
U											0	0		0	0
V			0	1	0		1	0	0		0	0		0	0
W			0	0	0		3	0	0		0	0		0	0
X			0	0	0		3	0	0		0	2		0	2
Y			0	0	0		1	0	0		0	0		0	2
Z			0	0	0		1	0	0		0	0		0	0
AA			0	0	0		1	0	0		0	0		0	0
BB			0	0	0		1	0	0		0	2		0	0

Appendix 8 – MGCCMFP classroom results

	Problem		A_1	B_1	C_1		A_2	B_2	C_2		D_1	E_1		D_2	E_2
Student															
A			2	3	0		1	0	0		0	2		0	1
B			2	3	2		2	2	0		0	2		0	1
C			2	0	0		0	0	0		0	0		0	0
D			2	3	0		2	2	0		0	1		0	1
E			2	0	0		1	0	0		0	0		0	1
F			0	0	2		2	0	0		0	0		0	0
G			2	3	0		2	3	0		0	0		0	1
H			2	0	0		0	0	0		0	0		0	1
I			0	3	0		2	1	0		0	1		0	1
J			0	3	0		2	1	0		0	0		0	1
K			2	3	0		1	3	0		0	1		0	0
L			2	2	0		2	3	0		0	1		0	0
M			2	0	0		2	0	0		0	0		0	1
N			0	3	0		0	3	0		0	1		0	0
O			2	0	0		0	1	0		0	1		0	1
P			2	3	0		2	3	0		0	1		0	0
Q			2	3	0		2	3	0		0	1		0	1
R			2	0	0		2	3	2		0	1		0	1
S			2	3	0		2	2	0		0	0		0	1
T			0	3	0		2	3	0		0	1		0	1
U			0	0	0		2	0	0		0	0		0	1
V			2	0	0		2	2	0		0	0		0	0
W			2	0	0		2	0	0		0	1		0	0
X			0	0	0		2	1	0		0	1		0	1

Appendix 9 – MFRN classroom results

	Problem		A_1	B_1	C_1		A_2	B_2	C_2		D_1	E_1		D_2	E_2
Student															
A			0	0			0	0							
B			2	0	0		3	0			0	0		0	
C			2		0		3	0			0	2			0
D			0	2	0		3	2	0		0	1		0	1
E			2	0	0		0	0	0		0	1		0	0
F			2	0	0		3	2	0		0	3		0	3
G			1		0				0		0	1			0
H			0	2	0		0	2	0		0	0		0	0
I			0	0	0		2	0	0		0	0		0	0
J			0	0											
K			1	0	0		0	0	0		0	1		0	0
L			0	2			0	0	0		0	0		0	0
M			1	0	0		0	0	0		0	1		0	2
N			2	1			0	0	0		0	3		0	3
O			2	0			0	0	0		0	2		0	0
P			2	0	0		3	0	0		0	1		0	0
Q			0	0	0		0	0	0		0	3		0	1
R			0	0			0					0			

Appendix 10 – MJNRL classroom results

	Problem		A_1	B_1	C_1		A_2	B_2	C_2		D_1	E_1		D_2	E_2
Student															
A			2	3	0		3	3	0		2	2		2	0
B			2	0	0		2	0	0		2	2		0	2
C			2	2	0		3	2	0		×	×		×	×
D			2	3	0		3	2	0		×	×		×	×
E			2	2	2		3	3	0		2	3		3	3
F			2	2	2		3	0	2		×	×		×	×
G			2	0	0		3	0	0		×	×		×	×
H			1	0	0		3	3	0		0	0		0	0
I			2	2	0		3	2	0		2	2		2	2
J			1	0	0		3	3	0		2	0		0	3
K			2	3	0		3	2	0		×	×		×	×
L			2	2	0		3	3	0		2	0		2	3
M			2	3	0		3	3	2		2	3		2	3
N			2	0	0		0	0	0		2	2		0	2
O			2	0	0		2	2	0		2	2		2	0
P			2	3	1		3	2	0		2	3		0	1
Q			2	2	0		3	0	0		2	0		2	0
R			2	0	0		3	0	0		×	×		×	×
S			2	2	0		3	2	0		2	0		0	3
T			1	0	0		3	2	0		0	3		0	1
U			2	2	0		3	0	2		×	×		×	×
V			2	3	0		3	2	0		×	×		×	×
W			2	3	0		2	3	0		0	0		0	1
X			2	0	0		3	0	0		×	×		×	×
Y			2	2	0		3	2	0		2	3		0	3
Z			2	2	0		3	2	0		0	0		0	0
AA			2	2	0		3	3	0		×	×		×	×
BB			2	2	0		2	3	0		0	3		0	2

Appendix 11 – MPJJS classroom results

	Problem		A_1	B_1	C_1		A_2	B_2	C_2		D_1	E_1		D_2	E_2
Student															
A			2	0	0		3	0	0		0	0		0	0
B			2	0	0		3	0	0		0	3		2	3
C			0	0	0		0	0	0		0	0		0	0
D			2	2	0		3	0	0		0	0		0	0
E			2	2	0		3	3	0		0	0		0	0
F			0	0	0		3	0	0		0	3		0	1
G			2	0	0		0	0	0		0	1		0	1
H			0	0	0		3	2	0		0	0		0	0
I			2	2	0		3	0	0		0	0		0	0
J			2	0	0		3	0	0		0	3		2	3
K			2	2	0		3	3	0		0	3		0	1
L			0	0	0		3	0	0		0	0		0	0
M			2	2	0		0	2	0		0	3		0	3
N			2	2	0		3	0	0		0	3		0	0
O			0	2	0		2	3	0		0	1		2	1
P			2	2	0		2	0	0		0	3		0	0
Q			2	0	0		3	0	0		0	0		0	0
R			2	0	0		3	0	0		0	0		0	0
S			1	2	0		3	0	0		0	1		0	2

Appendix 12 – PCOA classroom results

	Problem		A_1	B_1	C_1		A_2	B_2	C_2		D_1	E_1		D_2	E_2
Student															
A			2	0	2		3	0	0		0	2		0	0
B			0	0	2		3	0	0		0	2		0	0
C			1	2	0		3	2	0		0	3		0	3
D			2	3	2		3	3	0		0	3		0	3
E			1	2	2		3	0	0		0	3		0	3
F			2	0	0		3	3	0		0	3		2	3
G			0	2	0		0	2	0		0	3		0	3
H			0	0	0		0	0	0		0	2		0	3
I			2	0	0		0	1	0		0	0		0	0
J			0	0	0		0	0	0		0	0		0	0
K			0	0	0		0	0	0		0	3		0	3
L			0	0	2		0	0	0		0	2		0	3
M			0	0	0		3	2	0		0	0		0	0
N			2	3	2		3	3	0		0	3		0	3
O			0	0	0		3	0	0		0	0		0	0

Appendix 13 – PCMLEC classroom results

	Problem		A_1	B_1	C_1		A_2	B_2	C_2		D_1	E_1		D_2	E_2
Student															
A			1	0	0		3	0	0		AM	AM		AM	AM
B			2	2	0		3	2	0		0	2		0	0
C			2	2	0		3	2	0		0	2		0	0
D			2	0	2		3	0	2		0	2		0	1
E			2	0	2		3	0	0		0	2		0	0
F			2	0	2		2	0	2		0	0		0	2
G			2	2	0		3	3	2		3	2		3	2
H			2	0	0		3	2	0		3	2		0	2
I			2	0	0		3	0	0		0	0		0	2
J			2	0	0		3	2	0		0	2		0	2
K			2	0	0		3	0	0		0	2		0	3
L			2	0	0		2	0	0		0	2		0	3
M			2	2	0		2	2	0		3	0		0	3
N			2	0	0		2	0	0		0	0		0	2
O			2	3	2		3	3	2		0	0		3	2
P			2	0	0		3	0	0		F	F		F	F
Q			2	3	0		3	3	2		0	2		2	3
R			2	0	0		3	0	2		0	0		0	0
S			2	2	0		3	3	0		F	F		F	F
T			2	2	0		3	3	2		0	2		0	3
U			2	2	0		2	2	3		3	3		2	3
V			2	2	0		3	2	0		F	F		F	F
W			2	0	0		3	0	0		0	3		2	3
X			2	0	0		3	0	0		3	0		0	0
Y			2	0	0		2	0	0		3	3		2	2

AM: Stopped attending lessons
F: Absent

Appendix 14(A) – RMVF classroom results (A)

	Problem		A_1	B_1	C_1		A_2	B_2	C_2		D_1	E_1		D_2	E_2
Student															
A			2	3			3	0				1		0	2
B			2	2	0		3	0			1	1		0	3
C															
D			2	3	0		3	3	0		1	0		0	0
E			2	2	0		3	0	0		0	1		0	3
F			2	0			3	2	0		1			1	0
G			2	0	0		3	1			0	0		0	1
H			2	2			0	0			0	0		0	1
I			1	0	0		2	0			1	2		0	0
J			2	0			3	0			1	2		0	0
K			2	2	0		3	0	0		1	0		2	3
L			2	0			3	0			1	1		0	0
M			2	1			3	0			1	0		2	3
N			2	2	0		2	3			1			2	1
O			2	0	0		3	0	0		0	0		0	0
P			2	0	0		3	2						0	0
Q			2	3	0		3	3	0		1	0		2	0
R			2	2	0		3	2			1			1	2

Appendix 14(B) – RMVF classroom results (B)

	Question		A_1	B_1	C_1		A_2	B_2	C_2		D_1	E_1		D_2	E_2
Student															
A			2	0	0		1	1			0	1		0	0
B			2	1			3							0	0
C			0	0	0		3	0	0						
D			2	2			3	0	0		1	2		0	2
E			2	1	2		3				1	2		0	0
F			2	0			2	3	3		1	0		0	2
G			2	3			3	3			0	2		0	
H			2	1			2	3	0		1	0		0	0
I			2	1	0		3	3			1	1		0	3
J			2	1			3	3			1	2		0	0
K			2	0	0		3				0	0		0	0
L			2	1			3	3			0	2		0	0
M			2	3			3	3	0		1	2		0	2
N			2	0			3					2			0
O			2	1	1		3				1				0
P			2	0	0		3	3	0		1	0		0	0
Q			2	0			3				1	0		0	0
R			0	0			3	0			0	0		0	0
S			2	0	1		3				1	1		0	
T			2				3	1				1		0	0

Appendix 15 – RMSSV classroom results

	Problem		A_1	B_1	C_1		A_2	B_2	C_2		D_1	E_1		D_2	E_2
Student															
A			2	0	0		3	3	0		0	0		0	1
B			2	0	0		3	0	0		0	0		0	3
C			2	3	0		3	3	0		0	3		0	3
D			1	0	0		3	0	0		0	0		0	1
E			2	0	0		3	0	0		0	2		0	0
F			2	0	0		3	0	0		0	0		0	0
G			0	0	0		3	0	0		0	1		0	0
H			2	0	0		0	0	0		0	2		0	2
I			2	2	0		3	0	2		0	2		0	3
J			0	1	0		2	0	0		0	2		2	0
K			2	2	0		3	3	0		3	3		2	3
L			2	0	0		3	0	0		0	2		0	0
M			0	0	0		3	0	0		0	0		0	0
N			2	0	0		3	0	0		0	3		0	0
O			0	0	0		0	0	0		0	3		0	0
P			1	0	0		3	0	1		0	3		3	1
Q											3	3		0	3

Appendix 16 – TCRO classroom results

Classification grid for the problems given to 7th grade students

	Question		A_1	B_1	C_1		A_2	B_2	C_2		D_1	E_1		D_2	E_2
Student															
A			2	0	0		1	0	0		1	0		0	0
B			2	0	0		2	0	0		1	0		0	0
C			2	0	0		2	0	0		1	0		0	0
D			2	0	0		2	0	0		1	1		0	0
E			2	1	0		2	0	0		1	0		0	1
F			2	2	0		1	1	0		1	0		0	0
G			2	2	0		2	0	0		1	1		0	0
H			2	0	0		2	0	0		1	0		0	0
I			2	0	0		2	0	0		1	0		0	0
J			2	0	0		2	0	0		1	0		0	0
K			2	2	0		1	0	0		1	1		0	0
L			2	0	0		2	0	0		0	0		0	1
M			2	0	0		1	0	0		1	0		0	0
N			2	2	0		2	1	0		1	2		0	1
O			2	2	0		0	0	0		1	1		0	0

ANNEX

Annex 1

CPJustiça Centro Protocolar de Formação Profissional para o Sector da Justiça	**Centro Educativo dos Olivais** ***Citizenship and employability*** **Monitoring the performance of students (evaluation)**

Unit: ____
Student: ____

Discipline			**Performance**			
Attitude	**Relation with pairs**	**Relation with teachers and other educational agents**	**Interest**	**Adaptability**	**Creativity**	**Autonomy**

Unit: ____
Student: ____

Discipline			**Performance**			
Attitude	**Relation with pairs**	**Relation with teachers and other educational agents**	**Interest**	**Adaptability**	**Creativity**	**Autonomy**

Validations

Unit	**Day**	**Unit**	**Day**	**Unit**	**Day**	**Unit**	**Day**

NS - Not Satisfactory: (Weak, 1 and 2): the student does not aim to pursue a correct behaviour nor follow the teacher's orientations, thus compromising the results expected from the work developed during the teaching and learning session.

S - Satisfactory: (Regular, 3): the student reveals a behaviour which can be defined as passive and / or instable, with a lack of awareness of the results which are expected of him/her during the teaching and learning session.

A – Active (Good, 4): the student shows signs of commitment and reveals attempts of metacognition during the lesson, so as to benefit from it and develop understanding about the contents during the teaching and learning session.

PA – Proactive (Very Good, 4 + and 5): the student shows a consistent and solid aptitude about the subject, and his/her performance is decisive for the development of work during the teaching and learning session.

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Annex 2

GUIDE FOR PROBLEM SOLVING

"If I do not fear the error that is because I am always ready to correct it."
Bento de Jesus Caraça

Name ____

Class ______ Date ___/___/_____ Name of the Problem ______________________________

1. READING AND ANALYSING THE PROBLEM

1.1 Did I carefully read the whole statement of the problem?
Yes □ **No** □ I will read the statement again.

1.2 Did I understand the problem?
Yes □
No □
- I do not know the meaning of one or more words. □
  I seek help (dictionary/teacher).
- The statement is long and/or has much information. □
  I underline what I consider to be the most important.

1.3 What is the unknown of the problem?

____

1.4 What are the data provided by the statement which may help me solve the problem?

____

1.5 Are there rules, limitations or special cases that should be taken into consideration? If there are, which?

____

2. DEVISING A PLAN

2.1 Do I have a general plan to solve the problem? **Yes** □ **No** □

2.2 Have I ever solved a similar problem?

Yes □ What strategies did I use to solve this problem?

____

No □ Where should I start? Choose at least one of the following possibilities:

□ I can make a sketch, diagram, table or take notes to organise the data.

□ I can divide the problem in several parts.

□ I can relate the data with the unknown of the problem.

□ I can predict the answer by using mental calculation.

□ I solve the problem from the end to the beginning.

3. IMPLEMENTING THE PLAN

Clear presentation of the operations that I think are the most correct to find the solution.

4. SOLUTION VERIFICATION

I found a solution! Is it the answer to the problem? Let me check, taking into consideration that there are problems which may have several solutions, problems which have only one solution and problems with no solutions at all!

4.1 What did I do to check the validity of my solution?

____

____

4.2 Does my solution satisfy the statement of the problem?

Yes □ The solution is:

____

____

No □ My solution does not make sense in the context of the problem.

a) I found an error and corrected it.

b) I followed a different plan to solve the problem.

End.

Annex 3

Sample Statements of the SELF-EFFICACY (SE) scale

1.	I am one of the best students in Mathematics.	1	2	3	4	5
2.	I believe that I have a lot of weaknesses in Mathematics.	1	2	3	4	5
3.	Compared to other students, I am a weak student in Mathematics.	1	2	3	4	5
4.	Mathematics is not one of my strengths.	1	2	3	4	5
5.	I can usually help my classmates when they ask me for help with problem solving.	1	2	3	4	5
6.	I can usually solve any mathematical problem.	1	2	3	4	5
7.	I do not feel sure about my self in problem solving.	1	2	3	4	5
8.	When I start solving a mathematical problem, I usually feel that I won't to be able to find a solution.	1	2	3	4	5
9.	I can easily solve two-step problems.	1	2	3	4	5
10.	I have difficulties in solving one-step problems	1	2	3	4	5

Sample statements of the ATTITUDES TOWARDS MATHEMATICS (ATM) scale

1.	I am interested in Mathematics!	1	2	3	4	5
2.	Mathematics is boring!	1	2	3	4	5
3.	I wouldn't study Mathematics if it were optional.	1	2	3	4	5
4.	Mathematics thrills me! It's my favorite subject!	1	2	3	4	5
5.	I get anxious when doing Mathematics.	1	2	3	4	5
6.	I do not like school Mathematics.	1	2	3	4	5
7.	I detest Mathematics and avoid it at all times!	1	2	3	4	5
8.	Mathematics is useful for everybody's life.	1	2	3	4	5
9.	I enjoy the struggle to solve a mathematical problem.	1	2	3	4	5
10.	I like problem solving.	1	2	3	4	5

1=strongly disagree, 2 disagree, 3=neither agree nor disagree, 4=agree, 5=strongly agree

Maria Nicolaidou and George Philippou

Annex 4

HEURISTICS IN PROBLEM SOLVING
FOR THE TEACHING AND LEARNING OF MATHEMATICS
Educational Sciences Doctorate Project

Nuno Álvaro Ferreira Rodrigues

PROBLEM SOLVING BELIEFS QUESTIONNAIRE

Write the last 4 digits (Identity Card or Citizen Card) _ _ _ _

Birthday: ___ / ___ / _____ Gender: ______ Grade: ________

Next you will read a set of sentences with multiple choice answers. No option is more correct than another(s) and there are no wrong answers. The important is that the answers are a consequence of your thought(s).

The **questionnaire is confidential,** with the purpose to study standard students' attitudes when facing different problem solving situations.

Read each sentence and choose the answer which is **closest to your usual attitude**.

There are four answers available for each question. You must choose your most usual attitude: *"Very Much"*, *"More or Less"*, *"Sometimes"* and *"Not Really"*.

For each sentence draw a **single cross** (X) in the column that most suits you.
Please answer all the questions.

	VERY MUCH	**MORE OR LESS**	**SOMETIMES**	**NOT REALLY**
1. I like doing problem solving.	☐	☐	☐	☐
2. The time spent solving problems doesn't matter when I'm trying to find the solution.	☐	☐	☐	☐
3. When I reach a solution I always check my reasoning and calculations.	☐	☐	☐	☐
4. My justifications are well organised so as to be easily understood.	☐	☐	☐	☐
5. I like when my teacher sees me doing problem solving well.	☐	☐	☐	☐
6. I enjoy solving problems on the board and correcting resolutions.	☐	☐	☐	☐
7. Learning to solve problems can be useful for my daily life / professional future.	☐	☐	☐	☐

	VERY MUCH	MORE OR LESS	SOMETIMES	NOT REALLY
8. I believe problem solving is a good mental exercise because I learn to think.	☐	☐	☐	☐
9. When I'm engaged in doing problem solving I like to decide what to do.	☐	☐	☐	☐
10. I feel embarrassed when I don't know how to solve a problem.	☐	☐	☐	☐
11. Problem solving makes me anxious.	☐	☐	☐	☐
12. I read the problem statement carefully.	☐	☐	☐	☐
13. I should be more swift and effective in problem solving.	☐	☐	☐	☐
14. I need help to decide what to do.	☐	☐	☐	☐
15. Mistakes make me bad humoured.	☐	☐	☐	☐
16. When I have difficulties I know that I can do something to improve.	☐	☐	☐	☐
17. Before doing any calculations I analyse the problem.	☐	☐	☐	☐
18. I believe I do problem solving well.	☐	☐	☐	☐
19. This is a funny activity.	☐	☐	☐	☐
20. Problem solving is exhausting.	☐	☐	☐	☐
21. If I realise that the question is complex, I give up.	☐	☐	☐	☐
22. Even when I check the solution I still don't know if it is right or wrong.	☐	☐	☐	☐

Thanks for your collaboration!

Annex 5

Personal Mathematics Teaching Efficacy (PMTE)

I will find better ways to teach mathematics
I won't be able to effectively monitor mathematical activities[a]
I will generally teach mathematics ineffectively[a]
I will be able to answer students' mathematics questions
I won't willingly be observed by a supervisor when teaching mathematics[a]
I won't teach mathematics as well as most subjects, even if I try very hard[a]
I know how to teach mathematical concepts effectively
I understand mathematical concepts well enough for effective teaching
I will find it difficult to use manipulative math materials to explain why mathematics works[a]
I wonder if I will ever have the necessary skills to teach mathematics[a]
I will be at a loss as to how to help the students which reveal difficulty to understand concepts[a]
I will welcome students' questions
I don't know how to engage children in mathematics[a]

Notes: Items are abbreviated for presentation. [a]Items were reversed scored in order to produce consistent values between positively and negatively worded items.

Mathematics Teaching Outcome Expectancy (MTOE)

Improved marks in Mathematics are due to effective teaching approaches
When a low-achieving child progresses in mathematics, it is usually due to extra attention paid by the teacher
The teacher is generally responsible for the achievement of the students
If parents note an increased interest in mathematics, it is due to the teacher's performance
When a student does better than usual in mathematics, it is due to the teacher's extra effort
Underachievement is due to ineffective mathematics teaching
The inadequacy of a student's mathematics background can be overcome by good teaching
Student achievement in Mathematics is directly related to the teacher's effectiveness in teaching

Note: Items are abbreviated for presentation.

Annex 6

QUESTIONNAIRE FOR THE TEACHERS REGARDING THEIR PROFESSIONAL ACTIVITY

In the scope of a study on problem solving in the teaching and learning of Mathematics, we kindly ask you to answer this questionnaire. It is composed of a series of statements which admit different answers. No answer is more correct than another and there are no wrong answers. It is important that the answers reflect what each teacher really thinks or usually does in the context of their professional activity.

The answers are anonymous and confidential. The aim of this questionnaire is to study teachers' opinions and practices regarding teaching, as well as the promotion of the learning of Mathematics, especially in what concerns problem solving.

After reading each statement, mark with an X the column which best shows your attitude, usual situation or personal opinion. From question 1 to question 40 use the following scale to rate your answers: Strongly Agree (**SA**), Agree (**A**), Indifferent / Not Sure (**I**), Disagree (**D**), Strongly Disagree (**SD**); from question 41 to question 65 use the following scale to rate your answers: Almost Never (**AN**), Sometimes (**S**), Frequently (**F**), Almost Always (**AA**).

Statements		SA	A	I	D	SD
1. When a student achieves a better performance than usual in Mathematics, it is because the teacher made an extra effort.						
2. I am always looking for better ways to teach Mathematics.						
3. Even if I try really hard, I will not be as efficient as the best teachers I know.						
4. When students' marks improve in Mathematics, it is often due to a more effective teaching approach.						
5. I know how to efficiently teach Mathematical concepts.						
6. I am not very efficient in what concerns the monitoring of learning activities.						
7. Bad results in Mathematics are often due to inadequate teaching.						
8. In general, the teaching of Mathematics is not adequate.						
9. The lack of prerequisites in the learning of Mathematics can be surpassed with adequate teaching.						
10. When a student with a poor performance suddenly has better results in Mathematics, it is usually due to extra-attention of the teacher.						
11. I have a sufficient command of the concepts to be an efficient Mathematics teacher.						
12. Usually, the teacher is responsible for the performance of the students.						
13. The performance of the students in Mathematics is directly related to the efficacy of the teaching.						
14. If parents comment that their child is revealing more interest in the Mathematics taught at school, it is probably due to the performance of the teacher.						
15. I experience difficulties in applying manageable materials to explain the applicability of mathematics to my students.						
16. I can usually answer the questions posed by my students.						
17. I question myself whether I have the necessary skills to teach Mathematics.						
18. If I had the chance, I would invite different pedagogical entities of the education system to assess my way of teaching Mathematics.						
19. When a student reveals difficulties to understand a mathematical concept, I usually help them understand that concept.						
20. I promote a question-friendly environment in my classroom.						
21. I do not know how to lure students into studying Mathematics.						
22. I am very stressed out when I am going to teach Mathematics to a class of problematic / low-performance students.						
23. I like Mathematics but I dislike teaching it.						
24. Mathematics is an interesting subject that I like to teach.						
25. Mathematics is a fun and fascinating subject.						
26. When I teach Mathematics, I feel secure. I find it stimulating.						
27. Some concepts are difficult to teach if I do not check the manual.						
28. I feel insecure when I have to solve an unknown problem.						
29. Teaching Mathematics to students who do not learn easily makes me restless, unhappy, angry and impatient.						
30. I feel good when I am teaching Mathematics.						
31. If I do not comply with my lesson plans, it is as if I am lost in a labyrinth.						
32. Mathematics is something I fully appreciate.						
33. I feel anguished when a student says *"I don't understand"*.						

34. I face Mathematics with a feeling of indecision which results from the fear of not being able to teach efficiently.						
35. I enjoy Mathematics, independently of my students' learning.						
36. I feel pleasure when I teach Mathematics.						
37. Thinking about having to correct Mathematics exercises make me nervous.						
38. I remember that I did not like Mathematics in Basic Education.						
39. I prefer teaching Mathematics to doing any other school task.						
40. I feel at ease when I am teaching Mathematics because I like studying Mathematics.						

Statements		**AN**	**S**	**F**	**AA**
41. I enjoy Mathematics because it has many practical applications.					
42. I am a mathematician. Nevertheless, I know that I am not a good Mathematics teacher.					
43. It is impossible to learn Mathematics if students do not like it.					
44. I make sure that my students develop individual work in the classroom.					
45. I explain in detail what my students have to do to solve the problems.					
46. At the end of a problem solving class, I establish a debate in the classroom so that students share their solutions and strategies.					
47. My students can use calculators.					
48. I encourage my students to cooperatively work in small groups.					
49. I present open and imaginative problems in the classroom and provide a minimum of indications as to how to solve them.					
50. I encourage my students to take notes of their own problem solving procedures and methods.					
51. I encourage my students to present their mathematical problems.					
52. I regularly provide a set of problems so that my students can choose one they would like to solve.					
53. I let my students solve the same problem in one or more lessons.					
54. I use problems to show the students that there is knowledge, skills and procedures that they need to master.					
55. I give my students problems which may be applied in other contexts.					
56. I hand out consistent materials for students who require them.					
57. I provide the class with a problem solving model.					
58. I discuss useful strategies for problem solving (organising lists, drawing diagrams, …)					
59. I discuss problem solving procedures (*i.e.*, devising a plan, following the plan, checking calculations, …)					
60. I use problems from the school context or which are related to the students' experiences.					
61. I provide open problems so that students may autonomously explore mathematical situations.					
62. I provide exercises which enable the students to practise their skills.					
63. I introduce new problems but also use problems which the students already know.					
64. I promote problem solving in the classroom to *"relax"*.					
65. I believe that the act of learning the contents is more important than the way they are worked by the students.					

Annex 7

ASSESSMENT OF PROBLEM SOLVING STRATEGIES

1. Consider the multiplications:

$$12\,345\,679 \times 18 = 222\,222\,222$$
$$12\,345\,679 \times 27 = 333\,333\,333$$
$$12\,345\,679 \times 54 = 666\,666\,666$$

What number should be multiplied by 12 345 679 in order to obtain 999 999 999?

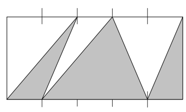
2. If a digital clock marks from 0:00 to 23:59, how many times a day does it show all equal digits?

3. A promotional campaign allows the change of 4 empty 1-litre bottles for a 1-litre bottle full of milk. How many litres of milk can a person who has 43 empty bottles obtain?

4. In an empty box there were several balls of different colours: 5 blue balls, 3 yellow, 2 white and 1 black. Renato took 3 balls from the box. Knowing that the balls taken were neither blue, nor yellow, nor black, can we say that:

A) The three balls have the same colour? B) The 3 balls are red?
C) 1 ball is red and 2 are white? D) 1 ball is white and 2 are red?
E) At least 1 of the balls is red?

5. Considering that the area of the rectangle is 12, what is the area of the shaded part?



6. The number 10 can be written as the sum of 2 prime numbers in two different ways:
10 = 5 + 5 and 10 = 7 + 3. How many ways are there to express the number 25 as the sum of two prime numbers?

7. A given 2-digit number N is the square of a natural number. If the order of the digits is inverted, an odd number is obtained. The difference between the 2 numbers is the cube of a natural number. Can we say that the sum of the digits of N is:

A) 7? B) 10? C) 13? D) 9? E) 11?

8. If natural numbers are put in columns, as can be seen in the table, under which letter will the number 2,000 be?

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
1		**2**		**3**		**4**		**5**
	9		**8**		**7**		**6**	
10		**11**		**12**		**13**		**14**
	18		**17**		**16**		**15**	
19		**20**		**21**		**...**		**...**

9. The emir Abdel Azir was famous for several reasons. He had more than 39 children, including several twins. In fact, the historian Ahmed Aab says in one of his writings that all the children of the emir were twins, except for 39; all were triplets, except for 39; all were quadruplets, except for 39. How many children did the emir Abdel Azir have?

10. Four friends go visit a museum and one of them decides to go in without paying for the ticket.
A controller shows up and wants to know which of them entered without paying.

– It wasn't me, says Benjamin. – It was Carlos, says Mario.
– It was Pedro, says Carlos. – Mario is wrong, says Pedro.

Only one of them is lying. Who did not pay for their ticket to the museum?

A) Mario. B) Pedro. C) Benjamin. D) Carlos.

E) It is not possible to find a solution for the problem, for the statement lacks information.

11. How many whole positive numbers under 1,000,000 are there whose cubes end in 1?

12. The 61 candidates approved for a tender all had different marks. They were divided into 2 classes, according to the marks obtained: the first 31 were put in class A and the remaining 30 in class B. The means of the 2 classes were calculated. However, later it was decided that the last candidate to be put in class A should move to class B. Hence:

A) The mean of class A increased, but the mean of class B decreased.
B) The mean of class A decreased, but the mean of class B increased.
C) The means of both classes increased.
D) The means of both classes decreased.
E) The means of both classes can either increase or decrease, depending on the marks of the candidates.

End.

Annex 8

The assessment and performance criteria were previously established by a group of 3rd Cycle teachers involved in the workshop. The teachers gave their students a battery of questions with an increasing level of difficulty. The students were allowed 15 minutes to answer each question. The questions were thought to be applied in a 45-minute lesson.

Hence, for question A_1:

Assessment	Performance Criteria
2	Provides an adequate strategy and gives a correct answer.
1	Provides an adequate strategy but does not complete it.
0	Gives another answer or does not answer.

The following assessment criteria were established for the remaining questions:

Assessment	Performance Criteria
3	Provides an adequate strategy and gives a correct answer.
2	Provides an adequate strategy but does not complete it.
1	Does not provide a strategy but gives a correct answer.
0	Gives another answer or does not answer.

QUESTIONS FOR THE ASSESSMENT OF
EXERCISE / PROBLEM SOLVING STRATEGIES

MATHEMATICS / PROBLEM SOLVING – Time allowed: 15 *min*

Name: ____

Age: ______ Class: ______

You cannot use a calculator.
Write ALL the calculations performed and ideas that you had for each question!

A_1: What is the value of 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10?

B_1: What is the result of the sum of the first 50 natural numbers?

C_1: What is the value of $x - y$,
if $x = 1^2 + 2^2 + 3^2 + \cdots + 105^2$ and $y = 1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + 104 \times 106$.

MATHEMATICS / PROBLEM SOLVING – Time allowed: 15 *min*

Name: ____

Age: ______ Class: ______

You cannot use a calculator.
Write ALL the calculations performed and ideas that you had for each question!

+

A_2: What is the value of 12 + 23 + 34 + 45 + 56 + 67 + 78 + 89?

B_2: What is the result of the sum of the first 100 natural numbers?

C_2: What is the value of $x - y$,
if $x = 1^2 + 2^2 + 3^2 + \cdots + 2005^2$ and $y = 1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + 2004 \times 2006$.

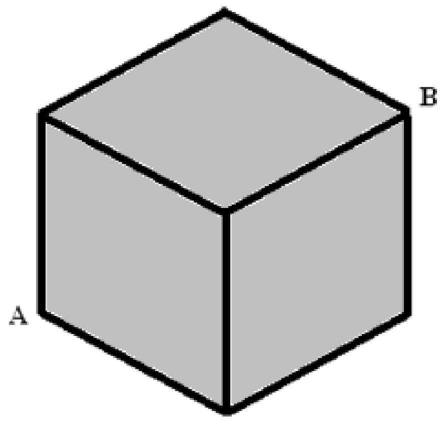
MATHEMATICS / PROBLEM SOLVING – Time allowed 15 *min*

Name: ____

Age: ______ Class: ______

You cannot use a calculator.
Write ALL the calculations performed and ideas that you had for each question!

D_1: Consider a massive cube whose edge measures 1 m. What is the length of the trajectory which corresponds to the smallest distance between point A and point B?



E_1: When the first round of a group of the Champions League was complete, each team had played once against each of the other teams. The scores were as follows: A (7 points); B (4 points); C (3 points); D (3 points). As 3 points are awarded for each win and 1 point for each draw, what was the outcome of the game between teams A and D?

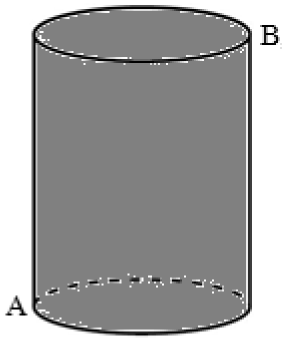
MATHEMATICS / PROBLEM SOLVING – Time allowed: 15 *min*

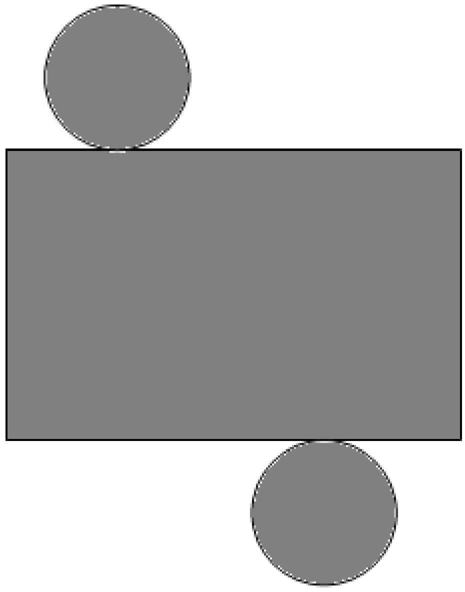
Name: ____

Age: ______ Class: ______

You cannot use a calculator.
Write ALL the calculations performed and ideas that you had for each question!

D_2: Consider a massive cylinder with a 1-metre radius and 6 metres of height. What is the length of the trajectory which corresponds to the smallest distance between point A and point B?





E_2: When the first round of a group of the Champions League was complete, each team had played once against each of the other teams. The scores were as follows: A (7 points); B (4 points); C (3 points); D (3 points). As 3 points are awarded for each win and 1 point for each draw, what was the outcome of the game between teams A and D?

	A	B	C	D
A	×			
B	×	×		
C	×	×	×	
D	×	×	×	×