Weighted Sums of Associated Random Variables

Tonguç Çağın

Departamento de Matemática

Faculdade de Ciências e Tecnologia

Universidade de Coimbra

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Abstract

Sums of random variables have always attracted a lot of interest as their asymptotic behaviour raises relevant theoretical challenges. Moreover, many statistical procedures are described by such sums. Thus, there is a natural interest in considering the convergence of $T_n = \sum_{i=1}^{n} a_{ni} X_i$, where the variables X_i are centered. Therefore this thesis focuses on weighted sums of associated random variables which verify a particular dependence structure and on moderate deviations of non-weighted sums of associated and strictly stationary random variables. Two problems relevant with above mentioned subjects are considered.

At the first problem we study the convergence of $T_n = \sum_{i=1}^{n} a_{ni} X_i$ weighted sums normalized by $n^{1/p}$, $p \in (1,2)$ where the random variables X_i are associated and have moments of order somewhat larger than p. Notice that it is well known that the existence of the p-th order moment is an optimal assumption even for independent variables. This requirement has been extended to non-weighted sums. As what concerns weighted sums some extensions have been proved, requiring the existence of higher order, still less than 2, moments. We could relax this moment assumption getting closer the p-th order moment requirement. We also consider the Marcinkiewicz-Zygmund law with assumptions on the 2-dimensional analogue of tail probabilities of the random variables relaxing in this case the assumption on the decay rate on the covariances.

Later a different approach is taken to the same problem. A truncation technique is used together with coupling with independent variables, which allows a relaxation of the assumptions on the weights. Moreover, this coupling allows not only for the proof of almost sure results and but enables to identify convergence rates. The assumptions on p, that now include the case $p < 1$, excluded from earlier results for positively associated variables, depend on the asymptotic behaviour of the weights, as usual. Also we give a direct comparison with the characterizations previously available, showing that the scope of applicability of our results does not overlap with known conditions for the same asymptotic results.

At the second problem, we present a moderate deviation in the non-logarithmic scale for sums of associated and strictly stationary random variables with finite moments of order larger than 2. The control of this dependence structure relies on the decay rate of the covariances for which we assume a relatively mild polynomial decay rate. The proof combines a coupling argument together with a suitable use of a Berry-Esséen inequality.

Keywords: almost sure convergence, associated random variables, Berry-Esséen inequality, convergence rates, moderate deviations, weighted sums.

Resumo

O estudo de propriedades assintóticas de somas de variáveis aleatórias atraiu desde sempre bastante atenção dadas as interessantes e não triviais questões que o seu tratamento levanta. Além disso, muitos dos procedimentos estatísticos correntes dependem de somas de variáveis aleatórias. É, portanto, natural o interesse no estudo da convergência de somas do tipo $T_n = \sum_{i=1}^n a_{ni} X_i$, com X_i variáveis centradas. A presente dissertação estuda assim condições de convergência para somas ponderadas de variáveis aleatórias dependentes. Aborda-se também o problema de caracterização de probabilidades do tipo desvios moderados, considerando agora somas não ponderadas, apresenta um resultado do tipo desvios moderados.

A primeira questão estudada procura condições para a convergência das somas pesadas $T_n = \sum_{i=1}^n a_{ni} X_i$ normalizadas por $n^{1/p}$, $p \in (1, 2)$, admitindo que as variáveis X_i são associadas e têm momentos finitos de ordem um pouco maior do que p. Recorde-se que, para variáveis independentes, a existência de momentos de ordem p é a caracterização ótima para a convergência. Sabe-se que esta hipótese é suficiente para a convergência de somas não pesadas de variáveis associadas. No que respeita a somas pesadas são conhecidos resultados que garantem a convergência exigindo a existência de momentos de ordem superior a p, embora ainda inferior a 2. Apresenta-se um relaxamento da ordem do momento exigido, aproximando-nos da caracterização ótima p. Demonstram-se ainda leis dos grandes números do tipo Marcinkiewicz-Zygmund sob condições nas probabilidades bidimensionais de cauda, o que permite ainda aligeirar as hipóteses acerca da velocidade de decrescimento das covariâncias.

O problema da convergência de somas ponderadas de variáveis associadas é retomado em seguida utilizando uma técnica distinta: truncagem das variáveis acompanhada de um emparelhamento adequado com variáveis independentes. Esta abordagem permite imediatamente um relaxamento de algumas hipóteses algo restritivas sobre o comportamento dos pesos. Além disso, este emparelhamento permite não só demonstrar caracterizações de convergência quase certa, mas também das suas velocidades. As hipóteses colocadas permitem agora a normalização das somas ponderadas por $n^{1/p}$ com $p < 1$, excluído da abordagem descrita acima. O expoente de normalização depende, como é habitual neste tipo de resultados do comportamento assintótico dos pesos e da ordem dos momentos das variáveis. Apresenta-se finalmente uma comparação com as caracterizações obtidas na primeira abordagem, que mostra que os dois conjuntos de hipóteses parecem não se sobrepor.

No capítulo final apresenta-se uma caracterização exata para probabilidades de desvios moderados para variáveis associadas, estritamente estacionárias e com momentos finitos de ordem $q > 2$. Neste caso o controlo da estrutura de dependência é obtido à custa de uma hipótese de velocidade de decrescimento polinomial das covariâncias, combinada com uma utilização adequada de uma desigualdade do tipo Berry-Esséen.

Palavras-chave: convergência quase certa variáveis aleatórias associadas, desigualdade de Berry-Esséen, velocidade de convergência, desvios moderados, somas ponderadas.

Contents

Introduction

This thesis focuses on two problems that are related with weighted sums of associated random variables and moderate deviations of non-weighted sums of associated and strictly stationary random variables respectively.

Sums of random variables have always attracted a lot of interest as their asymptotic behaviour raises relevant theoretical challenges. Moreover, many statistical procedures depend on such sums. Thus, there is a natural interest in considering the convergence of $T_n = \sum_{i=1}^{n} a_{ni} X_i$, where the variables X_i are centered. Applications of this subject mostly includes random variables which are not centered but during theoretical approach taking them as centered does not cause any kind of loss from generality.

For $a_{ni} = 1$ and independent and identically distributed random variables X_i Baum and Katz [5] proved the Marcinkiewicz-Zygmund strong law of large numbers, that is, that $n^{-1/p} S_n \longrightarrow 0$ where $S_n = \sum_{i=1}^{n} X_i$ and $p \in [1, 2)$, almost surely if and only if $\mathrm{E}\,|X_1|^p < \infty$.

Chow [12] and Cuzick [15] considered variables with finite moments of order larger than 1 and weights satisfying certain summation conditions, to be specified later, in order to extend the Marcinkiewicz-Zygmund law with $p = 1$. This was later extended by Cheng [11] and Bai and Cheng [3] to other values of $p \in (1, 2)$ requiring the existence of moments of order between p and 2.

Later, positively associated random variables were considered by Louhichi [26] for constant weights. Later, contributions by Ko and Kim [24], Baek, Park, Chung and Seo [2], Cai [10], Qiu and Chen [44] or Shen, Wang, Yang and Hu [49], who considered negatively dependent random variables, followed. Positively associated random variables seem to be more challenging as the variance of the sums increases with respect to the independent framework and many of these asymptotic results rely on a suitable control of the growth of the variances of sums. Oliveira [38] took the results of Louhichi [26], again for positively dependent random variables, and extended them to weighted sums.

In the second chapter, which is based on the published paper 'A note on weighted sums of associated random variables' by Çağın and Oliveira [13], as the first problem, we extend the results in [38], relaxing the moment assumption on the random variables, getting nearer of the p-th order moment assumption used by Louhichi [26] to prove the convergence for constant weights, while strengthening the assumption on the decay rate of the covariances. We also consider the Marcinkiewicz-Zygmund law with assumptions on the 2-dimensional analogue of tail probabilities of the random variables relaxing in this case the assumption on the decay rate on the covariances, but strengthening the moment condition.

In the third chapter, which is based on the pre-printed paper 'On convergence rates for weighted sums of associated random variables' by the same authors [14], we study a different approach to the problem studied in Chapter 2. We use a truncation technique together with coupling with independent variables, a technique, which is commonly used in the literature of associated random variables (see for example Ioannides and Roussas [23] and Oliveira [37] which allows a relaxation of the assumptions on the weights. Moreover, this coupling allows not only for the proof of almost sure results, but enables to identify convergence rates. The assumptions on the moments order p, that now include the case $p < 1$, excluded from earlier results for positively associated variables, depend on the asymptotic behaviour of the weights, as usual. Also we give a direct comparison with the characterizations previously available, showing that the scope of applicability of our results does not overlap with known conditions for the same asymptotic results.

In the fourth chapter, which is based on the submitted paper 'A moderate deviation for associated variables' by Çağın, Oliveira and Torrado, as the third problem, we consider a moderate deviation in the non-logarithmic scale for non-weighted sums for associated and strictly stationary random variables with finite moments of order $q > 2$. The control of the dependence structure relies on the decay rate of the covariances for which we assume a relatively mild polynomial decay rate.

1. Preliminary results on sums of random variables

Sums of random variables have always attracted a lot of interest as their asymptotic behaviour raises relevant theoretical challenges. It has always been a central subject in the probabilistic literature. Moreover, many linear statistics are written as weighted sums of random variables, raising thus the interest in the characterization of the asymptotics of such sums, conveniently normalized.

First, sums of random variables in the form of $S_n = \sum_{i=1}^{n} X_i$ were considered. For independent and identically distributed variables Baum and Katz [5] proved the Marcinkiewicz-Zygmund strong law of large numbers, that is, that $n^{-1/p} S_n \longrightarrow 0$ almost surely, $p \in [1, 2)$, if and only if $E|X_1|^p < \infty$. For $p < 1$, Marcinkiewicz-Zygmund law takes a totally different structure. Being out of our spectrum of interest, this case is omitted and case for $p \geq 1$ is focused on.

1.1 Sums of independent random variables with weights

As a natural extension of the above mentioned summation, sums with weights were considered i.e. $T_n = \sum_{i=1}^{n} a_{ni} X_i$, where the variables X_i are centered and where weights, a_{ni}, satisfy $a_{ni} \geq 0$, $i \leq n$, $n \geq 1$, were considered.

Chow [12] and Cuzick [15] considered variables such that $E|X_1|^\beta < \infty$ and weights satisfying

$$\sup_{n\geq 1} \frac{1}{n} \sum_{i=1}^{n} a_{n,i}^{\alpha} < \infty$$

for some $1 < \alpha \leq \infty$ where $\alpha^{-1} + \beta^{-1} = 1$, to prove the Marcinkiewicz-Zygmund law with $p = 1$. Due to its prominence in the development of the subject and being an important reference in our studies, we would like to refer to the results by Cuzick [15].

Theorem 1.1.1. *(Theorem 1.1. in Cuzick [15]) Assume* $X, X_1, X_2, \ldots$ *are i.i.d.,* $EX = 0$ *and* $T_n = \sum_{i=1}^{n} a_{ni} X_i$*, and assume*

$$\sup_n \left(\frac{1}{n} \sum_{i=1}^{n} |a_{ni}|^q \right)^{1/q} < \infty \quad \textit{for some} \quad 1 < q \leq \infty \tag{1.1}$$

If $E|X|^p < \infty$ *for* $p^{-1} + q^{-1} = 1$*, then*

$$\frac{1}{n} T_n \to 0 \quad \textit{almost surely.}$$

In particular if $\sup |a_{ni}| < \infty$*, then*

$$E|X| < \infty \Rightarrow \frac{1}{n} T_n \to 0 \quad \textit{almost surely.}$$

In particular, when $q = \infty$ *we interpret the inequality* (1.1) *as* $\sup |a_{ni}| < \infty$*.*

The result is also true when $q = 1$ *under the additional assumption that*

$$\lim_n \sup_{i \leq n} |a_{ni}| \frac{1}{n} \log n = 0.$$

Following theorem stands out from the line of the previously quoted results as well as the two following results since it features weights which are not deterministic but random variables that satisfy certain regulatory moment conditions. Notice that the normalizing sequence is still $1/n$.

Theorem 1.1.2. *(Theorem 1.2. in Cuzick [15])Assume* $X, X_1, X_2, \ldots$ *are i.i.d. and* a_{ni} *is an i.i.d. array independent of the* X_i*. Let* $T_n = \sum_{i=1}^{n} a_{ni} X_i$*. If* $E|X|^p < \infty$ *and* $E|a_{ni}|^q < \infty$ *for* $q \geq 2$ *and* $p^{-1} + q^{-1} = 1$ *then*

$$\frac{1}{n} T_n \to EXEa_{11} \quad \textit{almost surely.} \tag{1.2}$$

Furthermore, if only $E|X| < \infty$*, then* (1.2) *holds if we replace the moment assumption on the* a_{ni}*'s by* $\sup |a_{ni}| < \infty$ *almost surely.*

We would like to underline that analogs of theorem 1.1.2 holds for more general normalizing sequences b_n other than $b_n = n$. Following cited results constitute examples to this fact.

Theorem 1.1.3. *(Theorem 2.1. in Cuzick [15]) Assume* $X, X_1, X_2, \ldots$ *are i.i.d. and let* $T_n = \sum_{i=1}^n a_{ni} X_i$. *Also assume*

$$\sup_n \left(\frac{1}{n} \sum_{i=1}^n |a_{ni}|^q \right)^{1/q} < \infty$$

for some $1 \le q \le \infty$ *and that* $b_n/n \to \infty$. *Let* $p^{-1} + q^{-1} = 1$. *Then* $B(n) \equiv b_n^2/n^{2-2/p}$ *is increasing and*

$$EB^{-1}(X^2) < \infty \tag{1.3}$$

implies $T_n/b_n \to 0$ *almost surely. If* $b_n = n^\alpha, \quad \alpha > 1$, *then* (1.3) *is equivalent to* $E|X|^{p/(1+p(\alpha-1))} < \infty$. *If additionally* $|a_{ni}|$ *are bounded then* (1.3) *reduces to* $E|X|^{1/\alpha} < \infty$.

Theorem 1.1.4. *(Theorem 2.2. in Cuzick [15]) Let* $T_n = \sum_{i=1}^n a_{ni} X_i$ *where* $X, X_1, X_2, \ldots$ *is a mean zero i.i.d. sequence and* a_{ni} *are uniformly bounded constants. Assume* $B(u)$ *is regularly varying at infinity with index* $\frac{1}{2} \le \alpha < 1$ *and that*

$$\frac{B(u)}{u^{1/2} \log u} \to \infty.$$

If

$$EB^{-1}|X| < \infty \tag{1.4}$$

and

$$\frac{B^{-1}(x \log x)}{\log x} G(x) \to 0 \quad as \quad x \to \infty \tag{1.5}$$

where $G(x) = P(|X| \ge x)$, *then for* $b_n = B(n)$, $T_n/b_n \to 0$ *almost surely.*

Corollary 1.1.5. *(Corollary 2.2. in Cuzick [15]) If* (1.4) *and* (1.5) *are replaced by*

$$E|X|^{1/\alpha} < \infty$$

and

$$x^{1/\alpha} (\log^{1/\alpha - 1} x) G(x) \to 0,$$

respectively, then

$$\frac{1}{n^\alpha} T_n \to 0 \quad almost\, surely.$$

The need for condition $b_n/(n \log n)^{1/2} \to \infty$ is made clear by the following analog of the law of the iterated logarithm for randomly re-signed partial sums. Only a single logarithm is needed here:

Theorem 1.1.6. *(Theorem 2.3. in Cuzick [15]) Assume $X, X_1, X_2, \ldots$ are i.i.d. with $\sigma^2 = EX^2$ and let $T_n = \sum_{i=1}^n \varepsilon_{ni} X_i$ where ε_{ni} is an independent Rademacher array. Then*

$$\limsup_{n\to\infty} \frac{1}{(2\sigma^{1/2} n \log n)^{1/2}} T_n = 1 \quad \textit{almost surely.}$$

Results by Chow [12] and Cuzick [15] were extended by Cheng [11] and Bai and Cheng [3] to other values of $p \in (1,2)$. Results in the latter paper are as follows:

Theorem 1.1.7. *(Theorem 2.1. in Bai, Cheng [3]) Let $T_n = \sum_{i=1}^n a_{ni} X_i$ be a weighted sum where $X_1, X_2, \ldots$ are independent and identically distributed random variables with $EX_i = 0$. Set $1/p = 1/\alpha + 1/\beta$, for $1 < \alpha, \beta < \infty$ and assume for some $1 < p < 2$ and $1 < \alpha$, $\beta < \infty$ we have the moment condition $E|X|^\beta < \infty$. Also assume*

$$A_\alpha = \limsup A_{\alpha,n} < \infty \quad \textit{where} \quad A_{\alpha,n}^\alpha = \frac{1}{n} \sum_{i=1}^n |a_{ni}|^\alpha \tag{1.6}$$

holds. Then

$$\frac{1}{n^{1/p}} T_n \to 0 \quad \textit{almost surely.} \tag{1.7}$$

Conversely, if (1.7) is true for any coefficient arrays satisfying (1.6), then

$$E|X|^\beta < \infty \quad \textit{and} \quad EX = 0.$$

Proof. **Sufficiency:**

We start by making following definitions for all $1 \le i \le n$:

$$X_i^c = X_i \mathbb{I}_{[|X_i|^\beta > n]},$$
$$X_i^{'} = X_i \mathbb{I}_{[\frac{n^{1/\beta}}{(\log n)^{3(\alpha-1)}} < |X_i| \le n^{1/\beta}]},$$
$$\bar{X}_i = X_i \mathbb{I}_{[|X_i| \le \frac{n^{1/\beta}}{(\log n)^{3(\alpha-1)}}]}.$$

Further define:

$$a_{ni}^{'} = a_{ni} \mathbb{I}_{[|a_{ni}| > \frac{n^{1/\alpha}}{\log^2 n}]}, \quad \textit{for} \quad 1 < \alpha < \infty,$$
$$\bar{a}_{ni} = a_{ni} - a_{ni}^{'}, \quad \textit{for} \quad \textit{all} \quad 1 \le i \le n.$$

Using above defined expressions, set

$$T_n^c = \sum_{i=1}^n a_{ni} X_i^c,$$

$$T_n^{'} = \sum_{i=1}^{n} a_{ni}^{'} X_i^{'},$$

$$T_n^{*} = \sum_{i=1}^{n} a_{ni}^{'} \bar{X}_i,$$

$$\bar{T}_n = \sum_{i=1}^{n} \bar{a}_{ni}(X_i^{'} + \bar{X}_i) = \sum_{i=1}^{n} \bar{a}_{ni}(X_i - X_i^{c}).$$

By definition

$$T_n = T_n^c + T_n^{'} + T_n^{*} + \bar{T}_n. \tag{1.8}$$

We show first that $T_n^c/n^{1/p} \to 0$ almost surely. Since $1/p = 1/\alpha + 1/\beta$ and $\beta = \alpha/(\alpha-1)\{(1+\beta(1-1/p))\}$, it holds that

$$|X_i^c| \le |X_i^c|^{\beta(\alpha-1)/\alpha} n^{-(1-1/p)}.$$

By the Hölder inequality and the moment condition $E|X|^\beta < \infty$,

$$n^{-1/p}|T_n^c| \le n^{-1} \sum_{i=1}^{n} |a_{ni}||X_i^c|^{\beta(\alpha-1)/\alpha} \le A_{\alpha,n} \left(\frac{1}{n} \sum_{i=1}^{n} |X_i^c|^{\beta} \right)^{(\alpha-1)/\alpha} \to 0 \quad a.s. \tag{1.9}$$

Secondly, for $T_n^{'}$, by the definitions of $A_{\alpha,n}$ and $X_i^{'}$, we have

$$\begin{aligned} n^{-1/p}|T_n^{'}| &\le \left\{ \max_{1\le i\le n} |a_{ni}^{'} X_i^{'}| + \left(\sum_{i=1}^{n} |a_{ni}^{'} X_i^{'}| \right) \mathbb{I}_{[\sharp\{i: a_{ni}^{'} X_i^{'} \neq 0\} \ge 2]} \right\} \\ &\le n^{-1/\beta} \max_{1\le i\le n} |X_i| + n^{(1-1/\alpha)} \mathbb{I}_{[\sharp\{i: a_{ni}^{'} X_i^{'} \neq 0\} \ge 2]}. \end{aligned} \tag{1.10}$$

Since $E|X|^\beta \le \infty$, the first term of the last sum converges zero almost surely. For the last term on the right hand side, there are at most $O(\log^{2\alpha} n)$ many nonzero $a_{ni}^{'}$ by the definitions of $A_{\alpha,n}$ and $a_{ni}^{'}$, and so, by the definition of $X_i^{'}$,

$$\begin{aligned} P[\sharp\{i : a_{ni}^{'} X_i^{'} \neq 0\} \ge 2] &\le P\left(\bigcup_{i \neq j} \{a_{ni}^{'} X_i^{'} \neq 0, a_{nj}^{'} X_j^{'} \neq 0\} \right) \\ &\le [O(\log n)]^{4\alpha} P^2[|X_i| > n^{1/\beta}(\log n)^{-3(\alpha-1)}] \\ &= O[(\log n)^{4\alpha + 6\beta(\alpha-1)}] n^{-2}. \end{aligned}$$

By this estimate and Markov's inequality, the probability that the last term of (1.10) is positive is a general term of a convergent series in $n \ge 1$. Therefore, the left hand side of (1.10) converges to zero almost surely. Furtermore, associated with (1.9) and (1.10), we find that

$$n^{-1/p}|E(T_n^c + T_n^{'})| \leq \frac{n^{-1/p}E|X|^\beta \sum_{i=1}^n |a_{ni}|}{[n^{1/\beta}(\log n)^{-3(\alpha-1)}]^{\beta-1}}$$
$$\leq A_{\alpha,n}E|X^\beta|n^{-1/\alpha}(\log n)^{3(\alpha-1)(\beta-1)} \to 0 \qquad (1.11)$$

Thirdly, by the definitions of $a_{ni}^{'}$ and $\bar{X}_i$, we can estimate $T_n^* - ET_n^*$ by

$$n^{-1/p}|T_n^* - ET_n^*| \leq \frac{2\sum_{i=1}^n |a_{ni}|^\alpha \{n^{1/\beta}/(\log n)^{3(\alpha-1)}\}}{n^{1/p}(n^{1/\alpha}\log^{-2} n)^{\alpha-1}}$$
$$= \frac{2}{n(\log n)^{\alpha-1}}\sum_{i=1}^n |a_{ni}|^\alpha \to 0. \qquad (1.12)$$

Finally, let $Y_{ni} = \bar{a}_{ni}(X_i - X_i^c)$. By $1/p = 1/\alpha + 1/\beta$, $1 < p < 2$, we have

$$E\sum_{i=1}^n Y_{ni}^2 \leq nA_{\alpha\wedge 2,n}^{\alpha\wedge 2}(n^{1/\alpha}\log^{-2} n)^{(2-\alpha)^+} n^{(2-\beta)^+/\beta} \| X \|_{\beta\wedge 2}^{\beta\wedge 2} = O(\max(n^{2/\alpha}, n^{2/\beta}, n))$$

where $a \wedge b = min(a,b)$. On the other hand, for any $t > 0$, $tn^{1/p}|Y_{ni}| \leq tn^{2/p}\log^{-2} n$ and $\max\{n^{2/\alpha}, n^{/\beta}, n\} = o(n^{2/p}\log^{-2} n)$.

It follows by Bernstein's inequality (see [7], Theorem 1.2.) that

$$P(|\bar{T}_n - E\bar{T}_n| > tn^{1/p}) = P\left(|\sum_{i=1}^n (Y_{ni} - EY_{ni})| > tn^{1/p}\right)$$
$$\leq 2\exp\left(\frac{-t^2 n^{2/p}}{tn^{2/p}\log^{-2} n}\right), \qquad (1.13)$$

which is summable in n. The proof of sufficiency part of Theorem 1.1.7 is complete in the view of (1.8) through (1.13)

Necessity:

Suppose (1.7) is true for any weights sequence satisfying (1.6). Choose, for each $n, a_{n1} = \cdots = a_{n,n-1} = 0$ and $a_{nn} = n^{1/\alpha}$. Then, by (1.7) we have

$$\frac{1}{n^{1/\beta}}X_n \to 0 \quad almost\, surely,$$

which implies that $E|X|^\beta < \infty$. Since $\beta > 1$, EX exists. Moreover, by the sufficiency part of Theorem (1.1.7), we have

$$\frac{1}{n^{1/p}}(T_n - ET_n) \to 0 \quad almost\, surely$$

for any weight sequence satisfying (1.6). Therefore,

$$\frac{1}{n^{1/p}} ET_n = \frac{1}{n^{1/p}} \sum_{i=1}^{n} a_{ni} EX \to 0$$

for any weight sequence a_{ni} satisfying (1.6). Selecting $a_{ni} = 1$, we show what $EX = 0$. ■

It is now assumed that random variable $|X|^\gamma$ has a finite moment generating function. Or in another words we assume that for some h, $\gamma > 0$

$$E \exp(h|X|^\gamma) < \infty. \tag{1.14}$$

This corresponds to the case with and arbitrarily large β in Theorem 1.1.7. The following result, thus, complements the results of Cuzick [15].

Theorem 1.1.8. *(Theorem 2.2. in Bai, Cheng [3]) Let* $T_n = \sum_{i=1}^n a_{ni} X_i$ *be a weighted sum. Suppose that* (1.6) *holds for* $\alpha \in (0, 2)$ *and* (1.14) *holds. Then for* $0 < \alpha \leq 1$ *and* $b_n = n^{1/\alpha} \log^{1/\gamma} n$

$$\limsup \frac{|T_n|}{b_n} \leq h^{-1/\gamma} A_\alpha \quad a.s., \tag{1.15}$$

moreover, for $1 < \alpha < 2$, $b_n = n^{1/\alpha} (\log n)^{1/\gamma + \gamma(\alpha - 1)/\alpha(1+\gamma)}$, *and* $EX = 0$,

$$\limsup \frac{|T_n|}{b_n} = 0 \quad a.s.. \tag{1.16}$$

Conversely, if (1.15) *holds when* $0 < \alpha \leq 1$ *for all weights sequences satisfying* (1.6), *then necessarily* (1.14) *holds,* $E(h'|X|^\gamma) < \infty$ *for any* h', $0 < h' < h$*; if* (1.16) *holds when* $1 < \alpha < 2$*, then* $E|X|^\nu < \infty$ *for any* $h > 0$*, where* $1/\nu = 1/\gamma + \gamma(\alpha - 1)/\alpha(1+\gamma)$.

Proof of this theorem is an involved one and it is not crucial for the results of this thesis, hence excluded from the text for sake of simplicity.

Improvement from the results of Cuzick [15] to Bai and Cheng [3] is the relaxing of the moment assumptions. Also the normalizing sequence is more general. Still, the assumption for the independence of the random variables is essential throughout all of them. It is natural to consider dependent random variables. This will be our next subject.

1.2 Sums of dependent random variables without weights

Here we shift to a different kind of sequence of random variables which requires us to introduce the definition of positively associated random variables. This will be important during the following parts.

Definition 1.2.1. *Let $X_1, X_2, ...$ be a sequence of random variables. If for every subcollection $X_{i_1}, ..., X_{i_n}$ and every pair of coordinatewise non decreasing functions h, $k : \mathbb{R}^n \to \mathbb{R}$*

$$\mathrm{Cov}(h(X_{i_1}, ..., X_{i_n}),\, k(X_{i_1}, ..., X_{i_n})) \geq 0,$$

whenever the covariance is defined, then $X_1, X_2, ...$ are called positively associated random variables.

Non weighted sums of positively associated random variables were considered by Louhichi [26]. We are going to present her results with the additional assumption of random variables being centered (i.e. $EX = 0$) whenever it is convenient for our later use.

Theorem 1.2.2. *(Theorem 1 in Louhichi [26]) Let p be a fixed real number in $[1, 2)$. Let $X_1, X_2, ...$ be a sequence of centered and associated random variables. Suppose that there exists a positive random variable X such that $EX^p < \infty$ and that $\sup_{i>0} P(|X_i| > x) \leq P(X > x)$, for any positive x. If*

$$\sum_{1 \leq i < j \leq \infty} \int_{j^{1/p}}^{\infty} v^{-3} G_{i,j}(v) dv < \infty \tag{1.17}$$

where

$$G_{i,j}(v) = \mathrm{Cov}(g_v(X_i), g_v(X_j)), \quad g_v(u) = \max(\min(u,\, v),\, -v)$$

then

$$\lim_{n\to\infty} \frac{1}{n^{1/p}} S_n = 0, \textit{ almost surely}. \tag{1.18}$$

It is worth noticing that we are requiring only the existence of moments order less than 2. From this results, the following corollaries are easily derived.

Corollary 1.2.3. *(Remark on Theorem 1 in Louhichi [26]) For weakly stationary and associated sequences condition* (1.17) *holds as soon as*

$$\sum_{r=1}^{\infty} \int_{(r+1)^{1/p}}^{\infty} v^{p-3} G_{0,r}(v) dv < \infty.$$

For random variables with finite variance, above theorem yields the following result, providing rates of convergence in the strong law of large numbers of Birkel [6].

Corollary 1.2.4. *(Corollary 1 in Louhichi [26]) Let $X_1, X_2, ...$ be a sequence of centered and associated random variables that fulfills the requirement of above theorem. Suppose moreover that $X_1, X_2, ...$ has finite variance. If*

$$\sum_{j=1}^{\infty} j^{-2/p}\mathrm{Cov}(X_j,\, S_{j-1}) < \infty, \quad \textit{for some} \quad p \in [1, 2),$$

with the convention that $S_0 = 0$, then

$$\lim_{n\to\infty} \frac{1}{n^{1/p}} S_n = 0, \quad \textit{almost surely}.$$

Proof of Theorem 1.2.2 relies on a few technical arguments that will be explained in detail later when we present our own contribution on Chapter 2(see 2.2.1). Therefore for now, we find it sufficient to give just the main idea of the proof of the above stated theorem. In order to prove this statement Louhichi uses the following lemma (and several other already well known ones in the literature):

Lemma 1.2.5. *(Lemma 1 in Louhichi [26]) Let $X_1, X_2, ...$ be a sequence of associated random variables. Let $S_n^* = \sup_{k\le n} S_k$ where $S_n = \sum_{i=1}^{n}(X_i - EX_i)$. Suppose that there exists a positive random variable X such that for any positive x, $\sup_{i>0} P(|X_i| > x) \le P(X > x)$. Then for any positive real numbers x and M*

$$P(S_n^* \ge x) \le \frac{4n}{x^2} EX^2\mathbb{I}_{[X\le M]} + \frac{4n}{x} EX\mathbb{I}_{[X>M]} + \frac{4nM^2}{x^2} P(X > M) + \frac{8}{x^2} \sum_{1\le i<j\le n} G_{i,j}(M)$$

Louhichi's proof uses Borel-Cantelli Lemma to prove (1.18). A direct usage would mean showing $\sum_i P(S_n > \varepsilon n^{1/p}) < \infty$. Instead, we replace S_n by the larger S_n^*, as defined in 1.2.5, which is an increasing sequence of random variables. Taking this into account, it is enough to prove

$$\sum_{n\ge 0} n^{-1} P(S_n^* \ge \epsilon n^{1/p}) < \infty.$$

1.3 Sums of positively associated random variables with weights

Now we go on citing some results by Oliveira [38] who, combining the methods described in the previous sections, considered weighted sums of positively associated random variables. He shows the following.

Theorem 1.3.1. *(Theorem 4.1. in Oliveira [38]) Let $X_1, X_2, ...$ be centered and identically distributed associated random variables such that*

$$E|X_1|^{p(\alpha+2)/\alpha} < \infty, \quad \text{for some } p \in (1,2),\ \alpha > \frac{2p}{2-p},$$

$$\sum_{1 \leq i < j < \infty} \int_{j^{1/p}}^{\infty} v^{-3+2\frac{p}{\alpha}} G_{i,j}(v)dv < \infty.$$

Assume that the weights satisfy

$$a_{n,i} \geq 0, \quad i \leq n,\ n \geq 1, \quad \text{and } a_{k,j} \geq a_{k-1,j} \quad \text{for each } k, j \in \mathbb{N}$$

and that $\sup_{n\in\mathbb{N}} A_{n,\alpha} < \infty$ *where* $A_{n,\alpha}$ *is defined in the same way with* (1.6) *. Then*

$$\frac{1}{n^{1/p}} T_n \to 0 \quad a.s..$$

where $T_n = \sum_{i=1}^{n} a_{n,i} X_i$

Remark 1.3.2. *(Remark 4.3., Oliveira [38]) If we allow $\alpha \to \infty$ in the same assumptions of the above theorem these reduce to the assumption of Theorem 1.2.2. As mentioned before, Louhichi's framework corresponds to case of constant weight so, for every $\alpha > 0$, $A_{n,\alpha}$ is equal to the constant defining the weight; thus we are really allowed to let $\alpha \to \infty$. That is, above theorem really extends Louhichi's results.*

For an even more general normalizer, he shows the following.

Theorem 1.3.3. *(Theorem 4.4., Oliveira [38]) Let $X_1, X_2, ...$ be centered and identically distributed associated random variables such that*

$$EX_1^2 < \infty$$

$$E\left(\frac{X_1^2}{\log^{\frac{2}{\gamma}-1}|X_1|}\right) < \infty, \quad \text{for some } \gamma \in (0,2),$$

$$\sum_{1 \leq i < j < \infty} \int_{j^{1/\beta}}^{\infty} \frac{1}{v^{\beta+1}} G_{i,j}(v)dv < \infty,$$

where $\beta > 0$ *if* $\alpha < 2$ *and* $\beta \in (0, \frac{2\alpha}{\alpha-2})$ *if* $\alpha > 2$. *Assume that* $\alpha > 1$ *and that the weights satisfy*

$$a_{n,i} \geq 0, \quad i \leq n,\, n \geq 1, \quad \textit{and } a_{k,j} \geq a_{k-1,j} \quad \textit{for each } k, j \in \mathbb{N}$$

and $\sup_{n\in\mathbb{N}} A_{n,\alpha} < \infty$. *Define* $q = \frac{2\alpha}{\alpha+2}$. *Then*

$$\frac{1}{n^{1/q} \log^{1/\gamma} n} T_n \to 0, \quad a.s..$$

1.4 Contributions on this thesis

1.4.1 Weighted sums of associated random variables

The results on sums of independent random variables were firs extended by considering weighted sums and late, by dropping the independence assumption but considering no weights. A first extension in merging these two extensions, that is, considering weighted sums of dependent random variables, was made in Oliveira [38].

At the second chapter of this thesis we extend the results in [38], relaxing the moment assumption on the random variables, still assuming the existence of moments somewhat larger than p, approaching the p-th order moment assumption used by Louhichi [26] to prove the convergence of weighted sums of associated random variables normalized by constant weights, $n^{1/p}$, $p \in (1, 2)$, while strengthening the assumption on the decay rate of the covariances. We also consider the Marcinkiewicz-Zygmund law with assumptions on the 2-dimensional analogue of tail probabilities of the random variables relaxing in this case the assumption on the decay rate on the covariances, but strengthening the moment condition.

Besides moment conditions we assume a convenient behaviour either on truncated covariances or on joint tail probabilities. Our results extend analogous characterizations known for sums of independent or negatively dependent random variables.

1.4.2 Convergence rates for weighted sums of associated random variables

At the third chapter, we study the convergence of weighted sums of associated random variables assuming only the existence of moments of order $p < 2$. We utilize a truncation technique together with coupling with independent variables which allows a relaxation of the assumptions on the weights. Truncation of random variables is a common technique already present in the associated literature for the proof of exponential inequalities (see Ioannides and Roussas [23] and Oliveira [37]). The assumptions on p, which depends on the asymptotic behaviour of the weights, as usual, now includes the case $p < 1$, which was excluded from earlier results for positively associated variables.

Also, we give a direct comparison with the characterizations previously available, showing that the scope of applicability of the results obtained in this chapter does not overlap with previously known conditions for the same asymptotic results.

In Sect. 3.1 we describe the framework and useful results, Sect. 3.3 presents the main results, and compares with results in Çağın and Oliveira [13], while Sect. 3.2 states versions of these results in a reduced setting, but proves the main steps for the final theorems.

1.4.3 A moderate deviation for associated random variables

Chapter 4 evolves around the concept of moderate deviations. Since this subject is different than the previously mentioned ones, we will give a short account of relevant early results on this subject.

Moderate deviations are an important topic in many theoretical or applied statistical areas. We consider a moderate deviation for associated and strictly stationary random variables with finite moments of order $q > 2$. The control of this dependence structure relies on the decay rate of the covariances for which we assume a relatively mild polynomial decay rate. The proof combines a coupling argument together with a suitable use of a Berry-Esséen inequality.

As mentioned before, sums of random variables have always been a central subject in the probabilistic literature, with a special interest on their asymptotic. Among results on this topic the important Central Limit Theorems (CLT) describes the limiting distributional behaviour of such sums, providing useful approximate descriptions of the tail probabilities. These, besides their natural theoretical interest, are extremely relevant in statistical applications.

There is, however, a limitation inherent to the properties of convergence in distribution, requiring that the tails considered through the limiting process should behave like the variance. More specifically, if the random variables X_n, $n \geq 1$, are assumed centered and we define $S_n = X_1 + \cdots + X_n$, $s_n^2 = ES_n^2$, the CLT provide the approximation of $\mathrm{P}(S_n > x\, s_n)$ by $N(x) = 1 - \Phi(x)$, for $x > 0$ fixed, where Φ is the distribution function of a standard Gaussian variable. If we allow x to depend on n, converging to infinity, then the above approximation is known as a moderate or large deviation, depending on how fast x grows to infinity, moderate deviations corresponding to the case where $x = O(s_n)$ (throughout the text me may use $x \sim s_n$ to represent the same approximation). Remark that the approximating function N is no longer necessarily the tail of a standard Gaussian, depending on the growth rate of x to infinity.

First large deviations were proved by Linnik [25], Ibragimov and Linnik [22], Nagaev [30];

Theorem 1.4.1. *(Theorem 1.9. in Navaev [30]) Let $X_1, X_2, ..., X_n$ be identically distributed and suppose that $1 - F(x) = l(x)x^{-t}(1 + o(1))$ as $x \to \infty$, where $l(x)$ is a slowly varying function and $t > 2$. If, in addition, $EX_1 = 0$, $\sigma^2 = 1$, and $E|X_1|^{2+\delta} < \infty$ for some $\delta > 0$, then*

$$P(S_n \geq x) = [1 - \Phi(x/n^{\frac{1}{2}})][1 + o(1)] + n[1 - F(x)](1 + o(1))$$

for $n \to \infty$ and $x \geq n^{\frac{1}{2}}$.

and Nagaev [31] or Rozovski [45] for independent and identically distributed variables. We refer the reader to the survey paper by Nagaev [32] for a nice account of these early results. The techniques of proof were much based on suitable exponential bounds, the so called Fuk-Nagaev inequalities, on the tail probabilities. A typical result in Nagaev [32], states that

$$\mathrm{P}(S_n > x\, s_n) = (1 - \Phi(x) + n\mathrm{P}(X_1 > x\, s_n))\,(1 + o(1)), \tag{1.19}$$

provided that $x \geq 1$, $s_n = n^{1/2}$ and the right tail of the X_n's is a regularly varying function.

Extensions of such results have been recently proved by Peligrad, Sang, Zhong and Wu [43] considering weighted sums $\tilde{S}_n = \sum_i c_{n,i} X_i$ instead of S_n. These authors prove a result similar to (1.19) under essentially the same assumptions on the random variables (i.i.d. and regularly varying tails) and a regularity condition on the weights:

$\max_i c_{n,i}/\mathrm{E}\tilde{S}_n^2 \longrightarrow 0$. The proof of this extension relies on moderate or large deviations for triangular arrays of random variables and convenient strong approximations between the tails of $\tilde{S}_n$ and the sums of tails of the X_n's, much in the same spirit of the proof technique used in Theorem 1.9 in Nagaev [32].

Going back to early results, moderate or large deviations for triangular arrays of row-wise independent variables were considered by Rubin and Sethuraman [46], Amosova [1], Slastnikov [50] or, more recently, by Frolov [18].

All the results mentioned so far characterize the tail probabilities directly. Concerning large deviations, that is, x growing fast to infinity, a lot of attention was given to the logarithms of the tail probabilities instead, thus providing exponential bounds for the tail probabilities themselves. The bound for these logarithms appears then as the Fenchel-Légendre transform of the normalized logarithm of the Laplace transform of S_n (notice that we are now back to non-weighted sums). A good account of results in this direction can be found in the book by Dembo and Zeitouni [16].

The interest on logarithmic tails meant that there are much fewer results available in the non-logarithmic scale in recent literature, particularly for weighted sums. Another recent direction of development is concerned with dependent variables. Here, available results seem even more scarce. Looking at large deviations, some results were proved by Nummelin [36], Bryc [8] or Bryc and Dembo [9] considering mixing variables or, Henriques and Oliveira [21] for associated random variables. Here the interest was on logarithmic scale results and the proof techniques relied on suitable exponential bounds and required a rather fast decay on the coefficients characterizing the dependence structure, meaning they should decrease faster than geometrically.

More recently, for mixing variables Merlevède, Peligrad and Rio [28] relaxed the assumption on the mixing coefficients, requiring just the geometric decay to prove a large deviation. Their proof technique, called by the authors a "Cantor set construction", adapts the block decomposition of sums, popular for proving CLT, to large deviations. These authors have more recently extended their results to other forms of weak dependent variables (see Merlevède, Peligrad and Rio [29]). Efforts in the non-logarithmic scale for dependent variables were made by Grama [19], Grama and Haeusler [20] for martingales, Wu and Zhao [54] for stationary processes, Tang [52] for negatively dependent variables or Liu [27] for negatively dependent heavy tailed variables.

At the fourth chapter of this thesis, we present a moderate deviation in the non-logarithmic scale for sums of associated and strictly stationary random variables with finite moments of order larger than 2. The control of this dependence structure relies on the decay rate of the covariances for which we assume a relatively mild polynomial decay rate. The proof combines a coupling argument together with a suitable use of a Berry-Esséen inequality. In Sect. 4.1 we give some definitions and recall some auxiliary results, in Sect. 4.3 we prove the main result and a corollary with an assumption that identifies more explicitly the behaviour of the relevant dependence coefficients, and finally in Sect. 4.4 we give an application to moving averages of our main result.

2. A note on weighted sums of associated random variables

In this chapter we prove the convergence of weighted sums of associated random variables normalized by $n^{1/p}$, $p \in (1, 2)$, assuming the existence of moments somewhat larger than p, depending on the behaviour of the weights, improving on previous results by getting closer to the moment assumption used for the case of constant weights. Besides moment conditions we assume a convenient behaviour either on truncated covariances or on joint tail probabilities.

Our results extend analogous characterizations known for sums of independent or negatively dependent random variables.

2.1 Framework and preliminaries

Let X_n, $n \geq 1$, be a sequence of random variables and define partial sums $S_n = \sum_{i=1}^n X_i$ and weighted partial sums $T_n = \sum_{i=1}^n a_{n,i} X_i$, where $a_{n,i} \geq 0$, $i \leq n$, $n \geq 1$. The variables X_n, $n \geq 1$, are assumed to be positively associated (or simply 'associated' in our framework), that is, for any $m \geq 1$ and any two real-valued coordinatewise nondecreasing functions f and g,

$$\mathrm{Cov}\Big(f\left(X_1, \ldots, X_m\right), g\left(X_1, \ldots, X_m\right)\Big) \geq 0,$$

whenever this covariance exists. It is well known that the covariance structure of associated random variables characterizes their asymptotics, so it is natural to seek assumptions on the covariances.

In this chapter we will be interested in the case where second order moments do not exist, so we will avoid using covariances directly, using them only through truncation. For this later argument, define, for each $v > 0$, the nondecreasing function $g_v(u) = \max(\min(u, v), -v)$, which performs the truncation at level v, and introduce, for each $n \geq 1$, the random variables $\bar{X}_n = g_v(X_n)$ and $\tilde{X}_n = X_n - \bar{X}_n$.

It is easily checked that both these families of random variables are associated, as they are nondecreasing transformations of the original ones. Define next the weighted sums of the truncated variables: for each $n \geq 1$, $\bar{T}_n = \sum_{i=1}^n a_{n,i}(\bar{X}_i - \mathrm{E}\bar{X}_i)$ and $\tilde{T}_n = \sum_{i=1}^n a_{n,i}(\tilde{X}_i - \mathrm{E}\tilde{X}_i)$, and the maxima $T_n^* = \max_{k\leq n} |T_k|$ and $\bar{T}_n^* = \max_{k\leq n} \left|\bar{T}_k\right|$. To handle covariances define, for each $i, j \geq 1$, $\Delta_{i,j}(x,y) = \mathrm{P}(X_i \geq x, X_j \geq y) - \mathrm{P}(X_i \geq x)\mathrm{P}(X_j \geq y)$. Of course, $\mathrm{Cov}(X_i, X_j) = \int_{\mathbb{R}^2} \Delta_{i,j}(x,y)\, dxdy$. Moreover,

$$G_{i,j}(v) = \mathrm{Cov}(\bar{X}_i, \bar{X}_j) = \int_{[-v,v]^2} \Delta_{i,j}(x,y)\, dxdy. \qquad (2.1)$$

The control of moments of maxima of partial sums is a crucial argument throughout. For nonweighted sums it was proved by Newman and Wright [34] that $\mathrm{E}\left(\max_{k\leq n} S_k^2\right) \leq \mathrm{E}S_n^2$.

This maximal inequality is one of the key ingredients used by Louhichi [26] to control tail probabilities of maxima of sums of associated random variables and then prove that $n^{-1/p}S_n \longrightarrow 0$ a.s., where $p \in [1,2)$ when one only has p-th order moments. For weighted sums, the following extension of this maximal inequality was proved by Oliveira [38].

Lemma 2.1.1. *Let X_n, $n \geq 1$, be centered and associated random variables. Assume the coefficients are such that*

$$a_{n,i} \geq 0, \qquad \textit{and} \qquad a_{n,i} \geq a_{n-1,i}, \quad i < n,\ n \geq 1. \qquad (2.2)$$

Then $\mathrm{E}\left(\max_{k\leq n} T_k^2\right) \leq \mathrm{E}(T_n^2)$.

We will need some more assumptions on the weights. Define, for each $\alpha > 0$, $A_{n,\alpha}^\alpha = n^{-1}\sum_{i=1}^n |a_{ni}|^\alpha$. These coefficients are considered in [2, 3, 10, 15, 24, 44, 51], assuming them to be either bounded or convergent.

Finally, we recall the following extension of Lemma 1 in Louhichi [26] proved by Oliveira [38].

Lemma 2.1.2. *Let X_n, $n \geq 1$, be centered and identically distributed associated random variables and assume the weights satisfy (2.2). Then, for every $\alpha > 1$, $x \in \mathbb{R}$ and $v > 0$,*

$$\begin{aligned} \mathrm{P}(T_n^* > x) \;\leq\; & \frac{8}{x^2} n^{1+2/\alpha} A_{n,\alpha}^2 \mathrm{E}\left(X_1^2 \mathbb{I}_{|X_1|\leq v}\right) + \frac{8}{x^2} n^{1+2/\alpha} A_{n,\alpha}^2 v^2 \mathrm{P}(|X_1| > v) \\ & + \frac{16}{x^2} n^{2/\alpha} A_{n,\alpha}^2 \sum_{1\leq i<j\leq n} G_{i,j}(v) + \frac{4}{x} n A_{n,\alpha} \mathrm{E}\left(|X_1|\, \mathbb{I}_{|X_1|>v}\right). \end{aligned} \qquad (2.3)$$

Proof. Taking into account

$$T_n^* \leq \bar{T}_n^* + \sum_{i=1}^{n} a_{n,i} \left(\left| \tilde{X}_i \right| + \mathrm{E} \left| \tilde{X}_i \right| \right)$$

and Markov's inequality, we have

$$\begin{aligned} \mathrm{P}(T_n^* > x) &\leq \mathrm{P}\left(\bar{T}_n^* > \frac{x}{2}\right) + \frac{4}{x}\sum_{i=1}^{n} a_{n,i}\mathrm{E}\left(\left|\tilde{X}_i\right|\right) \\ &\leq \frac{4}{x^2}\mathrm{E}\left((\bar{T}_n^*)^2\right) + \frac{4}{x}\sum_{i=1}^{n} a_{n,i}\mathrm{E}\left(\left|\tilde{X}_i\right|\right) \\ &\leq \frac{8}{x^2}\mathrm{E}\left(\bar{T}_n^2\right) + \frac{4}{x}\sum_{i=1}^{n} a_{n,i}\mathrm{E}\left(\left|\tilde{X}_i\right|\right). \end{aligned}$$

Remembering $\max_{i\leq n} |a_{n,i}| \leq n^{1/\alpha} A_{n,\alpha}$ and $\sum_{i=1}^n |a_{n,i}| \leq nA_{n,\alpha}$, the last term is bounded above by

$$\frac{4}{x}\mathrm{E}\left(\left|\tilde{X}_1\right|\right)\sum_i a_{n,i} \leq \frac{4}{x}\mathrm{E}\left(|X_1|\, \mathbb{I}_{|X_1|>v}\right) nA_{n,\alpha}.$$

For the upper bound of $\mathrm{E}\left(\bar{T}_n^2\right)$ use

$$\begin{aligned} \mathrm{E}\left(\bar{T}_n^2\right) &= \sum_{i,j=1}^{n} a_{n,i}a_{n,j}G_{i,j}(v) \\ &\leq \max_{i\leq n} a_{n,i}^2 \sum_{i,j=1}^{n} G_{i,j}(v) \leq n^{2/\alpha}A_{n,\alpha}^2 \sum_{i,j=1}^{n} G_{i,j}(v). \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &\sum_{i,j=1}^{n} G_{i,j}(v) \leq n\mathrm{E}\left(\bar{X}_1^2\right) + 2\sum_{1\leq i<j\leq n} G_{i,j}(v) \\ &\quad = n\mathrm{E}\left(X_1^2\mathbb{I}_{|X_1|\leq v}\right) + nv^2\mathrm{P}(|X_1| > v) + 2\sum_{1\leq i<j\leq n} G_{i,j}(v). \end{aligned}$$

□

2.2 Some Marcinkiewicz-Zygmund strong laws

We now prove the almost sure convergence of $n^{-1/p}T_n$ based on the Borel-Cantelli Lemma. Instead of considering T_n directly, we replace it by the larger T_n^*, which is an increasing sequence. For this increasing sequence T_n^*, the use of the Borel-Cantelli Lemma may be reduced to proving $\sum_n n^{-1}\mathrm{P}(T_n^* > \varepsilon n^{1/p}) < \infty$ (see, for example, Yang, Su and Yu [55]).

Theorem 2.2.1. *Let X_n, $n \geq 1$, be centered and identically distributed associated random variables. Let $p \in (1,2)$. Assume the weights satisfy (2.2) and $\sup_{n\geq 1} A_{n,\alpha} < \infty$, for some $\alpha > \frac{2p}{2-p}$. Further, assume that $\mathrm{E}\,|X_1|^{p\frac{\alpha-2}{\alpha-2p}} < \infty$. If*

$$\sum_{1\leq i<j<\infty} \int_{j^{(\alpha-2p)/(\alpha p)}}^{\infty} v^{-2\frac{\alpha-p}{\alpha-2p}-1}\, G_{i,j}(v)\, dv < \infty, \tag{2.5}$$

then $n^{-1/p}\, T_n \longrightarrow 0$ almost surely.

Proof. The proof follows similar arguments as in Theorem 4.1 in Oliveira [38]. Taking into account (2.3), with $v = n^{1/q}$, where q is to be specified later, we find that

$$\begin{aligned}
\frac{1}{n}\mathrm{P}(T_n^* > \varepsilon n^{1/p}) \quad \leq \quad & \frac{8n^{2/\alpha-2/p}}{\varepsilon^2} A_{n,\alpha}^2 \mathrm{E}\left(X_1^2 \mathbb{I}_{|X_1|\leq n^{1/q}}\right) \\
& +\frac{8n^{2/\alpha-2/p+2/q}}{\varepsilon^2} A_{n,\alpha}^2 \mathrm{P}(|X_1| > n^{1/q}) \\
& +\frac{16n^{2/\alpha-2/p-1}}{\varepsilon^2} A_{n,\alpha}^2 \sum_{1\leq i<j\leq n} G_{i,j}(n^{1/q}) \\
& +\tfrac{4n^{-1/p}}{\varepsilon} A_{n,\alpha} \mathrm{E}\left(|X_1|\, \mathbb{I}_{|X_1|>n^{1/q}}\right).
\end{aligned}$$

The remaining argument is to prove that this upper bound defines a convergent series. Taking into account that $A_{n,\alpha}$ is bounded, we may drop these terms. Notice that $\alpha > \frac{2p}{2-p}$ is equivalent to $\frac{2}{\alpha} - \frac{2}{p} < -1$. Using Fubini's Theorem we easily find that:

$$\sum_{n=1}^{\infty} n^{2/\alpha-2/p}\mathrm{E}\left(X_1^2\mathbb{I}_{|X_1|^q\leq n}\right) = \mathrm{E}\left(X_1^2 \sum_{n=|X_1|^q}^{\infty} n^{2/\alpha-2/p}\right) \leq c_1\mathrm{E}\left|X_1\right|^{q(1+2/\alpha-2/p)+2},$$

$$\sum_{n=1}^{\infty} n^{2/\alpha-2/p+2/q}\mathrm{E}\left(\mathbb{I}_{|X_1|^q>n}\right) = \mathrm{E}\left(\sum_{n=1}^{|X_1|^q} n^{2/\alpha-2/p+2/q}\right) \leq c_2\mathrm{E}\left|X_1\right|^{q(1+2/\alpha-2/p)+2},$$

$$\sum_{n=1}^{\infty} n^{-1/p}\mathrm{E}\left(|X_1|\,\mathbb{I}_{|X_1|^q>n}\right) \leq c_3\mathrm{E}\left|X_1\right|^{q(1-1/p)+1}. \tag{2.6}$$

The constants c_1, c_2 and c_3 used above only depend on p, q and α. As $1+\frac{2}{\alpha}-\frac{2}{p}<0$, in order to consider the lowest moment assumption possible on the variables, the first two terms above imply that we want to choose q as large as possible. On the other hand, as $1-\frac{1}{p}>0$, the last term implies that we should choose q as small as possible. It is clear that for small values of q we have $q\left(1-\frac{1}{p}\right)+1<q\left(1+\frac{2}{\alpha}-\frac{2}{p}\right)+2$, so we choose q such that these two expressions coincide, that is, $q=\frac{\alpha p}{\alpha-2p}$. Notice that $\alpha>\frac{2p}{2-p}$, with $p\in(1,2)$, implies that $\alpha>2p$, so the above choice for q is positive. It is now straightforward to verify that the moments considered above are of order $p\frac{\alpha-2}{\alpha-2p}$, thus finite.

Finally we control the term depending on the covariances. Again, using Fubini's Theorem we may write

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{2/\alpha-2/p-1}\sum_{1\leq i<j\leq n} G_{i,j}(n^{1/q}) \\
&\quad= \sum_{1\leq i<j<\infty}\iint\sum_{n>j} n^{2/\alpha-2/p-1}\,\mathbb{I}_{n>\max(|x|^q,|y|^q,j)}\,\Delta_{i,j}(x,y)\,dxdy \\
&\quad\leq c_4\sum_{1\leq i<j<\infty}\iint\Big(\max(|x|^q,|y|^q,j)\Big)^{2/\alpha-2/p}\Delta_{i,j}(x,y)\,dxdy \\
&\quad= c_4\sum_{1\leq i<j<\infty}\iint\int_0^{j^{2/\alpha-2/p}}\mathbb{I}_{|x|\leq u^{-\frac{\alpha p}{2q(\alpha-p)}}}\,\mathbb{I}_{|y|\leq u^{-\frac{\alpha p}{2q(\alpha-p)}}}\,du\,\Delta_{i,j}(x,y)\,dxdy \\
&\quad= \frac{2q(\alpha-p)c_4}{\alpha p}\sum_{1\leq i<j<\infty}\int_{j^{1/q}}^{\infty} v^{-2q\frac{\alpha-p}{\alpha p}-1}G_{i,j}(v)\,dv<\infty,
\end{aligned} \tag{2.7}$$

taking into account (2.5), where c_4 depends only on p and α, so the proof is concluded. □

Remark 2.2.2. *Notice that* $\alpha > \frac{2p}{2-p}$, *as assumed in Theorem 2.2.1, implies that* $p\frac{\alpha-2}{\alpha-2p} < 2$, *thus we are still not assuming second order moments.*

Remark 2.2.3. *In Theorem 4.1 in Oliveira [38] the moment considered was* $p\frac{\alpha+2}{\alpha}$. *It is easily seen that* $\alpha > \frac{2p}{2-p}$ *implies that* $p\frac{\alpha+2}{\alpha} > p\frac{\alpha-2}{\alpha-2p}$, *thus we are improving somewhat the moment assumption. As what regards the integrability assumption (2.5), in [38] the exponent of the polynomial term in the integrand was* $-3+2\frac{p}{\alpha} > -2\frac{\alpha-p}{\alpha-2p}-1$, *thus the present integrability assumption is a little stronger. The difference between these exponents is equal to* $4p\frac{p-\alpha}{\alpha(\alpha-2p)}$, *thus of order* α^{-1}.

Remark 2.2.4. *To compare this result with Louhichi's [26] conditions for nonweighted sums, notice that allowing* $\alpha \longrightarrow \infty$ *in the assumptions of Theorem 2.2.1 we are lead to assume the existence of p-th order moments and the exponent in the integrability condition converges to* -3, *that is, we find the assumptions of Theorem 1 in [26].*

It is easy to adapt the integrability assumption (2.5) to the case where the random variables are stationary.

Corollary 2.2.5. *Let* X_n, $n \geq 1$, *be centered and stationary associated random variables. Let* $p \in (1,2)$. *Assume the weights satisfy (2.2) and* $\sup_{n\geq 1} A_{n,\alpha} < \infty$, *for some* $\alpha > \frac{2p}{2-p}$. *Put* $\beta = \frac{2(\alpha-p)}{\alpha-2p}+1$. *If* $\mathrm{E}\,|X_1|^{p\frac{\alpha-2}{\alpha-2p}} < \infty$ *and*

$$\sum_{n=1}^{\infty} \int_{(n+1)^{(\alpha-2p)/(\alpha p)}}^{\infty} v^{\frac{\alpha p}{\alpha-2p}-\beta}\, G_{0,n}(v)\, dv < \infty, \tag{2.8}$$

then $n^{-1/p}\,T_n \longrightarrow 0$ *almost surely.*

We present next an application of the above result, extending Corollary 4 of Louhichi [26]. Let ε_n, $n \in \mathbb{Z}$, be stationary, centered and associated random variables, ϕ_n, $n \geq 0$, positive real numbers and define $X_n = \sum_{i=0}^{\infty} \phi_i \varepsilon_{n-i}$. The random variables X_n are associated and stationary. If the variables ε_n have finite moments of order s, $\sum_{i=0}^{\infty} \phi_i^{\rho s} < \infty$ and $\sum_{i=0}^{\infty} \phi_i^{(1-\rho)s/(s-1)} < \infty$, for some $\rho \in (0,1)$, then, using Hölder inequality, it follows $\mathrm{E}\,|X_n|^s < \infty$. Write now $U_n = \sum_{i=0}^{n} \phi_i \varepsilon_{n-i}$ and $V_n = \sum_{i=n+1}^{\infty} \phi_i \varepsilon_{n-i}$. Then

$$\begin{aligned} G_{0,n}(v) &= \mathrm{Cov}(g_v(X_0), g_v(X_n)) \\ &= \mathrm{Cov}(g_v(X_0), g_v(U_n+V_n)-g_v(V_n)) + \mathrm{Cov}(g_v(X_0), g_v(V_n)). \end{aligned}$$

Taking into account that $v > 0$ and $|g_v(y)| \leq |y|$ it follows that, given $\gamma \in (0,1)$,

$$\begin{aligned} G_{0,n}(v) &\leq 2\left(\mathrm{E}\,|g_v(X_0)V_n| + \mathrm{E}\,|g_v(X_0)|\,\mathrm{E}\,|V_n|\right) \\ &\leq 2\left(\mathrm{E}(\min(v,|X_0|)\,|V_n|) + \mathrm{E}(\min(v,|X_0|))\mathrm{E}\,|V_n|\right) \\ &\leq 2v^\gamma\left(\mathrm{E}(|X_0|^{1-\gamma}\,|V_n|) + \mathrm{E}\,|X_0|^{1-\gamma}\,\mathrm{E}\,|V_n|\right). \end{aligned}$$

Using now Hölder inequality for a suitable $r > 1$, it follows that

$$G_{0,n}(v) \leq 4v^\gamma(\mathrm{E}\,|V_n|^r)^{1/r}(\mathrm{E}\,|X_0|^{(1-\gamma)r/(r-1)})^{(r-1)/r}.$$

It is easily verified that

$$\mathrm{E}\,|V_n|^r \leq \left(\sum_{i=n+1}^{\infty} \phi_i^{(1-\rho)r/(r-1)}\right)^{(r-1)/r} \left(\sum_{i=n+1}^{\infty} \phi_i^{\rho r}\right) \mathrm{E}\,|\varepsilon_0|^r.$$

Assume that the moments of X_0 and ε_0 above are finite. Then, with β defined as in Corollary 2.2.5, the following upper bound holds,

$$\sum_{n=1}^{\infty} \int_{n^{(\alpha-2p)/(\alpha p)}}^{\infty} v^{\frac{\alpha p}{\alpha-2p}-\beta}\, G_{0,n}(v)\, dv \leq c'\,(\mathrm{E}\,|V_n|^r)^{1/r} \sum_{n=1}^{\infty} \int_{n^{(\alpha-2p)/(\alpha p)}}^{\infty} v^{\frac{\alpha p}{\alpha-2p}-\beta+\gamma}\, dv.$$

If $\gamma < \frac{2(\alpha-p)-\alpha p}{\alpha-2p}$, so that the integrals above converge, it follows that (2.8) holds whenever,

$$\begin{aligned} &\sum_{n=1}^{\infty} \int_{n^{(\alpha-2p)/(\alpha p)}}^{\infty} v^{\frac{\alpha p}{\alpha-2p}-\beta}\, G_{0,n}(v)\, dv \\ &\quad\leq c_1' \sum_{n=1}^{\infty} n^{\gamma\frac{\alpha-2p}{\alpha p}-\frac{2(\alpha-p)}{\alpha p}+1} \left(\sum_{i=n+1}^{\infty} \phi_i^{(1-\rho)r/(r-1)}\right)^{(r-1)/r^2} \left(\sum_{i=n+1}^{\infty} \phi_i^{\rho r}\right)^{1/r} (\mathrm{E}\,|\varepsilon_0|^r)^{1/r} < \infty. \end{aligned}$$

So, finally, the above condition implies that $n^{-1/p}\sum_{i=1}^{n} a_{n,i}X_i \longrightarrow 0$ for every choice of weights satisfying (2.2) and $\sup_{n\geq 1} A_{n,\alpha} < \infty$. If we assume that $\phi_n \sim n^{-a}$, for some $a > 1$, $\mathrm{E}\,|\varepsilon_0|^r < \infty$ and choose $\rho \in (1/(ar), 1-(r-1)/(ar))$, both the series above defined using the coefficients ϕ_n are convergent and then (2.8) is satisfied if we can choose $\gamma \in (0,1)$ such that

$$\gamma\frac{\alpha-2p}{\alpha p} - \frac{2(\alpha-p)}{\alpha p} - a\left(\frac{1-\rho}{r}+\rho\right) + \frac{2r-1}{r^2} < -2.$$

Choosing $r = p(\alpha-2)/(\alpha-2p)$, meaning the existence of the moment assumed to be finite in Theorem 2.2.1 and Corollary 2.2.5, the condition rewrites as

$$\gamma\frac{\alpha-2p}{\alpha p} - a\left(\frac{(1-\rho)(\alpha-2p)}{p(\alpha-2)}+\rho\right) < \frac{2(\alpha-p)}{\alpha p} - 2 - \frac{2(\alpha-2p)}{p(\alpha-2)} + \frac{(\alpha-2p)^2}{p^2(\alpha-2)^2}.$$

Allowing $\alpha \longrightarrow +\infty$, which corresponds to the case studied in Louhichi [26], means that we should find $a > \frac{p(\gamma+2)-1}{p(1-\rho+p\rho)}$. The most favorable choice is $\rho = 1$. The condition that follows on the convergence for the coefficients ϕ_n defining the moving average is somewhat stronger than what is assumed in Corollary 4 in Louhichi [26], which is $a > 2 - 1/p$, which is essentially what corresponds to the choice $\rho = 0$. But this stronger assumption is due to the fact that we are assuming the ε_n to be dependent, so an extra effort must be made in order to control the moments of the variables X_n.

The statement of Theorem 2.2.1 assumes a moment condition and adjusts the integrability condition on the truncated covariances to get the convergence. One may be interested in doing the opposite, that is, assume an integrability condition on the truncated variables and describe which moments should be required. Assume that for some $\beta > 0$ and a suitable $q > 0$ we have

$$\sum_{1\leq i<j<\infty} \int_{j^{1/q}}^{\infty} v^{-\beta}\, G_{i,j}(v)\, dv < \infty. \tag{2.9}$$

We now choose q conveniently. Comparing with (2.7) we need that $2q\frac{\alpha-p}{\alpha p}+1 \geq \beta$ or, equivalently, $q \geq \frac{p\alpha(\beta-1)}{2(\alpha-p)}$. Assume that $\alpha > \frac{2p}{2-p}$, which is equivalent to $\frac{2}{\alpha} - \frac{2}{p} < -1$ and implies that $\alpha > 2p$. So, if $\beta \in [0,1]$ the above condition is verified for every choice of $q > 0$, thus, as seen in the proof of Theorem 2.2.1, the choice $q^* = \frac{\alpha p}{\alpha-2p}$ optimizes the moment assumption, requiring the existence of the absolute moment of order $p^* = p\frac{\alpha-2}{\alpha-2p}$. Because of the integration region in (2.9) we need to assume that $q \geq q^*$. If $\beta > 1$, we look at $\frac{\alpha p}{\alpha-2p} - \frac{p\alpha(\beta-1)}{2(\alpha-p)}$. As we assumed that $\alpha > 2p$ it is easily seen that the sign of this difference is equal to the sign of $(3-\beta)\alpha - 2p(2-\beta)$. If $\beta \in (1,2]$ this means that the sign is positive if $\alpha > 2p\frac{2-\beta}{3-\beta} = 2p\left(1 - \frac{1}{3-\beta}\right)$ which always holds. Thus, the optimization of the moments is achieved by the choice $q^* = \frac{\alpha p}{\alpha-2p}$. If $\beta \in (2,3]$ the above difference is always nonnegative, so we choose again $q^* = \frac{\alpha p}{\alpha-2p}$. Now, if $\beta > 3$, $\frac{\alpha p}{\alpha-2p} - \frac{p\alpha(\beta-1)}{2(\alpha-p)} \geq 0$ is equivalent to $\alpha \leq 2p\frac{\beta-2}{\beta-3} = 2p\left(1 + \frac{1}{\beta-3}\right)$. So, when $\beta > 3$, if $2p < \alpha \leq 2p\left(1 + \frac{1}{\beta-3}\right)$ we should also choose $q^* = \frac{\alpha p}{\alpha-2p}$. Finally, when $\beta > 3$ and $\alpha > 2p\left(1 + \frac{1}{\beta-3}\right)$ we must assume the finiteness of the largest of the moments appearing in (2.6), where q^* is taken to be $\frac{p\alpha(\beta-1)}{2(\alpha-p)}$. Thus we have proved the following statement.

Theorem 2.2.6. *Let X_n, $n \geq 1$, be centered and identically distributed associated random variables. Assume the weights satisfy (2.2) and $\sup_{n\geq 1} A_{n,\alpha} < \infty$. Further, assume that $p \in (1,2)$ and $\alpha > \frac{2p}{2-p}$ are satisfied.*

Define q^* and p^* as

- *if $\beta \leq 3$ or if $\beta > 3$ and $\alpha \in \left(2p,\, 2p\left(1+\frac{1}{\beta-3}\right)\right]$, $q^* = \frac{\alpha p}{\alpha-2p}$ and $p^* = p\frac{\alpha-2}{\alpha-2p}$,*
- *if $\beta > 3$ and $\alpha > 2p\left(1+\frac{1}{\beta-3}\right)$, $q^* = \frac{p\alpha(\beta-1)}{2(\alpha-p)}$ and $p^* = 1+\frac{\alpha(\beta-1)(p-1)}{2(\alpha-p)}$.*

If (2.9) is satisfied with $q \geq q^*$ and $\mathrm{E}\,|X_1|^{p^*} < \infty$ then $n^{-1/p}T_n \longrightarrow 0$ almost surely.

We will now look for assumptions on the functions $\Delta_{i,j}$ rather than on the truncated covariances. Remark that the $\Delta_{i,j}$ may also be interpreted as covariances: $\Delta_{i,j} = \mathrm{Cov}\left(\mathbb{I}_{[x,+\infty)}(X_i), \mathbb{I}_{[y,+\infty)}(X_j)\right)$. It follows from Sadikova [47] that, if the random variables have bounded density and covariances do exist that $\Delta_{i,j}(x,y) \leq c\mathrm{Cov}^{1/3}(X_i,X_j)$, where $c>0$ is a constant depending only on the density function. This made natural to seek for assumptions on the $\Delta_{i,j}$ while studying the asymptotics of empirical processes based on associated random variables, as in Yu [56], Shao and Yu [48] or Oliveira and Suquet [40, 42]. Moreover, the $\Delta_{i,j}(x,y)$ play, in dimension two, the role of the tail probabilities usually considered in the one dimensional framework. So, we will now consider the following assumption on the limit behaviour of $\Delta_{i,j}$:

$$\sup_{i,j\geq 1} \Delta_{i,j}(x,y) = O\left(\max(|x|,|y|)^{-a}\right), \qquad \text{as } \max(|x|,|y|) \longrightarrow +\infty. \tag{2.10}$$

Thus, outside of some $[-j_0,j_0]^2$ we may assume that all the $\Delta_{i,j}$ are, up to the multiplication by some constant c_0, that does not depend on i or j, bounded above by $\max(|x|,|y|)^{-a}$. Thus

$$G_{i,j}(v) \leq 4j_0^2 + 4c_0 \int_{j^{1/q}}^{\infty}\int_{-x}^{x} x^{-a}\,dy\,dx = 4j_0^2 + \frac{4c_0}{2-a}\left(v^{2-a} - j_0^{2-a}\right). \tag{2.11}$$

Remember that $\mathrm{Cov}(X_i,X_j) = G_{i,j}(+\infty)$. Looking at the expression above, if we allow $v \longrightarrow +\infty$ we have convergence to a finite limit whenever $a>2$. Thus, the most interesting case for us corresponds to $0<a\leq 2$, so that we do not have finite covariances between the random variables.

Theorem 2.2.7. *Let X_n, $n\geq 1$, be centered and identically distributed associated random variables. Assume the weights satisfy (2.2) and $\sup_{n\geq 1} A_{n,\alpha} < \infty$. Let $p\in(1,2)$ and $\alpha > \frac{2p}{2-p}$. Assume that (2.10) is satisfied for some $a\in(0,2]$ and (2.9) holds for some $q>0$ and $\beta > 3-a+2q$. If $\mathrm{E}\,|X_1|^{p^*} < \infty$, where $p^* = \max\left(q\left(1+\frac{2}{\alpha}-\frac{2}{p}\right)+2, q\left(1-\frac{1}{p}\right)+1\right)$ then $n^{-1/p}T_n \longrightarrow 0$ almost surely.*

Proof. Using (2.11) to compute the integral in (2.9), one easily finds that, as $\beta > 3 - a + 2q > 3 - a$,

$$\int_{j^{1/q}}^{\infty} v^{-\beta} G_{i,j}(v)\, dv \leq c_0' j^{(1-\beta)/q} + j^{3-(\beta+a)} q,$$

where c_0' does not depend on i or j. Thus inserting this upper bound in (2.9) and taking into account the summation, we have a convergent series if both $1 + \frac{1-\beta}{q} < -1$ and $1 + \frac{3-(\beta+a)}{q} < -1$. But these two inequalities follow from $\beta > 3 - a + 2q$. As the summations in (2.6) are finite due to our moment assumptions, the proof is concluded. □

Remark 2.2.8. *The above statement allows to consider $\beta < 3$ in (2.9). This was out of reach in Theorem 2.2.1. However, the moment assumed to be finite is of order $p^* = \max\left(q\left(1 + \frac{2}{\alpha} - \frac{2}{p}\right) + 2, q\left(1 - \frac{1}{p}\right) + 1\right)$. It is easily seen that if $q > \frac{\alpha p}{\alpha - 2p}$, then $p^* = q\left(1 - \frac{1}{p}\right) + 1$. The difference between this order and the one considered in Theorem 2.2.1 has the same sign as $(\alpha - 2p)q - p\alpha \geq 0$ for the range of values for q where this applies. Likewise, if $q < \frac{\alpha p}{\alpha - 2p}$ then the difference of order moments has the same sign as $(p-2)\alpha^2 + 2p\alpha(3-p) - 2p^2 > 0$ for the range of values for q considered. Thus, the moment condition assumed in Theorem 2.2.7 is always stronger than the one in Theorem 2.2.1.*

3. Convergence rates for weighted sums of associated variables

In this chapter we study the convergence of weighted sums of associated random variables assuming only the existence of moments of order $p < 2$. We use a truncation technique together with coupling with independent variables, which allows a relaxation of the assumptions on the weights. Moreover, this coupling allows not only for the proof of almost sure results and but enables to identify convergence rates. The assumptions on p, that now include the case $p < 1$, excluded from earlier results for positively associated variables, depend on the asymptotic behaviour of the weights, as usual. We give a direct comparison with the characterizations previously available, showing that the scope of applicability of our results does not overlap with known conditions for the same asymptotic results.

Our frame work is as follow: In Sect. 3.1 we describe the framework and useful results, Sect. 3.3 presents the main results, and compares with results in Çağın and Oliveira [13], while Sect. 3.2 states versions of these results in a reduced setting, but proves the main steps for the final theorems.

3.1 Definitions and preliminary results

Let us assume that the X_n, $n \geq 1$, are centered and associated random variables and denote $S_n = X_1 + \cdots + X_n$. Let $a_{n,i}$, $i = 1, \ldots, n$, $n \geq 1$, be non negative real numbers and define, for some $\alpha > 1$, $A_{n,\alpha}^{\alpha} = n^{-1} \sum_{i=1}^{n} |a_{n,i}|^{\alpha}$. We will be interested in the convergence of $T_n = \sum_{i=}^{n} a_{n,i} X_i$ assuming that

$$A_\alpha = \sup_n A_{n,\alpha} < \infty. \tag{3.1}$$

This is the only condition on the weights throughout this chapter, thus relaxing the assumption on the weights when compared to Oliveira [38] or Çağın and Oliveira [13]. Remark that, due to the nonnegativity of the weights, the variables T_n, $n \geq 1$, are associated. Define the usual Cox-Grimmett coefficients

$$u(n) = \sup_{k\geq 1} \sum_{j:|k-j|\geq n} \mathrm{Cov}(X_j, X_k). \tag{3.2}$$

If the random variables are stationary, then $u(n) = 2\sum_{j=n+1}^{\infty} \mathrm{Cov}(X_1, X_j)$.

Consider p_n a sequence of natural numbers such that $p_n < \frac{n}{2}$, r_n the largest integer less or equal to $\frac{n}{2p_n}$, and define the variables

$$Y_{j,n} = \sum_{i=(j-1)p_n+1}^{jp_n} a_{n,i} X_i, \qquad j = 1, \ldots, 2r_n.$$

These random variables are associated, due to the nonnegativity of the weights. Moreover, if the variables X_n are uniformly bounded by $c > 0$, then it is obvious that $|Y_{j,n}| \leq cA_\alpha n^{1/\alpha} p_n$. Finally, put

$$T_{n,od} = \sum_{j=1}^{r_n} Y_{2j-1,n} \quad \text{and} \quad T_{n,ev} = \sum_{j=1}^{r_n} Y_{2j,n}.$$

We prove first an easy but useful upper bound.

Lemma 3.1.1. *Assume the variables X_n, $n \geq 1$, are associated, stationary, centered, bounded (by $c > 0$) and $u(0) < \infty$. Then $\mathrm{E}(S_n^2) \leq 2c^* n$, where $c^* = c^2 + u(0)$.*

Proof. Using the stationarity, it follows easily that

$$\mathrm{E}(S_n^2) = n\mathrm{Var}(X_1) + 2\sum_{j=1}^{n-1} (n-j)\mathrm{Cov}(X_1, X_{j+1}) \leq 2nc^2 + 2nu(0)$$

. □

The next result is an extension of Lemma 3.1 in Oliveira [37].

Lemma 3.1.2. *Assume the variables X_n, $n \geq 1$, are centered, associated, stationary, bounded (by $c > 0$), $u(0) < \infty$ and the nonnegative weights satisfy (3.1). If $d_n \geq 1$ and $0 < \lambda < \frac{d_n - 1}{d_n} \frac{1}{cA_\alpha n^{1/\alpha} p_n}$, then*

$$\prod_{j=1}^{r_n} \mathrm{E}\left(e^{\lambda Y_{2j-1,n}}\right) \leq \exp\left(\lambda^2 c^* A_\alpha^2 n^{1+2/\alpha} d_n\right)$$

and

$$\prod_{j=1}^{r_n} \mathrm{E}\left(e^{\lambda Y_{2j,n}}\right) \leq \exp\left(\lambda^2 c^* A_\alpha^2 n^{1+2/\alpha} d_n\right).$$

where $c^* = c^2 + u(0)$.

Proof. As remarked above, as the variables X_n are bounded, we have that $|Y_{j,n}| \le cA_\alpha n^{1/\alpha} p_n$. So, using a Taylor expansion it follows that

$$\mathrm{E}\left(e^{\lambda Y_{2j-1,n}}\right) \le 1 + \lambda^2 \mathrm{E}\left(Y_{2j-1,n}^2\right) \sum_{k=2}^{\infty} (cA_\alpha \lambda n^{1/\alpha} p_n)^{k-2}.$$

Now, $\mathrm{E}\left(Y_{2j-1,n}^2\right) = \sum_{\ell,\ell'} a_{n,l} a_{n,\ell'} \mathrm{Cov}(X_\ell, X_{\ell'}) \le n^{2/\alpha} A_\alpha^2 \mathrm{E}(S_{p_n}^2)$, due to the stationarity and the nonnegativity of the weights and covariances. So, applying Lemma 3.1.1, it follows that

$$\mathrm{E}\left(e^{\lambda Y_{2j-1,n}}\right) \le 1 + \frac{2\lambda^2 c^* A_\alpha^2 n^{2/\alpha} p_n}{1 - cA_\alpha \lambda n^{1/\alpha} p_n} \le \exp\left(2\lambda^2 c^* A_\alpha^2 n^{2/\alpha} p_n d_n\right).$$

To conclude the proof multiply the upper bounds and remember $2r_n p_n \le n$. □

A basic tool for the analysis of convergence and rates is the following inequality due to Dewan and Prakasa Rao [17].

Theorem 3.1.3. *Assume $X_1, \ldots, X_n$ are centered, associated and uniformly bounded (by $c > 0$). Then, for every $\lambda > 0$,*

$$\left| \mathrm{E} e^{\lambda \sum_j X_j} - \prod_j \mathrm{E} e^{\lambda X_j} \right| \le \frac{1}{2} \lambda^2 e^{c\lambda n} \sum_{j \ne k} \mathrm{Cov}(X_j, X_k). \tag{3.3}$$

3.2 The case of uniformly bounded variables

We assume first that there exists some $c > 0$ such that, with probability 1, $|X_n| \le c$, for every $n \ge 1$. This allows for a direct use of the results proved above. We start by deriving an upper bound for the tail probabilities for the summations defined above.

Lemma 3.2.1. *Assume the variables X_n, $n \ge 1$, are centered, associated, stationary and bounded (by $c > 0$) and $u(0) < \infty$. If the nonnegative weights satisfy (3.1), $d_n \ge 1$ and $0 < \lambda < \frac{d_n - 1}{d_n} \frac{1}{cA_\alpha n^{1/\alpha} p_n}$, then, for every $\varepsilon > 0$ and n large enough,*

$$\begin{aligned} \mathrm{P}(T_{n,od} > n^{1/p}\varepsilon) \;\le\; & \frac{1}{4}\lambda^2 n^{1+2/\alpha} A_\alpha^2 \exp\left(\tfrac{1}{2} c n^{1+1/\alpha} A_\alpha \lambda - \lambda n^{1/p}\varepsilon\right) u(p_n) \\ & + \exp\left(\lambda^2 c^* A_\alpha^2 n^{1+2/\alpha} d_n - \lambda n^{1/p}\varepsilon\right). \end{aligned} \tag{3.4}$$

An analogous inequality for $\mathrm{P}(T_{n,ev} > n^{1/p}\varepsilon)$ also holds.

Proof. If we apply (3.3) to $T_{n,od}$ we find

$$\left|\mathrm{E}e^{\lambda T_{n,od}} - \prod_j \mathrm{E}e^{\lambda Y_{n,2j-1}}\right| \le \frac{1}{2}\lambda^2 \exp\left(cA_\alpha r_n p_n n^{1/\alpha}\lambda\right) \sum_{j\ne j'} \mathrm{Cov}(Y_{n,j}, Y_{n,j'}). \tag{3.5}$$

Now, it is obvious that each $0 \le a_{n,i} \le n^{1/\alpha} A_{n,\alpha}$, thus

$$\mathrm{Cov}(Y_{n,j}, Y_{n,j'}) \le \sum_{\ell,\ell'} a_{n,\ell}a_{n,\ell'}\mathrm{Cov}(X_\ell, X_{\ell'}) \le n^{2/\alpha}A_\alpha^2 \sum_{\ell,\ell'} \mathrm{Cov}(X_\ell, X_{\ell'}).$$

Put $Y_{n,j}^* = \sum_{\ell=(j-1)p_n+1}^{jp_n} X_\ell$, $j = 1, \ldots, r_n$. Then we have just verified that

$$\mathrm{Cov}(Y_{n,j}, Y_{n,j'}) \le n^{2/\alpha}A_\alpha^2 \mathrm{Cov}(Y_{n,j}^*, Y_{n,j'}^*).$$

Using twice the stationarity of the random variables we obtain

$$\sum_{j\ne j'} \mathrm{Cov}(Y_{n,j}^*, Y_{n,j'}^*) = 2\sum_{j=1}^{r_n-1}(r_n - j)\mathrm{Cov}(Y_{n,1}^*, Y_{n,2j-1}^*)$$

and

$$\begin{aligned}
&\mathrm{Cov}(Y_{n,1}^*, Y_{n,2j-1}^*)\\
&\quad\le \sum_{\ell=0}^{p_n-1}(p_n-\ell)\mathrm{Cov}(X_1, X_{2jp_n+\ell+1}) + \sum_{\ell=1}^{p_n-1}(p_n-\ell)\mathrm{Cov}(X_\ell, X_{2jp_n+1})\\
&\quad\le p_n \sum_{\ell=(2j-1)p_n+2}^{(2j+1)p_n} \mathrm{Cov}(X_1, X_\ell).
\end{aligned}$$

Inserting this inequality in (3.5) we find

$$\begin{aligned}
&\left|\mathrm{E}e^{\lambda T_{n,od}} - \prod_j \mathrm{E}e^{\lambda Y_{n,2j-1}}\right|\\
&\quad\le \frac{1}{2}\lambda^2 n^{2/\alpha}A_\alpha^2 r_n p_n \exp\left(\tfrac{1}{2}cn^{1+1/\alpha}A_\alpha\lambda\right)\sum_{\ell=p_n+2}^{2r_n-1}\mathrm{Cov}(X_1, X_\ell)\\
&\quad\le \frac{1}{4}\lambda^2 n^{1+2/\alpha}A_\alpha^2 \exp\left(\tfrac{1}{2}cn^{1+1/\alpha}A_\alpha\lambda\right) u(p_n+2).
\end{aligned}$$

We can now use this together with Markov's inequality to find that, for every $\varepsilon > 0$,

$$\begin{aligned}
\mathrm{P}(T_{n,od} > n^{1/p}\varepsilon) &\le e^{-\lambda n^{1/p}\varepsilon}\left|\mathrm{E}e^{\lambda T_{n,od}} - \prod_j \mathrm{E}e^{\lambda Y_{n,2j-1}}\right| + e^{-\lambda n^{1/p}\varepsilon}\prod_j \mathrm{E}e^{\lambda Y_{n,2j-1}}\\
&\le \frac{1}{4}\lambda^2 n^{1+2/\alpha}A_\alpha^2 \exp\left(\tfrac{1}{2}cn^{1+1/\alpha}A_\alpha\lambda - \lambda n^{1/p}\varepsilon\right) u(p_n+2)\\
&\qquad + \exp\left(\lambda^2 c^* A_\alpha^2 n^{1+2/\alpha}d_n - \lambda n^{1/p}\varepsilon\right),
\end{aligned}$$

and remember that $u(p_n+2) \le u(p_n)$, due to the nonnegativity of the covariances. □

3.2.1 Almost sure convergence

We prove two different versions of the almost sure of $n^{-1/p}T_n$, depending on the Cox-Grimmett coefficients being decreasing at polynomial or geometric rate.

Theorem 3.2.2. *Assume the random variables X_n, $n \geq 1$, are centered, associated, stationary and bounded (by $c > 0$). Assume that $p < 1$ and $\alpha > 1$ are such that $\frac{1}{p} - \frac{1}{\alpha} \geq 1$ and $u(n) \sim n^{-a}$, for some $a > 0$. If the nonnegative weights satisfy (3.1), then, with probability 1, $n^{-1/p}T_n \longrightarrow 0$.*

Proof. Consider the decomposition of T_n into the blocks $Y_{n,j}$ defined previously, taking $p_n \sim n^\theta$, for some $\theta \in (0,1)$. It is obviously enough to prove that both $n^{-1/p}T_{n,od}$ and $n^{-1/p}T_{n,ev}$ converge almost surely to 0. As these terms are analogous we will concentrate on the former, starting from (3.4). A minimization of the exponent on the second term of the upper bound in (3.4) leads to the choice

$$\lambda = \frac{\varepsilon}{2c^* A_\alpha^2} \frac{n^{1/p-1-2/\alpha}}{d_n}, \tag{3.6}$$

meaning that

$$\exp\left(\lambda^2 c^* A_\alpha^2 n^{1+2/\alpha} d_n - \lambda n^{1/p}\varepsilon\right) = \exp\left(-\frac{\varepsilon^2 n^{2/p-1-2/\alpha}}{4c^* A_\alpha^2 d_n}\right).$$

Assume that, for some $\beta > 1$,

$$\frac{\varepsilon^2 n^{2/p-1-2/\alpha}}{4c^* A_\alpha^2 d_n} = \beta \log n \quad \Leftrightarrow \quad d_n = \frac{\varepsilon^2}{4c^* A_\alpha^2 \beta} \frac{n^{2/p-1-2/\alpha}}{\log n}. \tag{3.7}$$

As $\frac{1}{p} - \frac{1}{\alpha} > 1$, it follows that, for n large enough, we have $d_n > 1$ as required in Lemma 3.1.2. In order to use Lemma 3.1.2 we also need to verify that the condition on λ is satisfied: $\lambda < \frac{d_n - 1}{d_n} \frac{1}{cA_\alpha n^{1/\alpha} p_n}$. Replacing the above choices for λ and d_n, this condition on λ is satisfied if

$$\varepsilon^{-1} \leq \frac{1}{2cA_\alpha \beta} \frac{n^{1/p-1/\alpha}}{n^\theta \log n}. \tag{3.8}$$

As $\theta < 1 \leq \frac{1}{p} - \frac{1}{\alpha}$ this upper bound grows to infinity, so this inequality is satisfied for n large enough.

We consider now the first term in (3.4), the term involving the Cox-Grimmett coefficients. The exponent in this term is

$$cn^{1+1/\alpha} A_\alpha \lambda - \lambda n^{1/p}\varepsilon = \frac{c\varepsilon}{2c^* A_\alpha} \frac{n^{1/p-1/\alpha}}{d_n} - \frac{\varepsilon^2}{2c^* A_\alpha^2} \frac{n^{2/p-1-2/\alpha}}{d_n}.$$

The second term above is, up to multiplication by 2, the exponent that was found after the optimization with respect to λ of the exponent on the second term of (3.4). So, to control the upper bound (3.4) we can factor this part of the exponential, leaving to control, after substituting the expression for d_n,

$$\frac{1}{4}\lambda^2 n^{1+2/\alpha} A_\alpha^2 \exp\left(\frac{2cA_\alpha\beta}{\varepsilon} n^{1/\alpha-1/p+1}\log n\right) u(p_n). \tag{3.9}$$

As the term that we factored defines a convergent series, it is enough to verify that (3.9) is bounded. Further, the polynomial term in (3.9) is clearly dominated by the exponential, thus we may drop it, verifying only that

$$\exp\left(\frac{2cA_\alpha\beta}{\varepsilon} n^{1/\alpha-1/p+1}\log n\right) u(p_n) \leq c_0, \tag{3.10}$$

for some $c_0 > 0$. Taking logarithms and taking into account the choice for $p_n \sim n^\theta$, the above inequality is equivalent to $\frac{2cA_\alpha\beta}{\varepsilon} n^{1/\alpha-1/p+1}\log n - a\theta\log n$ having a finite upper bound. But then, this a consequence of the assumption on p and α, as $\frac{1}{p}-\frac{1}{\alpha} \geq 1$ implies that the exponent on the first term is not positive, so this term converges to 0. □

We may relax somewhat the assumptions on p and α if the covariances decrease faster.

Theorem 3.2.3. *Assume the random variables X_n, $n \geq 1$, are centered, associated, strictly stationary and bounded (by $c > 0$). Assume that $p < 2$ and $\alpha > 1$ are such that $\frac{1}{p} - \frac{1}{\alpha} > \frac{1}{2}$ and $u(n) \sim \rho^{-n}$, for some $\rho \in (0, 1)$. If the nonnegative weights satisfy (3.1), then, with probability 1, $n^{-1/p}T_n \longrightarrow 0$.*

Proof. Follow the proof of Theorem 3.2.2, choosing $\max(0, \frac{1}{p}-\frac{1}{\alpha}+1) < \theta < \frac{1}{p}-\frac{1}{\alpha}$, until (3.10). Remark that the assumption on p and α ensures that such a choice for θ is possible. Now the boundedness required in (3.10) is equivalent to $\frac{2cA_\alpha\beta}{\varepsilon} n^{1/\alpha-1/p+1}\log n - n^\theta \log\rho$ being bounded above. But this follows from $\theta > \frac{1}{p} - \frac{1}{\alpha} + 1$ and $\rho \in (0, 1)$. □

3.2.2 Convergence rates

A small modification of the previous arguments allows, for the case of geometric decreasing Cox-Grimmett coefficients, the identification of a convergence rate for the almost sure convergence just proved.

Theorem 3.2.4. *Assume the random variables X_n, $n \geq 1$, are centered, associated, strictly stationary and bounded (by $c > 0$). Assume that $p < 2$ and $\alpha > 1$ are such that $\frac{1}{p} - \frac{1}{\alpha} > \frac{1}{2}$ and $u(n) \sim \rho^{-n}$, for some $\rho \in (0,1)$. If the nonnegative weights satisfy (3.1), then, with probability 1, $n^{-1/p}T_n \longrightarrow 0$ with convergence rate $\frac{\log n}{n^{1/p-1/\alpha-1/2-\delta}}$, for arbitrarily small $\delta > 0$.*

Proof. We again start as in the proof of Theorem 3.2.2 choosing $\theta = \frac{1}{2} + \delta$, with $0 < \delta < \frac{1}{p} - \frac{1}{\alpha} - \frac{1}{2}$ and $p_n \sim n^\theta$. Now, on (3.7), allow ε to depend on n:

$$\varepsilon_n^2 = \frac{4\beta c^* A_\alpha^2 d_n \log n}{n^{2/p-1-2/\alpha}}.$$

The verification of the assumptions of Lemma 3.1.2, given above by (3.8), becomes now:

$$\frac{n^{1/p-1/2-1/\alpha}}{2(\beta c^*)^{1/2} A_\alpha d_n^{1/2} (\log n)^{1/2}} \leq \frac{1}{2c\beta A_\alpha} \frac{n^{1/p-1/\alpha}}{n^\theta \log n},$$

which is equivalent to $d_n \geq \frac{c^2\beta}{c^*} n^{2\theta-1} \log n \sim n^{2\delta} \log n$. Thus, as we are interested in a slow growing sequence, we choose $d_n \sim n^{2\delta} \log n$. So, $\varepsilon_n^2 \sim n^{2\delta-2/p+2/\alpha+1} (\log n)^2 \longrightarrow 0$, given the choice for δ. To complete the proof, it is enough to bound $\exp(cn^{1+1/\alpha}\lambda)u(p_n)$. It is easily verified that $n^{1+1/\alpha}\lambda \sim n^{1/2-\delta}$, so the term we need to bound is of order $n^{1/2-\delta} + n^\theta \log \rho = n^{\frac{1}{2}-\delta} + n^{1/2+\delta} \log \rho$. But, this is an immediate consequence of $\rho \in (0,1)$ and $\delta > 0$, so the proof is concluded. □

Remark 3.2.5. *The above argument does not hold if the decrease rate of the Cox-Grimmett coefficients is only polynomial. Indeed, in this case we would be driven to bound $n^{1/2-\delta} + a(\frac{1}{2} + \delta) \log n$, which is always unbounded as $\frac{1}{2} + \delta > 0$.*

3.3 The general case

For general sequences of random variables we need an extension of Lemma 3.2.1. For this purpose we will introduce a truncation on the random variables, which can be analysed using the results in the previous section, and control the remaining tails.

Let c_n, $n \geq 1$, be a sequence of nonnegative real numbers such that $c_n \longrightarrow +\infty$ and define, for each $i, n \geq 1$,

$$\begin{aligned} X_{1,i,n} &= -c_n \mathbb{I}_{(-\infty,-c_n)}(X_i) + X_i \mathbb{I}_{[-c_n,c_n]}(X_i) + c_n \mathbb{I}_{(c_n,+\infty)}(X_i), \\ X_{2,i,n} &= (X_i - c_n)\mathbb{I}_{(c_n,+\infty)}(X_i), \qquad X_{3,i,n} = (X_i + c_n)\mathbb{I}_{(-\infty,-c_n)}(X_i), \end{aligned} \tag{3.11}$$

where $\mathbb{I}_A$ represents the characteristic function of the set A. Notice that the above transformations are monotonous, so these new families of variables are still associated. Morevoer, it is obvious that, for each $n \geq 1$ fixed, the variables $X_{1,1,n}, \ldots, X_{1,n,n}$ are uniformly bounded. Consider, as before, a sequence of natural numbers p_n such that, for each $n \geq 1$, $p_n < \frac{n}{2}$ and define r_n as the largest integer less or equal to $\frac{n}{2p_n}$. For $q = 1, 2, 3$, and $j = 1, \ldots, 2r_n$, define

$$Y_{q,j,n} = \sum_{\ell=(j-1)p_n+1}^{jp_n} a_{n,i}\Big(X_{q,\ell,n} - \mathrm{E}(X_{q,\ell,n})\Big), \tag{3.12}$$

and

$$T_{q,n,od} = \sum_{j=1}^{r_n} Y_{q,2j-1,n}, \qquad T_{q,n,ev} = \sum_{j=1}^{r_n} Y_{q,2j,n}, \tag{3.13}$$

For $q = 2, 3$, assuming the variables are identically distributed, we have the following upper bound,

$$\begin{aligned} &\mathrm{P}\left(\left|\sum_{i=1}^{n} a_{n,i}\Big(X_{q,i,n} - \mathrm{E}(X_{q,i,n})\Big)\right| > n^{1/p}\varepsilon\right) \\ &\qquad \leq n\mathrm{P}\left(|X_{q,1,n} - \mathrm{E}(X_{q,1,n})| > \frac{n^{1/p-1}\varepsilon}{A_\alpha}\right) \\ &\qquad \leq \frac{n^{3-2/p}A_\alpha^2}{\varepsilon^2}\mathrm{Var}(X_{q,1,n}) \leq \frac{n^{3-2/p}A_\alpha^2}{\varepsilon^2}\mathrm{E}(X_{q,1,n}^2). \end{aligned}$$

The following result is an easy extension of Lemma 4.1 in [37].

Lemma 3.3.1. *Let $X_1, X_2, \ldots$ be strictly stationary random variables such that there exists $\delta > 0$ satisfying $\sup_{|t|\leq\delta} \mathrm{E}(e^{tX_1}) \leq M_\delta < +\infty$. Then, for $t \in (0, \delta]$,*

$$\mathrm{P}\left(\left|\sum_{i=1}^{n} a_{n,i}\Big(X_{q,i,n} - \mathrm{E}(X_{q,i,n})\Big)\right| > n^{1/p}\varepsilon\right) \leq \frac{2M_\delta A_\alpha^2 n^{3-2/p} e^{-tc_n}}{t^2\varepsilon^2}, \qquad q = 2, 3. \tag{3.14}$$

3.3.1 Almost sure convergence and rates

We may now prove the extensions of the results proved for uniformly bounded sequences of random variables. The main argument in the proofs in Sect. 3.2 was the control of the exponent in the exponential upper bounds found. The bound obtained in (3.14) is, essentially, of the same form, depending on the choice of the truncation sequence. So, we will obtain the same characterizations for the almost sure convergence and for its rate, as in the case of uniformly bounded sequences of random variables. Remark that, due to the association of the variables,

$$\begin{aligned}
&\mathrm{Cov}(X_{1,1,n}, X_{1,j,n}) \\
&\quad = \iint_{[-c_n,c_n]^2} \mathrm{P}(X_1 > u, X_j > v) - \mathrm{P}(X_1 > u)\mathrm{P}(X_j > v)\, dudv \\
&\quad \leq \iint_{\mathbb{R}^2} \mathrm{P}(X_1 > u, X_j > v) - \mathrm{P}(X_1 > u)\mathrm{P}(X_j > v)\, dudv = \mathrm{Cov}(X_1, X_j).
\end{aligned}$$

Obviously, this inequality holds even if $\mathrm{Cov}(X_1, X_j)$ is not finite.

Theorem 3.3.2. *Assume the random variables X_n, $n \geq 1$, are centered, associated and strictly stationary. Assume that $p < 1$ and $\alpha > 1$ are such that $\frac{1}{p} - \frac{1}{\alpha} > 1$ and $u(n) \sim n^{-a}$, for some $a > 0$. If the nonnegative weights satisfy (3.1), then, with probability 1, $n^{-1/p}T_n \longrightarrow 0$.*

Proof. To control the tail terms, that is, $T_{q,n,od}$ and $T_{q,n,ev}$, for $q = 2, 3$, choose the truncation sequence $c_n = \log n$ and $t = \beta > 4 - \frac{2}{p}$. Thus according to Lemma 3.3.1, the probabilities depending on these variables are bounded above by a convergent series. Concerning the remaining term, follow the proof of Theorem 3.2.2 but keep in mind that the constants c and c^* now depend on n. According to the comment immediately after Lemma 3.1.1, we have $c_n^* = c_n^2 + u(0) \sim (\log n)^2$. Thus, instead of (3.6), we find the choice

$$\lambda = \frac{n^{1/p-1-2/\alpha}\varepsilon}{2c_n^* A_\alpha^2 d_n} \sim \frac{n^{1/p-1-2/\alpha}\varepsilon}{(\log n)^2 d_n},$$

and

$$\frac{n^{2/p-1-2/\alpha}\varepsilon^2}{4c_n^* A_\alpha^2 d_n} = \beta \log n \quad \Leftrightarrow \quad d_n = \frac{\varepsilon^2}{4c_n^*\beta A_\alpha^2}\,\frac{n^{2/p-1-2/\alpha}}{\log n} \sim \frac{n^{2/p-1-2/\alpha}}{(\log n)^3}.$$

The condition on λ required by Lemma 3.1.2 translates now into

$$\varepsilon^{-1} \leq \frac{n^{1/p-1/\alpha}}{2c_n\beta A_\alpha n^\theta \log n} \sim \frac{n^{1/p-1/\alpha-\theta}}{(\log n)^2}.$$

Thus, up to a logarithmic factor, we find an upper bound with the same behaviour as the one found in (3.8), so the argument used in course of proof of Theorem 3.2.2 still applies. Remark also that the present choice for d_n also only changes with respect to the one made in the proof of Theorem 3.2.2 by the introduction of a logarithmic factor in the denominator. Thus the fact that d_n becomes larger that 1, for n large enough, is not affected. The same holds for the term corresponding to (3.10). Indeed, the exponent we need to control takes now the form $c_n n^{1+1/\alpha}\lambda \sim n^{1-1/p+1/\alpha}(\log n)^2$, that is, the same we found before multiplied by a logarithmic factor that, as is easily verified, does not affect the remaining argument of the proof. $\square$

For sake of completeness we state the results corresponding to Theorems 3.2.3 and 3.2.4. We do not include proofs as these are modifications of the corresponding ones exactly as done for Theorem 3.3.2.

Theorem 3.3.3. *Assume the random variables X_n, $n \geq 1$, are centered, associated and strictly stationary. Assume that $p < 2$ and $\alpha > 1$ are such that $\frac{1}{p} - \frac{1}{\alpha} > \frac{1}{2}$ and $u(n) \sim \rho^{-n}$, for some $\rho \in (0,1)$. If the nonnegative weights satisfy (3.1), then, with probability 1, $n^{-1/p}T_n \longrightarrow 0$.*

Theorem 3.3.4. *Assume the random variables X_n, $n \geq 1$, are centered, associated and strictly stationary. Assume that $p < 2$ and $\alpha > 1$ are such that $\frac{1}{p} - \frac{1}{\alpha} > \frac{1}{2}$ and $u(n) \sim \rho^{-n}$, for some $\rho \in (0,1)$. If the nonnegative weights satisfy (3.1), then, with probability 1, $n^{-1/p}T_n \longrightarrow 0$ with convergence rate $\frac{\log n}{n^{1/p-1/\alpha-1/2-\delta}}$, for arbitrarily small $\delta > 0$.*

The above statements include an assumption on the Cox-Grimmett coefficients of the original untruncated variables. In fact, this assumption, which implies the existence of second order moments, may be relaxed, as we only need the coefficients corresponding to the truncated variables defined as, assuming already the stationarity of the variables,

$$u^*(n) = 2\sum_{j=n+1}^{\infty} \mathrm{Cov}(X_{1,1,n}, X_{1,j,n}).$$

Taking into account the inequality between the covariances, it is obvious that $u^*(n) \leq u(n)$. Of course, this choice for the statements would imply a definition for the truncating sequence on the statement.

3.3.2 Comparing with previous results

Theorem 3.3.2 above extends Corollary 3.5 in Çağın and Oliveira [13]. Indeed, in [13] it is assumed that $p > 1$ due to the technicalities of the proof, while here we are assuming $p < 1$. This later case would imply, with respect to the framework of [13], the need to assume the existence of moments of order 2, which was what was trying to be avoided in [13]. In the present case, as we are dealing with bounded variables or using truncation, this is not a problem. In order to be somewhat more precise on the relations of the present results and those in [13] we need some more notation, extending the truncation in (3.11). Given $v > 0$ and $i \geq 1$, define $X_{1,i,v} = -v\mathbb{I}_{(-\infty,-v)}(X_i) + X_i\mathbb{I}_{[-v,v]}(X_i) + v\mathbb{I}_{(v,+\infty)}(X_i)$ and $G_i(v) = \mathrm{Cov}(X_{1,1,v}, X_{1,i,v})$. Now the assumption on the Cox-Grimmett coefficients in Theorem 3.3.2 rewrites as

$$u^*(n) = 2 \sum_{j=n+1} G_j(\log n) \sim n^{-a}, \quad a > 0.$$

A translation of this decay rate directly into the covariances is achieved if we assume that $G_j(v) \sim e^{-(a+1)v}$, thus a geometric decay rate for the covariances. Moreover, we may still verify that Corollary 3.5 in Çağın and Oliveira [13] does not overlap with Theorem 3.3.2, considering the version with an assumption on the truncated Cox-Grimmett coefficients $u^*(n)$ instead. In fact, the result in [13] assumes that

$$\sum_{n=1}^{\infty} \int_{(n+1)^{(\alpha-2p)/(\alpha p)}} v^{\frac{\alpha(p-1)}{\alpha-2p}-2} G_n(v)\, dv < \infty.$$

If $G_j(v) \sim e^{-(a+1)v}$, this condition is equivalent to

$$\sum_{n=1}^{\infty} \int_{(a+1)(n+1)^{(\alpha-2p)/(\alpha p)}} t^{\frac{\alpha(p-1)}{\alpha-2p}-2} e^{-t}\, dt < \infty.$$

The convergence of the series above is equivalent to the finiteness of the integral

$$\int_1^{\infty} \int_{(a+1)(n+1)^{(\alpha-2p)/(\alpha p)}} t^{\frac{\alpha(p-1)}{\alpha-2p}-2} e^{-t}\, dt\, dx,$$

which, after inverting the integration order is bounded above by

$$\int_{2^{(\alpha-2p)/(\alpha p)}(a+1)}^{\infty} t^{(2\alpha p-\alpha)/(\alpha-2p)-2} e^{-t}\, dt \leq \Gamma\left(\frac{2\alpha p-\alpha}{\alpha-2p} - 1\right),$$

where Γ represents the Euler Gamma function, and this is finite if the argument is positive, that is, if $\alpha p > \alpha - p$ or, equivalently, if $\frac{1}{p} - \frac{1}{\alpha} < 1$, the reverse inequality of what is assumed in Theorem 3.3.2.

4. A moderate deviation for associated random variables

In this chapter we consider a moderate deviation for associated and strictly stationary random variables with finite moments of order $q > 2$. The control of this dependence structure relies on the decay rate of the covariances for which we assume a relatively mild polynomial decay rate. We present a moderate deviation in the non-logarithmic scale for sums of associated random variables. Outline of the chapter is as follows: In Section 2 we give some definitions and recall some auxiliary results, in Section 3 we prove the main result and a corollary with an assumption that identifies more explicitly the behaviour of the relevant dependence coefficients. The proof combines a coupling argument together with a suitable use of a Berry-Esséen inequality; finally in Section 4 we give an application to moving averages of our main result.

4.1 Framework and auxiliary results

To define appropriately our framework let X_n, $n \geq 1$, be strictly stationary centered and associated random variables with finite variances. Denote $S_n = X_1 + \cdots + X_n$ and $s_n^2 = \mathrm{E} S_n^2$. Recall that association means that for any $m \geq 1$ and any two real-valued coordinatewise nondecreasing functions f and g,

$$\mathrm{Cov}\Big(f\left(X_1, \ldots, X_m\right), g\left(X_1, \ldots, X_m\right)\Big) \geq 0,$$

whenever this covariance exists. It is well known that the covariance structure of associated random variables characterizes their asymptotics, so it is natural to seek assumptions on the covariances. A common assumption when proving CLT is $\frac{1}{n} s_n^2 \longrightarrow \sigma^2 > 0$ (see, for example, Newman and Wright [34, 35] or Oliveira and Suquet [40, 41]), so we will be assuming this is fulfilled in the sequel. Notice this assumption implies that $s_n^2 \sim n$. Finally, recall the Cox-Grimmett coefficients, currently used to control dependence for associated random variables (remember they are assumed stationary):

$$u(n) = \sum_{k=n}^{\infty} \mathrm{Cov}(X_1, X_k). \tag{4.1}$$

Our proof will rely on a suitable approximation to independent variables which will be chosen so that these satisfy the moderate deviation we want to extend. We quote next a result by Frolov [18] providing a moderate deviation for triangular arrays of row-wise independent random variables. This will be the tool to prove the moderate deviation for the approximating variables.

Theorem 4.1.1 (Theorem 1.1 in Frolov [18]). *Let $X_{n,k}$, $k = 1, \ldots, k_n$, $n \geq 1$, be an array of row-wise independent variables with $F_{n,k}(y) = \mathrm{P}(X_{n,k} \leq y)$, $\mathrm{E}X_{n,k} = 0$ and $\mathrm{E}X_{n,k}^2 = \sigma_{n,k}^2 < \infty$. Denote $T_n = \sum_{k=1}^{k_n} X_{n,k}$ and $B_n = \sum_{k=1}^{k_n} \sigma_{n,k}^2$. For $q > 2$, let $\beta_{n,k} = \int_0^{\infty} y^q \, F_{n,k}(dy) < +\infty$. Define*

$$M_n = \sum_{k=1}^{k_n} \beta_{n,k} \quad \text{and} \quad L_n = B_n^{-q/2} M_n.$$

Assume that $L_n \longrightarrow 0$, and that, for each $\varepsilon > 0$,

$$\Lambda_n(x) = x^4 B_n^{-1} \sum_{k=1}^{n} \int_{-\infty}^{-\varepsilon\sqrt{B_n}/x^5} y^2 \, F_{n,k}(dy) \longrightarrow 0. \tag{4.2}$$

If $x \longrightarrow +\infty$ such that $x^2 - 2\log(L_n^{-1}) - (q-1)\,\mathrm{loglog}(L_n^{-1}) \longrightarrow -\infty$ then

$$\mathrm{P}(T_n \geq x B_n^{1/2}) \sim \frac{1}{\sqrt{2\pi}\, x} \, e^{-x^2/2}.$$

Remark that, using standard Gaussian approximations, from the conclusion of this theorem follows easily that $\mathrm{P}(T_n \geq x B_n^{1/2}) = (1 - \Phi(x))(1 + o(1))$, where Φ stands for the distribution function of a standard Gaussian variable.

Finally, we will be dealing with integration of squares of sums of random variables that we will need to decompose. The following result describes how we can do this and control the original integral.

Lemma 4.1.2 (Lemma 4 in Utev [53]). *Let U_n, $n \geq 1$, be random variables. Then, for every $\varepsilon > 0$ and $n \geq 1$,*

$$\int_{\{|\sum_{i=1}^{n} U_i| > \varepsilon n\}} \left(\sum_{i=1}^{n} U_i \right)^2 dP \leq 2n \sum_{i=1}^{n} \int_{\{|U_i| > \varepsilon/2\}} U_i^2 \, dP.$$

4.2 A general moderate deviation

We now state the moderate deviation for associated random variables. Besides moment conditions we will require a suitable decrease rate on the Cox-Grimmett coefficients (4.1). To state and prove our main result we need some preparatory definitions. Consider an increasing sequence of integers $p_n < \frac{n}{2}$ and define r_n as the largest integer that is less or equal to $\frac{n}{2p_n}$. Decompose $S_n = X_1 + \cdots + X_n$ into blocks, each summing p_n variables. For this purpose, define

$$Y_{j,n} = \sum_{\ell=(j-1)p_n+1}^{jp_n} X_\ell, \quad j = 1, \ldots, r_n,$$

which obviously verify

$$S_n = Y_{1,n} + \cdots + Y_{2r_n,n} + \sum_{\ell=2r_np_n+1}^{n} X_\ell.$$

The final term is a residual block summing at most $2p_n - 1$ variables. Finally, put

$$Z_{n,od} = \sum_{j=1}^{r_n} Y_{2j-1,n} \qquad \text{and} \qquad Z_{n,ev} = \sum_{j=1}^{r_n} Y_{2j,n}.$$

Define now a family of coupling variables: $Y^*_{j,n}$, $j = 1, \ldots, r_n$, are independent random variables such that $Y^*_{j,n}$ has the same distribution as $Y_{j,n}$. Remark that, if the original variables X_n are strictly stationary, the $Y^*_{j,n}$, $j = 1, \ldots, 2r_n$, are identically distributed. Moreover, in such case, $E(Y^*_{j,n})^2 = s^2_{p_n}$. Further, denote

$$Z^*_{n,od} = \sum_{j=1}^{r_n} Y^*_{2j-1,n}$$

and analogously for $Z^*_{n,ev}$.

Theorem 4.2.1. *Let X_n, $n \geq 1$, be strictly stationary centered and associated random variables. Let $S_n = X_1 + \cdots + X_n$, $s_n^2 = \mathrm{E}S_n^2$. Assume that*

(A1) *the random variables X_n have finite moments of order $q > 2$;*

(A2) $x_n^2 = 2\gamma \log n$, *for some* $\gamma < \frac{q}{2} - 1$;

(A3) $\frac{1}{n} s_n^2 \longrightarrow \sigma^2$ *for some* $\sigma^2 < \infty$;

(A4) $u(n) = O\left(n^{-\frac{1+3\gamma}{1-\alpha}}\right)$, *where* $\alpha \in \left(\max\left(\frac{1}{2} + \frac{1}{q}, \frac{1}{2} + \frac{1+2\gamma}{2(q-1)}\right), 1\right)$;

(A5) $|\mathrm{P}\left(S_n > 2x_n s_n\right) - 2\mathrm{P}\left(Z_{n,od} > x_n s_n\right)| = O\left(n^{-\gamma}\right)$.

Then

$$\mathrm{P}(S_n > 2x_n s_n) = (1 - \Phi(x_n))(1 + o(1)). \tag{4.3}$$

Proof. The proof of the theorem follows the more or less classical steps after the decomposition of S_n into blocks and coupling these blocks with variables with the same distribution but independent: 1. prove the moderate deviation for the coupling variables; 2. control the difference between the original blocks and the coupling ones; 3. prove the residual block converges to zero at the appropriate rate; 4. finally, approximate the convenient tail probabilities. To complete this plan we need to be more specific about the sequence p_n used for the construction of the blocks. We will assume that $p_n \sim n^{1-\alpha}$, where $\alpha \in (0,1)$ is given by **(A4)** (remark that the assumption on γ in **(A2)** ensures that a choice $\alpha < 1$ is indeed possible).

Step 1. To accomplish this step we apply Theorem 4.1.1 to the random variables $Y_{j,n}^*$ defining each of the summations $Z_{n,od}^*$ and $Z_{n,ev}^*$. We shall concentrate on $Z_{n,od}^*$, as the other summation is analogous. Now, as mentioned above, $Z_{n,od}^*$ is a sum of identically distributed random variables. It follows from **(A1)**, that the moment assumption required by Theorem 4.1.1 on the variables $Y_{j,n}^*$ is satisfied. Referring to the notation of Theorem 4.1.1, we have $B_n = r_n s_{p_n}^2 \sim n\sigma^2$ (this corresponds to our s_n^2), $M_n = r_n \mathrm{E}(Y_{j,n}^q \mathbb{I}_{Y_{j,n}\geq 0})$ and $L_n = r_n B_n^{-q/2} \mathrm{E}(Y_{j,n}^q \mathbb{I}_{Y_{j,n}\geq 0}) \sim r_n n^{-q/2} p_n^q = n^{(1-\alpha)(q-1)+1-q/2}$. The exponent in this last expression is rewritten as $\frac{q}{2} - \alpha(q-1) < -\gamma < 0$, as follows from **(A4)**, thus $L_n \longrightarrow 0$, as required by Theorem 4.1.1. Moreover, $x_n^2 - 2\log L_n^{-1} \sim n^{2\gamma} - n^{2\alpha(q-1)-q}$, again from **(A4)**, $\alpha > \frac{1}{2} + \frac{1+2\gamma}{2(q-1)}$, so it follows that $2\alpha(q-1) - q > 2\gamma$, thus $x_n^2 - 2\log L_n^{-1} \longrightarrow -\infty$, hence satisfying the assumption on x_n in Theorem 4.1.1.

Concerning (4.2), a Lindeberg like assumption in Theorem 4.1.1, notice that when applied to the $Y_{j,n}^*$ variables, remembering that $B_n \sim n$ and all the terms in the summation are identical, it may be rewritten as

$$x_n^4 E\left(Y_{j,n}^2 \mathbb{I}_{(-\infty, -\varepsilon s_n/x_n^5)}(Y_{j,n})\right) \longrightarrow 0.$$

(We do not include the $*$ as the mathematical expectation above only depends on the moments of each variable). Of course, we may replace s_n by $n^{1/2}$. Enlarging the integration set, we obviously have the upper bound

$$x_n^4 E\left(Y_{j,n}^2 \mathbb{I}_{\left(-\infty, -\varepsilon n^{1/2}/x_n^5\right)}(Y_{j,n})\right) \leq x_n^4 E\left(Y_{j,n}^2 \mathbb{I}_{|Y_{j,n}| > \varepsilon n^{1/2}/x_n^5}\right).$$

The integrand above is the square of a sum of random variables, so we need to separate the random variables in this square. This may be accomplished using Lemma 4.1.2. Remembering that the X_n variables are identically distributed, one easily obtains that

$$\begin{aligned}
& x_n^4 E\left(Y_{j,n}^2 \mathbb{I}_{|Y_{j,n}|>\varepsilon n^{1/2}/x_n^5}\right) \\
& \leq 2x_n^4 p_n^2 \int_{\left\{|X_i|>\varepsilon n^{1/2}/(2p_n x_n^5)\right\}} X_i^2 \, dP \\
& \leq 2x_n^4 p_n^2 \left(E\left|X_1\right|^q\right)^{2/q} \left(P\left(|X_1| > \frac{\varepsilon n^{1/2}}{2p_n x_n^5}\right)\right)^{1-2/q} \\
& \leq 2x_n^4 p_n^2 \left(E\left|X_1\right|^q\right)^{2/q} \left(E\left|X_1\right|^q \left(\frac{2p_n x_n^5}{\varepsilon n^{1/2}}\right)^q\right)^{1-2/q} \\
& = 2E\left|X_1\right|^q \frac{2^{q-2} p_n^q x_n^{5q-6}}{\varepsilon^{q-2} n^{(q-2)/2}} .
\end{aligned}$$

Taking into account **(A2)**, x_n grows to infinity at a logarithmic rate, thus the behaviour of the term above is driven by the polynomial factors. We have chosen $p_n \sim n^{1-\alpha}$, so it follows that

$$\frac{p_n^q x_n^{5q-6}}{n^{(q-2)/2}} \sim \frac{p_n^q (\log n)^{5q/2-3}}{n^{(q-2)/2}} \sim n^{q(1/2-\alpha)+1} (\log n)^{5q/2-3} \longrightarrow 0,$$

since $q(\frac{1}{2}-\alpha)+1<0$, as follows from **(A4)**. Then, from Theorem 4.1.1 it follows that (remember $\mathrm{E}(Y_{j,n}^2) = s_{p_n}^2$)

$$P\left(Z_{n,od}^* > x_n s_{p_n} \sqrt{r_n}\right) \sim \frac{1}{\sqrt{2\pi}\, x_n} e^{-x_n^2/2}.$$

Step 2. Denote by G_1 the distribution function of $Z_{n,od}$, by G_2 the distribution function when the summands are assumed independent, that is, the distribution function of $Z_{n,od}^*$, and by φ_1 and φ_2 the corresponding characteristic functions:

$$\varphi_1(t) = E\left(e^{itZ_{n,od}}\right), \qquad \text{and} \qquad \varphi_2(t) = \prod_{j=1}^{r_n} E\left(e^{itY_{j,n}}\right).$$

The classical Berry-Esséen inequality shows that

$$\sup_{x\in\mathbb{R}} |G_1(x) - G_2(x)| \leq c_1 \int_{-T}^{T} \frac{|\varphi_1(t) - \varphi_2(t)|}{|t|} \, dt + \frac{c_2}{T}, \qquad \text{for every } T>0,$$

where c_1 and c_2 are constants independent of T. It follows from Newman's inequality for characteristic functions of associated variables (Theorem 1 in Newman [33]) that

$$|\varphi_1(t) - \varphi_2(t)| \leq \frac{t^2}{2} \sum_{j\neq k} \mathrm{Cov}\left(Y_{j,n}, Y_{k,n}\right).$$

As the X_n are stationary, it still follows that

$$\sum_{j\neq k} \mathrm{Cov}\left(Y_{j,n}, Y_{k,n}\right) \leq n \sum_{\ell=p_n+2}^{+\infty} \mathrm{Cov}\left(X_1, X_\ell\right) = nu(p_n+2) \leq nu(p_n),$$

referring to the Cox-Grimmett coefficients, as the covariances are nonnegative. Inserting this into the Berry-Esséen bound one finds

$$\sup_{x\in\mathbb{R}} |G_1(x) - G_2(x)| \leq \frac{c_1}{2}\int_{-T}^{T} nu(p_n)\,|t|\;dt + \frac{c_2}{T} \leq \frac{c_1}{2} nu(p_n)T^2 + \frac{c_2}{T},$$

So, by choosing $T \sim (nu(p_n))^{-1/3}$, we find an upper bound of order $(nu(p_n))^{1/3}$. Using now the choice $p_n \sim n^{1-\alpha}$ and taking into account **(A4)**, it follows that $(nu(p_n))^{1/3} \sim n^{-\gamma}$. Given the behaviour of x_n described in assumption **(A2)**, it follows that $x^{-1}e^{-x^2/2} \sim n^{-\gamma}$, hence, we have $(nu(p_n))^{1/3} = O\left(x^{-1}e^{-x^2/2}\right)$, which controls the convergence rate of the approximation between the actual variables and the coupling ones.

Step 3. We prove that the residual block defines probabilities that converge to zero faster than the terms considered in the previous steps. Remember that it follows from **(A3)** that $s_n \sim n^{1/2}$. Thus, as the variables X_ℓ are identically distributed,

$$\begin{aligned}\mathrm{P}&\left(\sum_{\ell=2r_np_n+1}^{n} X_\ell > x_n s_n\right)\\ &\leq \sum_{\ell=2r_np_n}^{n} \mathrm{P}\left(X_\ell > \frac{x_n n^{1/2}}{n-2r_np_n}\right) \leq \frac{(n-2r_np_n)^{q+1}}{x_n n^{q/2}}\,\mathrm{E}\,|X_1|^q\,.\end{aligned}$$

As $x_n^2 = 2\gamma\log n$ it is enough to verify that

$$\frac{(n-2r_np_n)^{q+1}}{n^{q/2}} \leq \frac{2^{q+1}p_n^{q+1}}{n^{q/2}} \sim n^{(q+1)(1-\alpha)-q/2}.$$

Now $(q+1)(1-\alpha) - \frac{q}{2} > (q-1)(1-\alpha) - \frac{q}{2} > \gamma$, as follows from the **(A4)**, so

$$\mathrm{P}\left(\sum_{\ell=2r_np_n+1}^{n} X_\ell > x_n s_n\right) = O\left(n^{-\gamma}\right).$$

Step 4. In the previous steps we controlled the behaviour of $\mathrm{P}(Z_{n,od} > x_n s_{p_n}\sqrt{r_n})$, but we are interested in probabilities of the form $\mathrm{P}(S_n > x_n s_n)$. The difference between these two terms is controlled at the appropriate convergence rate by **(A5)**.

■

Remark 4.2.2. *Assumption* **(A5)** *is not a very natural one. We give an example showing that it is indeed achievable. We have assumed the* X_n *to be stationary, so* $Z_{n,od}$ *and* $Z_{n,ev}$ *have the same distribution. Assume, for simplicity, that* $S_n = Z_{n,od} + Z_{n,ev}$*, that is, the residual term does not exist (remember we have already shown that this residual term is negligible). So one could look at*

$$\mathrm{P}\left(S_n > 2xs_n\right) - 2\mathrm{P}\left(Z_{n,od} > xs_n\right).$$

Assume $(Z_{n,od}, Z_{n,ev})$ *has Gaussian distribution with mean* $(0,0)$ *and* $\mathrm{Cov}(Z_{n,od}, Z_{n,ev}) = \rho_n$*. Remark that, as* $\mathrm{Var}(Z_{n,od}) \sim \mathrm{Var}(Z_{n,ev}) \sim \frac{n}{2}$*, we have* $\rho_n \leq \frac{n}{2}$*. It is easily verified that* S_n *is Gaussian with mean 0 and variance* $n + 2\rho_n$*. So, denoting by* Z *a standard Gaussian random variable:*

$$\begin{aligned} &\mathrm{P}\left(S_n > 2x\sqrt{n+2\rho_n}\right) - 2\mathrm{P}\left(Z_{n,od} > x\sqrt{n+2\rho_n}\right) \\ &\qquad = \mathrm{P}(Z > 2x) - \mathrm{P}\left(Z > x\sqrt{\frac{2n+4\rho_n}{n}}\right). \end{aligned} \tag{4.4}$$

As we have already remarked, as $x \longrightarrow +\infty$*,*

$$\mathrm{P}(Z > x) \sim \frac{1}{\sqrt{2\pi}\,x} e^{-x^2/2}.$$

Using this approximation on (4.4) and multiplying by $xe^{x^2/2}$*, we find*

$$\begin{aligned} &\left[\mathrm{P}\left(S_n > 2x\sqrt{n+2\rho_n}\right) - 2\mathrm{P}\left(Z_{n,od} > x\sqrt{n+2\rho_n}\right)\right] \frac{x}{e^{-x^2/2}} \\ &\qquad \sim \frac{\exp\left(-3x^2/2\right)}{2\sqrt{2\pi}} + \frac{\exp\left(-x^2(1/2 + 2\rho_n/n)\right)}{\sqrt{2\pi}\sqrt{2+4\rho_n/n}}. \end{aligned}$$

As both exponents are negative, this remains bounded, so that ***(A4)*** *is fulfilled.*

Remark 4.2.3. *Still about* **(A5)***. One can easily see that the argument above is a lot more restrictive if we compare*

$$\left|\mathrm{P}\left(S_n > xs_n\right) - 2\mathrm{P}\left(Z_{n,od} > \frac{xs_n}{2}\right)\right|.$$

Indeed, repeating the approximations for the Gaussian variable as above, one could only conclude about the boundedness of this difference if $\rho_n \geq \frac{n}{2}$*. Now remember that* ρ_n *represents the covariance of two random variables with variances equal to* $\frac{n}{2}$*, so in order to make these two requirements compatible we would need that* $\rho_n \sim \frac{n}{2}$*, thus reducing significantly the possibility of choices for* ρ_n*.*

Assumption **(A4)** describes the decrease rate for the Cox-Grimmett coefficients depending on a parameter that is used for tuning the technical construction needed for the proof. It is useful to have a version of the result with an assumption independent from these tuning parameters.

Corollary 4.2.4. *The result in Theorem 4.2.1 holds if we replace* **(A4)** *by*

(A4′) $u(n) = O\left(n^{-\theta}\right)$*, where* $\theta > (1+3\gamma)\max\left(2+\frac{4}{q-2}, 2+\frac{4\gamma+2}{q-2\gamma-2}\right)$.

Proof. With respect to the proof of Theorem 4.2.1, it is enough to verify that $\theta > \frac{1+3\gamma}{1-\alpha}$, where $\alpha > \max\left(\frac{1}{2}+\frac{1}{q}, \frac{1}{2}+\frac{1+2\gamma}{2(q-1)}\right)$. From here follow immediate bounds for $1-\alpha$ that we plug in the above expression to find the given condition for he choice of the parameter θ. ■

Remark 4.2.5. *Notice that the assumption on the Cox-Grimmett coefficients, in either form* **(A4)** *or* **(A4′)***, is much milder than what was assumed in Henriques and Oliveira [21] to prove a large deviation principle:* $\mathrm{Cov}\left(X_1, X_n\right) = a_0 \exp\left(-n(\log n)^{1+a}\right)$*, where* $a_0 > 0$ *and* $a > 0$*.*

Let us get back to a discussion about assumption **(A5)**, seeking for more a natural sufficient condition. According to Remark 4.2.2, when the distributions are Gaussian **(A5)** is satisfied. So, one way to look for more natura conditions is to try to control the distance with respect to Gaussian distributions using Berry-Esséen bounds.

Theorem 4.2.6. *Let* X_n*,* $n \geq 1$*, be strictly stationary centered and associated random variables. Let* $S_n = X_1 + \cdots + X_n$*,* $s_n^2 = \mathrm{E}S_n^2$*. Assume that* **(A1)**–**(A5)** *in Theorem 4.2.1 are satisfied with* $q > 2$ *and* $\gamma < \min(\frac{1}{5}, \frac{q}{2}-1)$ *Then (4.3) holds.*

Proof. We need to verify that **(A5)** is satified. For this purpose introduce Gaussian centered variables $\widehat{S}_n$, $\widehat{Z}_{n,od}$ and $\widehat{Z}_{n,ev}$ with variances $\mathrm{E}S_n^2$, $\mathrm{E}Z_{n,od}^2$ and $\mathrm{E}Z_{n,ev}^2$, respectively, and such that $\mathrm{Cov}(\widehat{Z}_{n,od}, \widehat{Z}_{n,ev}) = \mathrm{Cov}(Z_{n,od}, Z_{n,ev})$, and decompose

$$\begin{aligned} &\left|\mathrm{P}\left(S_n > 2x_n s_n\right) - 2\mathrm{P}\left(Z_{n,od} > x_n s_n\right)\right| \\ &\quad \leq \left|\mathrm{P}\left(S_n > 2x_n s_n\right) - \mathrm{P}\left(\widehat{S}_n > 2x_n s_n\right)\right| + \left|\mathrm{P}\left(\widehat{S}_n > 2x_n s_n\right) - 2\mathrm{P}\left(\widehat{Z}_{n,od} > x_n s_n\right)\right| \\ &\qquad + \left|\mathrm{P}\left(\widehat{Z}_{n,od} > x_n s_n\right) - 2\mathrm{P}\left(Z_{n,od} > x_n s_n\right)\right|. \end{aligned}$$

Remark 4.2.2 shows that the middle term above $\left|\mathrm{P}\left(S_n^* > 2x_n s_n\right) - 2\mathrm{P}\left(Z_{n,od}^* > x_n s_n\right)\right| = O\left(n^{-\gamma}\right)$. As the variables satisfy the Central Limit Theorem, the remaining terms may be bounded by the Berry-Esséen inequality. Now, taking into account Corollary 4.14 in Oliveira [39], the convergence rate for these terms is of order $n^{-1/5}$.

Hence, $|\mathrm{P}(S_n > 2x_n s_n) - 2\mathrm{P}(Z_{n,od} > x_n s_n)|$ is of the same order as the slowest term, that is $n^{-\gamma}$, thus **(A5)** is satisfied, so the conclusion of Theorem 4.2.1 holds, that is, (4.3) is verified. ■

4.3 Main result

The result stated in Theorem 4.2.6 is a sort of a worst case scenario concerning the approximation to the Gaussian distribution. We may improve on our Theorem 4.2.1 if we are more precise about the convergence rate in assumption **(A3)**. To accomplish this we need first to prove an adapted version of the Berry-Esséen bound for the approximation of distribution functions in the Central Limit Theorem.

Theorem 4.3.1. *Let X_n, $n \geq 1$, be strictly stationary centered and associated random variables with finite moments of order 3. Let $S_n = X_1 + \cdots + X_n$, $s_n^2 = \mathrm{E}S_n^2$ and assume that $\frac{1}{n}s_n^2 \longrightarrow \sigma^2 < \infty$. If p_n and r_n are sequences as defined in the beginning of Section 4.2, then, for n large enough,*

$$|\mathrm{P}(S_n \leq x s_n) - \Phi(x)| \leq T^2\left(1 - \frac{2r_n s_{p_n}^2}{s_n^2}\right) + \frac{24}{\pi\sqrt{2\pi}\,T} + 4\sqrt{\pi}\,c_1' e^{c_1'/(2c_1^2)} \frac{r_n \mathrm{E}\left|Y_{j,n}\right|^3}{s_n^3}, \tag{4.5}$$

where $\Phi(\cdot)$ is the distribution function of the standard Gaussian distribution, $T = \frac{s_{p_n}^2 s_n}{4\mathrm{E}\left|Y_{j,n}^3\right|}$ and $c_1, c_1' > 0$ are constants that do not depend on the random variables.

Proof. Using the classical Berry-Esséen bound we have for every $T > 0$ (see, for example, Theorem A.1 in [39]),

$$|\mathrm{P}(S_n \leq x s_n) - \Phi(x)| \leq \frac{1}{\pi}\int_{-T}^{T} \frac{1}{|t|}\left|\varphi_{S_n}(\tfrac{t}{s_n}) - e^{-t^2/2}\right| dt + \frac{24}{\pi\sqrt{2\pi}\,T},$$

where φ_{S_n} represents that characteristic function of S_n. To bound the integral above remember that $S_n = Y_{1,n} + \cdots + Y_{2r_n,n}$ and add and subtract the terms $\prod_{j=1}^{2r_n} \mathrm{E}e^{\frac{it}{s_n}Y_{j,n}}$ and $e^{-r_n t^2 s_{p_n}^2/s_n^2}$ inside the absolute value and separate the corresponding three integrals, and that, due to the strict stationarity, the blocks $Y_{j,n}$ have the same distribution as S_{p_n}. Now, using Newman's inequality for characteristic functions (Theorem 1 in Newman [33]), for the first integral obtained it follows immediately that,

$$\begin{aligned}
&\int_{-T}^{T} \frac{1}{|t|}\left|\mathrm{E}\exp\left(\tfrac{it}{s_n}\sum_{j=1}^{2r_n} Y_{j,n}\right) - \prod_{j=1}^{2r_n} \mathrm{E}e^{\frac{it}{s_n}Y_{j,n}}\right| dt \\
&\quad \leq \frac{1}{2}\int_{-T}^{T} \frac{1}{|t|}\sum_{j\neq j'} \frac{t^2}{s_n^2}\mathrm{Cov}(Y_{j,n}, Y_{j',n})\,dt = \frac{T^2}{2}\left(1 - \frac{2r_n s_{p_n}^2}{s_n^2}\right).
\end{aligned}$$

The third integral is also easily bounded. Indeed, using $|e^x - e^y| \le |x-y|$,

$$\int_{-T}^{T} \frac{1}{|t|} \left| e^{-r_n t^2 s_{p_n}^2/s_n^2} - e^{-t^2/2} \right| dt \le \frac{T^2}{2}\left(1 - \frac{2r_n s_{p_n}^2}{s_n^2}\right).$$

We have thus obtained the first two terms in the upper bound in (4.5). The remaining integral to analyse is

$$\int_{-T}^{T} \frac{1}{|t|} \left| \prod_{j=1}^{2r_n} \mathrm{E} e^{\frac{it}{s_n} Y_{j,n}} - e^{-r_n t^2 s_{p_n}^2/s_n^2} \right| dt = \int_{-T}^{T} \frac{1}{|t|} \left| \prod_{j=1}^{2r_n} \varphi_{Y_{j,n}}\left(\tfrac{t}{s_n}\right) - e^{-r_n t^2 s_{p_n}^2/s_n^2} \right| dt,$$

where $\varphi_{Y_{j,n}}$ is the characteristic function of $Y_{j,n}$. Let W_j, $j = 1, \ldots, 2r_n$, be random variables with the same distribution as $Y_{j,n}$ such that the two variables are independent. Then, for each $j = 1, \ldots, r_n$, $\mathrm{E}(W_j - Y_{j,n}) = 0$, $\mathrm{Var}(W_j - Y_{j,n}) = 2s_{p_n}^2$ and $\mathrm{E}\left|W_j - y_{j,n}\right|^3 \le 8\mathrm{E}\left|Y_{j,n}\right|^3$. Hence, for some $\theta \in (-1,1)$,

$$\left|\varphi_{Y_{j,n}}\left(\tfrac{t}{s_n}\right)\right|^2 = \varphi_{W_j - Y_{j,n}}\left(\tfrac{t}{s_n}\right) \le 1 - \frac{s_{p_n}^2 t^2}{s_n^2} + \frac{4\theta}{3}\frac{|t|^3 \mathrm{E}\left|Y_{j,n}\right|^3}{s_n^3} \le \exp\left(-\frac{s_{p_n}^2 t^2}{s_n^2} + \frac{4\theta}{3}\frac{|t|^3 \mathrm{E}\left|Y_{j,n}\right|^3}{s_n^3}\right),$$

and

$$\left|\prod_{j=1}^{2r_n} \varphi_{Y_{j,n}}\left(\tfrac{t}{s_n}\right)\right|^2 \le \exp\left(-\frac{2r_n t^2 s_{p_n}^2}{s_n^2} + \frac{8\theta}{3}\frac{|t|^3 \mathrm{E}\left|Y_{j,n}\right|^3}{s_n^3}\right).$$

Assume that $|t| \le T = \frac{s_{p_n}^2 s_n^2}{4\mathrm{E}\left|Y_{j,n}^3\right|}$. Then $\frac{8\theta}{3}\frac{|t|^3 \mathrm{E}|Y_{j,n}|^3}{s_n^3} \le \frac{2r_n t^2 s_{p_n}^2}{3s_n^2}$, thus

$$\left|\varphi_{Y_{j,n}}\left(\tfrac{t}{s_n}\right) - e^{-r_n t^2 s_{p_n}^2/s_n^2}\right| \le \exp\left(-\frac{2r_n t^2 s_{p_n}^2}{3s_n^2}\right) + \exp\left(-\frac{r_n t^2 s_{p_n}^2}{s_n^2}\right) \le 2\exp\left(-\frac{2r_n t^2 s_{p_n}^2}{3s_n^2}\right). \tag{4.6}$$

Another Taylor expansion gives, for some $\theta \in (-1,1)$,

$$\varphi_{Y_{j,n}}\left(\tfrac{t}{s_n}\right) = 1 - \frac{t^2 s_{p_n}^2}{2s_n^2} + \theta\frac{|t|^3 \mathrm{E}\left|Y_{j,n}\right|^3}{6s_n^3}. \tag{4.7}$$

If we assume now that $|t| \le \frac{s_n}{c_1\left(2r_n \mathrm{E}|Y_{j,n}|^3\right)^{1/3}}$, it follows from the previous inequality that

$$\left|\varphi_{Y_{j,n}}\left(\tfrac{t}{s_n}\right) - 1\right| \le \frac{1}{2(2r_n)^{2/3}c_1^2} + \frac{1}{12c_1^3 r_n},$$

which is, for n large enough, arbitrarily small, thus the characteristic function is bounded away from 0 for $|t| \le \frac{s_n}{c_1\left(2r_n \mathrm{E}|Y_{j,n}|^3\right)^{1/3}}$. Moreover, from (4.7) and taking into account the upper bound for $|t|$, it follows that

$$\left|\varphi_{Y_{j,n}}\left(\tfrac{t}{s_n}\right) - 1\right|^2 \le \frac{t^4 s_{p_n}^4}{2s_n^4} + \frac{t^6(\mathrm{E}\left|Y_{j,n}\right|^3)^2}{18s_n^6} \le \frac{|t|^3 \mathrm{E}\left|Y_{j,n}\right|^3}{s_n^3}\frac{1+18c_1^2}{36c_1^3}.$$

As the characteristic functions are bounded away from 0, we may take their logarithms, for which we find that, for some $\theta, \gamma \in (-1,1)$,

$$\log \varphi_{Y_{j,n}}\left(\tfrac{t}{s_n}\right) = -\frac{t^2 s_{p_n}^2}{2s_n^2} + \theta \frac{|t|^3 \operatorname{E}|Y_{j,n}|^3}{6s_n^3} + \gamma \frac{1+18c_1^2}{36c_1^3} \frac{|t|^3 \operatorname{E}|Y_{j,n}|^3}{s_n^3} = -\frac{t^2 s_{p_n}^2}{2s_n^2} + \eta \frac{|t|^3 \operatorname{E}|Y_{j,n}|^3}{2s_n^3},$$

where $\eta = \frac{\theta}{3} + \gamma \frac{1+18c_1^2}{18c_1^3}$. If we define $c_1' = \frac{1}{3} + \frac{1+18c_1^2}{18c_1^3}$, we have $|\eta| \leq c_1' \leq 1$, for c_1 conveniently chosen. Summing these bound for the logarithms, we find that

$$\log \varphi_{S_n}\left(\tfrac{t}{s_n}\right) = -\frac{r_n t^2 s_{p_n}^2}{s_n^2} + \eta \frac{r_n |t|^3 \operatorname{E}|Y_{j,n}|^3}{s_n^3},$$

and

$$\begin{aligned} \left| \prod_{j=1}^{2r_n} \operatorname{E} e^{\frac{it}{s_n} Y_{j,n}} - e^{-r_n t^2 s_{p_n}^2 / s_n^2} \right| &\leq e^{-r_n t^2 s_{p_n}^2/s_n^2} \left| e^{c_1' r_n |t|^3 \operatorname{E}|Y_{j,n}|^3 / s_n^3} - 1 \right| \\ &\leq \tfrac{c_1' r_n |t|^3 \operatorname{E}|Y_{j,n}|^3}{s_n^3} e^{c_1' r_n |t|^3 \operatorname{E}|Y_{j,n}|^3/s_n^3} e^{-r_n t^2 s_{p_n}^2/s_n^2}. \end{aligned}$$

In order to get an unified upper bound with (4.6) we choose the constant c_1 such that, for $|t| > \frac{s_n}{c_1 \left(2r_n \operatorname{E}|Y_{j,n}|^3\right)^{1/3}}$

$$\frac{c_1' r_n |t|^3 \operatorname{E}|Y_{j,n}|^3}{s_n^3} \geq \frac{c_1'}{2c_1^3} = \frac{6c_1^3 + 18c_1^2 + 1}{36c_1^6} \geq 2,$$

which is fulfilled if $c_1 < .7621$. For such a constant we have thus that

$$\left| \prod_{j=1}^{2r_n} \operatorname{E} e^{\frac{it}{s_n} Y_{j,n}} - e^{-r_n t^2 s_{p_n}^2/s_n^2} \right| \leq \frac{c_1' r_n |t|^3 \operatorname{E}|Y_{j,n}|^3}{s_n^3} e^{c_1' r_n |t|^3 \operatorname{E}|Y_{j,n}|^3/s_n^3} e^{-r_n t^2 s_{p_n}^2/s_n^2},$$

for every $|t| \leq T = \frac{s_{p_n}^2 s_n}{4\operatorname{E}\left|Y_{j,n}^3\right|}$. Taking into account this variation for t, it still follows that $\frac{c_1' r_n |t|^3 \operatorname{E}|Y_{j,n}|^3}{s_n^3} \leq \frac{c_1'}{2c_1^2}$. Furthermore, as $\frac{1}{n} s_n^2 \longrightarrow \sigma^2$ it follows $\frac{r_n s_{p_n}^2}{s_n} \longrightarrow 1$, hence, for n large enough, we have $\frac{1}{4} < \frac{r_n s_{p_n}^2}{s_n} < 1$, so that $e^{-r_n t^2 s_{p_n}^2/s_n^2} \leq e^{-t^2/4}$. Inserting this bounds in the integral we find the remaining upper bound in (4.5). ■

Theorem 4.3.1 show that the rate of the convergence $\frac{1}{n} s_n^2 \longrightarrow \sigma^2$ can play an important role on simplifying assumption **(A5)** in Theorem 4.2.1.

Theorem 4.3.2. *Let X_n, $n \geq 1$, be strictly stationary centered and associated random variables. Let $S_n = X_1 + \cdots + X_n$, $s_n^2 = \mathrm{E}S_n^2$. Assume that*

(B1) *the random variables X_n have finite moments of order $q \geq 3$;*

(B2) $x_n^2 = 2\gamma \log n$, *for some* $\gamma < \min\left(\frac{1}{2}, \frac{q}{2} - 1\right)$*;*

(B3) *for some* $\sigma^2 < \infty$, $\left|\frac{1}{n}s_n^2 - \sigma^2\right| = O(n^\beta)$, *for some* $\beta < 0$*;*

(B4) $u(n) = O\left(n^{-\frac{1+3\gamma}{1-\alpha}}\right)$, *where* $\alpha \in \left(\frac{3}{4} + \frac{\gamma}{2}, 1\right)$*;*

Then

$$\mathrm{P}(S_n > 2x_n s_n) = (1 - \Phi(x_n))(1 + o(1)). \tag{4.8}$$

Proof. We follow the arguments in the proof of Theorem 4.2.1, with $p_n \sim n^{1-\alpha}$. This produces a convergence term of order $n^{-\gamma}$. Now, we have to verify that the approximation to the Gaussian is, at least, as fast as the arte $n^{-\gamma}$. With respect to the proof Theorem 4.3.1 remark, from **(B3)** it follows that $\left|1 - \frac{2r_n s_{p_n}^2}{s_n^2}\right| = O(n^{\beta(1-\alpha)})$. Moreover, we have $T = \frac{s_{p_n}^2 s_n}{4\mathrm{E}|Y_{j,n}|^3} \sim n^{1/2}p_n^{-2} \sim n^{2\alpha-3/2} \longrightarrow \infty$, as $\alpha > \frac{3}{4}$. This implies that the two first terms in the upper bound in (4.5) are of order $T^2 n^{\beta(1-\alpha)} \sim n^{4\alpha-3+\beta(1-\alpha)}$ and $T^{-1} \sim n^{3/2-2\alpha}$. It follows from **(B4)** that both $4\alpha - 3 + \beta(1-\alpha) < -\gamma$ and $3/2 - 2\alpha < -\gamma$, thus converging faster that the order $n^{-\gamma}$ that comes from the arguments in course of proof of Theorem 4.2.1. Finally, the last term in the upper bound in (4.5) is easily verified to be of order $n^{3/2-2\alpha}$, line the term corresponding to T^{-1}, so the proof is concluded. ∎

Finally, we may state a result in the same spirit as Corollary 4.2.4. We state it without proof, as this is a very simple replication of the argument used to prove Corollary 4.2.4.

Corollary 4.3.3. *The result in Theorem 4.3.2 holds if we replace* **(B4)** *by*

(B4′) $u(n) = O\left(n^{-\theta}\right)$, *where* $\theta > 4 + \frac{20\gamma}{1-2\gamma}$.

4.4 An application

As an application of the previous results, consider a moving average model $X_n = \sum_{i=1}^{\infty} \phi_i \varepsilon_{n-i}$, where the ε_n are independent and identically distributed with mean 0, variance 1 and finite moments of order $q > 2$, and $\phi_n > 0$, so the X_n, $n \geq 1$, are associated. Using Hölder inequality it easily follows that X_n has finite moment of order q, for some $\rho \in (0,1)$, $\sum_{i=1}^{\infty} \phi_i^{\rho q} < \infty$ and $\sum_{i=1}^{\infty} \phi_i^{(1-\rho)q/(q-1)} < \infty$. Concerning the covariances, whose control is needed in order to verify **(A4′)**, it is easily verified that

$$\mathrm{Cov}(X_1, X_n) = \sum_{i=1}^{\infty} \phi_i \phi_{n-1+i} \leq \left(\sum_{i=n}^{\infty} \phi_i^{\tau} \right)^{1/\tau} \left(\sum_{i=1}^{\infty} \phi_i^{\tau'} \right)^{1/\tau'}, \tag{4.9}$$

where $\tau, \tau' > 1$ are such that $\tau^{-1} + (\tau')^{-1} = 1$. So, **(A4′)** is verified if the moving average coefficients satisfy, for some $\tau > 1$,

$$\begin{aligned}
&\phi_n \longrightarrow 0, \\
&\sum_{i=1}^{\infty} \phi_i^{s} < \infty, \text{ where } s = \min\left(\rho q, \frac{(1-\rho)q}{q-1}, \frac{\tau}{\tau - 1} \right), \ \rho \in (0,1), \\
&\sum_{i=n}^{\infty} \phi_i^{\tau} \sim n^{-\theta\tau}, \ \theta > (1+3\gamma) \max\left(2 + \frac{4}{q-2}, 2 + \frac{4\gamma+2}{q-2\gamma-2} \right).
\end{aligned}$$

Assume now that the coefficients verify $\phi_n \sim n^{-a}$, for some $a > 0$. We need to adjust the choice of the decrease rate, that is, the exponent a, in order to meet the requirements discussed above. To have the appropriate finite moment of order q for the X_n we need to ensure the convergence of the above mentioned series. This follows if we can choose $\rho \in (0,1)$ such that $a\rho q > 1$ and $\frac{(1-\rho)aq}{q-1} > 1$, that is $\frac{1}{aq} < \rho < 1 - \frac{aq}{q-1}$. Such a choice is always possible as soon as $a > 1$. Inserting now the behaviour of the ϕ_i in (4.9) it follows that $\mathrm{Cov}(X_1, X_n) \sim n^{-(a+1/\tau)}$, so that the Cox-Grimmett coefficient $u(n) \sim n^{-(a+1+1/\tau)}$, where $\tau > 1$ is arbitrarily chosen. Thus, in order to verify **(A4′)** we must require $a + 1 + \frac{1}{\tau} > (1+3\gamma)\max\left(2 + \frac{4}{q-2}, 2 + \frac{4\gamma+2}{q-2\gamma-2}\right)$, where $0 < \gamma < \frac{q}{2} - 1$ and $q > 2$. So, finally, taking into account the liberty to choose τ, it is enough to require that $a > (1+3\gamma)\max\left(2 + \frac{4}{q-2}, 2 + \frac{4\gamma+2}{q-2\gamma-2}\right) - 2$. A condition based on the more usable Corollary 4.3.3 would require $a + 1 + \frac{1}{\tau} > 4 + \frac{20\gamma}{1-2\gamma}$ or, using the liberty to choose τ, $a > 2 + \frac{20\gamma}{1-2\gamma}$ and we should remember that in this case we must have $0 < \gamma < \min\left(\frac{1}{2}, \frac{q}{2} - 1\right)$ and $q \geq 3$.

5. Conclusions and future work

Studying convergence rates of weighted sums of associated random variables and moderate deviations of sums of associated and strictly stationary random variables were the objectives of this dissertation.

We showed the convergence of weighted sums of associated random variables normalized by $n^{1/p}$ where $p \in (1,2)$ assuming moments larger than p. Some previous results by Oliveira [38] are extended by relaxing the moment assumption on the random variables, approaching the $p{-}th$ order moment assumptions used by Louhichi [26] to prove the convergence for constant weights, while strengthening the assumption on the decay rate of the covariances. We also considered the Marcinkiewicz-Zygmund law with assumptions on the 2-dimensional analogue of tail probabilities of the random variables relaxing in this case the assumption on the decay rate on the covariances. Our results extended analogous characterizations known for sums of independent or negatively dependent random variables.

We utilized a truncation technique together with coupling with independent variables which allows a relaxation of the assumptions on the weights. The assumptions on p, which depends on the asymptotic behaviour of the weights, as usual, now includes the case $p < 1$, which was excluded from earlier results for positively associated variables. Also, we gave a direct comparison with the characterizations previously available, showing that the scope of applicability of the results obtained in this chapter does not overlap with previously known conditions for the same asymptotic results.

Finally, we presented a moderate deviation in the non-logarithmic scale for sums of associated and strictly stationary random variables with finite moments of order larger than 2.

As sums of associated random variables are concerned; a possible improvement on the regularity conditions of the coefficients is not likely to be achieved. Rather one should look for characterizations of the asymptotics of weighted sums for more general normalizations, in the same sprit as explored by Cuzick [15], that is considering logarithmic factors in the normalizing sequence. Also, searching for more general description of normalizing sequences allowing to prove almost sure results should be addressed.

Still about weighted sums consideration of random weights could be an interesting problem. Both previous problems could contribute to some results on characterisation of consistency of non-parametric regression estimators based on associated sample.

As what concerns moderate deviation these results must be accompanied by large deviation counterparts. This will require a rather different approach to the problem. Having such characterization will allow for some explicit contributions for approximating a few risk measures that are popular in application literature.

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