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MODELING AND NUMERICAL ANALYSIS IN OPTION MARKET WITH MEMORY

Dissertação de Doutoramento na área científica de Economia, orientada pelo Professor Doutor José Augusto Mendes Ferreira e Professor Doutor Hélder Miguel Correia Virtuoso Sebastião e apresentada à Faculdade de Economia da Universidade de Coimbra.

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FEUC FACULDADE DE ECONOMIA
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Tese de Doutoramento em Economia, apresentada à Faculdade de

Economia da Universidade de Coimbra para obtenção do grau de Doutor

Orientadores: Professor Doutor José Augusto Mendes Ferreira e Professor Doutor Hélder Miguel Correia Virtuoso Sebastião

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I dedicate this work with great affection to the three most important persons in my life. My mother Arlete, who is now with our Great Father. To my forever friend, my big daddy José Carlos. And to my beautiful Rachel Ragaglia, owner of my love.

Dedico com muito carinho este trabalho às três pessoas mais importantes de minha vida. À minha mãe Arlete, que hoje está junto de nosso Pai Maior. Ao meu amigo de sempre, meu grande pai José Carlos. E à minha linda Rachel Ragaglia, dona do meu amor.

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Resumo

Esta tese apresenta uma nova proposta para a modelação do comportamento dos preços dos activos financeiros, cujo aspecto mais relevante é a inclusão de estruturas de memória. De forma a motivar novos modelos de avaliação de opções, a presente tese inicia-se com a apresentação de vários modelos para os activos subjacentes, começando com uma formalização mais clássica e aumentando o nível de complexidade até aos modelos mais sofisticados onde já se encontra presente a memória no preço e na volatilidade do activo. Tendo por base estes modelos, são posteriormente derivados os modelos de avaliação de opções tradicionais. Estes modelos são representados por sistemas diferenciais sem solução analítica, o que implica a utilização de métodos numéricos. Na presente tese foi adotado o método de Galerkin. Para a solução do problema algébrico foram utilizados dois métodos: o *Interative Method of Successive Over-Relaxation - SOR* e o algoritmo de Picard.

Ao longo da tese existe um esforço matemático assinalável para modelar os preços das opções na presença de estruturas de memória (processo JTDD). E, dada a necessidade de teoria estatística, em especial do Lema de Itô, é apresentada a formulação deste Lema para o processo JTDD com coeficientes não constantes. Também de igual importância na área do cálculo estocástico, é aqui demonstrada a exponencial estocástica para aquele processo.

Um ponto retratado na presente tese, muito relevante para a economia financeira, é a possibilidade das séries dos preços dos activos financeiros apre-

sentarem algum tipo de persistência, sobretudo para periodicidades intradiárias. A memória nas séries financeiras é aqui captada através da consideração de processos de telégrafo.

Abstract

This thesis provides a new proposal for modeling the dynamics of financial prices that takes into account memory structures. In order to motivate new option pricing models, this thesis presents several models for the price of the underlying asset, beginning with the classical models and increasing the complexity until more sophisticated models with memory in price and in volatility. In this framework, new pricing models are derived for plain vanilla options. These models are represented by differential systems with no analytical solution, and therefore they impose the use of numerical methods. Here it is adopted the Galerkin method, and the solution of the algebraic problem is found using two methods: the Interactive Method of Successive Over-Relaxation (SOR) and Picard algorithm.

Throughout this thesis there is a significant mathematical effort aiming to model option prices in the presence of memory structures (JTDD process). Given the need for statistical theory, in particular for the Itô's Lemma, here it is shown the formulation of the Itô's lemma for JTDD process with non-constant coefficients. Also of great importance in the stochastic calculus field, the thesis presents a demonstration of the exponential stochastic process for JTDD.

An important issue for financial economics is the possibility that series have some kind of persistence, at least in a high frequency setting. In this thesis this aspect of the prices time series is captured through the consideration of Telegraph processes.

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1 Introduction

What is the fair value of a financial option? This is probably one of the most important questions in the modern finance framework. However, although this question maintain its pertinence today, it is quite old, and it can be traced back to the beginning of the XX century, when it can be found in the work of Louis Bachelier[2] in 1900. Only in the late 1960's Black, Scholes and Merton provided a satisfactory solution to the problem, for which they have received a Nobel Prize in economics.

After Black and Scholes[3], there were several seminal studies on this topic. For instance, in 1973 Merton[4] presented a model for option pricing on a stock that pays dividends, showing also that an American call option on a stock that pays no dividend, can be priced as an European call option. Cox and Ross[5] presented option pricing models considering other stochastic processes than the geometric Brownian motion for the underlying asset. More recently, in 2002, Lisa Borland[6] brought back this issue, supporting her research on findings in nonextensive statistical mechanics by Tsallis[7] in 1988 and Curado and Tsallis[8] in 1991. According to Tsallis the random walk process can be interpreted as a generalized Wiener process, governed by a Tsallis probability distribution, for the entropy with nonextensive thermodynamic parameter "q". Borland[6] presented a generalized form of the Black and Scholes model, for which she derives the partial differential equations and closed form solutions for European options. The author shows that when "q tends to 1" the geometric Brownian motion is recovered and thus the model recovers the basic theory developed by Black and Scholes[3].

In 1976, Merton[9] proposed a dynamic process for the underlying asset that essentially reassembles a geometric Brownian motion with random jumps with a Poisson distribution. These jumps were interpreted as a component of non-systematic risk. In this framework, jumps model the arrival of new information, unanticipated by the financial markets, and imply discontinuities in the underlying asset price process.

Recent research in option pricing brought into discussion another level of complexity by considering some kind of memory structure in the price of the underlying asset. This new research path intends to achieve a more realistic way of modeling the data features of financial prices. In fact, in a high frequency setting prices have memory mainly due to market microstructure features and volatility has a hyperbolic decaying rate. Based on the ideas originally presented by Di Crescenzo and Pellerey[10], where *Geometric Telegraph Processes* are proposed as way to describe the price dynamics of risky assets, Nikita Ratanov[11, 12, 13], presented a new class of models of financial prices based. These models are constructed in a no-arbitrage and complete world, if the price's jumps have a certain correspondence with the behaviour of interest rates, the model can be complete without assuming the existence of another asset with the same sources of randomness. These three articles present detailed descriptions of the *Telegraph Processes*. In paper [11] the author proposes a Jump-telegraph model for the dynamics of the underlying asset. Ratanov assumed that the volatility is equal to zero, which is, in fact, a completely inadequate assumption if one intends to model financial prices. In the other two papers, the author tackles the problem of European options and derives explicit pricing formulas, however also here there are some problems. For instance, in [12] it is assumed that the price of the underlying asset has a stochastic component governed by a telegraph process, but this process is completely unrealistic because it does not accommodates some main features of financial data. Probably, the most important criticism to the work of Ratanov, which is in fact the main criticism to the Black and Scholes model, is that the author assumes that the price volatility is a constant.

Currently, the majority of academic work is no longer based on the unrealistic assumption that the volatility of the underlying asset is constant over time. In most cases, volatility is viewed as having a deterministic and a stochastic parts. Sometimes, the stochastic component explicitly considers its temporal dependence, i.e. volatility has memory. Nevertheless, models with this feature usually do not reach nice numerical results. For example, this happens in the "finite memory" model of Arriojas at el. [14] and in the "infinite memory" model of Kazmerchuk et al. [15].This model may be represented by a Markov system with more state variables than the number of risky assets.

Engle[16] in 1982 presented ARCH-type models (again, a Nobel Prize in economics). Posteriorly in 1986, Bollerslev[17] generalizes the dynamics of the volatility process, introducing the GARCH process as a model with good forescast abilities. Also in 1986, Stephen J. Taylor[18] proposed the concept that the volatility itself followed a stochastic process. Hull and White[19] in 1987 examined the option pricing problem when the underlying asset has stochastic volatility, being the option price resolved in series form in the case of independent stochastic volatility.

In 2000, Broadie et al.[20] presents a formulation for pricing American options, when the underlying asset has stochastic dividends and volatility. Because the theoretical model is very complex, the authors introduce a non-parametric approach that allows the analysis of the short-form and the testing of the decision on early exercises, as it is defined by Harvey and Whaley[21, 22] and Fleming and Whaley[23].

In 1998, Hobson and Rogers[24] proposed an original class of models of the price process of a financial security in continuous time setting, considering a non-constant volatility. They defined volatility in terms of exponentially weighted moments of historic log-price and thus the instantaneous volatility is driven by the same stochastic factors as the process price. Therefore, unlike many other models of stochastic volatility, it is not necessary to introduce new sources of randomness. In 2004, Di Francesco and Pascucci[25]presented a complete model with stochastic volatility in the sense of Hobson and Rogers, concluding that, in this framework, option prices are solutions to degenerate partial differential equations obtained from the inclusion of other state variables describing the dependence on the past prices of the underlying asset. In 2009, Foschi and Pascucci[26] empirically tested option pricing models with Hobson and Rogers' volatility, achieving positive results, as they could replicate the observe "smile efffect" and the patterns of the implied volatility structure. A calibration procedure based on an ad-hoc numerical scheme for "hypoelliptic PDEs" is proposed and used for the performance analysis of the model price, using a data on the $S\&P500$ option prices.

The early exercise feature, present in the American style options, has proven to be of great complexity has it implies the nonexistence of analytical pricing solutions and therefore it became necessary to use numerical methods.

Boyle[27] introduced the Monte Carlo simulation method to evaluate options. However, there are many studies that consider this technology inapplicable to American options, since their algorithm specification involves determining the optimal strategy of early exercise, by an action of dynamic "backwards" programming.

In 1977 and 1978, Brennan and Schwartz[28, 29] applied finite difference methods to solve the partial differential equations for an American option. Cox et al.[30] used a discrete process in time and binomial in space for approximate the continuous process. Johnson[31] and Geske and Johnson[32] showed how to value a American put option using the extrapolation method of Richardson. Duffie[33] presented the Crank-Nicholson algorithm for finite differences for the calculation of debentures and options on them. Hull and White[34] suggested changes in the explicit finite difference scheme, similar to that presented by Courtadon[35], ensuring that options have positive transition probability, in allusion to the trinomial model. According to Hull[36] the finite difference methods are commonly used to price exotic derivatives, including European and American options on debentures and interest rates. However, these methods are very complicated and "expensive" computationally, as its complexity increases substantially with the number of variables that determine the option value, simply because the mesh becomes multidimensional.

Broadie and Glasserman[37] presented a simulation algorithm to estimate the American option prices, providing point estimates of error and confidence intervals for the true price value. Oliveira[38] and Rochman[39] presented comparisons of various numerical methods for solving this problem. Marcozzi, et al.[40] used radial basis functions, and Song Wang[41] applied the volume method. The use of finite element methods is presented in my thesis, Thomaz[42] defended at LNCC - Brazil and also in my recent work[43], presented at the 5th Portuguese Finance Network Conference. In this paper I present a formulation for an option on various underlying assets (multi-assets) and its numerical solution using finite element methods in order to study the influence of the definition of the range of the underlying asset price for the numerical solution of option pricing on one and two assets.

The derivation of the option pricing formulae can be based on an equivalent local martingale measure or Itô Lemma, therefore this work presents the formulation of two models for pricing European and American options considering memory in the price of the underlying asset. For this task it was necessary to use some stochastic calculus tools, which are shown in Chapter 2. Chapter 3 presents a numerical simulation of SDE, and a method for studying its convergence. Chapters 4 and 5 give a general idea of underlying asset models under scrutiny, namely in Chapter 4 there is an overview of the classical model introduced by Black and Scholes, as well as other models suggested in the literature that overcome some critics and flaws of the classical model. Chapter 5 presents the stochastic processes of the underlying asset that serve the basis for new option prices models. In fact, it was necessary to reformulate the Itô's lemma for this new classes of stochastic Processes. This is conducted in Chapter 6, with the proof of the Itô's lemma for the jump-telegraph-diffusion-drift stochastic process. Chapter 7 derives the option pricing formulae, in the presence of this new underlying asset pricing processes, motivates the use of the Galerkin method for solving the resulting differential equations system.

2 Concepts in Stochastic Calculus

This chapter introduces some concepts and properties on non-random functions, stochastic processes and stochastic calculus. All functions are defined on $[0, +\infty[$. The main purpose is to prepare the groundwork for a discussion on modeling underlying assets and options prices.

This chapter also presents the Brownian motion or Winner process , which plays an important role in the classic valuation problem of the underlying asset. Pricing models for derivative assets are usually formulated in continuous time, but these models are normally applied in discrete small time intervals.

2.1 Stochastic Processes

Time series of financial prices are commonly interpreted as a sequence of random variables, in continuous time, i.e. as a stochastic process. A basic block for modeling prices of financial assets is the Brownian motion or Winner process.

The Brownian motion or Winner process, $W(t)$, is a continuous random process, that can be used to model the cumulative effect of a pure white noise. This process satisfies the following assumptions:

1. **Independence of Increments:** $W(t) - W(s)$, for $t > s$, is independent of the past, that is, of $W(u)$, for $u \in [0, s]$, or of $\mathfrak{F}_s$, the σ-field generated by $W(u)$, $u \in [0, s]$.

2. **Normal Increments:** $W(t) - W(s)$, $u < s,$ has a normal distribution with zero mean and variance $t - s.$

   This assumption implies (taking $s = 0$) that $W(t) - W(0)$ has a $N(0, t)$ distribution.

3. **Continuity of Paths:** $W : t \to W(t)$ is continuous.

Let $W(t)$ be a Wiener process and $\Delta W(t)$ an increment corresponding to a time increment Δt. Then

1. $\Delta W(t) = \varepsilon(t)\sqrt{\Delta t}$, where $\varepsilon(t)$ is a random variable that follows a standard normal distribution with zero mean and unitary standard deviation;

2. The random variables $\varepsilon(t), t \geq 0,$ are not serially correlated, that is $E(\varepsilon(t)\varepsilon(t-1)) = 0$ for $t \neq 0.$

When the time interval Δt becomes infinitesimally small, we can represent the incremental changes in the Wiener process, dW, in continuous time, as

$$dW(t) = \varepsilon(t)\sqrt{dt}. \tag{2.1}$$

A standard Brownian motion with trend is an extension of the above process, being represented by the following stochastic equation

$$dS(t) = \mu dt + \sigma dW(t)\ , \quad \text{for a given } S(0), \tag{2.2}$$

where μ is the trend (or growth) parameter, σ is the variance parameter and $S = \{S(t)\}_{t \geq 0}$ is the stochastic process, for example, of an asset price. For any time interval Δt, the increment in $S(t)$, has a Normal distribution, with mean $E(\Delta S(t)) = \mu \Delta t$, and variance $Var(\Delta S(t)) = \sigma^2 \Delta t$.

2.2 Variations of Stochastic Processes

This section discusses some concepts in stochastic calculus that will be used in the formulation of both models for the underlying asset and for the option prices. First, it presents functions variations and then stochastic process variations, which are the foundations for the next section.

Definition 2.2.1 (First Variation of Function)**.** *If f is a function of a real variable, its variation on $[0,T]$ is defined as*

$$FV_{[0,T]}(f) = \sup \sum_{k=0}^{n-1} |f(t_{k+1}^n) - f(t_k^n)| \ , \tag{2.3}$$

where the supremum is taken over all partitions $\pi_n = \{t_0^n, t_1^n, \cdots, t_n^n\}$,

$$0 = t_0^n \leq t_1^n \leq \cdots \leq t_n^n = T \ ,$$

of $[0,T]$.

By the triangle inequality, the sums in (2.3) increase as new points are added to the partitions. Therefore, the variation of f is given by

$$FV_{[0,T]}(f) = \lim_{||\pi_n|| \to 0} \sum_{k=0}^{n-1} |f(t_{k+1}^n) - f(t_k^n)| \ . \tag{2.4}$$

where $||\pi_n||$ is defined as

$$||\pi_n|| = \max_{k=0,\cdots,n-1} (t_{k+1}^n - t_k^n) \ ,$$

If $FV_{[0,T]}(f)$ is finite then f is said to be a function of finite variation on $[0,T]$.

Next there are two examples that motivate the first variation of a function as defined in 2.2.1.

Example 2.2.1. *If f is differentiable and f' is integrable, then the Mean Value Theorem implies that in each subinterval $[t_k, t_{k+1}]$, there is a point t_k^* such that*

$$f(t_{k+1}^n) - f(t_k^n) = f'(t_k^*)(t_{k+1}^n - t_k^n).$$

Consequently

$$\sum_{k=0}^{n-1} |f(t_{k+1}^n) - f(t_k^n)| = \sum_{k=0}^{n-1} |f'(t_k^*)|(t_{k+1}^n - t_k^n)$$

and

$$FV_{[0,T]}(f) = \lim_{||\pi_n|| \to 0} \sum_{k=0}^{n-1} |f'(t_k^*)|(t_{k+1}^n - t_k^n) = \int_0^T |f'(t)|dt.$$

Example 2.2.2. *Let f be a regular right-continuous (càdlàg) function or regular left-continuous (cáglád) in $[0,T]$ that changes only in jumps:*

$$f(t) = \sum_{0 \leq s \leq t} |\Delta f(s)| \, , \tag{2.5}$$

where $\Delta f(t) = f(t_+) - f(t_-)$, with

$$f(t_-) = \lim_{s \uparrow t} f(s) \quad (f \textit{ is a left-continuous function}) \, ,$$

and

$$f(t_+) = \lim_{s \downarrow t} f(s) \quad (f \textit{ is a right-continuous function}) \, .$$

Then it is easy to see from the definition that

$$FV_{[0,T]}(f) = \sum_{0 \leq s \leq t} |\Delta f(s)| \, . \tag{2.6}$$

Definition 2.2.2 (Finite/Bounded Variation of a Function)**.** *Let f be of finite variation if $FV_t(f) < \infty$ for all t. Then function f is of bounded variation if $sup_t FV_t(f) < \infty$. In other words, for all t, $FV_t(f) < C$, where C is a constant independent of t.*

Definition 2.2.3 (Quadratic Variation of a Function)**.** *If f is a function of real variable, its quadratic variation on $[0,T]$ is defined as*

$$\langle f, f \rangle_t = \lim_{||\pi_n|| \to 0} \sum_{k=0}^{n-1} \Big[f(t_{k+1}^n) - f(t_k^n) \Big]^2 \, , \tag{2.7}$$

if the limit exists, where π_n and $||\pi_n||$ have the same meaning as in Definition 2.2.1

Theorem 2.2.1. *If f is continuous and of finite variation on $[0, T]$, then its quadratic variation on $[0, T]$ is zero.*
Proof: *See Klebaner[44]*

It can be easily proved that if f is differentiable and f' is integrable, then

$$\langle f, f\rangle_t = 0 \ .$$

In fact, since

$$\sum_{k=0}^{n-1}\Big[f(t_{k+1}^n) - f(t_k^n)\Big]^2 = \sum_{k=0}^{n-1}\Big[f'(t_k^*)\Big]^2 (t_{k+1}^n - t_k^n)^2 \le ||\pi_n|| \sum_{k=0}^{n-1}\Big[f'(t_k^*)\Big]^2 (t_{k+1}^n - t_k^n)$$

then

$$\langle f, f\rangle \le \lim_{||\pi_n||\to 0} ||\pi_n|| \lim_{||\pi_n||\to 0} \sum_{k=0}^{n-1}\Big[f'(t_k^*)\Big]^2 (t_{k+1}^n - t_k^n) = \lim_{||\pi_n||\to 0} ||\pi_n|| \int_0^T \Big[f'(t)\Big]^2 dt \ ,$$

where the last limit equals to zero.

Definition 2.2.4 (Quadratic Covariation Between Two Function)**.** *If f and g are real variable functions, the quadratic covariation (or simply covariation) of f and g on $[0, T]$ is defined (when it exists) as*

$$\langle f, g\rangle_t = \lim_{||\pi_n||\to 0} \sum_{k=0}^{n-1}\Big[f(t_{k+1}^n) - f(t_k^n)\Big]\Big[g(t_{k+1}^n) - g(t_k^n)\Big] \ , \tag{2.8}$$

where π_n and $||\pi_n||$ have the same meaning as in the previous definitions.

Theorem 2.2.2. *If f is continuous and g is of finite variation on $[0, T]$, then their covariation on $[0, T]$ is zero, i.e.*

$$\langle f, g\rangle_t = 0 \ .$$

The proof of this result is similar to the proof of Theorem 2.2.1.

The previous definitions are presented hereafter in the context of stochastic processes with the same probability space $(\Omega, \mathfrak{F}, \mathbb{P})$.

The following definition of a quadratic variation of a stochastic process is given in Protter[45].

Definition 2.2.5 (Quadratic Variation of Stochastic Process)**.** *Suppose that $Y(t)$ is a real-valued stochastic process defined on a probability space ($\Omega, \mathfrak{F}, \mathbb{P}$) and with time index t ranging over the non-negative real numbers. Its quadratic variation is the process defined by*

$$\langle Y, Y\rangle_t = \lim_{||\pi_n||\to 0} \sum_{k=0}^{n-1} \Big[Y(t_{k+1}) - Y(t_k) \Big]^2 , \tag{2.9}$$

where π ranges over all partitions of the interval $[0, t]$ and the norm of the partition π is the mesh size. This limit, if it exists, is defined using convergence in probability.

Definition 2.2.6 (Quadratic Covariation of Stochastic Process)**.** *The quadratic covariation of two processes X and Y is defined by*

$$\langle X, Y\rangle_t = \lim_{||\pi_n||\to 0} \sum_{k=0}^{n-1} \Big[X(t_{k+1}) - X(t_k) \Big] \Big[Y(t_{k+1}) - Y(t_k) \Big] . \tag{2.10}$$

The quadratic variation and covariation of processes X and Y are usually denoted by

$$\langle Y, Y\rangle_t = \int_0^t [dY]^2 \quad \text{and} \quad \langle X, Y\rangle_t = \int_0^t dX dY$$

Equivalently it can be used the differential notations

$$d\langle Y, Y\rangle_t = [dY(t)]^2 \quad \text{and} \quad d\langle X, Y\rangle_t = dX(t)dY(t) .$$

Definition 2.2.7 (Finite Variation of Stochastic Process)**.** *A process Y is said to have finite variation if it has bounded variation over every finite time interval (with probability 1).*

Remark 2.2.1 (Properties of the Quadratic Variation of a Stochastic Process)**.** *Here are stated and explained the fundamental properties of the quadratic variation process, proofs are omitted as they can be found in Klebaner[44].*

1. *If $Y(t)$ is a semimartingale, then $\langle Y, Y\rangle_t$ exists and is an adapted process.*

2. *It is clear from the definition 2.2.5 that quadratic variation over non-overlapping intervals is the sum of the quadratic variation over each interval. As such,* $\langle Y, Y \rangle_t$ *is a non-decreasing function of* t*. Consequently* $\langle Y, Y \rangle_t$ *is a function of finite variation.*

3. *It follows from the Definition 2.2.6, SDE (2.10), that* $\langle X, Y \rangle_t$ *is bilinear and symmetric, that is,* $\langle X, Y \rangle_t = \langle Y, X \rangle_t$ *and*

$$\langle \alpha X + Y, \beta U + V \rangle_t = \alpha\beta\langle X, U \rangle_t + \alpha\langle X, V \rangle_t + \beta\langle Y, U \rangle_t + \langle Y, V \rangle_t \, . \tag{2.11}$$

4. *Polarization identity*

$$\langle X, Y \rangle_t = \frac{1}{4}\Big[\langle X + Y, X + Y \rangle_t - \langle X - Y, X - Y \rangle_t\Big] \, , \tag{2.12}$$

which can be written as

$$\langle X, Y \rangle_t = \frac{1}{2}\Big[\langle X + Y, X + Y \rangle_t - \langle X, X \rangle_t - \langle Y, Y \rangle_t\Big] \, . \tag{2.13}$$

This property follows directly from the previous one.

5. $\langle X, Y \rangle_t$ *is a regular right-continuous (càdlàg) function with finite variation. This follows from the polarization identity, as* $\langle X, Y \rangle_t$ *is the difference of two increasing functions.*

6. *The jumps of the quadratic covariation process occur only at points where both processes have jumps,*

$$\Delta\langle X, Y \rangle_t = \Delta X(t)\Delta Y(t) \, .$$

7. *If one of the processes,* X *or* Y*, has finite variation, then*

$$\Delta\langle X, Y \rangle_t = \sum_{s \le t} \Delta X(s)\Delta Y(s) \, .$$

Notice that although the summation is taken over all s *not exceeding* t*, there are at most a countable number of terms different from zero.*

Remark 2.2.2. *The first variation of a Brwonian motion, $W(t)$, on $[0,T]$ is not finite.*

This classical result shown in Lévy [46], is also presented in Bingham and Kiesel[47]. However, an alternative proof of this result is presented here.

Theorem 2.2.3 (Lévy). *The quadratic variation of a Brownian motion over $[0,T]$ exists and equals T, in mean square (and hence in probability):*

$$\langle W, W\rangle_t = T \tag{2.14}$$

Proof: *Let $\{t_0, t_1, \cdots, t_n\}$ be a partition of $[0,T]$. To simplify the presentation consider $D_k = W(t_{k+1}) - W(t_k)$. Then Q_π is the quadratic variation, such that*

$$Q_\pi = \sum_{k=0}^{n-1} D_k^2 \, .$$

Then

$$Q_\pi - T = \sum_{k=0}^{n-1} \Big[D_k^2 - [t_{k+1} - t_k] \Big] \, .$$

Consider an individual summand $D_k^2 - [t_{k+1} - t_k]$. As

$$E\Big(D_k^2 - [t_{k+1} - t_k] \Big) = 0 \, ,$$

it follows that

$$E(Q_\pi - T) = E\left(\sum_{k=0}^{n-1} \Big[D_k^2 - [t_{k+1} - t_k] \Big] \right) = 0.$$

For $j \neq k$, the terms $D_j^2 - [t_{j+1} - t_j]$ and $D_k^2 - [t_{k+1} - t_k]$ are independent because *the increments of* W *are independent. So*

$$\begin{aligned} Var(Q_\pi - T) &= \sum_{k=0}^{n-1} Var\Big[D_k^2 - [t_{k+1} - t_k]\Big] \\ &= \sum_{k=0}^{n-1} E\Big[D_k^4 - 2[t_{k+1} - t_k]D_k^2 - [t_{k+1} - t_k]^2\Big] \\ &= \sum_{k=0}^{n-1} \Big[3[t_{k+1} - t_k]^2 - 2[t_{k+1} - t_k]^2 + [t_{k+1} - t_k]^2\Big] \\ &= 2\sum_{k=0}^{n-1} \Big[t_{k+1} - t_k\Big]^2 \\ &\leq 2||\pi_n||T \ . \end{aligned}$$

Thus

$$E(Q_\pi - T) = 0, \quad \textit{and} \quad Var(Q_\pi - T) \leq 2||\pi_n||T.$$

As $||\pi_n|| \to 0$*,* $Var(Q_\pi - T) \to 0$*, and*

$$Q_\pi - T \xrightarrow[||\pi_n|| \to 0]{L^2} 0 \ .$$

This proof is concluded once one considers that the L^2 *convergence implies the convergence in probability. The definition of convergence concepts can be seen in Arnold[48].*

□

As corollary of this result, the quadratic variation of a Brownian motion is not finite. It can be shown that the variation of order p is finite if and only if $p > 2$.

Remark 2.2.3 (Differentiable Representation)**.** *Consider that*

$$E\Big(\Big[W(t_{k+1}) - W(t_k)\Big]^2 - [t_{k+1} - t_k]\Big) = 0 \ .$$

and, as it has been showed above, that

$$Var\Big(\Big[W(t_{k+1}) - W(t_k)\Big]^2 - [t_{k+1} - t_k]\Big) = 2[t_{k+1} - t_k]^2 \ .$$

Then when $[t_{k+1} - t_k]$ *is small,* $[t_{k+1} - t_k]^2$ *is very small, implying the approximate equation*

$$\Big[W(t_{k+1}) - W(t_k)\Big]^2 \simeq t_{k+1} - t_k$$

which may be informally written as

$$dW(t)dW(t) = dt.$$

Note that Brownian motion paths are not differentiable in the ordinary sense of calculus. Therefore Ito calculus must be used instead.

The Brownian process applied to financial series has known limitations, such as infinite first variation and independence of log-returns increments that leads to a pathologic behaviour of asset prices. In other words, the transition density of the Brownian motion satisfies the heat equation, which is characterized by infinite propagation speed, reflecting a pathology induced by the mathematical model.

The prototype of a stochastic processes with finite variation is the telegraph process (see Goldstein [49] and Kac [50]) that describes the position of a particle moving on the real line, alternatively with constant velocity $+v$ or $-v$. The changes of direction are governed by a homogeneous Poisson process $N(t)$ with rate $\lambda > 0$.

The Telegraph process[1] is defined by

$$X(t) = V(0) \int_0^t (-1)^{N(\nu)} d\nu \ , \quad t > 0 \ , \tag{2.15}$$

where the initial velocity $V(0)$ assumes the values $\pm v$ with equal probability and independence of $\{N(t)\}_{t>0}$.

The description of the price dynamics of a financial asset can be performed using the following telegraph Process: Let $N_+ = \{N_+(t)\}_{t\geq 0}$ and $N_- =$

[1]In the literature, this process is alternatively called the Telegraph process or the Telegrapher's process.

$\{N_-(t)\}_{t\geq 0}$ be two counting Poisson processes with alternating intensities λ_+, λ_-, λ_+, $\cdots$ and λ_-, λ_+, λ_-, $\cdots$, respectively; that is, as $\Delta t \to 0$,

$$\begin{aligned}
\mathbb{P}(N_+(t+\Delta t) = 2n+1 | N_+(t) = 2n) &= \lambda_+ \Delta t + o(\Delta t), \\
\mathbb{P}(N_+(t+\Delta t) = 2n+2 | N_+(t) = 2n+1) &= \lambda_- \Delta t + o(\Delta t), \\
\mathbb{P}(N_-(t+\Delta t) = 2n+1 | N_-(t) = 2n) &= \lambda_- \Delta t + o(\Delta t), \\
\mathbb{P}(N_-(t+\Delta t) = 2n+2 | N_-(t) = 2n+1) &= \lambda_+ \Delta t + o(\Delta t).
\end{aligned}$$

where $n = 0, 1, \cdots$, $o(\Delta t)$ is the "Landau Notation" or "Asymptotic Notation", defined by $f = o(\Delta t)$ meaning that "$f/\Delta t \to 0$" (see Hardy and Wright[51]).

It is also assumed that all stochastic processes subscribed by $+$ or $-$ are adapted to the filtrations generated by N_+ and N_-, respectively, and have right continuous trajectories.

Let $g_+(t) = (-1)^{N_+(t)}$ and $g_-(t) = -(-1)^{N_-(t)}$, and define the Telegraph process with states (ν_+, λ_+) and (ν_-, λ_-) as

$$X_+(t) = \int_0^t \nu_{g_+(\tau)} d\tau \tag{2.16}$$

and

$$X_-(t) = \int_0^t \nu_{g_-(\tau)} d\tau. \tag{2.17}$$

Theorem 2.2.4 (Quadratic Variation of Telegraph Process). *If $X_\pm$ is a Telegraph process then*

$$\langle X_\pm, X_\pm \rangle_t = 0.$$

Proof: *Replacing the Telegraph process defined in (2.16) or (2.17),*

$$FV_{[0,T]}(X_\pm) = \lim_{||\Pi_n|| \to 0} \sum_{k=0}^{n-1} \left| \int_0^{t_{k+1}} \nu_{g_\pm(\tau)} d\tau - \int_0^{t_k} \nu_{g_\pm(\tau)} d\tau \right| .$$

Therefore

$$\begin{aligned}
\sum_{k=0}^{n-1}\left|\int_0^{t_{k+1}}\nu_{g_\pm(\tau)}d\tau-\int_0^{t_k}\nu_{g_\pm(\tau)}d\tau\right| &\leq \sum_{k=0}^{n-1}\left|\int_{t_k}^{t_{k+1}}\nu_{g_\pm(\tau)}d\tau\right| \\
&\leq \sum_{k=0}^{n-1}\left|\int_{t_k}^{t_{k+1}}\max(|\nu_+|,|\nu_-|)d\tau\right| \\
&\leq |\max(|\nu_+|,|\nu_-|)|\sum_{k=0}^{n-1}[t_{k+1}-t_k] \\
&\leq |C|T
\end{aligned}$$

Hence, one can conclude that the Telegraph process has finite variation. Finally, the proof in concluded once Theorem 2.2.1 is considered .

□

2.3 Stochastic exponential

This section presents some concepts and results (see Klebaner[44]) that play a crucial role in supporting the main results of the Chapter 5, on models asset models with memory in the underlying asset.

Because the models used for the dynamics in time of assets pricing are stochastic differential equation (SDE) where the analytical solution is a stochastic exponential, therefore the interest on these concepts, especially for the semimartigale case.

Definition 2.3.1 (Semimartingales)**.** *A regular right-continuous with left limits (càdlàg) adapted process is a semimartingale if it can be represented as a sum of two processes: a local martingale $M(t)$ and a process of finite variation $A(t)$, with $M(0) = A(0) = 0$, and*

$$S(t) = S(0) + M(t) + A(t) \;. \tag{2.18}$$

Remark 2.3.1. *For a semimartingale X, the jump process ΔX is defined by*

$$\Delta X(t) = X(t) - X(t_-) \,, \tag{2.19}$$

and represents the jump at point t. If X is continuous, then of course, $\Delta X = 0$.

In general, stochastic integrals with respect to martingales are only local martingales rather than true martingales. This is the main reason for introducing local martingales. It is also common to see, in this context, the use of stoppings and truncations for the expectations computation. These ideas motivate the following

Definition 2.3.2. *A property of a stochastic process $X(t)$ is said to hold locally if there exists a sequence of stopping times τ_n, called the localizing sequence, such that $\tau_n \uparrow \infty$ as $n \to \infty$ and for each n the stopped processes $X(t \wedge \tau_n)$ has this property.*

Local martingales are defined by localizing the martingale property.

Definition 2.3.3. *An adapted process $M(t)$ is called a local martingale if there exists a sequence of stopping times τ_n, such that $\tau_n \uparrow \infty$ and for each n the stopped processes $M(t \wedge \tau_n)$ is a uniformly integrable martingale in t.*

Theorem 2.3.1. *Let $Y(t)$ be a real semimartingale such that $Y(0) = 0$, then the stochastic process $Z = \{Z(t)\}_{,t\geq 0}$ for*

$$\begin{aligned} Z(t) &= Exp\left\{ Y(t) - \frac{1}{2}\langle Y, Y\rangle_t + \frac{1}{2}\sum_{\nu\in(0,t]} [\Delta Y((\nu))]^2 \right\} \times \\ &\times \prod_{\nu\in]0,t]} \Big[1 + \Delta Y(\nu)\Big]\; Exp\Big\{-\Delta Y(\nu)\Big\} \,, \quad t \geq 0. \end{aligned} \tag{2.20}$$

And it follows that Z is a semimartingale cádlág which verifies the equation

$$Z(t) = 1 + \int_{]0,t]} Z(\nu-) dY(\nu) \,, \quad t \geq 0 \,,$$

i.e. $dZ(t) = Z(t_-) dY_{\pm}(t)$.

Proof: *See Protter[45].*

Definition 2.3.4 (Predictable Process). *$H(t)$ is predictable if it is one of the following:*

1. *A left-continuous adapted process, in particular, a continuous adapted process.*
2. *A limit (almost sure, in probability) of a left-continuous adapted processes.*
3. *A regular right-continuous process such that, for any stopping time τ, $H(\tau)$ is $\mathfrak{F}_\tau$-measurable, being the σ-field generated by the sets $A \cap \{\tau < t\}$, where $A \in \mathfrak{F}_t$.*
4. *A Borel-measurable function of a predictable process.*

Example 2.3.1. *The Poisson process $N(t)$ is right-continuous and is obviously adapted to its natural filtration. It can be shown that it is not predictable. However, its left-continuous modification $N(t_-) = lim_{s\uparrow t} N(s)$ is predictable, because it is adapted and left-continuous (item 2 of the Definition 2.3.4). Any measurable function (even right-continuous) of $N(t_-)$ is also predictable (item 4 of the Definition 2.3.4.*

Definition 2.3.5 (Stochastic Integrals with respect to Semimartingales). *Let S be a semimartingale with representation*

$$S(t) = S(0) + M(t) + A(t) \ , \tag{2.21}$$

where $M(t)$ is a local martingale and $A(t)$ is a finite variation process. Let $H(t)$ be a predictable process such that

$$\int_0^t |H(t)| dV_A(t) < \infty \tag{2.22}$$

where $V_A(t)$ is the variation process of $A(t)$ and

$$\sqrt{\int_0^t H^2(t) d\langle M, M\rangle_t} \quad \text{is locally integrable} \ . \tag{2.23}$$

Then the stochastic integral is defined as the sum of integrals,

$$\int_0^t H^2(t)dS(t) = \int_0^t H^2(t)dM(t) + \int_0^t H^2(t)dA(t) \ . \tag{2.24}$$

As in Klebaner[44], if $M(t)$ is a continuous local martingale then the stochastic integral is well defined

$$\int_0^t H^2(t)d\langle M, M\rangle_t < \infty \quad \text{a.s.} \ .$$

Corollary 2.3.1.1. *If $Y(t)$ is a semimartingale of finite variation, then $Y^{cm}(t) \equiv 0$.*

Consider that $Y(t)$ is a semimartingale and that $\langle Y, Y\rangle^{cont}$ denotes the path-by-path continuous part of $\langle Y, Y\rangle$. Klebaner[44], page 232, presented

$$\langle Y, Y\rangle_t = \langle Y, Y\rangle_t^{cont} + \sum_{0<\nu\leq t} [\Delta Y(\nu)]^2 \ . \tag{2.25}$$

Analogously, if $\langle X, Y\rangle^{cont}$ denotes the path-by-path continuous part of $\langle X, Y\rangle$ then

$$\sum_{\nu\leq t} [\Delta Y(\nu)]^2 < \infty$$

The next result establishes the existence and uniqueness of the solution of the SDE.

Theorem 2.3.2 (Stochastic Exponential of Semimartingale)**.** *Let $Y(t)$ be a semimartingale. Then the stochastic equation*

$$U(t) = 1 + \int_0^t U(\nu-)dY(\nu) \tag{2.26}$$

has a unique solution, given by

$$U(t) = \varepsilon_t(Y) = e^{Y(t)-Y(0)-\frac{1}{2}\langle Y,Y\rangle^{cont}(t)} \prod_{\nu\leq t}[1+\Delta Y(\nu)]e^{-\Delta Y(\nu)} \ . \tag{2.27}$$

This is called the stochastic exponential of $Y(t)$ and can be stated in an equivalent form by using the quadratic variation where $\varepsilon_t(Y(t))$ is given by

$$\varepsilon_t(Y) = e^{Y(t)-Y(0)-\frac{1}{2}\langle Y,Y\rangle_t} \prod_{\nu\leq t}[1+\Delta Y(\nu)]e^{-\Delta Y(\nu)+\frac{1}{2}[\Delta Y(\nu)]^2} \ . \tag{2.28}$$

Proof: *See Klebaner[44].*

Through this work it is frequent to consider SDE, where the existence and uniqueness of its solution are guaranteed by the following theorem.

Theorem 2.3.3 (Existence and Uniqueness of the Solution of a SDE)**.** *Let $Y(t)$ be such that*

$$dY(t) = f(Y(t), t)dt + G(Y(t), t)dM(t) \ , \quad Y(0) \textit{ is given} \tag{2.29}$$

and $M(t) = \{M(t)\}_{t\geq 0}$ is a stochastic process. Consider the following conditions:

1. *The coefficients are locally Lipschitz in x uniformly in t, that is, for every T and N, there is a constant K depending only on T and N such that for all $|x|$, $|y|$. N and all $0 \leq t \leq T$*

$$|f(x,t) - f(y,t)| + |G(x,t) - G(y,t)| < K|x - y| \, , \tag{2.30}$$

2. *The coefficients satisfy the linear growth condition*

$$|f(x,t)| + |G(x,t)| < K[1 + |x|] \, , \tag{2.31}$$

3. *$Y(0)$ is independent of ($M(t), 0 \leq t \leq T$), and $EY^2(0) < \infty$.*

If these conditions are satisfied then there exists a unique strong solution $Y(t)$ of (2.29). $Y(t)$ has continuous paths, moreover

$$E\Big\{\sup_{0\leq t\leq T} Y^2(t)\Big\} < C\Big[1 + E\{Y^2(0)\}\Big] \, , \tag{2.32}$$

where the constant C depends only on K and T.

Proof: *See Klebaner[44].*

3 Numerical Simulation of SDE

Often when working with SDE it is impossible to find analytical solutions, and therefore one must use a numerical method to find the approximate solutions.

Progressively, the SDE has become a widely used technology to model and solve various problems in economics and finance. A classic example is the use of the Geometric Brownian motion (SDE) for modeling stock dynamics.

This chapter deals with applications of SDE in assets pricing problems. Since some of these equations have no analytical solution, this chapter also conducts an examination on the numerical analysis (study of the convergence) of the approximate solution, in time discretization, by Euler-Maruyama method and Milstein method. The SDE (2.29) is used here as an working example. For the particular case $f(Y(t),t) = \mu Y(t)$, $G(Y(t),t) = \sigma Y(t)$ and $M(t)$ is Brownian Motion, one obtains the analytical solution given by

$$Y(t) = Y(0)\, exp\left\{\left[\mu - \frac{1}{2}\sigma^2\right] t + M(t)\right\} \ , \tag{3.1}$$

shown in Arnold[48] and Thomaz[52].

So, this study is particularly focused on the presentation of the numerical solution of the SDE, and on the study of convergence to the analytical solution, with the illustration of some numerical results.

3.1 Euler-Maruyama Method

Consider that in interval $t \in [0, T]$, there is the partition

$$0 = t_0 < \cdots < t_N = T$$

with $\Delta_n = t_{n+1} - t_n$, not necessarily uniform. Let $Y_n = Y(t_n)$ be an approximation function defined by the *Euler method*

$$Y_{n+1} = Y_n + f(Y_n, t_n)\ \Delta_n + G(Y_n, t_n)\Big[W(t_{n+1}) - W(t_n)\Big]\ , \tag{3.2}$$

for $n = 0, \cdots, N-1$ and Y_0 is known. Let δ be given as $\delta = \max_n \Delta_n$. In (3.2) f and G are defined according to the SDE (2.29) with $f(Y(t), t) = \mu Y(t)$, $G(Y(t), t) = \sigma Y(t)$. Additionally, consider the uniform grids $t_n = t_0 + n\delta$, with $\delta = \Delta_n \equiv \Delta = (T - t_0)/N$ for some integer N large enough so that $\delta \in (0, 1)$.

The sequence of Euler approximation values to Y_n for $n = 0, \ldots, N$, defined by (3.2), are computed as in the deterministic case. The main difference is that there is need to generate the random increments $\Delta W_n = W(t_{n+1}) - W(t_n)$ for $n = 0, 1, \cdots, N-1$, of the Wiener process. These increments can be generated by a random number generator for independent Gaussian pseudo-random numbers. Here it is used the Polar Marsaglia generator presented in Kloeden and Platen [53].

3.2 Milstein Method

This section presents a discrete time approximation called the Milstein method. Its order of convergence is grater than the one of the Euler-Maruyama method, which, in fact, is the simplest of all numerical methods.

Already the Milstein approximation, in the one-dimensional case, is given for only a sum of terms of the expansion of Ito-Taylor,

$$G'GI_{(1,1)} = \frac{1}{2}G'G\Big[(\Delta W_n)^2 - \Delta_n\Big]$$

in the Euler-Maruyama approximation, then we get the Milstein Method

$$Y_{n+1} = Y_n + f\Delta_n + G\Delta W_n + \frac{1}{2}G'G\Big[(\Delta W_n)^2 - \Delta_n\Big], \tag{3.3}$$

In addition to these methods of Milstein and Euler-Maruyama, there are others presented in the literature applied to the stochastic problem of the initial value. For example, the *Heun* method (Mcshane[54], Saito and Mitsui[55]), the *Derivative-free* method (Klonden and Platen[56]), the *FRKI* method (Newton[57]), The *Improved 3-Stage RK* method (Mcshane [54] or Saito and Mitsui[55]), the *Taylor* method (Milshtein[58] or Klonden and Platen[56]), the *ERKI* method (Newton[57]) and the method of *Local Linearization* (Biscay et al.[59] or Jimenez [60]).

3.3 Interpolation of the Approximation of the Discrete Time

The numerical methods of Euler-Maruyama (3.2) and Milstein (3.4) determine values of the approximating process at discrete times only. If required, more precise values can then be determined at intermediate instants using an appropriate interpolation method.

The approximated values Y_n at intermediate instants can be determined by an appropriate interpolation method. The simplest one is the piecewise constant interpolation with nt $Y(t) = Y_{n_t}$, $t \in [0, \infty)$, where n_t is an integer defined by $n_t = Max(0, 1, \cdots, N : t_n \leq t)$, that is the largest "n" for which t_n does not exceed t.

By a linear interpolation procedure, one obtains an approximation $Y(t)$ for $t \in [t_n, t_{n+1}]$, defined by

$$Y(t) = Y_n + \Big[t - t_n\Big]\frac{Y_{n+1} - Y_n}{\Delta_n}. \tag{3.4}$$

This procedure is often used because of its simplicity and continuity.

The emphasis is on the approximated values at given discrete instants. In fact, it is impossible to reproduce the finer structure of sample paths of an Ito's process as they have the irregularity property of the sample paths of the driving Wiener's process; in particular they are not differentiable.

3.4 Convergence

In assessing the quality of a numerical scheme it is necessary to have some kind of measure of how well the method approximates the analytical solution, i.e. a procedure to measure how the numerical solution converges to the analytical solution. In a stochastic environment, unlike the deterministic case, there is more than one method for measuring convergence. We consider the two main types of convergence, namely weak and strong convergence.

3.4.1 Strong Convergence

The strong convergence method derives from the concept of absolute error, which is just the expectation of the absolute value of the difference between the numerical approximation and the analytical solution in time "t", i.e. $E|X(t) - Y(t_n)|$. One says that a discrete approximation in time $Y(t_n)$, to the exact solution $X(t)$ of a SDE, converges in the strong sense with rate $\gamma > 0$ if there is a constant $C < \infty$ such that

$$E|X(t) - Y(t_n)| \leq C\Delta^{\gamma} , \tag{3.5}$$

for all the fixed steps $\Delta \in (0, 1)$. The strong convergence provides a measure of approximation of the individual sample trajectories to the solution of a SDE.

3.4.2 Weak Convergence

Strong convergence is computationally expensive and it is not a necessary condition to compute the expected values, only information on the type of solution is needed. For example, if and only if one is interested in computing the moments of the solution $X(t)$, then there is no need to know how the individual paths approach $X(t)$. This leads to the concept of weak convergence. One says that a discrete approximation in time, $Y(t_n)$, of a solution $X(t)$ of a SDE converges in the weak sense with rate $\gamma > 0$ if, for any polynomial g , there is a constant $C < \infty$ such that

$$|E(g(X(t))) - E(g(Y(t_n)))| \leq C\Delta^{\gamma} , \tag{3.6}$$

for all fixed steps $\Delta \in (0,1)$, provided that they are functional. This criterion provides an error measure for the mean, variance or any other moment that is necessary to calculate. It is usually easier and faster to implement numerical methods subject to the condition of weak convergence. Therefore, it is important, when dealing with an identification problem (if possible), to know if a good approximation of the path is required or if an approximation of a functional solution is sufficient.

Remark 3.4.1. *The expected strong convergence theoretical rate for the Euler-Maruyama approximation is* $\gamma = 1/2$ *, while for the Milstein approximation is* $\gamma = 1$. *Thus, the Milstein approximation is a better approximation technique in the presence of discretization because it does not requires a high refinement of the mesh for the numerical solution. This reduces the computational cost and leads to more accurate responses.*

3.4.3 Study of Convergence

The simulations presented in this section aim to verify the convergence of the numerical solution given by the Euler-Maruyama method (3.2) and by the

Milstein method (3.4) for the Classic Model - Black & Scholes (2.29) which has an analytical solution (3.1).

The expected theoretical rate of strong convergence is $\gamma = 1/2$ using the Euler-Maruyama Method applied to the discretization of time for (2.29).

The computational simulations presented hereafter assume the following parameters: $h = 0.001$, $\mu = 0.15$, $\sigma = 0.30$, with initial value $S_0 = 100$. It was also adopted a Δ^γ , for time Steps, equal to 16, 32, 64, 128 and 256.

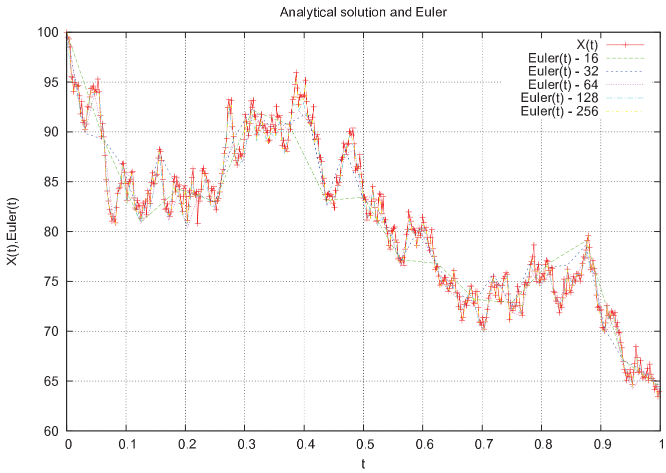

Figure 3.1: Analytical solution and Euler Approximation

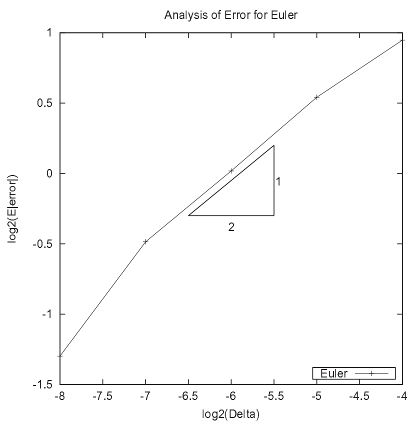

Figure 3.2: Error analysis of the Euler Approximation

Figure 3.1 shows one path for the analytical solution of (2.29) and its approximated values obtained from the Euler-Maruyama method. Figure 3.2 presents a graphical analysis of the approximation error, and it confirms a convergence rate of $1/2$, as predict by the theory, since this is the slope of the line in the space $log_2(\Delta) \times log_2(E|error|)$.

Figure 3.3 presents a different sample trajectory than 3.1, for the same parameters. However, the same convergence can be seen in Figure 3.4.

The next figures present a comparative analysis of the Euler-Maruyama and Milstein approximation methods, performed by a simulation using the previous parameters.

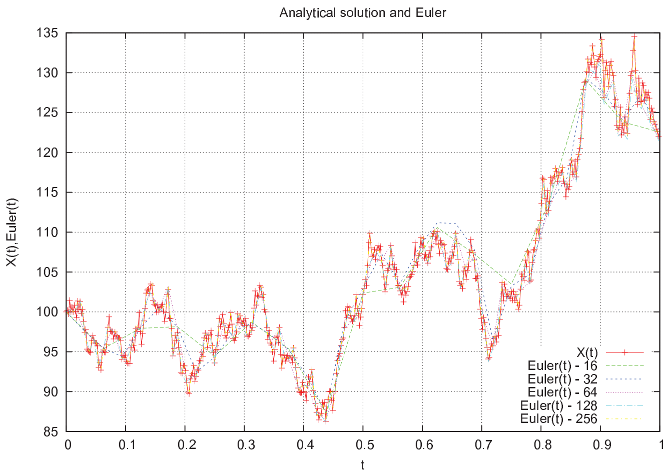

Figure 3.3: Analytical solution and Euler Approximation

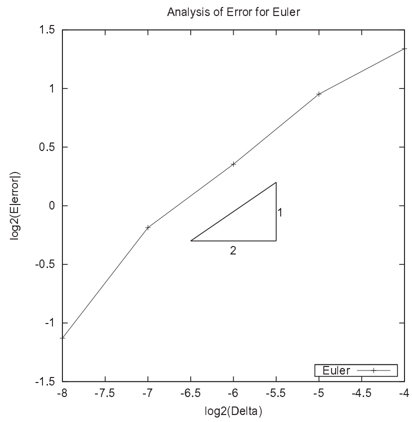

Figure 3.4: Error analysis of the Euler Approximation

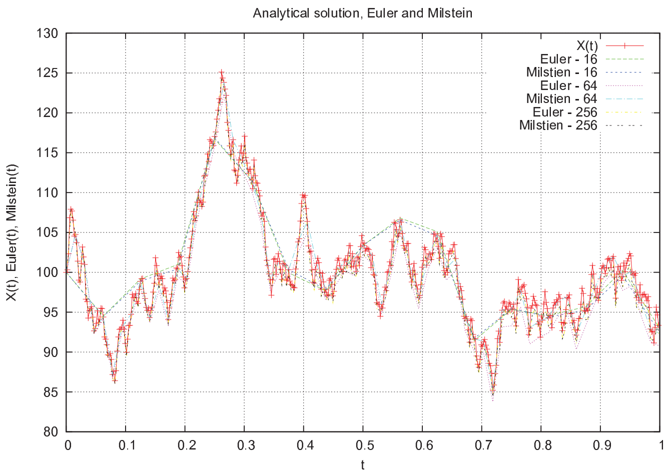

Figure 3.5: Analytical solution, Euler and Milstein Approximation

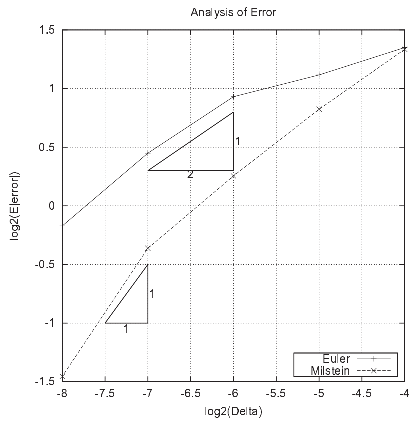

Figure 3.6: Error analysis of the Euler and Milstein Approximation

Figure 3.6 confirmes the convergence at a rate of the $1/2$ for the Euler-Maruyama approximation and at a rate of 1 for Milstein approximation, as can be seen by the slope of line corresponding to the plot $log_2(\Delta)\times log_2(E|error|)$. Already in Figure 3.5 present the path for the analytical solution of and its approximated values obtained from the Euler-Maruyama and Milstein approximation methods.

4 Modeling Financial Assets

The analysis of the price dynamics of underlying assets is a crucial issue in pricing financial options. Since the simplified Brownian stochastic differential equation proposed by Black and Scholes [3], several models were proposed in the literature. The aim of this chapter is to present the most significant asset pricing models that were introduced to overcome the limitations of the Black and Scholes model.

4.1 Classic Model - Black & Scholes

The most popular model to describe the time evolution of asset prices is the one proposed in 1973, by Black and Scholes [3] where the authors establish SDE for the asset price depending on a Brownian motion process (also called a diffusion process). This chapter begins by evoking the concept of Brownian motion.

Assuming that the underlying asset $S(t)$ follows a geometric Brownian motion with constant drift μ and volatility σ, Black and Scholes[3] introduced the equation

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \ , \quad \text{where } S(0) \text{ is given} \tag{4.1}$$

to describe the time evolution of the price of an asset traded in the spot market.

Figure 4.1 shows historical daily closing prices, since 26-11-2008 until 24-11-2009, of International Business Machines Corp. (IBM corp.) stocks traded at the New York Stock Exchange (NYSE). It appears that, over time, this asset has a positive expected rate of return, which can be interpreted as the long run investment compensation.

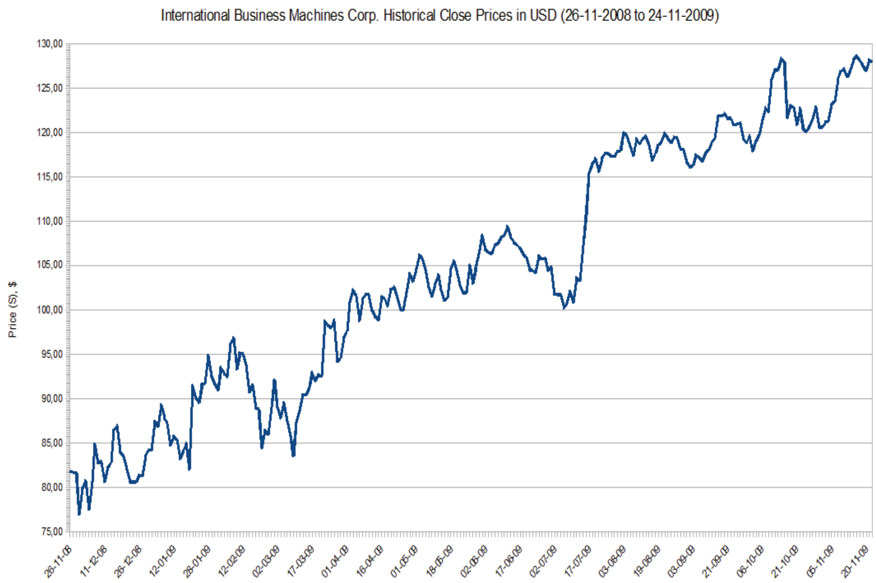

Figure 4.1: Historical close prices of IBM stocks traded at NYSE (26-11-2008 to 24-11-2009).

The equation (4.1) can then be used to model a stock price that fluctuates randomly. Its solution admits the representation

$$S(t) = S(0)\ exp \left\{ \left[\mu - \frac{1}{2}\sigma^2 \right] t + W(t) \right\} . \tag{4.2}$$

In order to illustrate the trajectory dependence on parameters μ and σ of an asset price given by (4.2), some simulations were done with different parameters but with the same noise, so trajectories change only due to changes in the parameters.

Assume that the drift parameter, i.e. the expected rate of return (μ), is 0.30. Figure 4.2 shows a trajectory for $\sigma = 0.50$, while Figure 4.3 shows a trajectory for $\sigma = 0.30$. Obviously, because the volatility is a measure of dispersion, the first case presents a greater dispersion. In other words, the path presented in Figure 4.2 is more jagged than the one in Figure 4.3.

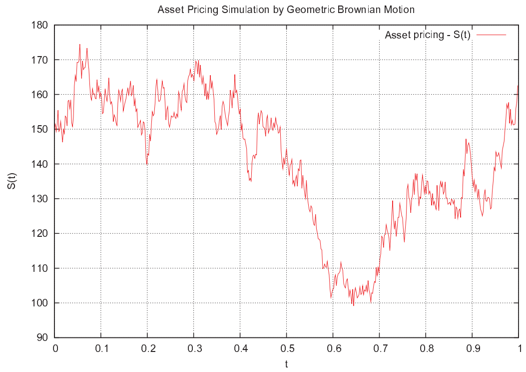

Figure 4.2: Simulation of a geometric Brownian motion with $\mu = 0.30$ and $\sigma = 0.50$.

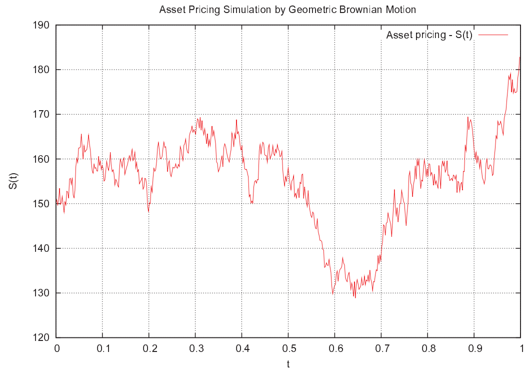

Figure 4.3: Simulation of a geometric Brownian motion with $\mu = 0.30$ and $\sigma = 0.30$.

The following figures present new simulations, by setting $\sigma = 0.50$ and varying the drift parameter. The trend is more pronounced in Figure 4.5, for $\mu = 0.40$ than in Figure 4.4 for $\mu = 0.10$, while the variability is the same.

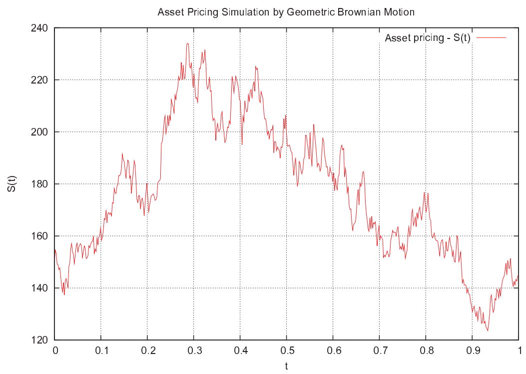

Figure 4.4: Simulation of a geometric Brownian motion with $\mu = 0.10$ and $\sigma = 0.50$.

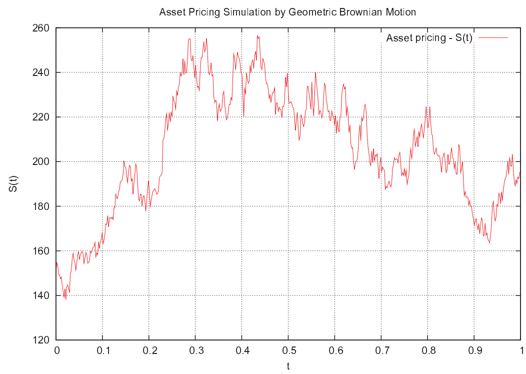

Figure 4.5: Simulation of a geometric Brownian motion with $\mu = 0.40$ and $\sigma = 0.50$.

The study of stochastic calculus, especially SDE with white noises was firstly presented in Langevin[61] in 1908, where the author analyzed the Brownian motion. However, only in 1951, Itô[62] defined the improper integral, which raised up the formalization of the SDE theory. Subsequently, several have been the contributions to the development of what is today known as stochastic calculus especially SDE. Accordingly, stochastic calculus has been increasingly used for modeling problems in economics and finance. Some seminal works are those of Skorokhod[63], McKean[64], Gikhman and Skorokhod[65], Dynkin[66], Arnold[48], Oksendal[67], Kloeden and Platen[53] and Look[68].

4.2 Some improvements on the classical model

The SDE (4.1) or its solution (4.2) has been largely used to model asset prices in spot markets. However there exist some pertinent differences between the behavior of real prices and simulated prices. The pathologic behavior of (4.2) is well illustrated in the literature and it is consequence, for instance, of the following properties: the trajectories of asset prices are continuous, $S(t)$ is independent of its history and parameters μ and σ do not depend on the asset prices.

Consequently, it is commonly accepted that the dynamics of asset prices cannot be described by the geometric Brownian motion with constant drift and volatility. Several sophisticated theoretical constructions have been presented in the literature to capture the real features of asset prices dynamics.

The next sections present some of these models that intent to avoid the pathological behavior induced by the Black and Scholes model (4.1). Namely the model proposed by Merton in [9] which is basically characterized by the existence of jumps, the models with stochastic volatility, especially the one proposed in Hobson and Roger[24] and finally the models introduced by Di Crescenzo and Pellerey[10] and by Ratanov[11, 12, 13] which are characterized by telegraph processes.

4.2.1 Jump-Diffusion Model

Several undesirable properties have been observed when the SDE (4.1) is used to model the time dynamics of the prices of financial assets. The first one was pointed out, for instance in Tankov and Voltchkova[69], and it is related with the continuous trajectories. In a model with continuous paths, the price process behaves locally like a Brownian motion and the probability that the asset moves by a large amount over a short period of time is very small, unless one fixes an unrealistically high value for the volatility. Therefore, in the short run modeled prices have smaller moves than those observed in real markets.

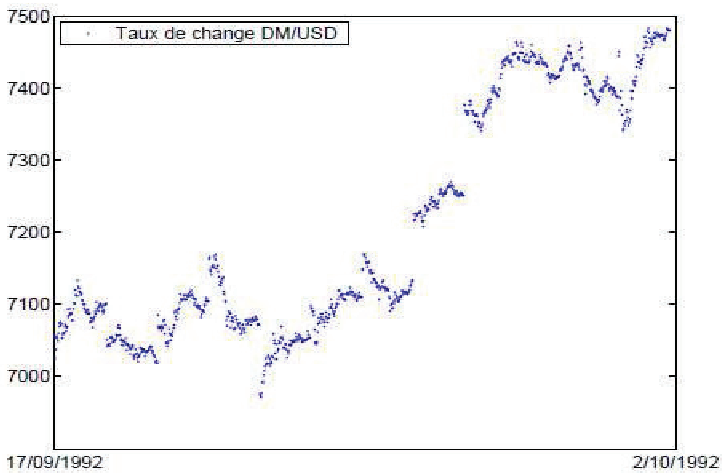

Figure 4.6: Jumps in the trajectory of DM/USD exchange rate, sampled at 5-minute intervals.

For example, consider the time evolution of the DM/USD exchange rate, with 5-minute time resolution, over a two-week period in 1992 taken from Tankov and Voltchkova[69] (see Figure 4.6). One can observe that there exists at least three instants where the rate jump 100bp within a 5-minute interval which appears as discontinuities in the price trajectory. Price movements like these ones clearly cannot be described by a diffusion model with continuous paths, therefore it was necessary to construct a stochastic process which admits discontinuities in its trajectories.

In order to incorporate discontinuities in the price process, Merton in [9] assumed the partition of the asset price process into three components: a linear drift, a Brownian motion representing "normal" price variations, and a compound Poisson process that generates an "abnormal" change (jump) in prices due to the arrival of new information. The jump value is determined by sampling from an independent and identically distributed (i.i.d.) random variable. The asset price is then described by the following SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) + dJ(t) \ , \tag{4.3}$$

where μ is the instantaneous expected return on the asset, σ is the instantaneous volatility of the return, conditional on the inexistence of "news" (i.e., the Poisson event does not occur) and $dW(t)$ is a standard Gauss-Wiener process. In (4.3) the process $J(t)$ is given by

$$J(t) = \sum_{i=1}^{N(t)} \Big[V_i - 1\Big]$$

where $N(t)$ is a Poisson process with rate λ (the mean number of arrivals per unit time), V_i is a sequence of independent identically distributed (i.i.d.) nonnegative random variables. Hence, $J(t)$ is the independent Poisson process described by

1. $\mathbb{P}\Big\{\textit{the event does not occur in the time interval}[t, t + dt]\Big\} = 1 - \lambda dt + O(dt)$,
2. $\mathbb{P}\Big\{\textit{the event occurs once in the time interval } [t, t + dt]\Big\} = \lambda dt + O(dt)$,
3. $\mathbb{P}\Big\{\textit{the event occurs more than once in the time interval } [t, t + dt]\Big\} = O(dt)$

where $O(dt)$ is the asymptotic order symbol defined by $\psi(dt) = O(dt)$ if

$$\lim_{dt \to 0} \left[\frac{\psi(dt)}{dt}\right] = 0 \ .$$

It is assumed that dJ and dW are independent.

If $\lambda = 0$ (and therefore, $dJ = 0$), then the return dynamics would be identical to those of Black and Scholes[3] and Merton[4].

The solution of the SDE (4.3) is shown in Kou [70] and [71] as

$$S(t) = S(0)Exp\left\{\left[\mu - \frac{1}{2}\sigma^2\right]t + \sigma W(t)\right\}\prod_{i=1}^{N(t)}[V_i]\ . \tag{4.4}$$

The stochastic process $S(t)$ is usually called the Jump-Diffusion process.

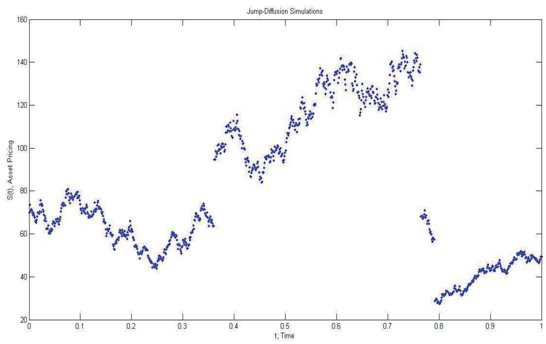

Figure 4.7: A trajectory of (4.4) with $\mu = 0.20$, $\sigma = 0.50$ and $\lambda = 1.0$.

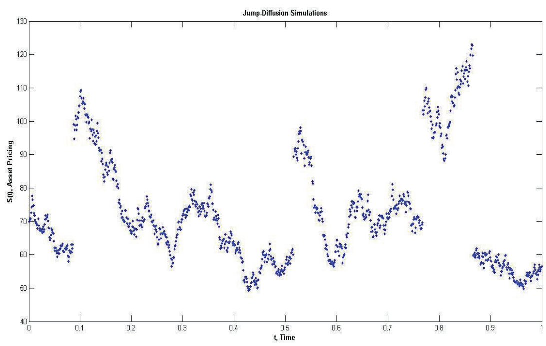

Figure 4.8: A trajectory of (4.4) with $\mu = 0.20$, $\sigma = 0.50$ and $\lambda = 1.0$

Figures 4.7 and 4.8 display two trajectories of the jump-diffusion process (4.4) with expected rate of return $\mu = 0.20$, volatility $\sigma = 0.50$ and a rate of the Poisson process $\lambda = 1.0$. Although the paremters are the same, the two trajectories seem quite different, even in relation to the number of realized jumps.

From a risk management perspective, the inclusion of jumps in the model allows to quantify and take into account the risk of an abrupt stock price movement over short time intervals which appear to be non-existent in the Brownian framework. For the purpose of option pricing, Merton assumed that the jumps are log-normally distributed. This special case renders estimation and hypothesis testing tractable and has become the most important representation of the jump-diffusion process. Moreover, by adding discontinuous jumps to the Black-Scholes model and choosing the appropriate parameters of the jump process, log-normal jump models can accommodate volatility smiles.

From the point of view of hedging, continuous models of stock price generally lead to a complete market or to a market which can be made complete by adding one or two additional instruments. Since in such market every terminal payoff can be exactly replicated, options are redundant assets, and the existence of traded options becomes a puzzling issue. The mystery is easily solved by allowing discontinuities: in real markets, due to the presence of jumps in prices, perfect hedging is impossible and options enable the market participants to hedge risks that cannot be hedged by using the underlying asset only (Tankov and Voltchkova[69]).

The jump-diffusion models seem more realistic than the Black-Scholes model (4.1), however some questions remain unanswered. The main problem with jump-diffusion models is that they cannot capture the volatility clustering effects. This has been the main motivation for the stochastic volatility models ([71]) presented in the next section.

4.2.2 Stochastic Volatility

This section starst by introducing some motivation to the need of stochastic volatility in modeling asset prices. To illustrate empirically the stylized facts that motivate the need of stochastic volatility, Billio and Sartore[1] present examples of three European stock indexes: the FTSE100, the CAC40 and the MIB30, which are market indexes for the London, Paris and Milan equity markets, respectivelly. These series run from 4^{th} January 1999 to 12^{th} August 2002, yielding 899 daily observations.

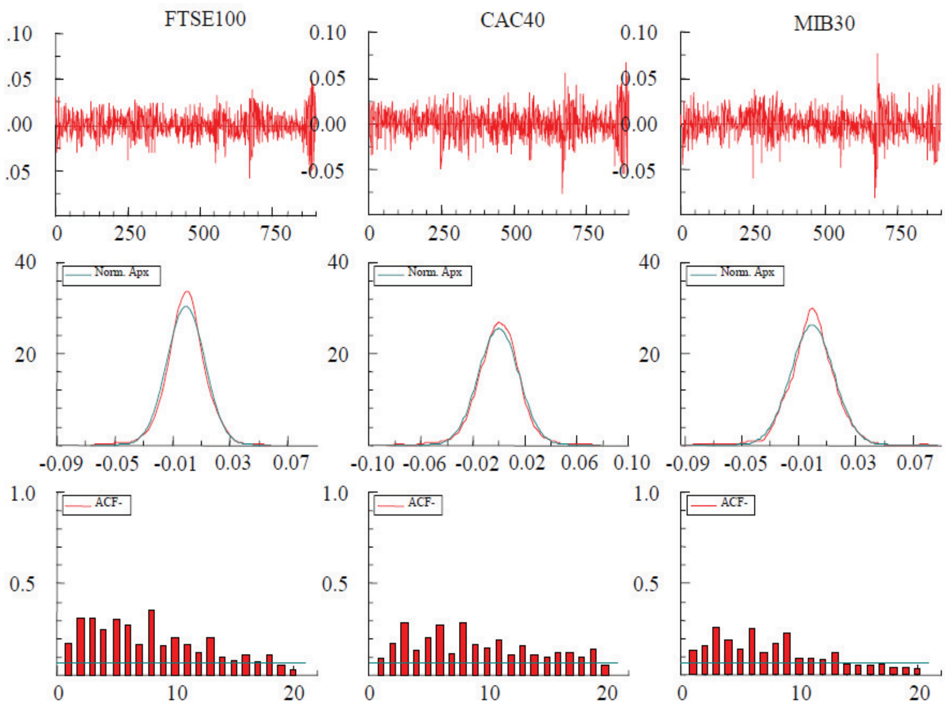

Figure 4.9: Daily returns on three European stock indexes: FTSE100, CAC40 and MIB30 (raw returns, nonparametric density estimate and normal approximation and correlogram of squared returns)(Billio and Sartore [1])

Figure 4.9 presents graphically some properties of these three series. The raw returns time series suggest that there are periods of volatility clustering, that is, days of large price movements are followed by days with the same characteristics. This is confirmed by the a correlogram on squared returns series, which shows significant correlations at quite extended lag lengths. Figure 4.9 also gives a density estimate of the unconditional distribution of returns series together with the corresponding normal approximation, which suggest that returns series are leptokurtic.

These stylized facts can be summarized as follows: non-significant serial correlation in the levels of return, volatility clustering, which implies a significant and positive serial correlation in the squared returns series, heavy tails and persistence in volatility.

There is strong evidence that the volatility depends on the present asset price (leverage effect) and on the past price realization (volatility persistent) as well as on other parameters of the market like time or maturity. Several approaches have been proposed in the literature to resolve the limitations of the constant volatility models: in the first one the volatility was assumed to be a deterministic function of time and current asset prices $S(t)$. In this case the asset price is given by (4.1) where volatility is such that $\sigma = \sigma(S(t), t)$, this is, $S(t)$ is the solution of the following SDE

$$dS(t) = \mu S(t)dt + \sigma(S(t), t)S(t)dW(t) \ , \quad t > 0 \ , \quad S(0) \text{ is given} \ , \tag{4.5}$$

and, as before, $W(t)$ denotes a Winner process. See for instance Cox and Ross[5], Geske[72], Rubinstein [73] and Bensoussan et al. [74]. In this case the market is complete (see Harrison and Pilska[75]). Another class of models was introduced taking into account that the volatility depends on the asset price but with some delay, that is, $S(t)$ satisfies

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma(S(t-\tau), t)S(t)dW(t) \ , \quad t > 0 \ , \qquad (4.6) \\ & S(t) \text{ is given for } t \in [-\tau, 0] \ . \end{aligned}$$

This type of models were proposed in Arriojas at al. [14], Kazmerchuk et al [15] and Lee at al. [76].

In the asset price models (4.5) and (4.6) there is only one source of randomness given by $W(t)$. However the volatility randomness can be induced by another source. One of the first models of this class is the one proposed in Hull and White [19]. The authors assume that both the underlying asset $S(t)$ and the variance $\sigma^2(t)$ follow a geometric diffusion process

$$dS(t) = \mu S(t)dt + \sigma(t)S(t)dW_1(t) \tag{4.7}$$

$$d\sigma^2(t) = \phi\sigma^2(t)dt + \varepsilon\sigma^2(t)dW_2(t) \tag{4.8}$$

such that ρ is the correlation coeficient between the two Brownian motions $dW_1(t)$, $dW_2(t)$ which is assumed to be constant with modulus less than one. The parameter ϕ denotes the expected rate of volatility and ε represents the volatility of the volatility. In Hull and White [19] it was considered that $\rho \equiv 0$.

The solution of SDE (4.8) with initial conditions $\sigma^2(0)$ admits the representation

$$\sigma^2(t) = \sigma^2(0)\ Exp\left\{\left[\phi - \frac{1}{2}\varepsilon^2\right] t + \varepsilon W_2(t)\right\}\ .$$

For $S(t)$ one must use numerical methods because it has no known analytical solution. For instance, one may obtain the numerical solution for $S(t)$ combining Euler's method (3.2) and the interpolation method (3.4), where we have

$$f(S_n, t_n) = \mu S(t) \quad \text{and} \quad G(S_n, t_n) = \sigma(t)S(t)\ .$$

The figures that follow illustrate the volatility and corresponding asset prices defined by the Hull-White model with parameters: $\phi = 0.20$, $\varepsilon = 0.50$ or 0.30 and $\mu = 0.20$.

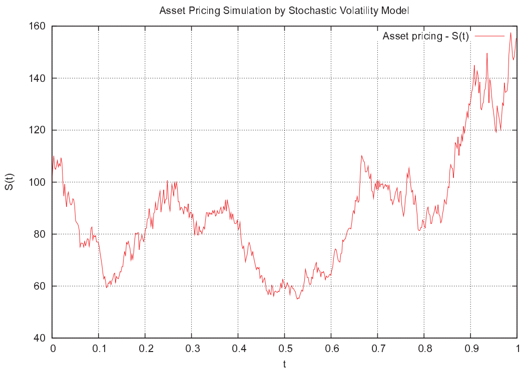

Figure 4.10: Asset pricing simulation by Hull and White model with $\mu = 0.20$ and volatility by Figura 4.11.

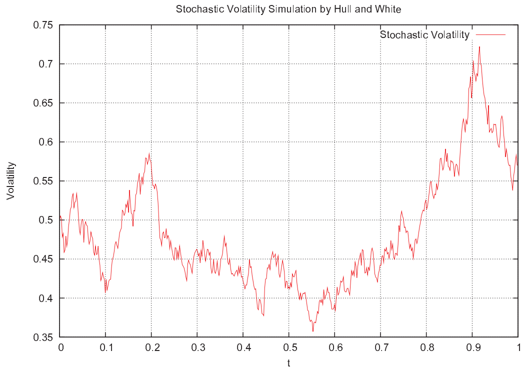

Figure 4.11: Simulated volatility by Hull and White model with $\phi = 0.20$ and $\varepsilon = 0.50$.

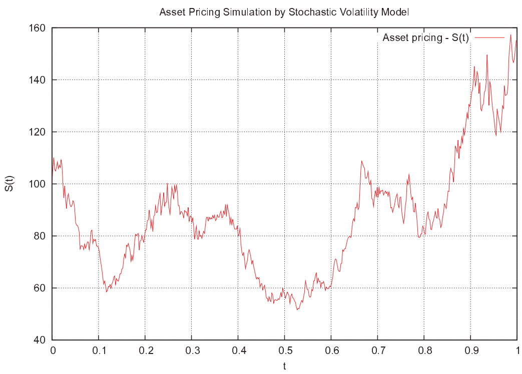

Figure 4.12: Asset pricing simulation by Hull and White model with $\mu = 0.20$ and volatility by Figura 4.13.

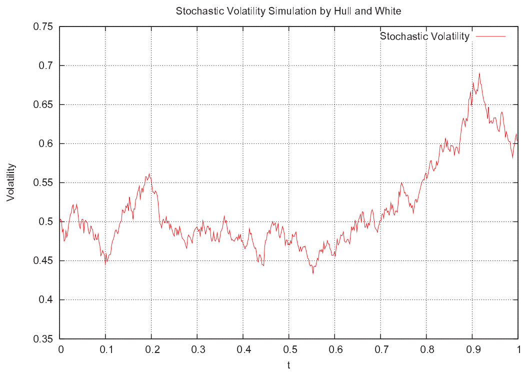

Figure 4.13: Simulated volatility by Hull and White model with $\phi = 0.20$ and $\varepsilon = 0.30$.

It can be seen that there are time intervals with high and low volatility, thus creating volatility clusters (this is also visible in in Figures 4.14 and 4.15, that display the simulated returns given the two sets of parameters).

One can also observe that in the Figure 4.11 for $\varepsilon = 0.50$ there is a greater dispersion in the trajectory than in 4.13 where $\varepsilon = 0.30$. Thus, trajectory is steepest in Figure 4.11 than in Figure 4.13.

The increase in the volatility parameter of the volatility, leads to an increase in the price dispersion. One can observe this effect through the simulation of the two processes by computing continuous returns.

$$Y(t) = \log\left(\frac{S(t)}{S(t-1)}\right)$$

where $S(t)$ is the asset price in time t. This simulations are presented in Figures 4.14 and 4.15.

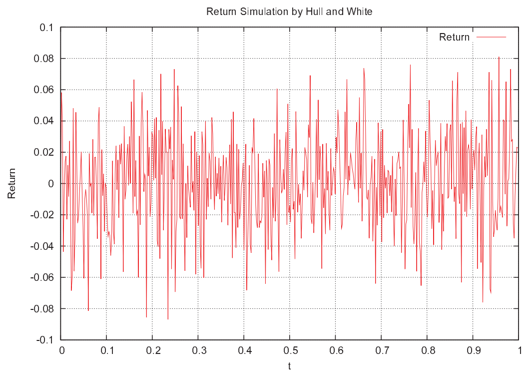

Figure 4.14: Simulated returns corresponding of the asset price in Figure 4.10.

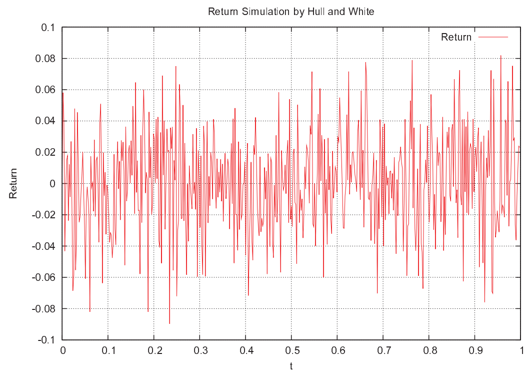

Figure 4.15: Simulated returns corresponding of the asset price in Figure 4.12.

Heston[77] proposed a mean reverting square root process for the volatility process. In this model, the price $S(t)$ and the squared volatility processes $\sigma^2(t)$, are presented in the risk neutral measure, by the following system of SDE's

$$dS(t) = \mu S(t)dt + \sigma(t)S(t)dW_1\ , \quad t > 0\ \text{, with} \quad S(0)\ \text{known}\ , \tag{4.9}$$

$$d\sigma^2(t) = \gamma\Big[\phi - \sigma^2(t)\Big]dt + \varepsilon\sigma(t)dW_2\ , \quad t > 0\ \text{, with}\ \sigma^2(0)\ \text{known}\ , \tag{4.10}$$

where ϕ is the mean long-term volatility that satisfies $\rho \neq 0$, γ denotes the rate at which the volatility reverts toward its long-term mean. Therefore the variance is a stochastic process such that exhibits a tendency to revert towards a long-term mean ϕ at a rate γ. It also exhibits a volatility proportional to the square root of its level and the source of its randomness is correlated (with correlation ρ) with the randomness of the underlying's price processes.

The SDEs (4.9) and (4.10) have unknown analytical solutions, and therefore it must be approximated numerically using, for instance, the linear interpolation

(3.4) with $\sigma_n = \sigma(t_n)$ as defined by the Euler-Maruyama's method (3.2) and an initial condition $\sigma(0)$. So

$$f(\sigma_n, t_n) = \gamma\left[\phi - \sigma_n\right]$$

and

$$G(\sigma_n, t_n) = \varepsilon\sigma_n \ .$$

For $S(t)$ with dynamics (4.10) and a given initial condition $S(0)$ the method implies that

$$f(S_n, t_n) = \mu S_n \quad \text{and} \quad G(t_n, Y_n) = \sigma(t_n)S_n \ ,$$

The numerical results for S_n and σ_n, plotted in the next figures, were obtained combining the linear interpolation procedure (3.4) with the Euler-Maruyama method (3.2). Figures 4.16 to 4.19 illustrate the behavior of the asset price $S(t)$ and volatility $\sigma(t)$ according to the Heston model (4.9), (4.10) when $\phi = 0.20$, $\varepsilon = 0.50$ or 0.30, $\mu = 0.20$ and $\gamma = 0.50$, wich implies an adjustment in the volatility of 50%.

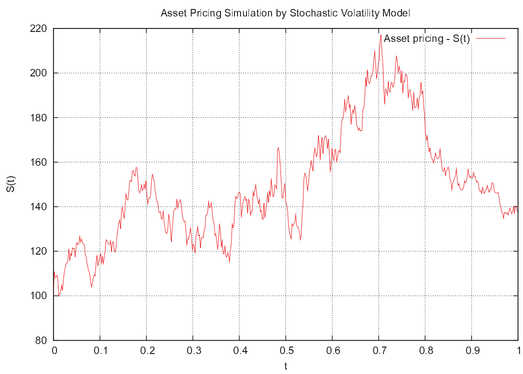

Figure 4.16: Asset pricing simulation by Heston model for $\mu = 0.20$.

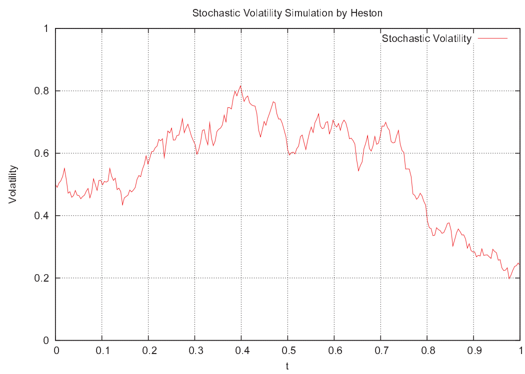

Figure 4.17: Simulated volatility by Heston model for $\phi = 0.20$ and $\varepsilon = 0.50$.

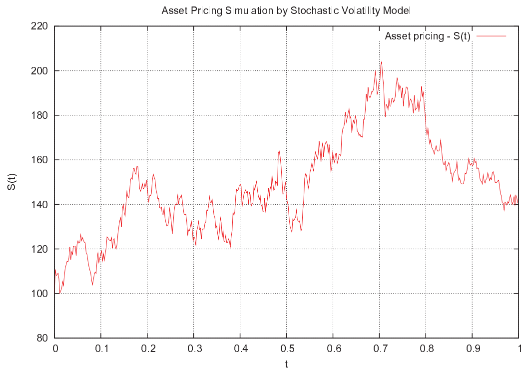

Figure 4.18: Asset pricing simulation by Heston model for $\mu = 0.20$.

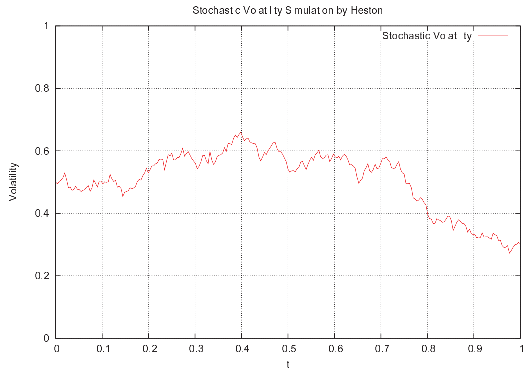

Figure 4.19: Simulated volatility by Heston model for $\phi = 0.20$ and $\varepsilon = 0.30$.

Figures 4.20 and 4.21, show the return corresponding of the asset price paths of Figures 4.16 and 4.17.

The path in Figure 4.17, with parameter $\varepsilon = 0.50$, presents a higher dispersion than in Figure 4.19 with $\varepsilon = 0.30$. Additionally, an increase in the volatility parameter of the volatility leads to an increase in the dispersion of the underlying asset prices, as can observed in Figure 4.16 and Figure 4.18.

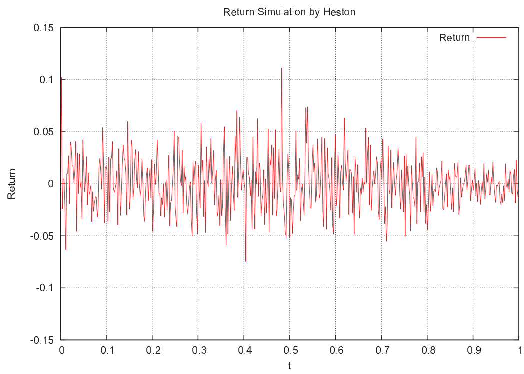

Figure 4.20: Simulated Return corresponding of the asset price in Figure 4.16.

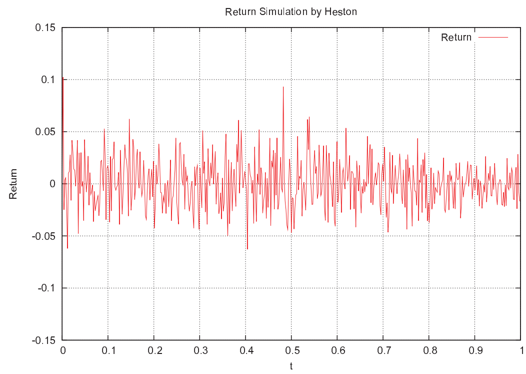

Figure 4.21: Simulated Return corresponding of the asset price in Figure 4.18.

As in the Hull and White model, the Heston model have moments of high and low volatility, creating volatility clustering as can be seen in the compounded return in Figures 4.20 and 4.21.

Several extensions of these two stochastic volatility models were proposed in the literature (see for instance, two examples presented by in Andersson[78] for the Swedish option market).

A particular issue in the Hull-White and Heston models deserves some additional comments. In those models the asset price and the volatility are computed at the same time level. But in the delayed asset pricing models, the volatility depends on the asset price at some point in the past. Hence a certain pointwise memory effect is introduced. The literature presents some evidence that the asset prices dynamics do not depend only on their current values but also on past values (not point wisely) (see, for example, Akgiray[79], Scheinkman and LeBaron[80] and Kind at al [81]). One of the most important model where asset prices dynamics do not depend only on their current values, but also on past values is the one proposed in Hobson and Rogers[24]. This model can be seen as a generalizations of the so-called level-dependent volatility models, where the volatility is usually a function of time and current price level and therefore market is complete. Hobson and Rogers[24] assume, that the volatility depends on the past prices of the risky assets. Consequently the mathematical description of the market behavior is enriched and it reproduces correlations and dependencies which are observed in practice.

Let $Z(t)$ be the discounted log-price process defined by

$$Z(t) = log(e^{-rt}\, S(t))\ , \tag{4.11}$$

where r is the (constant) risk-free interest rate, as introduced in Hobson and Rogers[24]. The authors define the offset function of order m, $D^{(m)}(t)$, by

$$D^{(m)}(t) = \theta \int_0^{+\infty} e^{-\lambda v} \Big[Z(t) - Z(t-v) \Big]^m dv, \quad \lambda > 0.$$

where the parameter λ describes the rate at which past information is discounted, and thus describes the weight of historic observations.

In what follows consider the particular case $m = 1$. In this case

$$D(t) = Z(t) - \int_0^{+\infty} \theta e^{-\lambda v} \, Z(t-v)dv, \quad \lambda > 0\,, \tag{4.12}$$

is equivalent to

$$D(t) = Z(t) - \theta \int_{-\infty}^{t} e^{-\lambda(t-v)} Z(v)\, dv\ .$$

So $D(t)$ is decomposed as the deviation of the current price from an exponentially weighted average of past records, such that θ determines the horizon of the "*moving time window*" of the integral on the right. For bigger values of this parameter, $S(t)$ is more dependent on the recent past, while small values almost identify the offset increments with price changes. Obviously in this case a level dependent volatility assumption would be numerically more convenient.

As in the discrete-time ARCH and GARCH environments, this is designed to reflect the perception that large movements of the asset price in the past tend to forecast higher future volatility. It can also be shown that the model encompasses a wide variety of smiles and skews, and can account for volatility term structures as the average of prices evolves through time.

It is assumed that the dynamics of $Z(t)$ are described by the following SDE

$$dZ(t) = \mu(D(t))\, dt + \sigma(D(t))\, dW_1(t),$$

where $W_1(t)$ is a Brownian motion and $\sigma(.)$ and $\mu(.)$ are deterministic functions of $D(t)$. For $D(t)$ Hobson and Rogers[24] establish that

$$dD(t) = \Big[\mu(D(t)) - \theta D(t)\Big]\, dt + \sigma(D(t))\, dW_1(t). \tag{4.13}$$

Considering

$$\beta(D(t)) = \frac{1}{2}\sigma(D(t)) + \frac{\mu(D(t))}{\sigma(D(t))}\ ,$$

(4.13) can be rewritten in the equivalent form

$$dD(t) = -\left[\frac{1}{2}\sigma^2(D(t)) + \theta D(t)\right] dt + \sigma(D(t))\, dW_2(t)\ , \tag{4.14}$$

where

$$W_2(t) = W_1(t) + \int_0^t \beta(D(v))\, dv \; .$$

Consequently the asset price $S(t)$ is described by an Itô process of the form

$$dS(t) = \Big[\mu(D(t)) + r\Big]\; S(t)dt + \sigma(D(t))\; S(t)dW_1(t) \tag{4.15}$$

The SDEs (4.14) and (4.15) do not have known analytical solutions, and the solutions must be approximated by numerical methods. For instance, one may use the linear interpolation (3.4) where Y_n is defined by the Euler-Maruyama method (3.2) for $D(t)$ in (4.14) and an initial condition $D(0)$, such that

$$f(D_n, t_n) = -\left[\frac{1}{2}\sigma^2(D_n) + \theta D_n\right]$$

and

$$G(D_n, t_n) = \sigma(D_n) \; ,$$

and for $S(t)$ with dynamics (4.15) and a given initial condition $S(0)$ such that

$$f(S_n, t_n) = \Big[\mu(D(t_n)) + r\Big] S_n$$

and

$$G(S_n, t_n) = \sigma(D(t_n))S_n \; .$$

In our numerical experiments we take

$$\mu(D(t_n)) = \frac{1}{n}\sum_{t=t_0}^{t_n} D(t) \tag{4.16}$$

and

$$\sigma(D_n) = \min\left\{\eta\sqrt{1 + D_n^2}, N\right\} \; , \tag{4.17}$$

being (4.17) suggested in Hobson and Rogers[24]. In (4.17) N, ε and η are positive constants where η is minimal level of implied volatility. The constant ε is a scaling parameter introduced to take into account the influence of the initial offset in the volatility function.

The next figures plot two simulations of asset prices by Hobson and Roger model and the corresponding deviation of the current price for volatility models. The parameters are $r = 0.2$, $\eta = 0.40$ or 0.50, $\varepsilon = 5.0$, and a large constant N. The rate at which past information gets discounted into the offset function is $\theta = 1$.

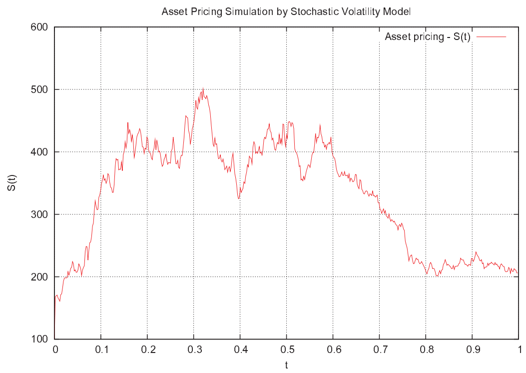

Figure 4.22: Asset pricing simulation by Hobson & Roger model for deviation simulation in Figure 4.23.

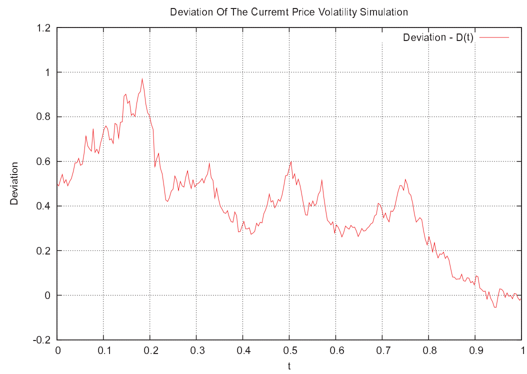

Figure 4.23: Simulated of deviation of the current price for volatility model with $\eta = 0.40$.

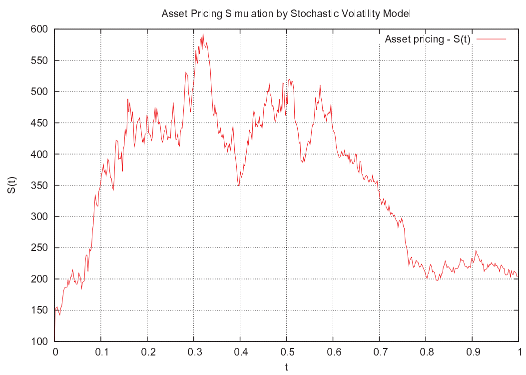

Figure 4.24: Asset pricing simulation by Hobson & Roger model for deviation simulation in Figure 4.25.

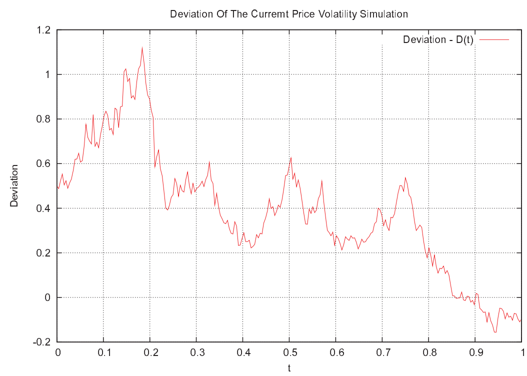

Figure 4.25: Simulated of deviation of the current price for volatility model with $\eta = 0.50$.

As one can see in Figure 4.25, when the parameter η (minimal level of implied volatility) has its value equal to 0.50 there is a greater dispersion in the deviation values, as compared with Figure 4.23 with $\eta = 0.40$, meaning that prices have higher volatility. When comparing Figures 4.25 and 4.23 one also observe that there is a change in the value of the asset price in function of the difference in the η value.

Figures 4.26 and 4.27 show the return processes corresponding to the asset price processes of Figures 4.22 and 4.24. As expected, returns in Figure 4.26 have higher volatility than those in Figure 4.27, this volatility increase is due to the higher value of the parameter η.

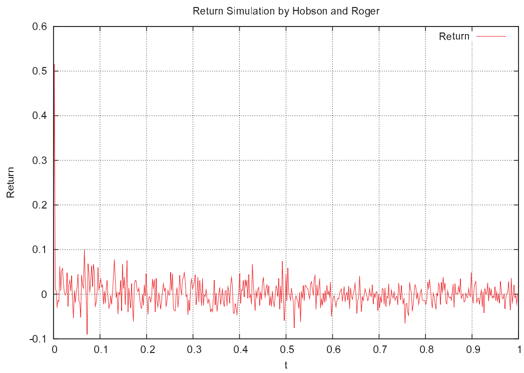

Figure 4.26: Simulated Return corresponding of the asset price in Figure 4.22.

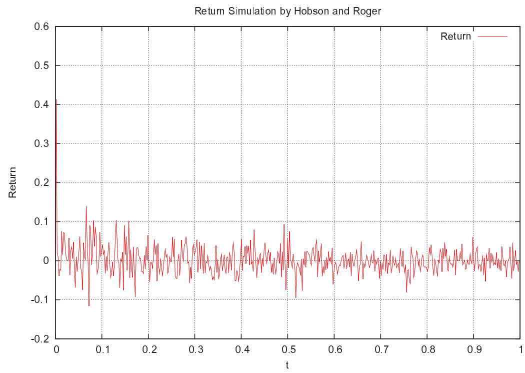

Figure 4.27: Simulated Return corresponding of the asset price in Figure 4.24.

In fact, the Hobson and Rogers model is equivalent to a 2-dimensional Markov model. Thus the problem of pricing and hedging a derivative asset leads to the solution of a linear PDE (Hallulli and Vargiolu[82]). The Hobson and Rogers model has two qualities: Firstly, it is potentially able to reproduce smiles, skews of different directions, and volatility term structures, and secondly, it preserves the completeness of the market. Since no exogenous source of risk is added, the classical arbitrage pricing and hedging theory is appliable. This last feature constitutes an advantage over fully stochastic volatility processes, where arbitrage considerations are not sufficient to identify uniquely the "*risk premia*" (Platania and Rogers[83]).

Other models for the asset price, with the delayed effect, based in the one proposed by Hobson and Rogers have been proposed in the literature (see, for instance, V. Hallulli and Vargiolu[84, 85]), but the Di Francesco and Pascucci[25] model deserves a special reference. The authors propose a complete model with stochastic volatility in the sense of Hobson and Rogers[24], such that the options are the solutions to degenerate partial differential equations obtained by the inclusion of other state variables describing the dependence on past prices of the underlying asset. Foschi and Pascucci[26] have tested empirically with option prices assuming a volatility structure as in Hobson and Rogers[24] market completeness, and were able to reproduce the"smile" and observed patterns of implied volatility structure.

Andersson[78] presents a stochastic volatility model where the volatility changes randomly according to some SDE or some discrete random process. Our main reference for this theory is Jean-Pierre et al. [86]. This type of stochastic volatility model introduces more random sources then the number of traded assets and therefore the market is not complete. Pricing in a market with stochastic volatility is thus an incomplete market problem, which means that it does not exist a unique martingale measure, and the derivative cannot be perfectly hedged with just the underlying asset and a riskless asset.

4.2.3 Geometric Telegrapher's Process

As pointed out before, the Brownian process has some limitations like infinite first and second variations, independence of of the log-returns increments, which lead to a pathologic behavior of asset prices. To avoid such limitations in financial market several authors have considered the telegraph process to model the dynamics of the underlying asset price. Among these authors stand out Di Crescenzo and Pellerey[10] that assume that $S(t)$ evolves in time according to the following process

$$S(t) = S_0 \, exp\Big\{\alpha t + \sigma X(t)\Big\} \ , \quad t \geq 0. \tag{4.18}$$

where $\alpha = \mu - \frac{1}{2}\sigma^2$, $\sigma > 0$, $X(t)$ is the Telegraph process (2.15) and $V(0) = 1$. Given that $X(t)$ has a bounded first variation, then $S(t)$ in (4.18) has also a bounded first variation. This seems a more realistic way to model paths of financial prices, however it is still not a satisfactory one, because in this process the drift and volatility are still constant over time.

Figure 4.28 displays a trajectory of the geometric Telegraph process with parameters $\mu = 0.30$, $\sigma = 0.50$ and intensity of Poisson process equal to 2. In Figure 4.29 there is a trajectory for the same μ and σ but with an intensity of Poisson process equal to 3. The changes of direction are governed by a homogeneous Poisson process $N(t)$ with rate $\lambda > 0$. When λ increases from 2 to 3, changes occur more frequently.

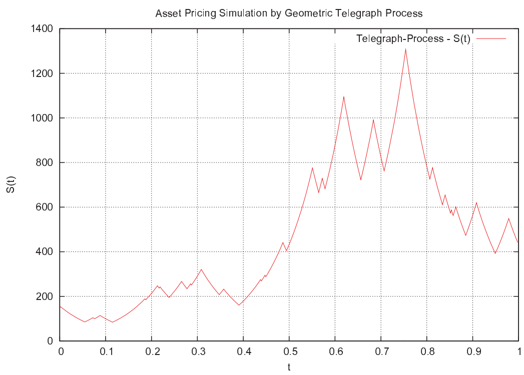

Figure 4.28: Asset Pricing Simulation by Geometric Telegraph Process (4.18) with intensities of Poisson process is 2.

Figure 4.29: Asset Pricing Simulation by Geometric Telegraph Process (4.18) with intensities of Poisson process is 3.

Ratanov[11, 12, 13] introduces a new class of models where the Telegraph process of Di Crescenzo and Pellerey[10] is replaced by a Jump Telegraph process. This new Telegraph process is characterized by two intensities ($\lambda_{\pm}$) and two velocities ($c_{\pm}$). The next chapter presents in detail this new model.

5 Asset Models with Memory in Price and Volatility

The Geometric Telegraph Process (4.18) studied in Section 3.2.3, arises in the literature to improve some features of the Telegraph model proposed by Crescenzo and Pellerey[10]. It should be pointed out that the first process describes a random motion with finite velocity and it is usually suggested as an alternative to classical diffusion model.

The aim of this chapter is to improve the models studied in Ratanov[11, 12, 13]. In fact these models are characterized by constant volatility and they do not take into account the drift (measure of the average rate of growth of the asset price). Also, to include the stochastic volatility we follow the approach presented in Hobson and Roger[24].

Section 5.1 displays a natural extension of the SDE (4.1) for the asset price where the drift is included and the volatility is constant and same particular cases are present in Section 5.2. The stochastic volatility is considered in Section 5.3.

5.1 The JTDD-Process For Asset Pricing

Here, it is admitted that the model for the dynamics of the underlying asset returns is given by the Jump Telegraph Diffusion Drift Process (JTDD-process),

$$dS(t) = \mu S(t)dt + \sigma S(t_-)dW(t) + \sigma S(t_-)dX_\pm(t) + S(t_-)dJ_\pm(t) \ , \tag{5.1}$$

where σ is the volatility of the asset's price $S(t)$ and μ is the expected return of asset, $S(t)$. It is assumed that the price process is right-continuous.

Let $\nu_\pm$, $r_\pm$, $h_\pm$ be real numbers such that $\nu_+ > \nu_-$, $r_\pm \geq 0$ and $h_\pm > -1$. Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space, and let $\lambda_\pm$ be positive numbers. The dynamics of the underlying asset in (5.1) incorporates the following processes:

- A pure Jump process $J_\pm = \{J_\pm(t)\}_{t\geq 0}$ with jumps at the Poisson times τ_j, $j = 1, 2, \cdots$, with alternating jumps of sizes $h_\pm \in (-1, \infty)$, defined by

$$J_+(t) = \int_0^t h_{g_+(\tau)} dN_+(\tau) = \sum_{j=1}^{N_+(t)} h_{g_+(\tau_j-)} \tag{5.2}$$

  and

$$J_-(t) = \int_0^t h_{g_-(\tau)} dN_-(\tau) = \sum_{j=1}^{N_-(t)} h_{g_-(\tau_j-)}, \tag{5.3}$$

  where $g_+(t) = (-1)^{N^+(t)}$ and $g_-(t) = -(-1)^{N^-(t)}$.

- The Telegraph process $X_\pm(t) = \{X_\pm(t)\}_{t\geq 0}$ with velocity $\nu_\pm$ defined by (2.16) and (2.17), and

- A pure Diffusion process (Wiener's process) for $W(t) = \{W(t)\}_{t\geq 0}$.

Let $r_\pm \geq 0$ be the riskless interest rate which is in the initial state $+ \ or \ -$. The riskless asset is given by the exponential $B(t) = e^{Y_\pm(t)}$ of the process

$$Y_\pm = \{Y_\pm\}_{t\geq 0} = \left\{ \int_0^t r_\pm d\tau \right\}_{t\geq 0} ,$$

where the interest rates $r_\pm > 0$ and $+$ or $-$ indicate the initial market state. Here again $Y_\pm = \{Y_\pm(t)\}_{t\geq 0}$ is a Telegraph process with velocity values $r_\pm$.

In view of such trajectories, the market is set up as a continuous process that evolves with velocities ν_+ or ν_-, changes the direction of movement from $\nu_\pm$ to $\nu_\mp$, and exhibits jumps of sizes $h_\pm$ whenever velocity changes. The different parameters for up and down movements lead to a gain/loss asymmetry.

Ratanov[13] presents the following characterization:

Theorem 5.1.1. *Jump-Telegraph-Diffusion process (JTD-process) is a martingale if and only if*

$$\lambda_+ h_+ = -\nu_+ \quad \textit{and} \quad \lambda_- h_- = -\nu_- \ . \tag{5.4}$$

Theorem 2.3.2 is then used to establish the solution of (5.1). This result agrees to the representation

$$S(t) = S_0 \ EXP\left\{\left[\mu - \frac{1}{2}\sigma^2\right] t + \sigma W(t)\right\} \varepsilon_t\Big\{\sigma X_\pm(t) + J_\pm(t)\Big\} \ ,$$

where $S_0 = S(0)$ and $\varepsilon_t(\cdot)$ denotes the stochastic exponential. Additionally

$$\varepsilon_t\Big\{\sigma X_\pm(t) + J_\pm(t)\Big\} = e^{X_\pm(t)} K_\pm(t) \ ,$$

$$K_m(t) = \prod_{\tau\leq t}\Big[1 + \Delta J_\pm(t)\Big] = \prod_{j=1}^{N_\pm(t)}\Big[1 + h_{g_\pm(\tau_j -)}\Big].$$

Where τ_j , $j \geq 1$ are the jumping times of the Poisson processes $N_\pm$.

With $Z = \sigma X_\pm + J_\pm$ one gets

$$\varepsilon_t\Big\{Z\Big\} = e^{Z(t) - \frac{1}{2}\langle Z,Z\rangle^{cont}(t)} \prod_{0<\tau\leq t}\Big[1 + \Delta Z(\tau)\Big] e^{-\Delta Z(\tau)}.$$

As

$$\langle Z, Z\rangle^{cont} = \langle \sigma X_\pm + J_\pm \ , \ \sigma X_\pm + J_\pm\rangle^{cont} = 0$$

and

$$\begin{aligned}\varepsilon_t\Big\{\sigma X_\pm + J_\pm\Big\} &= e^{\sigma X_\pm + J_\pm}\prod_{0<\tau\le t}\Big[1+\Delta J_\pm(\tau)\Big]e^{-\Delta J_\pm(\tau)}\\ &= e^{\sigma X_\pm + J_\pm}e^{-J_\pm(t)}\prod_{0<\tau\le t}\Big[1+\Delta J_\pm(\tau)\Big]\\ &= e^{\sigma X_\pm}\prod_{0<\tau\le t}\Big[1+\Delta J_\pm(\tau)\Big]\,.\end{aligned}$$

Finally one obtains

$$S(t) = S_0\ EXP\left\{\left[\mu - \frac{1}{2}\sigma^2\right]t + \sigma W(t) + \sigma X_\pm(t)\right\}K_\pm(t), \tag{5.5}$$

with

$$K_\pm(t) = \prod_{j=1}^{N_\pm(t)}\Big[1 + h_{g_\pm(\tau_j -)}\Big].$$

Hereafter, it is proven that one may in fact apply Theorem 2.3.2 to legitimize the above construction. Note that $(\Omega, \mathfrak{F}, P)$ is a complete probability space, with a complete filtration $\mathfrak{F} = (\mathfrak{F}_t, t \le 0)$, generated by the Wiener process $W(t)$ and $N_\pm$, and all stochastic processes with index $+$ (or $-$) are adapted to filtration generated by N_+ (or N_-).

Theorem 5.1.2 (Stochastic Exponential of JTDD-Process)**.** *The stochastic equation of JTDD-process (5.1), has a unique solution, given by (5.5). Moreover this solution is a semimartingale.*

Proof: *The SDE (5.1) admits the following representation*

$$dS(t) = S(t_-)dY_\pm(t)\ , \quad S(0)\ \textit{is given}\ ,$$

with

$$Y_\pm(t) = M(t) + A_\pm(t) \tag{5.6}$$

where

$$M(t) = \sigma W(t)\ , \quad A_\pm(t) = \mu t + \sigma X_\pm(t) + J_\pm(t)\ , \quad t \ge 0\ .$$

Assume the existence of the integration for SDE (5.1). For all $t > 0$*,*

$$\int_0^t S^2(x)dx < +\infty \quad a.s.$$

and

$$\int_0^t |S(x)|dV_{A_\pm} < +\infty \quad a.s. \ ,$$

where $V_{A_\pm}$ *is the variation process of* $A_\pm$*.*

Note that the definition (2.3.1) of semimartingales is verified for (5.6) due to the fact that the Wiener process $W(t)$*, with* $W(0) = 0$ *a.s., is a local martingale because it is a martingale with continuous trajectories (see definition (2.3.2) and (2.3.3)). According to this condition,* $M(t) = \sigma W(t)$ *is also a local martingale.*

The process $A_\pm$ *is of finite variation due to the fact that* $\sigma X_\pm(t)$ *and* $J_\pm(t)$ *are of finite variation (*$\sigma X_\pm(t)$ *is continuous and* $J_\pm(t)$ *is pure jump) and* μt *is a continuous and monotonic function.*

Moreover,

$$M(0) = \sigma W(0) = 0 \quad a.s. \ , \quad A_\pm(0) = \sigma X_\pm(0) + J_\pm(0) = 0 \ .$$

and

$$Y_\pm(t) = Y_\pm(0) + M(t) + A_\pm(t) \quad with \quad Y_\pm(0) = 0 \ , \quad a.s.$$

Applying Theorem 2.3.1, which was presented for instance in Protter[45], one can show that

$$\begin{aligned} S(t) &= Exp\left\{ Y_\pm(t) - \frac{1}{2}\langle Y_\pm, Y_\pm\rangle_t + \frac{1}{2}\sum_{\nu\in(0,t]}[\Delta Y_\pm((\nu))]^2 \right\} \times \\ &\times \prod_{\nu\in]0,t]} \Big[1 + \Delta Y_\pm(\nu)\Big] \ Exp\Big\{-\Delta Y_\pm(\nu)\Big\} \ , \quad t \geq 0, \end{aligned} \tag{5.7}$$

with $Y_\pm(t)$ *given by (5.6). For this last stochastic process it is defined that*

$$\Delta Y_\pm(\nu) = Y_\pm(\nu) - Y_\pm(\nu) = \Delta J_\pm(t) \ ,$$

μt *is a continuous function and* $\sigma X_\pm(t)$ *and* $\sigma W(t)$ *are processes with continuous trajectories.*

The quadratic variation process of $Y(t)$, $\langle Y_\pm, Y_\pm\rangle_t$, is given by

$$\langle Y_\pm, Y_\pm\rangle_t = \langle M + A_\pm, M + A_\pm\rangle_t$$

and applying the property of Remark 2.2.1, then

$$\begin{aligned}\langle Y_\pm, Y_\pm\rangle_t &= \langle M, M\rangle_t + \langle M, A_\pm\rangle_t + \langle A_\pm, A_\pm\rangle_t \\ &= \sigma^2\langle W, W\rangle_t + 2\sum_{\nu\le t}\Delta A_\pm(\nu)\Delta M(\nu) + \sum_{\nu\le t}[\Delta A_\pm(\nu)]^2 \ .\end{aligned}$$

Moreover, as $\Delta M(\nu) = 0$, then

$$\langle Y_\pm, Y_\pm\rangle_t = \sigma^2 t + \sum_{\nu\le t}[\Delta J_\pm(\nu)]^2 \ .$$

Considering the previous conclusions in (5.7), one obtains

$$\begin{aligned}S(t) &= Exp\left\{Y_\pm(t) - \frac{1}{2}\sigma^2 t - \frac{1}{2}\sum_{\nu\le t}[\Delta J_\pm((\nu))]^2 + \frac{1}{2}\sum_{\nu\le t}[\Delta J_\pm((\nu))]^2\right\} \times \\ &\quad \times \prod_{\nu\in]0,t]}\Big[1 + \Delta J_\pm(\nu)\Big] Exp\Big\{-\Delta J_\pm(\nu)\Big\} \ ,\end{aligned}$$

which is equivalent to

$$\begin{aligned}Z(t) &= Exp\left\{\left[\mu - \frac{\sigma^2}{2}\right] t + \sigma W(t) + X_\pm(t) + J_\pm(t)\right\} \times \\ &\quad \times \prod_{\nu\in]0,t]}\Big[1 + \Delta J_\pm(\nu)\Big] \ Exp\Big\{-\sum_{\nu\in]0,t]}\Delta J_\pm(\nu)\Big\} \\ &= Exp\left\{\left[\mu - \frac{\sigma^2}{2}\right] t + \sigma W(t) + X_\pm(t)\right\} \times \prod_{\nu\in]0,t]}\Big[1 + \Delta J_\pm(\nu)\Big] \ .\end{aligned}$$

Furthermore, the SDE $dS(t) = S(t_-)dY_\pm(t)$ can be written as

$$dS(t) = \sigma S(t_-)dW(t) + \sigma S(t_-)dA_\pm(t)$$

because

$$\int_0^t S(\nu-)dY_\pm(\nu) = \int_0^t S(\nu-)dM(\nu) + \int_0^t S(\nu-)dA_\pm(\nu)$$

where the last integral coincides with the Lebesgue-stiltjes integral because $A_\pm$ is of finite variation.

□

Figure 5.1 plots a trajectory of the JTDD-process for an underlying asset with the intensity of the Poisson process equal to 4, $\sigma = 0.70$ and $\mu = 0.30$.

Figure 5.1: Asset Prices Simulation by JTDD-process.

In this model, the parameter of the expected rate of return is included, but still volatility is assumed to be constant. Thus, in the next section this model is extended by including the Hobson and Roger model for the volatility, which, in facts, results in giving a memory structure to the volatility.

5.2 Same Particular Cases

A particular case of (5.1) is presented in Ratanov[11, 12] where the price of the risky asset $S(t)$ is described by

$$dS(t) = S(t-)d\Big\{X_{\pm}(t) + J_{\pm}(t)\Big\} \ . \tag{5.8}$$

The solution of the initial stochastic differential problem (5.8) with an initial condition $S(0)$ was presented in Ratanov[11, 12] as

$$S(t) = S(0)\, e^{X(t)} \prod_{j=1}^{N_{\pm}(t)} \Big[1 + h_{g_{\pm}(\tau_j -)}\Big] \tag{5.9}$$

and its behavior is hereafter illustrated.

Figure 5.2 plots a trajectory of the Jump Telegraph process (5.9) with parameter of intensity of Poisson process, λ, equal to 4 and jumps of sizes $h_{\pm}$ equal to 0.5.

Figure 5.2: Evolution of the Jump Telegraph Process (5.9) for $\lambda = 4$ and $h_{\pm} = \pm 0.5$

The comparison between the model of Ratanov and the model of DiCrenzo, highlights that the main difference is the discontinuity (the jumps).

It is clear that the mathematical model (5.9) cannot be used to describe the evolution of the asset prices in a real market. In fact, the parameter that characterize the asset prices market like the volatility and the expected rate return are not included in the model.

To obtain a more realistic mathematical model to describe the asset price evolution, a diffusion process was introduced in the previous model (5.8) in Ratanov[13], that is

$$dS(t) = S(t-)d\Big\{X_{\pm}(t) + J_{\pm}(t) + D_{\pm}(t)\Big\} , \tag{5.10}$$

where $D_{\pm}(t)$ is given by

$$D_{\pm}(t) = \int_0^t \sigma_{\pm} dW(\tau) ,$$

where $\sigma_\pm$ is a real parameter. Therefore, this model can be interpreted as a diffusion process with Markov switching (see Ratanov[13]).

The stochastic exponential of JTD-process (5.10) with the initial condition $S(0)$ has the form

$$S(t) = S(0)\, exp\left\{X_\pm(t) + D_\pm(t) - \frac{1}{2}\int_0^t \sigma_\pm^2 d\tau\right\} \prod_{j=1}^{N_\pm(t)} \left[1 + h_{g_\pm(\tau_j -)}\right]. \qquad (5.11)$$

The behaviour of (5.11) is illustrated in Figure 5.3, with the parameter of Poisson intensity, λ, equal to 4, the volatility parameter ($\sigma_\pm$) equal to 0.40 and jumps of sizes $h_\pm$ equal to 0.5. Figure 5.4 presents a trajectory for the same parameters, except that (σ) is 0.70.

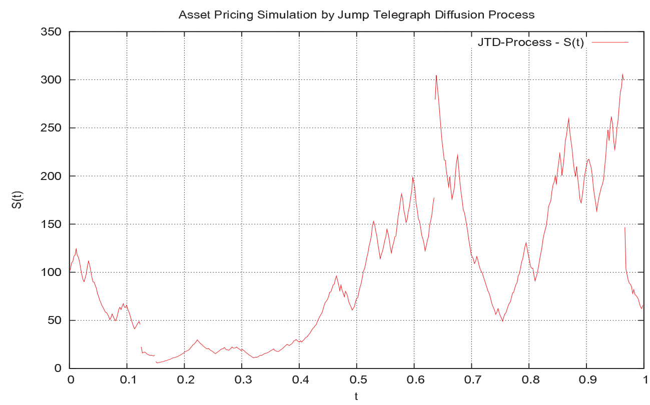

Figure 5.3: Evolution of the JTD Process (5.11) with $\lambda = 4$, $\sigma_{\pm} = 0.40$ and $h_{\pm} = \pm 0.5$.

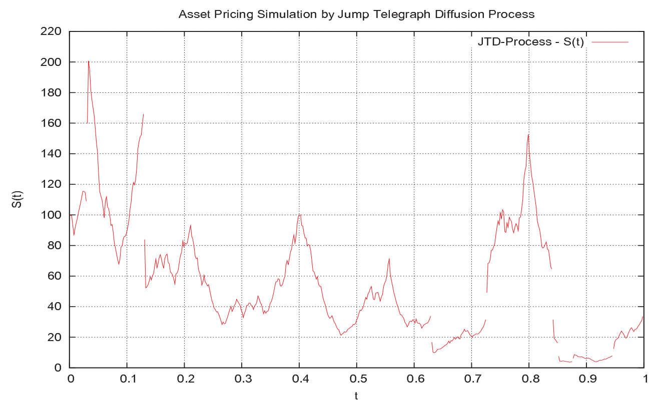

Figure 5.4: Evolution of the JTD Process (5.11) for $\lambda = 4$, $\sigma_{\pm} = 0.70$ and $h_{\pm} = \pm 0.5$.

One can conclude that changes in the volatility parameter changes the pattern of trajectory. More precisely, when $\sigma_{\pm} = 0.40$ the asset price behavior is dominated by Telegraph process, when $\sigma_{\pm} = 0.70$ the diffusion process is the dominant pattern.

In the stochastic process $S(t)$ given by (5.11) the volatility was included. Nevertheless $S(t)$ does not depend on the expected rate of return. It is obvious that the mathematical model of Ratanov needs to be modified in order to include this parameter. Another handicap of the Ratanov's model is that it assumes a constant volatility. More realistic models can be constructed considering, for instance, the approaches of Cox and Ross[5] or Hobson and Rogers[24].

The next section presents new mathematical models for the asset price where the volatility is described as following the approach of Hobson and Rogers.

5.3 JTDD-Process with Memory in the Volatility

Consider the JTDD-process (5.1) for the asset prices but with a random volatility as in Hobson and Rogers[24]. More precisely, assume that the volatility is described using the approach introduced by these authors but with the randomness introduced as in $S(t)$.

As before, let $Z(t)$ the be defined by (4.11). Now suppose that

$$dZ(t) = \mu(D(t))\, dt + \sigma(D(t))\, dW_1(t) + \sigma(D(t))\, dX_\pm(t) + dJ_\pm(t)\ , \tag{5.12}$$

where $D(t)$ is given by (4.12) and $\sigma^2(D(t))$ is the Hobson and Roger volatility of the price of asset $S(t)$, and $\mu(D(t))$ is the expected return.

So, the dynamics of the asset price $S(t)$ is governed by the SDE

$$\begin{aligned} dS(t) &= \Big[\mu(D(t)) + r\Big] S(t)dt + \sigma(D(t))\, S(t_-)dW_1(t) + \\ &\quad + \sigma(D(t))\, S(t_-)dX_\pm(t) + S(t_-)dJ_\pm(t) \end{aligned} \tag{5.13}$$

It can be shown that $D(t)$ holds the following SDE

$$\begin{aligned} dD(t) &= \Big[\mu(D(t)) - \theta D(t)\Big]\, dt + \sigma(D(t))\, dW_1(t) + \\ &\quad + \sigma(D(t))\, dX_\pm(t) + dJ_\pm(t)\ . \end{aligned} \tag{5.14}$$

and consequently

$$\begin{aligned} dD(t) &= -\left[\frac{1}{2}\sigma^2(D(t)) + \theta D(t)\right]\, dt + \sigma(D(t))\, dW_2(t) + \\ &\quad + \sigma(D(t))\, dX_\pm(t) + dJ_\pm(t)\ . \end{aligned} \tag{5.15}$$

The SDEs (5.13) and (5.15) have unknown analytical solution, and its solution may be computed at least numerically using, for instance, the linear interpolation (3.4) where Y_n is defined by the Euler-Maruyama's method

$$\begin{aligned} Y_{n+1} &= Y_n + f(Y_n, t_n)\,\Delta_n + G_1(Y_n, t_n)\Big[W_2(t_{n+1}) - W_2(t_n)\Big] + \\ &\quad + G_2(Y_n, t_n)\Big[X_\pm(t_{n+1}) - X_\pm(t_n)\Big] + \\ &\quad + G_3(Y_n, t_n)\Big[J_\pm(t_{n+1}) - J_\pm(t_n)\Big]\ . \end{aligned} \tag{5.16}$$

For $S(t)$ in (5.13) and initial conditions $S(0)$ one has

$$f(Y_n, t_n) = \Big[\mu(D(t_n)) + r\Big]Y_n \quad \text{and} \quad G_1(Y_n, t_n) = G_2(Y_n, t_n) = \sigma(D(t_n))Y_n\ ,$$

$$G_3(Y_n, t_n) = Y_n\ .$$

for $D(t)$ in (5.15) and initial condition $D(0)$ have

$$f(Y_n, t_n) = -\left[\frac{1}{2}\sigma^2(Y_n) + \theta Y_n\right] \quad \text{and} \quad G_1(Y_n, t_n) = G_2(Y_n, t_n) = \sigma(Y_n)\ ,$$

$$G_3(Y_n, t_n) = 1\ .$$

In the following numerical experiments it is used (4.17) as suggested in Hobson and Rogers[24].

Figures 5.5 and 5.6, show simulated values of deviation of the current price for volatility models and corresponding asset prices by the JTDD-process with Hobson and Roger models for volatility. The following parameters are used: $r = 0.1$, $S_0 = 150.00$, $D(0) = 0.5$, $h_+ = 0.5$, $h_- = -0.5$, $\eta = 0.40$, $\varepsilon = 0.1$ and a large constant N. The rate at which the past information gets discounted into the offset function is $\theta = 1$. The parameter of intensity of the Poisson process is $\lambda = 6$.

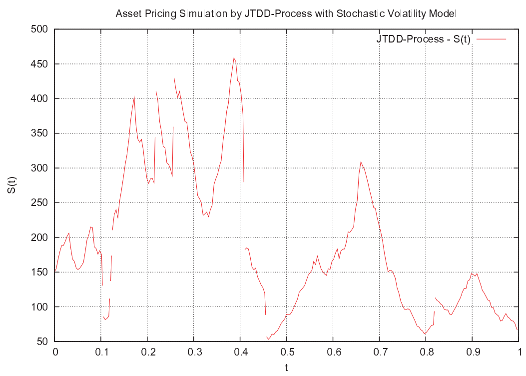

Figure 5.5: Asset pricing simulation by JTDD-Process with Memory in the Volatility (Hobson & Roger) for deviation simulation in Figure 5.6.

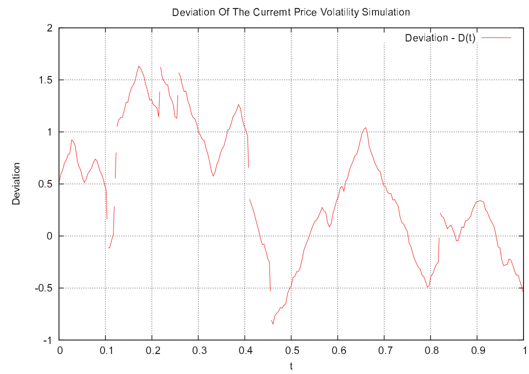

Figure 5.6: Simulated values of deviation of the current price for volatility model (Hobson & Roger) with $\lambda = 6$ and $\varepsilon = 0.1$.

Figures 5.7 and 5.8 present a new simulation with the same parameters, except that the λ is equal to 3.

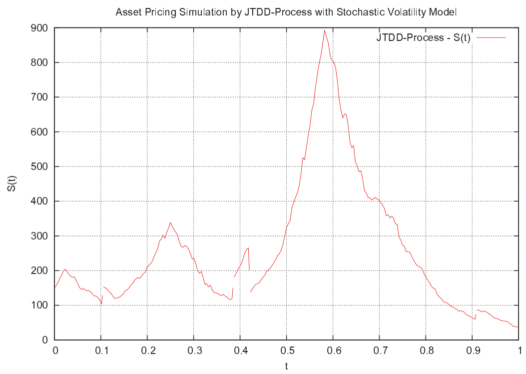

Figure 5.7: Asset pricing simulation by JTDD-Process with Memory in the Volatility (Hobson & Roger) for deviation simulation in Figure 5.8.

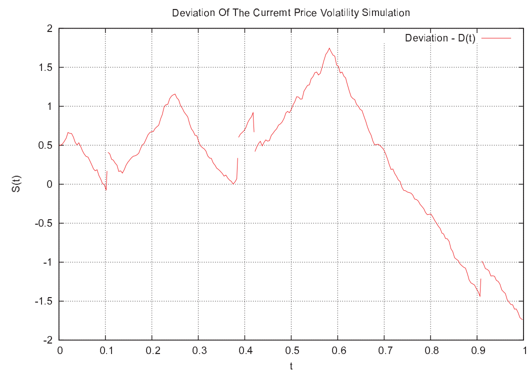

Figure 5.8: Simulated values of deviation of the current price for volatility model (Hobson & Roger) with $\lambda = 3$ and $\varepsilon = 0.1$.

Finally, Figures 5.9 and 5.10 present another simulation, but this time with a value of 2 for the ε.

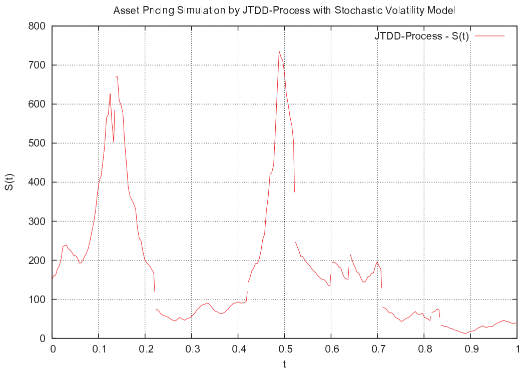

Figure 5.9: Asset pricing simulation by JTDD-Process with Memory in the Volatility (Hobson & Roger) for deviation simulation in Figure 5.8.

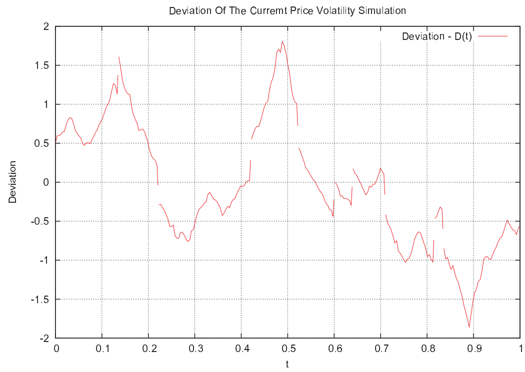

Figure 5.10: Simulated values of deviation of the current price for volatility model (Hobson & Roger) with $\lambda = 6$ and $\varepsilon = 2$.

In Figure 5.5 with $\lambda = 6$ and Figure 5.7 with $\lambda = 3$, as expected one observes the existence of a variation in the number of jumps, plus change in trajectory due to the telegraph process. However, when one changes the parameter ε it is apparent that the price oscillation has changed, due to the fact that the Hobson and Roger[24] volatility takes larger values, as in Figures 5.5 with $\varepsilon = 0.1$ and 5.9 with $\varepsilon = 2$.

6 Itô's Lemma

The Itô's Lemma is the key ingredient in the establishment of partial differential equations for option pricing. Denoting $\mathcal{C}^{m,n}$ as the set of functions $F(x,y)$ such that $\dfrac{\partial^m F}{\partial x^m}$ and $\dfrac{\partial^n F}{\partial y^n}$ are continuous, then the classic formulation of the Itôt's Lemma is the following:

Lemma 6.0.1 (Itô's Lemma)**.** *If $Y(t)$ is a continuous semimartingale and $F \in \mathcal{C}^{2,1}$, then*

$$F(Y(t),t) = F(Y(0),0) + \int_0^t \frac{\partial F}{\partial t} dt + \int_0^t \frac{\partial F}{\partial Y} dY + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial Y^2} d < Y, Y >_t \ .$$

The main goal of the present chapter is the construction of the Itô's lemma for functions of the generalization of the JTDD process (5.1) introduced in the last chapter.

The approach followed here was presented, for instance, in Cont and Tankov[87] and Hanson[88]. This last author proposes a chain rule for calculating the differential of a composite process $F(S(t),t)$. The author begins by interpreting the differential as an infinitesimal increment and recognizing that, since the poisson jumps are instantaneous, it cannot be interpreted as a change in continuous time. Thus, a critical assumption in deriving the chain rule is that the continuous changes and jump changes can be calculated independently.

Therefore, following the assumption of independence, the overall JTDD process is spited into a TDD process and Jump process and then the Itô's Lemmas is developed for each part. Section 6.1 presents the quadratic variation of the TDD process, under convenient assumptions, necessary to deduce the Itô's Lemma for the TDD process. Section 6.2 combines the results and shows the Itô's Lemma for the JTDD process.

6.1 Quadratic variation of the TDD-Process

Our aim is to establish the Itô's lemma for the JTDD-process.

$$\begin{aligned} S(t) &= S(0) + \int_0^t A(S(\tau),\tau)d\tau + \int_0^t B(S(\tau),\tau)dW(\tau) + \\ &+ \int_0^t C(S(\tau),\tau)dX_{\pm}(\tau) + \int_0^t D(S(\tau),\tau)dJ_{\pm}(\tau)\ , \end{aligned} \tag{6.1}$$

where W is a standard Brownian motion and $X_{\pm}$ is a Telegraph process (5.2) for positive past information or (5.3) for negative past information. We observe that the processes W and $X_{\pm}$ are independent and adapted with respect to the filtration $\mathfrak{F} = \sigma(\sigma(W(\tau), \tau \leq t) \cup \sigma(N(\tau), \tau \leq t))$.

We assume that the coefficients $A(S(\tau),\tau)$, $B(S(\tau),\tau)$ and $C(S(\tau),\tau)$ are $\mathfrak{F}$-predictable processes and Lipschitz with respect to the first argument.

We replace the SDE (5.1) by this new SDE in (6.1) to include in the dynamic of the asset price non-constant volatility and non-constant drift as in Heston[77], Comte and Renault[89], Hobson and Rogers[24], Francesco and Rogers[25], Foschi and Pascucci[26] and Arriojas et al. [14] among others.

We remark that the existence and uniqueness of the solution of (6.1) for constant coefficients was established in the previous section. Moreover, for $C(S(\tau),\tau)$ and $D(S(\tau),\tau)$ are zero, the conditions on $A(S(\tau),\tau)$ and $B(S(\tau),\tau)$ that lead to a unique solution can be seen in Theorem 5.1.2. Numerical evidence allows one to believe that (6.1) is a unique solution provided that convenient smoothness conditions on the coefficients $A(S(\tau),\tau)$, $B(S(\tau),\tau)$, $C(S(\tau),\tau)$, $D(S(\tau),\tau)$ and on the initial condition are assumed. However, the existence and uniqueness of this problem will not be discussed in this work.

Consider (6.1) without the jump process, that is, with $D(S(\tau),\tau) = 0$. In order to compute its quadratic variation one may adopt the following natural assumptions on the coefficients A, B and C

$$\int_0^t |A(S(\tau),\tau)|d\tau < \infty \; , \tag{6.2}$$

$$\int_0^t \Big[B(S(\tau),\tau)\Big]^2 d\tau < \infty \tag{6.3}$$

and

$$\int_0^t \Big[C(S(\tau),\tau)\Big]^2 d\tau < \infty \; . \tag{6.4}$$

As it will be shown hereafter, these assumptions are crucial for the establishment of the quadratic variations of TDD-process. In order to simplify the presentation, let's consider the following notations

$$Int_1(t) = \int_0^t A(S(\tau),\tau)d\tau \; ,$$

$$Int_2(t) = \int_0^t B(S(\tau),\tau)dW(\tau)$$

and

$$Int_3(t) = \int_0^t C(S(\tau),\tau)dX_{\pm}(\tau) \; .$$

If one proves that $Int_2(t)$ is a local martingale and $Int_1(t) + Int_3(t)$ has finite variation, then one may conclude that $S(t)$ is a semimartingale (see Definition (2.3.1)). For $Int_1(t) + Int_3(t)$ one has

$$\begin{aligned} FV_{[0,T]}(Int_1(t) + Int_3(t)) &= \lim_{||\pi_n||\to 0} \Bigg[\sum_{k=0}^{n-1} |Int_1(t_{k+1}) - Int_1(t_k)| + \\ &+ \sum_{k=0}^{n-1} |Int_3(t_{k+1}) - Int_3(t_k)|\Bigg] \\ &= \lim_{||\pi_n||} \Big[Sum_1 + Sum_2\Big] \; . \end{aligned}$$

As

$$Sum_1 \le \sum_{k=0}^{n-1} \left| \int_{t_k}^{t_{k+1}} A(S(\tau),\tau)d\tau \right|$$

and

$$\sum_{k=0}^{n-1}\left|\int_{t_k}^{t_{k+1}} A(S(\tau),\tau)d\tau\right| = \int_0^T |A(S(\tau),\tau)|d\tau < +\infty$$

then

$$Sum_1 < +\infty.$$

For sum_2 one obtains successively

$$\begin{aligned}
I_2 &= \sum_{k=0}^{n-1}\left|\int_0^{t_{k+1}} C(S(\tau),\tau)dX_\pm(\tau) - \int_0^{t_k} C(S(\tau),\tau)dX_\pm(\tau)\right| \\
&\leq \sum_{k=0}^{n-1}\left|\int_{t_k}^{t_{k+1}} C(S(\tau),\tau)dX_\pm(\tau)\right| \\
&\leq \sum_{k=0}^{n-1}\left|\int_{t_k}^{t_{k+1}} C(S(\tau),\tau)\nu_{g_\pm(\tau)}d\tau\right| \\
&\leq \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \Big|C(S(\tau),\tau)\Big|\Big|\nu_{g_\pm(\tau)}\Big|d\tau \\
&\leq \left[\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\Big[C(S(\tau),\tau)\Big]^2 d\tau\right]^{\frac{1}{2}}\left[\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\Big[\nu_{g_\pm(\tau)}\Big]^2 d\tau\right]^{\frac{1}{2}} \\
&\leq \left[\int_0^T\Big[C(S(\tau),\tau)\Big]^2 d\tau\right]^{1/2}\Big[\max(|\nu_+|,|\nu_-|)\Big]\sqrt{T}\,.
\end{aligned}$$

Consequently

$$Sum_2 < +\infty\ .$$

Consider now $Int_2(t)$. As

$$\int_0^T B(S(\tau),\tau)d\tau < +\infty\ ,$$

$Int_2(t)$ is an Itô integral and then it is a local martingale (see Definitions 2.3.2 and 2.3.3 and Remark 2.2.1)

Now, consider the computation of the quadratic variation of $S(t)$. One has

$$\begin{aligned}
\langle S,S\rangle_t &= \langle Int_1,Int_1\rangle_t + \langle Int_2,Int_2\rangle_t + \langle Int_3,Int_3\rangle_t + 2\langle Int_1,Int_2\rangle_t + \\
&\quad + 2\langle Int_1,Int_3\rangle_t + 2\langle Int_2,Int_3\rangle_t \qquad (6.5)
\end{aligned}$$

As $Int_1(t)$ is given by a Riemann-Stieltjes integral, $Int_1(t)$ is differentiable with

$$Int_1(t)' = A(S(t), t) \; .$$

Consequently

$$\langle Int_1, Int_1 \rangle_t = 0 \; .$$

As mentioned before, $Int_2(t)$ is an Itô integral which implies that

$$\langle Int_2, Int_2 \rangle_t = \int_0^t \Big[B(S(\tau), \tau) \Big]^2 d\tau \; .$$

It was proven that $Int_3(t)$ has finite variation and it can be seen under the assumption (6.4) that it is continuous. So

$$\langle Int_3, Int_3 \rangle_t = 0 \; .$$

For the covariances between $Int_1(t)$, $Int_2(t)$ and $Int_3(t)$, it is direct to observe that

$$\langle Int_1, Int_2 \rangle_t = 0 \; ,$$

because, under the assumption (6.2), $Int_1(t)$ has finite variation and $Int_2(t)$ is continuous. Under the assumption (6.4), $Int_3(t)$ is continuous and then

$$\langle Int_1, Int_3 \rangle_t = 0.$$

One also has

$$\langle Int_2, Int_3 \rangle_t = 0 \; .$$

The previous considerations are summarized in the next result:

Theorem 6.1.1. *Under the assumptions (6.2), (6.3) and (6.4),* $S(t)$ *defined by (4.1) with* $D = 0$ *is a semimartingale and*

$$\langle S, S \rangle_t = \int_0^t \Big[B(S(\tau), \tau) \Big]^2 d\tau \; . \tag{6.6}$$

6.2 Ito's Lemma for JTDD-Process

In order to obtain an Itô's Lemma for the complete process (JTDD-process) in (6.1), one follows the approach considered in the literature as in Cont and Tankov[87] and Hanson[88].

The model for dynamic of underlying assets return (6.1) can be rewritten in the equivalent form

$$dS(t) = d_{cont}\, S(t) + d_{jump}\, S(t)\ ,$$

where the continuous and discontinuous parts, $d_{cont}(t), d_{jump}(t)$, respectively, are given by

$$d_{cont}\, S(t) = A(S(t),t)dt + B(S(t),t)dW(t) + C(S(t),t)dX_{\pm}(t)$$

and

$$d_{jump}\, S(t) = D(S(t),t)dJ_{\pm}(t)\ .$$

Thus, the change of a function depending on the state process $S(t)$, $dF_{\pm}(S(t),t)$, can be decomposed into the sum of continuous and discontinuous changes.

Now, one may proceed with the formalization of the Itô's lemma for the TDD-process. As $S(t)$ is a semimartingale and $F_{\pm} \in \mathcal{C}^{2,1}$, then $F(S(t),t)$ is a semimartingale (see in Klebaner[44]). According to Lemma 6.0.1

$$F_{\pm}(S(t),t) = F_{\pm}(S(0),0) + \int_0^t \frac{\partial F_{\pm}}{\partial t}dt + \int_0^t \frac{\partial F_{\pm}}{\partial S}dS + \frac{1}{2}\int_0^t \frac{\partial^2 F_{\pm}}{\partial S^2}d\langle S,S\rangle_t \qquad (6.7)$$

and, from (6.6), it follows that

$$F_{\pm}(S,t) - F_{\pm}(S(0),0) = \int_0^t \frac{\partial F_{\pm}}{\partial t}dt + \int_0^t \frac{\partial F_{\pm}}{\partial S}dS + \frac{1}{2}\int_0^t B^2\frac{\partial^2 F_{\pm}}{\partial S^2}dt. \qquad (6.8)$$

Using now (6.1) with $D(S(t),t) = 0$ in (6.8), one obtains

$$\begin{aligned} F_\pm(S(t),t) - F_\pm(S(0),0) &= \int_0^t \frac{\partial F_\pm}{\partial t} dt + \int_0^t A \frac{\partial F_\pm}{\partial S} d\tau + \int_0^t B \frac{\partial F_\pm}{\partial S} dW(\tau) + \\ &+ \int_0^t C \frac{\partial F_\pm}{\partial S} dX_\pm(\tau) + \frac{1}{2} \int_0^t B^2 \frac{\partial^2 F_\pm}{\partial S^2} d\tau \\ &= \int_0^t \left[\frac{\partial F_\pm}{\partial t} + A \frac{\partial F_\pm}{\partial S} + \frac{1}{2} B^2 \frac{\partial^2 F_\pm}{\partial S^2} \right] d\tau + \\ &+ \int_0^t B \frac{\partial F_\pm}{\partial S} dW(\tau) + \int_0^t C \frac{\partial F_\pm}{\partial S} dX_\pm(\tau) . \end{aligned}$$

Theorem 6.2.1 (Itô's Lemma for TDD Process)**.** *Let $S(t)$ be defined by (6.1) with $D = 0$ and $F \in C^{2,1}$. Under the assumptions (6.2), (6.3) and (6.4),*

$$\begin{aligned} dF_\pm(S(t),t) &= \left[\frac{\partial F_\pm}{\partial t} + A \frac{\partial F_\pm}{\partial S} + \frac{1}{2} B^2 \frac{\partial^2 F_\pm}{\partial S^2} \right] dt + B \frac{\partial F_\pm}{\partial S} dW(t) + \\ &+ C \frac{\partial F_\pm}{\partial S} dX_\pm(t) . \end{aligned} \tag{6.9}$$

Consider now the Itô's lemma for the Jump process defined by (6.1) with $B(S(\tau),\tau) = C(S(\tau),\tau) = 0$, where the Jump Process, $J_\pm(t)$, is a compound Poisson Process and a drift term is the continuous part of $S(t)$. If $F_\pm \in \mathcal{C}^{1,1}$, using the Proposition for Itô formula for jump-diffusion process (Proposition 8.14 of Cont and Tankov[87]), we can write $F_\pm(S(t),t) - F_\pm(S(0),0)$ as

$$\begin{aligned} F_\pm(S(t),t) &- F_\pm(S(0),0) = \int_0^t \frac{\partial F_\pm}{\partial \tau} d\tau + \\ &+ \sum_{0 \le \tau \le t,\, h_{g_\pm(\tau)} \neq 0} \Big[F_\pm(S(\tau-) + D\, h_{g_\pm(\tau)}, \tau) - F(S(\tau-),\tau) \Big] \end{aligned}$$

which leads to the following result:

Theorem 6.2.2 (Itô's Lemma for Jump Process)**.** *Let $S(t)$ be defined by (6.1) with $B = C = 0$ and $F_\pm \in C^{2,1}$. Under the assumption (6.2), then*

$$dF_\pm(S(t),t) = \frac{\partial F_\pm}{\partial t} dt + \Big[F_\pm(S(t-) + D h_{g_\pm(t)}, t) - F_\pm(S(t-),t) \Big] dJ_\pm(t) . \tag{6.10}$$

Let $F_\pm$ be $\mathcal{C}^{2,1}$ and let τ_i, $i = 1, 2, \cdots$ be the jump times of $S(t)$. Between the the two consecutive time jumps τ_i and τ_{i+1} the evolution of $S(t)$ is described by the TDD-process. As $dS(t) = d_{cont}S(t)$ on this interval, applying the Itô's formula (6.9) one gets

$$d_{cont}F_\pm(S(t),t) = \left[\frac{\partial F_\pm}{\partial t} + A\frac{\partial F_\pm}{\partial S} + \frac{1}{2}B^2\frac{\partial^2 F_\pm}{\partial S^2}\right]dt + B\frac{\partial F_\pm}{\partial S}dW(t) + C\frac{\partial F}{\partial S}dX_\pm(t) \ .$$

The discontinuous change follows from the transformation of the jump in $S(t)$ at time t given the previous time of the jump in the composite function $F_\pm(S(t),t)$,

$$d_{jump}F_\pm(S(t),t) = \Big[F_\pm(S(t-) + Dh_{g_\pm(t)}, t) - F_\pm(S(t-),t)\Big]dJ_\pm(t) \ .$$

Combining the continuous and discontinuous process,

$$\begin{aligned} dF_\pm(S(t),t) &= \left[\frac{\partial F_\pm}{\partial t} + A\frac{\partial F_\pm}{\partial S} + \frac{1}{2}B^2\frac{\partial^2 F_\pm}{\partial S^2}\right]dt + B\frac{\partial F_\pm}{\partial S}dW(t) + \\ &+ C\frac{\partial F_\pm}{\partial S}dX_\pm(t) + \Big[F_\pm(S(t-) + Dh_{g_\pm(t)}, t) - F_\pm(S(t-),t)\Big]dJ_\pm(t) \end{aligned}$$

Theorem 6.2.3 (Itô's Lemma for the JTDD Process)**.** *Let $S(t)$ be defined by (6.1) with $F \in C^{2,1}$. Under the assumptions (6.2), (6.3) and (6.4),*

$$\begin{aligned} dF_\pm(S(t),t) &= \left[\frac{\partial F_\pm}{\partial t} + A\frac{\partial F_\pm}{\partial S} + \frac{1}{2}B^2\frac{\partial^2 F_\pm}{\partial S^2}\right]dt + B\frac{\partial F_\pm}{\partial S}dW(t) + \\ &+ C\frac{\partial F_\pm}{\partial S}dX_\pm(t) + \qquad\qquad (6.11) \\ &+ \Big[F_\pm(S(t-) + Dh_{g_\pm(t)}, t) - F_\pm(S(t-),t)\Big]dJ_\pm(t) \ . \end{aligned}$$

We recall that martingales and measures are critical to the risk-neutral valuation. A martingale is a zero-drift stochastic process. Any variable following a martingale has the simplifying property that its expected value at any future time equals its value today.

We remark that if the dynamic of $S(t)$ is described by (6.1), then $F_{\pm}(S(t),t)$ is not a martingale. To construct related martingales one has to calculate the exact representation of expectations $E(X_{\pm}(t)|\mathfrak{F}_s)$ and $E(J_{\pm}(t)|\mathfrak{F}_s)$ (see Ratanov[90])

$$\begin{aligned} E\Big(X_{\pm}(t)|\mathfrak{F}_s\Big) &= X_{\pm}(s) + \frac{\nu_{+}\lambda_{-} + \nu_{-}\lambda_{+}}{\lambda_{-}+\lambda_{+}}[t-s] + \\ &+\lambda_{\pm}\left[\frac{\nu_{\pm}+\nu_{\mp}}{\lambda_{-}+\lambda_{+}}\right]\left[\frac{1-e^{-(\lambda_{-}+\lambda_{+})(s-t)}}{\lambda_{-}+\lambda_{+}}\right] \\ &\neq X_{\pm}(s)\ , \end{aligned}$$

$$\begin{aligned} E\Big(J_{\pm}(t)|\mathfrak{F}_s\Big) &= J_{\pm}(s) + \frac{\lambda_{-}\lambda_{+}}{\lambda_{-}+\lambda_{+}}[h_{-}+h_{-}][t-s] + \\ &+\lambda_{\pm}\left[\frac{\lambda_{\pm}h_{\pm}-\lambda_{\mp}h_{\mp}}{\lambda_{-}+\lambda_{+}}\right]\left[\frac{1-e^{-(\lambda_{-}+\lambda_{+})(s-t)}}{\lambda_{-}+\lambda_{+}}\right] \\ &\neq J_{\pm}(s)\ , \end{aligned}$$

for $t > s$. In particular, for $\lambda_{\pm} = \lambda$, $h_{\pm} = h$, $\nu_{\pm} = \nu$ and

$$E\Big(X_{\pm}(t)|\mathfrak{F}_s\Big) = X_{\pm}(s) + \nu[t-s]\ ,$$

$$E\Big(J_{\pm}(t)|\mathfrak{F}_s\Big) = J_{\pm}(s) + \lambda h[t-s]\ .$$

Therefore it is quite important to modify or to compensate the processes $X_{\pm}(t)$ and $J_{\pm}(t)$ in order to obtain martingales that lead to a martingale $F_{\pm}(S(t),t)$ If one defines

$$M1_{\pm}(t) = X_{\pm}(t) - \nu t \tag{6.12}$$

and

$$M2_{\pm}(t) = J_{\pm}(t) - \lambda h t\ , \tag{6.13}$$

then $E\Big(M1_{\pm}(t)|\mathfrak{F}\Big) = M1_{\pm}(s)$ and $E\Big(M2_{\pm}(t)|\mathfrak{F}\Big) = M2_{\pm}(s)$, that is $M1_{\pm}(t) + M2_{\pm}(t)$ is a martingale.

7 Option Pricing with Memory in the Underlying Asset

7.1 Introduction

The works of Black and Scholes[3] and Merton[4] were pioneers on the use of Itô's Lemma to establish the PDE's for option pricing. Since then, the literature has been quite fruitful on the use of such approach to study the evolution of option prices (see, for instance, Wilmott et al. [91], Neftci[92], Hanson[88] and Cont and Takov[87] and the references contained in these two last books).

The well-known Black-Scholes equation was established under the following assumptions (Black and Scholes[3]):

1. The short-term interest rate is known and constant through time.

2. The stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price. Thus the distribution of possible stock prices at the end of any finite interval is log-normal. The variance rate of the return on the stock is constant.

3. The stock pays no dividends or other distributions.

4. The option is "European", that is, it can only be exercised at maturity.

5. There are no transaction costs in buying or selling the stock or the option.

6. Its possible to borrow any fraction of the price of a security to buy or to hold it, at short time interest rate.

7. There are no penalties to short selling. A seller who does not own a security will simply accept the price of the security and will agree to settle with the buyer at some future date by paying him an amount equal to the price of the security on that day.

8. let $S(t)$ the price of the stock at time t.

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t)\ , \quad S(0) \text{ is given .}$$

This equation was then used separately for pricing European put and call options using the so called delta-hedge portfolio theory. The two problems can be stated as follows:

PROBLEM EP: Find $p \in C^{2,1}\Big(\mathbb{R}^+ \times (0,T]\Big)$ such that

$$\frac{\partial p}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 p}{\partial S^2} - rS\frac{\partial p}{\partial S} + rp = 0 \ \text{ in } \ \mathbb{R}^+ \times (0,T],$$

with the initial condition

$$p(S,0) = (E-S)^+ = Max\{E-S,0\},\ S \in \mathbb{R}^+,$$

and with the boundary conditions

$$p(0,T-\tau) = Ee^{-r\tau} \tag{7.1}$$

$$\lim_{S\to\infty} p(S,\tau) = 0,\ \tau \in (0,T]. \tag{7.2}$$

PROBLEM EC: Find $c \in C^{2,1}\Big(\mathbb{R}^+ \times (0,T]\Big)$ such that

$$\frac{\partial c}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} - rS\frac{\partial c}{\partial S} + rc = 0 \ \text{ in } \ \mathbb{R}^+ \times (0,T],$$

with the initial condition

$$c(S,0) = (S-E)^+ = Max\{S-E,0\},\ S \in \mathbb{R}^+$$

and with the boundary conditions

$$c(0, T-\tau) = 0, \ \tau \in (0, T] \tag{7.3}$$

$$\lim_{S\to\infty} c(S, T-\tau) - S = 0, \ \tau \in (0, T]. \tag{7.4}$$

Hereafter, the purpose is to establish an initial boundary value problem for $V = (V^+, V^-)$ where V^+ and V^- are European option prices when the asset price satisfies (5.1), that is (6.1) with $A(S(t), t) = S(t)\mu$, $B(S(t), t) = C(S(t), t) = S(t_-)\sigma$ and $D(S(t), t) = S(t_-)$ but with the compensated processes (6.12) and (6.13). The approach that is used here is similar to the one used to establish Black-Scholes equations: delta-hedge portfolio theory and the Itô's lemma.

Section 7.2 establishes the initial boundary value problem for V when V^+ and V^- are European put or call options prices, respectively. The mathematical problem for American options is presented in Section 7.3. The initial boundary value problem for European options is solved numerically using the so called Finite Element Method in Section 7.4. More precisely, this section introduces the variational problem for European options which is solved discretizing in space using finite element methods and in time with the implicit Euler's method. Section 7.5 applies the approach introduced in Section 7.4 to American options. Finally, Section 7.6 presents several numerical illustrations.

7.2 Mathematical Model for European Options

Let $V^+(S(t), t)$ be an European option price with initial state information on up prices (positive information) and let $V^-(S(t), t)$ be the European option price with initial state information on down prices (negative information). Let one assume that the underlying asset is described by (5.1), where $X\pm(t)$ and $J_\pm(t)$ are replaced by the compensated processes (6.12) and (6.13), respectively.

Considering that V^+ satisfies the assumptions of Theorem (6.2.3), one has

$$\begin{aligned} dV^+ &= \left[\frac{\partial V^+}{\partial t} + \frac{\partial V^+}{\partial S}\mu S(t) + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 V^+}{\partial S^2}\right] dt + \frac{\partial V^+}{\partial S}\sigma S(t_-)dW(t) + \\ &\quad + \frac{\partial V^+}{\partial S}\sigma S(t_-)dX_+ + \Big[V^+(S + Sh_{g+(t)}, t) - V^+(S,t)\Big] dJ_+ \ . \end{aligned}$$

A jump occurs if $g_+(t) = -1$. Then

$$\begin{aligned} dV^+ &= \left[\frac{\partial V^+}{\partial t} + \frac{\partial V^+}{\partial S}\mu S(t) + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 V^+}{\partial S^2}\right] dt + \frac{\partial V^+}{\partial S}\sigma S(t_-)dW(t) + \\ &\quad + \frac{\partial V^+}{\partial S}\sigma S(t_-)dX_+ + \Big[V^+(S + Sh_-, t) - V^+(S,t)\Big] dJ_+ \ . \end{aligned}$$

To obtain the system of PDE's shown in Ratanov[12] for the particular case of European options when the underlying asset is modeled as a JT process, one needs to impose that

$$V^+(S + Sh_-, t) = V^-(S + Sh_+, t) \ .$$

Because there is the assumption of the compensated processes (6.12) and (6.13), then

$$dX_+(t) = dM1_+(t) + \nu dt \quad \text{and} \quad dJ_+(t) = dM2_+(t) + \lambda h dt \ ,$$

that leads to

$$\begin{aligned} dV^+ &= \left[\frac{\partial V^+}{\partial t} + \frac{\partial V^+}{\partial S}\Big[\mu + \nu\sigma\Big] S(t) + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 V^+}{\partial S^2} + \right. \\ &\quad \left. + \lambda h\Big[V^-(S + Sh_+, t) - V^+(S,t)\Big]\right] dt + \frac{\partial V^+}{\partial S}\sigma S(t_-)dW(t) + \\ &\quad + \frac{\partial V^+}{\partial S}\sigma S(t_-)dM1_+(t) + \\ &\quad + \Big[V^-(S + Sh_+, t) - V^+(S,t)\Big] dM2_+(t) \ . \end{aligned} \tag{7.5}$$

Let $\Pi^+ = V_1^+ - \Delta_2^+ V_2^+ - \Delta_1^+ S$ be the value of the hedge portfolio. Where V_1^+ is the European option price with initial state information of an up price and maturity date "T_1", V_2^+ is European option price with initial state information of the up price and maturity date "T_2". The formulation of the portfolio and the mathematical development presented here are based on the reasoning presented by Wilmott[91] for interest rate options.

Considering the increase in the value of the portfolio in a space-time, *keeping fixed* Δ_1^+ *and* Δ_2^+ rates, this implies that

$$d\Pi^+ = dV_1^+ - \Delta_2^+ dV_2^+ - \Delta_1^+ dS \, . \tag{7.6}$$

Using (7.5) in (7.6), we deduce

$$\begin{aligned} d\Pi^+ \; = \; & \Bigg[\frac{\partial V_1^+}{\partial t} + \frac{\partial V_1^+}{\partial S}\Big[\mu + \nu\sigma\Big] S(t) + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 V_1^+}{\partial S^2} + \\ & +\lambda h\Big[V_1^-(S + Sh_+, t) - V_1^+(S,t)\Big]\Bigg]dt + \frac{\partial V_1^+}{\partial S}\sigma S(t_-)dW(t) + \\ & + \frac{\partial V_1^+}{\partial S}\sigma S(t_-)dM1_+(t) + \Big[V_1^-(S + Sh_+, t) - V_1^+(S,t)\Big]dM2_+(t) - \\ & -\Delta_2^+ \Bigg\{\Bigg[\frac{\partial V_2^+}{\partial t} + \frac{\partial V_2^+}{\partial S}\Big[\mu + \nu\sigma\Big] S(t) + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 V_2^+}{\partial S^2} + \\ & +\lambda h\Big[V_2^-(S + Sh_+, t) - V_2^+(S,t)\Big]\Bigg]dt + \frac{\partial V_2^+}{\partial S}\sigma S(t_-)dW(t) + \\ & + \frac{\partial V_2^+}{\partial S}\sigma S(t_-)dM1_+(t) + \Big[V_2^-(S + Sh_+, t) - V_2^+(S,t)\Big]dM2_+(t) \Bigg\} - \\ & -\Delta_1^+ dS \, . \end{aligned}$$

Taking into account (5.1) and (6.12) and (6.13), then

$$\begin{aligned} d\Pi^+ \; = \; & \left[\frac{\partial V_1^+}{\partial t} - \Delta_2^+\frac{\partial V_2^+}{\partial t} + \left[\frac{\partial V_1^+}{\partial S} - \Delta_2^+\frac{\partial V_2^+}{\partial S} - \Delta_1^+\right]\mu S(t) + \right. \\ & + \left[\frac{\partial V_1^+}{\partial S} - \Delta_2^+\frac{\partial V_2^+}{\partial S} - \Delta_1^+\right]\nu\sigma S(t) + \frac{1}{2}\sigma^2 S^2(t)\left[\frac{\partial^2 V_1^+}{\partial S^2} - \Delta_2^+\frac{\partial^2 V_2^+}{\partial S^2}\right] + \\ & +\lambda h\Big\{\Big[V_1^-(S + h_+S, t) - V_1^+(S,t)\Big] - \Delta_2\Big[V_2^-(S + h_+S, t) - V_2^+(S,t)\Big] - \\ & \left. - \Delta_1 S(t)\Big\}\right]dt + \sigma S(t)\left[\frac{\partial V_1^+}{\partial S} - \Delta_2^+\frac{\partial V_2^+}{\partial S} - \Delta_1^+\right]d\Big\{W(t) + M1_+(t)\Big\} + \\ & +\Big\{\Big[V_1^-(S + h_+S, t) - V_1^+(S,t)\Big] - \Delta_2\Big[V_2^-(S + h_+S, t) - V_2^+(S,t)\Big] - \\ & - \Delta_1 S(t)\Big\}dM2_+(t) \, . \end{aligned}$$

Fixing now Δ_1^+ and Δ_2^+ by

$$\Delta_1^+ = \frac{\partial V_1^+}{\partial S} - \frac{S\frac{\partial V_1^+}{\partial S} - \Big[V_1^-(S+h_+S,t) - V_1^+(S,t)\Big]}{S\frac{\partial V_2^+}{\partial S} - \Big[V_2^-(S+h_+S,t) - V_1^+(S,t)\Big]}\frac{\partial V_2^+}{\partial S}$$

$$\Delta_2^+ = \frac{S\frac{\partial V_1^+}{\partial S} - V_1^-(S+h_+S,t) - V_1^+(S,t)\Big]}{S\frac{\partial V_2^+}{\partial S} - \Big[V_2^-(S+h_+S,t) - V_1^+(S,t)\Big]},$$

One obtains for $d\Pi^+$ the representation

$$\begin{aligned} d\Pi^+ &= \Bigg[\frac{\partial V_1^+}{\partial t} - \frac{S\frac{\partial V_1^+}{\partial S} - \Big[V_1^-(S+h_+S,t) - V_1^+(S,t)\Big]}{S\frac{\partial V_2^+}{\partial S} - \Big[V_2^-(S+h_+S,t) - V_1^+(S,t)\Big]}\frac{\partial V_2^+}{\partial t} + \\ &+ \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 V_1^+}{\partial S^2} - \\ &- \frac{1}{2}\sigma^2 S^2(t)\left[\frac{S\frac{\partial V_1^+}{\partial S} - \Big[V_1^-(S+h_+S,t) - V_1^+(S,t)\Big]}{S\frac{\partial V_2^+}{\partial S} - \Big[V_2^-(S+h_+S,t) - V_1^+(S,t)\Big]}\right]\frac{\partial^2 V_2^+}{\partial S^2}\Bigg] dt. \end{aligned} \tag{7.7}$$

Considering that the return of the portfolio Π^+ is constant, then, in equilibrium, it must be equal to the riskless interest rate r_+, which implies that

$$\frac{d\Pi^+}{dt} = r_+\Pi^+$$

and according to (7.7) that

$$\begin{aligned} rV_1^+ &- r\left[\frac{S\frac{\partial V_1^+}{\partial S} - \Big[V_1^-(S+h_+S,t) - V_1^+(S,t)\Big]}{S\frac{\partial V_2^+}{\partial S} - \Big[V_2^-(S+h_+S,t) - V_1^+(S,t)\Big]}\right] V_2^+ - r\frac{\partial V_1^+}{\partial S}S(t) - \\ &- r\left[\frac{S\frac{\partial V_1^+}{\partial S} - \Big[V_1^-(S+h_+S,t) - V_1^+(S,t)\Big]}{S\frac{\partial V_2^+}{\partial S} - \Big[V_2^-(S+h_+S,t) - V_1^+(S,t)\Big]}\right]\frac{\partial V_2^+}{\partial S}S = \frac{\partial V_1^+}{\partial t} - \\ &- \left[\frac{S\frac{\partial V_1^+}{\partial S} - \Big[V_1^-(S+h_+S,t) - V_1^+(S,t)\Big]}{S\frac{\partial V_2^+}{\partial S} - \Big[V_2^-(S+h_+S,t) - V_1^+(S,t)\Big]}\right]\frac{\partial V_2^+}{\partial t} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 V_1^+}{\partial S^2} - \\ &- \frac{1}{2}\sigma^2 S^2(t)\left[\frac{S\frac{\partial V_1^+}{\partial S} - \Big[V_1^-(S+h_+S,t) - V_1^+(S,t)\Big]}{S\frac{\partial V_2^+}{\partial S} - \Big[V_2^-(S+h_+S,t) - V_1^+(S,t)\Big]}\right]\frac{\partial^2 V_2^+}{\partial S^2}. \end{aligned} \tag{7.8}$$

Finally from (7.8) one obtains

$$\frac{\frac{\partial V_1^+}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1^+}{\partial S^2} - rV_1^+ + rS\frac{\partial V_1^+}{\partial S}}{S\frac{\partial V_1^+}{\partial S} - \Big[V_1^-(S+h_+S,t) - V_1^+(S,t)\Big]} = \frac{\frac{\partial V_2^+}{\partial t} + \frac{1}{2}S^2\sigma^2 \frac{\partial^2 V_2^+}{\partial S^2} - rV_2^+ + r\frac{\partial V_2^+}{\partial S}}{S\frac{\partial V_2^+}{\partial S} - \Big[V_2^-(S+h_+S,t) - V_2^+(S,t)\Big]} \tag{7.9}$$

The left-hand side of (7.9) depends on V_1^+ while the right-hand side depends on V_2^+. The only way to achieve (7.9) is that both sides of the equation are independent of the maturity date. Thus, dropping the subscript from "V", one gets

$$\frac{\frac{\partial V^+}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V^+}{\partial S^2} - rV_1^+ + r\frac{\partial V^+}{\partial S}S}{S\frac{\partial V^+}{\partial S} - \Big[V^-(S+h_+S,t) - V^+(S,t)\Big]} = A(S,t)\ ,$$

for some function $A(S,t)$. At this point, there is the need for an additional assumption. Therefore, consider that

$$A(S,t) = \gamma(S,t) + \lambda h.$$

where $\gamma(S,t)$ is the market price of risk, i.e. a measure of the reward-to-risk ratio of the market portfolio. Taking into account the expression for A one finally gets

$$\begin{aligned}\frac{\partial V^+}{\partial t} &+ \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V^+}{\partial S^2} - rV^+ + r\frac{\partial V^+}{\partial S}S(t) = \\ &= \left[S\frac{\partial V^+}{\partial S} - \Big[V^-(S+h_+S,t) - V^+(S,t)\Big]\right]\Big[\gamma(S,t) + \lambda h\Big]\end{aligned}$$

or equivalently

$$\begin{aligned}\frac{\partial V^+}{\partial t} &+ \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V^+}{\partial S^2} + \Big[r - \gamma(S,t) - \lambda h\Big]S(t)\frac{\partial V^+}{\partial S} = \\ &= \Big[r + \gamma(S,t) + \lambda h\Big]V^+ - \Big[\gamma(S,t) + \lambda h\Big]V^-(S+h_+S,t)\end{aligned} \tag{7.10}$$

which is the **equation for positive past information** that defines the differential system for European options.

Similarly to the positive past information case, the **equation for negative past information** can be obtained as

$$\begin{aligned}\frac{\partial V^-}{\partial t} &+ \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V^-}{\partial S^2} + \Big[r - \gamma(S,t) - \lambda h\Big]S(t)\frac{\partial V^-}{\partial S} = \\ &= \Big[r + \gamma(S,t) + \lambda h\Big]V^- - \Big[\gamma(S,t) + \lambda h\Big]V^+(S+h_-S,t)\end{aligned} \tag{7.11}$$

For the European put options the following boundary value problem with a final condition can be stated:

PROBLEM EPH: Find $p = p^{\pm} \in C^{2,1}\Big(\mathbb{R}^+ \times [0,T)\Big)$ depending on past information such that, in $\mathbb{R}^+ \times [0,T)$,

$$\begin{cases} \dfrac{\partial p^+}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 p^+}{\partial S^2} + [r - \gamma(S,t) - \lambda h]\dfrac{\partial p^+}{\partial S} - [r + \gamma(S,t) + \lambda h]p^+ \\ \qquad = -[\gamma(S,t) + \lambda h]p^-(S + h_+ S, t) \\ \\ \dfrac{\partial p^-}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 p^-}{\partial S^2} + [r - \gamma(S,t) - \lambda h]S\dfrac{\partial p^-}{\partial S} - [r + \gamma(S,t) + \lambda h]p^- \\ \qquad = -[\gamma(S,t) + \lambda h]p^+(S + h_- S, t) \end{cases} \tag{7.12}$$

with payoff conditions

$$\begin{aligned} p^+(S,T) &= Max\{E - S, 0\}\ ,\ S \in \mathbb{R}^+, \\ p^-(S,T) &= Max\{E - S, 0\}\ ,\ S \in \mathbb{R}^+, \end{aligned} \tag{7.13}$$

and boundary conditions

$$\begin{aligned} p^+(0,t) &= Ee^{-r_+[T-t]} \quad , \quad \lim_{S\to+\infty} p^+(S,t) = 0,\ t \in (0,T), \\ p^-(0,t) &= Ee^{-r_-[T-t]} \ \text{ and } \lim_{S\to+\infty} p^-(S,t) = 0,\ t \in (0,T). \end{aligned} \tag{7.14}$$

Similarly, for European call options the pricing problem may be stated using the following boundary value problem with a final condition:

PROBLEM ECH: Find $c = c^{\pm} \in C^{2,1}\Big(\mathbb{R}^+ \times [0,T)\Big)$ depending on past information such that, in $\mathbb{R}^+ \times [0,T)$,

$$\begin{cases} \dfrac{\partial c^+}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 c^+}{\partial S^2} + [r - \gamma(S,t) - \lambda h]S\dfrac{\partial c^+}{\partial S} - [r + \gamma(S,t) + \lambda h]c^+ \\ \qquad = -[\gamma(S,t) + \lambda h]c^-(S + h_+ S, t) \\ \\ \dfrac{\partial c^-}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 c^-}{\partial S^2} + [r - \gamma(S,t) - \lambda h]S\dfrac{\partial c^-}{\partial S} - [r + \gamma(S,t) + \lambda h]c^- \\ \qquad = -[\gamma(S,t) + \lambda h]c^+(S + h_- S, t) \end{cases} \tag{7.15}$$

with payoff conditions

$$\begin{aligned} c^+(S,T) &= Max\{S - E, 0\}\ ,\ S \in \mathbb{R}^+, \\ c^-(S,T) &= Max\{S - E, 0\}\ ,\ S \in \mathbb{R}^+ \end{aligned} \tag{7.16}$$

and boundary conditions

$$\begin{aligned} &c^+(0,t) = 0 \quad , \quad \lim_{S\to+\infty} c^+(S,t) \approx S \ , \ t \in (0,T) \\ &c^-(0,t) = 0 \quad \text{and} \quad \lim_{S\to+\infty} c^-(S,t) \approx S \ , \ t \in (0,T) \ . \end{aligned} \tag{7.17}$$

In order to solve numerically the two problems stated above one needs to replace the domain $\mathbb{R}_0^+ \times [0,T]$ by a bounded domain. If one assume S_{max} large enough, then the first domain may be replaced by $[0, S_{max}] \times [0,T]$ in the differential problems with the convenient modifications, and the boundary conditions (7.14) and (7.17) are replaced by the following

$$\begin{aligned} &p^+(0,t) = Ee^{-r_+[T-t]}, \quad p^+(S_{max},t) = 0 \ , \ t \in (0,T), \\ &p^-(0,t) = Ee^{-r_-[T-t]}, \quad p^-(S_{max},t) = 0 \ , \ t \in (0,T), \end{aligned} \tag{7.18}$$

and

$$\begin{aligned} &c^+(0,t) = 0 \quad , \quad c^+(S_{max},t) = S_{max} - Ee^{-r_+[T-t]} \ , \ t \in (0,T), \\ &c^-(0,t) = 0 \quad \text{and} \quad c^-(S_{max},t) = S_{max} - Ee^{-r_-[T-t]} \ , \ t \in (0,T), \end{aligned} \tag{7.19}$$

respectively.

7.3 Mathematical Model for American Options

This section deals with the mathematical formulation of pricing American options as a free boundary problem. Here, the same idea presented by Wilmott[91] to the standard Black and Scholes Framework is used.

First let one tackles the American put problem. If at any time $t^* < T$ the price of the underlying asset is $S^* < E(1 - e^{-r(T-t^*)})$, the put option must be exercised immediately, because the income L generated by the premature exercise satisfies

$$L = (E - S^*) > E(e^{-r(T-t^*)}) > p_*^+(S^*, t),$$

or

$$L = (E - S^*) > E(e^{-r(T-t^*)}) > p_*^-(S^*, t),$$

where $p^{+}(S^{*},t)$ is the price of the European put option with "positive past information" and with the exercise price E and maturity in $T-t^{*}$ years and $p^{-}(S^{*},t)$ is the price of the European put option with "negative past information" and with the exercise price E and maturity in $T-t^{*}$ years. Because the return is always $(E-S)^{+}$ in $t=T$ and this has the same value that the European option, so there is no portfolio that represents a better alternative that the premature exercise. Notice that in this case the price $P^{+}(S^{*},t^{*})$ or $P^{-}(S^{*},t^{*})$ should have the value $E-S^{*}$ in a no-arbitrage setting.

In particular, one has for $S^{*}=0$, $P^{+}(0,t^{*})=E$ or $P^{-}(0,t^{*})=E$ using the No-arbitrage argument, and then

$$P^{+}(S,t)\geq (E-S)^{+}=0,\quad \forall\, S\geq E,$$

or

$$P^{-}(S,t)\geq (E-S)^{+}=0,\quad \forall\, S\geq E.$$

It appears that for each instant of time t, the prices that give the premature exercise have interval $[0,S_f]$, whose upper limit is called the **optimal point of exercise**.

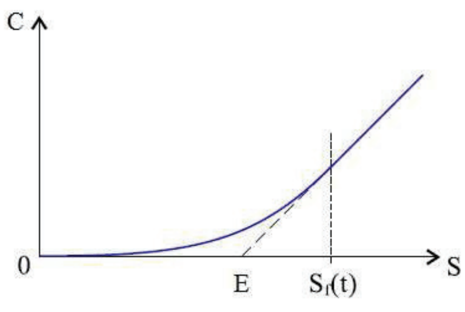

Figure 7.1: Call Option

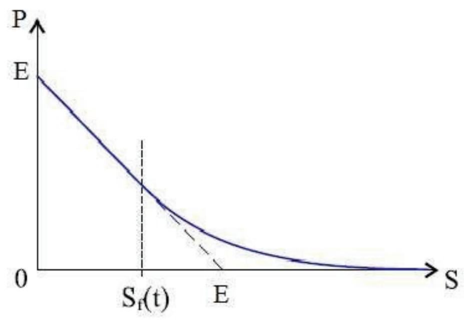

Figure 7.2: Put Option

The point S_f, divides the domain in a segment where the option is exercised, and in another where the immediate exercise is not optimal. Thus, the pricing problem of an American put option may be seen as a **free boundary problem**, where the free boundary is given by $S_f = S_f(t)$.

The **free boundary problem** can be formulate in two well defined regions for "positive past information", one where the option is exercised,

$$0 \leq S \leq S_f.$$

where we must exercise the option

$$p^+(S,t) = Max\{E - S, 0\} \ ,$$

$$\begin{aligned}\frac{\partial p^+}{\partial t} &+ \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 p^+}{\partial S^2} + [r - \gamma(S,t) - \lambda h] S \frac{\partial p^+}{\partial S} - \\ &- [r + \gamma(S,t) + \lambda h] P^+ \leq -[\gamma(S,t) + \lambda h] P^-(S + h_+ S, t) \ ;\end{aligned}$$

and another

$$S_f \leq S \leq \infty.$$

where there is no premature exercise of the option

$$P^+(S,t) > Max\{E - S, 0\} \ ,$$

$$\begin{aligned}\frac{\partial P^+}{\partial t} &+ \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P^+}{\partial S^2} + [r - \gamma(S,t) - \lambda h] S \frac{\partial P^+}{\partial S} - \\ &- [r + \gamma(S,t) + \lambda h] P^+(S,t) = -[\gamma(S,t) + \lambda h] P^-(S + h_+ S, t) \ .\end{aligned}$$

The conditions of the interface between the two regions of the domain in points S_f, the discussed in more detail from the point by Willmott[91], are given by

$$P^+(S_f,t) = Max\{E - S_f, 0\} \quad \text{and} \quad \frac{\partial P^+}{\partial S}(S_f,t) = -1 \ .$$

Besides the final time condition, i.e. the payoff function

$$P^+(S,T) = Max\{E - S, 0\} \ , \ s \in \mathbb{R}^+$$

and its behavior at the boundary point $S = 0$ and when $S \to \infty$ is given by

$$P^+(0,t) = Ee^{-r_+(T-t)} \quad \text{and} \quad \lim_{S\to\infty} P^+(S,t) = 0,\ t \in [0,T) \ .$$

Analogous problem can be established for put European options P^- with "negative past information".

The problem of pricing of the American put option is stated as

PROBLEM APH: Find $P^{\pm} \in C^{2,1}\Big(\mathbb{R}^+ \times [0,T)\Big)$ depending on past information, in $\mathbb{R}^+ \times [0,T)$, such that, for regions where early exercise is optimal, $0 \le S \le S_f$ and

$$P^+(S,t) = Max\{E-S,0\}\ , \quad P^-(S,t) = Max\{E-S,0\}$$

$$\begin{cases} \dfrac{\partial P^+}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 P^+}{\partial S^2} + [r-\gamma-\lambda h]S\dfrac{\partial P^+}{\partial S} - [r+\gamma+\lambda h]P^+ \\ \qquad \le -[\gamma+\lambda h]P^-(S+h_+S,t)\ , \\ \\ \dfrac{\partial P^-}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 P^-}{\partial S^2} + [r-\gamma-\lambda h]S\dfrac{\partial P^-}{\partial S} - [r+\gamma+\lambda h]P^- \\ \qquad \le -[\gamma+\lambda h]P^+(S+h_-S,t)\ , \end{cases} \tag{7.20}$$

For regions where early exercise is not optimal, $S_f \le S \le \infty$ and

$$P^+(S,t) > Max\{E-S,0\}\ , \quad P^-(S,t) > Max\{E-S,0\}$$

$$\begin{cases} \dfrac{\partial P^+}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 P^+}{\partial S^2} + [r-\gamma-\lambda h]S\dfrac{\partial P^+}{\partial S} - [r+\gamma+\lambda h]P^+ \\ \qquad = -[\gamma+\lambda h]P^-(S+h_+S,t)\ , \\ \\ \dfrac{\partial P^-}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 P^-}{\partial S^2} + [r-\gamma-\lambda h]S\dfrac{\partial P^-}{\partial S} - [r+\gamma+\lambda h]P^- \\ \qquad = -[\gamma+\lambda h]P^+(S+h_-S,t)\ , \end{cases} \tag{7.21}$$

The conditions of the interface between the two regions are

$$\begin{aligned} &P^{\pm}(S_f,t) = Max\{E-S_f,0\}\ , \\ &\frac{\partial P^{\pm}}{\partial S}(S_f,t) = -1, \end{aligned} \tag{7.22}$$

$P^{\pm}$ satisfies at $t = T$ the payoff condition,

$$P^{\pm}(S,T) = Max\{E - S, 0\} \ , \quad S \in \mathbb{R}^{+} \ , \tag{7.23}$$

and boundary conditions,

$$\begin{aligned} &P^{\pm}(0,t) = Ee^{-r_{+}(T-t)} \ , \\ &\lim_{S\to\infty} P^{\pm}(S,t) = 0 \ , \ t \in [0,T) \ . \end{aligned} \tag{7.24}$$

Similarly, the pricing problem of an American call option is stated as

PROBLEM ACH: Find $C^{\pm} \in C^{2,1}\Big(\mathbb{R}^{+} \times [0,T)\Big)$ depending on past information, in $\mathbb{R}^{+} \times [0,T)$, such that, for regions where early exercise is optimal, $0 \leq S \leq S_f$ and

$$C^{+}(S,t) > Max\{E - S, 0\} \ , \quad C^{-}(S,t) > Max\{E - S, 0\}$$

$$\begin{cases} \dfrac{\partial C^{+}}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 C^{+}}{\partial S^2} + [r - \gamma - \lambda h] S \dfrac{\partial C^{+}}{\partial S} - [r + \gamma + \lambda h] C^{+} \\ \qquad -[\gamma + \lambda h] C^{-}(S + h_{+}S, t) \ , \\ \\ \dfrac{\partial C^{-}}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 C^{-}}{\partial S^2} + [r - \gamma - \lambda h] S \dfrac{\partial C^{-}}{\partial S} - [r + \gamma + \lambda h] C^{-} \\ \qquad = -[\gamma + \lambda h] C^{+}(S + h_{-}S, t) \ , \end{cases} \tag{7.25}$$

For regions where early exercise is not optimal, $S_f \leq S \leq \infty$ and

$$C^{+}(S,t) = Max\{E - S, 0\} \ , \quad C^{-}(S,t) = Max\{E - S, 0\}$$

$$\begin{cases} \dfrac{\partial C^{+}}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 C^{+}}{\partial S^2} + [r - \gamma - \lambda h] S \dfrac{\partial C^{+}}{\partial S} - [r + \gamma + \lambda h] C^{+} \\ \qquad \leq -[\gamma + \lambda h] C^{-}(S + h_{+}S, t) \ , \\ \\ \dfrac{\partial C^{-}}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 C^{-}}{\partial S^2} + [r - \gamma - \lambda h] S \dfrac{\partial C^{-}}{\partial S} - [r + \gamma + \lambda h] C^{-} \\ \qquad \leq -[\gamma + \lambda h] C^{+}(S + h_{-}S, t) \ , \end{cases} \tag{7.26}$$

Hold interface conditions between the two regions, two boundary conditions at the free boundary,

$$\begin{aligned} &C^{\pm}(S_f, t) = (S_f - E)^{+} \ , \\ &\frac{\partial C^{\pm}}{\partial S}(S_f, t) = 1 \ , \end{aligned} \tag{7.27}$$

$C^{\pm}$ satisfies at $t = T$ the payoff conditions,

$$C^{\pm}(S,T) = Max\{S - E, 0\}\ ,\ S \in \mathbb{R}^{+}\ , \tag{7.28}$$

and boundary conditions,

$$\begin{aligned} &C^{\pm}(0,t) = 0\ ,\\ &\lim_{S\to\infty} C^{\pm}(S,t) - S = -Ee^{-r_{+}(T-t)}\ , t \in [0,T) \end{aligned} \tag{7.29}$$

7.4 Galerkin Method for European Option

This section proposes de use of finite element approximations in space domain (assets) and of finite differences in time domain for solving the option pricing problems.

Subsection 7.4.1 presents the Variational Formulation for European options with memory in the underlying assets. Subsections 7.4.2 and 7.4.3 shows the approximation by Finite Elements and the approximation in time, respectively, for European options with memory in the underlying assets.

7.4.1 Variational Formulation for European Option Pricing

In order to solve numerically the two problems presented in this section (problems ***EPH*** and ***ECH***) one needs to replace the domain $\mathbb{R}_0^+ \times [0,T]$ by a bounded domain. Let S_{max} be the highest price that the underlying asset may achieve in the spot market. An usual approach is to replace $\mathbb{R}_0^+ \times [0,T]$ by $[0, S_{max}] \times [0,T]$ in the differential problems with the convenient modifications, and then boundary conditions (7.14) and (7.17) are replaced by the following

$$\begin{aligned} &p^{\pm}(0,t) = Ee^{-r_{+}[T-t]},\\ &p^{\pm}(S_{max},t) = 0\ ,\ t \in (0,T), \end{aligned} \tag{7.30}$$

and

$$\begin{aligned} c^{\pm}(0,t) &= 0 \\ c^{\pm}(S_{max},t) &= S_{max} - Ee^{-r_{+}[T-t]} \ , \ t \in [0,T), \end{aligned} \tag{7.31}$$

respectively.

A natural question is the magnitude of S_{max}. According to Oliveira[38] and Thomaz[42], S_{max} should be fixed sufficiently large but one needs to take into account the accuracy of the numerical method used as well as the computational cost. It has been observed that values S_{max} between $1,5E$ and $3E$ provide good results.

An open question is the dependence of S_{max} on the parameters of the model as well as on the maximum error allowed.

Making the time variable change $\tau = T - t$, problem ***EPH*** is converted in the following initial boundary value problem:

PROBLEM IEPH: Finding $P^{\pm} \in C^{2,1}\Big((0,S_{max}) \times (0,T]\Big)$ depending on past information, in $(0,S_{max}) \times (0,T]$, such that

$$\begin{cases} \dfrac{\partial p^+}{\partial \tau} - \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 p^+}{\partial S^2} + [r-\gamma-\lambda h] S \dfrac{\partial p^+}{\partial S} - [r+\gamma+\lambda h] p^+ \\ \qquad = -[\gamma+\lambda h] p^-(S+h_+S,\tau) \\ \\ \dfrac{\partial p^-}{\partial \tau} - \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 p^-}{\partial S^2} + [r-\gamma-\lambda h] S \dfrac{\partial p^-}{\partial S} - [r+\gamma+\lambda h] p^- \\ \qquad = -[\gamma+\lambda h] p^+(S+h_-S,\tau) \end{cases} \tag{7.32}$$

with initial conditions,

$$p^{\pm}(S,0) = Max\{E-S,0\} \ , \ S \in (0,S_{max}) \ , \tag{7.33}$$

and boundary conditions,

$$\begin{aligned} p^{\pm}(0,\tau) &= Ee^{-r_{\pm}\tau} \ , \\ p^{\pm}(S_{max},\tau) &= 0, \tau \in (0,T) \ . \end{aligned} \tag{7.34}$$

The variable τ represents the time remaining to the maturity of the option, although the system presented here does not require the condition $\tau \leq T$, the model assumes that it is calculating the option price at this time limit. An initial boundary value problem that replaces problem ***IECH*** can be established considering in this case the boundary conditions

$$\begin{aligned} &c^{\pm}(0,\tau) = 0 \\ &c^{\pm}(S_{max},\tau) = S_{max} - Ee^{-r_{\pm}\tau}\ ,\ \tau \in (0,T]\ . \end{aligned} \tag{7.35}$$

In order to simplify, we assume that the market price of risk is zero, $\gamma = 0$, this means that the market players are all risk neutral.

To solve numerically the initial boundary value problem ***IEPH*** or its correspondent for call option problem, several approaches can be considered. The two main approaches are the Galerkin methods and the finite difference methods. As the last class of methods enable us to compute an approximation for the solution of the problem in its strong formulation, that is, in this case, in $C^{2,1}\Big([0,S_{max}]\ \times\ [0,T]\Big)$, we use in what follows a Galekin method that enable us to compute an approximation for the solution of in its weak formulation that we present now. We start to write the system (7.32) in vector equivalent form

$$\frac{\partial \mathbf{p}}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \mathbf{I}\,\frac{\partial^2 \mathbf{p}}{\partial S^2} - (r-\lambda h)S\mathbf{I}\frac{\partial \mathbf{p}}{\partial S} + (r+\lambda h)\mathbf{I}\mathbf{p} - \mathbf{F}(\mathbf{p}) = 0\ , \tag{7.36}$$

where $\mathbf{p} = \Big(p^+, p^-\Big)$, $\dfrac{\partial^i \mathbf{p}}{\partial S^i}$ denotes the vector whose components are $\dfrac{\partial^i p^{\pm}}{\partial S^i}$, $i\ =\ 1,2$, and $\mathbf{F}(\mathbf{p}) = \Big(F_+(p^-), F_-(p^+)\Big)$ where

$$F_+(S,\tau) = \begin{cases} \lambda h p^-([1+h_+]S,\tau),\ \text{for } S \in \left[0, \dfrac{1}{(1+h_+)S_{max}}\right] \\ 0 \text{ for } S \in \left[\dfrac{1}{(1+h_+)S_{max}}, S_{max}\right] \end{cases}$$

and

$$F_-(S,\tau) = \begin{cases} \lambda h p^+([1+h_-]S,\tau),\ \text{for } S \in \left[0, \dfrac{1}{(1+h_-)S_{max}}\right] \\ 0 \text{ for } S \in \left[\dfrac{1}{(1+h_-)S_{max}}, S_{max}\right] \end{cases} .$$

As S_{max} is fixed to be large, the replacement of $\Big(\lambda h p^-([1+h_+]S,\tau), \lambda h p^+([1+h_-]S,\tau)\Big)$ by $\mathbf{F}$ do not disturbs $\mathbf{p}$ in the region Ω of interest.

We use the notations: $\Omega = [0, S_{max}]$, $\partial\Omega = \{0, S_{max}\}$, $\mathbf{L}^2(\Omega)$, $\mathbf{H}^1(\Omega)$, and $\mathbf{H}_0^1(\Omega)$ denote the following functional spaces $\mathbf{L}^2(\Omega) = L^2(\Omega) \times L^2(\Omega)$, $\mathbf{H}^1(\Omega) = H^1(\Omega) \times H^1(\Omega)$ and $\mathbf{H}_0^1(\Omega) = H_0^1(\Omega) \times H_0^1(\Omega)$, respectively, where $L^2(\Omega)$, $H^1(\Omega)$ and $H_0^1(\Omega)$ denote the usual Sobolev spaces. In $H^1(\Omega)$ we consider the usual norm $\|.\|_{H^1(\Omega)}$. Let $L^2(0,T;\mathbf{H}^1(\Omega))$ be the space of functions $\mathbf{u} : [0,T] \rightarrow \mathbf{H}^1(\Omega))$ such that

$$\int_0^T \|\mathbf{u}(\tau)\|^2_{\mathbf{H}^1(\Omega)} \, d\tau < \infty \ ,$$

where, for $\mathbf{v} \in \mathbf{H}^1(\Omega)$, $\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} = \left(\|v_1\|^2_{H^1(\Omega)} + \|v_1\|^2_{H^1(\Omega)} \right)^{1/2}$. We consider the following subspace of $L^2(0,T;\mathbf{H}^1(\Omega))$

$$\mathbf{U} = \Big\{ \mathbf{v} \in L^2(0,T;\mathbf{H}^1(\Omega)) \ ; \ \mathbf{v}(0,\tau) = Ee^{-r\tau} \ , \ \mathbf{v}(S_{max},\tau) = 0 \Big\},$$

In $\mathbf{L}^2(\Omega)$ we consider the usual inner product

$$\Big(\mathbf{u}, \mathbf{v}\Big) = \int_\Omega \mathbf{u}(x)^t \mathbf{v}(x) \, dx \ , \tag{7.37}$$

where $\mathbf{u}(x)^t$ represents the transpose of $\mathbf{u}(x)$.

To define the so called variational problem associated with equation (7.36) complemented with the introduced boundary and initial conditions, we multiply, with respect the inner product (7.37), by a test function $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$. As

$$\frac{1}{2} S^2 \frac{\partial^2 \mathbf{p}}{\partial S^2} = \frac{1}{2} \frac{\partial}{\partial S} \left(S^2 \frac{\partial \mathbf{p}}{\partial S} \right) - S \frac{\partial \mathbf{p}}{\partial S} \ ,$$

that is

$$\begin{aligned} \left(\frac{\partial \mathbf{p}}{\partial \tau}, \mathbf{v} \right) - \left(\frac{1}{2} \sigma^2 \mathbf{I} \, \frac{\partial}{\partial S} \left(S^2 \frac{\partial \mathbf{p}}{\partial S} \right), \mathbf{v} \right) &- \left((r - \lambda h - \sigma^2) S \mathbf{I} \frac{\partial \mathbf{p}}{\partial S}, \mathbf{v} \right) \\ &+ \Big((r + \lambda h) \mathbf{I} \ \mathbf{p}, \mathbf{v} \Big) = \Big(\mathbf{F}, \mathbf{v} \Big) \ , \end{aligned} \tag{7.38}$$

using integration by parts in (7.38) for $\mathbf{v} = (v_1, v_2) \in \mathbf{H}_0^1(\Omega)$, one obtains

$$\left(\frac{\partial \mathbf{p}}{\partial \tau}, \mathbf{v} \right) + \left(\frac{1}{2} \sigma^2 \mathbf{I} \ S \frac{\partial \mathbf{p}}{\partial S}, S \frac{\partial \mathbf{v}}{\partial S} \right) - \left((r - \lambda h - \sigma^2) S \mathbf{I} \frac{\partial \mathbf{p}}{\partial S}, \mathbf{v} \right) + \Big((r + \lambda h) \mathbf{I} \ \mathbf{p}, \mathbf{v} \Big) = \Big(\mathbf{F}, \mathbf{v} \Big) \ ,$$

that leads to

$$\begin{aligned}&\left(\frac{\partial p^+}{\partial \tau}, v_1\right) + \left(\frac{\partial p^-}{\partial \tau}, v_2\right) + \frac{1}{2}\sigma^2 \left(\left(S\frac{\partial p^+}{\partial S}, S\frac{\partial v_1}{\partial S}\right) + \left(S\frac{\partial p^-}{\partial S}, S\frac{\partial v_2}{\partial S}\right)\right)\\ &-(r-\lambda h-\sigma^2)\left(\left(S\frac{\partial p^+}{\partial S}, v_1\right) + \left(S\frac{\partial p^-}{\partial S}, v_2\right)\right)\\ &+(r+\lambda h)\left((p^+, v_1) + (p^-, v_2)\right) - (F_+(p^-), v_1) - (F_-(p^+), v_2) = 0\end{aligned} \tag{7.39}$$

Thus, the pricing problem for put options can be formulated as follows

PROBLEM IEPHV: Finding $\mathbf{p} = (p^+, p^-) \in \mathbf{U}$ such that

$$\begin{aligned}\left(\frac{\partial p^+}{\partial \tau}, v_1\right) + \left(\frac{\partial p^-}{\partial \tau}, v_2\right) + a(\mathbf{p}, \mathbf{v}) = 0 \quad a.e. \text{ in } (0,T)\ ,\\ \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega),\end{aligned} \tag{7.40}$$

satisfying the initial condition,

$$\mathbf{p}(S,0) = (\max\{E-S\}, \max\{E-S\})$$

In (7.40), $a(\cdot,\cdot)$ denotes the bilinear form $a(\cdot,\cdot) : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R}$ defined by

$$a(\mathbf{p},\mathbf{v}) = a_1(p^+, v_1) + a_2(p^-, v_2) - \Big(F_+(p^-), v_1\Big) - \Big(F_-(p^+), v_2\Big)\ ,$$

for $\mathbf{p} = (p^+, p^-), \mathbf{v} = (v_1, v_2) \in \mathbf{H}^1(\Omega)$, and

$$\begin{aligned}a_1(p^+, v_1) = \ & \frac{1}{2}\sigma^2\left(S\frac{\partial p^+}{\partial S}, S\frac{\partial v_1}{\partial S}\right) - (r-\lambda h-\sigma^2)\left(S\frac{\partial p^+}{\partial S}, v_1\right)\\ &+(r+\lambda h)(p^+, v_1).\end{aligned} \tag{7.41}$$

The previous problem is nonhomogeneous in what concerns the boundary conditions. In what follows we replace the problem ***IEPHV*** by an homogeneous one. Let $\hat{\mathbf{p}} = (\hat{p}_1, \hat{p}_2)$ be defined by

$$\hat{p}_i = Ee^{-r\tau}\frac{S_{max}-S}{S_{max}}, i = 1, 2.$$

Then $\mathbf{w} = (w_1, w_2)$ defined by

$$\mathbf{w} = \mathbf{p} - \hat{\mathbf{p}}$$

belongs to $L^2(0,T;\mathbf{H}_0^1(\Omega))$ and satisfies the following variational equation

$$\begin{aligned}&\left(\frac{\partial w_1}{\partial \tau}, v_1\right) + \left(\frac{\partial w_2}{\partial \tau}, v_2\right) + a(\mathbf{w},\mathbf{v})\\ &= -\left(\frac{\partial \hat{p}_1}{\partial \tau}, v_1\right) - \left(\frac{\partial \hat{p}_2}{\partial \tau}, v_2\right) - a(\hat{\mathbf{p}},\mathbf{v})\end{aligned} \tag{7.42}$$

for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$.

We introduce now the following variational problem:

PROBLEM IEPHVH : Find $\mathbf{w} = (w_1, w_2) \in L^2(0,T;\mathbf{H}_0^1(\Omega))$ such that (7.42) holds almost everywhere in $(0,T)$ and for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, and $\mathbf{w}$ satisfies the initial condition

$$\mathbf{w}(0) = (\max\{E-S\} - E\frac{S_{max}-S}{S_{max}}, \max\{E-S\} - E\frac{S_{max}-S}{S_{max}}) \; .$$

Solving problem ***IEPHVH*** we obtain the solution of problem ***IEPHV*** considering

$$\mathbf{p} = \mathbf{w} + \hat{\mathbf{p}} \; .$$

The existence and uniqueness of solution of the variational problem ***IEPHVH*** can be seen in Brezis[93].

7.4.2 Approximation by Finite Element Methods

The Galerkin method enable us to obtain an approximation (continuous in time) for the solution of problem *IEPHVH* replacing the space $H_0^1(\Omega)$ by a finite dimension space. The central idea of finite element methods, which is a Galerkin method, is to define the mentioned finite dimension space with particular properties that lead to a significant reduction in the computational cost when compared with general Galerkin methods.

For the construction of an finite elements approximation, we build a partition of the domain in sub-regions **elements**, establishing points in this partition, called **nodes**, where the approximate solution will be evaluated and built with basis of polynomial functions in general associated with each node of the partition, and characterized by not only zero in the elements adjacent to the associated node. The determination of each basis function is obtained by requirement of a set of restrictions of the node.

Let T_h be a partition of $\overline{\Omega} = [0, S_{max}]$ into n elements k. Based on this partition we define the finite dimension space

$$\mathcal{U}_{h,0}^{k} = \{v_h \in C^0(\Omega) : v_h = 0 \text{ on } \partial\Omega, v_h|_k \in P_k(K)\},$$

where $P_k(K)$ is the polynomial set of degree $\leq k$ defined in K. In $\mathcal{U}_{h,0}^{k}$ we fix a basis $\{N_i, i = 2, \ldots, n-1\}$ where each N_i has compact support contained in a small set of elements. By $\mathbf{U}_{h,0}^{k}$ we denote the spaces $\mathcal{U}_{h,0}^{k} \times \mathcal{U}_{h,0}^{k}$.

The finite element method enable us to obtain an approximation (continuous in time) for the solution of problem *IEPHVH* replacing the space $\mathbf{H}_0^1(\Omega)$ by $\mathbf{U}_{h,0}^{k}$.
PROBLEM IEPHVHh: Finding $\mathbf{w}_h = (w_1, w_2) \in \mathbf{U}_{h,0} = \{\mathbf{v}_h \in L^2(0, T; \mathbf{U}_{h,0}^{k})\}$ depending on past information such that

$$\left(\frac{\partial w_1}{\partial \tau}, v_1\right) + \left(\frac{\partial w_2}{\partial \tau}, v_2\right) + a(\mathbf{w}, \mathbf{v}) = G(\mathbf{w}, \mathbf{v}) \tag{7.43}$$

with the initial condition

$$\mathbf{w}_h(0) = \left(\max\{E - S\} - E\frac{S_{max} - S}{S_{max}}, \max\{E - S\} - E\frac{S_{max} - S}{S_{max}}\right).$$

In (7.43) G is defined by

$$G(\hat{\mathbf{p}}, \mathbf{v}) = G_1(\hat{p}_1, v_1) + G_2(\hat{p}, v_2) - \Big(F_+(u_2), v_1\Big) - \Big(F_-(u_1), v_2\Big),$$

with

$$G_i(\hat{p}_i, v_i) = -\left(\frac{\partial \hat{p}_i}{\partial \tau}, v_i\right) - a_i(\hat{p}_i, v_i)$$

for $i = 1, 2$.

Let $\{N_i, i = 2, \ldots, n-1\}$ be a basis of $\mathcal{U}_{h,0}^{k}$ being each N_i associated with the node x_i and having support contained in a few elements of the partition.

Considering

$$w_1(S,\tau)=\sum_{i=2}^{n-1} w_{1,i}(\tau)N_i(S), \quad w_2(S,\tau)=\sum_{j=2}^{n-1} w_{2,j}(\tau)N_j(S), \tag{7.44}$$

in (7.43), we obtain

$$\begin{aligned}
&\sum_{i=2}^{n-1}\frac{dw_{1,i}}{d\tau}(\tau)(N_i,v_1)+\sum_{i=2}^{n-1}\frac{dw_{1,i}}{d\tau}(\tau)a_1(N_i,v_1)\\
&+\sum_{j=2}^{n-1}\frac{dw_{2,j}}{d\tau}\tau(\tau)(N_j,v_2)+\sum_{j=2}^{n-1}\frac{dw_{2,j}}{d\tau}(\tau)a_2(N_i,v_1)\\
&-\sum_{j=2}^{n-1}w_{2,j}(\tau)(F_+(N_j),v_1)-\sum_{i=2}^{n-1}w_{1,i}(\tau)(F_-(N_i),v_2)\\
&=G_1(\hat p_1,v_1)+G_2(\hat p,v_2)-(F_+(\hat p_2),v_1)-(F_-(\hat p_2),v_2)
\end{aligned} \tag{7.45}$$

for $(v_1,v_2)\in \mathbf{U}^k_{h,0}$.

Taking now in (7.45) $v_1=N_\ell, v_2=N_q$ for $\ell,q=2,\dots,n-1$, we deduce

$$\begin{aligned}
&\sum_{i=2}^{n-1}\frac{dw_{1,i}}{d\tau}(\tau)(N_i,N_\ell)+\sum_{i=2}^{n-1}\frac{dw_{1,i}}{d\tau}(\tau)a_1(N_i,N_\ell)\\
&+\sum_{j=2}^{n-1}\frac{dw_{2,j}}{d\tau}\tau(\tau)(N_j,N_q)+\sum_{j=2}^{n-1}\frac{dw_{2,j}}{d\tau}(\tau)a_2(N_i,N_q)\\
&-\sum_{j=2}^{n-1}w_{2,j}(\tau)(F_+(N_j),N_\ell)-\sum_{i=2}^{n-1}w_{1,i}(\tau)(F_-(N_i),N_q)\\
&=G_1(\hat p_1,N_\ell)+G_2(\hat p,N_q)-(F_+(\hat p_2),N_\ell)-(F_-(\hat p_2),N_q).
\end{aligned} \tag{7.46}$$

Ordinary differential system (7.46) can be rewritten using a matrix notation. In order to do that we introduce the following block matrices:

$$\mathcal{M}=\begin{bmatrix} M_1 & 0\\ 0 & M_2\end{bmatrix}, \quad \mathcal{A}=\begin{bmatrix} A_1 & 0\\ 0 & A_2\end{bmatrix}, \quad \mathcal{F}=\begin{bmatrix} F_1 & 0\\ 0 & F_2\end{bmatrix},$$

where

$$M_{1,i,\ell}=(N_i,N_\ell),\ M_{2,j,q}=(N_j,N_q)\ ,$$

$$A_{1,i,\ell}=a_1(N_i,N_\ell),\ A_{2,j,q}=a_2(N_j,N_q)\ ,$$

and

$$F_{1,i,q} = (F_-(N_i), N_q),\ F_{2,i,q} = (F_+(N_j), N_\ell)\ .$$

By $\mathcal{G}(\tau)$ we represent the vector with components defined by the right-hand side of (7.46).

Let $\mathcal{W}(\tau)$ be the vector of the components of $w_{1,i}(\tau)$, $i = 2, \ldots, n-1$, $w_{2,j}(\tau)$, $j = 2, \ldots, n-1$. Thus $\mathcal{W}(\tau)$ is solution of the ordinary differential system

$$\mathcal{M}\frac{d\mathcal{W}}{d\tau}(\tau) + \mathcal{A}\mathcal{W}(\tau) - \mathcal{F}\mathcal{W}(\tau) = \mathcal{G}(\tau),\ \tau \in (0, T], \tag{7.47}$$

complemented with an initial condition $\mathcal{W}(0)$ which is the vector of the components of $\mathrm{w}_h(0)$ with respect to the fixed basis.

7.4.3 Approximation in Time

Let $\partial_\tau p(\tau)$ be an approximation of implicit finite difference of first-order for $\partial p(\tau)/\partial\tau$. Dividing the interval $[0, T]$ into sub-intervals $[\tau_{m-1}, \tau_m]$, where $\tau_m = m\Delta\tau$, $m = 0, 1, 2, \cdots, m_{max}$, as $\tau_0 = 0$ and $\tau_{m_{max}} = T$, the implicit Euler scheme to approximate of $\partial p(\tau)/\partial\tau$ is given by

$$\frac{\partial p^\pm}{\partial\tau}(\tau_m) \approx \partial_\tau p^{m,\pm} = \frac{p^{m+1,\pm} - p^{m,\pm}}{\Delta\tau}.$$

Thus, the approximation by finite differences is

PROBLEM IEPHhm: For $m = 0, 1, 2, \cdots$ we finding $p_h^m(S,\tau)$, where $p_h^m(S,\tau) = p_h^{m,+}(S,\tau)$ and $p_h^m(S,\tau) = p_h^{m,-}(S,\tau)$ depending on past information, and $p_h^m(S,\tau) \in \mathcal{U}_\tau^h$ such that

$$\Big(\partial_\tau \mathbf{p}_h^m, \mathbf{v}_h\Big) + a_1(\mathbf{p}_h^{m+1}, \mathbf{v}_h) = \Big(\mathbf{F}_h^m, \mathbf{v}_h\Big)\ , \quad \forall \mathbf{v} \in \mathcal{U}_0^h\ ,$$

with initial condition

$$\Big(\mathbf{p}_h^0, \mathbf{v}_h\Big) = \Big(\mathbf{p}_h(S,0), \mathbf{v}_h\Big) = \begin{bmatrix} \Big(p_h^+(S,0), v_h^+\Big) \\ \\ \Big(p_h^-(S,0), v_h^-\Big) \end{bmatrix}\ , \qquad \forall \mathbf{v}_h \in \mathcal{U}_0^h,$$

with the bilinear form given by

$$\begin{aligned} a_1(\mathbf{p}_h^{m+1}, \mathbf{v}_h) &= \frac{1}{2}\sigma^2 \left(S(\tau)\frac{\partial \mathbf{p}_h^{m+1}}{\partial S}, S(\tau)\frac{\partial \mathbf{v}_h}{\partial S} \right) - \\ &-(r - \lambda h - \sigma^2) \left(S(\tau)\frac{\partial \mathbf{p}_h^{m+1}}{\partial S}, \mathbf{v}_h \right) + (r + \lambda h)\left(\mathbf{p}_h^{m+1}, \mathbf{v}_h \right) \end{aligned}$$

Considering the definition of $\partial_\tau p(\tau)$ and introducing the boundary conditions the *PROBLEM IEPHhm*, results in the following system of algebraic equations for each $m = 0, 1, 2, \cdots$,

$$(M + \Delta\tau K)\mathbf{P}^{m+1} = \Delta\tau \mathbf{R}^{m+1} - M\mathbf{P}^m,$$

M is the mass matrix, K the rigidity matrix, R^{m+1} is the vector of nodal actions in the moment $m+1$, resulting in the imposition of the boundary conditions, $\mathbf{P}^m$ and $\mathbf{P}^{m+1}$ are nodal values in the moments m and $m+1$, respectively.

7.5 Galerkin Method for American option

7.5.1 Variational Formulation

Defining the subset of functions limited below by $g(S)$, which is the payoff value, which is for a put option $g(S) = (E-S)^+$ and for call option $g(S) = (S-E)^+$, as

$$\mathcal{K}_{\mathcal{V}} = \left\{ \mathbf{P} \in \mathbf{U}(\Omega) \; ; \; \mathbf{P}(S, \tau) \geq g(S) \right\} . \quad (7.48)$$

The system (7.25) can be written in vector form as

$$\frac{\partial \mathbf{P}}{\partial \tau} - \frac{1}{2}\,\sigma^2 S^2 \mathbf{I}\frac{\partial^2 \mathbf{P}}{\partial S^2} - (r - \lambda h)S\mathbf{I}\frac{\partial \mathbf{P}}{\partial S} + (r - \lambda h)\mathbf{I}\mathbf{P} \geq \mathbf{F}(\mathbf{P}) \quad (7.49)$$

and the system (7.26) as

$$\frac{\partial \mathbf{P}}{\partial \tau} - \frac{1}{2}\,\sigma^2 \mathbf{I}\, S^2\frac{\partial^2 \mathbf{P}}{\partial S^2} - (r - \lambda h)S\mathbf{I}\frac{\partial \mathbf{P}}{\partial S} + (r - \lambda h)\mathbf{I}\mathbf{P} = \mathbf{F}(\mathbf{P})\ , \quad (7.50)$$

where $\mathbf{P} = \left(P^+, P^-\right)$, $\dfrac{\partial^i \mathbf{P}}{\partial S^i}$ denotes the vector whose components are $\dfrac{\partial^i P^{\pm}}{\partial S^i}$, $i = 1, 2$, and $\mathbf{F}(\mathbf{P}) = \left(F_+(P^-), F_-(P^+)\right)$ where

$$F_+(S,\tau) = \begin{cases} \lambda h P^-([1+h_+]S,\tau), \text{ for } S \in \left[0, \dfrac{1}{(1+h_+)S_{max}}\right] \\ 0 \text{ for } S \in \left[\dfrac{1}{(1+h_+)S_{max}}, S_{max}\right] \end{cases}$$

and

$$F_-(S,\tau) = \begin{cases} \lambda h P^+([1+h_-]S,\tau), \text{ for } S \in \left[0, \dfrac{1}{(1+h_-)S_{max}}\right] \\ 0 \text{ for } S \in \left[\dfrac{1}{(1+h_-)S_{max}}, S_{max}\right] \end{cases} .$$

As S_{max} is fixed large, the replacement of $\left(\lambda h P^-([1+h_+]S,\tau), \lambda h P^+([1+h_-]S,\tau)\right)$ by $\mathbf{F}$ do not disturbs $\mathbf{P}$ in the region of Ω of interest.

To define the so called variational problem associated with equations (7.49) and (7.50) complemented with the introduced boundary and inicial conditions, we multiply, with respect the inner product (7.37), by $(\mathbf{v} - \mathbf{P})$, where function $\mathbf{v} \in \mathcal{K}_{\mathcal{V}}$. As

$$\frac{1}{2} S^2 \frac{\partial^2 \mathbf{P}}{\partial S^2} = \frac{1}{2}\frac{\partial}{\partial S}\left\{S^2 \frac{\partial \mathbf{P}}{\partial S}\right\} - S\frac{\partial \mathbf{P}}{\partial S} ,$$

this results in

$$\begin{aligned} &\left(\frac{\partial \mathbf{P}}{\partial \tau}, \mathbf{v} - \mathbf{P}\right) - \left(\frac{1}{2}\sigma^2 \mathbf{I}\, \frac{\partial}{\partial S}\left(S^2 \frac{\partial \mathbf{P}}{\partial S}\right), \mathbf{v} - \mathbf{P}\right) \\ &- \left((r - \lambda h - \sigma^2) S \mathbf{I} \frac{\partial \mathbf{P}}{\partial S}, \mathbf{v} - \mathbf{P}\right) \\ &+ \Big((r + \lambda h)\mathbf{I}\, \mathbf{P}, \mathbf{v} - \mathbf{P}\Big) \geq \Big(\mathbf{F}, \mathbf{v} - \mathbf{P}\Big) , \end{aligned} \tag{7.51}$$

using integration by parts in (7.51) for $\mathbf{v} = (v_1, v_2) \in \mathcal{K}_{\mathcal{V}}$, we obtain

$$\begin{aligned} \left(\frac{\partial \mathbf{P}}{\partial \tau}, \mathbf{v} - \mathbf{P}\right) + \left(\frac{1}{2}\sigma^2 \mathbf{I}\, S\frac{\partial \mathbf{P}}{\partial S}, S\frac{\partial}{\partial S}(\mathbf{v} - \mathbf{P})\right) - \left((r - \lambda h - \sigma^2) S \mathbf{I} \frac{\partial \mathbf{P}}{\partial S}, \mathbf{v} - \mathbf{P}\right) \\ + \Big((r + \lambda h)\mathbf{I}\, \mathbf{P}, \mathbf{v} - \mathbf{P}\Big) \geq \Big(\mathbf{F}, \mathbf{v} - \mathbf{P}\Big) , \end{aligned}$$

that leads to

$$\begin{aligned}
&\left(\frac{\partial P^+}{\partial \tau}, v_1 - P^+\right) + \left(\frac{\partial P^-}{\partial \tau}, v_2 - P^-\right)\\
&+\frac{1}{2}\sigma^2 \left(\left(S\frac{\partial P^+}{\partial S}, S\frac{\partial v_1 - P^+}{\partial S}\right) + \left(S\frac{\partial P^-}{\partial S}, S\frac{\partial v_2 - P^-}{\partial S}\right)\right)\\
&-(r-\lambda h - \sigma^2)\left(\left(S\frac{\partial P^+}{\partial S}, v_1 - P^+\right) + \left(S\frac{\partial P^-}{\partial S}, v_2 - P^-\right)\right)\\
&+(r+\lambda h)\Big((P^+, v_1 - P^+) + (P^-, v_2 - P^-)\Big)\\
&-\Big(F_+(P^-), v_1 - P^+\Big) - \Big(F_-(P^+), v_2 - P^-\Big) \geq 0
\end{aligned} \tag{7.52}$$

Thus, the pricing problem for an American put option pricing can be formulated as follows

PROBLEM IAPHV: Finding $\mathbf{P} = (P^+, P^-) \in \mathcal{K}_\mathcal{V}$ such that

$$\begin{aligned}
\left(\tfrac{\partial P^+}{\partial \tau}, v_1 - P^+\right) + \left(\tfrac{\partial P^-}{\partial \tau}, v_2 - P^-\right) + a_A(\mathbf{P}, \mathbf{v} - \mathbf{P}) \geq 0 \ ,\\
a.e. \text{ in } (0,T) \ , \quad \forall \ \mathbf{v} \in \mathcal{K}_\mathcal{V},
\end{aligned} \tag{7.53}$$

satisfying the initial condition

$$\mathbf{P}(S,0) = \Big(P^+(S,0), P^-(S,0)\Big) = \Big(max\{E-S\}, max\{E-S\}\Big)$$

In (7.53), $a_A(\cdot,\cdot)$ denotes the bilinear form and defined by

$$\begin{aligned}
a_A(\mathbf{P}, \mathbf{v} - \mathbf{P}) = a_{A1}(P^+, v_1 - P^+) + a_{A2}(P^-, v_2 - P^-)\\
-\Big(F_+(P^-), v_1 - P^+\Big) - \Big(F_-(P^+), v_2 - P^-\Big) \ ,
\end{aligned}$$

for $\mathbf{P} = (P^+, P^-)$, $\mathbf{v} = (v_1, v_2) \in \mathcal{K}_\mathcal{V}$, and

$$\begin{aligned}
a_{A1}(p^+, v_1 - P^+) = \ & \frac{1}{2}\sigma^2 \left(S\frac{\partial p^+}{\partial S}, S\frac{\partial}{\partial S}(v_1 - P^+)\right)\\
& -(r - \lambda h - \sigma^2)\left(S\frac{\partial p^+}{\partial S}, v_1 - P^+\right)\\
& +(r+\lambda h)(p^+, v_1 - P^+).
\end{aligned} \tag{7.54}$$

The previous problem is nonhomogeneous in what concerns the boundary conditions. In what follows we replace problem ***IAPHV*** by an homogeneous one. Let $\hat{\mathbf{P}} = (\hat{P}_1, \hat{P}_2)$ be defined by

$$\hat{P}_i = Ee^{-r\tau}\frac{S_{max} - S}{S_{max}} \ , \quad i = 1, 2.$$

Then $\mathbf{w}_A = (w_{A,1}, w_{A,2})$ defined by

$$\mathbf{w}_A = \mathbf{P} - \hat{\mathbf{P}}$$

belongs to $L^2(0,T;\mathbf{H}_0^1(\Omega))$ and satisfies the following variational inequation

$$\begin{aligned} &\left(\frac{\partial w_{A,1}}{\partial \tau}, v_1 - P^+\right) + \left(\frac{\partial w_{A,2}}{\partial \tau}, v_2 - P^-\right) + a_A(\mathbf{w}_A, \mathbf{v} - \mathbf{P}) \\ &\geq -\left(\frac{\partial \hat{P}_1}{\partial \tau}, v_1 - P^+\right) - \left(\frac{\partial \hat{P}_2}{\partial \tau}, v_2 - P^-\right) - a_A(\hat{\mathbf{P}}, \mathbf{v} - \mathbf{P}) \end{aligned} \tag{7.55}$$

for all $\mathbf{v} \in \mathcal{K}_{\mathcal{V}}$.

We introduce now the following variational problem:

PROBLEM IAPHVH : Find $\mathbf{w}_A = (w_{A,1}, w_{A,2}) \in L^2(0,T;\mathbf{H}_0^1(\Omega))$ such that (7.55) holds almost everywhere in $(0,T)$ and for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, and $\mathbf{w}_A$ satisfies the initial condition

$$\mathbf{w}_A(0) = \left(\max\{E-S\} - E\frac{S_{max}-S}{S_{max}}, \max\{E-S\} - E\frac{S_{max}-S}{S_{max}}\right) .$$

Solving problem ***IAPHVH*** we obtain the solution of problem ***IAPHV*** considering

$$\mathbf{P} = \mathbf{w}_A + \hat{\mathbf{P}} .$$

The existence and uniqueness of solution of the variational problem ***IAPHVH*** can be seen in Brezis[93].

7.5.2 Approximation by Finite Elements

In order to construct a finite element approximation, one defines the approximation $\mathcal{K}_{\mathcal{V}}^h$ for the set $\mathcal{K}_{\mathcal{V}}$,

$$\mathcal{K}_{\mathcal{V}}^h = \left\{P_h(S,\tau) \in \mathbf{U}_{h,0}^k; P_h(S,\tau) \geq g(S)\right\}, \tag{7.56}$$

i.e. the inequality constraint will be checked only at the nodal points of the finite element mesh.

By using the *Galerkin method* in *PROBLEM IAPHVH* us to obtain an approximation (continuous in time) for the solution, as follows

PROBLEM IAPHVHh: Finding $\mathbf{w}_A = (w_{A,1}, w_{A,2}) \in \mathcal{K}_{\mathcal{V}}^h$ depending on past information such that

$$\left(\frac{\partial w_{A,1}}{\partial \tau}, v_1 - P^+\right) + \left(\frac{\partial w_{A,2}}{\partial \tau}, v_2 - P^-\right) + a_A(\mathbf{w}_A, \mathbf{v} - \mathbf{P}) \geq G(\mathbf{w}_A, \mathbf{v} - \mathbf{P}) \quad \textbf{(7.57)}$$

with the initial condition

$$\mathbf{w}_h(0) = \left(\max\{E - S\} - E\frac{S_{max} - S}{S_{max}}, \max\{E - S\} - E\frac{S_{max} - S}{S_{max}}\right) .$$

In (7.57) G is defined by

$$G(\hat{\mathbf{P}}, \mathbf{v}-\mathbf{P}) = G_1(\hat{P}_1, v_1 - P^+) + G_2(\hat{P}, v_2 - P^-) - \Big(F_+(u_2), v_1 - P^+\Big) - \Big(F_-(u_1), v_2 - P^-\Big) ,$$

with

$$G_i(\hat{P}_i, v_i - P^{j(i)}) = -\left(\frac{\partial \hat{P}_i}{\partial \tau}, v_i - P^{j(i)}\right) - a_{Ai}(\hat{p}_i, v_i - P^{j(i)})$$

for $i = 1, 2$, $j(i = 1) = +$ or $j(i = 2) = -$.

7.5.3 Approximation in Time

The approximation of a parabolic variational inequality is similar to the case of an elliptical if one considers a discretisation in time, i.e. approximating the time domain $[0, T]$ for a partition $0 = \tau_1 < \tau_2 < \cdots < \tau_{m_{max}} = T$ for uniform intervals with time $\Delta\tau$. The implicit Euler scheme to approximate of $\partial P(\tau)/\partial\tau$ is given by

$$\frac{\partial P^{\pm}}{\partial \tau}(\tau_m) \approx \partial_\tau P^{m,\pm} = \frac{P^{m+1,\pm} - P^{m,\pm}}{\Delta\tau}.$$

Thus, the approach to the American put options problem is

PROBLEM IAPHhm: For $m = 0, 1, 2, \cdots$ we find $\mathbf{P}_h^m \in \mathcal{K}_{\mathcal{V}}^h$ satisfying

$$(\partial_\tau \mathbf{P}^m, \mathbf{v}_h - \mathbf{P}^m) + a_{A1}(\mathbf{P}_h^{m+1}, \mathbf{v}_h - \mathbf{P}^m) \geq \Big(\mathbf{F}_h, \mathbf{v}_h - \mathbf{P}^m\Big) , \quad \forall\, \mathbf{v}_h \in \mathcal{K}_{\mathcal{V}}^h,$$

satisfying the initial condition

$$\left(\mathbf{P}_h^0, \mathbf{v}_h\right) = \left(\mathbf{P}_h(S,0), \mathbf{v}_h\right) = \begin{bmatrix} \left(P_h^+(S,0), v_h^+\right) \\ \\ \left(P_h^-(S,0), v_h^-\right) \end{bmatrix}, \quad \forall\, \mathbf{v}_h \in \mathcal{K}_{\mathcal{V}}^h,$$

with the bilinear form given by

$$\begin{aligned} a_{A1}(\mathbf{P}_h^{m+1}, \mathbf{v}_h - \mathbf{P}^{m+1}) \quad &= \quad \frac{1}{2}\,\sigma^2 \left(S\frac{\partial \mathbf{P}_h^{m+1}}{\partial S}, S\frac{\partial \mathbf{v}_h}{\partial S}\right) - (r - \lambda h - \sigma^2)\left(S\frac{\partial \mathbf{P}_h^{m+1}}{\partial S}, \mathbf{v}_h - \mathbf{P}^{m+1}\right) + \\ &+ \quad (r + \lambda h)\left(\mathbf{P}_h^{m+1}, \mathbf{v}_h - \mathbf{P}^{m+1}\right) \end{aligned}$$

7.6 Numerical Simulation

This section presents the results of numerical experiments in order to illustrate some important aspects concerning the application of numerical methods in financial markets. More precisely, the use of finite element methods (Galerkin Method) for numerical solutions of equations and inequalities corresponding to European and American options with memory in the underlying assets.

The implementation code for the finite element method is written in Matlab in the spirit of C. Cartensen et al. in series [94, 95, 96] and more J.T. Oden [97]

One remarks that the purpose of these simulations is to provide an easier overview of the theoretical results presented throughout this work.

The numerical results were obtained with the Picard's Algorithm with the method of Successive Over-Relaxation - SOR(ω) for European option pricing. For American options, this last method was replaced by the method of Successive Over-Relaxation SOR(ω) with projection on the convex set.

7.6.1 Examples: Recovering the Standard Black and Scholes Model

One uses the following economic parameters: $\$36.00$ is the initial price of underlying asset, the exercise price is $\$40$, the interest rate is $6\%/year$, and the time of maturity is one year. Here, in this example, the volatility is constant (σ) at a level of $30\%/year$. The memory parameters are all zero.

For implementing the Finite Element Method one need to add the following parameters: 51 is the number of steps of time and 51 is the number of steps of underlying asset (space).

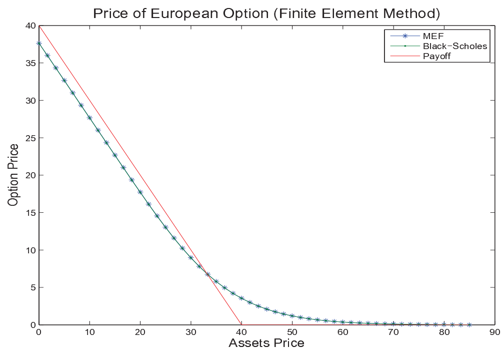

Figure 7.3: European put option in the Black and Scholes framework

As one can see in figure, Figure 7.3, the Black and Scholes model has been recovered, i.e. the numerical values found by the Finite Element Method are equal to those obtained via the Black and Scholes formula. A detailed study of the convergence of Numerical Solution for the Black-Scholes formula is presented in the Thomaz master's thesis [42]. The next figure, Figure 7.4, shows the option price as a function of price and time to maturity.

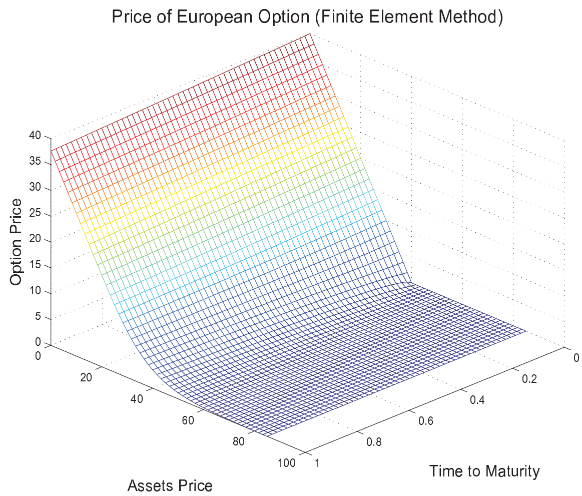

Figure 7.4: European option in a Black-Scholes framework with respect to underlying price and time to maturity

7.6.2 Examples: Memory in Assets

In this example, we present the solutions for an European and American put and call options, assuming memory in the underlying asset.

The economic parameters for the European put or call option are the following: $\$30.00$ is the initial price of the underlying asset, the exercise price is $\$80$, the interest rate is $21\%/year$ and the time to maturity is one year.

Here, in this example, the volatility is constant (σ) at $50\%/year$. The discontinuity parameter of the underlying asset is $\lambda = 0.1$ and the parameter of jump's size is $h = 1.0$.

Furthermore, the numerical method's parameters are: 51 for the number of steps of time and 41 for the number of steps for the underlying asset (space). Figure 7.5 shows the option price as a function of the underlying asset price, while Figure 7.6 and Figure 7.7 show the option price in a three dimension context as a function of price and time to maturity.

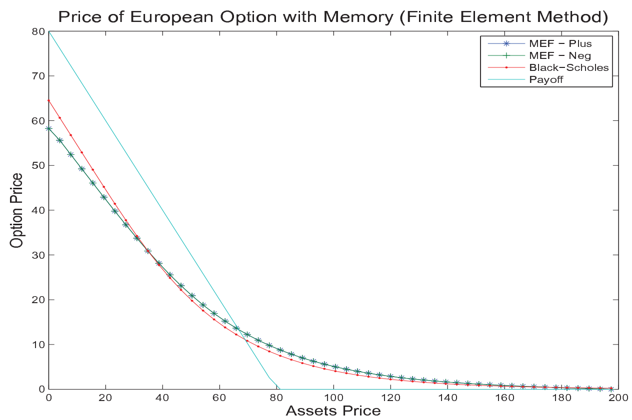

Figure 7.5: Option prices with memory in the underlying asset

Option pricing with memory assets with positive and negative information along time,

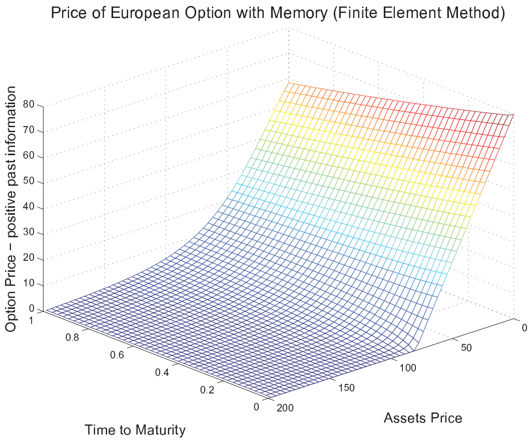

Figure 7.6: Option prices with memory in the underlying asset - positive information.

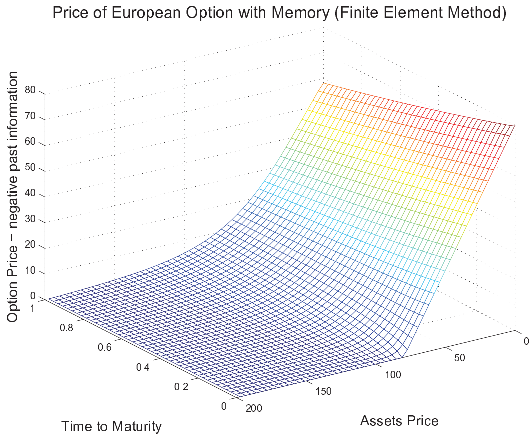

Figure 7.7: Option prices with memory in the underlying asset - negative information.

Changing to $\lambda = 0.05$ as the discontinuity parameter of the underlying asset, found

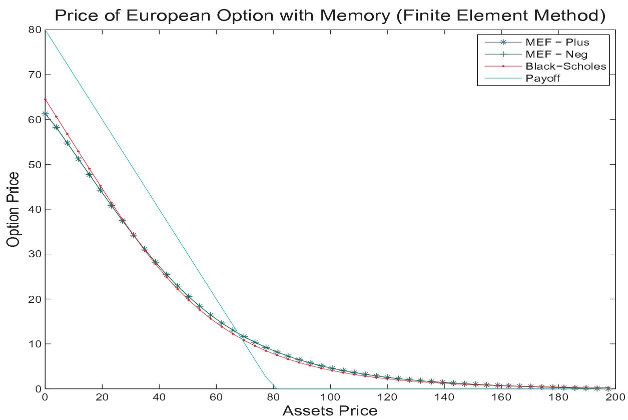

Figure 7.8: Option pricing with memory assets in time of maturity.

Over time of option pricing with memory assets with positive and negative information,

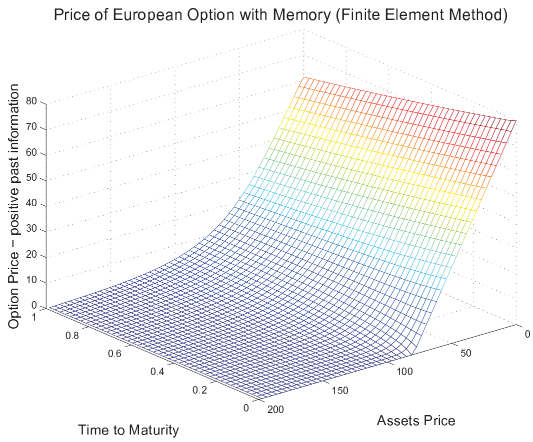

Figure 7.9: Option pricing with memory assets - positive information.

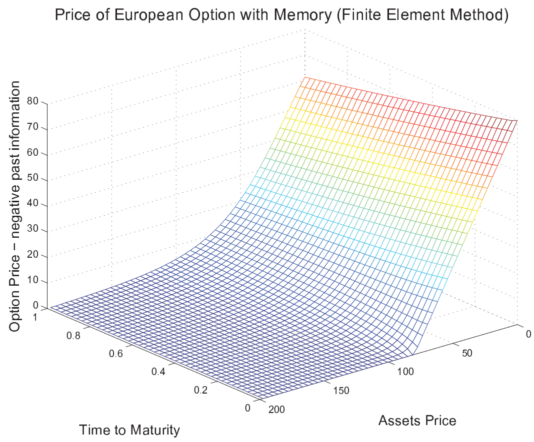

Figure 7.10: Option pricing with memory assets - negative information.

If $\lambda = 0.0$ is discontinuity parameter of the underlying asset recovering the Standard Black and Scholes Model.

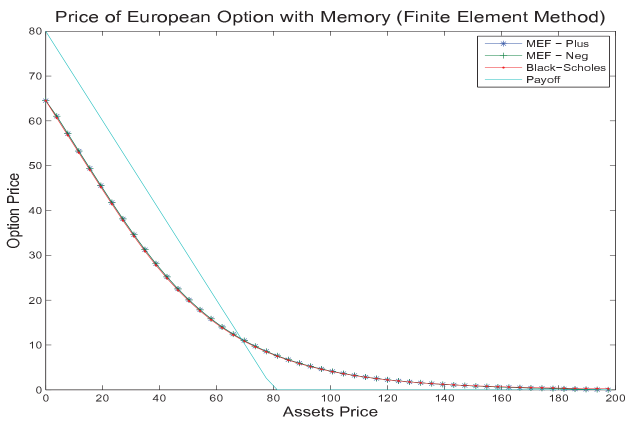

Figure 7.11: Option pricing with memory assets in time of maturity.

American put option with the same parameters with $\lambda = 0.1$ is discontinuity parameter of the underlying asset.

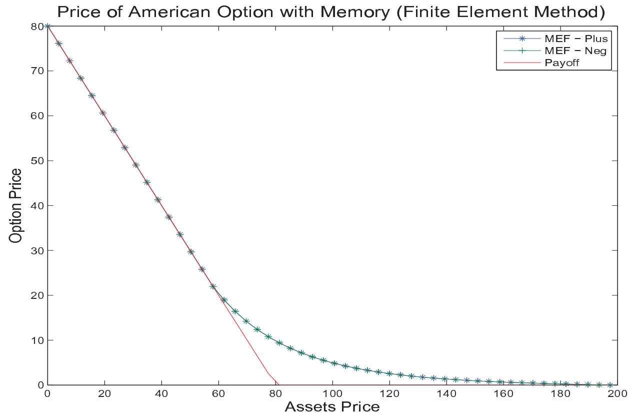

Figure 7.12: Option pricing with memory assets in time of maturity.

Over time of option pricing with memory assets with positive and negative information,

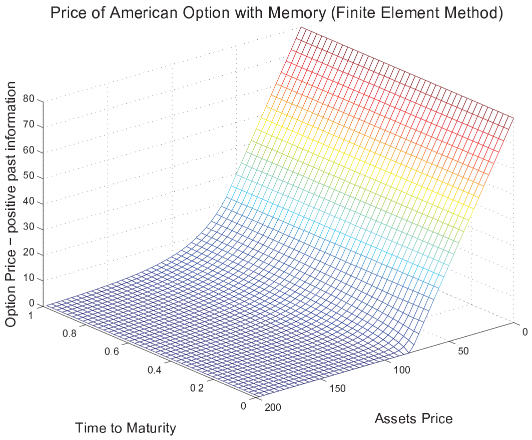

Figure 7.13: Option pricing with memory assets - positive past information.

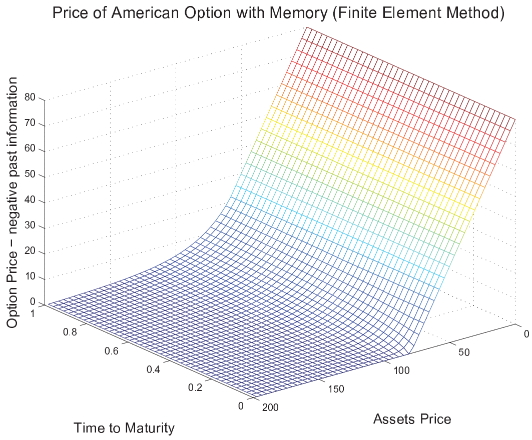

Figure 7.14: Option pricing with memory assets - negative past information.

Now, let us work with call options. The parameters are: $\$30.00$ is the initial price of underlying asset, exercise price is $\$100$, interest rate is $23\%/year$. Here in this example the volatility is constant (σ) de $57\%/year$ and the time to maturity of the option is one year.

Furthermore we have $\lambda = 0.1$ as the discontinuity parameter of the underlying asset and $h = 1.0$ is parameter of jump's size.

The numerical method's parameters are: 51 is the number of step of time, 41 is the number of underlying asset (space).

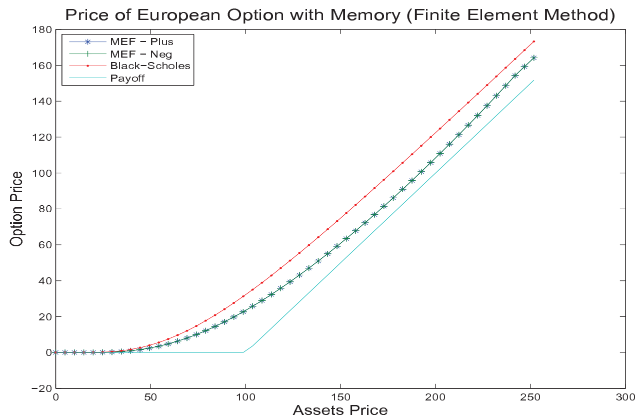

Figure 7.15: Option pricing with memory assets in time of maturity.

Over time of option pricing with memory assets with positive and negative information,

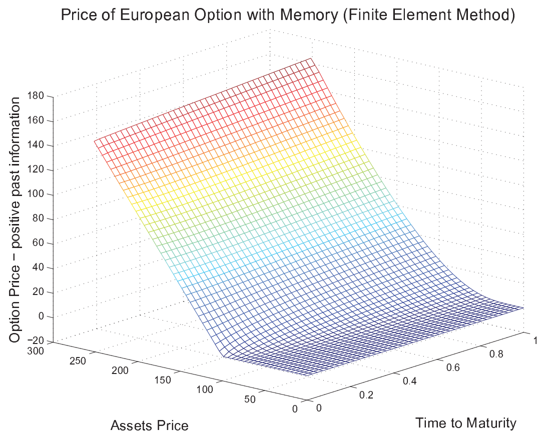

Figure 7.16: Option pricing with memory assets - positive information.

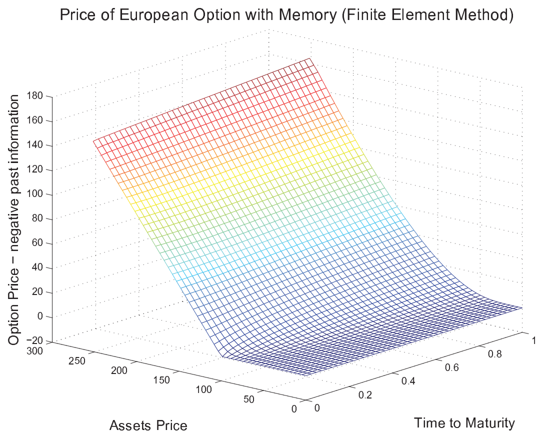

Figure 7.17: Option pricing with memory assets - negative information.

If $\lambda = 0.0$ is discontinuity parameter of the underlying asset recovering the Standard Black and Scholes Model.

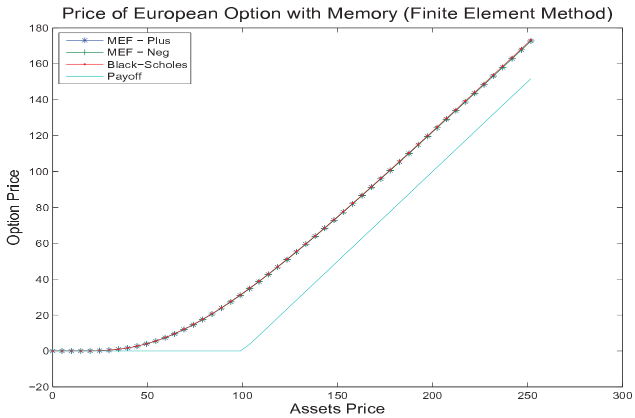

Figure 7.18: Option pricing with memory assets in time of maturity.

As can be seen in Figure 7.18, as was expected, the new model presented in this work restores the traditional modeling when the parameters of memory are equal to zero. When this parameters are nonzero values, has a new model that try to get a better match to the market.

8 Conclusion

8.1 Concluding Remarks

Mainly, this work presents new models for pricing options considering memory in the underlying asset process. These new models are specially based on the works of Di Crescenzo and Pellerey[10], Ratanov[11, 12, 13] and Hobson and Rogers[24]. However other more specific results are also presented here.

This work shows new stochastic calculus tools which are quite important as they have direct application in the pricing framework. For instance the Levy theorem (Theorem 2.2.3), the stochastic integral for Semimartigales (Definition 2.3.5), Theorem of Exponential Stochastic of the Semimartigales (Theorem 2.3.2) and the Theorem of Existence and Uniqueness of a solution for a special SDE (Theorem 2.3.3). It also focuses on the implementation of numerical methods analysing SDE.

Although there has been a proliferous production of literature on option pricing, the main models for the underlying asset, since the Black and Scholes[3] model, are highlighted here focusing on their main characteristics, the SDEs, the analytical or numerical solutions and examples of price trajectories.

The aim of the study of underlying asset process was to produce the theoretical background to motive other processes. Firstly, it is presented the Jump-Telegraph-Diffusion-Drift processes model (JTDD process), which basically is a generalization of the model studied in Ratanov[11, 12, 13], by the inclusion of the drift. Consequently, it is shown the Exponential Stochastic of JTDD-process (Theorem 5.1.2) which is essential to formalize the analytical solution for pricing assets with memory. Secondly, the level of complexity was increased by considering stochastic volatility. Here the work was conducted on the ground of Hobson and Roger[24] framework. Therefore, the final result was a pricing model with a complex memory structure, which is present not only in the volatility but also in the price itself - The JTDD-Process with Memory in the Volatility.

The use of the Jump-Telegraph-Diffusion-Drift processes (JTDD process) To model the dynamics of the underlying asset created the need to develop a new Ito's Lemma, resulting in different systems according to the option's type: systems of equations (7.12) and (7.15) refers to the European put and call options, respectively; while the systems of differential inequalities (7.20, 7.21) and (7.25, 7.26) are the pricing formulae for American put and call options, respectively.

The solution of the previous systems of PDEs were approximated using a combination between the Galerkin method and the Implicit Euler's method. The fully discrete problem was numerically solved using the Picard's algorithm.

A direct extension for this work would be to consider the underlying asset as described in (5.12) with a random volatility in the Hobson and Rogers[24] sense. Other extension would be to consider stochastic interest rates, therefore resulting on a three-factor model. Finally, although this is a theoretical work one should not disregard the importance of the empirical work. In fact, it is essencial for the calibration of volatility function, as in Foschi and Pascucci[26].

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