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Asymptotic limits for the doubly nonlinear equation

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Abstract

This thesis is concerned with the asymptotic limits of the solutions of the homogeneous Dirichlet problem associated to a doubly nonlinear evolution equation of the form $u_t = \Delta_p u^m + g$, as the parameters p and m go to infinity. This equation combines the nonlinearities of the porous medium equation, which corresponds to the case $p = 2$, and the p-Laplace equation, which corresponds to the case $m = 1$. The main contribution we give is to generalize some of the results known for the asymptotic limits of the solutions of initial-value problems associated to the porous medium equation, as m tends to infinity, and to the p-Laplace equation, as p goes to infinity. The motivation for the study of the limiting behaviour of solutions to these equations arises from their potential physical applications, as they serve as mathematical models for physical problems in several fields, for example in the study of non-Newtonian fluids, turbulent flow of a gas in porous media and glaciology. Moreover, under certain conditions on the initial data, they give rise at the limit to problems with completely different properties, with important physical applications of their own and which require novel analytical approaches.

We will address the limits in p and m separately and in sequence, eventually completing a convergence diagram for the problem. As far as we know, not much has been done on the asymptotic behaviour of the solutions of the doubly nonlinear equation, when both $m \neq 1$ and $p \neq 2$. In particular, the complete convergence diagram is a novelty.

We associate to the doubly nonlinear equation an integrable source term g and integrable nonnegative initial data. To analyze the limit when $p \to \infty$, we take any bounded domain of $\mathbb{R}^N$ with smooth boundary. To evaluate the limit as m goes to infinity, we further assume that the domain is either a bounded interval of the real line or a ball of radius R, in which case we also assume that the initial data is radial. We prove, under the additional assumptions on the domain and initial data stated above, that the equation satisfied at the limit is independent of the order in which we take the limits in p and m. We achieve the complete diagram for the regular limit of the solutions, but we also present some results regarding the singular limits of the solutions of the doubly nonlinear equation as p and m tend to infinity. The nonlinear semigroup approach will be employed to pass to the limit.

Keywords: doubly nonlinear equation, asymptotic limit, singular limit.

Resumo

O objetivo deste trabalho é investigar os limites assimptóticos das soluções do problema de Dirichlet homogéneo associado à equação de evolução duplamente não linear $u_t = \Delta_p u^m + g$, quando os parâmetros p e m tendem para infinito. Esta equação combina a não linearidade da equação dos meios porosos, que corresponde ao caso $p = 2$, com a não linearidade da equação de p-Laplace, que corresponde ao caso $m = 1$. A contribuição principal deste trabalho é generalizar alguns dos resultados conhecidos para os limites assimptóticos das soluções de problemas de valor inicial associados à equação dos meios porosos, quando m tende para infinito e à equação de p-Laplace, quando p tende para infinito. A motivação para o estudo do comportamento no limite das soluções destas equações radica nas suas aplicações físicas, uma vez que constituem modelos matemáticos para problemas físicos em diferentes contextos, por exemplo no estudo dos fluidos não Newtonianos, do fluxo turbulento de um gás em meios porosos e em glaciologia. Adicionalmente, sob certas condições iniciais, encontramos no limite problemas com propriedades completamente diferentes, com aplicações físicas que são interessantes por si sós e que exigem uma abordagem analítica inovadora.

Estudaremos os limites em p e m separadamente e em sequência, eventualmente completando um diagrama de convergência para o problema. Tanto quanto sabemos, muito pouco tem sido feito sobre o comportamento assimptótico das soluções da equação duplamente não linear, quando $m \neq 1$ e $p \neq 2$ simultaneamente. Em particular, o diagrama de convergência completo é uma novidade.

Associamos à equação duplamente não linear um termo g integrável e um valor inicial não negativo e também integrável. Para analisar o limite quando $p \to \infty$, consideramos qualquer domínio limitado de $\mathbb{R}^N$ com fronteira regular. Para determinar o limite quando m tende para infinito, assumimos que o domínio é um intervalo limitado da recta real ou uma bola de raio R, e neste último caso assumimos também que o valor inicial é radial. Provamos, sob as condições adicionais no domínio e no valor inicial referidas, que a equação satisfeita no limite é independente da ordem pela qual tomamos os limites em p e m. Além de obtermos o diagrama completo para o limite regular das soluções, apresentamos alguns resultados relacionados com o limite singular da equação duplamente não linear quando p e m tendem para infinito, usando a teoria dos semigrupos não lineares.

Palavras-chave: equação duplamente não linear, limite assimptótico, limite singular.

To my parents.

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Notation

$\lvert A\rvert$	Lebesgue measure of a set A
χ_A	characteristic function of a set A
$\operatorname{supp}(f)$	support of a function f
f^+, f^-	$\max(f,0)$, $\max(-f,0)$
$f\wedge g$, $f\vee g$	$\inf(f,g)$, $\sup(f,g)$
$B(x,r)$	ball in $\mathbb{R}^N$ of centre x and radius r
$\to$	strong convergence
$\rightharpoonup$	weak convergence
$\overset{*}{\rightharpoonup}$	weak star convergence

$\mathrm{sign}(r)$ is the signum graph:

$$\mathrm{sign}(r) = \begin{cases} -1 & \text{if } r<0,\\ [-1,+1] & \text{if } r=0,\\ 1 & \text{if } r>0. \end{cases}$$

$\mathrm{sign}_0(r)$ is the single-valued restriction:

$$\mathrm{sign}_0(r) = \begin{cases} -1 & \text{if } r<0,\\ 0 & \text{if } r=0,\\ 1 & \text{if } r>0. \end{cases}$$

$\mathrm{sign}^+(r)$ is the restriction to the positive part:

$$\mathrm{sign}^+(r) = \begin{cases} -1 & \text{if } r<0,\\ [0,+1] & \text{if } r=0,\\ 0 & \text{if } r>0. \end{cases}$$

$\mathrm{sign}_0^+(r)$ is the single-valued restriction to the positive part

$$\mathrm{sign}_0^+(r) = \begin{cases} 0 & \text{if } r\le 0,\\ 1 & \text{if } r>0. \end{cases}$$

Contents

Introduction

In this thesis, we study the limiting behaviour of the solutions of the Dirichlet problem associated to the doubly nonlinear equation

$$u_t = \Delta_p u^m + g, \text{ with } m(p-1) > 1, \tag{DNE}$$

as the parameters p and m tend to infinity, where

$$\Delta_p w := \operatorname{div}(|\nabla w|^{p-2} \nabla w)$$

is the p-Laplace operator. This important parabolic equation includes the (degenerate) parabolic p-Laplace equation, for $m = 1$,

$$u_t = \Delta_p u, \text{ with } p > 2, \tag{PLE}$$

and the porous medium equation, for $p = 2$,

$$u_t = \Delta u^m, \text{ with } m > 1, \tag{PME}$$

both of which are prototypes of diffusion equations and are extensively studied in the literature.

The equation in (DNE) is known as the doubly nonlinear diffusion equation since its diffusion coefficient $D(u, \nabla u) = m^{p-1} u^{(m-1)(p-1)} |\nabla u|^{p-2}$ exhibits a double nonlinearity, depending on both u and its gradient ∇u. It also possesses a double degeneracy, for the slow diffusion case $m(p-1) > 1$, as its diffusion coefficient vanishes at points where $|\nabla u| = 0$ or $u = 0$. The study of this class of nonlinear evolution equations is motivated by their physical applications. They are used as mathematical models for physical problems in many fields, for example in the study of non-Newtonian fluids [35], turbulent flow of a gas in porous media ([37]) and glaciology ([22],[32]).

Since the early eighties, extensive work has been done for the asymptotic limit of initial-value problems associated to (PLE) and (PME), as the parameters p and m tend to infinity, respectively. However, very few references appear in the literature on the asymptotic limits of the general (DNE) in the case that both $m \neq 1$ and $p \neq 2$.

As far as we know, only some progress has been made, especially in the case when the domain is the whole of $\mathbb{R}^N$, under restrictive conditions on the initial datum, for the asymptotic limit with respect to the parameter m, when p is fixed (see [15], [33] and [31]).

In the present work, we consider the following homogeneous Dirichlet problem for (DNE)

$$\begin{cases} u_t = \Delta_p u^m + g & \text{on } (0,T)\times\Omega \\ \\ u^m = 0 & \text{on } (0,T)\times\partial\Omega \\ \\ u(0,\cdot) = u_0(\cdot) & \text{on } \Omega, \end{cases} \tag{0.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, with initial datum $u_0 \in L^1(\Omega)$ and source term $g \in L^1(\Omega_T)$.

Notice that for (DNE), variations of the parameters p and m affect directly the diffusion coefficient

$$D(u, \nabla u) \approx |\nabla u^m|^{p-2} u^{m-1} \approx u^{(m-1)(p-1)} |\nabla u|^{p-2}.$$

We can see that formally, fixing m, $D(u,\nabla u)$ converges to infinity for the points where $|\nabla u^m| > 1$ and to zero for those where $|\nabla u^m| < 1$, as p tends to infinity. Similarly, fixing p, $D(u,\nabla u)$ converges to infinity for points where $|u| > 1$ and to zero for those where $|u| < 1$, when m tends to infinity. We will see that these four regions, $\{|\nabla u^m| > 1\}$, $\{|\nabla u^m| < 1\}$, $\{|u| < 1\}$ and $\{|u| > 1\}$, will play an important role in studying the asymptotic limits and in many cases we will need to study them separately.

Our aim is to shed some light into the complete picture by generalizing some of the results known for the prototype equations (PLE) and (PME) to the doubly nonlinear equation, by means provided by nonlinear semigroup theory. Since (DNE) inherits many of the characteristic features of (PLE) and (PME), the outline of the theory will be similar. However, the appearance of the double degeneracy will offer new challenges that will require using different techniques.

We will address the limits in p and m separately and in sequence, eventually completing a convergence diagram for the problem. To be precise, we prove under certain additional conditions on the initial data, the existence of the limit of solutions of problem (0.1), as p tends to infinity, and the equation it satisfies at the limit, hence generalizing the results found in [27] and [4]. We show as well, assuming that either Ω is a bounded interval of the real line or a ball in $\mathbb{R}^N$, in which case we also assume that

u_0 is radial, that the solutions of (0.1) converge as the parameter m tends to infinity and present the equation satisfied by this limit. We see that this equation coincides with the one obtained in [33], where the asymptotic behaviour as m tends to infinity of the solutions of the associated Cauchy problem to (DNE) was studied. However, we highlight that working in a bounded domain requires using different techniques to pass to the limit. Finally, we study the convergence of the solutions of the corresponding limit equations as the exponents m and p tend to infinity, respectively, establishing that once both parameters have been taken to infinity, the equation we obtain is the same, independently of the order in which we take the limits in p and m. For the original results, we refer to [5].

In a broader sense, the study of asymptotic limits of partial differential equations (pdes) attracts a lot of attention due to its physical interest. Indeed, several physical and mechanical problems are modeled by perturbations of pdes, which are, in many cases, given by parameters. Understanding the dependence of the pdes on the variations of these parameters allows us to solve questions regarding important properties they satisfy. Moreover, it may occur that variations of the evolution of the pdes with respect to changes of the parameters are so significant, that in the limit the problem may be completely different in nature. From the physical point of view, this means that for large values of the parameters, there appear important phenomena, which although intuitive, must be rigorously understood in terms of the mathematical model.

This kind of problems also hold mathematical interest on their own in the study of singular limits of homogeneous semigroups following the work of Bénilan. The notion of convergence of nonlinear semigroups, as presented by Brézis and Pazy [20], provides the appropriate framework for studying the asymptotic limits of large classes of evolution problems. In fact, for a sequence of evolution problems governed by multivalued m-accretive operators, we can use nonlinear semigroup theory to understand how the solutions of these problems depend continuously on the sequence of operators. Specifically, let us consider the following sequence of problems

$$(u_k)_t + A_k u_k \ni g \ \text{ in } \ (0,\infty), \ \ u_k(0) = u_0, \qquad (0.2)_k$$

for m-accretive operators A_k in a Banach space X and assume that there exists an m-accretive operator A such that $A_k \to A$ in the sense of resolvents. Then, we can conclude, as long as $u_0 \in \overline{D(A)}$, that there exists a function u such that

$$u_k \to u \ \text{ in } \ C(0,\infty;X),$$

and u is the solution of

$$u_t + Au \ni g \ \text{ in } \ (0,\infty), \quad u(0) = u_0. \tag{0.2}$$

This powerful tool allowed to determine the convergence of solutions of evolution problems, as well as the equation satisfied at the limit, merely from the convergence of the sequence of operators, as long as the initial data was compatible with the limiting m-accretive operator A. However, if $u_0 \in X\backslash\overline{D(A)}$, then the limit problem (0.2) is not well posed. The limit of the solutions of the sequence of problems may not exist.

Nevertheless, it turns out that for a large class of specific problems the limit exists and it is interesting to understand the properties of the limit solution. When this occurs, we say that there exists a singular limit. For the limit problem to be well posed, there must be a discontinuity in the passage to the limit with respect to the initial inconsistent data. The problem at the limit will then have a new initial data which is adapted to the limit operator. We would have the following problem at the limit

$$u_t + Au \ni g \ \text{ in } \ (0,\infty), \ u(0) = \underline{u_0}, \tag*{(0.2)$'$}$$

where now $\underline{u_0} \in \overline{D(A)}$. However, to this day, it is still unclear what are the features of this new initial data. For a long time, it was believed that this $\underline{u_0}$ was the projection of u_0 onto the closure of the domain of the limit operator A. This is the case when the sequence of operators A_k are the Yosida approximation of the limit operator A, $A_k := (I - (I + kA)^{-1})/k$, where A is assumed to be maximal monotone in a Hilbert space [18]. It also holds for certain approximations of homogenous m-accretive operators in the case that the sequence of solutions of the approximate problems are non-negative [14]. However, this does not hold in general, as it was shown to be false for the limit problem as p tends to infinity of the Dirichlet problem associated to (PLE), even for the convergence of non-negative solutions [27].

Outline of the thesis

In chapter 1, we collect several definitions and results that we will need throughout the thesis. We start by compiling some standard results on Sobolev, Lebesgue and BV spaces. Then we introduce some properties of accretive and m-accretive operators, focusing in particular on subdifferential operators, as well as the evolution equations associated to these operators. We present the concepts of mild and integral solutions and finally provide an overview on the study of the convergence of the operators and their corresponding semigroups.

In the first section of chapter 2, we gather the results known on the asymptotic limit of the Cauchy problem associated to (PLE), when p tends to infinity, as well as its physical significance. This section will have two main parts, which depend on the condition imposed on the initial datum u_0, i.e., whether $\|\nabla u_0\|_{L^\infty(\mathbb{R}^N)} \leq 1$ or $\|\nabla u_0\|_{L^\infty(\mathbb{R}^N)} > 1$. We emphasize that the past studies on limiting behaviour of the parabolic p-Laplace equation were carried out considering the domain to be the whole of $\mathbb{R}^N$ and solutions in the weak sense. To study the asymptotic behaviour of the problem in (0.1) we will work on bounded domains and solutions in the mild sense. Under these conditions, we will generalize the results obtained for (PLE) using the classical results of the nonlinear semigroup theory.

Chapter 3 will follow a similar outline to chapter 2, only now we will analyze the asymptotic limit with respect to the parameter m. We briefly give an overview of the results obtained for the asymptotic limit of (PME), which has been extensively studied in the literature. Furthermore, we discuss the main results which have been proved for the Cauchy-Dirichlet problem associated to (DNE). Most of these results will be adaptable to the case of bounded domains, except for a compactness result. Our contribution will focus on certain BV estimates which will allow us to prove the convergence of non-negative solutions of the Dirichlet problem, when m tends to infinity, and the equation it satisfies at the limit, assuming that the domain is a bounded interval in the real line or a ball, in which case we also assume that the initial data is radial.

In chapter 4, we study the asymptotic behaviour of the solutions of the limit equations obtained in chapter 2 and 3. In the first section we analyze the behaviour, as m tends to infinity, of the limit obtained in chapter 2, for the regular case. Similarly, in the second section, we examine the asymptotic behaviour, as p goes to infinity, of the regular limit obtained in chapter 3. Hence, in the final section, we already have all the ingredients necessary to complete the convergence diagram in the regular case. We prove that the equation satisfied in the limit is the same, independently of the order of the limits in p and m.

We conclude with a summary of the main results obtained and the problems that still remain open and will be an interesting direction for research in the future.

1. Preliminaries

In this chapter, we will recall some definitions and properties of function spaces as well as classical results of nonlinear semigroup theory. Standard references for the material presented here, for the nonlinear semigroup theory, are [7],[13] and [40].

1.1 Function spaces

Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$. For $1 \leq p \leq \infty$, we denote by $L^p(\Omega)$ the space of Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ such that, if $p < \infty$,

$$\|u\|_{L^p(\Omega)} = \|u\|_p = \left(\int_\Omega |u|^p dx\right)^{1/p} < \infty,$$

and, for $p = \infty$,

$$\|u\|_{L^\infty(\Omega)} = \|u\|_\infty = \operatorname*{ess\,sup}_\Omega |u| < \infty.$$

Let us denote by $L^p_{loc}(\Omega)$ the space of Lebesgue measurable functions u such that $\|u\|_{L^p(K)} < \infty$, for all compact subsets $K \subset \Omega$. For $u \in C^1(\Omega)$, denote by $\frac{\partial u}{\partial x_i}$ (or simply u_{x_i}), its partial derivative and by $\nabla u = (u_{x_1}, \cdots, u_{x_N})$ its gradient.

The Sobolev space $W^{1,p}(\Omega)$ with $1 \leq p \leq \infty$, is the space of functions $u \in L^p(\Omega)$, whose generalized derivatives or derivatives in the distribution sense u_{x_i}, belong to $L^p(\Omega)$ for all $i = 1, \cdots, N$, namely $\nabla u \in (L^p(\Omega))^N$, endowed with the natural norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{1,p} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

$W^{1,p}_0(\Omega)$ denotes the closure of $C^\infty_0(\Omega)$ under this norm. A function $u \in W^{1,p}_{loc}(\Omega)$ if $\|u\|_{W^{1,p}(K)} < \infty$, for every compact subset $K \subset \Omega$. We recall that, for $1 < p < \infty$, the dual space of $L^p(\Omega)$ is identified with $L^{p'}(\Omega)$, where $p' = \frac{p}{p-1}$ is the conjugate of p.

We recall the following Sobolev embedding for functions in $W^{1,p}_0$, $1 \leq p < N$.

Theorem 1.1.1. Let $u \in W_0^{1,p}(\Omega)$ for some $p \in [1, N)$. Then there exists a constant $C = C(N, p)$ such that

$$\|u\|_{q,\Omega} \leq C\|\nabla u\|_{p,\Omega},$$

for each $q \in [1, p^*]$, where p^* is the Sobolev conjugate of p, defined as

$$p^* := \frac{Np}{N-p}.$$

In particular, for all $p \in [1, N)$

$$\|u\|_{p,\Omega} \leq C\|\nabla u\|_{p,\Omega}.$$

This last estimate is referred to as the Poincaré's inequality for functions whose trace on the boundary of Ω is zero.

Let us consider as well some properties of the space $BV(\Omega)$ as defined below:

Definition 1.1.1. A function $u \in L^1(\Omega)$ has bounded variation in Ω if

$$\int_\Omega |\nabla u| := \sup\left\{\int_\Omega u \operatorname{div}\varphi : \ \varphi \in C_0^\infty(\Omega; \mathbb{R}^N) \ \text{ and } \ \|\varphi\|_\infty \leq 1 \ \text{ in } \ \Omega\right\} < \infty.$$

We write

$$BV(\Omega)$$

to denote the space of functions of bounded variation.

Remark 1.1.1. If $u \in BV(\Omega)$ and ∇u is the gradient of u in the sense of distributions, then ∇u is a vector valued Radon measure and $\int_\Omega |\nabla u|$ is the total variation of ∇u on Ω.

$BV(\Omega)$ is a Banach space endowed with the norm

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + \int_\Omega |\nabla u|.$$

Theorem 1.1.2. (Lower semicontinuity) Let Ω be an open set and let $\{u_j\}$ be a sequence of functions in $BV(\Omega)$ which converges in $L^1(\Omega)$ to a function u. Then

$$\int_\Omega |\nabla u| \leq \liminf_{j\to\infty} \int_\Omega |\nabla u_j|.$$

Theorem 1.1.3. (Compactness) Let Ω be open and bounded with $\partial\Omega$ Lipschitz. Assume $\{u_j\}$ is a sequence in $BV(\Omega)$ that satisfies

$$\sup_j \|u_j\|_{BV(\Omega)} < \infty,$$

then there exists a subsequence $\{u_{j_k}\}$ and a function $u \in BV(\Omega)$ such that

$$u_{j_k} \to u \ \text{ in } \ L^1(\Omega) \ \text{ as } \ k \to \infty.$$

For $0 < T < \infty$, let us denote by Ω_T the cylindrical domain $(0,T) \times \Omega$. The space $L^r(0,T; L^p(\Omega))$ for $r, p \geq 1$ is the collection of functions $u(x,t)$, defined and measurable in Ω_T, such that for almost every t, $0 < t < T$, the functions $u \in L^p(\Omega)$ and

$$\|u\|_{r,p,\Omega_T} = \left(\int_0^T \left(\int_\Omega |u(x,t)|^p dx \right)^{r/p} dt \right)^{1/r} < \infty.$$

Also, $u \in L^r_{loc}(0,T; L^p_{loc}(\Omega))$, if for every compact subset $K \subset \Omega$ and every subinterval $[t_1, t_2] \subset (0,T]$

$$\int_{t_1}^{t_2} \left(\int_K |u|^p dx \right)^{r/p} dt < \infty.$$

Whenever $r = p$, we set $L^p(0,T; L^p(\Omega)) = L^p(\Omega_T)$. These definitions are extended in the obvious way when either p or r are infinity.

The parabolic Sobolev space $L^r(0,T; W^{1,p}(\Omega))$ is the space of functions $u(x,t)$, such that for almost every t, $0 < t < T$, the functions $u \in W^{1,p}(\Omega)$ and

$$\int_0^T \left(\int_\Omega |u|^p + |\nabla u|^p \right)^{r/p} dt < \infty.$$

The space $C(0,T; L^p(\Omega))$ is defined as the space of all measurable functions u on Ω_T such that for all $t \in [0,T]$, $u(t,\cdot) \in L^p(\Omega)$ and $u(t,\cdot)$ is a continuous function from $[0,T]$ to $L^q(\Omega)$, that is

$$\lim_{h \to 0} \|u(t+h,\cdot) - u(t,\cdot)\|_{p,\Omega} = 0.$$

1.2 Accretive operators

Let A be an operator (possibly multivalued) $A : X \to \mathcal{P}(X)$ acting on a Banach space X endowed with a norm denoted by $\|\cdot\|$. The domain of A is defined as $D(A) = \{x \in X \mid Ax \neq \emptyset\}$ and its range as $R(A) = \bigcup_{x \in D(A)} Ax$.

An operator A is *single-valued* if Ax is a singleton for all $x \in D(A)$. The operator A can be identified with its graph in $X \times X$ as follows: $(x,y) \in A$ if and only if $x \in D(A)$ and $y \in Ax$. $\overline{A}$ will denote the closure in $X \times X$ of the graph of A.

Definition 1.2.1. An operator A in X is *accretive* if

$$\|x - \hat{x}\| \leq \|x - \hat{x} + \lambda(y - \hat{y})\| \text{ whenever } \lambda \geq 0, \text{ and } (x,y), (\hat{x},\hat{y}) \in A.$$

In practice it is useful to reformulate the definition of accretive operators in terms of its *resolvent operator* $(I + \lambda A)^{-1}$ which we will denote by J^A_λ and the bracket $[\cdot,\cdot]$ which will be defined below.

Definition 1.2.2. For every $x, y \in X$, we define the *bracket* $[\cdot, \cdot] : X \times X \to \mathbb{R}$ by

$$[x, y] = \inf_{\lambda > 0} [x, y]_\lambda = \inf_{\lambda > 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda}.$$

Proposition 1.2.1. The following are equivalent

(*i*) A is accretive,

(*ii*) J_λ^A is a single-valued contraction operator which means that it satisfies the following inequality for any $x, y \in D(J_\lambda^A)$ and $\lambda \geq 0$:

$$\|J_\lambda^A x - J_\lambda^A y\| \leq \|x - y\|,$$

(*iii*) $[x - \hat{x}, y - \hat{y}] \geq 0$ whenever $(x, y), (\hat{x}, \hat{y}) \in A$.

If X is a Banach lattice, in which case we can define the positive part of an element $x \in X$, we can also define $[\cdot, \cdot]^+$ and the corresponding T-accretive operators as follows.

Definition 1.2.3. For every $x, y \in X$, $[\cdot, \cdot]^+ : X \times X \to \mathbb{R}$ is

$$[x, y]^+ = \inf_{\lambda > 0} \frac{\|(x + \lambda y)^+\| - \|(x)^+\|}{\lambda}.$$

Definition 1.2.4. An operator A in X is T*-accretive* if one of the following equivalent properties hold:

(*i*) $\|(x - \hat{x})^+\| \leq \|(x - \hat{x} + \lambda(y - \hat{y}))^+\|$ whenever $\lambda \geq 0$, and $(x, y), (\hat{x}, \hat{y}) \in A$.

(*ii*) J_λ^A is a single-valued T-contraction operator, which means that it satisfies the following inequality for any $x, y \in D(J_\lambda^A)$ and $\lambda \geq 0$:

$$\|(J_\lambda^A x - J_\lambda^A y)^+\| \leq \|(x - y)^+\|.$$

(*iii*) $[x - \hat{x}, y - \hat{y}]^+ \geq 0$ whenever $(x, y), (\hat{x}, \hat{y}) \in A$.

We will need as well a stronger concept that ensures the existence of a unique solution of $x + \lambda A x = y$ for all $y \in X$ and $\lambda > 0$.

Definition 1.2.5. An operator A in X is m*-accretive* if it verifies one of the following equivalent properties:

(*i*) For all $\lambda > 0$, J_λ^A is an everywhere defined contraction.

(ii) A is accretive and there exists $\lambda > 0$ such that $R(I + \lambda A) = X$.

(iii) A is accretive and for all $\lambda > 0$, $R(I + \lambda A) = X$.

Definition 1.2.6. An operator A of X is *m-T-accretive* if it is T-accretive and there exists $\lambda > 0$ such that $R(I + \lambda A) = X$.

Now let Ω be any open domain in $\mathbb{R}^N$ and $X = L^1(\Omega)$. Using the definition of accretive and T-accretive operators we can see that the following propositions hold.

Proposition 1.2.2.

1. An operator A in $L^1(\Omega)$ is accretive if and only if one of the following equivalent properties holds for all $(u, v), (\hat{u}, \hat{v}) \in A$:

   (i)
$$\int_\Omega \operatorname{sign}_0(u - \hat{u})(v - \hat{v})dx + \int_{\{u=\hat{u}\}} |v - \hat{v}|dx \geq 0.$$

   (ii) There exists $\alpha \in L^\infty(\Omega)$, $\alpha(x) \in \operatorname{sign}(u(x) - \hat{u}(x))$ a.e. in Ω such that
$$\int_\Omega \alpha(x)(v - \hat{v})dx \geq 0.$$

2. An operator A in $L^1(\Omega)$ is T-accretive if and only if one of the following equivalent properties holds for all $(u, v), (\hat{u}, \hat{v}) \in A$:

   (i)
$$\int_{\{u=\hat{u}\}} (v - \hat{v})^+ dx + \int_{\{u>\hat{u}\}} (v - \hat{v})dx \geq 0.$$

   (ii) There exists $\alpha \in L^\infty(\Omega)$, $\alpha(x) \in \operatorname{sign}^+(u(x) - \hat{u}(x))$ a.e. in Ω such that
$$\int_\Omega \alpha(x)(v - \hat{v})dx \geq 0.$$

Let us note that in the case of $L^p(\Omega)$ spaces, $1 \leq p \leq \infty$, T-accretivity implies accretivity since the norm satisfies

$$\|x^+\| \leq \|y^+\| \text{ and } \|x^-\| \leq \|y^-\| \text{ implies } \|x\| \leq \|y\|.$$

For all $u, v \in L^1(\Omega)$, we denote $u \ll v$ if and only if

$$\int_\Omega j(u)dx \leq \int_\Omega j(v)dx \text{ for all } j : \mathbb{R} \to [0, \infty] \text{ convex, l.s.c and } j(0) = 0.$$

Definition 1.2.7. An operator A in $L^1(\Omega)$ is *completely accretive* if it verifies one of the following equivalent conditions:

(*i*)

$$u - \hat{u} \ll u - \hat{u} + \lambda(v - \hat{v}),\ \forall\,(u,v),\ (\hat{u},\hat{v}) \in A \ \text{ and } \ \lambda > 0.$$

(*ii*)

$$\int_{\{u-\hat{u}>k\}} (v-\hat{v})dx \geq 0 \geq \int_{\{u-\hat{u}<-k\}} (v-\hat{v})dx,\ \forall\,(u,v),\ (\hat{u},\hat{v}) \in A \ \text{ and } \ k > 0.$$

(*iii*)

$$\int_\Omega (v-\hat{v})h(u-\hat{u})dx \geq 0,\ \forall\,(u,v),\ (\hat{u},\hat{v}) \in A \ \text{ and } \ h \in H_0,$$

where

$$H_0 := \{h \in C^1(\mathbb{R});\ h(0) = 0 \text{ and } 0 \leq h' \leq 1\}. \tag{1.1}$$

It is clear that a completely accretive operator is accretive for all the norms $L^p(\Omega)$ with $1 \leq p \leq \infty$.

1.3 Subdifferential operators

We will examine a particular class of operators called subdifferential operators of convex, proper, lower semicontinuous (l.s.c) functions.

Let us consider X a real Banach space and Φ a convex function. A function $\Phi : X \to (-\infty, \infty]$ is said to be proper if it is not identically equal to $+\infty$ (that is, if its effective domain $D(\Phi)$ defined by $D(\Phi) = \{u \in X : \Phi(u) < +\infty\ \}$ is non-empty).

Definition 1.3.1. Let $\Phi : X \to (-\infty, \infty]$ be proper and convex on a Banach space X. The subdifferential of Φ is a possibly multi-valued operator $\partial\Phi : X \to 2^{X^*}$ defined by $(u, u^*) \in \partial\Phi$, i.e., $u \in D(\partial\Phi)$, $u^* \in \partial\Phi(u)$ if

$$\Phi(v) \geq \Phi(u) + {}_{X^*}\langle u^*, v - u\rangle_X \quad \forall\, v \in D(\Phi),$$

with $D(\partial\Phi) = \{u \in X \ :\ \partial\Phi(u) \neq \emptyset\}$, where ${}_{X^*}\langle\cdot,\cdot\rangle_X$ denotes the natural duality between X and X^*.

Clearly then $D(\partial\Phi) \subset D(\Phi)$.

In the case that X is a Hilbert space H we have that

$$u \in D(\partial\Phi) \ \text{ and } \ w \in \partial\Phi(u),$$

provided

$$\Phi(v) \geq \Phi(u) + (w, v - u),$$

for all $v \in H$, where $(\cdot, \cdot)$ is the inner product in H. It also holds that $\overline{D(\partial\Phi)} = \overline{D(\Phi)}$.

Given C a closed convex subset of H, we define the indicator function of C by:

$$I_C(u) = \begin{cases} 0 & \text{if } u \in C, \\ \\ +\infty & \text{if } u \notin C. \end{cases} \tag{1.2}$$

Then the subdifferential of I_C is characterized by

$$v \in \partial I_C(u) \Longleftrightarrow u \in C \text{ and } (v, w - u) \leq 0 \quad \forall\, w \in C. \tag{1.3}$$

Maximal accretive operators in Hilbert spaces more commonly known as maximal monotone operators are defined below. In the particular case when $H = \mathbb{R}$ they are called *maximal monotone graphs*.

Definition 1.3.2. An operator A in a Hilbert space H is called *maximal monotone* if it satisfies the following properties:
(i) for every u_1,$u_2 \in D(A)$ and every $v_1 \in A(u_1)$, $v_2 \in A(u_2)$ we have

$$(v_1 - v_2, u_1 - u_2) \geq 0.$$

(ii) Its graph is a maximal element among all monotone operators in H.

It is important to note that even though any accretive operator has a maximal accretive extension by Zorn's lemma and m-accretive operators are maximal elements in the set of accretive operators, there are accretive operators which do not admit any m-accretive extension. Therefore in a general Banach space these concepts do not coincide, although they do coincide in a Hilbert space.

Proposition 1.3.1. Let Φ be a proper convex function in H. If Φ is lower semicontinuous, then its subdifferential $\partial\Phi$ is a maximal monotone operator.

Denote by $M(\Omega)$ the space of (a.e. equivalence classes) of measurable mappings from Ω into $\mathbb{R}$. Let us now consider X a linear subspace of $M(\Omega)$ and $\Phi : X \to (-\infty, +\infty]$. We define as in [12] the operator $\partial_X\Phi$ in X by

$$\begin{cases} v \in \partial_X\Phi(u) \Longleftrightarrow u \in D(\Phi),\ v \in X \text{ and} \\ \\ \Phi(w) \geq \Phi(u) + \displaystyle\int (w - u)v, \quad \text{for } w \in X \text{ with } (w - u)v \in L^1(\Omega), \end{cases} \tag{1.4}$$

where $D(\Phi)$ continues to be the effective domain of Φ.

Remark 1.3.1. Notice that in the case that $X \subset L^2(\Omega)$, $\partial_X \Phi = (\partial\hat{\Phi})_X$ where $\partial\hat{\Phi}$ is the subdifferential in $L^2(\Omega)$, as defined above, of the extension $\hat{\Phi}$ of Φ to $L^2(\Omega)$ which is $+\infty$ on $L^2(\Omega)\backslash X$.

The following result was proved in [12] for more general spaces.

Lemma 1.3.2. Let X be any $L^p(\Omega)$ space, $1 \leq p \leq \infty$ and $\Phi : X \to (-\infty, +\infty]$. Assume that

$$\Phi(u + h(\hat{u} - u)) + \Phi(\hat{u} - h(\hat{u} - u)) \leq \Phi(u) + \Phi(\hat{u}), \quad \text{for } u, \hat{u} \in X, \tag{1.5}$$

holds for $h \in H_0$. Then $\partial_X \Phi$ is completely accretive.

Let $\Phi = I_C$ be the indicator function as defined in (1.2). In this case $\partial_X \Phi$ is the graph

$$\left\{(u, v) \in C \times X : \int (u - w)v \geq 0 \text{ for } w \in C \text{ with } (u - w)v \in L^1(\Omega)\right\}, \tag{1.6}$$

and property (1.5) for $h \in \hat{H}_0$ is then exactly

$$u, \hat{u} \in C, \; h \in \hat{H}_0 \Longrightarrow u + h(\hat{u} - u) \in C,$$

where

$$\hat{H}_0 = \{h \in Lip(\mathbb{R}) : \; h(0) = 0 \text{ and } 0 \leq h' \leq 1 \text{ a.e.}\},$$

the closure of H_0 in $C(\mathbb{R})$.

Proposition 1.3.3. Let A be a completely accretive operator in $L^1(\Omega)$ and φ a maximal monotone graph then $A \circ \varphi$ is T-accretive in $L^1(\Omega)$.

Proof. $A \circ \varphi$ is T-accretive in $L^1(\Omega)$, by proposition 1.2.2, if whenever $(u, v), (\hat{u}, \hat{v}) \in A \circ \varphi$ then

$$\int_{\{u=\hat{u}\}} (v - \hat{v})^+ dx + \int_{\{u>\hat{u}\}} (v - \hat{v}) dx \geq 0.$$

Therefore, it is enough to prove that the second term is non-negative.

$$v \in A(\varphi(u)) \text{ if there exists } w \in \varphi(u) \text{ such that } v \in Aw.$$

By the complete accretivity of A, for $(u, w), (\hat{u}, \hat{w}) \in A$,

$$\int_\Omega (v - \hat{v})h(w - \hat{w}) \geq 0, \; \forall\, h \in H_0.$$

Take h as a smooth approximation of $(r-\epsilon)^+ \wedge \epsilon$, divide by $\epsilon > 0$ and let $\epsilon \to 0$ to obtain

$$\int_{\{w>\hat{w}\}} (v-\hat{v}) = \int_\Omega (v-\hat{v})\mathrm{sign}_0^+(w-\hat{w}) \geq 0.$$

On the other hand by the maximal monotonicity of φ we have $\{u > \hat{u}\} = \{w > \hat{w}\}$ and the result follows. $\square$

We will also need the following lemma by Brézis [19].

Lemma 1.3.4. Let A be a maximal monotone operator on a Hilbert space H. Let Z_n and W_n be measurable functions from Ω (a finite measure space) onto H. Assume $Z_n \to Z$ a.e. on Ω and $W_n \rightharpoonup W$ weakly in $L^1(\Omega; H)$. If $W_n(x) \in A(Z_n(x))$ a.e. on Ω, then $W(x) \in A(Z(x))$ a.e. on Ω.

1.4 Evolution equation associated to accretive operators

Let A be an operator in X, $u_0 \in X$ and $g \in L^1(0,T;X)$. We consider the following Cauchy problem

$$\begin{cases} \dfrac{du}{dt} + Au \ni g \quad \text{on } [0,T], \\ \\ u(0) = u_0. \end{cases} \tag{P}$$

Let us recall some results from non-linear semigroup theory applied to obtain existence of solutions of abstract differential equations as the one in (P).

1.4.1 Mild solutions

Let us revisit the method of *implicit time discretization* (ITD) which allows us to find, under certain conditions on the operator A, approximations of solutions to the ODE problem in (P).

Indeed, for every partition $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_n \leq T\}$, we can consider the discretized system

$$\frac{u_i - u_{i-1}}{\epsilon_{i-1}} + Au_i \ni g_i \quad i = 1, \cdots, n, \tag{1.7}$$

where $\epsilon_{i-1} = t_i - t_{i-1}$ is the time step and $g_1, g_2, \cdots, g_n$ is a discretization of g adapted to the partition $\mathcal{P}$, which satisfies

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|g(s) - g_i\| ds \leq \epsilon.$$

Let us denote such ϵ-*discretization* of the problem (P) by $D_A(t_0, \cdots, t_n : g_1, \cdots, g_n)$. Equation (1.7) can be rewritten as

$$u_{i-1} + \epsilon_{i-1} g_i \in u_i + \epsilon_{i-1} A u_i. \tag{1.8}$$

Using the definition of J_λ^A, the resolvent of A, (1.8) determines the values of u_i by

$$u_i = J_{\epsilon_{i-1}}(u_{i-1} + \epsilon_{i-1} g_i), \quad i = 1, \cdots, n.$$

This is well defined if and only if $u_{i-1} + \epsilon_{i-1} g_i \in R(I + \lambda A)$ for all $i = 1, \cdots, n$, which holds in particular for m-accretive operators. Therefore, given u_0, we can successively find the values of u_i from its previous values u_{i-1}. We can then define the piecewise constant function $u_\epsilon : [0, T] \to X$ by $u_\epsilon(0) = u_0$ and $u_\epsilon(t) = u_i = u(t_i)$ for $t_{i-1} < t \leq t_i$, and retrieve from the discrete set $\{u_i\}$ a function u_ϵ defined for all $t \in [0, T]$. Such a function is considered an ϵ-*approximate* solution of (P).

The concept of mild solutions can be formulated as follows.

Definition 1.4.1. Let $g \in L^1(\Omega_T)$. A mild solution of (P) is a function $u \in C(0, T; X)$ that is obtained as a uniform limit of the ϵ-discretization $D_A(t_0, \cdots, t_n : g_1, \cdots, g_n)$ of the problem. Namely, for every $\epsilon > 0$, there exists an ϵ-discretization with solution u_ϵ in $[0, T]$ and

$$\|u(t) - u_\epsilon(t)\| < \epsilon, \text{ for } t_0 \leq t \leq t_n.$$

Theorem 1.4.1. (Continuity properties of mild solutions) Let A be an accretive operator in X and let u be a mild solution of (P) for $g \equiv 0$ on $[0, T]$.

(i) If v is an ϵ-approximate solution of (P) on $[0, T]$ with $[0, s]$ in its domain $0 \leq t \leq T$, and $(x, y) \in A$, then

$$\|u(t) - v(s)\| \leq \|u(0) - x\| + \|v(0) - x\| + \|y\||t - s| + 3\|y\|\sqrt{\epsilon}\sqrt{T + \epsilon} + 3\epsilon.$$

(ii) If $(x, y) \in A$, then

$$\|u(t) - u(s)\| \leq 2\|x - u(0)\| + \|y\||t - s| \text{ for } 0 \leq s \leq T.$$

(iii) If $\hat{u}$ is a mild solution of $\hat{u}' + A\hat{u} \ni 0$ on $[0, T]$, then

$$\|u(t) - \hat{u}(t)\| \leq \|u(0) - \hat{u}(0)\| \text{ for } 0 \leq t \leq T.$$

If $g \equiv 0$, an accretive operator A that satisfies the *range condition*, i.e., $\overline{D(A)} \subset R(I + \lambda A)$ for all $\lambda > 0$, generates a strongly continuous semigroup of contraction. A strongly continuous semigroup is a functional object that plays an important role in the theory of existence of mild solutions and it is defined below.

Definition 1.4.2. Let D be a subset of X. A family of mappings $S(t)$, $t \geq 0$, of D into itself satisfying

(i) $S(t+s)x = S(t)S(s)x$ for $t, s \geq 0$ and $x \in D$, and

(ii) $\lim_{t\to 0} S(t)x = x$ for $x \in D$,

is called a *strongly continuous semigroup* on D.

Crandall and Liggett [23] proved the following:

Theorem 1.4.2. Let A be an accretive operator in X that satisfies the range condition. Then, for any $u_0 \in \overline{D(A)}$, the limit

$$\lim_{n\to\infty} (J^A_{t/n})^n u_0 = \lim_{n\to\infty} \left(I + \frac{t}{n} A \right)^{-n} u_0 = S(t)u_0 \tag{1.9}$$

exists uniformly on compact subsets of $[0, \infty)$. Moreover, the family of operators $S(t)$, $t > 0$ is a strongly continuous semigroup of contractive mappings of $\overline{D(A)} \subset X$.

Formula (1.9) is called the Crandall-Ligget exponential formula for the nonlinear semigroup generated by $-A$. Furthermore, by Brézis and Pazy [19], the problem (P) with $g \equiv 0$ admits a unique mild solution $u \in C(0, T; X)$ such that $u(0) = u_0$ and such a solution satisfies the exponential formula. Therefore, the solution can be represented as follows:

$$u(t) = \lim_{n\to\infty} (I + \frac{t}{n} A)^{-n} u_0 := e^{-tA} u_0.$$

For a general $g \in L^1(0, T; X)$ the following theorem holds.

Theorem 1.4.3. Let A be an m-accretive operator acting on X and $g \in L^1(0, T; X)$. Then for all $u_0 \in \overline{D(A)}$ there exists a unique mild solution of (P).

1.4.2 From mild solutions to integral solutions

Theoretically, it is difficult to determine from the definition whether a given function is a mild solution of $u' + Au \ni g$. However, mild solutions satisfy a family of integral inequalities which give us a more direct characterization of these solutions and that lead us to define integral solutions.

Definition 1.4.3. Let A be an accretive operator and $g \in L^1(0,T;X)$. A function $u \in C(0,T;X)$ is an *integral solution* of $u' + Au \ni g$ on $[0,T]$ if it satisfies

$$\|u(t) - x\| - \|u(s) - x\| \leq \int_s^t [u(\tau) - x, g(\tau) - y] d\tau,$$

for every $(x,y) \in A$ and $0 \leq s \leq t \leq T$. An integral solution of the initial-value problem $v' + Av \ni g$, $v(0) = x$, on $[0,T]$ is an integral solution u of the relation $v' + Av \ni g$ which satisfies $u(0) = x$.

Mild and integral solutions are connected in the following way.

Theorem 1.4.4. Let A be an m-accretive operator in X and $g \in L^1(0,T;X)$. Then for every $x \in \overline{D(A)}$, the initial value problem (P) has a unique integral solution. Moreover, this integral solution is the mild solution.

Theorem 1.4.5. Let A be an accretive operator and $f, g \in L^1(0,T;X)$. If v is an integral solution of $v' + Av \ni f$ on $[0,T]$ and u is a mild solution of $u' + Au \ni g$ on $[0,T]$, then

$$\frac{d}{dt}\|u(t) - v(t)\| \leq [u(t) - v(t), g(t) - f(t)] \text{ in } \mathcal{D}'(0,T).$$

The following lemma will allow us to rewrite the inequality above in an "integrated form".

Lemma 1.4.6. (Generalized Gronwall lemma) Let $T > 0$, $\varphi \in C([0,T])$ and $\psi \in L^1(0,T)$. Then the following assertions are equivalent

(i)

$$\varphi(t) - \varphi(s) \leq \int_s^t \psi(\tau) d\tau, \text{ for } 0 \leq s \leq t \leq T,$$

(ii)

$$\int_0^T \left(\varphi(t) \frac{d}{dt}\xi(t) + \psi(t)\xi(t) \right) dt \geq 0 \text{ for } \xi \in \mathcal{D}(0,T)^+,$$

which means by definition that $\varphi' \leq \psi$ in $\mathcal{D}'(0,T)$.

Remark 1.4.1. By the previous lemma, under the conditions of Theorem 1.4.5, we have

$$\begin{aligned} \|u(t)-v(t)\| - \|u(s)-v(s)\| &\leq \int_s^t [u(\tau)-v(\tau), g(\tau)-f(\tau)]d\tau \\ &\leq \int_s^t \|g(\tau)-f(\tau)\|d\tau, \end{aligned} \tag{1.10}$$

for $0 \leq s \leq t \leq T$.

1.5 Convergence of semigroups

Let $(A_k)_{k>0}$ be a family of operators on X, then $\liminf_{k\to\infty} A_k$ is defined by $(x,y) \in \liminf_{k\to\infty} A_k$ if there exits $(x_k, y_k) \in A_k$ such that $x_k \to x$ and $y_k \to y$ in X.

Proposition 1.5.1. Let A_k be m-accretive operators for $k = 1, 2, \cdots, \infty$, D be a dense set in X, $\lambda > 0$ and recall that we set $J_\lambda^{A_k} = (I + \lambda A_k)^{-1}$, for $k = 1, 2, \cdots, \infty$. Then the following statements are equivalent:

(i) $\liminf_{k\to\infty} A_k = A_\infty$.

(ii) $\liminf_{k\to\infty} A_k \supseteq A_\infty$.

(iii) $\lim_{k\to\infty} J_\lambda^{A_k} z = J_\lambda^{A_\infty} z,\ \ \forall\, z \in D$.

(iv) For some $\lambda_0 > 0$, $\lim_{k\to\infty} J_{\lambda_0}^{A_k} z = J_{\lambda_0}^{A_\infty} z,\ \ \forall\, z \in D$.

Proof. $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$ are self-evident.

$(ii) \Rightarrow (iii)$ By the accretivity of A_k, given

$$(x,y) \in A_\infty \subset \liminf_{k\to\infty} A_k,$$

then

$$\lim_{k\to\infty} (I + \lambda A_k)^{-1}(x + \lambda y) = x. \tag{1.11}$$

As A_∞ is m-accretive, for $z \in X$ we may uniquely write $z = x + \lambda y$ and therefore $x = (I + \lambda A_\infty)^{-1} z$. This, together with (1.11), gives (iii).

$(iv) \Rightarrow (i)$ First notice that, if for any $z \in D$, (iv) is satisfied, then it must hold for any element in X, since $J_{\lambda_0}^{A_k}$, $k = 1, \cdots, \infty$, are contractive operators defined everywhere.

Let now $(x, y) \in \liminf_{k\to\infty} A_k$. Then, by (1.11) and (iv), we have that

$$x = \lim_{k\to\infty} (I + \lambda_0 A_k)^{-1}(x + \lambda_0 y) = (I + \lambda_0 A_\infty)^{-1}(x + \lambda_0 y).$$

Therefore $(x, y) \in A_\infty$ and $A_\infty \supset \liminf_{k\to\infty} A_k$. On the other hand, defining

$$x_k = (I + \lambda_0 A_k)^{-1}(x + \lambda_0 y), \quad y_k = \lambda_0^{-1}(x + \lambda_0 y - x_k),$$

it is clear that $(x_k, y_k) \in A_k$, and from (iv), it follows that $x_k \to x$ and $y_k \to y$ as $k \to \infty$.

Then $(x, y) \in \liminf_{k\to\infty} A_k$ and $A_\infty \subset \liminf_{k\to\infty} A_k$. Thus $A_\infty = \liminf_{k\to\infty} A_k$. $\square$

Let A_k be m-accretive operators, $k = 1, 2, \cdots, \infty$, $u_{0_k} \in \overline{D(A_k)}$, $g_k \in L^1(0, T; X)$, and consider the family of Cauchy problems

$$\begin{cases} \dfrac{du_k}{dt} + A_k u_k \ni g_k \quad \text{on } (0, T) \\ \\ u_k(0) = u_{0_k}, \end{cases} \tag{P_k}$$

where the solutions u_k are taken in the mild sense. We are interested in when the convergence of the operators in the resolvent sense implies the convergence of the solutions of the Cauchy problem.

Theorem 1.5.2. Let X be a Banach space, A_k a sequence of accretive operators on X, $k = 1, 2. \cdots, \infty$, $u_{0_k} \in \overline{D(A_k)}$ and u_k a mild solution of (P_k) with $g_k \equiv 0$. Then, if

(i) $A_\infty \subseteq \liminf_{k\to\infty} A_k$,

and

(ii) $u_{0_k} \to u_{0_\infty}$ in X,

then

$$u_k \to u_\infty \text{ in } C(0, T; X), \text{ when } k \to \infty.$$

Proof. Since u_∞ is a mild solution of (P_∞), fixing $\epsilon > 0$, we know that there exists a solution v_∞ of an ϵ-discretization $D_{A_\infty}(0 = t_0, \cdots, t_n : e_1, \cdots, e_n)$ of the problem on $[0, T]$ satisfying

$$\|v_\infty(0) - u_{0_\infty}\| < \epsilon.$$

Let $w_i \in A_\infty v_\infty(t_i)$, $1 \leq i \leq n$, be given by

$$\frac{v_\infty(t_i) - v_\infty(t_{i-1})}{t_i - t_{i-1}} + w_i = e_i,\ i = 1, 2, \cdots, n.$$

Since $A_\infty \subset \liminf_{k\to\infty} A_k$, there are sequences $(x_i^k, y_i^k) \in A_k$, $1 \le i \le n$, such that $x_i^k \to v_\infty(t_i)$ and $y_i^k \to w_i$ as $k \to \infty$. Setting $x_0^k = u_{0_k}$ and defining e_i^k by

$$\frac{x_i^k - x_{i-1}^k}{t_k - t_{k-1}} + y_i^k = e_i^k,\ 1 \le i \le n,$$

and

$$v_k(t) = \begin{cases} x_0^k = u_{0_k} & \text{for } t = t_0 = 0 \\ \\ x_i^k & \text{for } t_{i-1} \le t \le t_i,\ 1 \le i \le n, \end{cases}$$

then v_k is a solution of an ϵ'-discretization $D_{A_k}(0 = t_0, \cdots, t_n : e_1^k, \cdots, e_n^k)$ of (P_k) on $[0, T]$. Moreover, for every $\delta > 0$, letting $k > M(\delta)$,

$$\|v_k(t) - v_\infty(t)\| \le \delta + \epsilon.$$

Once again, since $A_\infty \subset \liminf_{k\to\infty} A_k$, considering any $(p_\infty, q_\infty) \in A_\infty$, we can choose $(p_k, q_k) \in A_k$ such that $p_k \to p_\infty$ and $q_k \to q_\infty$.

Using also Theorem 1.4.1 (i), (ii), and $u_{0_k} \to u_{0_\infty}$ as $k \to \infty$ for arbitrary $\delta > 0$, we obtain

$$\limsup_{k\to\infty} \sup_{0\le t\le T} \|u_k(t) - u_\infty(t)\| \le \delta\|u_{0_\infty} - p_\infty\| + \|q_\infty\|(2\epsilon + 8(\epsilon(T + 2\epsilon))^{1/2}) + 7\epsilon.$$

Recalling that $u_{0_\infty} \in \overline{D(A_\infty)}$ and $q_\infty \in A_\infty p_\infty$ is arbitrary, we can consider p_∞ such that $p_\infty \to u_{0_\infty}$ and take $\epsilon \to 0$ to conclude that u_k tends to u_∞ uniformly on $[0, T]$. □

Theorem 1.5.3. Let X be a Banach space, A_k a sequence of m-accretive operators on X, $k = 1, 2. \cdots, \infty$, $u_{0_k} \in \overline{D(A_k)}$ and u_k a mild solution of (P_k). Then, if

(i) $A_\infty \subseteq \liminf_{k\to\infty} A_k$,

(ii) $u_{0_k} \to u_{0_\infty}$ in X,

(iii) $g_k \to g_\infty$ in $L^1(0, T; X)$,

then

$$u_k \to u_\infty \text{ in } C(0, T; X), \text{ when } k \to \infty.$$

Proof. Let us denote the mild solution of $u + A_k u \ni g$, with $u(0) = u_0$, $k = 1, \cdots, \infty$, by $E_k(u_0, g)$. This defines E_k as a mapping

$$E_k : \overline{D(A_k)} \times L^1(0,T;X) \longmapsto C(0,T;X).$$

Suppose now that g is a step function such that $u_k' + A_k u_k \ni g$, $k = 1, 2, \cdots, \infty$, then it follows by the previous theorem that $E_k(u_{0_k}, g) \to E_k(u_{0_\infty}, g)$ uniformly in $[0,T]$. Indeed on each interval of constancy of the step function we can consider $\hat{A}_k = A_k - z$, where $z \in X$ is the value of g on each time interval and the result follows. For general $g_k \in L^1(0,T;X)$ we have, by (1.10),

$$\begin{aligned} \|E_k(u_{0_k}, g_k)(t) - E_\infty(u_{0_\infty}, g_\infty)(t)\| \ &\leq \ \|E_k(u_{0_k}, g)(t) - E_\infty(u_{0_\infty, g})(t)\| \\ &+ \ \int_0^T \|g(t) - g_k(t)\| dt + \int_0^T \|g(t) - g_\infty(t)\| dt. \end{aligned}$$

Therefore

$$\limsup_{k\to\infty} \sup_{0\leq t\leq T} \|E_k(u_{0_k}, g_k)(t) - E_\infty(u_{0_\infty}, g_\infty)\| \leq 2 \int_0^T \|g(t) - g_\infty\| dt,$$

for every step function g. Since step functions are dense in $L^1(0,T;X)$, the proof is complete.

□

However, in general, $\overline{D(A_\infty)} \neq \bigcap_{k\geq 1} \overline{D(A_k)}$, and therefore, only if $u_{0_\infty} \in \overline{D(A_\infty)}$, we have the certainty, by Theorem 1.4.3, that the solution u_∞, to which the sequence of solutions $(u_k)_{k\geq 0}$ converges, is the unique mild solution of

$$\begin{cases} \dfrac{du_\infty}{dt} + A_\infty u_\infty \ni g_\infty \quad \text{in } [0,T) \\ \\ u_\infty(0) = u_{0_\infty}. \end{cases}$$

Now if $u_{0_\infty} \notin \overline{D(A_\infty)}$ and the limit u_k exists, then the limit is singular, since an initial boundary layer at $t = 0$ appears in the passage to the limit.

More generally, we can consider the following problem

$$\begin{cases} \dfrac{du_k}{dt} + A_k u_k \ni F(u_k) \quad \text{on } (0,T) \\ \\ u_k(0) = u_{0_k}, \end{cases} \tag{$P_k(F)$}$$

where F is a continuous perturbation in X, and as before A_k is a sequence of m-accretive operators, $k = 1, 2, \cdots, \infty$, $u_{0_k} \in \overline{D(A_k)}$. The problem $(P_k(F))$ has a unique mild solution. Moreover, we have the following result.

Theorem 1.5.4. Let X be a Banach space, A_k a sequence of m-accretive operators on X and $F : X \to X$ continuous and bounded such that $F + kI$ is accretive where $k \in \mathbb{R}$ and I is the identity in X, and $u_{0_k} \in \overline{D(A_k)}$, $k = 1, 2, \cdots, \infty$. Denote by u_k the unique mild solution of $(P_k(F))$. Then, if

$$(I + A_k)^{-1}x \to (I + A_\infty)^{-1}x \text{ in } X, \text{ when } k \to \infty,$$

and

$$u_{0_k} \to u_{0_\infty} \text{ in } X,$$

we have that

$$u_k \to u_\infty \text{ in } C(0, T; X), \text{ when } k \to \infty.$$

Let us as well point out a property of the resolvents that we will need later on.

Proposition 1.5.5. Let D be a dense subset in X and A_k, A be accretive operators such that, as $k \to \infty$,

$$J_\lambda^{A_k} f \to J_\lambda^A f, \quad \forall\, f \in D.$$

If $\overline{A}_k$ is m-accretive, then $\overline{A}$ is m-accretive and

$$J_\lambda^{\overline{A}_k} f \to J_\lambda^{\overline{A}} f, \quad \forall\, f \in X. \tag{1.12}$$

Proof. Let us consider $x = J_\lambda^{\overline{A}_k} f$, $f \in D$. Then $f = x + \lambda y$ for some $y \in \overline{A}_k x$ and therefore there exist $(x_n, y_n) \in A_k$ such that, denoting $f_n = x_n + \lambda y_n$, $f_n \to f$ in X as $n \to \infty$. Then

$$\|J_\lambda^{\overline{A}_k} f - J_\lambda^{A} f\| \le \|x - x_n\| + \|f_n - f\| \to 0 \text{ as } n \to \infty.$$

Similarly for A which is also accretive. Hence, for all $\epsilon > 0$,

$$\|J_\lambda^{\overline{A}_k} f - J_\lambda^{\overline{A}}\| \le \|J_\lambda^{\overline{A}_k} f - J_\lambda^{A_k} f\| + \|J_\lambda^{A_k} f - J_\lambda^{A} f\| + \|J_\lambda^{A} f - J_\lambda^{\overline{A}} f\| \le \epsilon, \quad \forall\, f \in D. \tag{1.13}$$

It is then easy to see that (1.12) holds for all $f \in X$.

Now let us take an arbitrary $z \in X$, since $\overline{R(I + A_k)} = X$ by the m-accretivity of $\overline{A}_k$ there exists $z_n \subset R(I + A_k)$ such that $z_n \to z$ as $n \to \infty$ in X and there exist $x_k = (I + A_k)^{-1} z_n$. Then $x_k \to x$, where $x = J_\lambda^A z_n$, and therefore $z_n \in R(I + A)$ and $\overline{A}$ is m-accretive. $\square$

2. Limit of solutions of (DNE) as $p \to \infty$

We wish to study the following doubly nonlinear diffusion problem $(DNE)_{p,m}$,

$$\begin{cases} (u_{p,m})_t = \Delta_p u_{p,m}^m + g & \text{in } (0,\infty) \times \Omega \\ \\ u_{p,m}^m = 0 & \text{on } (0,\infty) \times \partial\Omega \\ \\ u_{p,m}(0,\cdot) = u_0(\cdot) & \text{on } \Omega, \end{cases} \tag{2.1}$$

where

$$\Delta_p w = \operatorname{div}(|\nabla w|^{p-2} \nabla w)$$

is the p-Laplace operator. We will devote this section to identifying the limit u_m of the family of solutions $u_{p,m}$ of $(DNE)_{p,m}$, as p goes to infinity, as well as the equation satisfied at the limit by u_m. The main problem will be to find a suitable topology, which must be weak enough and therefore have sufficient compactness properties to allow us to pass to the limit, but at the same time be strong enough to pass important properties onto the function u_m, which enable us to find the equation that is satisfied at the limit and in what sense.

2.1 Asymptotic limit for the (PLE)

For $m = 1$, the equation in $(DNE)_{p,m}$ reduces to the parabolic p-Laplace equation (PLE)

$$(u_p)_t - \Delta_p u_p = g.$$

Therefore, we will give a short summary of the main results in terms of the asymptotic limit of solutions of initial-value problems for this equation, when p goes to infinity.

As already highlighted by Aronsson et al. in [4], the p-Laplacian is a prototype of a "fast/slow" diffusion operator, in the sense that its nonlinear diffusion coefficient

$|\nabla u|^{p-2}$, for large but finite values of p, is very large within the region $\{|\nabla u| > 1+\delta\}$ for each small $\delta > 0$, and therefore there is a very rapid movement of mass, whereas it is very small within the region $\{|\nabla u| < 1 - \delta\}$, which implies there is very little material being transported. The set $\{1 - \delta \le |\nabla u| \le 1 + \delta\}$ is an intermediate zone. The diffusion coefficient tends to infinity above the level $\{|\nabla u| = 1\}$ and to zero below it, as p tends to infinity, and we encounter at the limit an "infinitely fast/infinitely slow" diffusion operator, which is interesting not only from a mathematical stand point, but also for its physical interpretation.

In 1995, the authors in [4] discovered that the highly nonlinear evolution problem for the p-Laplacian reduces to a much simpler problem in the limit that in fact has an important physical meaning, as it not only provided a mathematical framework for sandcone models proposed earlier by Aronsson [3], but it also helped in understanding the structure of sandcones growing and interacting with each other as they are being fed by point sources ([4], see also [41]). Also, other physical interpretations have been given to the evolution problem at the limit, for example, a Bean's critical-state model for type II superconductivity ([6], [43] and [44]) and river networks ([41], see also [24]). We refer to [17], [39] and [38] for the limiting behaviour of the variable exponent p-Laplacian and to [1] and [2] for the limit as $p \to \infty$ of the nonlocal analogous of the p-Laplace equation.

The authors in [4] considered the following Cauchy problem for the parabolic p-Laplace equation:

$$\begin{cases} (u_p)_t - \operatorname{div}\left(|\nabla u_p|^{p-2}\nabla u_p\right) = g_p & \text{in } (0,\infty)\times\mathbb{R}^N \\ \\ u_p = u_0 & \text{on } \{t=0\}\times\mathbb{R}^N, \end{cases} \tag{2.2}$$

where $N+1 \le p < \infty$ and u_0 is a Lipschitz function with compact support, satisfying

$$\|\nabla u_0\|_{L^\infty(\mathbb{R}^N)} \le 1. \tag{2.3}$$

Assuming the basic physical condition that a sandpile is stable if and only if its slope (determined by its resting angle) is everywhere less than or equal to one, the condition (2.3) for the distribution u_0, which represents the height of the sandpile at $t = 0$, implies that the initial sand heaps are stable.

The function g_p is smooth, with compact support in $\mathbb{R}^N \times [0, T]$ for each $T > 0$. Furthermore g_p is a smooth approximation to the time-varying measures

$$g = \sum_{k=1}^{m} g_k(t)\delta_{d_k}(x), \tag{2.4}$$

where δ_{d_k} denotes a Dirac mass at the point d_k and the functions $\{g_k\}_{k=1}^m$ are non-negative and Lipschitz. Here, g represents a given source term which physically can be interpreted as adding material to the evolution system. In this case, where g has the structure (2.4), the measure g records the point sources at the locations $\{d_k\}_{k=1}^m$ such that the sand is added to the pile at the rate $g_k(t) \geq 0$ for $t \geq 0$, $k = 1, \cdots, m$.

The following reinterpretation of the p-parabolic problem in (2.2), was used:

$$\begin{cases} g_p - (u_p)_t = \partial I_p(u_p) & \text{a.e. } t > 0 \\ \\ u_p = u_0 & t = 0, \end{cases}$$

where ∂I_p denotes the single-valued subdifferential of the functional I_p:

$$I_p(v) = \begin{cases} \dfrac{1}{p} \displaystyle\int_{\mathbb{R}^N} |\nabla v|^p dx & \text{if } v \in L^2(\mathbb{R}^N),\ |\nabla v| \in L^p(\mathbb{R}^N), \\ \\ +\infty & \text{otherwise.} \end{cases} \tag{2.5}$$

The passage to the limit under the conditions listed above was completely solved in [4] and the main results are summarized in the following proposition:

Proposition 2.1.1. Consider the Cauchy problem for the parabolic p-Laplacian in (2.2) with conditions on the initial value u_0 and source term g_p as explained above. Then we can extract a subsequence $\{p_i\}$, p_i tending to infinity, and a limit u such that, for each $T > 0$,

$$\begin{cases} u_{p_i} \to u \text{ a.e. and in } L^2(0, T; \mathbb{R}^N) \\ \\ \nabla u_{p_i} \rightharpoonup \nabla u, \quad (u_{p_i})_t \rightharpoonup u_t \text{ weakly in } L^2(0, T; \mathbb{R}^N), \end{cases} \tag{2.6}$$

and the limit function u satisfies

$$\begin{cases} g - u_t \in \partial I_\infty(u) & \text{a.e. } t > 0 \\ \\ u = u_0 & t = 0, \end{cases} \tag{2.7}$$

where ∂I_∞ is the subdifferential of the convex functional

$$I_\infty(v) = \begin{cases} 0 & \text{if } v \in \mathbb{K}, \\ \\ +\infty & \text{otherwise,} \end{cases}$$

for

$$\mathbb{K} = \{w \in L^2(\mathbb{R}^N) : \ |\nabla w| \leq 1 \text{ a.e.}\}.$$

Moreover the problem in (2.7) has a unique solution u with the explicit form

$$u(t,x) = \max\left(0, z_1(t) - |x-d_1|, \cdots, z_m(t) - |x-d_m|\right),$$

where the non-negative height functions $\{z_k(t)\}_{k=1}^m$ satisfy a certain coupled system of ODEs. It then follows that u determines the height of a pile of noncohesive sand which will grow continuously as long as the slope does not exceed one. As more sand falls into the pile, added by the source term g, the slope locally increases until it reaches the peak value one, beyond which the pile becomes unstable and suddenly pours down.

We can also interpret these results using the terminology of non-linear semigroup theory introduced in the Preliminaries as done in [14]. Taking $X = L^2(\mathbb{R}^N)$, let us define the m-accretive operator A_p by

$$A_p u := -\Delta_p u = \partial I_p(u),$$

for u belonging to

$$D(A_p) = \{u \in L^2(\mathbb{R}^N) : \ \Delta_p u \in L^2(\mathbb{R}^N)\}.$$

Let us also define

$$A_\infty(u) := \partial I_\infty(u),$$

where

$$\overline{D(A_\infty)} = \mathbb{K}.$$

Note that since $u_0 \in \overline{D(A_\infty)}$, by theorem 1.5.3 of the Preliminaries, the result of proposition 2.1.1 reduces to proving that

$$(I + A_p)^{-1} f \to (I + A_\infty)^{-1} f \ \text{ in } \ L^2(\mathbb{R}^N), \tag{2.8}$$

which follows by [14]. In other words, if we consider the stationary problem associated to the operator A_p, $z_p := J_\lambda^{A_p}(f)$, for $f \in L^2(\mathbb{R}^N)$, and considering without loss of generality $\lambda = 1$, i.e.,

$$z_p + A_p(z_p) = f \ \text{ in } \ \mathbb{R}^N,$$

it would be enough to show that there exists a limit function z such that

$$z_p \to z \ \text{ in } \ L^2(\mathbb{R}^N),$$

and z is the unique solution of

$$f \in z + A_\infty(z).$$

Note also that, by the definition of I_∞, we have that

$$v \in \partial I_\infty(u) \Longleftrightarrow u \in \mathbb{K} \text{ and } 0 \geq \int v(w-u)dx, \ \forall\, w \in \mathbb{K}.$$

We can conclude that if $f \in \mathbb{K}$, then

$$(I + A_p)^{-1} f \to f \text{ in } L^2(\mathbb{R}^N).$$

It follows, considering $g \equiv 0$ and $u_0 \in \mathbb{K}$, that the unique solution of (2.7) is the trivial solution $u \equiv u_0$, which is of course time-independent.

Let us now present a shortened version of the main results in [27], where once again the Cauchy problem for the parabolic p-Laplacian is considered. However, unlike in [4], Evans, Feldman and Gariepy considered an initial condition u_0 that does not belong to $\mathbb{K}$ and $g \equiv 0$, i.e.,

$$\begin{cases} (u_p)_t - \Delta_p u_p = 0 & \text{in } (0,\infty) \times \mathbb{R}^N \\ \\ u_p = u_0 & \text{on } \{t = 0\} \times \mathbb{R}^N, \end{cases} \tag{2.9}$$

where $u_0 : \mathbb{R}^N \to \mathbb{R}$ has compact support, is nonnegative and Lipschitz, with

$$||\nabla u_0||_{L^\infty(\mathbb{R}^N)} = L > 1. \tag{2.10}$$

Therefore, in this case, there is no source term, which means that no more sand will be added to the pile and condition (2.10) implies that the authors consider an initial configuration which is unstable.

The following result for the limit, as p goes to infinity, was proved.

Proposition 2.1.2. Consider the initial value problem in (2.9) with initial value u_0 as described above. Then, there exists a subsequence $p_j \to \infty$ and a Lipschitz function u such that

$$\begin{cases} u_{p_j} \to u \text{ uniformly on compact subsets of } [0,\infty) \times \mathbb{R}^N, \\ \\ \nabla u_{p_j} \rightharpoonup \nabla u \text{ weakly star in } L^\infty((0,T) \times \mathbb{R}^N), \\ \\ (u_{p_j})_t \rightharpoonup u_t \text{ weakly star } L^\infty((0,T) \times \mathbb{R}^N), \end{cases}$$

and the limit function u satisfies

$$|\nabla u| \leq 1 \text{ a.e.} \tag{2.11}$$

A function v_j that "stretches" the time variable is defined as follows

$$v_j(t,x) = t u_{p_j}\left(\frac{t^{p_j-1}}{p_j-1}, x\right), \quad \text{for } 0 \le t \le 1, \; x \in \mathbb{R}^N. \tag{2.12}$$

By (2.9), v_j solves

$$(v_j)_t - \Delta_p v_j = \frac{v_j}{t} \text{ in } (\tau, 1) \times \mathbb{R}^N, \tag{2.13}$$

where $\tau = \frac{1}{L}$, and the authors were able to conclude that there exists a function v such that

$$v_j \to v \text{ uniformly on } [\tau, 1] \times \mathbb{R}^N, \tag{2.14}$$

and v solves

$$\begin{cases} \dfrac{v}{t} - v_t \in \partial I_\infty(v) & (\tau \le t \le 1) \\ \\ v = v_0 & (t = \tau), \end{cases} \tag{2.15}$$

where $v_0 = \tau u_0$. Finally, by the time transformation in (2.12), (2.14) and the uniform convergence of $u_{p_j}\left(\frac{1}{p_j-1}, x\right)$ to u, it was obtained that

$$u(x) = v(1,x).$$

As the authors pointed out, physically it is natural that the sand particles would rapidly reorganize themselves to reach a state of stability, represented by u, which is independent of the initial conditions. Once the critical state has been reached, the motion stops. The mapping $u_0 \mapsto u$ records the final state of repose of the sandpile after various avalanches.

Remark 2.1.1. The subdifferential $\partial I_\infty(u)$ is not defined for $u \notin \mathbb{K}$. Since $u_0 \notin \overline{D(A_\infty)} = \overline{D(\partial I_\infty)}$ and yet a limit of the family of solutions u_p as p tends to infinity exists, we are dealing with what is called a singular perturbation problem. The singularity arises as a boundary layer appears in a neighbourhood of $t = 0$ in the passage to the limit. This boundary layer is explained as a period during which the solution rapidly changes before reaching its stable profile. In the particular case of the parabolic p-Laplacian in (2.9) this is seen clearly, as the property of the initial value (2.10) is not compatible with the property (2.11) of the limiting function u.

The main tool used to overcome this incompatibility and the more delicate transformation of $u_0 \mapsto u$ is the stretching of the time variable in (2.12). Note that it allows a regularization of the problem in the sense that the new problem (2.13) satisfies the conditions of proposition 2.1.1. More specifically the initial value $v_j(\tau) \to \tau u_0 \in \mathbb{K}$. Such a scaling argument was generalized in [14] to the setting of abstract nonlinear evolution equations governed by homogeneous accretive operators.

Remark 2.1.2. It is known that the resolvent operator $(I+\lambda\partial I_\infty)^{-1}$ is equal to $Proj_{\mathbb{K}}$ which denotes the projection onto the closed set $\mathbb{K}$. Therefore

$$(I+\lambda\partial I_\infty)^{-1}u_0 = Proj_{\mathbb{K}}u_0.$$

However, it was proved in [27] that there exists an initial data u_0 with compact support such that

$$Proj_{\mathbb{K}}u_0 \neq v(x,1).$$

Therefore the sequence of solutions u_p converges to a time-independent profile $u(x) = v(x,1)$ which is not the projection of the initial data u_0 onto the closure of the domain of A_∞.

2.2 Properties of solutions of the doubly nonlinear diffusion equation

We will now review some results regarding the existence and properties of solutions of the problem $(DNE)_{p,m}$ in (2.1), proved in [33].

Let us define the nonlinear operator $A_{p,m}$ in $L^1(\Omega)$ by

$$\begin{gathered} A_{p,m}u = -\Delta_p u^m, \\ \mathrm{D}\left(A_{p,m}\right) = \{\ u \in L^\infty(\Omega):\ u^m \in W_0^{1,p}(\Omega) \text{ and } \ \Delta_p u^m \in L^1(\Omega)\ \}, \end{gathered} \tag{2.16}$$

where r^m denotes $|r|^{m-1}r$ for all $r \in \mathbb{R}$.

Let us now consider the functional $\Phi_p : L^2(\Omega) \to [0,+\infty]$ defined as follows

$$\Phi_p(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla u|^p dx & \text{if } \ u \in W_0^{1,p}(\Omega) \cap L^2(\Omega) \\ \\ +\infty & \text{otherwise.} \end{cases} \tag{2.17}$$

The functional Φ_p is convex, proper, l.s.c, $\Phi_p(0) = 0$ and for all $h \in H_0$, where H_0 is given by

$$H_0 := \{h \in C^1(\mathbb{R}):\ h(0) = 0 \text{ and } 0 \leq h' \leq 1\}, \tag{2.18}$$

we have

$$\Phi_p(w + h(\hat{w} - w)) + \Phi_p(\hat{w} - h(\hat{w} - w)) \leq \Phi_p(w) + \Phi_p(\hat{w}),\ \forall\, w, \hat{w} \in L^2(\Omega). \tag{2.19}$$

As already noted in [33], $v = A_{p,m}(u)$ in $\mathcal{D}'(\Omega)$ if and only if

$$\begin{cases} u \in L^\infty(\Omega), \ u^m \in W_0^{1,p}(\Omega) \cap L^2(\Omega) \\ \\ \Phi_p(\eta) \geq \Phi_p(u^m) + \displaystyle\int_\Omega v(\eta - u^m)dx, \ \forall\, \eta \in L^\infty(\Omega). \end{cases} \tag{2.20}$$

The following proposition, proved in [8], holds for more general operators in domains not necessarily bounded. For completeness, we will provide a simplified version which applies to the operator $A_{p,m}$ in a bounded domain.

Proposition 2.2.1. For all $m \geq 1$, we have:

(i) $A_{p,m}$ is T-accretive in $L^1(\Omega)$.

(ii) $R(I + \lambda A_{p,m}) \supseteq L^\infty(\Omega)$ for all $\lambda > 0$.

(iii) $D(A_{p,m})$ is dense in $L^1(\Omega)$.

(iv) $(I + \lambda A_{p,m})^{-1} f \ll f$ for all $\lambda > 0$ and $f \in L^\infty(\Omega)$.

(v) For all $u, \hat{u} \in D(A_{p,m})$, we have

$$\int_\Omega (A_{p,m}u - A_{p,m}\hat{u})h(u^m - \hat{u}^m) \geq 0, \ \forall\, h \in H_0,$$

where H_0 is given by (2.18).

Proof. Let us consider $u \in D(A_{p,m})$, then $u^m + h(\gamma - u^m) \in L^\infty(\Omega)$ for $h \in H_0$ and any $\gamma \in L^2(\Omega)$. We have by (2.20), that for $v = A_{p,m}u$,

$$\Phi_p(u^m + h(\gamma - u^m)) \geq \Phi_p(u^m) + \int vh(\gamma - u^m). \tag{2.21}$$

Using as well (2.19), with $w = u^m$ and $\hat{w} = \gamma$, we obtain that

$$\Phi_p(\gamma) \geq \Phi_p(\gamma - h(\gamma - u^m)) + \int vh(\gamma - u^m). \tag{2.22}$$

Similarly, taking $\hat{v} = A_{p,m}\hat{u}$ and $\hat{h}(r) = -h(-r)$, we have that

$$\Phi_p(\gamma) \geq \Phi_p(\gamma - \hat{h}(\gamma - \hat{u}^m)) + \int \hat{v}\hat{h}(\gamma - \hat{u}^m), \ \forall\, \gamma\, L^2(\Omega).$$

Considering in the previous expression $\gamma = u^m$ then

$$\Phi_p(u^m) \geq \Phi_p(u^m + h(\hat{u}^m - u^m)) - \int \hat{v}h(\hat{u}^m - u^m). \tag{2.23}$$

Therefore, by (2.21) with $\gamma = \hat{u}^m$ and (2.23), we have

$$\begin{aligned} \int v h(\hat{u}^m - u^m) &\leq \Phi_p(u^m + h(\hat{u}^m - u^m)) - \Phi_p(u^m) \\ &\leq \int \hat{v} h(\hat{u}^m - u^m), \end{aligned}$$

and (v) is satisfied.

Take h as a smooth approximation of $(r-\epsilon)^+ \wedge \epsilon$, divide by $\epsilon > 0$ and let $\epsilon \to 0$ to obtain

$$\int_{\{\hat{u}>u\}} \hat{v} - v = \int (\hat{v} - v)\mathrm{sign}_0^+(\hat{u}^m - u^m) > 0.$$

By proposition 1.2.2, (i) is satisfied.

Let us now denote $u := (I+\lambda A_{p,m})^{-1} f$ and $v = A_{p,m}u$, then $f = u + \lambda v$. Therefore, to prove (iv), we need to see that

$$u \ll u + \lambda v \quad \forall\, \lambda > 0.$$

By proposition 2.2 in [12], it is enough to prove that

$$\int_{\{u<-k\}} v \leq 0 \leq \int_{\{u>k\}} v, \ \text{ for } \ k > 0. \tag{2.24}$$

By (2.22), with $\hat{h}(r) = -h(-r)$,

$$\Phi_p(\gamma) > \Phi_p(\gamma - \hat{h}(\gamma - u^m)) + \int v\hat{h}(\gamma - u^m).$$

Then, taking $\gamma = 0$,

$$\int h(u^m)v > \Phi_p(h(u^m)) \geq 0.$$

Consider h as a smooth approximation of $(r-k^m)^+ \wedge \epsilon$ (resp. $-[\,(-r+(-k)^m)^+ \wedge \epsilon\,]$) divide by $\epsilon > 0$ and let $\epsilon \to 0$. Thus (2.24) is proved.

By standard variational arguments, we know that for all $f \in L^2(\Omega)$ there exists a unique solution $u \in L^2(\Omega)$ such that $u^m \in W_0^{1,p}(\Omega)$ and it satisfies

$$f = u - \Delta_p u^m \ \text{ in } \ \mathcal{D}'(\Omega).$$

Then, considering $f \in L^\infty(\Omega)$, by (iv), we have that $\|u\|_\infty \leq \|f\|_\infty$ and (ii) follows.

To prove (iii), it is enough to show that

$$\{u \in L^\infty(\Omega) : \Phi_p(u^m) < \infty\} \subseteq \overline{D(A_{p,m})}.$$

By (ii), for an arbitrary $u \in L^\infty(\Omega)$, there exists a $u_\lambda \in D(A_{p,m})$ such that $u_\lambda = (I+\lambda A_{p,m})^{-1}u$. We have also by (iv) that $\|u_\lambda\|_p \leq \|u\|_p$ for $1 < p < \infty$. Then $u_\lambda \rightharpoonup \overline{u}$

in $L^p(\Omega)$ for any $1 < p < \infty$. In this case, by the definition of u_λ, $v = \dfrac{u - u_\lambda}{\lambda} = A_{p,m}u$, and therefore by (2.20)

$$\Phi_p(u^m) \geq \Phi_p(u_\lambda^m) + \int \frac{u - u_\lambda}{\lambda}(u^m - u_\lambda^m).$$

Taking $\lambda \to 0$, we have

$$\int (u - \overline{u})(u^m - \overline{u}^m) \leq 0,$$

and so $u = \overline{u}$. Thus $u_\lambda \to u$ in $L^1(\Omega)$. □

By the previous proposition, $\overline{A}_{p,m}$ (the closure of $A_{p,m}$ in $L^1(\Omega)$) is m-T-accretive in $L^1(\Omega)$ and by standard nonlinear semigroup theory results we have the following result.

Corollary 2.2.2. For all $u_0 \in L^1(\Omega)$, $T > 0$ and $g \in L^1(\Omega_T)$, there exists a unique mild solution of the following evolution problem

$$\begin{cases} u_t + A_{p,m}u = g \quad \text{on } [0, T), \\ \\ u(0) = u_0, \end{cases} \tag{2.25}$$

and therefore of $(DNE)_{p,m}$.

Furthermore $\overline{A}_{p,m}$ generates a nonlinear semigroup of contraction in $L^1(\Omega)$ denoted by $S_{p,m}(t)$. Using also the regularity of mild solutions, it was proved as well in [33] that $(DNE)_{p,m}$ for $g \equiv 0$ has a solution in the following sense.

Proposition 2.2.3. Given $u_0 \in L^1(\Omega)$, $1 < p < \infty$, $m > \dfrac{1}{p-1}$ and $g \equiv 0$, we have

(i) $u(t) = S_{p,m}(t)u_0$ is the unique solution of $(DNE)_{p,m}$ in the following sense:

$$\begin{cases} u \in C(0, \infty; L^1(\Omega)) \cap L^\infty([\delta, \infty) \times \Omega) \cap W^{1,\infty}(\delta, \infty; L^1(\Omega)) \\ \\ u^m \in L^\infty(\delta, \infty; W_0^{1,p}(\Omega)),\ \forall \delta > 0,\ u_m(0) = u_0 \\ \\ \dfrac{d}{dt}\displaystyle\int_\Omega uv + \int_\Omega |\nabla u^m|^{p-2}\nabla u^m \cdot \nabla v = 0 \ \text{ in } \mathcal{D}'(0, \infty),\ \forall\, v \in W_0^{1,p}(\Omega). \end{cases} \tag{2.26}$$

(ii) If $u_0, \hat{u}_0 \in L^1(\Omega)$ and $u, \hat{u}$ are the corresponding solutions, then

$$\int_\Omega (u(t) - \hat{u}(t))^+ dx \leq \int_\Omega (u_0 - \hat{u}_0)^+ dx, \quad \forall\, t > 0.$$

2.3 Regular limit of the (DNE) when $p \to \infty$

In the present section, we generalize the results in [4] to the doubly nonlinear diffusion problem $(\mathrm{DNE})_{p,m}$ in (2.1), using the nonlinear semigroup approach.

We see, by (2.20), that the operator $A_{p,m}$ continues to "act as a subdifferential" even when $m \neq 1$. It seems reasonable then, that when p goes to infinity, the operator $A_{\infty,m}$, obtained at the limit, will also "act as a subdifferential" of an indicator function of a convex set $\tilde{\mathbb{K}}$. Indeed, we know by (2.8) that

$$I_\infty \subseteq \liminf_{p\to\infty} I_p. \tag{2.27}$$

Let us now define $\Phi_\infty : L^1(\Omega) \to [0, +\infty]$ by

$$\Phi_\infty(u) = \begin{cases} 0 & u \in \tilde{\mathbb{K}} = \{\xi : |\nabla \xi| \leq 1 \ \text{ a.e.}\} \\ \\ +\infty & \text{otherwise.} \end{cases}$$

Then

$$\partial_{L^1}\Phi_\infty = \left\{(u,v) \in \tilde{\mathbb{K}} \times L^1(\Omega) : \ \int (u-w)v \geq 0 \ \text{ for } \ w \in \tilde{\mathbb{K}} \ \text{ with } \ (u-w)v \in L^1(\Omega)\right\}.$$

We would expect $A_{p,m}$ to converge to some operator $A_{\infty,m}$ that acts as $\partial_{L^1}\Phi_\infty(u^m)$. Actually, we will prove that $A_{\infty,m}$ behaves as follows:

$$v \in A_{\infty,m}u \Longleftrightarrow \begin{cases} u, v \in L^1(\Omega), \ u^m \in W^{1,\infty}(\Omega) \cap C_0(\overline{\Omega}), \\ \\ u^m \in \tilde{\mathbb{K}} \ \text{ and } \ 0 \geq \displaystyle\int_\Omega v(\xi - u^m)dx \ \ \forall \xi \in \tilde{\mathbb{K}}, \end{cases} \tag{2.28}$$

where

$$\tilde{\mathbb{K}} := \{\xi \in L^1(\Omega) : \ |\nabla \xi| \leq 1 \ \text{ a.e.}\}. \tag{2.29}$$

Let us at this point focus on the stationary equation associated to the operator $A_{p,m}$, i.e., $z_{p,m} := (I + A_{p,m})^{-1} f$ for $f \in L^\infty(\Omega)$, since we will be interested in the properties of the resolvent operator to pass to the limit.

By proposition 2.2.1, we see that for every $f \in L^\infty(\Omega)$, $z_{p,m}$ is the unique solution of the problem

$$\begin{cases} z_{p,m} - \Delta_p z_{p,m}^m = f & \text{on } \Omega \\ \\ z_{p,m} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.30}$$

in the following sense

$$\begin{cases} z_{p,m} \in L^\infty(\Omega),\ \ z_{p,m}^m \in W_0^{1,p}(\Omega) \ \text{ and} \\ \\ -\Delta_p z_{p,m}^m = f - z_{p,m} \,\text{in }\ \mathcal{D}'(\Omega). \end{cases} \tag{2.31}$$

We will need the following a priori estimates, uniform with respect to p, for the elliptic equation in (2.30) associated to the problem $(DNE)_{p,m}$.

Lemma 2.3.1. If $z_{p,m}$ is the solution of (2.30) for $f \in L^\infty(\Omega)$, then $z_{p,m}^m$ is uniformly bounded in $W_0^{1,q}(\Omega)$ for any $q > 1$.

Proof. Let us denote $z_{p,m}$ simply by z_p. We know by (2.31) that z_p satisfies

$$\int_\Omega |\nabla z_p^m|^{p-2}\nabla z_p^m \cdot \nabla\varphi dx = \int_\Omega (f - z_p)\varphi dx, \quad \forall\, \varphi \in \mathcal{D}(\Omega). \tag{2.32}$$

By density, (2.32) continues to hold for all $\varphi \in W_0^{1,p}(\Omega)$ and we can substitute $\varphi = z_p^m$ in the previous expression. Then:

$$\begin{aligned} \int_\Omega |\nabla z_p^m|^p dx &= \int_\Omega (f - z_p) z_p^m dx \leq ||f - z_p||_{L^\infty(\Omega)} ||z_p^m||_{L^1(\Omega)} \\ \\ &\leq C||f - z_p||_{L^\infty(\Omega)} ||\nabla z_p^m||_{L^1(\Omega)} \\ \\ &\leq 2C||f||_{L^\infty(\Omega)} \int_\Omega |\nabla z_p^m| dx \\ &\leq C||f||_{L^\infty(\Omega)} \left(\int_\Omega |\nabla z_p^m|^p\right)^{1/p} |\Omega|^{1-1/p}. \end{aligned}$$

The second inequality is due to Poincaré's inequality, with $p = 1$, and the third is due to the fact that $z_p \ll f$ by proposition 2.2.1 (iv), since then we have that $||z_p||_{L^\infty} \leq ||f||_{L^\infty}$. Therefore

$$\left(\int_\Omega |\nabla z_p^m|^p dx\right)^{1/p} \leq (C||f||_{L^\infty(\Omega)})^{1/(p-1)} |\Omega|^{1/p}.$$

On the other hand, by Hölder's inequality,

$$||\nabla z_p^m||_q \leq ||\nabla z_p^m||_p |\Omega|^{1/q - 1/p},$$

for any $p > q$ and we finally obtain that $\{z_p^m\}$ is uniformly bounded in $W_0^{1,q}(\Omega)$ for any $q > 1$. $\square$

As a result of lemma 2.3.1, for a subsequence $\{p_i\}$, there exists a function w_m such that, when $p_i \to \infty$,

$$z_{p_i,m}^m \rightharpoonup w_m \ \text{ in } \ W^{1,q}(\Omega) \text{ for any } q > 1. \tag{2.33}$$

Moreover, by Sobolev embedding,

$$z_{p_i,m}^m \to w_m \ \text{ in } \ L^q(\Omega) \ \text{ as } \ p \to +\infty. \tag{2.34}$$

Hence, passing if necessary to yet another subsequence, we would get

$$z_{p_i,m}^m \to w_m \ \text{ a.e..} \tag{2.35}$$

Denoting $z_m := w_m^{1/m}$, then

$$z_{p_i,m} \to z_m \ \text{ a.e..} \tag{2.36}$$

By (2.33), $z_m^m \in W^{1,\infty}(\Omega) \cap C_0(\overline{\Omega})$, since taking $q > N$, $W_0^{1,q}(\Omega) = W^{1,q}(\Omega) \cap C_0(\overline{\Omega})$. We will also need the following property of the limiting function z_m.

Lemma 2.3.2. The limit function z_m in (2.36) satisfies the following estimate

$$\|\nabla z_m^m\|_{L^\infty(\Omega)} \leq 1. \tag{2.37}$$

Proof. We can prove this in exactly the same way as lemma 3.2 in [27]. Fixing $\eta > 0$, and denoting

$$A_\eta = \{x \in \Omega \mid |\nabla z_m^m| \geq 1 + \eta\},$$

then

$$\begin{aligned} (1+\eta)|A_\eta| \ &\leq \ \int_{A_\eta} |\nabla z_m^m| dx \ \leq \ \liminf_{p_i \to +\infty} \int_{A_\eta} |D z_{p_i,m}^m| dx && (2.38) \\ &\leq \ \liminf_{p_i \to +\infty} \left(\int_\Omega |\nabla z_{p_i,m}^m|^{p_i} dx \right)^{1/p_i} |A_\eta|^{1-1/p_i} \leq |A_\eta|, && (2.39) \end{aligned}$$

where the last inequality is a consequence of lemma 2.3.1. Therefore $|A_\eta| = 0$ and (2.37) is satisfied. □

To apply the classical results of the nonlinear semigroup theory we will need the following result.

Lemma 2.3.3. For all $f \in L^\infty(\Omega)$ and $\lambda > 0$, we obtain, when $p \to \infty$,

$$(I + \lambda A_{p,m})^{-1} f \to (I + \lambda A_{\infty,m})^{-1} f \ \text{ in } \ L^1(\Omega). \tag{2.40}$$

Proof. We need to check that there exists a function z_m such that

$$z_{p,m} \to z_m \text{ in } L^1(\Omega), \tag{2.41}$$

and z_m is the unique solution of

$$f - z_m \in A_{\infty,m}(z_m), \ \forall f \in L^\infty(\Omega). \tag{2.42}$$

Let us consider $\xi \in \tilde{\mathbb{K}}$. By approximation, we may assume that ξ has compact support. Therefore, since $f - z_{p,m} \in A_{p,m}(z_{p,m})$, according to (2.17) and (2.20),

$$\begin{aligned} \frac{1}{p}\int_\Omega |\nabla \xi|^p dx &\geq \frac{1}{p}\int_\Omega |\nabla z_{p,m}^m|^p dx + \int_\Omega (f - z_{p,m})(\xi - z_{p,m}^m) dx \\ &\geq \int_\Omega (f - z_{p,m})(\xi - z_{p,m}^m) dx. \end{aligned}$$

It follows, taking the limit as p goes to infinity, that

$$0 \geq \lim_{p\to\infty} \int_\Omega (f - z_{p,m})(\xi - z_{p,m}^m).$$

Since $z_{p,m} \ll f$, then

$$\|z_{p,m}\|_r \leq \|f\|_r, \text{ for any } 1 \leq r \leq \infty. \tag{2.43}$$

Therefore in particular for q', the conjugate of q,

$$z_{p,m} \rightharpoonup z_m \text{ in } L^{q'}(\Omega).$$

We have as well, by (2.34), that

$$z_{p,m}^m \to z_m^m \text{ in } L^q(\Omega), \ \forall q > 1.$$

Then, when $p \to \infty$,

$$0 \geq \int_\Omega (f - z_m)(\xi - z_m^m) dx, \quad \forall \xi \in \tilde{\mathbb{K}}, \tag{2.44}$$

which proves (2.42); (2.41) follows from (2.36) and (2.43).

Let us prove the uniqueness of z_m. Let us suppose that both $z_{m,1}, z_{m,2}$ satisfy the limit equation, i.e.,

$$0 \geq \int_\Omega (f - z_{m,1})(\xi - z_{m,1}^m) dx \tag{2.45}$$

and

$$0 \geq \int_\Omega (f - z_{m,2})(\xi - z_{m,2}^m) dx. \tag{2.46}$$

We then substitute $\xi = z_{m,i}^m$, $i = 1, 2$, in the respective inequalities. Notice that by lemma 2.3.2, $z_m^m \in \tilde{\mathbb{K}}$ and this is a valid choice. Add (2.45) to (2.46) to obtain

$$\int_\Omega (z_{m,1} - z_{m,2})(z_{m,1}^m - z_{m,2}^m)dx \leq 0$$

which would then give us $z_{m,1} = z_{m,2}$. $\square$

Corollary 2.3.4. The operator $\overline{A}_{\infty,m}$ is m-T-accretive in $L^1(\Omega)$; furthermore for all $f \in L^1(\Omega)$ and $\lambda > 0$, when $p \to \infty$,

$$(I + \lambda \overline{A}_{p,m})^{-1} f \to (I + \lambda \overline{A}_{\infty,m})^{-1} f \text{ in } L^1(\Omega). \tag{2.47}$$

Proof. By lemma 2.3.3 and proposition 1.5.5, it is enough to prove that $A_\infty^{(m)}$ is T-accretive. Let us denote by $A^{(1)} := \partial_{L^1}\Phi_\infty$ and considering $\phi_m(r) = |r|^{m-1}r$ then

$$A_{\infty,m} \subseteq A^{(1)} \circ \phi_m,$$

in the sense that

$$v \in A_{\infty,m}u \Longrightarrow u, v \in L^1(\Omega),\ \exists\, w \in \phi_m(u) \text{ such that } v \in A^{(1)}w.$$

We also have that

$$u,\ \hat{u} \in \tilde{\mathbb{K}},\ \ h \in H_0 \Longrightarrow u + h(\hat{u} - u) \in \tilde{\mathbb{K}},$$

therefore $A^{(1)}$ is a completely accretive operator by lemma 1.3.2, and since ϕ_m is a maximal monotone graph, then $A_{\infty,m}$ is T- accretive by proposition 1.3.3. $\square$

The main theorem in this section then follows.

Theorem 2.3.5. Consider the problem $(DNE)_{p,m}$ in (2.1), where Ω is a bounded domain in $\mathbb{R}^N$, $u_0 \in L^1(\Omega)$, $u_0^m \in \tilde{\mathbb{K}}$ and $g \in L^1(\Omega_T)$. Then there exists a subsequence p_i, tending to infinity, and a function u_m such that, for each $T > 0$,

$$u_{p_i,m} \to u_m \text{ in } C([0,T]; L^1(\Omega)),$$

and u_m is the unique mild solution of

$$\begin{cases} g - (u_m)_t \in A_{\infty,m}(u_m) & \text{in } (0,T] \times \Omega \\ \\ u_m = u_0, & \{t = 0\} \times \Omega, \end{cases} \tag{2.48}$$

where $A_{\infty,m}$ is given by (2.28)-(2.29).

Proof. By corollary 2.3.4, we have the convergence in the resolvent sense of the m-accretive operator $\overline{A}_{p,m}$ to the m-accretive operator $\overline{A}_{\infty,\infty}$, as p tends to infinity. Since we also have that $u_0 \in \overline{D(A_{\infty,m})}$, all the hypotheses of theorem 1.5.3 are fulfilled and the result follows. $\square$

In the particular case that $g \equiv 0$ and $u_0^m \in \tilde{\mathbb{K}}$, then the family of solutions of $(DNE)_{p,m}$ converges to the initial data by the following results.

Proposition 2.3.6. For all $f \in L^1(\Omega)$ such that $f^m \in \tilde{\mathbb{K}}$, then as $p \to \infty$, we obtain

$$(I + \lambda A_{p,m})^{-1} f \to f \text{ in } L^1(\Omega),$$

for all $\lambda > 0$.

Proof. If $z_{p,m}$ is a solution of (2.30), we know by (2.17) and (2.20), that by approximation, for all $\xi \in \tilde{\mathbb{K}}$,

$$\frac{1}{p}\int |\nabla \xi|^p dx \geq \frac{1}{p}\int |\nabla z_{p,m}^m|^p dx + \int (f - z_{p,m})(\xi - z_{p,m}^m)dx.$$

Considering $\xi = f^m$, then

$$\frac{1}{p}\int |\nabla f^m|^p dx \geq \int (f - z_{p,m})(f^m - z_{p,m}^m).$$

Take $p \to \infty$ to obtain

$$0 \geq \lim_{p\to\infty} \int (f - z_{p,m})(f^m - z_{p,m}^m)dx.$$

Since $z_{p,m} \ll f$, we know as well that there exists a function z_m such that

$$\begin{cases} z_{p,m} \rightharpoonup z_m \text{ in } L^{q'}(\Omega), \\ \\ z_{p,m}^m \to z_m^m \text{ in } L^q(\Omega), \end{cases}$$

for any $1 < q < \infty$. Hence

$$0 \geq \int (f - z_m)(f^m - z_m^m)dx,$$

and $f = z_m$ a.e. in Ω. $\square$

Lemma 2.3.7. Let $m > \dfrac{1}{p-1}$, $u_0 \in L^1(\Omega)$ and $u_{p,m}$ a solution of $(DNE)_{p,m}$ with $g \equiv 0$. If $u_0^m \in \tilde{\mathbb{K}}$ then, when $p \to \infty$, we have

$$u_{p,m} \to u_0 \text{ in } C(0,T; L^1(\Omega)).$$

Proof. Let us consider the m-accretive operator A_m defined by

$$A_m u = 0, \text{ and } D(A_m) = \{u \in L^1(\Omega) : \ u^m \in \tilde{\mathbb{K}}\}.$$

Then, by the previous proposition, for all $u_m \in D(A_m)$,

$$(I + \lambda A_{p,m})^{-1} u_m \to u_m \text{ in } L^1(\Omega),$$

and therefore

$$A_m \subseteq \liminf_{p\to\infty} A_{p,m}.$$

Hence, by theorem 1.5.2, if $u_0^m \in \tilde{\mathbb{K}}$ then

$$u_{p,m} \to \underline{u} \text{ in } C(0,T;L^1(\Omega)),$$

where $\underline{u}$ is the unique mild solution of

$$\begin{cases} u_t + A_m u = 0 \text{ in } (0,\infty) \\ \\ u(0) = u_0. \end{cases}$$

Thus $\underline{u} \equiv u_0$. □

2.4 Singular limit of the (DNE) when $p \to \infty$

Here, we generalize the results in [27] for the problem $(DNE)_{p,m}$ in (2.1) and therefore study the behaviour at the limit when $u_0^m \notin \tilde{\mathbb{K}}$. For this, we will consider the natural rescaling, taking into account what has been done for the problem $(DNE)_{p,m}$, when $m = 1$ in [27] and for $p = 2$ in [16], which is the following

$$v_{j,m}(t,x) = t u_{p_j,m}\left(\frac{t^{m(p_j-1)}}{m(p_j-1)}, x\right), \ (0 \le t \le 1). \tag{2.49}$$

We will consider the problem $(DNE)_{p,m}$, for which the source term $g \equiv 0$, u_0 is nonnegative and u_0^m is Lipschitz with

$$||\nabla u_0^m||_{L^\infty(\Omega)} = L > 1.$$

Considering now $u \in D(A_{p,m})$ and $\lambda > 0$, we have that

$$\lambda u \in D(A_{p,m}) \text{ and } A_{p,m}(\lambda u) = \lambda^\beta A_{p,m}(u),$$

where $\beta = m(p-1)$. Therefore $A_{p,m}$ is a homogeneous operator and since the proper rescaling and the definition of $A_{\infty,m}$ have been established, as well as the result of theorem 2.3.5, proposition 2.3.6, and lemma 2.3.7, the passage to the limit in this case is solved by the methods in [14]. I will recall for completeness the results there using the structure in [16].

We will need the following lemma.

Lemma 2.4.1. As $p_j \to \infty$, we have that, when $t < \tau$, where $\tau = \frac{1}{L^{1/m}}$,

$$u_{p_j,m}\left(\frac{t^{m(p_j-1)}}{m(p_j-1)}, x\right) \to u_0(x) \text{ in } L^1(\Omega).$$

Proof. Denote by $\tilde{u}_{p,m}$ the unique solution of $(DNE)_{p,m}$ with $g \equiv 0$ and initial condition $\tilde{u}_0 = \tau u_0$. Notice that in this case $\tilde{u}_0^m \in \tilde{\mathbb{K}}$. Then, by proposition 2.3.6, when $f^m \in \tilde{\mathbb{K}}$

$$z_{p,m} := (I + A_{p,m})^{-1} f \to f \text{ in } L^1(\Omega).$$

In particular, if $f = \tau u_0$, then as $p \to \infty$,

$$\frac{1}{\tau} z_{p,m} \to u_0, \tag{2.50}$$

and also

$$A_{p,m}(z_{p,m}) \to 0 \text{ in } L^1(\Omega). \tag{2.51}$$

Let us now denote $t_p := \dfrac{t^{m(p-1)}}{m(p-1)}$. We know, by the contractive property of the generated semigroup, that

$$\left\| e^{-t_p \overline{A}_{p,m}} u_0 - e^{-t_p \overline{A}_{p,m}} \frac{1}{\tau} z_{p,m} \right\|_1 \le \left\| u_0 - \frac{1}{\tau} z_{p,m} \right\|_1. \tag{2.52}$$

Using the homogeneity of the operator $A_p^{(m)}$, we also obtain

$$\begin{aligned} \left\| e^{-t_p \overline{A}_{p,m}} \frac{1}{\tau} z_{p,m} - \frac{1}{\tau} z_{p,m} \right\|_1 &\le \left\| t_p \overline{A}_{p,m}\left(\frac{1}{\tau} z_{p,m}\right) \right\|_1 \\ &\le \left\| \frac{1}{m(p-1)} \left(\frac{t}{\tau}\right)^{m(p-1)} \overline{A}_{p,m}(z_{p,m}) \right\|_1. \end{aligned} \tag{2.53}$$

Therefore by (2.52) and (2.53) and recalling the exponential representation of the solution $u_{p_j,m}(t_{p_j}) = e^{-t_{p_j} \overline{A}_{p_j,m}} u_0$

$$||u_{p_j,m}(t_{p_j}) - u_0||_1 \le 2 \left\| u_0 - \frac{1}{\tau} z_{p_j,m} \right\|_1 + \frac{1}{m(p_j-1)} \left(\frac{t}{\tau}\right)^{m(p_j-1)} ||\overline{A}_{p_j,m}(z_{p_j,m})||_1.$$

By (2.50) and (2.51) we get the desired convergence when $p_j \to \infty$. □

The main theorem reads as follows.

Theorem 2.4.2. Let $v_{j,m}$ be as in (2.49), where $u_{p_j,m}$ is the solution of the problem $(DNE)_{p,m}$ in (2.1), for $g \equiv 0$ and for initial condition u_0 as described above. Then, there exists a limit function v_m such that, when $j \to \infty$, we have

$$v_{j,m} \to v_m \ \text{ in } \ C(0,T; L^1(\Omega)),$$

and v_m satisfies the following properties:

(i) $v_m(t) = tu_0$ for any $t \in [0, \tau]$ where $\tau = 1/L^{1/m} < 1$,

(ii) v_m is the unique mild solution of the evolution problem

$$\begin{cases} \dfrac{v_m}{t} - (v_m)_t \in A_{\infty,m} v_m & (\tau < t \leq 1) \\ \\ v_m = v_0 = \tau u_0 & (t = \tau). \end{cases} \tag{2.54}$$

Proof. By the definition of $v_{j,m}$ in (2.49) and by $(DNE)_{p,m}$, $v_{j,m}$ satisfies

$$\begin{cases} (v_{j,m})_t - \Delta_p(v_{j,m}^m) = \dfrac{v_{j,m}}{t} & (\tau < t \leq 1) \\ \\ v_{j,m}(x, \tau) = \tau u_{p_j,m}(x, \tau_{p_j}) & (t = \tau). \end{cases}$$

By lemma 2.4.1,

$$v_{j,m}(x, \tau) \to \tau u_0,$$

when $j \to \infty$, and certainly $(\tau u_0)^m \in \tilde{\mathbb{K}}$. Therefore, since

$$(I + \lambda A_{p,m})^{-1} f \to (I + \lambda A_{\infty,m})^{-1} f, \quad \forall\, f \in L^1(\Omega),$$

holds by lemma 2.3.3, by theorem 1.5.4,

$$v_{j,m} \to v_m \ \text{ in } \ C(0,T; L^1(\Omega)),$$

and now it is clear that (i) holds and that v_m satisfies (2.54). $\square$

Corollary 2.4.3. Let $u_{p,m}$ be the solution of problem $(DNE)_{p,m}$ in (2.1), with $g \equiv 0$ and for initial conditions u_0 as described above. Then, there exists a subsequence $\{p_j\}$ such that, as $p_j \to \infty$, we have

$$u_{p_j,m}(t) \to v_m(1) \ \text{ in } \ L^1(\Omega),$$

where v_m is given by theorem 2.4.2.

Proof. It is straightforward, by the transformation in (2.49), that

$$u_{p_j,m}(t) = \frac{v_{j,m}((m(p_j-1)t)^{1/m(p_j-1)})}{(m(p_j-1)t)^{1/m(p_j-1)}}.$$

Since $(m(p_j-1)t)^{1/(m(p_j-1))} \to 1$ as $p_j \to \infty$, then $u_{p_j,m}(t) \to v_m(1)$. $\square$

Remark 2.4.1. When $m = 1$, the operator $A_{p,m}$ reduces to the p-Laplace operator defined in $L^1(\Omega)$, which restricted to $L^2(\Omega)$ coincides with the subdifferential ∂I_p in bounded domains. The same argument applies to show that the limit operator $A_{\infty,m}$ coincides with ∂I_∞, when $m = 1$ and restricted to $L^2(\Omega)$. Therefore, theorem 2.3.5 serves as a generalization (in the mild sense) of proposition 2.1.1. Similarly, the results of theorem 2.4.2 and corollary 2.4.3 are a generalization of the results in proposition 2.1.2 and the equation satisfied in the limit as described in (2.15).

3. Limit of solutions of (DNE) when $m \to +\infty$

We intend to study in this section the asymptotic behaviour of the family of solutions $u_{p,m}$ of problem $(DNE)_{p,m}$ as defined in (2.1), as the parameter m goes to infinity. We will see that the passage to the limit is more delicate in this case in comparison with the study when p goes to infinity.

3.1 Asymptotic behaviour for the (PME)

When $p = 2$, the equation in $(DNE)_{p,m}$ simplifies to the porous medium equation (PME)

$$(u_m)_t - \Delta u_m^m = g.$$

This equation, for $g \equiv 0$, is a prototype of evolution equations of the form

$$(u_m)_t = \Delta \phi_m(u_m), \tag{3.1}$$

where ϕ_m is a monotone graph. In the early eighties, Bénilan and Crandall raised the question of the continuous dependence of solutions of initial-value problems for (3.1) as functions of the nonlinearity ϕ_m. Namely, given a sequence ϕ_m which converges in the sense of graphs to a maximal monotone graph ϕ_∞, what is the behaviour at the limit of the family of solutions u_m. Since the nonlinearity $|u|^{m-1}u$ converges in the sense of graphs to the multivalued maximal monotone graph ϕ_∞ defined as

$$\phi_\infty(r) = \begin{cases} \emptyset & \text{if } r < -1 \\ (-\infty, 0] & \text{if } r = -1 \\ \{0\} & \text{if } |r| < 1 \\ [0, +\infty) & \text{if } r = 1 \\ \emptyset & \text{if } r > 1, \end{cases} \tag{3.2}$$

then, by the results proved in [11], it follows that the solutions u_m of the Dirichlet problem associated to the porous medium equation converge in $C(0,\infty;L^1(\Omega))$ to u, where u is the unique solution of

$$\begin{cases} u_t = \Delta w \quad \text{ in } (0,\infty) \\ \\ u(0) = u_0, \end{cases} \tag{3.3}$$

with $w \in \phi_\infty(u)$. The unique solution u of problem (3.3) is the trivial solution $u \equiv u_0$, then

$$u_m \to u_0 \text{ in } C(0,\infty;L^1(\Omega)).$$

Note that (3.3) is only well-posed as long as $\|u_0\|_\infty \leq 1$. There was then a growing interest in what occurs when the initial data u_0 takes values outside $[-1,1]$. It was Elliot, Herrero, King and Ockendon in [25] that first conjectured that the problem for the porous medium equation with inconsistent initial values develops "mesas" at the limit as $m \to \infty$. This term mesa has been used to describe a pattern at the limit that resembles the features of this landscape of the far west, a flat top surface with relatively steep sides.

Important progress was made throughout the eighties to rigorously solve this question. During this period, Caffarelli and Friedman proved that for a bounded initial data with very strong geometric assumptions, the family of solutions u_m converges in the weak-star topology of $L^\infty(\Omega)$ to a function $\underline{u}$ which solves a "mesa problem" with plateau of height one ([21], see also [28], [29] and [42]). Precisely, $\underline{u}$ equals one on a set, which is characterized as the noncoincidence set of a variational inequality, and equals the initial data outside that set.

The authors also noted the physical interest in the porous medium equation for large values of the parameter m, given that the equation appears in several physical problems, among them the spreading of a liquid film under gravity [34] for $m = 3$ and a radiation in ionized gases for $m \in (5.5, 6.5)$ [45].

It was in 1989 that the problem was completely solved in [9] for non-negative initial data. The following result was obtained for the Dirichlet porous medium problem

$$\begin{cases} (u_m)_t = \Delta |u_m|^{m-1} u_m & \text{in } (0,\infty) \times \Omega \\ \\ u_m = 0 & \text{on } (0,\infty) \times \partial\Omega \\ \\ u_m(0) = u_0 & \text{on } \Omega, \end{cases} \tag{3.4}$$

where Ω is an open domain of $\mathbb{R}^N$ not necessarily bounded, $m \geq 1$, and $u_0 \in L^1(\Omega)$.

Theorem 3.1.1. Let u_m be the solution of problem (3.4) with initial value $u_0 \in L^1(\Omega)$, $u_0 \geq 0$. Then there exists a time-independent limit function $\underline{u_0}$ such that, when $m \to \infty$,

$$u_m \to \underline{u_0} = u_0 \chi_{[\underline{w}=0]} + \chi_{[\underline{w}>0]} \text{ in } L^1(\Omega),$$

uniformly for t in a compact set in $(0, \infty)$, where $\underline{w}$ satisfies

$$\underline{w} \in H^2(\Omega) \cap H_0^1(\Omega), \quad \underline{w} \geq 0, \quad 0 \leq \Delta \underline{w} + u_0 \leq 1 \text{ in } \mathcal{D}'(\Omega), \quad \underline{w}(\Delta \underline{w} + u_0 - 1) = 0,$$

or equivalently $\underline{w}$ is the solution of the mesa problem

$$\underline{u_0}, \underline{w} \in L^1(\Omega)^+, \quad \mathrm{sign}(\underline{w}) - \Delta \underline{w} \ni u_0 \text{ in } \mathcal{D}'(\Omega), \quad \underline{u_0} \in \mathrm{sign}(\underline{w}). \tag{3.5}$$

Let us define $A^{(\infty)}$ as follows:

$$z \in A^{(\infty)}(v) \Longleftrightarrow \begin{cases} v, z \in L^1(\Omega), \ \exists\, w \in H_0^1(\Omega), \ v \in sign(w) \text{ a.e. on } \Omega \\ \\ \text{and } \displaystyle\int_\Omega \nabla w . \nabla \xi = \int_\Omega z\, \xi, \ \forall \xi \in H_0^1(\Omega) \cap L^\infty(\Omega). \end{cases}$$

Then the following holds.

Corollary 3.1.2. Under the hypothesis of the previous theorem

$$\underline{u_0} = (I + A^{(\infty)})^{-1} u_0 = Proj^{H^{-1}}_{\overline{D(A^{(\infty)})}} u_0,$$

where $Proj^{H^{-1}}_{\overline{D(A^{(\infty)})}} u_0$ is the projection of u_0 onto the convex set $\overline{D(A^{(\infty)})}$ by the $H^{-1}(\Omega)$ norm.

The nonlinear diffusion coefficient of the porous medium equation, u^{m-1}, for large but finite values of m, is very large at all points where $u > 1$, hence all mass in that region tends to be quickly diffused into regions of lesser concentration. In the limit, the diffusion coefficient tends to infinity above the level $u = 1$ and zero below it. It makes sense that the region $\{u > 1\}$ instantaneously collapses and the region $\{u < 1\}$ tends not to be affected, as there is no evolution below $u = 1$, and we have a convergence to a stationary profile. The limit is singular, when $u_0 > 1$, and a discontinuity arises in the neighbourhood of $t = 0$, after which the limit profile becomes compatible with ϕ_∞.

Notice that the limit configuration is time-independent since there is a trivial boundary condition and no reaction term. In the case that there is a non-trivial Dirichlet boundary condition, an evolutionary problem is obtained at the limit, as shown in [30] by Gil and Quirós. Indeed, in the following result, it was shown by the authors that the limit function is a solution of a Hele-Shaw problem.

Theorem 3.1.3. Let u_0 be measurable, bounded and compactly supported, g the trace of a function in $W^{1,2}(\Omega) \cap L^\infty(\Omega)$ and (u_m, w_m) be the solution of

$$\begin{cases} u_t = \Delta w, \quad w = u^m \\ \\ w = g \text{ on } \partial\Omega \\ \\ u(x,0) = \underline{u_0}. \end{cases}$$

Then, when $m \to \infty$, we have

$$u_m(t,\cdot) \to \underline{u}(t,\cdot) \text{ in } L^1(\Omega) \text{ for all } t > 0$$

$$w_m \to \underline{w} \text{ in } L^1((T_1,T_2) \times \Omega) \text{ for all } T_2 > T_1 > 0$$

where $(\underline{u}, \underline{w})$ satisfies the following Hele-Shaw evolution problem

$$\begin{cases} u_t = \Delta w, \ w \in \phi_\infty(u) \\ \\ w = g \text{ on } \partial\Omega \\ \\ u(x,0) = \underline{u_0}, \end{cases} \tag{3.6}$$

where $\underline{u_0}$ is as defined in theorem 3.1.1.

Both the proof of theorem 3.1.1 and its generalization in [30], rely heavily on the following regularizing effect for solutions of the porous medium equation [10],

$$-u_t \leq \frac{u}{(m-1)t} \text{ in } \Omega_T,$$

which holds only if $u_0 \geq 0$. The problem still remained of what occurs for initial data of changing sign. Then in 2003, inspired by the work of Evans et al. for the singular limit of the p-Laplacian, the following result was obtained by Bénilan and Igbida in [16].

Theorem 3.1.4. Let u_m be the solution of problem (3.4) for $u_0 \in L^\infty(\Omega)$, no longer required to be non-negative. Then, there exists a function z such that, as m tends to infinity,

$$u_m \to z(1) \text{ in } L^1(\Omega) \text{ uniformly for } t \text{ in a compact set of } (0,\infty),$$

where z satisfies the following properties

(i) $z(t) = tu_0$ for any $t \in [0, a]$, where

$$a = \begin{cases} 1 & \text{if } \|u_0\|_\infty \le 1 \\ \\ 1/\|u_0\|_\infty & \text{if } \|u_0\|_\infty > 1, \end{cases}$$

(ii) z is the unique mild solution of the evolution problem

$$\begin{cases} z_t + A^{(\infty)} z \ni z/t \quad \text{in } (a, \infty) \\ \\ z(a) = a u_0. \end{cases}$$

In this case, we have that

$$(I + A^{(\infty)})^{-1} u_0 = \underline{u}_0 = u_0 \chi_{[\underline{w}=0]} + \chi_{[\underline{w}>0]} - \chi_{[\underline{w}<0]},$$

where $\underline{w} \in \phi_\infty(\underline{u}_0)$, now not necessarily non-negative, is still the solution of the mesa problem

$$\operatorname{sign}(\underline{w}) - \Delta \underline{w} \ni u_0.$$

However, as explained in [14], in general $z(1) \neq \underline{u}_0$.

3.2 Regular limit of (DNE) when $m \to +\infty$

Recall the operator $A_{p,m}$ as already defined in (2.16) and let V denote $W_0^{1,p}(\Omega)$ or $W^{1,p}(\mathbb{R}^N)$ depending on whether the domain Ω is bounded or the whole of $\mathbb{R}^N$. When $\Omega = \mathbb{R}^N$, it was proved in [33] that $A_{p,m}$ converges in the resolvent sense to the operator $A_{p,\infty}$, which behaves as follows:

$$v \in A_{p,\infty} u \Longleftrightarrow \begin{cases} u, v \in L^1(\Omega),\ \exists\, w \in V,\ u \in \operatorname{sign}(w) \ \text{ a.e. in } \Omega \\ \\ \text{and } \ -\Delta_p w = v \ \text{ in } \mathcal{D}'(\Omega). \end{cases} \tag{3.7}$$

We would like to see that this result continues to hold for bounded domains. We will show that the convergence holds for the following particular cases:

(i) Ω is a bounded interval in $\mathbb{R}$,

(ii) $\Omega = B(0, R)$ and f is radial and nonnegative.

We recall that we denote $z_{p,m} := (I + A_{p,m})^{-1} f$, for $f \in L^\infty(\Omega)$, and once again examine the stationary problem associated to the operator $A_{p,m}$, which is as follows

$$\begin{cases} z_{p,m} - \Delta_p z_{p,m}^m = f & \text{in } \Omega \\ \\ z_{p,m} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Remark 3.2.1. To prove the result in $\mathbb{R}^N$, it was used that $J_\lambda^{A_{p,m}}$ is invariant by translation, together with the L^1- contraction properties of solutions of (3.8), to obtain

$$\|z_{p,m}\|_1 \leq \|f\|_1,$$

$$\|z_{p,m}(x+h) - z_{p,m}(x)\|_1 \leq \|f(x+h) - f(x)\|_1 \quad \forall\, h > 0.$$

Therefore $z_{p,m}$ is relatively compact in $L^1_{loc}(\mathbb{R}^N)$. Since we are interested in working in bounded domains, we no longer have the translation invariance to make use of, and we need a different compactness result. The equivalent compactness result for bounded domains is more difficult to obtain. In this case, we will need to restrict even further the choice of domain. We emphasize that all the other results in [33], used for the convergence of the operators, apply to general open domains in $\mathbb{R}^N$, not necessarily bounded. We will recall these results below and their proofs, for bounded domains, for completeness.

We will see first that by the following results, also from [33], the convergence of the operator $A_{p,m}$ in the resolvent sense holds if $\|f\|_\infty \leq 1$.

Proposition 3.2.1. [33, Lemma 2.4] If $||f||_\infty \leq 1$, then when $m \to \infty$, we have

$$(I + \lambda A_{p,m})^{-1} f \to f \ \text{ in } \ L^1(\Omega),$$

for all $\lambda > 0$.

Proof. If $z_{p,m}$ is a solution of (3.8) then

$$\int_\Omega |\nabla z_{p,m}^m|^{p-2} \nabla z_{p,m}^m \cdot \nabla \varphi dx = \int_\Omega (f - z_{p,m}) \varphi dx, \ \forall\, \varphi \in \mathcal{D}(\Omega). \tag{3.9}$$

Let us first consider f such that

$$\|f\|_\infty \leq c < 1. \tag{3.10}$$

Then

$$\int_\Omega |z_{p,m}^m| dx \leq \|z_{p,m}^m\|_\infty |\Omega| \leq \|f\|_\infty^m |\Omega| < c^m |\Omega| \to 0.$$

Hence

$$z_{p,m}^m \to 0 \text{ in } L^1(\Omega).$$

Now by density we can take $\varphi = z_{p,m}^m$ in (3.9) and we obtain

$$\begin{aligned} \int_\Omega |\nabla z_{p,m}^m|^p dx &= \int_\Omega (f - z_{p,m}) z_{p,m}^m dx \\ &\leq 2\|f\|_\infty \|z_{p,m}^m\|_1 \end{aligned}$$

and

$$\nabla z_{p,m}^m \to 0 \text{ in } L^p(\Omega).$$

Therefore, by (3.9),

$$z_{p,m} \to f \text{ in } L^1(\Omega) \text{ as } m \to \infty. \tag{3.11}$$

If $\|f\|_\infty \leq 1$, we can consider a sequence f_n in $L^1(\Omega)$ which verifies (3.10) such that $f_n \to f$ as $n \to \infty$ in $L^1(\Omega)$. Then, by the accretivity of $A_{p,m}$ in $L^1(\Omega)$, it follows that (3.11) continues to hold in this case. □

This result is then enough to prove that $u_{p,m}$ converges to the initial data u_0, if $g \equiv 0$ and $\|u_0\|_\infty \leq 1$.

Lemma 3.2.2. [33, Proposition 2.3] Let $m > 1/(p-1)$, $u_0 \in L^1(\Omega)$ and $u_{p,m}$ be a solution of $(DNE)_{p,m}$ in (2.1) with $g \equiv 0$. If $\|u_0\|_\infty \leq 1$, then when $m \to \infty$ we have

$$u_{p,m} \to u_0 \text{ in } C(0,T; L^1(\Omega)),$$

for all $\lambda > 0$.

Proof. Let A be the m-accretive operator defined by

$$Au = 0 \text{ and } D(A) = \{u \in L^1(\Omega) : \|u_0\|_\infty \leq 1\}.$$

Then, by the previous result, we have

$$(I + \lambda A_{p,m})^{-1} u \to u \text{ in } L^1(\Omega) \text{ when } m \to \infty,$$

for all $u \in D(A)$ and

$$A \subseteq \liminf_{m\to\infty} A_{p,m}.$$

As $u \equiv u_0$ is the unique mild solution of

$$\begin{cases} \dfrac{du}{dt} + Au = 0 \quad \text{on } [0,\infty) \\ \\ u(0) = u_0, \end{cases} \tag{3.12}$$

the result follows. □

Just as in the previous chapter we will need certain uniform bounds in m to pass to the limit.

Lemma 3.2.3. If $z_{p,m}$ is the solution of (3.8) then $z_{p,m}^m$ is bounded in $W_0^{1,p}(\Omega)$.

Proof. In the same way as in lemma 2.3.1 we can see that if $z_{p,m}$ is a solution of (3.8) then

$$\int_\Omega |\nabla z_{p,m}^m|^p dx \leq C\|f\|_\infty \left(\int_\Omega |\nabla z_{p,m}^m|^p\right)^{1/p} |\Omega|^{1-1/p},$$

and therefore

$$\int_\Omega |\nabla z_{p,m}^m|^p dx \leq (c\|f\|_\infty)^{\frac{p}{p-1}} |\Omega|.$$

Hence $z_{p,m}^m$ is uniformly bounded in $W_0^{1,p}(\Omega)$. □

We will at this point need to restrict our choice of domain. Let us first consider the problem in (3.8) in one dimension, where we momentarily suppress the subscripts m and p:

$$\begin{cases} z - (|(z^m)_x|^{p-2}(z^m)_x)_x = f & \text{in } I \text{ an interval of } \mathbb{R} \\ \\ z = 0 & \text{on } \partial I. \end{cases} \tag{3.13}$$

Consider as well $\rho \in C_0^\infty(I)$, $\rho \geq 0$, $\displaystyle\int \rho = 1$ and for any function k let us define the convolution

$$\rho_\epsilon * k(x) = \int \rho_\epsilon(x-y)k(y)dy, \ \epsilon > 0,$$

where $\rho_\epsilon(y) = \rho(y/\epsilon)/\epsilon$. Adapting accordingly the results in [26], which apply for the doubly nonlinear diffusion equation in $(DNE)_{p,m}$ in one dimension, we have the following result.

Theorem 3.2.4. Let z be the unique solution of (3.13) for $f \geq 0$, $f \in L^\infty(I)$. For $p > 4$, $m > 0$ there exists a smooth approximation $\Psi_\epsilon(z,b)$ to $\Psi(b) = b|b|^{p-2}$ with

$$\Psi_\epsilon(z,b) = b|b|^{p-2} + \frac{n\epsilon}{m} z^{n-m} b,$$

where $n = (p-1)(m+1) - 1$, such that for $f_\epsilon = \epsilon + \rho_\epsilon * f$ and $z_{0_\epsilon} = \epsilon$, the problem

$$\begin{cases} z_\epsilon - (|(z_\epsilon^m)_x|^{p-2}(z_\epsilon^m)_x)_x - \epsilon(z_\epsilon^n)_{xx} = z_\epsilon - (\Psi_\epsilon(z_\epsilon,(z_\epsilon^m)_x))_x = f_\epsilon & \text{in } I \\ \\ z_\epsilon = z_{0_\epsilon} & \text{on } \partial I, \end{cases} \tag{3.14}$$

has a unique solution $z_\epsilon \in C^\infty(I)$ satisfying (*i*) $0 < \epsilon < z_\epsilon < \epsilon + \|f\|_\infty$.

(*ii*) z_ϵ converges uniformly in compact subsets of I to z.

(*iii*) $(z_\epsilon^m)_x \to (z^m)_x$ as $\epsilon \to 0$ a.e. $x \in I$.

Proof. It is easy to see that given the choice of f_ϵ, then

$$0 < \epsilon \leq f_\epsilon \leq \epsilon + \|f\|_\infty.$$

Taking into account as well the choice of initial data $z_{0_\epsilon} = \epsilon$, then (*i*) follows by the maximum principle. Furthermore, we have that

$$0 < c \leq (\Psi_\epsilon)_b = (p-1)|b|^{p-2} + \frac{n\epsilon}{m} z_\epsilon^{n-m} \leq C,$$

where c and C depend only on ϵ and $\|f\|_\infty$. Hence the equation in (3.13) is uniformly elliptic and by the general theory of quasilinear elliptic partial differential equations (see, for e.g., [36]), we obtain the existence of a smooth solution z_ϵ of problem (3.14). The convergence of the solution z_ϵ of (3.14) to the solution z of (3.13) as ϵ tends to 0, as well as the convergence in (*iii*) follow as in [26]. □

Remark 3.2.2. If $p < 4$, then the previous theorem continues to hold. However, depending on the relationship between p and m, a different approximation operator Ψ_ϵ would be needed to pass to the limit in ϵ.

Theorem 3.2.5. Let $z_{p,m}$ be a solution of (3.8). If one of the following conditions is satisfied:

(*i*) $\Omega = I$ is a bounded interval in $\mathbb{R}$ and $f \in L^\infty(\Omega)$ is nonnegative,

(*ii*) $\Omega = B(0, R)$ and $f \in L^\infty(\Omega)$ is radial and nonnegative,

then the total variation of $z_{p,m}$ is uniformly bounded.

Proof.

(*i*) By theorem 3.2.4, there exists a smooth approximation of the solution $z_{m,p}$ of (3.8), which we will continue to denote by z_ϵ. We differentiate (3.14) with respect to x to obtain

$$(z_\epsilon)_x - \left(|(z_\epsilon^m)_x|^{p-2}(z_\epsilon^m)_x + \frac{n\epsilon}{m} z_\epsilon^{n-m}(z_\epsilon^m)_x \right)_{xx} = (f_\epsilon)_x.$$

Let $b_\epsilon = (z_\epsilon^m)_x$ and consider a sequence of functions which satisfy $h_\delta \in C^\infty(\mathbb{R})$, $h'_\delta \geq 0$, $0 = h_\delta(0) \leq |h_\delta| \leq 1$. Multiply by $h_\delta(b_\epsilon)$ and integrate over I to get

$$\begin{aligned}
\int_I h_\delta(b_\epsilon)(z_\epsilon)_x &\leq \int_I \Big(|b_\epsilon|^{p-2}b_\epsilon + \frac{n\epsilon}{m} z_\epsilon^{n-m} b_\epsilon\Big)_{xx} h_\delta(b_\epsilon) + \int_I (f_\epsilon)_x h_\delta(b_\epsilon) \\
&\leq -\int_I \Big(|b_\epsilon|^{p-2}b_\epsilon + \frac{n\epsilon}{m} z_\epsilon^{n-m} b_\epsilon\Big)_{x} (h_\delta(b_\epsilon))_x + \sum_{\partial I} h_\delta(b_\epsilon)(z_\epsilon - f_\epsilon) \\
&\quad + \int_I |(f_\epsilon)_x| \\
&\leq -\int_I \Big(|b_\epsilon|^{p-2}b_\epsilon + \frac{n\epsilon}{m} z_\epsilon^{n-m} b_\epsilon\Big)_{b_\epsilon} (b_\epsilon)_x h'_\delta(b_\epsilon)(b_\epsilon)_x \\
&\quad + \sum_{\partial I}(|z_{0_\epsilon}| + |f_\epsilon|) + \int_I |(f_\epsilon)_x| \\
&\leq \|(f_\epsilon)_x\|_1 + \sum_{\partial I}(|z_{0_\epsilon}| + |f_\epsilon|).
\end{aligned}$$

Taking h_δ such that $h_\delta(r) \to \operatorname{sign}_0(r)$ as $\delta \to 0$, then

$$\int_I |(z_\epsilon)_x| \leq \int_I |(f_\epsilon)_x| + \sum_{\partial I}(|z_{0_\epsilon}| + |f_\epsilon|).$$

Hence, by the lower semicontinuity of the seminorm in BV, we have, as $\epsilon \to 0$,

$$\int_I |(z_{p,m})_x| \leq \int_I |(f)_x|.$$

(ii) Since f is radial, that is, $f(x) = l(|x|)$ and $J_\lambda^{A_{p,m}}$ is invariant by rotation, then the solution of (3.8) is radial and there exists $v_{p,m}$ such that $z_{p,m}(x) = v_{p,m}(|x|)$ and verifies

$$\begin{cases} v - \dfrac{(r^{N-1}|(v^m)_r|^{p-2}(v^m)_r)_r}{r^{N-1}} = l \quad \text{in } (0,R) \\ \\ v(0) = v(R) = 0. \end{cases} \tag{3.15}$$

As $\displaystyle\int_{B(0,R)} |\nabla z_{p,m}(x)|dx$ and $\displaystyle\int_0^R |(v_{m,p})_r| r^{N-1} dr$ differ only by a constant which is independent of m then it is enough to prove the uniform bound of the second.

We can then take a smooth approximation of the problem as in the previous case

$$\begin{cases} r^{N-1}v_\epsilon - \Big(r^{N-1}|(v_\epsilon^m)_r|^{p-2}(v_\epsilon^m)_r + \epsilon\frac{n}{m}v_\epsilon^{n-m}(v_\epsilon^m)_r\Big)_r = l_\epsilon r^{N-1} \quad \text{in } (0,R) \\ \\ v_\epsilon(0) = v_\epsilon(R) = v_{0_\epsilon}. \end{cases} \tag{3.16}$$

We differentiate the equation in (3.16) with respect to r to obtain

$$(r^{N-1}v_\epsilon)_r - \left(r^{N-1}|(v_\epsilon^m)_r|^{p-2}(v_\epsilon^m)_r + \epsilon\frac{n}{m}v_\epsilon^{n-m}(v_\epsilon^m)_r\right)_{rr} = (l_\epsilon r^{N-1})_r.$$

Denote now $b_\epsilon = (v_\epsilon^m)_r$ and h_δ as above, multiply the above equation by $h_\delta(b_\epsilon)$ and integrate over $(0,R)$ to obtain

$$\begin{aligned}
\int_0^R (r^{N-1}v_\epsilon)_r h_\delta(b_\epsilon)dr &= \int_0^R \left(r^{N-1}|b_\epsilon|^{p-2}b_\epsilon + \epsilon\frac{n}{m}v_\epsilon^{n-m}b_\epsilon\right)_{rr} h_\delta(b_\epsilon)dr \\
&\quad + \int_0^R (l_\epsilon r^{N-1})_r h_\delta(b_\epsilon)dr \\
&\leq -\int_0^R \left(r^{N-1}|b_\epsilon|^{p-2}b_\epsilon + \epsilon\frac{n}{m}v_\epsilon^{n-m}b_\epsilon\right)_r (h_\delta(b_\epsilon))_r \\
&\quad + \sum_{\partial I} h_\delta(b_\epsilon)\left(r^{N-1}|b_\epsilon|^{p-2}b_\epsilon + \epsilon\frac{n}{m}v_\epsilon^{n-m}b_\epsilon\right)_r \\
&\quad + \int_0^R |(l_\epsilon r^{N-1})_r| \\
&\leq R^{N-1}(v_{0_\epsilon} - l_\epsilon(R)) + \int_0^R |(l_\epsilon r^{N-1})_r|dr.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^R (v_\epsilon)_r r^{N-1}h_\delta(b_\epsilon)dr &= \int_0^R (v_\epsilon r^{N-1})_r h_\delta(b_\epsilon)dr - \int_0^R (r^{N-1})_r v_\epsilon h_\delta(b_\epsilon)dr \\
&\leq R^{N-1}(v_{0_\epsilon} - l_\epsilon(R)) + \int_0^R |(l_\epsilon r^{N-1})_r|dr \\
&\quad - \int_0^R (r^{N-1})_r v_\epsilon h_\delta(b_\epsilon)dr \\
&\leq R^{N-1}(v_{0_\epsilon} - l_\epsilon(R)) + \int_0^R |(l_\epsilon)_r| r^{N-1}dr + C(R)\|l_\epsilon\|_\infty.
\end{aligned}$$

Taking h_δ such that $h_\delta(r) \to \text{sign}_0(r)$ as $\delta \to 0$, then

$$\int_0^R |(v_\epsilon)_r| r^{N-1}dr \leq R^{N-1}(v_{0_\epsilon} - l_\epsilon(R)) + \int_0^R |(l_\epsilon)_r| r^{N-1}dr + C(R)\|l_\epsilon\|_\infty.$$

Hence as $\epsilon \to 0$

$$\int_0^R |(v_{p,m})_r| r^{N-1}dr \leq \int_0^R |l_r| r^{N-1}dr + C(R)\|l\|_\infty.$$

$\square$

Lemma 3.2.6. [33, Lemma 2.11] For all $m \geq 1$, let $w_m \in W_0^{1,p}(\Omega)$ and $g_m \in L^1(\Omega)$ be such that

$$-\Delta_p w_m = g_m \ \text{ in } \ \mathcal{D}'(\Omega). \tag{3.17}$$

If there exists $w_\infty \in W_0^{1,p}(\Omega)$ and $g_\infty \in L^1(\Omega)$ such that, when $m \to \infty$, we have

$$g_m \to g_\infty \ \text{ in } \ L^1(\Omega), \tag{3.18}$$

$$w_m \rightharpoonup w_\infty \ \text{ in } \ W^{1,p}(\Omega), \tag{3.19}$$

$$g_m w_m \to g_\infty w_\infty \ \text{ in } \ L^1(\Omega), \tag{3.20}$$

then

$$-\Delta_p w_\infty = g_\infty \ \text{ in } \ \mathcal{D}'(\Omega), \tag{3.21}$$

and furthermore, we have

$$\nabla w_m \to \nabla w_\infty \text{ , in } \ (L^p(\Omega))^N, \text{ when } \ m \to \infty. \tag{3.22}$$

Proof. By (3.19), we have that

$$w_m \to w_\infty \ \text{ in } \ L^p(\Omega).$$

Moreover, there exists $h \in (L^{p'}(\Omega))^N$ such that

$$|\nabla w_m|^{p-2}\nabla w_m \rightharpoonup h \ \text{ in } \ (L^{p'}(\Omega))^N. \tag{3.23}$$

This, together with (3.18), gives

$$-\operatorname{div} h = g_\infty \ \text{ in } \ \mathcal{D}'(\Omega). \tag{3.24}$$

It is therefore enough to show that

$$h = |\nabla w_\infty|^{p-2}\nabla w_\infty \ \text{ a.e. in } \Omega. \tag{3.25}$$

We claim that $\forall \, \eta \in \mathbb{R}^N$

$$\frac{1}{p}|\nabla w_\infty|^p + h \cdot \eta - \frac{1}{p}|\eta|^p \leq h \cdot \nabla w_\infty,$$

and therefore (3.25) is satisfied.

Let us fix $\xi \in \mathcal{D}(\Omega)$, $\xi \geq 0$ and $\eta \in (L^\infty(\Omega))^N$. We have by the convexity of $\dfrac{|r|^p}{p}$ that

$$\frac{1}{p}\int_\Omega |\nabla w_m|^p \xi + \int_\Omega |\nabla w_m|^{p-2}\nabla w_m \cdot \eta\xi - \frac{1}{p}\int_\Omega |\eta|^p \xi \leq \int_\Omega |\nabla w_m|^p \xi. \tag{3.26}$$

On the other hand, by (3.17), taking as test function ξw_m and taking the limit as $m \to \infty$, we obtain

$$\begin{aligned}
\lim_{m\to\infty} \int_\Omega |\nabla w_m|^p \xi &= \lim_{m\to\infty} \int_\Omega g_m w_m \xi - \lim_{m\to\infty} \int_\Omega |\nabla w_m|^{p-2} \nabla w_m \cdot w_m \nabla \xi \\
&= \int_\Omega g_\infty w_\infty \xi - \int_\Omega h \cdot w_\infty \nabla \xi \\
&= \int_\Omega h \cdot \nabla(w_\infty \xi) - \int_\Omega h \cdot w_\infty \nabla \xi \\
&= \int_\Omega h \cdot \xi \nabla w_\infty. \qquad (3.27)
\end{aligned}$$

The second equality follows from the hypotheses (3.18)-(3.20) and the third from (3.24). By (3.19) and (3.23), we also have

$$\begin{aligned}
&\frac{1}{p} \int_\Omega |\nabla w_\infty|^p \xi \leq \liminf_{m\to\infty} \frac{1}{p} \int_\Omega |\nabla w_m|^p \xi, \\
&\lim_{m\to\infty} \int_\Omega |\nabla w_m|^{p-2} \nabla w_m \cdot \eta \xi = \int_\Omega h \cdot \eta \xi.
\end{aligned} \qquad (3.28)$$

Using (3.27) and (3.28) to pass to the limit in (3.26), as $m \to \infty$, we obtain

$$\frac{1}{p} \int_\Omega |\nabla w_\infty|^p \xi + \int_\Omega h \cdot \eta \xi - \frac{1}{p} \int_\Omega |\eta|^p \xi \leq \int_\Omega h \cdot \xi \nabla w_\infty,$$

and the claim is true. To prove (3.22) recall that we have that

$$\nabla w_m \rightharpoonup \nabla w_\infty \ \text{ in } \ L^p(\Omega)$$

and by (3.27), since $h = |\nabla w_\infty|^{p-2} \nabla w_\infty$,

$$|\nabla w_m| \to |\nabla w_\infty| \ \text{ in } \ L^p(\Omega). \qquad (3.29)$$

□

We are now ready to prove the convergence in the resolvent sense of the operator $A_p^{(m)}$ as m tends to infinity.

Lemma 3.2.7. Let one of the following conditions be satisfied:

(i) Ω is a bounded interval in $\mathbb{R}$,

(ii) $\Omega = B(0, R)$ and f is radial, then for $f \in L^\infty(\Omega)$, $f \geq 0$ and $\lambda > 0$, when $m \to \infty$, we have that

$$(I + \lambda A_{p,m})^{-1} f \to (I + \lambda A_{p,\infty})^{-1} f \ \text{ in } \ L^1(\Omega).$$

Proof. Let $f \in L^\infty(\Omega)$ and $z_{p,m}$ be a solution of (3.8), by theorem 3.2.5, if conditions (i) or (ii) are satisfied, then there exists z_p such that

$$z_{p,m} \to z_p \ \text{ in } \ L^1(\Omega), \tag{3.30}$$

and by lemma 3.2.3 there exists some w_p such that

$$z_{p,m}^m \rightharpoonup w_p \ \text{ in } \ W_0^{1,p}(\Omega). \tag{3.31}$$

Therefore $z_p \in \operatorname{sign}(w_p)$ a.e. in Ω. Since we also have that

$$\|z_{p,m}\|_\infty \leq \|f\|_\infty,$$

then

$$(f - z_{p,m})(z_{p,m})^m \to (f - z_p) w_p \ \text{ in } \ L^1(\Omega)$$

and all the hypothesis of lemma 3.2.6 are satisfied, from which we obtain

$$-\Delta_p w_p = f - z_p \ \text{ in } \ \mathcal{D}'(\Omega)$$

and

$$\nabla z_{p,m}^m \to \nabla w_p \ \text{ in } \ L^p(\Omega).$$

$\square$

Corollary 3.2.8. [33] Let one of the following conditions be satisfied:

(i) Ω is a bounded interval in $\mathbb{R}$ and $f \in L^1(\Omega)$ such that $f \geq 0$,

(ii) $\Omega = B(0, R)$ and $f \in L^1(\Omega)$ such that f is radial and $f \geq 0$,

then $\overline{A}_{p,\infty}$ is m-T-accretive and for all $\lambda > 0$, when $m \to \infty$, we have

$$(I + \lambda \overline{A}_{p,m})^{-1} f \to (I + \lambda \overline{A}_{p,\infty})^{-1} f \ \text{ in } \ L^1(\Omega),$$

where $\overline{A}_{p,\infty}$, for $p > N$, is defined as follows

$$v \in \overline{A}_{p,\infty} u \Leftrightarrow \begin{cases} u, v \in L^1(\Omega), \exists\, w \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega), \ u \in \operatorname{sign}(w) \ \text{ a.e. } \Omega \ \text{ and} \\ \\ -\Delta_p w = v \ \text{ a.e. in } \ \Omega. \end{cases}$$

Proof. Similarly to corollary 2.3.4 in Chapter 2, it is enough to prove that $A_{p,\infty}$ is T-accretive which follows since $A_{p,\infty} \subseteq A_1\phi_\infty$, where $\phi_\infty = \text{sign}^{-1}$ is a maximal monotone graph and A_1 is the single-valued completely accretive operator defined by

$$A_1 u = -\Delta_p u,$$

$$D(A_1) = \{u \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega);\ \Delta_p u \in L^1(\Omega)\}.$$

□

The main theorem then follows.

Theorem 3.2.9. Let $u_{p,m}$ be the solution of the problem $(DNE)_{p,m}$ in (2.1), where $g \in L^1(\Omega_T)$ and one of the following conditions is satisfied:

(i) Ω is a bounded interval in $\mathbb{R}$ and $0 \leq u_0 \leq 1$

(ii) $\Omega = B(0, R)$ and u_0 is radial such that $0 \leq u_0 \leq 1$.

Then, there exists a function u_p such that, when $m \to \infty$, for each $T > 0$,

$$u_{p,m} \to u_p \ \text{ in } \ C(0, T; L^1(\Omega))$$

and u_p is the unique mild solution of

$$\begin{cases} (u_p)_t + A_{p,\infty}(u_p) \ni g & \text{in } \ \Omega \times [0, T] \\ \\ u_p(0) = u_0, \end{cases}$$

where $A_{p,\infty}$ is given by (3.7).

Proof. Since u_0 belongs to $\overline{D(A_{p,\infty})}$, the result follows from corollary 3.2.8 and theorem 1.5.3. □

3.3 Singular limit of the (DNE) when $m \to \infty$ and a conjecture

Given the result in theorem 3.1.1 for the porous medium equation, it is no wonder that the following was conjectured in [15] for solutions $u_{p,m}$ of the general Dirichlet problem $(DNE)_{p,m}$ in (2.1), with $g \equiv 0$:

Conjecture Let $u_0 \in L^1(\Omega)$, $u_0 \geq 0$. Then there exists a limit function $\underline{u_{0,p}}$ such that

$$u_{p,m} \to \underline{u_{0,p}} = u_0 \chi_{[\underline{w_p}=0]} + \chi_{[\underline{w_p}>0]}, \tag{3.32}$$

and $\underline{w_p}$ is the solution of the mesa problem

$$\begin{cases} \operatorname{sign}(\underline{w_p}) - \Delta_p \underline{w_p} \ni u_0 & \text{in } \mathcal{D}'(\Omega) \\ \\ \underline{w_p} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.33}$$

It was proved in [33] that the conjecture holds in the following particular cases:

1. Ω is an open domain in $\mathbb{R}^N$ not necessarily bounded and $||u_0||_\infty \leq 1$. It was proved that
$$u_{p,m} \to u_0 \text{ in } C(0,T;L^1(\Omega)).$$
The conjecture then holds since for $||u_0||_\infty \leq 1$, $\underline{w_p} \equiv 0$ is the unique solution of (3.33).

2. $\Omega = \mathbb{R}^N$ and u_0 is radial decreasing, i.e., $u_0(x) = h(|x|)$ and $h : [0,\infty) \to [0,\infty)$ is decreasing. It was also assumed that $h(0) > 1$.

   Then, when $m \to \infty$,
$$u_{p,m} \to \underline{u_{0,p}} = u_0 \chi_{[|x| \geq a]} + \chi_{[|x| \leq a]} \text{ in } C(0,\infty;L^1(\Omega)),$$
where $a > 0$ is given by
$$\int_0^1 h(ar) dr^N = 1.$$
The conjecture holds since $[\underline{w_p} = 0] = [|x| \geq a]$.

3. $\Omega = B(0,R)$ and u_0 is radial decreasing.

Remark 3.3.1. With the exception of case 3, none of the equations satisfied at the limit depend on p.

The following result was also proved in [33] for $\Omega = \mathbb{R}$.

Theorem 3.3.1. Let $u_{p,m}$ be the solution of $(DNE)_{p,m}$ in (2.1) for $g \equiv 0$, $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $u_0 \geq 0$. For all $\underline{u} \in L^1(\mathbb{R})$ such that $u_{p,m_k} \to \underline{u}$ in $C(0,\infty;L^1(\mathbb{R}))$ when $m_k \to \infty$, there exists $A \subseteq \mathbb{R}$ a bounded open set verifying

$$\exists\, \underline{W} \in C(\mathbb{R}), \quad \underline{W}' = u_0 - 1 \text{ a.e. in } A, \ \underline{W} = 0 \text{ a.e. in } \mathbb{R} - A,$$

such that

$$\underline{u} = \chi_A + u_0 \chi_{\mathbb{R} \setminus A} \text{ a.e. in } \mathbb{R}.$$

Remark 3.3.2. Even though this result shows that $\underline{u}$ is a mesa, it was not possible to relate the set A with $\{w_p > 0\}$ where w_p is the solution of

$$-\Delta_p w_p + \text{sign} w_p \ni u_0 \text{ in } \mathcal{D}'(\mathbb{R}).$$

On the other hand, we can apply the method in [14] for homogeneous accretive operators, in the same way as in the case of the convergence with respect to p, to prove the existence of the singular limit and the equation it satisfies. We consider the same stretching of the time variable:

$$v_{p,j}(t,x) = t u_{p,m_j}\left(\frac{t^{m_j(p-1)}}{m_j(p-1)}, x\right). \tag{3.34}$$

If we consider that there is no reaction term, we have the following result.

Theorem 3.3.2. Consider $u_{p,m}$ the solution of $(DNE)_{p,m}$ in (2.1) with $g \equiv 0$. Suppose moreover that one of the following conditions is satisfied:

(i) Ω is a bounded interval in $\mathbb{R}$ and $u_0 \geq 0$, $\|u_0\|_\infty = M > 1$,

(ii) $\Omega = B(0,R)$ and u_0 is radial such that $u_0 \geq 0$, $\|u_0\|_\infty = M > 1$.

Then, as $j \to \infty$,

$$v_{p,j} \to v_p \text{ in } C(0,\infty; L^1(\Omega)), \tag{3.35}$$

where v_p is given by

(i) $v_p(t) = tu_0$ for any $t \in [0,b]$, and $b = 1/M$,

(ii) v_p is the unique mild solution of the evolution problem

$$\begin{cases} (v_p)_t + A_{p,\infty} v_p \ni \dfrac{v_p}{t} & \text{in } (b,\infty) \\ \\ v_p(b) = bu_0, \end{cases} \tag{3.36}$$

and finally, when $m_j \to \infty$,

$$u_{p,m_j} \to v_p(1) \text{ in } L^1(\Omega) \text{ uniformly for } t \text{ in a compact set of } (0,\infty)\,. \tag{3.37}$$

Proof. Let $\hat{u}_{p,m}$ be the unique mild solution of $(DNE)_{p,m}$ with $g \equiv 0$ and initial condition $\hat{u}_0 = bu_0$, i.e., $\hat{u}_0 \in \overline{D(A_{p,\infty})}$. Then, by proposition 3.2.1, when $m \to \infty$,

$$\frac{1}{b} z_{p,m} \to u_0 \text{ in } L^1(\Omega)$$

and also

$$A_{p,m}z_{p,m} \to 0 \ \text{ in } \ L^1(\Omega).$$

Denoting $t_{m_j} := \dfrac{t^{m_j(p-1)}}{m_j(p-1)}$, then it follows by the same scheme as in the proof of lemma 2.4.1 that, when $m_j \to \infty$,

$$u_{p,m_j}(x, t_{m_j}) \to u_0(x) \ \text{ in } \ L^1(\Omega).$$

Hence, by the rescaling in (3.34),

$$v_{p,j}(x, b) \to bu_0, \ \text{ as } \ j \to \infty.$$

Using as well the result of corollary 3.2.8, we see that the hypotheses of theorem 1.5.4 are satisfied and thus (3.35) holds, where v_p is the unique mild solution of (3.36). The convergence in (3.37) follows as in corollary 2.4.3.

□

We will show below what is the relation between the results of theorem 3.3.2 and the conjecture, using a similar structure to the proof of proposition 1 in [14]. Let us observe that, formally, we have that the function v_p, obtained at the limit in theorem 3.3.2, satisfies the following relation

$$\frac{v_p(t,\cdot)}{t} - (v_p)_t(t,\cdot) \in A_{p,\infty}v_p(t,\cdot).$$

Hence, there exists $\hat{w}_p$ such that $\hat{w}_p(t,\cdot) \in W_0^{1,p}(\Omega)$, $v_p(t,\cdot) \in \text{sign}(\hat{w}_p(t,\cdot))$ and

$$-\Delta_p \hat{w}_p(t,\cdot) = \frac{v_p}{t}(t,\cdot) - (v_p(t,\cdot))_t \ \text{ in } \ \mathcal{D}'(\Omega). \tag{3.38}$$

Furthermore, we have that $v_p(\cdot, t)$ is decreasing with respect to t. Indeed, since $u_0 \geq 0$ then $v_p \geq 0$ and hence either $v_p = 1$ or by (3.38) $\frac{v_p(t,\cdot)}{t} = (v_p(t,\cdot))_t$.

(*i*) In the case that $\Omega = I$ is a bounded interval in $\mathbb{R}$, then for any $\varphi \in \mathcal{D}(I)$,

$$\begin{aligned}
\int_I (v_p(x,1) - u_0)\varphi dx &= \int_I \left(\int_b^1 \frac{d}{dt}\left(\frac{v_p}{t}\right)\right) \varphi dx \\
&= \int_I \left(\int_b^1 \frac{1}{t}\left((v_p)_t - \frac{v_p}{t}\right) dt \right) \varphi dx \\
&= -\int_I \left(\int_b^1 \frac{1}{t} |(\hat{w}_p)_x|^{p-1} (\hat{w}_p)_x dt \right) \varphi_x dx.
\end{aligned}$$

Let us denote

$$W := \int_b^1 \frac{1}{t} |(\hat{w}_p)_x|^{p-2} (\hat{w}_p)_x dt,$$

then, we obtain,

$$W_x = v_p(x,1) - u_0 \text{ in } \mathcal{D}'(I).$$

Now if $v_p(x,1) < 1$ then $0 \leq v_p(x,t) < 1$ for all $b < t < 1$. Hence, there exists $\theta < 1$ such that

$$v_{p,j} \leq \theta < 1.$$

Consider as well

$$W_{m_j}(x) := \int_b^1 \frac{1}{t} |(v_{p,j}^{m_j})_x|^{p-2} (v_{p,j}^{m_j})_x(x,t) dt.$$

Then, by [33, Lemma 3.5], we have that

$$\sup |W_{m_j}(x)| \leq C_p \left(\frac{M^{k+1}\theta^{m_j-k}}{c} \right)^{\frac{p-1}{c}} \frac{pc}{j(p-1)-1},$$

where C_p is a constant that depends only on p, $c = m(p-1)-1$ and $k \in (\frac{1}{p-1}, m)$. Taking k to infinity and then m_j, necessarily we have that $W \equiv 0$. Otherwise, if $v_p(x,1) = 1$, clearly

$$W_x = 1 - u_0 \text{ in } \mathcal{D}'(I).$$

We have then that an equivalent to theorem 3.3.1 continues to hold in a bounded interval, i.e., $v_p(x,1)$ is the solution of a mesa problem.

(ii) If $\Omega = B(0,R)$ and u_0 is non-negative and radial decreasing, then by (3.38),

$$\frac{1}{r^{N-1}} (r^{N-1} |(\hat{w}_p)_r|^{p-1})_r = -(v_p(t,r))_t + \frac{v_p(t,r)}{t} \text{ in } \mathcal{D}'(0,R),$$

and then we obtain that, for any $\varphi \in \mathcal{D}(0,R)$,

$$\begin{aligned}
\int_\Omega (v_p(r,1) - u_0)\varphi r^{N-1} dr &= \int_0^R \left(\int_b^1 \frac{d}{dt}\left(\frac{v_p}{t}\right) dt \right) \varphi r^{N-1} dr \\
&= \int_0^R \left(\int_b^1 \frac{1}{t} \left((v_p)_t - \frac{v_p}{t} \right) dt \right) \varphi r^{N-1} dr \\
&= -\int_\Omega \left(\int_b^1 \frac{1}{t} |(\hat{w}_p)_r|^{p-1} dt \right) (\varphi)_r r^{N-1} dr.
\end{aligned}$$

Let us denote w_p a function that satisfies the following

$$w_p(r) = \int_0^R \left(\int_b^1 \frac{1}{t} |(\hat{w}_p)_r(s,t)|^{p-1} dt \right)^{\frac{1}{p-1}} ds. \tag{3.39}$$

Therefore
$$v_p(r,1) - u_0 = \frac{1}{r^{N-1}}(r^{N-1}|(w_p)_r|^{p-1})_r \ \text{ in } \ \mathcal{D}'(0,R).$$
Denoting $\underline{u_{0,p}}(|x|) = v_p(r,1)$ and $\underline{w_p}(|x|) = w_p(r)$, we have

$$\Delta_p \underline{w_p} = \underline{u_{0,p}} - u_0 \ \text{ in } \ \mathcal{D}'(\Omega).$$

Hence, it is only left to show

$$\underline{u_{0,p}} \in \operatorname{sign}(\underline{w_p}) \ \text{ a.e. in } B(0,R). \tag{3.40}$$

Since $v_p(r,1) \in \operatorname{sign}(\hat{w}_p)$ then $0 \le v_p(r,1) \le 1$. Now, from (3.39), we have that $\underline{w_p} \ge 0$. Then, if $v_p(r,1) = 1$ it follows that $0 \le \underline{w_p} \in \phi_\infty(v_p(r,1))$. On the other hand, if $0 \le v_p(r,1) < 1$ then $0 \le v_p(r,t) < 1$ for all $b < t \le 1$. Hence, as in the previous case, we have that there exists $\theta < 1$ such that

$$v_{p,j} \le \theta < 1,$$

and considering

$$w_{p,m_j} = \int_0^R \left(\int_b^1 \frac{1}{t} |(v_{p,j}^{m_j})_r(s,t)|^{p-1} dt \right)^{\frac{1}{p-1}} ds,$$

then, by [33, Lemma 3.8],

$$w_{p,m_j}(r) \le R \left\{ \frac{cM^{k(p-1)}\theta^{(m-k)(p-1)}}{\delta^{p-1}(k(p-1)+2)} + \frac{2M\delta(p-1)}{k(p-1)-1} \right\}^{\frac{1}{p-1}},$$

for all $r \ge r_0 - \delta$, for any $\delta > 0$, where k and c are as in case (i) and $r_0 = \inf\{r \in \mathbb{R}_+ : \ v_p(r,1) < 1\}$. Taking k and then m_j to infinity, we obtain at the limit that $w_p \equiv 0$.

4. Asymptotic behaviour of the limit equations

In this chapter, we will analyze the asymptotic behaviour of the solutions of the limit equations obtained in chapters 2 and 3, when, respectively, m and p go to infinity. Taking into account the restrictions already imposed on the initial data, as well as the domain, we will be able to study the equations satisfied by the limits.

4.1 Asymptotic behaviour of the limit equation in p of the (DNE) when $m \to \infty$

Recall that in chapter 2 we obtained the convergence of the solutions $u_{p,m}$ of the problem $(DNE)_{p,m}$ in (2.1), as p tends to infinity. Depending on the conditions satisfied by the initial data u_0 and the source term g, we obtained a regular or singular limit and the equations satisfied by these limits. In this section, we study the asymptotic behaviour of the regular solutions u_m as m tends to infinity and the corresponding equation satisfied at the limit.

We have already seen that the operator $A_{p,m}$ converges, as p tends to infinity, to the operator $A_{\infty,m}$ given by

$$v \in A_{\infty,m}u \Longleftrightarrow \begin{cases} u, v \in L^1(\Omega),\ u^m \in W^{1,\infty}(\Omega) \cap C_0(\overline{\Omega}), \\ \\ u^m \in \tilde{\mathbb{K}} \text{ and } 0 \geq \displaystyle\int_\Omega v(\xi - u^m)dx, \quad \forall\, \xi \in \tilde{\mathbb{K}}, \end{cases} \tag{4.1}$$

with

$$\tilde{\mathbb{K}} := \{\xi \in L^1(\Omega) :\ |\nabla \xi| \leq 1 \text{ a.e.}\}. \tag{4.2}$$

As already noted, $\phi_m(r) = |r|^{m-1}r$ converges in the graph sense to the multivalued maximal monotone graph ϕ_∞, given by,

$$\phi_\infty(r) = \begin{cases} \emptyset & \text{if } r < -1 \\ \\ (-\infty, 0] & \text{if } r = -1 \\ \\ \{0\} & \text{if } |r| < 1 \\ \\ [0, +\infty) & \text{if } r = 1 \\ \\ \emptyset & \text{if } r > 1. \end{cases}$$

It seems then appropiate to expect that the operator $A_{\infty,m}$ converges in the resolvent sense to the operator $A_{\infty,\infty}$ defined in the following way

$$v \in A_{\infty,\infty}(u) \Longleftrightarrow \begin{cases} u, v \in L^1(\Omega),\ \exists\, w \in W^{1,\infty}(\Omega) \cap C_0(\overline{\Omega}) \text{ with } w \in \tilde{\mathbb{K}}, \\ \\ u \in \operatorname{sign}(w) \text{ and } 0 \geq \displaystyle\int_\Omega v(\xi - w)dx, \quad \forall\, \xi \in \tilde{\mathbb{K}}. \end{cases} \tag{4.3}$$

Under the conditions of theorem 2.3.5, the equation that the solution of $(DNE)_{p,m}$ satisfies in the limit, when $p \to \infty$, is

$$\begin{cases} (u_m)_t + A_{\infty,m}(u_m) \ni g \quad \text{in } \Omega \times (0, T] \\ \\ u_m(0) = u_{0_m}. \end{cases} \tag{4.4}$$

Let us show that indeed the operator $A_{\infty,m}$ converges in the resolvent sense to the operator $A_{\infty,\infty}$. For this, we look closer at the stationary problem associated to $A_{\infty,m}$. Denoting $z_m := (I + A_{\infty,m})^{-1}f$, the problem has a solution in the following sense

$$\begin{cases} z_m \in L^1(\Omega),\ z_m^m \in W^{1,\infty}(\Omega) \cap C_0(\overline{\Omega}), \\ \\ |\nabla z_m^m| \leq 1 \text{ a.e., } 0 \geq \displaystyle\int_\Omega (f - z_m)(\xi - z_m^m)dx,\ \forall\, \xi \in \tilde{\mathbb{K}}. \end{cases} \tag{4.5}$$

Lemma 4.1.1. Let Ω be a bounded domain, then for all $f \in L^\infty(\Omega)$ and $\lambda > 0$, we have,

$$(I + \lambda A_{\infty,m})^{-1}f \to (I + \lambda A_{\infty,\infty})^{-1}f \text{ in } L^1(\Omega),$$

when $m \to \infty$.

Proof. We would like to show that there exists a unique function $\underline{z}$ such that, when $m \to \infty$,

$$z_m \to \underline{z} \ \text{ in } \ L^1(\Omega) \tag{4.6}$$

and

$$\underline{z} \in \operatorname{sign}(w) \ \text{ such that } \ 0 \geq \int_\Omega (f - \underline{z})(\xi - w)dx, \ \forall \xi \in \tilde{\mathbb{K}}. \tag{4.7}$$

Since z_m^m is Lipschitz continuous, let us consider $y \in \partial\Omega$, then

$$|z_m^m(x)| \leq |z_m^m(x) - z_m^m(y)| + |z_m^m(y)| \leq \|\nabla z_m^m\|_\infty |x - y| \leq \operatorname{diam}(\Omega).$$

Therefore, there exists a subsequence $\{m_i\}$ such that, for some w,

$$z_{m_i}^{m_i} \to w \text{ uniformly} \tag{4.8}$$

and

$$\nabla z_{m_i}^{m_i} \overset{*}{\rightharpoonup} \nabla w \ \text{ in } \ L^\infty(\Omega : \mathbb{R}^N). \tag{4.9}$$

Moreover

$$\|\nabla w\|_\infty \leq \liminf_{m_i \to \infty} \|\nabla z_{m_i}^{m_i}\|_\infty \leq 1. \tag{4.10}$$

We will now use the Frechét-Kolmogorov's theorem to prove the relative compactness in $L^1(\Omega)$ of $\{z_m, \ m > 1\}$ and therefore the existence of a function $\underline{z}$ to which a subsequence of z_m converges in $L^1(\Omega)$. According to this result, since $\|z_m\|_1 \leq \|f\|_1$, to prove the relative compactness, it would be enough to prove that for every $y \in \mathbb{R}^N$ small enough and $\Omega' \subset\subset \Omega$ there exists a continuous function ψ such that

$$\sup_m \|z_m(x + y) - z_m(x)\|_{L^1(\Omega')} \leq \psi(y), \tag{4.11}$$

and

$$\lim_{|y| \to 0} \psi(y) = 0.$$

Let us consider the equation in (4.5). Then, for all $\Omega' \subset\subset \Omega$, $\xi_1, \ \xi_2 \in \tilde{\mathbb{K}}$ with $\operatorname{supp} \xi_i \subset \Omega'$ and $y \in \mathbb{R}^N$ such that $|y| < \operatorname{dist}(\operatorname{supp} \xi_i, \partial\Omega)$, $i = 1, 2$, the following holds

$$0 \geq \int_{\Omega'} (f(x) - z_m(x))(\xi_1(x) - z_m^m(x))dx \tag{4.12}$$

and

$$0 \geq \int_{\Omega'} (f(x + y) - z_m(x + y))(\xi_2(x) - z_m^m(x + y))dx. \tag{4.13}$$

Let us take a sequence of functions $h_\delta \in C^\infty(\Omega)$, $0 \leq h'_\delta \leq 1$, $0 = h_\delta(0) \leq |h_\delta| \leq 1$ and the following choices for ξ_1 and ξ_2

$$\xi_1(x) = h_\delta(z_m^m(x + y) - z_m^m(x)) + z_m^m(x),$$

and

$$\xi_2(x) = -h_\delta(z_m^m(x+y) - z_m^m(x)) + z_m^m(x+y).$$

Adding (4.12) and (4.13), we have that,

$$\begin{aligned}
&\int_{\Omega'} h_\delta(z_m^m(x+y) - z_m^m(x))(z_m(x+y) - z_m(x))dx \\
&\qquad \leq \int_{\Omega'} (f(x+y) - f(x))h_\delta(z_m^m(x+y) - z_m^m(x))dx \\
&\qquad \leq \int_{\Omega'} |f(x+y) - f(x)|dx.
\end{aligned}$$

Taking h_δ such that $h_\delta(r) \to \mathrm{sign}_0(r)$ as $\delta \to 0$,

$$\int_{\Omega'} |z_m(x+y) - z_m(x)|dx \leq \int_{\Omega'} |f(x+y) - f(x)|dx$$

and (4.11) is satisfied. By (4.6) and (4.8), it then follows that $\underline{z} \in \mathrm{sign}(w)$. Recall as well

$$\|z_m\|_\infty \leq \|f\|_\infty.$$

Therefore using also (4.8), taking the limit as m tends to infinity in (4.5), we get

$$0 \geq \lim_{m\to\infty} \int_\Omega (f - z_m)(\xi - z_m^m)dx = \int_\Omega (f - \underline{z})(\xi - w)dx.$$

To prove uniqueness, let us suppose that there exists two solutions, that is, $z_i \in \mathrm{sign}(w_i)$, $i = 1, 2$, which satisfy

$$0 \geq \int_\Omega (f - z_1)(\xi - w_1)dx \tag{4.14}$$

and

$$0 \geq \int_\Omega (f - z_2)(\xi - w_2)dx. \tag{4.15}$$

Substituting $\xi = w_2$ and $\xi = w_1$ respectively in (4.14) and (4.15), since $w \in \tilde{\mathbb{K}}$, by (4.10), we obtain

$$0 \geq \int_\Omega (z_1 - z_1)(w_1 - w_2)$$

and therefore the solution is unique.

□

Corollary 4.1.2. $\overline{A}_{\infty,\infty}$ is m-T-accretive and for all $f \in L^1(\Omega)$ and $\lambda > 0$, we obtain, when $m \to \infty$,

$$(I + \lambda\overline{A}_{\infty,m})^{-1} f \to (I + \lambda\overline{A}_{\infty,\infty})^{-1} f \ \text{ in } \ L^1(\Omega).$$

Proof. Similarly to corollary 2.3.4, it is enough to prove that $A_{\infty,\infty}$ is T-accretive, which follows since $A_{\infty,\infty} \subseteq A^{(1)} \circ \phi_\infty$, where we recall that $A^{(1)}$ is defined as

$$A^{(1)} = \left\{ (u,v) \in \tilde{\mathbb{K}} \times L^1(\Omega) : \int (u-w)v \geq 0 \text{ for } w \in \tilde{\mathbb{K}} \text{ with } (u-w)v \in L^1(\Omega) \right\}.$$

□

We are now ready to obtain the regular limit of the solutions u_m and the equation it satisfies, under the additional condition that $\|u_{0_m}\|_\infty \leq 1$. The following theorem holds.

Theorem 4.1.3. Consider the problem in (4.4), where u_{0_m} and g satisfy the same conditions as in theorem 2.3.5, as well as $\|u_{0_m}\|_\infty \leq 1$. Then, there exists a subsequence m_i tending to infinity, and a unique function $\underline{u}$ such that, for each $T > 0$,

$$u_{m_i} \to \underline{u} \text{ in } C(0,T;L^1(\Omega)),$$

with

$$u_{0_m} \to u_{0_\infty} \text{ in } L^1(\Omega),$$

and $\underline{u}$ is the unique mild solution of

$$\begin{cases} \underline{u}_t + A_{\infty,\infty}(\underline{u}) \ni g & \text{in } \Omega \times (0,T] \\ \\ \underline{u}(0) = u_{0_\infty}, \end{cases} \tag{4.16}$$

where $A_{\infty,\infty}$ is given by (4.3).

Proof. The result follows from lemma 4.1.1, corollary 4.1.2 and from the classical results in theorem 1.5.3, since $u_{0_m} \in \overline{D(A_{\infty,m})}$ and $u_{0_\infty} \in \overline{D(A_{\infty,\infty})}$. □

4.2 Asymptotic behaviour of the limit equation in m of the (DNE) when $p \to \infty$

In chapter 3, assuming certain conditions on the initial data u_0, source term g and the domain Ω, we obtained the convergence of solutions $u_{m,p}$ of $(DNE)_{p,m}$ as m tends to infinity. It was necessary to study separately the cases $\|u_0\|_\infty \leq 1$ and $\|u_0\|_\infty > 1$. For each case, we identified the equation that the corresponding solutions satisfy in the limit. We will henceforth in this section study the asymptotic behaviour of the regular solutions, as the parameter p goes to infinity.

It was already proved, under certain restrictive conditions, that the operator $A_{p,m}$ converges to $A_{p,\infty}$, defined as

$$v \in A_{p,\infty}u \Longleftrightarrow \begin{cases} u, v \in L^1(\Omega), \ \exists\, w \in W_0^{1,p}(\Omega), \ u \in \operatorname{sign}(w) \text{ a.e. in } \Omega \\ \\ \text{and } \ -\Delta_p w = v \ \text{ in } \ \mathcal{D}'(\Omega). \end{cases} \tag{4.17}$$

Given what we know about the behaviour of the p-Laplace operator when $p \to \infty$ (see (2.27)), it is natural to seek the convergence in the resolvent sense of the operator $A_{p,\infty}$ to the operator $A_{\infty,\infty}$, defined as follows:

$$v \in A_{\infty,\infty}u \Longleftrightarrow \begin{cases} u, v \in L^1(\Omega), \ \exists\, w \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega}) \ \text{ with } \ w \in \tilde{\mathbb{K}}, \ w = 0 \ \text{ on } \ \partial\Omega \\ \\ u \in \operatorname{sign}(w) \text{ a.e. in } \Omega \ \text{ and } \ 0 \geq \displaystyle\int_\Omega v(\xi - w)dx \ \forall\, \xi \in \tilde{\mathbb{K}}, \end{cases} \tag{4.18}$$

with

$$\tilde{\mathbb{K}} := \{\xi \in L^1(\Omega) : \ |\nabla \xi| \leq 1 \ \text{ a.e.}\}.$$

Hence, we will study the elliptic problem associated to the operator $A_{p,\infty}$. Denoting $z_p := (I + A_{p,\infty})^{-1} f$ for all $f \in L^\infty(\Omega)$, this problem has a solution in the following sense

$$\begin{cases} z_p \in L^1(\Omega), \ \exists\, w_p \in W_0^{1,p}(\Omega), \ z_p \in \operatorname{sign}(w_p) \text{ a.e. in } \Omega \\ \\ \text{and } \ -\Delta_p w_p = f - z_p \ \text{ in } \ \mathcal{D}'(\Omega). \end{cases} \tag{4.19}$$

Lemma 4.2.1. For all $f \in L^\infty(\Omega)$ and $\lambda > 0$, we have

$$(I + \lambda A_{p,\infty})^{-1} f \to (I + \lambda A_{\infty,\infty})^{-1} f \ \text{ in } \ L^1(\Omega),$$

when $p \to \infty$.

Proof. Recall that by proposition (2.2.1) (*iv*), for $f \in L^\infty(\Omega)$ and $z_{p,m}$ a solution of (2.30), we have

$$\|z_{p,m}\|_r \leq \|f\|_r \ \text{ for any } \ 1 \leq r \leq \infty,$$

and taking $m \to \infty$, it continues to hold

$$\|z_p\|_r \leq \|f\|_r \ \text{ for } \ 1 \leq r \leq \infty, \tag{4.20}$$

for $z_p = (I + A_{p,\infty})^{-1} f$. On the other hand, we see that there exists a w_p which is a solution of the equation in (4.19) and therefore satisfies

$$\int_\Omega |\nabla w_p|^{p-2} \nabla w_p \cdot \nabla \varphi dx = \int_\Omega (f - z_p)\varphi dx \ \ \forall\, \varphi \in \mathcal{D}(\Omega).$$

By density, we can consider $\varphi = w_p$ in the previous expression to obtain

$$\begin{aligned}\int_\Omega |\nabla w_p|^p dx &\leq \|f - z_p\|_\infty \|w_p\|_1\\ &\leq C\|f - z_p\|_\infty \|\nabla w_p\|_1\\ &\leq 2C\|f\|_{L^\infty(\Omega)} \left(\int_\Omega |\nabla w_p|^p dx\right)^{\frac{1}{p}} |\Omega|^{1-\frac{1}{p}}.\end{aligned}$$

The second inequality is due to Poincaré's inequality, with $p = 1$, and for the third we use (4.20) and Hölder's inequality. We have as well, by Hölder's inequality,

$$\|\nabla w_p\|_q \leq \|\nabla w_p\|_p |\Omega|^{\frac{1}{q}-\frac{1}{p}},$$

for any $p > q$ and we obtain that $\{w_p\}$ is uniformly bounded in $W_0^{1,q}(\Omega)$ for any $q > 1$. Hence, there exists a subsequence $\{p_i\}$ and a function w such that, when $p_i \to \infty$,

$$w_{p_i} \rightharpoonup w \text{ in } W^{1,q}(\Omega), \text{ for any } q > 1.$$

Thus, passing as necessary to yet another subsequence and relabeling, we deduce

$$\begin{cases} w_{p_i} \to w \text{ in } L^q(\Omega),\\ w_{p_i} \to w \text{ a.e.}\end{cases} \tag{4.21}$$

By the bound in (4.20),we have that there exists a function z such that, for q' the conjugate of q, when $p_i \to \infty$,

$$z_{p_i} \rightharpoonup z \text{ in } L^{q'}(\Omega).$$

Recalling that $z_p \in \operatorname{sign}(w_p)$, then by Lemma 1.3.4, it continues to hold in the limit that $z \in \operatorname{sign}(w)$. Moreover,

$$z_{p_i} \to z \text{ a.e.}$$

This together with (4.20) gives us the following strong convergence

$$z_{p_i} \to z \text{ in } L^1(\Omega). \tag{4.22}$$

Besides, by the equation in (4.19), we also have that, for all $\xi \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$,

$$\frac{1}{p}\int_\Omega |\nabla \xi|^p dx \geq \frac{1}{p}\int_\Omega |\nabla w_p|^p dx + \int_\Omega (f - z_p)(\xi - w_p)dx \geq \int_\Omega (f - z_p)(\xi - w_p)dx.$$

Taking $\xi \in \tilde{\mathbb{K}}$, assuming by approximation that ξ has compact support, we have by (4.21) and (4.20),

$$0 \geq \lim_{p_i \to \infty} \int_\Omega (f - z_{p_i})(\xi - w_{p_i})dx = \int_\Omega (f - z)(\xi - w)dx.$$

Moreover, we have that,

$$\|\nabla w\|_\infty \leq 1. \tag{4.23}$$

Indeed, as in lemma 2.3.2, taking $\eta > 0$ and denoting

$$A_\eta = \{x \in \Omega \mid |\nabla w| \geq 1 + \eta\},$$

then

$$(1+\eta)|A_\eta| \leq \int_{A_\eta} |\nabla w| dx \leq \liminf_{p\to\infty} \left(\int_\Omega |\nabla w_p|^p dx \right)^{1/p} |A_\eta|^{1-1/p} \leq |A_\eta|$$

and $|A_\eta| = 0$, showing that (4.23) holds.

Uniqueness follows as in the previous chapters, assuming that there exist solutions z_1, z_2 such that for the respective w_1, w_2 with $z_i \in \text{sign}(w_i)$, $i = 1, 2$, we have

$$0 \geq \int_\Omega (f - z_1)(\xi - w_1) dx,$$

and

$$0 \geq \int_\Omega (f - z_2)(\xi - w_2) dx.$$

Taking $\xi = w_2$ and $\xi = w_1$ respectively in the previous inequalities and adding we see that the solution to the equation must be unique. □

Similarly to corollary 4.1.2, we have the following result.

Corollary 4.2.2. $\overline{A}_{\infty,\infty}$ is m-accretive and for all $f \in L^1(\Omega)$ and $\lambda > 0$, when $p \to \infty$, we have

$$(I + \lambda \overline{A}_{p,\infty})^{-1} f \to (I + \lambda \overline{A}_{\infty,\infty})^{-1} f \text{ in } L^1(\Omega).$$

Whenever the conditions of theorem 3.2.9 are satisfied, the equation that the solution of $(DNE)_{p,m}$ satisfies in the limit, when $m \to \infty$, is

$$\begin{cases} (u_p)_t + A_{p,\infty}(u_p) \ni g \quad \text{in } \Omega \times [0,T] \\ \\ u_p(0) = u_0. \end{cases} \tag{4.24}$$

In the same way as in the previous chapters, we have the following result.

Theorem 4.2.3. Consider the problem (4.24), where $g \in L^1(\Omega_T)$, $A_{p,\infty}$ is defined in (4.17) and one of the following conditions is satisfied:

(i) Ω is a bounded interval in $\mathbb{R}$ and u_0 is nonnegative such that $\|u_0\|_\infty \leq 1$ or (*ii*) $\Omega = B(0,R)$ and u_0 is radial and nonnegative such that $\|u_0\|_\infty \leq 1$.

Then, there exists a function u such that, for each $T > 0$,

$$u_p \to u \text{ in } C(0,T;L^1(\Omega))$$

and u is the unique mild solution of

$$\begin{cases} u_t + A_{\infty,\infty} u \ni g & \text{in } \Omega \times [0,T] \\ \\ u(0) = u_0, \end{cases} \tag{4.25}$$

where $A_{\infty,\infty}$ is defined in (4.18).

Proof. The result holds using theorem 4.2.1 and corollary 4.2.2, as well as theorem 1.5.3, considering that $u_0 \in \overline{D(A_{\infty,\infty})}$. □

4.3 Complete diagram

Therefore, by the results of lemmas 2.3.3, 3.2.7, 4.1.1 and 4.2.1, the following is satisfied.

Lemma 4.3.1. Let one of the following conditions hold:

(*i*) Ω is a bounded interval in $\mathbb{R}$ and $f \in L^\infty(\Omega)$ is nonnegative,

(*ii*) Ω is a ball in $\mathbb{R}^N$ and $f \in L^\infty(\Omega)$ is radial and nonnegative.

Then, we have,

$$L^1 - \lim_{p\to\infty} \lim_{m\to\infty} (I + \lambda A_{p,m})^{-1} f = L^1 - \lim_{m\to\infty} \lim_{p\to\infty} (I + \lambda A_{p,m})^{-1} f = (I + \lambda A_{\infty,\infty})^{-1} f,$$

where we recall that $A_{p,m}$ is defined as

$$\begin{gathered} A_{p,m} u = -\Delta_p u^m, \\ \mathrm{D}\,(A_{p,m}) = \{\ u \in L^\infty(\Omega) : \ u^m \in W_0^{1,p}(\Omega) \text{ and } \ \Delta_p u^m \in L^1(\Omega)\ \}, \end{gathered} \tag{4.26}$$

and $A_{\infty,\infty}$ is given by

$$v \in A_{\infty,\infty}(u) \Longleftrightarrow \begin{cases} u, v \in L^1(\Omega),\ \exists\, w \in W^{1,\infty}(\Omega) \cap C_0(\overline{\Omega}) \text{ with } w \in \tilde{\mathbb{K}}, \\ \\ u \in \operatorname{sign}(w) \text{ and } 0 \geq \displaystyle\int_\Omega v(\xi - w)dx, \quad \forall\, \xi \in \tilde{\mathbb{K}}, \end{cases} \tag{4.27}$$

with

$$\tilde{\mathbb{K}} := \{\xi \in L^1(\Omega) : \ |\nabla \xi| \leq 1 \text{ a.e.}\}.$$

More importantly, if we consider one of the following hypotheses:

(i) Ω is a bounded interval in $\mathbb{R}$ and u_{0_m} is non-negative, $u_{0_m}^m \in \tilde{\mathbb{K}}$ and $\|u_{0_m}\|_\infty \leq 1$ or

(ii) $\Omega = B(0, R)$ and u_{0_m} is radial, non-negative such that $u_{0_m}^m \in \tilde{\mathbb{K}}$ and $\|u_{0_m}\|_\infty \leq 1$,

as well as $g \in L^1(\Omega_T)$, then the equation that the solutions $u_{p,m}$ of $(DNE)_{p,m}$ in (2.1) satisfy at the limit, when p and m tend to infinity, in sequence, is the same whichever limit we take first.

Hence, the results can be summarized in the convergence diagram below.

$$\begin{tikzcd}
\boxed{\begin{array}{c} (u_{p,m})_t + A_{p,m}u_{p,m} = g \\ u_{p,m}(0) = u_{0_m} \end{array}} \arrow[r, "p \to \infty"] \arrow[d, "m \to \infty"'] & \boxed{\begin{array}{c} (u_m)_t + A_{\infty,m}u_m \ni g \\ u_m(0) = u_{0_m} \end{array}} \arrow[d, "m \to \infty"] \\
\boxed{\begin{array}{c} (u_p)_t + A_{p,\infty}u_p \ni g \\ u_p(0) = u_{0_\infty} \end{array}} \arrow[r, "p \to \infty"'] & \boxed{\begin{array}{c} u_t + A_{\infty,\infty}u \ni g \\ u(0) = u_{0_\infty} \end{array}}
\end{tikzcd}$$

Figure 4.1: Complete convergence diagram

Conclusions

In this thesis, we considered a doubly nonlinear diffusion equation

$$u_t = \Delta_p u^m + g \quad \text{with} \quad m(p-1) > 1,$$

homogeneous Dirichlet boundary conditions, nonnegative integrable initial data u_0 and integrable source term g. In order to discuss the properties of the doubly nonlinear equation within the nonlinear semigroup theory we associated it to an accretive operator $A_{p,m}$ via the results in [8] and [33]. Once it was proved that $A_{p,m}$ converges in the resolvent sense to an operator denoted by $A_{\infty,m}$ as $p \to \infty$, the convergence of the mild solutions followed under consistent initial values ($u_0 \in \overline{D(A_\infty, m)}$), i.e., $\|\nabla u_0^m\|_\infty \leq 1$. This allowed to prove the convergence also in the case that $\|\nabla u_0^m\|_\infty > 1$ in a more abstract formulation. The results constituted a generalization of the results in [4] and [27] for the p-Laplace equation to the doubly nonlinear equation in the setting of mild solutions.

In the case of the asymptotic behaviour with respect to m, we showed that the results proved in [33] for the Cauchy problem associated to the doubly nonlinear equation apply as well to the Dirichlet problem in a bounded interval of the real line and in a ball, considering furthermore a radial initial data. The difficulty in this case lied in proving the convergence of the solutions of the associated elliptic problem in $L^1(\Omega)$, needed to prove the convergence of the operator $A_{p,m}$ to the operator $A_{p,\infty}$, and thus the convergence of the mild solutions for $u_0 \in \overline{D(A_{p,\infty})}$, i.e., $\|u_0\|_\infty \leq 1$.

In terms of future work in this matter, it would be interesting to prove that the convergence of the operators also hold for more general bounded domains, even for domains not necessarily bounded in $\mathbb{R}^N$. To this end, we would need to find a way to overcome the extra difficulty provided by the non-linearity of the p-Laplacian in several dimensions. More importantly, it would be very interesting to work further towards proving the conjecture, as formulated in [15] for the doubly nonlinear equation, in order to generalize the results known for the singular limit of the porous medium equation. In that case, one of the important steps used to prove the convergence to the stationary Hele-shaw problem is a Baiocchi type transformation necessary to pass from the evolution problem to the stationary one. It would probably be necessary to find out what is the equivalent one for the general doubly nonlinear equation.

After identifying the operators $A_{\infty,m}$ and $A_{p,\infty}$ to which $A_{p,m}$ converge, as p and m tend to infinity respectively, we prove that these operators converge to the operator $A_{\infty,\infty}$, completing the convergence diagram for the operators and hence of the associated mild solutions. In this case, only consistent initial data were considered. We concluded that under the combined conditions to ensure the convergence of the operators with respect to both the parameters p and m, the equation satisfied in the limit is the same, whichever limit we take first.

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