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Reliability Assessment and Reliability-Based Design of Plastic Shallow Curved Plates

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Abstract

A unified formalism allows the reliability-based design of stretching plates, plates in bending and shallow curved plates discretized in triangular finite elements to be treated in a similar fashion. The structural material is assumed to exhibit perfectly-plastic behaviour so that plastic collapse is the only possible failure mode. It consists of solving alternatively a reliability assessment (concave quadratic programming) and an optimal sizing problem (convex minimization) until the best solution is found. Numerical examples illustrate the procedure.

Keywords: shells, plasticity, reliability, concave programming, optimization.

1 Introduction

The finite element discretization is assumed to lead to smoothly continuous finite elements of constant thickness, the kinematic and static governing relations can be established according to the theory of curved plates. The shallow approximation consists in assuming that all fibers parallel to the middle surface and defined between the same two straight lines normal to the middle surface are of equal length. The curved plate approximation consists in assuming that the curvatures are constant and sufficiently small to allow for terms involving the product of curvatures to be neglected. The theory of shallow shells was established by Donnell (1934), Vlasof (1939) and Jenkins (1947) and implies solely the shallowness of the shell geometry while Maguerre (1938) implies a zero curvature shell surface.
The literature on finite element method formulation of plastic limit shell problems is very limited. A formulation for shallow shells was presented by Cyras, Karkauskas and Atkochyunas [1] et al and Kufner and Lee [2]. The later is an extension of the formulation developed by Anderheggen and Knopfel [3] for plastic limit problems. Da Fonseca [4] has shown the problem can be reduced to a pair of primal-dual linear

programs if a linearized yield condition is used. The general case of a three dimensional continuous structure could be studied and any particular case would then be derived by the introduction of the relevant simplifications, but for the sake of brevity the plate stretching problem is considered here. The formulation describing the fundamental relations of the problem reflects the finite element connectivity across the interelement boundaries and is thereby called kinematic description. The material is assumed stable in Drucker´s sense and the convex hypersurface is replaced by a set of convex hyperplanes. The optimum solution to the discretized problem may provide a bound to the solution to the continuous problem, but the existence and nature of such bound depends on the finite element modeling. Plastic limit analysis gives the collapse load which does not necessarily lead to the stochastically most relevant event. Moreover the contribution of other stochastically important mechanisms must be taken into account. Assessment of the reliability of a structure has to take into account that during its design life the structure is generally subjected to a number of varying loads and their combinations and its resistance may deteriorate with time. In this context the problem arises as how to evaluate the conditional failure probability of the shell given a certain load event. In order to avoid the difficult numerical integration of the probability distribution functions involved, the first order second moment approximation is employed. The reliability index is obtained from the limit state equation by minimizing a concave quadratic function over a linear domain, the local solution are vertices of the domain. A branch and bound technique is used to solve this MP and to enumerate local solutions which may also be important for the reliability assessment [5]. The reliability-based design consists of solving alternatively the reliability assessment and an optimal sizing problem (convex minimization) until the best solution is found. Two numerical examples are solved, namely the reliability–based design of a concrete pier (plate stretching) and of a reinforced concrete floor (thin flexural plate)

2 Problem Statement

2.1 Formulation

The following assumptions are made:
Loads and resistances are assumed to be random variables; statistical dependence among loads and among resistances are accounted for through the coefficients of correlation between loads and plastic moments; the necessary statistical information is assumed to be available; shear deformation neglected; local fracture, instability and other possible causes of failure are avoided; the load and rotations are deterministic.
The objective of reliability-based plastic optimization of concrete shells is usually to find the shell characteristics so that a specified reliability level against plastic collapse be provided and an adopted objective function be minimized. For a shell with known shape and whose thickness and boundary conditions are also known, the

reliability-based design problem reduces to finding the minimum average plastic moments of resistance (positive and negative) per unit length.
The solution method is divided in two alternating sub-procedures [6]:
a) an optimization procedure for the nonconvex inner problem that finds the stochastic most important mechanism and enumerates other relevant collapse modes for a given value of the design variables.
b) an optimization of the convex outer problem on the design variables that is the solution with least average plastic moments of resistance satisfying the reliability constraints.
The procedure is repeated until the vector of design variables converges. Since these two procedures are themselves mathematical programs, any suitable technique can be applied.

3 Governing Relations of Plate Stretching

3.1 Kinematic description

All these formulations state the limit analysis mathematical programs as direct applications of the limit analysis principles and the finite element models use as variables either displacements or stress-resultants at the shell middle surface. The single restriction on the loading forces applied upon the structure refers to the surface forces which are considered to be confined to the lateral forces perpendicular to the middle surface. The following two assumptions of the theory of shells are considered: straight lines which are normal to the middle surface remain straight during the deformation of the shell; the stress normal to the middle surface is negligible. As for problems of stretching and of bending of flat plates, surface forces applied upon the external faces parallel to the mid surface have to be taken either as surface forces acting on the interelement faces or as body forces acting in the middle surface.
In the formulation of the finite element method, three distinct levels: (i) the infinitesimal element level, (ii) the finite element level and (iii) the structural level are defined. For plastic collapse the interelement equilibrium may be achieved if new nodes are selected to define the independent stress field Γ_σ in terms of the nodal values σ^ε as follows:

$$\sigma = \Gamma_\sigma \, \sigma^e \tag{1}$$

The strain-resultant/displacement relations at the infinitesimal level are (by omitting the initial imposed strains)

$$- \Delta\varepsilon + D \, \Delta u = 0 \tag{2}$$

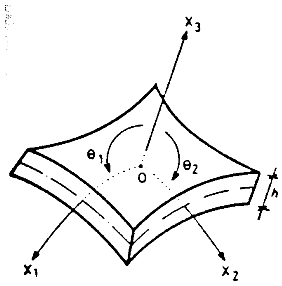


Figure 1 Rigid body rotation of the normal to the middle surface of the shell

where D is the 2d order differential operator. By dividing the finite element volume into subdomains associated with the control nodes, we have

$$\int_v \Gamma_\sigma^t \, \Delta\varepsilon \, dv = \Sigma_{j=1,c} \int_{vj} \Gamma_\sigma^t \, \Delta\varepsilon \, dv_j \tag{3}$$

The transposed field matrix Γ_σ^τ is assumed to have the constant value $(\Gamma_\sigma^\tau)_j$ inside each subdomain that it takes at the corresponding cj control node. The conditions of compatibility at finite element level read,

$$(\Gamma_\sigma^c)^t \, \Delta g^c = - E \, \Delta u^a = 0 \tag{4}$$

Where the total nodal strains and the compatibility matrix E are given by

$$\Delta g^c = \Sigma_{j=1,c} \int_{vj} \Delta\varepsilon \, dv \qquad ; \qquad E = \int_v \Gamma_\sigma^t \, D \, \Gamma_u \, dv \tag{5}$$

At the structural level, if the relevant coordinate transformation matrices T_σ^e and T_u^a are introduced, the compatibility equations become

$$(T_\sigma^e)^t \, (\Gamma_\sigma^c)^t \, T_g^c \, \Delta g^{cs} - (T_\sigma^e)^t \, E \, T_u^a \, \Delta u^{as} = 0 \tag{6}$$

Or in a more compact form

$$R^s \, \Delta g^{cs} - E^s \, \Delta u^{as} = 0 \tag{7}$$

3.2 Plasticity relations

Only the plastic phase of the structural material behavior has to be characterized. The conditions of yielding to occur may be defined by an inequality involving a function of the stress state. If such a hypersurface is replaced by a set of hyperplanes, the yield condition for every control node is given by,

$$Q^t \, T_\sigma^t \, \Gamma_\sigma \, \sigma^e - \sigma_* \leq 0 \tag{8}$$

If the conditions are stated for all cs nodes, their assembly can be written

$$\sigma^{cs} - s_*^{cs} = (Q^{cs})^t (R^s)^t \sigma^{es} - \sigma_*^{cs} \leq 0 \tag{9}$$

Where R^s is given as in (7). For a stable material the kinematic variables are defined by an associated flow rul, that for a control node is

$$\Delta g_j = \int_{vj} \Delta\varepsilon_j \, dv = \int_{vj} Q \, (\Delta\varepsilon_*)_j \, dv = Q \, (\Delta g^*)_j \tag{10}$$

Where are the total plastic parameters associated with the subdomain of unspecified volume inside which the material properties (expressed through matrix Q) are assumed constant. For all cs control nodes

$$\Delta g^{cs} = Q^{cs} \, \Delta g_*^{cs} \tag{11}$$

If the assumption of constant stress field and of constant strain field is introduced, the total plastic dissipation energy ΔD^{cs} becomes

$$\Delta D^{cs} = (\sigma^{cs})^t \, \Delta g^{cs} = (\sigma_*^{cs}) \, \Delta g_*^{cs} \geq 0 \tag{12}$$

For simplicity, superscripts s,c,a and e will be dropped.

4 Inner Problem

4.1 Reliability Assessment

The probability of failure via the k-th individual collapse mode pk can be obtained from the probability that a certain performance function Zk

$$Z_k = U_k - E_k = \mu_\sigma \, \Delta g - \mu_d \, \Delta u^d - \mu_l \, \Delta u^l \tag{13}$$

is negative. In eq. (13) U_k and E_k are the internal and external random works associated with the k collapse mode. Consistent with a first order second moment reliability analysis, the failure probability may be measured entirely by a function of the first and second moments of random parameters. It is assumed that safety with regard to plastic collapse via the failure mode k depends only on the reliability index β_k that is defined as the shortest distance from the origin to a failure surface in the reduced random variables coordinate system [7]

$$\beta_k = \mu_{Zk}/\sigma_{Zk} \tag{14}$$

4.2 Branch and Bound technique for quadratic concave minimization

By adding the finite element strains associated with stresses in half space represented by the same random variables, one has

$$\Delta g_*^+ = J_g^+ \; \Delta g_* \qquad ; \qquad \Delta g_*^- = J_g^- \; \Delta g_* \tag{15}$$

If the displacements of the point loads (or in the case of uniformly distributed load, deflections of the finite element centroids) linked to dead and live loading, respectively, are summed up,

$$\Delta u_*^d = J_u^d \; \Delta u \qquad ; \qquad \Delta u_*^l = J_u^l \; \Delta u \tag{16}$$

For statistically independent random normal variables, the mathematical program that gives the reliability index β associated with the stochastic most important collapse mode is

$$\beta = \frac{\mu_{\sigma+} \; \Delta g_*^+ - \mu_{\sigma-} \; \Delta g_*^- - \mu_d \; \Delta u_*^d - \mu_l \; \Delta u_*^l}{\sqrt{(\sigma_{\sigma+})^2 \; (\Delta g_*^+)^2 + (\sigma_{\sigma-})^2 \; (\Delta g_*^-)^2 - (\sigma_d)^2 \; (\Delta u_*^d)^2 - (\sigma_l)^2 \; (\Delta u_*^l)^2}} \tag{17}$$

Subject to the linear incidence equations, the compatibility relations and sign constraints on the variables. This mathematical program belongs to the class of fractional programming problems and shares its solutions with,

$$\text{Min} \; -1/\beta^2 = -(\sigma_{\sigma+})^2 (\Delta g_*^+)^2 - (\sigma_{\sigma-})^2 (\Delta g_*^-)^2 - (\sigma_d)^2 (\Delta u_*^d)^2 - (\sigma_l)^2 (\Delta u_*^l)^2 \tag{18}$$

Subject to sign constraints and

$$\mu_{\sigma+} \Delta g_*^+ - \mu_{\sigma-} \Delta g_*^- - \mu_d \Delta u_*^d - \mu_l \Delta u_*^l = 1 \tag{19}$$

That is a quadratic concave minimization. This type of problem cannot be solved by convex programming techniques because the solutions are vertices of the linear domain and it is intended to minimize a concave function. The global optimum of these programs gives the plastic deformations for the stochastic most important mechanism and the reduced random variables are

$$\sigma^{+'} = -\sigma_{\sigma+} \Delta g_*^+ \beta^2 \qquad ; \qquad \sigma^{-'} = -\sigma_{\sigma-} \Delta g_*^- \beta^2 \tag{20}$$

$$d' = -\sigma_d \Delta u_*^d \beta^2 \qquad ; \qquad l' = -\sigma_l \Delta u_*^l \beta^2 \tag{21}$$

The general nonconvex domain is transformed in the branch and bound (B&B) strategy into a sequence of intersecting convex domains by the use of convex underestimating functions. The two main ingredients are a combinatorial tree where the nodes are associated with linear programs and some upper and lower bounds to the final solution related to each node of the tree. For a quadratic concave function its convex underestimate is the affine function (linear plus constant) passing through the endpoints of the given function graph. Tight bounds on the nonlinear variables can be found by solving a multiple row linear program [5]. In the last two decades some new techniques appeared [8-10] (genetic algorithms, the differential evolution, ant colony technique, particle swarm optimization, the artificial immune system).

4.3 Multimode failure probability

Evaluation of the failure probability appearing in constraints of eq. (17) is one of the major concerns in the solution of reliability-based optimization problems. Since multiple integration with respect to random parameters must be executed and correlations of each failure mode must be known a priori, an exact calculation of the overall system probability is practically impossible without resorting to an approximation method. In general the admissible failure probability of the structure is very low. Cornell´s first order bounds could be employed but they are generally too wide. Improved bounds can be obtained by Vanmarke´s concept of failure mode decomposition of Ditlevsen [11] method of conditional bounding which take into account the probabilities of joint failure events. Collapse modes were found to be highly correlated in the plate stretching and thin flexural plate examples.

4.4 Thin Flexural Plates

The formulation hold unaltered just by re-interpretation of symbols for the classs of above mentioned discrete plastic models. The material is considered to satisfy a yield criterion formulated by Nielsen for reinforced concrete plates. In order to obtain linearized yield conditions, a safe linearization suggested by Wolfensberger [12] which considers an octahedron is adopted.

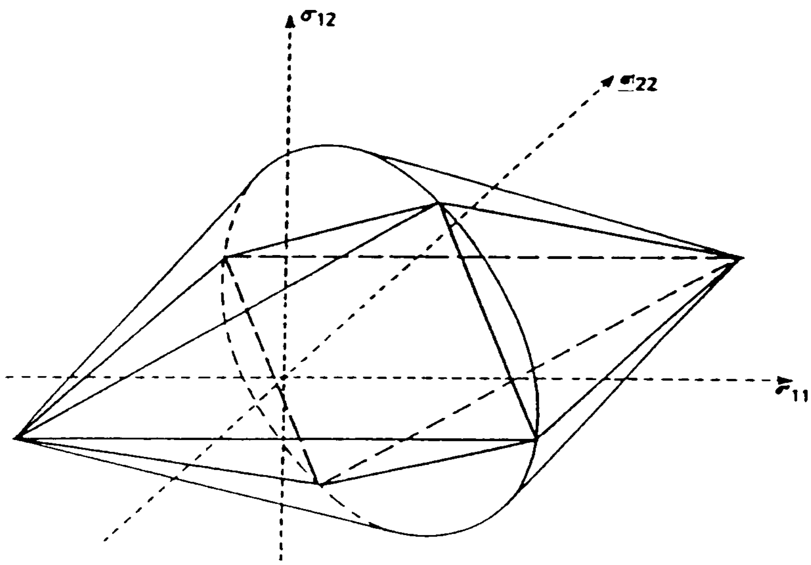


Figure 2 Eight yield planes for in-plane forces and bending

4.5 Shallow Curved Plates

The mathematical characterization of the plastic behavior of a general shell would require an yield criterion involving ten stress-resultants and ten strain-resultants. However as for flexural plates the effect of two transverse shear forces is generally negligible. Also since every finite element is shallow, the planar shear forces and the twisting moments are regarded as equal. The kinematic model under consideration can be taken as the superposition of stretching and bending models and the field functions required for the kinematic and static independent variables may be obtained through the superposition of the field functions defined for those models.
The yield criterion used considers separately the stretching and bending problems with no interaction between them. Both for the in-plane forces and for the moments, the linearized Nielsen criterion is adopted

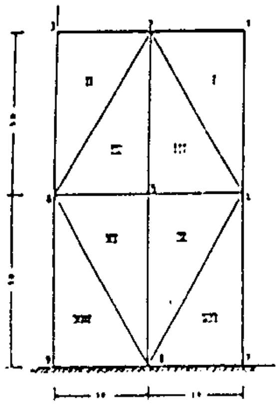
5 Outer Problem

In the context of reliability-based plastic optimization adopted here the objective is to minimize average resistance when the loading, the reliability level against plastic collapse and the shell thickness are prescribed. By fixing the design variables the inner problem gives the yield rotations and nodal displacements associated with the stochastic most important mechanism and other relevant modes. Since the approximation of the multimode constraint is convex, the outer problem can be solved by any convex programming technique. The design is updated and a new iteration performed repeating the process until convergence is achieved. Move limits are imposed at each iteration of the convex outer program.

6 Numerical Examples

6.1 Plate Stretching: Concrete Pier

The problem consists of determining average carrying capacity of concrete given the probability of failure for the concrete pier of a simply supporting bridge deck. If any change of geometry is neglected, the pier can be regarded as a plane stress state. Three nodal finite elements are used in order to define the compatibility matrix. A linear displacement field where nodes a are the corner nodes is assumed.
A constant stress field is considered, where the nodal stresses are at a single node anywhere inside the finite element. Thus, the single node e is coincident with the single control node c. Matrix Q is embodied in the yield criterion that considers the two-dimensional stress-space where the yield function is assumed to correspond to the ellipse of Fig.3. This example was solved in [4] for plastic limit analysis.



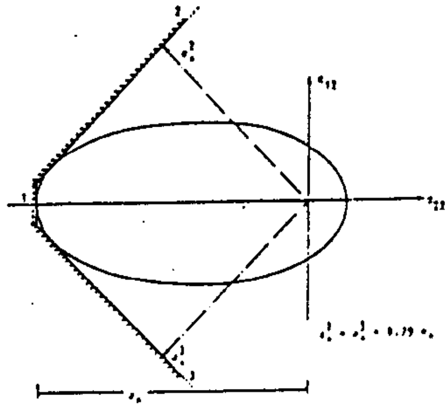
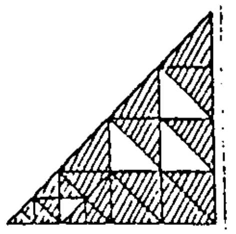


Figure 3 Finite element model   Yield function and linearizing planes

The unsafe linearization is performed by means of the three planes represented. The live load acting on top of the pier is $(\mu_l,\Omega_l)=(8kN/m^2,0.25)$. The dead load due to the bridge deck $(\mu_d,\Omega_d)=(3kN/m^2,0.10)$. The self weight of the pier is considered uncorrelated to the live load. Given a maximum probability of failure $p_f = 10^{-3}$ and the coefficient of variation of the carrying capacity of concrete $\Omega_s = 0.1$, the average carrying capacity of concrete μ_s is $20kN/m^2$.

6.2 Thin flexural plate reinforced concrete floor

Fig. 4 represents the finite element modeling of an octant of a uniformly loaded clamped square plate (side 10m) and a circular plate (diameter 10m), respectively. The plate discretization is done by means of triangular finite elements with a quadratic deflection field and thus six nodal displacement values must be specified (in these examples: vertical deflections at element corners and mid-side normal rotations) and a constant moment element is considered.
These examples were solved for plastic limit analysis in [3]. $(\mu_d,\Omega_d)=(10kNm/m,0.10)$ and $(\mu_l,\Omega_l)=(18kNm/m,0.25)$ are the dead and live transversal loading. Coefficients of variation $\Omega_m^+ = \Omega_m^-=0.05$ were stipulated. If a maximum probability of failure $p_f = 10^{-3}$ is required, the bending moment capacities $\mu_m^+=\mu_m^-=100kNm/m$ were found for the circular plate. These plastic capacities are associated with $p_f = 2.2\ 10^{-3}$ for the square plate.



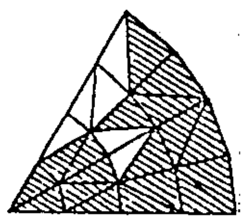


Figure 4 Square plate FE model   Circular plate FE model

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