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E TECNOLOGIA
UNIVERSIDADE DE COIMBRA

Public Facility Planning Models with Single and Multiple Services: Models, Solution Methods and Applications

Doctoral thesis

Thesis submitted to the Faculty of Sciences and Technology of the University of Coimbra in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the field of Civil Engineering, with specialization in Spatial Planning and Transportation.

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Coimbra, June 2012

Financial support

This work had the financial support of *Fundação para a Ciência e a Tecnologia* (FCT, Portugal) through doctoral degree grant SFRH/BD/12672/2003, which was co-financed by the European Social Fund under the Third Community Support Framework, Knowledge Society Operational Programme (POSC), Measure 1.2 – Advanced training.

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Fundação para a Ciência e a Tecnologia
MINISTÉRIO DA EDUCAÇÃO E CIÊNCIA

European Union
European Social Fund

Acknowledgements

I would like to thank all the people who contributed to make this thesis possible and to make my PhD experience rewarding. I first thank my supervisors, António Pais Antunes and Laurence Wolsey, for their guidance, availability and encouragement, and for the opportunity to learn from their knowledge and experience.

I also thank all colleagues and friends with whom I worked and socialized for making my PhD experience much more rewarding and enjoyable. For this I thank my colleagues at the Spatial Planning and Transportation Engineering Group (Department of Civil Engineering, University of Coimbra) and at CORE (Catholic University of Louvain). I also thank my former colleagues at the Figueira da Foz branch of the Portuguese Catholic University, where I was a lecturer when I started the thesis.

Finally, I thank my parents, my brother and other close friends for their continued support during the long period to complete the thesis.

Abstract

This thesis addresses the planning problem of reorganizing an existing network of public facilities, such as schools, hospitals or courts of justice, in response to structural changes in the demand for public services and to the need of improving the cost-effectiveness of service provision. "Public facility planning" is here understood as the activity consisting in making decisions on the number, location, type (in terms of the mix of services offered), and capacity of facilities supplying public services, and on their catchment areas (i.e. the population centers served by each facility).

Public facility planning problems are addressed in this thesis with mathematical programming (or optimization) models that aim to help decision makers arrive at efficient solutions in terms of costs to service providers and of quality of service to users in key components such as accessibility to facilities. More specifically, the optimization models studied here are discrete facility location models, formulated as mixed-integer linear programming (MILP or MIP) models. This thesis focuses on the following basic, single-service model and on extensions of it. The geographic setting is represented by a discrete set of population centers with known demands, a discrete set of sites where facilities can be located, and given travel distances (or times, or costs) between centers and sites. The problem is to locate facilities and assign centers to those facilities, so that each center is assigned to the closest facility, each facility satisfies minimum and maximum capacity constraints, and the total travel distance is minimized, i.e. accessibility to facilities is maximized.

The basic model described above, called the *capacitated median model*, captures relevant ingredients of public facility planning problems, but it has received little attention in the literature, particularly no hierarchical extension considering multiple services and multiple facility types has been presented, and no specialized exact solution method has been proposed.

The contributions of this thesis to the discrete facility location literature are the following:

- Formulation of optimization models combining multiple services, minimum and maximum capacity constraints, and constraints on the spatial pattern of assignments of users to facilities, extending previous hierarchical facility location models;

- Description of applications of models with single and multiple services to real-world problems of reorganizing networks of schools and courts of justice in Portugal;
- Development of new valid inequalities for the MIP formulation of the single service capacitated median model and proposal of an exact solution method, composed of a priori reformulation and branch-and-cut, that reduces solution times relatively to a generic MIP optimizer;
- Presentation of computational experiments on solving single service models with a modern generic MIP optimizer, including the fixed-charge capacitated facility location problem and the capacitated median model, in order to identify the most efficient formulation, among variants known from the literature, to solve these models to optimality without resorting to a specialized algorithm.

Resumo

Esta tese aborda o problema de planeamento de reorganizar uma rede existente de equipamentos colectivos, tais como escolas, hospitais ou tribunais, em resposta a alterações estruturais da procura de serviços públicos e à necessidade de melhorar a relação custo-eficácia da prestação de serviços. "Planeamento de equipamentos colectivos" entende-se aqui como a actividade que consiste na tomada de decisões sobre o número, localização, tipo (em termos do conjunto de serviços oferecidos) e capacidade dos equipamentos que fornecem serviços públicos, e sobre as suas áreas de influência (isto é, os aglomerados populacionais servidos por cada equipamento).

Os problemas de planeamento de equipamentos colectivos são abordados nesta tese com modelos de programação matemática (ou de optimização) que têm o propósito de ajudar os decisores a chegar a soluções eficientes em termos de custos para os prestadores de serviços e de qualidade de serviço para os utilizadores em componentes fulcrais como a acessibilidade aos equipamentos. Mais especificamente, os modelos de optimização aqui estudados são modelos de localização discreta de equipamentos, formulados como modelos de programação linear inteira mista. Esta tese foca-se no seguinte modelo básico com um único serviço e em extensões dele. O contexto geográfico é representado pelos seguintes dados: um conjunto discreto de centros de população com procura conhecida, um conjunto discreto de locais onde podem ser localizados equipamentos, e distâncias (ou tempos, ou custos) de viagem entre centros e locais. O problema consiste em localizar equipamentos e afectar os centros a esses equipamentos, de forma a que cada centro seja afectado ao equipamento mais próximo, cada equipamento satisfaça restrições de capacidade mínima e máxima, e a distância de viagem total seja minimizada, i.e. a acessibilidade aos equipamentos seja maximizada.

O modelo básico acima descrito, denominado *modelo da mediana com capacidades*, captura ingredientes relevantes dos problemas de planeamento de equipamentos colectivos mas tem recebido pouca atenção na literatura, nomeadamente não foram propostas extensões hierárquicas considerando múltiplos serviços e múltiplos tipos de equipamentos, e não foram propostos métodos exactos especializados para a sua resolução.

As contribuições desta tese para a literatura sobre localização discreta de equipamentos são as seguintes:

- Formulação de modelos de optimização combinando múltiplos serviços, restrições de capacidade mínima e máxima e restrições à configuração espacial

da afectação de utilizadores a equipamentos, que são extensões de anteriores modelos hierárquicos de localização de equipamentos;

- Descrição de aplicações de modelos com serviços únicos e múltiplos a problemas reais de reorganização de redes de escolas e de tribunais em Portugal;
- Desenvolvimento de novas desigualdades válidas para a formulação do modelo da mediana com capacidades com um único serviço, e proposta de um método exacto de resolução, composto de reformulação *a priori* e de *branch-and-cut*, que reduz os tempos de resolução relativamente a um optimizador genérico;
- Apresentação de experiências computacionais usando um optimizador genérico moderno para resolver modelos com serviços únicos, incluindo o problema de localização de equipamentos com custos fixos e capacidades e o modelo da mediana com capacidades, de forma a identificar a formulação mais eficiente, de entre variantes conhecidas da literatura, para resolver estes modelos até à optimalidade sem recorrer a um algoritmo especializado.

Contents

Chapter 1

Introduction

1.1 Context and objectives

This thesis addresses the planning problem of reorganizing an existing network of public facilities, such as schools, hospitals or courts of justice, in response to structural changes in the demand for public services and to the need of improving the cost-effectiveness of service provision. “Public facility planning” is here understood as the activity consisting in making decisions on the number, location, type (in terms of the mix of services offered), and capacity of facilities supplying public services, and on their catchment areas (i.e. the population centers served by each facility). These decisions are strategic in nature, as they are set in a large region (a municipality or larger region), will endure in a large temporal horizon (10 years or more), and influence other, more operational decisions, e.g. to deploy human resources and to organize public transportation networks. “Network of public facilities” is here defined as the set of facilities jointly providing public services to a region. The facilities are inter-related because they must provide the geographic coverage for the whole region and they may be of different types and need to coordinate the provision of multiple services (for example, health centers and hospitals offering primary and specialized health care, respectively).

Public facility planning problems are addressed in this thesis with mathematical programming (or optimization) models that aim to help decision makers arrive at efficient solutions in terms of costs to service providers and of quality of service to users in key components such as accessibility to facilities. More specifically, the optimization models studied here are discrete facility location models, formulated as mixed-integer linear programming (MILP or MIP) models. A large research effort has been dedicated to this type of models in the literature of operations research and other fields, as they have been shown to be flexible enough to incorporate fundamental components of many real-world planning problems, while at the same time being computationally tractable.

This thesis focuses on the following basic, single-service model and on extensions of it. The geographic setting is represented by a discrete set of population centers with known demands, a discrete set of sites where facilities can be located, and given travel distances (or times, or costs) between centers and sites. The problem is to locate facilities and assign centers to those facilities, so that each center is assigned to the closest facility, each facility satisfies minimum and maximum capacity constraints, and the total travel distance is minimized, i.e. accessibility to facilities is maximized.

The basic model described above captures relevant ingredients of public facility planning problems, but it has received little attention in the literature, particularly no hierarchical extension considering multiple services and multiple facility types has been presented, and no specialized exact solution method has been proposed.

In this context, the objectives of this thesis are as follows:

- Describe the application of single and multi-service models with the ingredients described above to real-world public facility planning problems. Three case studies are presented, addressing secondary schools, primary schools and courts of justice.
- Study formulations of these models that allow them to be solved efficiently using a general purpose MIP optimizer.
- Develop specialized exact solution methods for the single and multiple-service models that are able to solve large instances to optimality faster than current general purpose MIP optimizers. In the case of the single service model, the aim is to solve instances with 100 centers within 1 hour, and preferably much less.

The contributions of this thesis therefore relate, on the one hand, to the formulation of models extending previous models in the literature and the description of their application to real-world problems; and, on the other hand, to the development of efficient exact solution methods. The contributions are detailed in the conclusion of the thesis, together with a discussion of the degree of accomplishment of the objectives above.

1.2 Review of facility location models

A brief general literature review of facility location models and solution methods is now given, before discussing the basic assumptions of the models studied in this thesis in greater detail.

Facility location models are optimization models that determine the location of facilities in order to serve demands with known locations, according to some objective, such as minimizing the costs of serving all the demand or maximizing the quantity of demand served. Many applications exist in the public and private sectors. Examples in the public sector include the location of schools, hospitals, postal offices, waste treatment plants, emergency vehicle depots (e.g. ambulances or fire engines); examples in the private sector include the location of factories, warehouses and retail stores in supply chain networks, and the location of concentrators or antennas in telecommunications networks.

This thesis focuses on discrete location models, which assume a discrete set of demand locations and a discrete set of sites where facilities can be located. When applied in a geographic setting, these models require the following three preprocessing steps: i) partition the geographic territory into population centers or demand centers, represented by a point where demand is assumed to be concentrated; ii) enumerate the discrete sites where facilities can be located; iii) define the transportation cost (distance, time or monetary cost) relating each demand center and each site. Sites and centers in the model may represent the same or distinct geographic entities, depending on the chosen geographic level of aggregation and on legal, technical or other constraints that apply to the location of facilities. Transportation costs are assumed to be independent of the location decisions to be determined by the model. For example, they may be obtained by computing shortest paths on a representation of an underlying transportation network. In this case, the arcs existing in the network and their associated costs are taken as given, and it is assumed that changes in network flows induced by changes in facility locations do not influence network congestion significantly.

There are other types of location models, distinguished by the space where facilities can be located: continuous models, which allow locations at any point in a continuous space (e.g. the plane); and network models, which assume an underlying network of nodes and arcs and allow locations at any point in the network (nodes and interior points of arcs). However, discrete models are more suitable for practical applications of the type studied in this thesis. Two important reasons are (as also discussed by Hansen et al., 1987): i) while discrete models are apparently less general regarding candidate facility locations, in a practical application this can be overcome by an appropriate choice of geographic level of aggregation; moreover, candidate locations may indeed be restricted to a discrete set, e.g. due to zoning regulations; ii) many discrete models can be formulated as MIP models, which makes them flexible, allowing the incorporation of many economic and geographic features (e.g. fixed and variable costs, constraints on feasible locations, capacity constraints) while remaining computationally tractable, which might not be the case with continuous or network models.

Location models have been extensively studied since the 1960s in the operational research, management science, industrial engineering, economic geography and spatial (urban and regional) planning literatures. Among the many existing reviews and introductory references on facility location, the following were useful when developing the present thesis. ReVelle and Eiselt (2005) present a concise classification and review of continuous, network and discrete location models. Hansen et al. (1987) provide an extensive review of continuous, network and discrete location models. They present formulations, properties and solution methods and offer insights on the relationships between different basic models and on the economic interpretation of location models. Daskin (1995) provides a didactic textbook on modeling and solving discrete location models. Current et al. (2002) provide a large review of discrete location models, including formulations, properties, applications and heuristic solution methods. ReVelle (1987), Marianov and Serra (2002) and Peeters et al. (2002) discuss discrete models for public facility location, giving examples of practical applications and distinguishing models for public and private facility planning. Labbé and Louveaux (1997) present an annotated bibliography focusing on solution methods for several basic and extended discrete location models.

Public vs. Private facility planning

Models for public and private facility planning can be distinguished by the way they represent the trade-off between costs and benefits of location decisions, as discussed e.g. by ReVelle (1987), Hansen et al. (1987), and Eiselt and ReVelle (2005).

In models for private facility planning, typical objectives are to maximize profits (revenues minus the total fixed costs of installing facilities and variable operation and transportation costs) or to minimize total costs (equivalent to the previous objective if the total revenue is fixed). This is possible when both costs and benefits can be measured in monetary units and they are commensurable, which is the case when they fall on the same entity (a private company).

In models for public facility planning, usually no attempt is made to express costs and benefits in a single measure. First, benefits fall on users while facility costs fall on public entities. Second, it may be difficult or undesirable to measure benefits in monetary units (such as the value of increased accessibility or the value of lives saved by emergency vehicles). Thus, in models for public facility planning usually surrogate measures of benefits are used, not expressed in monetary units, and the trade-off between costs and benefits is represented with benefits in the objective and costs in constraints (or the converse). Additionally, defining the objective is less obvious in the public sector than in the private sector, as different decision makers may focus on different objectives, such as efficiency or equity. Different objectives are discussed further below.

Basic models

Basic discrete location models are presented next, in order to illustrate the representation of trade-offs discussed above, and to introduce ingredients common to more complex models. Different travel cost measures are used (distance, time or monetary cost), according to the classic definitions of these models.

- Uncapacitated facility location problem (UFLP): the problem is to locate facilities and to assign all demand centers to those facilities, in order to minimize the sum of fixed costs of installing facilities with variable costs of operation and transportation.
- *p*-median problem (PMP): the problem is to locate a given number (*p*) of facilities and to assign all demand centers to those facilities, in order to minimize the total demand-weighted travel distance.
- *p*-center problem (PCP): the problem is to locate a given number (*p*) of facilities and to assign all demand centers to those facilities, in order to minimize the maximum demand-weighted travel distance.
- Location set covering problem (LSCP): the problem is to locate facilities that cover all demand centers within a given time limit, in order to minimize the number of facilities (this problem becomes equivalent to the general set covering problem if facilities have distinct fixed costs and the objective is to minimize the total fixed cost).
- Maximal covering location problem (MCLP): the problem is to locate a given number (*p*) of facilities, in order to maximize the demand covered within a given time limit (not requiring all demand centers to be covered).

The UFLP can be considered a prototype for location models in the private sector. Both fixed facility costs and variable assignment costs are given and the number of facilities is endogenous to the model, resulting from the trade-off between the two types of cost. The other models can be considered prototypes for location models in the public sector. They represent trade-offs between non-commensurable user benefits and facility costs (expressed by the number of facilities).

The PMP and PCP are distinguished by their objectives: the PMP focuses on efficiency (minimize the total distance), while the PCP focuses on equity (minimize the maximum distance). The LCSP and MCLP are covering models and have applications in the location of depots for emergency vehicles, such as ambulances and fire engines. A demand center is said to be covered if at least one facility is installed within a given travel time limit. These models consider only facility location decisions, while the other models consider center-to-facility assignment decisions as well. This means that if a center is covered by more than one facility, a covering model does not distinguish which one provides the service.

Hierarchical models

Extended models considering multiple services and multiple facility types are particularly relevant for this thesis and are reviewed next. Facility location models are termed hierarchical when they involve the location of multiple types of facilities, jointly providing products or services to demand centers.

Two types of hierarchical models may be distinguished:

- Multiple service models, arising in applications to public facilities: centers have independent demands for multiple services, and there are multiple facility types, each type being defined by the mix of services offered; demand centers require an individual assignment to (or coverage by) a facility for each service type.

  An example is a two-level extension of the *p*-median model set in a health care context, where users have known demands for two service types, primary and specialized, and there are two facility types or levels with a nested service hierarchy: health centers for primary care only, hospitals for both primary and specialized care. The aim is to locate given numbers of the two facility types and assign users to facilities for each demand level, in order to satisfy all demand and minimize the total travel distance weighted by demand of both levels.

- Multi-level flow models, arising in applications to supply chains, communications networks, or solid waste disposal systems: centers have demands for one or more products, and there are multiple facility types organized into levels and installed in a network, such that products flow sequentially from each level to the next; each demand center requires assignment (for each product type) only to the facility level directly supplying it, but flows must be defined between all consecutive facility levels.

> An example is a two-level extension of the UFLP set in a supply chain context, where retail stores have known demands for a single product, and the aim is to locate factories and warehouses and define product flows from factories to warehouses and then to retail stores, in order to satisfy all demand and minimize total costs, including fixed facility costs, variable operation costs and transportation costs in the two levels of the supply chain.

Hierarchical models presented in this thesis are of the first type, and thus this is the most relevant here. Models of the first type have been formulated as extensions of the *p*-median and maximal covering models. Typically, service and facility types are organized into levels with a nested hierarchy, as in the example above – level-1 facilities supply level-1 services and are located relatively close to demand centers, while higher level facilities supply a matching high-level service and lower level ones, but few can be installed, requiring larger travel distances. Most applications reported in the literature refer to problems in the health care sector with 2 or 3 service levels.

Narula (1986) and Church and Eaton (1986) review models of the first type, with minisum (as in the *p*-median model) and covering objectives, respectively. Klose and Drexl (2005) and Melo et al. (2009) review models of the second type. Sahin and Sural (2007) review models of both types, updating the first two reviews above regarding models for public sector applications. The two types of models above correspond, respectively, to models with parallel and serial services in the classification of Church and Eaton (1986), and to multi-flow and single-flow models in the classification of Sahin and Sural (2007).

Static vs. Dynamic models, Deterministic vs. Stochastic models

Location models can also be distinguished by the way of treating time and uncertainty. Location models can be classified as static (or single-period) or dynamic (or multi-period) if, respectively, data and decisions are represented in a single point in time (e.g. 10 years into the future) or they are represented in multiple periods in a time horizon (e.g. periods of 1 or 5 years in a 10 year horizon). For single-period models, assuming a given model solution is to be implemented in practice, the timing of implementing changes to an existing public facility network is left outside the scope of the model, taking into account additional information not incorporated in the model, such as trends of demand evolution and budget availability. Multi-period location models consider the timing of facility locations relatively to the temporal evolution of demand, costs and budget availability. Decisions can include opening, expanding capacity or closing facilities. A typical objective in the private sector is to minimize the present value of total costs. Multi-period location models are reviewed by Owen and Daskin (1998) and Melo et al. (2009).

The models studied in this thesis are static, single period models with a time horizon of 10 years. This was judged to be appropriate for the applications studied since the time horizon is still relatively short, so that it is practical to devise the timing of changes to the existing public facility network outside the scope of the model. In addition, this avoided the need for reliable forecasts of the temporal evolution of demand and budget availability.

Location models can also be classified as deterministic or stochastic. Deterministic models assume demand and other data to be known with certainty. Stochastic models incorporate information on the uncertainty of data and aim to determine solutions that perform well under all possible data realizations, according to an objective derived from the objective of the deterministic version of the model, such as maximizing the expected performance or minimizing the worst-case performance. Stochastic facility locations models are reviewed by Owen and Daskin (1998) and Snyder (2006). It is useful to distinguish two types of uncertainty represented in stochastic location models: i) uncertainty in the operation of the system being modeled; ii) uncertainty in the data collected for use in the model, whether or not the system itself can be assumed to operate deterministically. Two examples of the first type that have received attention in the literature are congestion of facilities and failure or disruption of facilities (Snyder, 2006).

In this thesis only deterministic models are studied and the applications use data of a single scenario. This approach was judged to be appropriate for both applications studied (schools and courts of law) since decision makers were generally averse to commit to facility closure decisions unless this was shown to be reasonable even under optimistic demand forecasts. Therefore, an optimistic scenario of demand evolution in a 10 year horizon was adopted in both cases (as further discussed in the relevant chapters).

1.3 Solution methods

Most facility location models of practical interest, including all the basic models above, belong to the computational complexity class of NP-hard problems, which means that in the worst case computation times grow exponentially with instance size, and particular instances may be intractable (due to their size or to the type of data they contain). Nevertheless, many instances of location models of practical interest can be solved efficiently to optimality or near optimality with a careful choice of solution method among the several available ones. Next, we present a general classification of algorithms for solving MIP models, adapted from the classifications in sections II.4 and II.5 of Nemhauser and Wolsey (1988) and chapter 12 of Wolsey (1998).

Algorithms for solving MIP models may be classified as follows:

- Exact algorithms
- Approximate algorithms or heuristics
    - Providing a performance measure
    - Not providing a performance measure

An exact algorithm provides upon termination a provably optimal solution to any feasible instance. Exact algorithms may be further divided into general and special purpose algorithms. General purpose algorithms can be applied to any model that can be formulated as a MIP. An example is branch-and-bound based on linear programming (LP) relaxations (chapter 7 of Wolsey, 1998). Special purpose algorithms are dedicated to a particular model and exploit its structure in order to reduce the solution time or increase the size of instances that can be solved within a given time limit. An example is the DUALOC algorithm for the UFLP (Erlenkotter, 1978). Another example is a branch-and-cut algorithm (section 9.6 of Wolsey, 1998) embedding cutting-plane generation procedures dedicated to a particular model structure.

An approximate algorithm, or heuristic, provides a feasible but possibly non-optimal solution. Heuristics are designed to provide good solutions quickly, when exact algorithms have prohibitively large computation times. Heuristics can also be embedded into an exact algorithm based on branch-and-bound, in order to reduce its running time, by finding feasible solutions quicker and reducing the size of the search tree. Most heuristics are special purpose algorithms according to the definition above. Heuristics may be further divided into two types, according to whether or not they provide a performance measure for the particular instance being solved, that is, a bound on the deviation of the solution value relatively to the optimal solution value. Examples of heuristics providing a performance measure include Lagrangian-based heuristics (section 10.4 of Wolsey, 1998) and MIP-based heuristics (section 12.5 of Wolsey, 1998).

Heuristics not providing a performance bound can be classified into the following types (Blum and Roli, 2003): construction or greedy heuristics (that build a solution from scratch), improvement or local search heuristics (that take a solution given by a construction heuristic and find a better, locally optimal solution with respect to a given neighborhood structure), and metaheuristics (a framework composed of one or several construction heuristics, one or several improvement heuristics, and one or several mechanisms to explore the solution space more extensively and avoid focusing on a single local optimum). Metaheuristics include simulated annealing, tabu search, genetic algorithms, ant colony optimization, variable neighborhood search, and greedy randomized adaptive search procedures (GRASP).

Some heuristics have a known worst-case performance guarantee, which is an a priori bound on the deviation to the optimal value applicable to all instances (section 12.4 of Wolsey, 1988). In particular, "approximation algorithms" are polynomial time algorithms with a performance guarantee, and several of these have been proposed for facility location problems (Williamson and Shmoys, 2011). Although performance guarantees are of theoretical interest for the analysis of models and algorithms, their practical interest for solving a particular instance may be limited, since worst case deviations are generally large (e.g. 50% of the optimal value or more).

Branch-and-cut

Branch-and-cut based on LP relaxations is the method of choice in most generic MIP software packages, which justifies further discussion of this method. A branch-and-cut algorithm (described in section 9.6 of Wolsey, 1998) combines a branch-and-bound algorithm with cutting plane (or cut) generation throughout the branch-and-bound tree. It involves a trade-off between increasing the effort spent at each node (generating cuts and solving larger LP models) and reducing the number of nodes explored.

The ingredients of modern branch-and-cut algorithms include: 1) efficient and robust LP solvers; 2) presolve procedures to reduce and tighten the formulation (e.g. by dropping redundant constraints, fixing variables, tightening coefficients); 3) cut generation procedures using a variety of general purpose cut types (e.g. Gomory cuts) and structure-specific cut types (e.g. lifted knapsack cover cuts) to improve the dual bound; 4) heuristics to find and improve feasible solutions; 5) sophisticated strategies for branching node selection (e.g. hybrids of depth first and best-bound first strategies), for branching variable selection (e.g. strong branching), and for searching the tree in parallel, making use of the multiple cores or CPUs available in modern computers. All ingredients contribute to reducing the time to find good feasible solutions and the time to prove optimality.

General descriptions of MIP software implementing branch-and-cut algorithms are given by Atamturk and Savelsbergh (2005) and Lodi and Linderoth (2011). State-of-the-art commercial MIP software packages include: Xpress (Ashford, 2007; Laundy et al. 2009), CPLEX (Bixby et al., 2000; Bixby and Rothberg, 2007) and Gurobi (Bixby, 2011). The references cited describe the components of MIP solvers and their historical performance evolution.

The models studied in this thesis were solved exclusively with exact solution methods, either directly through the generic branch-and-cut algorithm implemented in a commercial MIP solver or with a specialized method extending that algorithm, e.g. through cut generation procedures for particular model structures. This has the advantage, relatively to other specialized exact algorithms, of leveraging the several components of generic MIP solvers, by benefiting from performance improvements in newer versions and by reducing development effort (e.g. presolve and branch-and-bound routines do not have to be duplicated). Next we further discuss the advantages of using exact rather than approximate methods.

Benefits of optimal vs. approximate solutions

It can be argued that it is reasonable to accept an approximate, near optimal solution, say within 1% of optimality, in a practical application where the computation effort is significantly higher to obtain an optimal solution. Since the data (demand, costs, etc.) almost inevitably will have an error larger than 1%, this renders the error in the objective function value larger than the optimality gap. Such an argument is made by Cordeau et al. (2006) in the context of solving a location model for supply chain network design.

On the other hand, in the case of models for strategic public facility planning, optimal solutions have advantages relatively to approximate ones beyond the gains measured by the objective function. Two arguments can be offered:

- In an application to strategic facility planning, the model typically has to be solved with different data for different scenarios and for sensitivity analysis. If solutions are approximate, the impact of different data may be difficult to distinguish from the effect of arbitrary or random choices in algorithm execution. This is especially important given that discrete location models may have feasible solutions with relatively close objective values but widely varying spatial configurations, in terms of the selected facility locations and assignments of users to facilities. If these objective values are within the optimality gap provided by the algorithm, it will be difficult to interpret the causal relation between model parameters and spatial configuration of solutions.

- In an application to strategic facility planning, typically there are stakeholders with different and possibly conflicting objectives. The purpose of the model, which is a simplified representation of reality, is to provide reference solutions and insights to be discussed by decision makers, in order to arrive at a solution to be adopted in practice. In this context, the discussion can be perturbed if approximate solutions are used rather than the best possible solutions under the model's assumptions.

The first argument was also made by Geoffrion and Powers (1980), in the context of facility location models for supply chain network design, and by ReVelle et al. (1970), in an early review of facility location models.

1.4 Modeling assumptions

In this section, we discuss the modeling assumptions of the basic, single-service model studied in this thesis, which also apply to multiple service models extending it. We also provide a more focused literature review.

First we recall the definition of the basic, single-service model, called here the *capacitated median model* due to its relationship with the *p*-median model, discussed below. The following data is given: a discrete set of centers where demand is concentrated; a discrete set of sites where facilities can be located; demand of each center; travel distances (or times, or costs) between centers and sites; minimum and maximum capacities of each facility, i.e. lower and upper bounds on the total demand served by each facility. The problem is to locate facilities and to assign centers to those facilities, with the objective of minimizing the total travel distance, and satisfying the following constraints: all demand of all centers is satisfied; each center is assigned to the closest facility, or to a single one of the closest facilities if several are equidistant; each facility satisfies the minimum and maximum capacity bounds. The problem can also include existing facilities, in which case location decisions determine both the installation of new facilities and the maintenance or closure of existing ones. Optionally, additional constraints may be included to impose: a maximum travel distance allowed for any center; a maximum number of new facilities to open; a maximum number of existing facilities to close.

In the discussion below, we assume a public facility planning context where demand centers correspond to population centers and users travel to facilities where service is provided. The single service in the model represents either a specific service or a group of services that can be aggregated for planning purposes. Demand is measured as the quantity of service in a given period of time (e.g. number of students attending school in a typical day; number of trips to a health care facility per year). Typically, the locations of demand centers represent places of residence, and travel costs are represented by distances or times computed with shortest paths on a model of the transport network.

We next discuss the assumptions of the model.

Location decision maker

There is a single, public authority responsible for defining facility locations, i.e. there is no competition between facilities located by distinct decision makers. This assumption does not rule out that distinct facilities may be financed and operated by distinct public and private entities.

Demand

Demand quantity is known (exogenous to the model) and is inelastic with respect to travel cost, which is assumed to be supported by users. If costs are charged to users at facilities, they are assumed to be equal at all facilities and demand quantities are assumed to already reflect them, and thus they are not represented in the model.

All the given demand has to be satisfied. Thus the model is appropriate for essential services, for which universal coverage is sought, such as mandatory education, as well as health care and justice services.

Single and closest assignment

The model considers two types of assignment constraints: 1) single assignment – all the demand from each center is assigned to the same facility, that is all the demand of each user is served by a single facility (e.g. a student does not attend different schools in the same year) and all users from a center are assigned to the same facility; 2) closest assignment – users from each center are assigned to the closest facility (or least-cost facility).

To be applicable, these constraints assume the following: 1) user preferences for facilities depend only on travel cost, while other attributes of facilities related to quality of service (or surrogates such as facility size) are perceived as indifferent; 2) all users within a center have homogeneous travel costs (and thus also homogeneous preferences for facilities, given the previous assumption); 3) assignment decisions are made by users according to their preferences or by a public entity taking user preferences into account; 4) for planning purposes, all users from a center are assigned to a single facility, even if several exist that are equally preferred.

Expanding on the third assumption, two cases regarding the assignment decision maker may be distinguished (as also discussed by Wagner and Falkson, 1975, and Hanjoul and Peeters, 1987): i) free choice by users (this is the case e.g. of post offices); ii) mandated assignment by a public authority (e.g. through a legal requirement based on place of residence; this is the case of health centers in Portugal). In the latter case, using closest assignment constraints guarantees that decisions by the public authority meet user preferences, in order to make assignments acceptable for the users.

Regarding the fourth assumption, note that closest assignment constraints imply single assignment if there is a single closest facility, but allow dividing users among equidistant facilities if they exist. Adding single assignment constraints avoids discriminating users from the same center in all cases.

Objective

The objective stated above is to minimize the total demand-weighted travel distance, which is equivalent to minimizing the average distance since all demand has to be satisfied.

This objective can be interpreted as the maximization of accessibility, if accessibility to facilities is defined as the average travel cost perceived by users to obtain service at facilities (alternative definitions of accessibility have been proposed, see e.g. Talen and Anselin, 1998). The objective function exactly measures accessibility with this definition under the model assumptions above (users patronize a single facility; users distinguish facilities only by travel cost and not by other facility attributes; travel costs are homogenous for all users from the same center; travel costs are represented by a measure of distance or time). However, the objective function is only a surrogate or proxy for the true accessibility (still with the definition given) in the sense that: i) users are aggregated into discrete centers in the model, thus there is a spatial aggregation error in the objective function; ii) the measure of travel cost used in the model represents only approximately the total perceived costs, including out-of-pocket costs and opportunity costs of time spent in travel, which in reality vary between different transport modes and different socio-economic population groups; iii) travel costs may be relative not only to the place of residence and the location of the service under analysis (as assumed in the model) but also to other destinations in multi-purpose trips (e.g. a school trip may be part of a trip chain home-school-work).

The objective focuses on efficiency and does not guarantee equity: users at large centers in central areas will be favored, while users at small centers in remote areas may be much worse off. To address equity, one approach is to limit the worst case possible for any user by adding a constraint on the maximum allowed travel cost. Such was the approach followed on this thesis. Another approach is to use an equity objective instead of the efficiency objective, or both in conjunction in a multi-objective model. Eiselt and Laporte (1995) and ReVelle and Eiselt (2005) discuss equity objectives, including the minimax objective of the *p*-center problem and so-called balancing objectives of minimizing travel cost deviations between centers.

Maximum capacity

Maximum capacity constraints represent either limited space availability to build new facilities or to expand existing ones, or a threshold to avoid diseconomies of scale in the operation of facilities (e.g. due to coordination-related management costs increasing with the quantity of services produced).

Minimum capacity

Minimum capacity constraints may be included for two reasons: i) technical requirements related to quality of service; ii) economic feasibility of individual facilities.

Regarding the first reason, in some applications a link can be established between providing at least a minimum quantity of service and the quality of that service.

- In the case of the health care sector, a minimum quantity of service may be required to guarantee diversity of experience and maintain the training level of professionals. In an example regarding mammogram screening centers cited by Vedat and Verter (2002), the U.S. Food and Drug Administration requires a radiologist to interpret at least 960 mammograms and a radiology technician to perform at least 200 mammograms in 24 months in order to retain their accreditations. In another example in Portugal, the guidelines of the Ministry of Health for planning cardiology services in hospital networks (DGS, 2001) indicate that a Heart Surgery Center should have a total activity of at least 650 surgical procedures per year, and each surgeon should perform at least 100 procedures per year in order to maintain competence.

- In the case of schools, a minimum number of students may provide benefits for student achievement and school environment. In Portugal, Council of Ministers Resolution 44-2010 (June 2010) defines criteria for reorganizing the school network. It states that primary schools with 20 students or less shall be closed and presents the following arguments: very small schools have lower student achievement scores than the national average; very small schools offer fewer opportunities for student education and teacher development due to limited opportunities for group work and social interaction; small schools usually are not equipped with a canteen, library and computer room. Additionally, the Ministry of Education established guidelines (MinEdu, 2000) defining minimum and maximum sizes for new primary and secondary schools, in terms of number of classrooms per school and number of students per classroom. These guidelines reflect concerns both with education quality and with cost efficiency.

Regarding the second reason, a minimum amount of service may be required for a facility to cover its fixed costs. This can be illustrated by a simplified break even analysis: assuming that a facility has annualized fixed costs (related to investment and operation), variable operation costs, and variable revenues, such that the government defines a maximum revenue per unit of service transferrable from public funds (possibly complemented by revenues charged to users; and such that unit revenues exceed unit variable costs), then there will be a minimum quantity of service for the facility to break even. Beyond this quantity, the unit revenue transferred would be decreased, so that the facility does not become profitable. In this setting, imposing a minimum capacity guarantees the economic feasibility of an individual facility, i.e. the facility is justified by being able to cover its fixed costs.

In the interpretation above, if fixed costs include the amortization of investment costs, the minimum capacity for new facilities will be higher than for existing ones whose investment is already amortized. On the other hand, if only fixed operation costs are considered (excluding amortizations), the minimum capacity may be equal for new and existing facilities. In the latter case, investment costs may be subject to a separate budget constraint, e.g. imposing a maximum number of new facilities.

Comparison with the *p*-median model

The capacitated median (CM) model shares with the *p*-median (PM) model the basic assumptions above regarding the location decision maker, demand, single and closest assignment, and the objective.

The PM model differs in that the number of facilities is a parameter and facility capacity is unrestricted. In comparison, in the CM model the number of open facilities is a model output (assuming no explicit constraints fixing that number are included as well), since the minimum and maximum capacity bounds impose implicit upper and lower bounds (respectively) on the number of facilities, with the accessibility-maximization objective driving solutions towards the upper bound. Additionally, in the PM model solutions naturally have the so-called single assignment and closest assignment properties (Krarup and Pruzan, 1990), that is, centers are fully assigned to a single, closest facility. In the CM model, due to the presence of capacity constraints, these properties must be enforced through explicit constraints. Thus, we can say that capacity constraints and explicit single and closest assignment constraints are the defining features of the CM model relatively to the PM model.

Previous applications and solution methods

The CM model, unlike the PM model, has rarely been dealt with in the literature. Carreras and Serra (1999) use the model without the maximum capacity constraints to address a pharmacy location problem in a rural region, and solve it through a tabu search heuristic. Kalcsics et al. (2002) use the model with minimum and maximum capacity constraints and a constraint on the number of facilities for designing balanced and compact sales territories, and solve it through a variable neighborhood search heuristic. Bigotte and Antunes (2007) present several heuristics to solve the model with minimum capacity constraints, including construction and improvement heuristics, a genetic algorithm and a tabu search heuristic. Related models, considering a given maximum distance for demand to be covered and not requiring all demand centers to be served, have also been proposed combining minimum capacity and closest assignment constraints. Verter and Lapierre (2002) present a model for locating preventive health care facilities with the objective of maximizing population coverage, and solve it with a commercial optimizer. Smith et al. (2009) present a model for locating primary health care facilities with the objective of maximizing the number of facilities satisfying minimum capacities, and solve it with a commercial optimizer.

1.5 Organization of the thesis

This thesis contains 5 chapters, besides the introduction and conclusion. These may be divided into two groups: chapters 2, 3 and 4 focus on the formulation and application of single and multiple-service variants of the capacitated median model; chapters 5 and 6 focus on solution methods for basic facility location models.

Chapter 2 describes an application of the basic, single-service capacitated median model to the location of secondary schools. The model is solved with a generic MIP optimizer. This chapter also describes how a Geographic Information System (GIS) was used in the practical implementation of the model, for data preparation and solution visualization. This implementation can be seen as the prototype of a Decision Support System embedding the model.

Chapter 3 describes an application of a hierarchical extension of the capacitated median model to the location of primary schools. The model is solved with a generic MIP optimizer.

Chapter 4 describes an application of hierarchical extensions of the capacitated median model to the districting and location of courts of justice. The models are solved with a generic MIP optimizer.

Chapter 5 presents a specialized exact solution method for the basic, single-service capacitated median model, consisting of a priori reformulation and branch-and-cut, exploiting previously known and new valid inequalities. The aim is to accelerate the solution of larger scale instances (of 100 centers or more) that still require relatively long computation times with a generic MIP optimizer (1 hour or more on a standard personal computer). Computational results are presented for a set of generated (abstract) instances.

Chapter 6 presents computational experiments on solving basic, single-service models with a modern MIP optimizer implementing a generic branch-and-cut algorithm. The models include the classic fixed-charge capacitated facility location problem and the capacitated median model. The aim is to test the effectiveness of well-known formulation variants, originally studied for the fixed-charge location problem.

Regarding the order of chapters, the following is remarked:

- Chapters 2, 3 and 4 are in chronological order of their development. Additionally, chapters 3 and 4 contain higher modeling complexity than chapter 2.

- Chapter 5 appears after the chapters with practical applications for the following reasons. First, instances addressed in chapters 2, 3 and 4 turned out to be relatively easy to solve with a generic MIP optimizer, and thus the specialized method of chapter 5 was not required. Second, the work in chapter 5 was developed after or in parallel with the previous chapters, aiming to provide an efficient solution method for larger instances that may arise in other applications.
- Chapter 6 appears last in the thesis for the following reasons. First, it includes results of computational experiments with fixed-charge facility location models, unrelated to the applications of public facility planning studied in the first chapters. Second, the formulations of the capacitated median model used in the previous chapters already reflect the results of this chapter, following preliminary tests of formulation variants with a small set of instances, which are corroborated in this chapter with more extensive tests.

The chapters were developed as stand-alone articles. The advantage is that the chapters are self-contained, including their own introduction, literature review, contributions and conclusions. The disadvantage is that some repetition can occur between chapters, especially in literature reviews. In addition, the nomenclature used in model formulations is not always homogeneous. In the presentation of the thesis, the following was adopted: chapters 2 and 3, which were published before completing subsequent work, are included as published, except that the titles were renamed to better fit the structure of the thesis, the expression “this article” was replaced by “this chapter”, and some footnotes were added to comment on relevant repetitions, inconsistencies and later developments; headings, equations, tables and figures are numbered globally; references are consolidated in a single section of the thesis.

Due to the time lapse between completing the original work and the final presentation of the thesis (see the next section), chapters 3 and 5 contain appendices written for the thesis. The appendix to chapter 3 contains an alternative formulation and a complementary literature review of path assignment constraints. A first appendix to chapter 5 compares computational results between Xpress 2005B, the MIP optimizer used for obtaining the original results, and Xpress 7.2, the latest version available when the thesis was completed (results with both versions were retained because they also illustrate the performance evolution of a generic MIP optimizer). A second appendix to chapter 5 discusses alternative formulations of closest assignment constraints (this discussion is complementary, but not essential, to the main content of the chapter).

1.6 Chronology, collaborations and publications

The work in this thesis was carried out in the periods October 2003 - July 2007 and August 2011 - June 2012. In the first period the large majority of the work was carried out and most results were obtained. In the intervening period the author interrupted work on the thesis. In the last period some results were finished and the remainder of the thesis was written.

In the course of the thesis, the following periods were spent at the Center for Operations Research and Econometrics (CORE) of the Catholic University of Louvain, at Louvain-la-Neuve, Belgium: 2,5 months in 2004 (1-Jun to 15-Aug); 2,5 months in 2005 (15-Aug to 31-Oct); 2 months in 2006 (23-Jul to 16-Sep). In these periods the author worked with Laurence Wolsey, and also with Dominique Peeters (mainly regarding chapter 2).

The applied work described in chapters 2, 3 and 4 was developed in parallel with and inspired by the following two studies.

- Educational Charter of the municipality of Coimbra – Planning models and solutions: study made in Jul-2003 to Oct-2006 at the Department of Civil Engineering of the University of Coimbra under contract with the Municipal Council of Coimbra. The authors were António Pais Antunes (coordinator) and João Teixeira. The reference of the final report is:

  Antunes, A. P. (Coord.), "Carta educativa do município de Coimbra 2006-2015", Câmara Municipal de Coimbra e Departamento de Engenharia Civil da Universidade de Coimbra, Outubro 2006. (In Portuguese).

- Proposal of revision of the Judiciary Map of Portugal: study made in Aug-2006 to Mar-2007 at the Department of Civil Engineering of the University of Coimbra under contract with the Ministry of Justice of the Portuguese government. The authors were António Pais Antunes (coordinator) and the PhD students João Bigotte, Hugo Repolho, and João Teixeira. The reference of the final report is:

  Antunes, A. P. (Coord.), "Proposta de revisão do mapa judiciário", Departamento de Engenharia Civil da Universidade de Coimbra, Março 2007. (In Portuguese).

The author of this thesis was responsible for the implementation of the facility location models used in both studies. Chapters 2 and 3 correspond to a work-in-progress report of the first study. The same models were used for the final report, although there were some changes in the data, scenarios analyzed, and model solutions. Chapter 4 corresponds to the final report of the second study.

Regarding chapter 6, the author also worked with Joaquim Júdice (University of Coimbra) and Pedro Martins (Polytechnic Institute of Coimbra). They had performed some computational experiments with the classic capacitated facility location problem (CFLP), finding that it was solved faster by a modern generic MIP optimizer using the so-called weak formulation rather than the strong one. These apparently surprising results were discussed with António Pais Antunes and then the author of this thesis was involved to perform further computational experiments, addressing the CFLP and also the capacitated median model.

Below is a list of publications and communications of the work carried out in the thesis. All communications were presented by the author of this thesis, except the one at ISOLDE XI, which was presented by António Pais Antunes.

Peer-reviewed publications:

- *(Chapter 2)* Teixeira, J., Antunes A. P., Peeters, D. (2007), "An optimization-based study on the redeployment of a secondary school network", Environment and Planning B 34 (2), 296-315.
- *(Chapter 3)* Teixeira, J. and Antunes, A. P. (2008), "A hierarchical location model for public facility planning", European Journal of Operational Research 185 (1), 92-104.

Communications in conferences:

- *(Chapter 3)* Teixeira, J., and Antunes, A. P. (2005), "School network planning – a case study", CUPUM'05 – Computers in Urban Planning and Urban Management, London, UK, July 2005.
- *(Chapter 3)* Teixeira, J., and Antunes, A. P. (2006), "Coupling GIS and optimization software in public facility planning", DMUCE 5 – Decision Making in Urban and Civil Engineering, Montreal, Canada, June 2006.
- *(Chapter 4)* Teixeira, J., Antunes, A. P. e Bigotte, J. (2008), "Aplicação de modelos de localização de equipamentos à revisão do mapa judiciário português", IO2008 – 13º Congresso da APDIO, Vila Real, Portugal, Março 2008. (In Portuguese).

- *(Chapter 4)* Antunes, A. P., Teixeira, J., Bigotte, J., Repolho, H. (2008), "Districting and location in the courts: the making of the new judiciary map of Portugal", ISOLDE XI – 11th International Symposium on Locational Decisions, Santa Barbara, CA, USA, June 2008.
- *(Chapter 5)* Teixeira, J., and Antunes, A. P. (2005), "The public facility planning problem: valid inequalities and computational experience", ISOLDE X – 10th International Symposium on Locational Decisions, Sevilla, Spain, June 2005.
- *(Chapter 5)* Teixeira, J., and Antunes, A. P. (2006), "Solving the capacitated median problem by a priori reformulation and branch-and-cut", Iberian Conference in Optimization, Coimbra, Portugal, November 2006.
- *(Chapter 6)* Teixeira, J., Antunes, A. P., Júdice, J., Martins, P. (2006), "Resolução de modelos de localização com software de optimização moderno: as formulações forte e fraca revisitadas", IO2006 – 12º Congresso da APDIO, Lisboa, Portugal, Outubro 2006. (In Portuguese).

Chapter 2

Application of the capacitated median model to the location of secondary schools

2.1 Introduction

In this chapter, we report the results of a study on secondary school planning in Coimbra, a medium size municipality of 320 km^2 and 150,000 inhabitants located in the center-littoral region of Portugal (Figure 2.1). The study was developed at the University of Coimbra within the framework of Coimbra's Educational Charter, a document currently being prepared to integrate the Municipal Development Plan for the period 2005-2015. The Educational Charter specifies the infrastructure, equipment, human and financial resources necessary for pre-school, primary and secondary education.

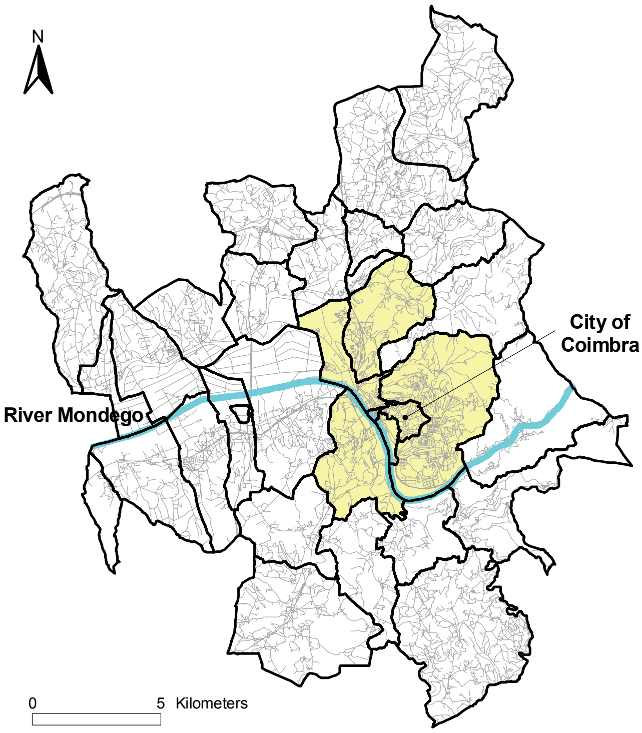


Figure 2.1: Municipality and communities of Coimbra

The importance of the Educational Charter derives from two reasons. First, current aggregate school capacity is excessive because of the strong decline of school-age population in the last two decades. Second, school typology needs to be changed according to a recent reorganization of the Portuguese educational system. Specifically, starting in 2005, the current nine- and three-year cycles of primary and secondary education will be converted to two six-year cycles. Within the reorganization, mandatory education will be expanded from primary to secondary education.[1]

The planning problem to be solved within the study consisted of defining the location, type and size of the schools composing Coimbra's secondary school network in 2015, the planning horizon of the Municipal Development Plan. A solution to the problem should meet a set of constraints imposed by the guidelines of the Ministry of Education for redeploying the school network, including: maximum travel distance of students to schools, maximum and minimum number of students per classroom, and maximum and minimum number of classrooms per school. These constraints seek to guarantee adequate accessibility, good pedagogical conditions (in terms of class size) and economic efficiency, that is, school occupation should justify operation costs and investments in equipment (laboratories, libraries, sports buildings, etc.).

The main tool used for the development of the study was a discrete facility location model based on the *p*-median model. The model is aimed at maximizing the accessibility of students to schools, and includes constraints to ensure that the students living in each population center are exclusively assigned to the closest school.

The contributions of our work are, first, the formulation of a model with some ingredients not previously found in the facility location literature addressing school network planning (specifically, the combination of maximum and minimum capacities and closest assignment constraints). Second, the use of this model for a practical application in Portugal. As far as we know, this is the first model incorporating all quantitative constraints present in the guidelines of the Ministry of Education regarding capacity and accessibility requirements.

This chapter is organized as follows. Sections 2 and 3 contain a presentation of the situation of the municipality of Coimbra with regard to educational demand, school

[1] Chapters 2 and 3 correspond to a non-final stage of the study. The reorganization described was envisioned by the reform of the education system according to proposal of law 74/IX of July 2003. Later in the study period, the government abandoned the changes to primary and secondary education cycles, but mandatory education was still expanded to secondary education and school typology changes were still considered in Coimbra.

facilities, and their expected long-term evolution. In Section 4, the planning problem to be solved within the study is presented in detail. In Section 5, the discrete facility location model developed to represent the planning problem is introduced, after a review of previous studies on school network planning. In Section 6, the results obtained with this model are presented and discussed. In Section 7, the GIS used for data handling and result analysis is concisely described. Finally, Section 8 contains a summary of the main conclusions of the study and a presentation of work to be developed in the near future within the framework of Coimbra's Educational Charter.

2.2 Current situation

In Portugal, non-higher education is composed of nine years of primary education (for ages 6-14) and three years of secondary education (for ages 15-17). Primary education (B) consists of three consecutive levels: B1, B2 or B3. Secondary education (S) is further divided in regular education and professional education. The latter is offered at specialized schools and was not included in the study. In the municipality of Coimbra, between 1998/99 and 2003/04, enrollments in primary and secondary education have decreased by 17% and 32%, respectively (Table 2.1). However, this decrease is smaller in the first levels, suggesting that the decline is slowing.

Table 2.1: Evolution of student enrollments in Coimbra in recent years

Level of education	Ages	Duration (years)	Number of students		
			1998/99	2003/04	Variation
B1	6-9	4	7331	6669	-9%
B2	10-11	2	4334	3990	-8%
B3	12-14	3	7667	5429	-29%
Primary	6-14	9	19332	16088	-17%
Secondary	15-17	3	8642	5876	-32%
Total	6-17	12	27974	21964	-21%

Coimbra's primary and secondary school network is composed of 102 public schools and 16 private schools. Currently, public schools are of types EB1, EB23 and ES (designations refer to education levels offered, see Table 2.2), while private schools may have other types, in general offering more levels than public schools (some of them offer the four levels). In Table 2.2, it can be seen that the EB1 type represents about 85% of the number of public schools, since they generally are small schools located very close to their students. However, their capacity is clearly smaller than that of EB23 or ES type schools, which are large schools that, in the past, served students from both Coimbra and neighboring municipalities.

As shown in Table 2.3, existing aggregate capacity is excessive for the current needs of the municipality (occupation of 75%), particularly in the case of secondary schools (occupation of 57%).

Table 2.2: Existing schools in Coimbra[2]

Type of school	Levels of education	Number of schools		Capacity (students)		
		Public	Private	Public	Private	Total
EB1	B1	86	7 EB1 + 9 EB23/ES	6350	1800	8150
EB23	B2+B3	9		6450	4360	10810
ES	S	7		8910	1380	10290
Total		102	16	21710	7540	29250

Note: the 9 EB23/ES private schools have mixed types:
1 EB1+EB23, 4 EB1+EB23+ES, 4 EB23+ES.

Table 2.3: Current aggregate school occupation in Coimbra

Type of school	Capacity (students)	Number of students (2003/04)	Occupation rate
EB1	8150	6669	82%
EB23	10810	9419	87%
ES	10290	5876	57%
Total	29250	21964	75%

[2] Public school capacity considers a maximum of 25 students per classroom for EB1 and EB23, and 30 for ES. Private school capacity was considered equal to occupation in 2003/04 (complete data on the number of classrooms became available only later in the study, after chapters 2 and 3 were written).

Some data in Table 2.2 was changed relatively to the published article to improve consistency with other data in chapters 2 and 3 (excluded small EB1 schools that had been closed in the previous year but that remained in the database; excluded 3 small, special-purpose schools not considered in the study; changed the maximum number of students per classroom in public EB23 schools from 30 to 25, as considered for the new type EB12). Table 2.3 and values cited in the text were updated accordingly. All other data in chapters 2 and 3 remains unchanged.

2.3 Future situation

For students entering school in 2004/05, there will be two cycles of primary education (B1 and B2) instead of three, and two cycles of secondary education (S1 and S2) instead of one. According to our forecast, the number of students in primary and secondary education in the year 2015 is expected to decrease by 11% and 18% relative to the present values of 2003/04 (Table 2.4).

Table 2.4: Number of students in 2015 (forecast)

Level of Education		Number of Students		
Current	Future	2003/04	2015	Variation
B1	B1	6669	6278	-6%
B2	B2	3990	3168	-21%
B1+B2	B1+B2	10659	9446	-11%
B3	S1	5429	4796	-12%
S	S2	5876	4475	-24%
B3+S	S1+S2	11305	9271	-18%

The number of students was forecast using communities (“freguesias”), the smallest level of administration in Portugal, as the geographic unit of analysis. The municipality of Coimbra comprises 31 communities (as shown in Figure 2.1).

The forecasting procedure used was as follows. First, the total population of the municipality was estimated for 2015 assuming that the annual demographic growth rate for the period 2001-2015 would be equal to the rate observed in 1991-2001. Second, the future population was distributed by communities taking into account their demographic evolution in the recent past and current trends in housing construction. Third, the school-age population for the municipality was estimated assuming that the percentage of 0-4 years old population would remain constant in the period 2001-2015, which means that the percentage of 0-19 population in 2015 would be approximately four times the current percentage of 0-4 population. This assumption meets the long-term optimistic (regarding the evolution of the birth rate) population forecast of the Portuguese Bureau of Statistics for the whole country (INE, 2003). Since in the last decade the demographic evolution of Coimbra exhibited a strong correlation with that of the whole country, the trend forecast for Portugal was adopted for the municipality of Coimbra. Fourth, at the community level, it was assumed that the proportion of young population (and in particular school-age population) would remain at the current level in 2015. This assumption was based on the observation that in both the Census of 1991 and the Census of 2001 the proportion of young population did not vary significantly across communities. Finally, the number of students for each community was calculated through the application of expected students/school-age population ratios. These ratios account for repetition and attendance of professional education in specialized schools, and were estimated according to the goals set out by education experts for the next decade in a study commissioned by the Ministry of Education (São Pedro et al., 2000).

The reorganization of primary and secondary education will be accompanied by a change in school typology. In the future, new schools and (adapted) existing schools should match types EB12 (primary schools, offering levels B1 and B2) and ES12 (secondary schools, offering levels S1 and S2, that correspond to current levels B3 and S, respectively).

For Coimbra, it was decided that existing public schools are to be converted to the new typology according to the following rules. Current ES schools are converted to ES12 (offering six instead of three years of education); current EB23 schools are converted to EB12 (offering six instead of five years); current EB1 schools remain dedicated to the first four years of primary education (i.e. level B1), if they have four or more classrooms. Smaller EB1 schools (about two thirds of existing schools, concentrating 40% of total EB1 capacity), which do not offer adequate pedagogic conditions, will either be closed or converted to kindergartens, to expand coverage of pre-school education. With regard to existing private schools, it was assumed that their typology will change to EB12, ES12 or EB12+ES12 depending on whether they currently offer primary education, secondary education, or both.

Because of this typology conversion, the aggregate occupation rate of existing schools in 2015 will be around 70% for both primary and secondary education (Table 2.5). That is, existing capacity will remain excessive in the future, in spite of the fact that small EB1 schools will be closed and that ES12 schools will offer two three-year cycles instead of one.[3]

[3] Later in the study period, the reorganization of education cycles was abandoned, as noted before. However, following a decision by the Municipal Council of Education of Coimbra, the study continued to consider a school typology conversion to EB12 (B1+B2) and E3S (B3+S) types, homologous to the EB12 and ES12 types, in order to use the existing school capacity more efficiently. With the typology conversion, the aggregate occupations of primary and secondary schools would become more balanced (Table 2.3 vs. Table 2.5), by using the capacity slack in secondary schools to compensate the capacity deficit experienced in some primary schools, while also allowing small EB1 schools to be closed.

Table 2.5: Capacity and expected occupation of existing schools after typology conversion[4]

Type of school	Capacity of schools (students)			Number of students (2015)	Occupation rate
	Public	Private	Total		
EB1+EB12	10175	3700	13875	9446	68%
ES12	8910	3840	12750	9271	73%

2.4 Planning problem

The planning problem addressed in this chapter regards the secondary school network. This problem was given priority because the spatial distribution of secondary schools is less balanced than that of primary schools (Figure 2.2).

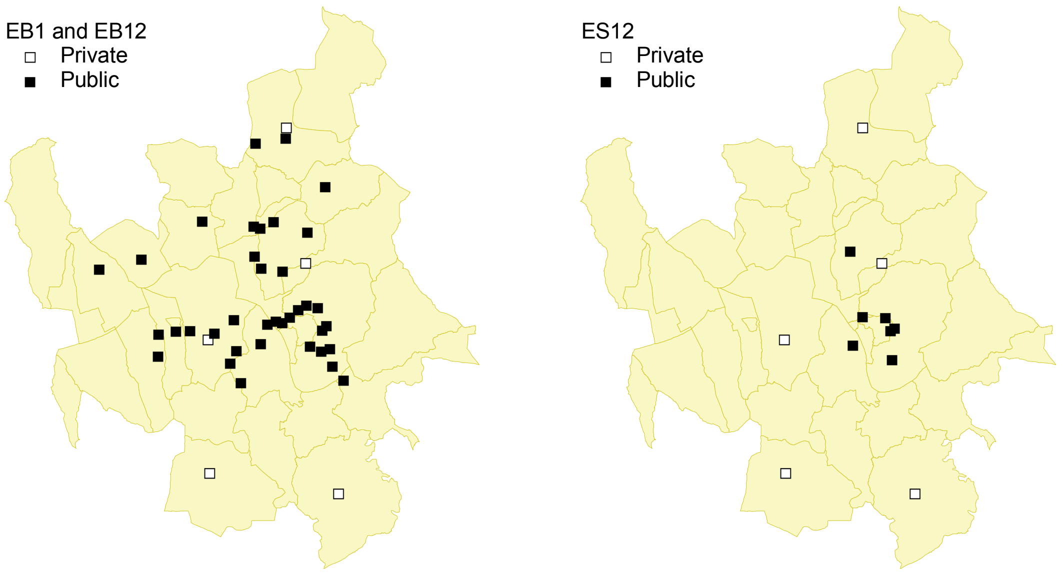


Figure 2.2: Location of existing schools[5]

[4] In Table 2.5, the capacity of private schools with both EB12 and ES12 types was allocated to each type according to the occupation in 2003/04 of the corresponding education levels.

[5] Figure 2.2 shows only schools included in the study, as discussed below (i.e. excludes public EB1 schools with less than 4 classrooms and non-subsidized private schools).

Given the existing school network and the typology conversion required by the reorganization of primary and secondary education, the planning problem to be solved consisted of defining the location, type and size of the schools composing Coimbra's secondary school network in 2015. The problem included decisions of closing existing schools and, possibly, building new schools. Indeed, despite the existing excess capacity, it could be advantageous to build new schools either to adjust the location of schools to recent housing developments or to replace existing small schools by larger ones, provided with better equipment (laboratories, libraries, sports buildings, etc.).

A solution to the problem should meet a set of constraints prescribed by the guidelines of the Portuguese Ministry of Education (MinEdu, 2000) for redeploying the school network. These constraints include maximum travel distance of students to schools and maximum and minimum numbers of students per classroom and classrooms per school.

Three objectives were pursued by the education authorities. First, all the population should be covered by either public schools or subsidized private schools. Second, the accessibility of students to schools should be maximized. Third, the changes to the existing network should be minimized, either because of scarce public budgets to build new schools or to avoid public reactions against school closure (particularly from parents and teachers). These objectives may of course be conflicting.

With regard to the first objective, existing schools included in the study were all current public schools and private schools subsidized by the Government, where students do not pay tuition fees. Other private schools, located in areas covered by public schools, which compete with public schools and are not subsidized, were left out of the study (currently there are three schools of this type, representing 10% of existing capacity).[6]

Finally, students should be exclusively assigned to the school nearest to their place of residence. Although not formally a part of the problem constraints, this rule was adopted because it leads to solutions that are easier to explain and to implement in a public facility planning context (see Section 2.5.3). What is more, as revealed through a survey carried out within the framework of Coimbra's Educational Charter (Canavarro

[6] More specifically, the existing 7 EB1 and 9 EB23/ES private schools would be converted to 7 EB1, 1 EB12, and 8 EB12+ES12 schools with the new typology. Of these, only 5 EB12+ES12 schools are subsidized. The Municipal Council of Education decided that the capacity of non-subsidized private schools would not be considered for planning purposes, that is, public and subsidized private schools should cover the total demand. The excluded private schools represent 15% of total EB1+EB12 capacity (excluding public EB1 schools with less than 4 classrooms) and 10% of total ES12 capacity as defined in Table 2.5.

et al., 2004), proximity to residence is, today, by far, the most important factor parents take into account when they choose schools for their children.

2.5 Optimization model

2.5.1 Literature review

The problem described in the previous section was represented by a mixed-integer linear optimization model, more specifically a discrete facility location model. Comprehensive surveys of facility location models are provided by Hansen et al. (1987), covering both continuous and discrete models and giving an economic interpretation of location models, Krarup and Pruzan (1990), focusing on discrete models, Labbé and Louveaux (1997), reviewing the specialized heuristic and exact solution procedures available for a range of model variants, and, more recently, Current et al. (2002), giving a broad review of recent discrete location model developments, applications, and heuristic solution procedures. References specifically discussing facility planning in the public sector include ReVelle (1987), Eiselt and Laporte (1995), and Marianov and Serra (2002).

The usual setting for the application of discrete facility location models is the following. Demand for the services provided by the facilities is measured in number of users (e.g., students) and is assumed to be concentrated in points named centers, which may represent regions, municipalities, towns or neighborhoods. Supply of facilities (e.g. schools) is assumed to be possible at specified points, named sites, which represent either one of the above geographical entities or specific plots of land. Centers and sites are connected by a transportation network.

We now present a brief review of previous optimization approaches to school network planning and management problems, applying discrete location or closely related models. Table 2.6 presents a summary of representative models in the literature since 1987, including model "ingredients", solution method (exact or heuristic), and practical application reported.

Models for school planning problems may be divided in two groups, based on the type of decisions considered. The first group addresses the management problem of assigning students to schools on a yearly basis. An existing school network is assumed and location decisions either are not considered or focus on school closure. On the other hand, assignment decisions are modeled in a detailed way, for instance taking into account student race (maintaining a racial balance across schools is an important issue in the USA). Examples are the models proposed by Schoepfle and Church (1991), a network flow model for minimizing travel distance, and by Church and Schoepfle (1993), a multiple-knapsack model for maximizing the assignments of students to schools of their choice. Church and Murray (1993) propose a multi-objective location model, including the minimization of travel distance, maximization of school district compactness and minimization of the number of students changing school from year to year. The model allows closing existing schools and includes constraints for balancing capacity occupation across open schools that improve on the formulation proposed in previous studies.

The second group of models addresses the planning problem of determining the location and capacity of schools in the medium to long term. These models include both location decisions (open/close schools) and assignment decisions (of students to schools). Antunes and Peeters (2001) and Greenleaf and Harrison (1987) propose multi-period location models seeking to optimize the schedule of network changes, that is, when to open/close schools and to expand/reduce capacity, so that total discounted costs (infrastructure, transportation, operation) are minimized. Multi-period models are useful, for example, when capacity requires expansion to meet short-term demand while medium-term demand is expected to decrease (Antunes, 1994), but involve a larger number of integer variables than single-period models and are considerably more difficult to solve. Pizzolato and Silva (1997) and Pizzolato et al. (2004) use a clustering approach to school planning based on the p-median model. Rather than determining the exact locations of new schools or the existing schools to close, a p-median model is first solved to find ideal school locations and the centers assigned to them. For each cluster formed in this way, aggregate demand is compared with existing capacity (of schools located in any of the centers of the cluster). In the first reference, the p-median model does not include capacity constraints, while in the second the total demand in each cluster may be at most the capacity of a standard school. Densham and Rushton (1996) present an interactive system to locate regional centers serving school districts, where regional centers are imposed a minimum capacity occupation. First, a p-median model is solved to locate regional centers and districts are allocated to their closest regional center, disregarding capacities. In a second phase, districts are reallocated (by user input or with the help of a heuristic) so that minimum capacities are satisfied.

Table 2.6: Representative examples of previous optimization approaches to school planning

Reference	Decisions	Model type and ingredients	Capacity constraints	Solution technique	Application
Antunes and Peeters (2001)	Location, Assignment	Multi-period location model, minimize costs, open/close facilities, capacity expansion/reduction	Maximum and minimum capacity	Heuristic (Simulated Annealing)	Primary and secondary school networks in Portugal with up to 29 centers, 38 sites and 3 time periods
Greenleaf and Harrison (1987)	Location, Assignment	Multi-period location model, minimize costs, open/close facilities	Maximum and minimum capacity	Heuristic (truncated search using an optimizer)	Primary and Secondary school networks in Pennsylvania (USA) with 16 centers, 11 schools and 5 time periods
Pizzolato and Silva (1997)	Location, Assignment	*p*-median model used to compare existing and ideal school locations	No capacity constraints	Heuristic (Interchange)	Primary school networks in Rio de Janeiro (Brazil) with up to 389 centers and 59 schools
Pizzolato et al. (2004)	Location, Assignment	As above, considering school capacities	Maximum capacity	Heuristic (Lagrangian)	Primary school network in Victoria (Brazil) with 271 centers and 51 schools
Densham and Rushton (1996)	Location, Assignment	*p*-median model, minimize travel distance	Minimum capacity	Heuristic (Interchange) plus interactive procedure	Define regional centers to serve school districts in Iowa (USA) with 480 school districts and 12 regional centers
Church and Murray (1993)	Location, Assignment	Multi-objective location model (minimize travel distance and others), allows school closure, balance capacity occupation and racial mix across schools	Maximum capacity	Exact (commercial optimizer)	Hypothetical network with 75 centers and 9 schools
Church and Schoepfle (1993)	Assignment	Multiple-knapsack model, maximize assignments of students to schools of their choice, balance capacity occupation and racial mix across schools	Maximum and minimum capacity	Exact (commercial optimizer)	No application reported.
Schoepfle and Church (1991)	Assignment	Network-flow model, minimize travel distance, balance race mix across schools	Maximum capacity	Exact (commercial optimizer)	Hypothetical network with 75 centers and 9 schools

The model proposed in our study includes location and assignment decisions and combines capacity and assignment constraints. Previous models in the school planning literature often include capacity constraints (in addition to maximum school capacities, minimum capacities may be imposed to ensure economic efficiency and/or balancing occupation across schools). However, unlike previous models, both single assignment and closest assignment constraints are included to ensure desirable properties of solutions, that is, all students in the same center must be assigned to the closest open school (discussed in detail in Section 2.5.3). We note that similar models have been proposed in other contexts: Carreras and Serra (1999) and Verter and Lapierre (2002) for the location of health care facilities, and Kalcsics et al. (2002) for designing sales territories. Regarding solution techniques, the harder location models reviewed above were solved with heuristic techniques, which provide only approximate solutions. Closest assignment constraints make location models much larger and much harder to solve. Fortunately, progress in mixed-integer linear optimization software in the last two decades has been very rapid (Atamturk and Savelsbergh, 2005), and it is now possible to solve exactly (to optimality) models large enough to be useful in practical applications, as is the case reported here.

2.5.2 Model formulation

The formulation of the model is based on the well-known *p*-median model (see Marianov and Serra (2002) for a recent survey). The objective of the model is to maximize the accessibility of students to schools, that is, to minimize the aggregate distance students need to travel to reach the school to which they are assigned.

Consider the following notation for data: I is the set of centers; J is the set of sites; $J^0 \subseteq J$ is the subset of sites with existing schools; u_i is the number of students originating from center i; d_{ij} is the travel distance between center i and site j; $Z0_j$ is the capacity of existing schools at site j (in number of students); $Z0_j^{\min}$ and $Z0_j^{\max}$ are the minimum occupied capacity and maximum expanded capacity of existing schools at site j; $Z_j^{\min}$ and $Z_j^{\max}$ are the minimum and maximum capacities of a new school installed at site j; p is the maximum number of new schools to open; q the maximum number of existing schools to close.

There are three sets of decision variables: assignment variables X_{ij}, location variables YE_j and YN_j, and capacity occupation variables ZE_j and ZN_j. They are defined as follows: X_{ij} equals 1 if center i is assigned to site j, and equals zero otherwise; YE_j equals 1 if existing schools at site j remain open, and equals zero otherwise; YN_j equals

1 if a new school is open at site j, and equals zero otherwise; ZE_j and ZN_j is the demand served at site j by existing and new schools, respectively.

The model is formulated as follows:

$$\text{Minimize } D = \sum_i \sum_j d_{ij} u_i X_{ij} \tag{2.1}$$

Subject to:

$$\sum_j X_{ij} = 1, \ \forall i \tag{2.2}$$

$$X_{ij} \leq YE_j + YN_j, \ \forall i, j \tag{2.3}$$

$$ZE_j + ZN_j = \sum_i u_i X_{ij}, \ \forall j \tag{2.4}$$

$$ZE_j \geq Z0_j^{\min} YE_j, \forall j \tag{2.5a}$$

$$ZE_j \leq Z0_j^{\max} YE_j, \forall j \tag{2.5b}$$

$$ZN_j \geq Z_j^{\min} YN_j, \forall j \tag{2.6a}$$

$$ZN_j \leq Z_j^{\max} YN_j, \forall j \tag{2.6b}$$

$$ZE_j \geq Z0_j YN_j, \ \forall j \in J^0 \tag{2.7}$$

$$\sum_{k \in N_{ij}} X_{ik} \geq YE_j, \ \ N_{ij} = \{k \in J \mid d_{ik} \leq d_{ij}\}, \ \forall i, \forall j \in J^0 \tag{2.8a}$$

$$\sum_{k \in N_{ij}} X_{ik} \geq YN_j, \ \ N_{ij} = \{k \in J \mid d_{ik} \leq d_{ij}\}, \ \forall i, \forall j \in J \tag{2.8b}$$

$$\sum_j YN_j \leq p \tag{2.9}$$

$$\sum_{j \in J^0} YE_j \geq \left| J^0 \right| - q \tag{2.10}$$

$$X_{ij} \in \{0,1\}, \forall i, j \tag{2.11}$$

$$YE_j, YN_j \in \{0,1\}, \forall j \tag{2.12}$$

The objective (2.1) expresses the minimization of aggregate travel distance. Constraints (2.2) guarantee that all students are assigned to schools. Constraints (2.3) ensure that students will not be assigned to sites where there are no schools (neither existing nor new). Constraints (2.4) define the occupied capacity of existing and new schools. Constraints (2.5a) and (2.5b) imply that, should existing facilities remain open, their occupation must satisfy maximum and minimum bounds. Constraints (2.6a) and (2.6b) define maximum and minimum capacity occupation for new schools. Constraints (2.7) imply that new capacity can only be added if existing capacity is used ($YN_j \leq YE_j$ is implied, which prevents opening a new school at a site where existing schools were closed). Constraints (2.8a) and (2.8b) are closest assignment constraints, i.e. they force students to be served by the closest open school, either existing or new. Constraints (2.9) and (2.10) bound the number of new schools to be opened and existing schools to be closed, respectively, where $\left|J^0\right|$ is the current number of sites with schools. Constraints (2.11) define assignment variables as binary, thus enforcing the single assignment property, i.e. each population center must be served by only one site. Constraints (2.12) define location variables as binary.

The differences between the model presented above and the *p*-median model are as follows: (i) new facilities can either be open at sites currently with no facilities or be co-located with an existing facility (in this way, significant capacity expansion can occur, should it be necessary); (ii) facilities have minimum and maximum capacities; (iii) single and closest assignment constraints are included (they are redundant in the *p*-median model).

The formulation of closest assignment constraints (2.8) was first introduced by Wagner and Falkson (1975). It has the advantage of remaining valid if a given center has two or more equidistant sites with open facilities. If this situation does not occur, single assignment is implied by closest assignment and the integrality constraints (2.11) could be linearly relaxed. Alternative formulations of closest assignment are analyzed by Gerrard and Church (1996).

2.5.3 Effect of single and closest assignment

Closest assignment constraints were included in the formulation since, in the presence of capacity constraints (maximum and/or minimum), optimum solutions do not have the so-called “closest assignment property”. This means that, in order to minimize total travel distance or transportation cost, it may not be optimal to assign all demand centers to the closest open facility. We consider a small example to illustrate the relevance of this property in the context of public facility planning and to show why it should be explicitly enforced.

A random instance with 20 population centers was generated to correspond to an average municipality in Portugal. Centers were uniformly generated in a circle with a 9 km radius. The total number of users was set at 10,000 (10% of a total population of

100,000), distributed according to Zipf's Law with calibration parameter equal to one (Zipf, 1949; Brakman et al., 2001). Sites are coincident with centers and distances were computed in kilometers rounded to one decimal place. Minimum capacity was set to 2,000 (thus, the maximum number of facilities is five). In this example there are no existing facilities.

This instance was solved with three variants of model (2.1)-(2.12): (i) without single assignment and closest assignment, that is, with the integrality constraints (2.11) relaxed and constraints (2.8) removed, respectively; (ii) with single assignment only, that is, with (2.8) removed; (iii) the full model with both sets of constraints. The optimum solutions are presented in Figure 2.3. Without single and closest assignment (panel i), the users of some centers are split between several facilities and small centers are served by distant facilities in order to guarantee minimum capacities. This happens for instance with the center with 556 users located on the left, which is partly served by the nearest site with an open facility and two further sites. Even with single assignment (panel ii) this situation is not eliminated, as the users of small centers might pass near to an open facility while traveling to the facility to which they are assigned, so that the minimum capacity of that facility is satisfied. The center with 556 users is now fully served by a facility in the middle of the region (at the center with 463 users), in spite of a nearby facility being open (at the center with 2780 users).

Introducing both single and closest assignment constraints eliminates undesirable configurations of assignments (panel iii). However, the objective function increases by 25% (Table 2.7). Furthermore, in this example, the number of facilities decreases when closest assignment constraints are added. This can be expected to occur in general because the number of feasible solutions diminishes. That is, single and closest assignment constraints prevent disadvantageous assignments of small centers for the sake of the global optimum. The resulting solutions are easier to interpret and to explain in a planning context, therefore being more easily accepted by the users.

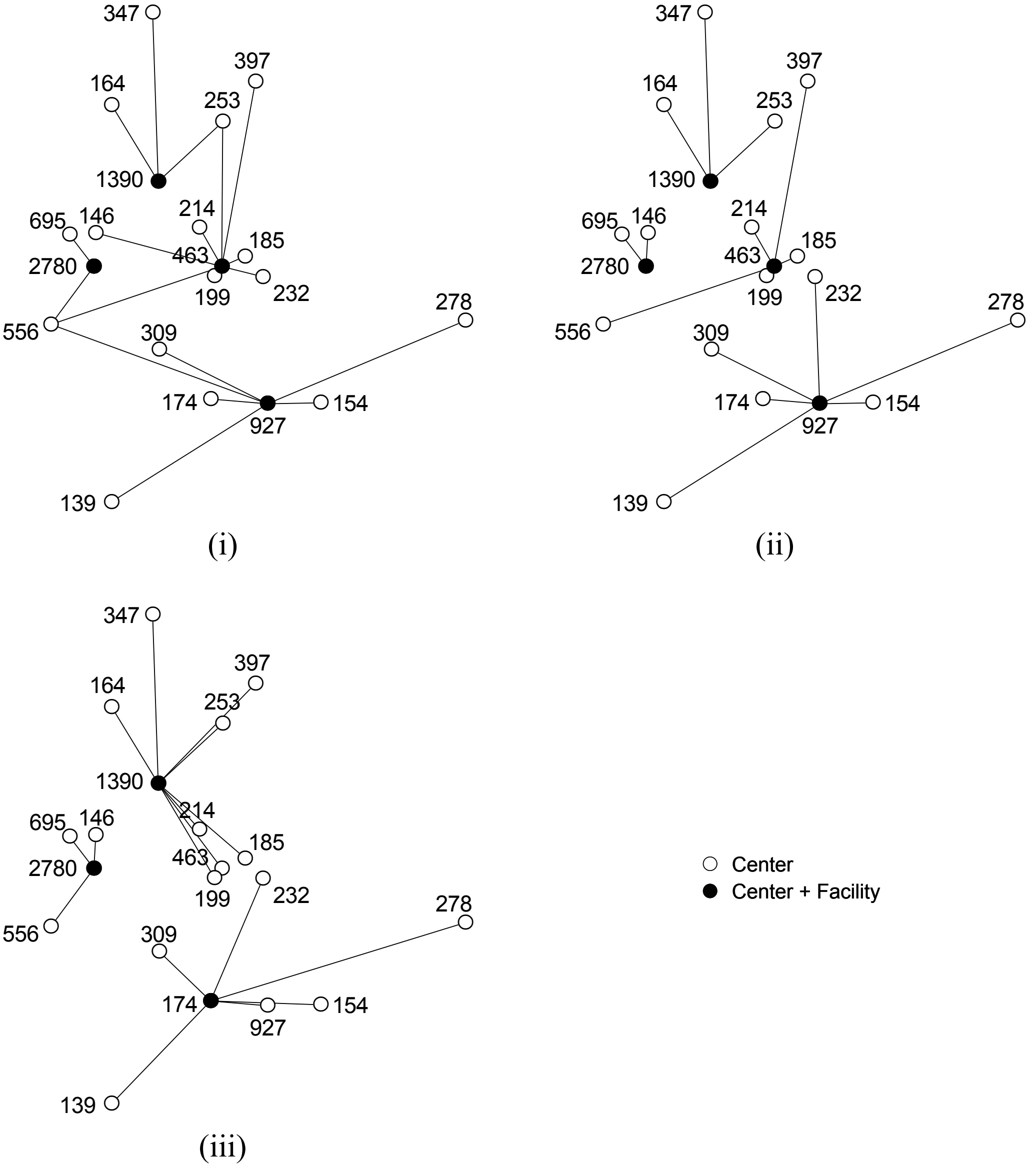


Figure 2.3: Solutions with and without single and closest assignment: (i) no assignment constraints, (ii) single assignment only, (iii) single and closest assignment

Table 2.7: Optimal value and number of facilities with and without single and closest assignment

Model	Number of facilities	Aggregate distance (km)	
		Value	Difference to minimum
No assignment constraints	4	14 482.7	0%
Single assignment only	4	16 237.4	12%
Single + Closest assignment	3	18 068.6	25%

2.5.4 Model data

The model of Section 2.5.2 was applied to Coimbra’s secondary schools by considering the number of students forecast for 2015 and the existing network of public and subsidized private schools. Students were assumed to be concentrated in 43 centers (Figure 2.4), corresponding to the 31 communities of the municipality of Coimbra, of which the five more densely populated were divided into three to four areas each. Existing schools, as well as possible new schools, were assumed to be located in 43 sites, coincident with the centers. For one particular site (Sé Nova-Combatentes) two existing schools were aggregated into a single school in the model, since they are located 500 meters from each other. Distances between centers were measured along the road network planned for 2015 (Figure 2.4).

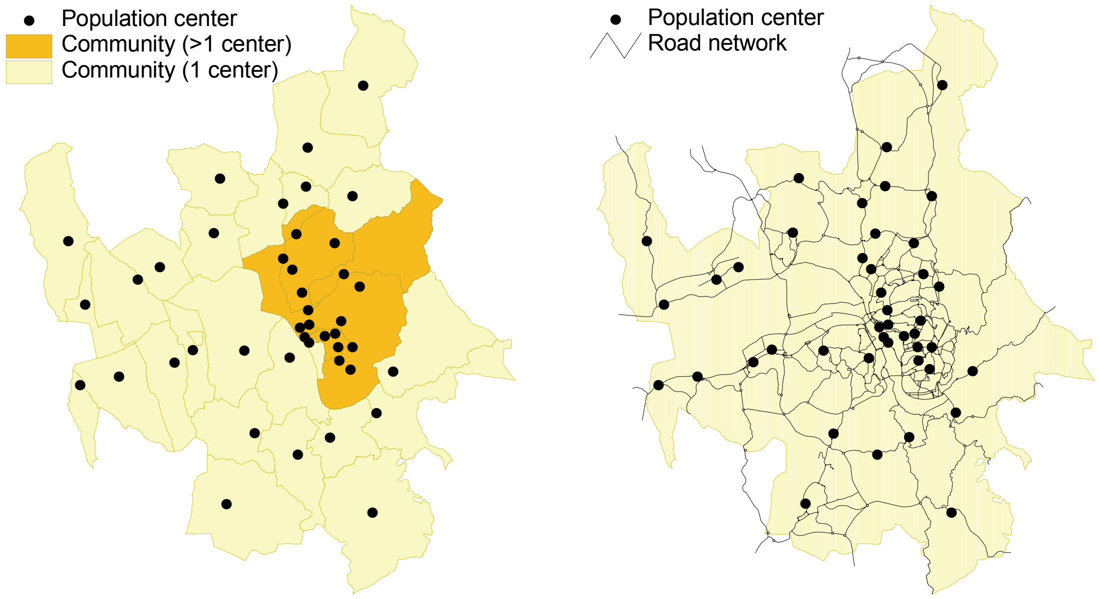


Figure 2.4: Population centers considered and road network used to compute distances

Minimum and maximum capacity occupation limits are presented in Table 2.8. For new schools these limits are set by the Ministry of Education. For existing ES schools it was assumed a minimum of 20 students per classroom, for both public and private schools. Public schools, which are all of large capacity, should contain at least 18 classes (three classes per year). For private schools it was assumed that the minimum number of classes they will offer is six (one class per year).

Table 2.8: Capacity and occupation parameters

Type of school	Maximum capacity			Minimum occupation		
	Number of rooms	Students per room	Students	Number of rooms	Students per room	Students
New public schools	39	30	1170	18	24	432
Existing public schools	current NR	30	-	18	20	360
Existing private schools	current NR	30	-	6	20	120

The maximum coverage distance was set to 12.5 km (to ensure travel time is below 30 minutes, at an average speed of 25 km/h by public transportation). This was considered in the model by setting to zero the assignment variables corresponding to pairs of centers and sites whose distance exceeded this limit.

2.6 Study Results

In this section, we present the results obtained for Coimbra's secondary school network with the model formulated in Section 2.5. Despite its large size (43 centers and sites, 2021 decision variables, of which 1935 are binary, and 5850 constraints), the model was solved with a commercial optimizer (XPRESS-MP version 2003G; Dash Optimization, 2002). Five alternative solutions were obtained by varying the number of new schools (parameter *p*) from zero to four, thus representing different trade-offs between school accessibility improvements and school network changes. Table 2.9 reports the resulting number of schools and aggregate accessibility. The chart in Figure 2.5 shows how aggregate accessibility improves as the number of schools increases. Solutions with less than two new schools were obtained by removing closest assignment constraints from the model, otherwise no feasible solutions could be found. It turned out that, in all solutions, the maximum number of school closures (parameter *q*) had no influence. With two or less new schools it is actually desirable to keep all schools open. With more than two new schools, no feasible solutions could be found without closing one school.

Table 2.9: Number of schools and accessibility of model solutions

Number of schools			Aggregate distance (km)	
New (*p*)	Closed (*q*)	Total	Value	Difference to maximum
0	0	11	21 098	0%
1	0	12	14 715	-23%
2	0	13	13 816	-39%
3	1	13	12 518	-48%
4	1	14	10 861	-51%

Note: solutions for *p*=0 and *p*=1 do not satisfy closest assignment constraints

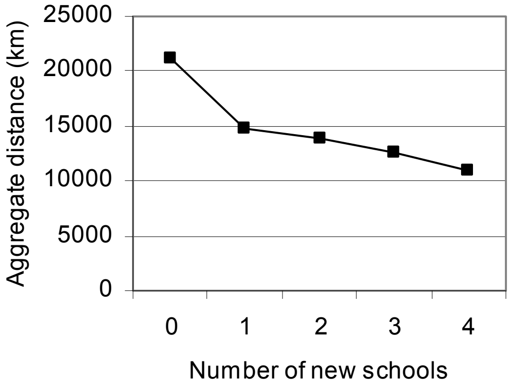


Figure 2.5: Aggregate distance as a function of the number of new schools (*p*)

School locations and student to school assignments for all solutions are represented in Figure 2.6. Solutions have the desirable property of being coherent (i.e. when adding one new school, the previously added schools remain in the same place)[7]. In the solution with two new schools (*p*=2), communities in the southwest of the municipality travel to a new school located in São Martinho do Bispo because that area of the municipality does not have enough students to justify a dedicated school there.

Future occupation of schools is summarized in Table 2.10. Public school occupation is high or at least acceptable (over 60%) in solutions with less than two new schools, except in the Sé Nova-Combatentes area (where two schools exist). In this area it would be possible to close one of the existing schools, guaranteeing 80% occupation for the school remaining open (instead of 40% for both).

[7] In chapter 4, the term coherency is employed with a different definition, applying to hierarchical models.

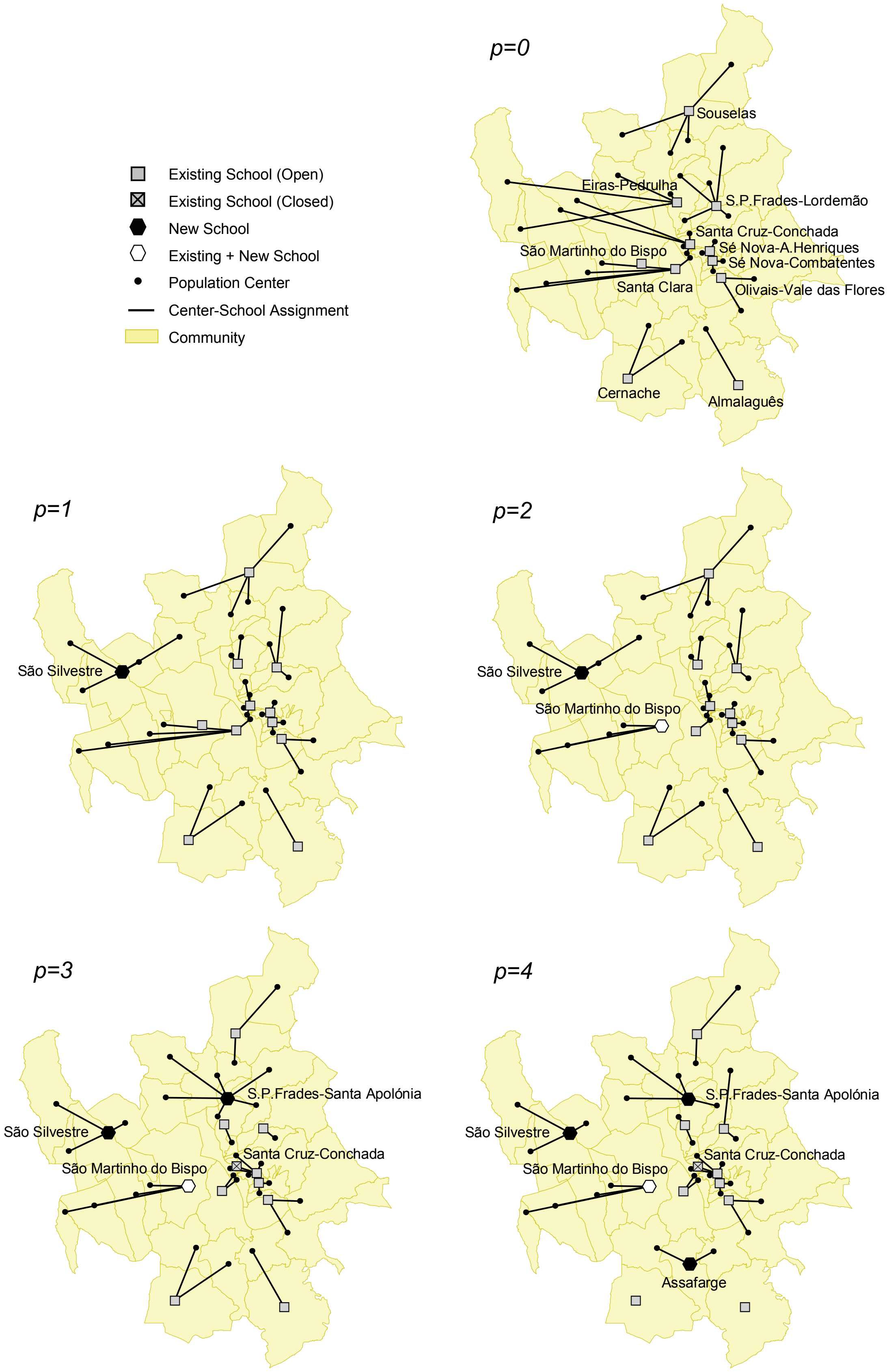


Figure 2.6: Model solutions obtained by varying the number of new schools (*p*)

For two or more new schools occupation is reduced to 40% in the Santa Clara school, and, for three or more, the school in Santa Cruz-Conchada is closed. Private school occupation was computed considering the capacity currently allocated to level B2 (which represents about 30% of total ES12 capacity in all schools). This explains the low occupation (under 40%) of some private schools in some solutions. With less than two new schools, occupation of existing private schools at S.P.Frades-Lordemão and São Martinho do Bispo is close to 100%.[8]

In conclusion, the largest improvement is obtained by adding one school to the current network. In this scenario, all existing schools guarantee sufficient occupation to remain open. Adding more schools will bring diminishing returns in terms of aggregate accessibility.

Table 2.10: Capacity occupation in model solutions

School	Status	Existing Capacity	Capacity Occupation (students)									
			p=4		p=3		p=2		p=1		p=0	
			Value	%	Value	%	Value	%	Value	%	Value	%
Almalaguês	Private	870	199	**23**	286	**33**	286	**33**	286	**33**	286	**33**
Assafarge	New	-	458	**-**	-	**-**	-	**-**	-	**-**	-	**-**
Cernache	Private	1620	228	**14**	599	**37**	599	**37**	599	**37**	599	**37**
Eiras-Pedrulha	Public	1260	1143	**91**	1029	**82**	1114	**88**	1114	**88**	1224	**97**
Olivais-Vale das Flores	Public	1260	917	**73**	917	**73**	917	**73**	917	**73**	917	**73**
S.P.Frades-Lordemão	Private	1260	382	**30**	277	**22**	582	**46**	582	**46**	1215	**96**
S.P.Frades-S.Apolónia	New	-	892	**-**	1111	**-**	-	**-**	-	**-**	-	**-**
Santa Clara	Public	1500	641	**43**	641	**43**	626	**42**	976	**65**	976	**65**
São Martinho do Bispo 1	Private	1020	917	**90**	917	**90**	917	**90**	999	**98**	999	**98**
São Martinho do Bispo 2	New	-	432	**-**	432	**-**	432	**-**	-	**-**	-	**-**
São Silvestre	New	-	620	**-**	620	**-**	808	**-**	808	**-**	-	**-**
Sé Nova-A.Henriques	Public	1440	1032	**72**	1032	**72**	895	**62**	895	**62**	895	**62**
Sé Nova-Combatentes	Public	2760	1040	**38**	1040	**38**	1040	**38**	1040	**38**	1040	**38**
Souselas	Private	1290	369	**29**	369	**29**	571	**44**	571	**44**	571	**44**
Santa Cruz-Conchada	Public	690	0	**0**	0	**0**	483	**70**	483	**70**	548	**79**

[8] These results were obtained with non-final data on private school capacity. First, the number of classrooms was estimated by dividing occupation in 2003/04 by 20 students per room (this over-estimated capacity and was revised later in the study). Second, the capacity of modeled private schools, all with both EB12 and ES12 types (considered separately in the models for primary and secondary schools), was assumed to be wholly available for the ES12 type. Later in the study, each of the EB12 and ES12 types was allocated 50% of the total number of classrooms. The new solutions had the same school locations, but some assignments were changed. In particular, some centers assigned to the S.P.Frades-Lordemão school were re-assigned to neighboring schools in scenarios with p<2; the demand of center São Martinho do Bispo, which became higher than the capacity of the school located there, was allowed (by disabling single and closest assignment constraints for this center) to be partly assigned to the neighboring, large school in Santa Clara.

2.7 Applying the model with a Geographic Information System

The study reported in this chapter relied heavily on a Geographic Information System (GIS) for data handling and result analysis. Geographic data available for the municipality of Coimbra included community boundaries and population centers, census tracts, the road network, and existing school locations.

The GIS was used to compute a distance matrix for Coimbra and to display the solutions obtained with the optimizer (XPRESS-MP) on a map. This was made through a prototype system based on Arcview GIS version 3.2 (ESRI, 2000). The system consists of a set of programs implemented with Arcview's scripting language (Avenue), and is available from the authors on request. Figure 2.7 shows the steps for applying the model in practice by using the GIS and the optimizer. Typically, the model is run using different model parameters (number of schools to open, etc.), and possibly different data (e.g. school capacities), for analyzing different scenarios.

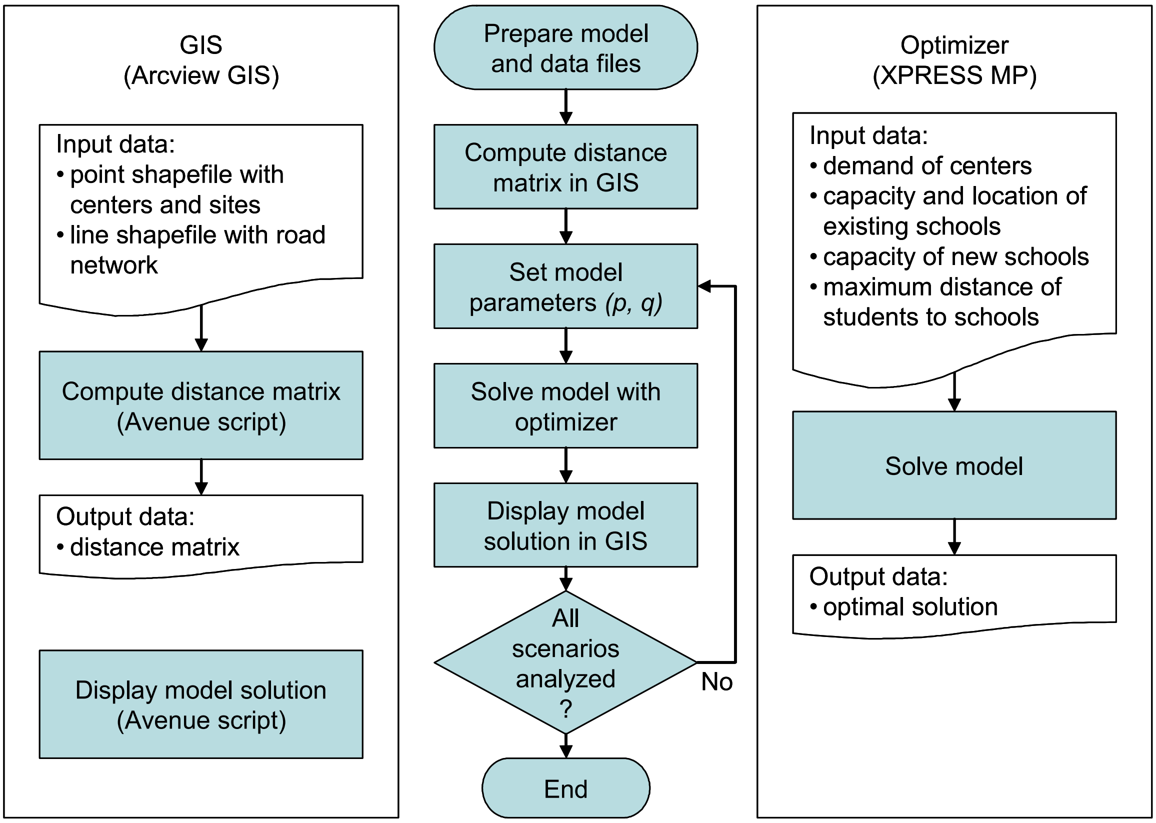
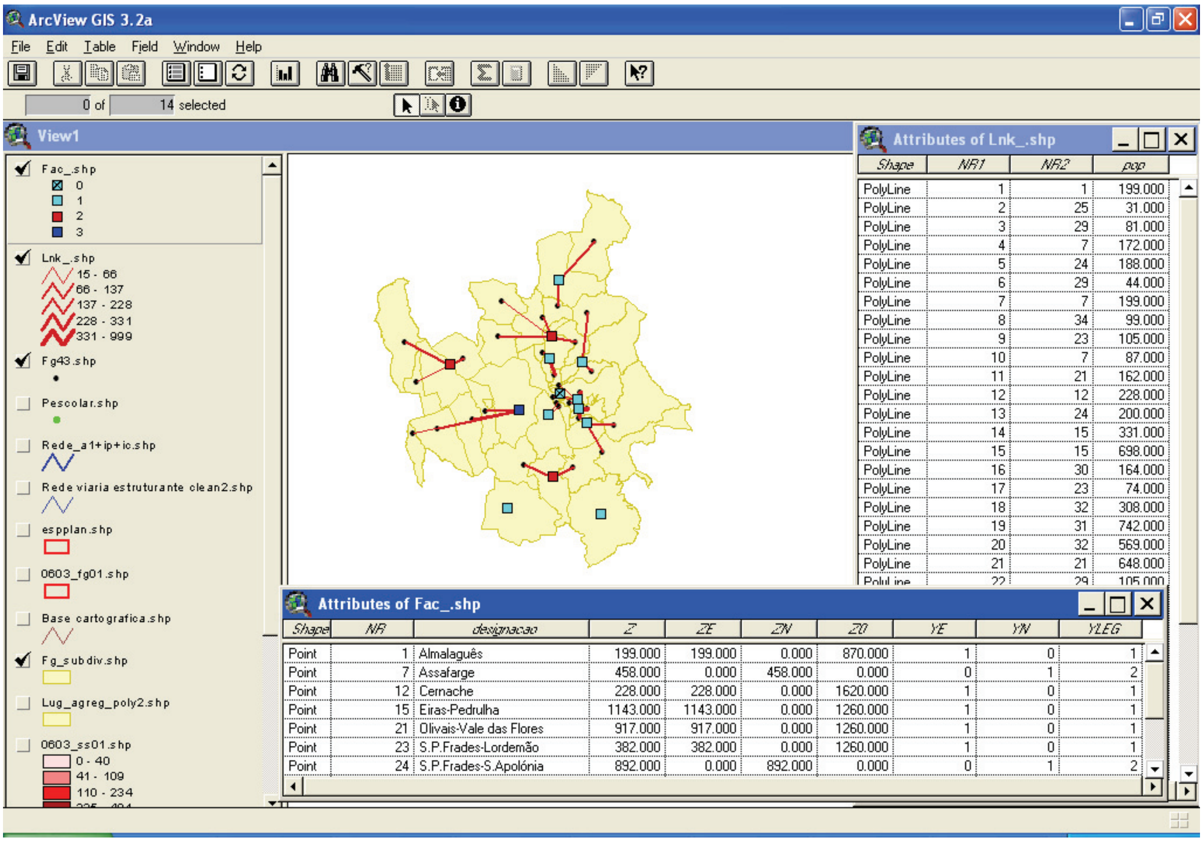

GIS
(Arcview GIS)
Input data:
• point shapefile with centers and sites
• line shapefile with road network
Compute distance matrix (Avenue script)
Output data:
• distance matrix
Display model solution (Avenue script)
Prepare model and data files
Compute distance matrix in GIS
Set model parameters (p, q)
Solve model with optimizer
Display model solution in GIS
All scenarios analyzed ?
No
End
Optimizer
(XPRESS MP)
Input data:
• demand of centers
• capacity and location of existing schools
• capacity of new schools
• maximum distance of students to schools
Solve model
Output data:
• optimal solution

Figure 2.7: Applying the model by using the GIS and the Optimizer

The distance matrix consists of all shortest path lengths between centers and sites and was computed over the main road network projected for 2015. This road network will comprise around 1,400 links and 1,000 intersections. The computation was made by scripting Arcview's Network Analyst extension, which includes built-in functions for finding shortest paths on networks. Although computing all shortest paths is an easily solvable problem (for which there are efficient methods, e.g. Floyd's algorithm), using a GIS is still convenient and saves burdensome work. For instance, it is not necessary to export the road network, which must include nodes for all road intersections, and to track the correspondence between network nodes and centers or sites.

The display of model solutions on a map largely facilitates the diagnosis of model errors and the interpretation of model results. Within the prototype system, solutions are exported from the optimizer as text files and read by a script in the GIS, where they are displayed as school locations and center assignments. In addition, all solution data (such as school occupation) can easily be displayed and inspected in tabular form (Figure 2.8).



Attributes of Lnk_.shp

Shape	NR1	NR2	pop
PolyLine	1	1	199.000
PolyLine	2	25	31.000
PolyLine	3	29	81.000
PolyLine	4	7	172.000
PolyLine	5	24	188.000
PolyLine	6	29	44.000
PolyLine	7	7	199.000
PolyLine	8	34	99.000
PolyLine	9	23	105.000
PolyLine	10	7	87.000
PolyLine	11	21	162.000
PolyLine	12	12	228.000
PolyLine	13	24	200.000
PolyLine	14	15	331.000
PolyLine	15	15	698.000
PolyLine	16	30	164.000
PolyLine	17	23	74.000
PolyLine	18	32	308.000
PolyLine	19	31	742.000
PolyLine	20	32	569.000
PolyLine	21	21	648.000
PolyLine	22	29	105.000

Attributes of Fac_.shp

Shape	NR	designacao	Z	ZE	ZN	Z0	YE	YN	YLEG
Point	1	Almalaguês	199.000	199.000	0.000	870.000	1	0	1
Point	7	Assafarge	458.000	0.000	458.000	0.000	0	1	2
Point	12	Cernache	228.000	228.000	0.000	1620.000	1	0	1
Point	15	Eiras-Pedrulha	1143.000	1143.000	0.000	1260.000	1	0	1
Point	21	Olivais-Vale das Flores	917.000	917.000	0.000	1260.000	1	0	1
Point	23	S.P.Frades-Lordemão	382.000	382.000	0.000	1260.000	1	0	1
Point	24	S.P.Frades-S.Apolónia	892.000	0.000	892.000	0.000	0	1	2

Figure 2.8: Cartographic solution display and values of non-zero model variables

2.8 Conclusion

In this chapter, we presented a study on secondary school planning made within the framework of Coimbra's Educational Charter. The main tool used for the study was a discrete facility location model based on the well-known *p*-median model. The model includes constraints on maximum and minimum capacity occupation, single assignment and closest assignment. Although models containing these ingredients were used before for other applications, such as the location of health services, the combination of capacity and assignment constraints had not been reported before in an application to school network planning. In addition, it was shown that a modern commercial optimizer is capable of solving models large enough to be useful in practice. The solutions found through the model clearly suited the needs of a real-world decision-making process about the future development of a school network, and were extremely helpful in the discussions held within Coimbra's Educational Council. This council involves the municipal administration, school administrations, parents' organizations and teacher unions, and is responsible for the approval of the Educational Charter. The multiple interests of all parties involved, often conflicting, make this decision-making process complex. The merit of the model presented here is to provide rational solutions as a basis for discussion. Specifically, it was found that, with only the existing schools, it cannot be guaranteed that students can be served by the closest school. To accommodate both the forecast demand and the school typology change (secondary schools will receive two three-year cycles of education instead of one), the best trade-off between accessibility improvement and network changes was found to be a single new school.

The study focused on secondary education, for which in Portugal a single type of school exists. In the near future, a similar study regarding primary education needs to be made. The model will have to be extended to take into account the existence of two types of school: EB1 (dedicated to the first cycle) and EB12 (offering the two cycles of primary education). This leads to a so-called hierarchical facility location model (Narula, 1986), with two demand levels and two facility levels. Most of previous models (e.g. Weaver and Church, 1991, and Galvão et al., 2002) do not consider facility capacities. Those that do (e.g. Eitan et al., 2001, and Galvão et al., 2006) do not include assignment constraints, such as closest assignment. Our future research will be directed towards the development of hierarchical models coupling capacity and assignment constraints, and the development of specialized solution procedures for solving large-scale hierarchical and non-hierarchical models still out of reach of current state-of-the-art commercial optimizers.

Chapter 3

Application of a hierarchical model to the location of primary schools

3.1 Introduction

During the last two decades, planning processes involving public facilities such as schools or hospitals became increasingly more complex, especially because of the participatory mechanisms they started to involve. Indeed, in the presence of stakeholders with different viewpoints and interests, planning solutions can only be widely agreed upon if they are the result of transparent, rational planning processes. When the number of possible planning solutions is very large, optimization models are indispensable decision-aid tools. Location models certainly are among the main optimization models to be used within public facility planning processes. These models are basically aimed at determining the most efficient locations for all types of facilities according to some objective or objectives (cost minimization, accessibility maximization, etc.). They are classified as continuous or discrete depending on whether the facilities can be located anywhere on the plane or in some points of the plane, specified in advance. In practical applications, planners often resort to discrete location models.

Location models have been extensively studied since the 1960s, in the operational research, management science, industrial engineering, economic geography and spatial planning literatures. ReVelle and Eiselt (2005) present a concise review of the main classes of continuous and discrete location models. ReVelle (1987) and Marianov and Serra (2002) discuss discrete models for public facility location. Daskin (1995) presents a didactic textbook on modeling and solving discrete location models. Labbé and Louveaux (1997) review specialized solution methods for basic and extended discrete location models.

In this chapter, we present a discrete hierarchical location model for public facility planning, considering several levels of demand and several types of facilities. The model is an extension of the well-known *p*-median model, which applies to facility location problems where the objective is to maximize the accessibility of users to facilities.

The model was developed within the framework of the Coimbra Educational Charter 2006-2015, to help making decisions on the redeployment of the primary school network of the municipality of Coimbra, Portugal. By law, all Portuguese municipalities must have an educational charter where the infrastructure, equipment, human, and financial resources necessary for pre-school, primary and secondary education are specified. The preparation of the educational charter of a municipality is advised by the education council of the municipality. This body integrates, among others, representatives of the local administration, the Ministry of Education, private school owners, public school administrations, teacher unions, and student parents, which often have different viewpoints and interests with regard to the evolution of school networks.

This chapter is organized as follows. In Section 2, we present the basic location models applicable when the objective is to maximize the accessibility of users to facilities. These models consider a single level of demand and a single type of facility. In Section 3, we discuss different user-to-facility assignment constraints, including a new type of constraints called path-assignment constraints. In Section 4, we present the hierarchical location model. In Section 5, we discuss the results obtained with this model for Coimbra's primary school network. Finally, in Section 6, we summarize the main contributions of the chapter, reflect on the application of the model in Coimbra, and identify some research needs to be fulfilled in the future.

3.2 Basic models

In this section, we present the basic location models upon which the hierarchical model is built. The simplest of these models is the p-median model (ReVelle and Swain, 1970), which can be stated as follows. We are given a set of demand centers $I = \{1,...,n\}$, where each center i has a demand u_i (number of users), a set of sites $J = \{1,...,m\}$, and travel costs c_{ij} for serving all the demand from center i at site j. Travel costs are defined as $c_{ij} = u_i \cdot d_{ij}$, where d_{ij} is the unit travel cost between center i and site j (or distance, if the unit cost is constant). The problem is to find the set of p facilities that should be open, and to determine which centers should be served from which facilities, so that the travel costs of serving all the demand from all centers is minimized. For formulating the model, we define two sets of decision variables: binary location variables y_j, where $y_j = 1$ if a facility is located (or "open") at site $j \in J$, and $y_j = 0$ otherwise; and assignment variables x_{ij} representing the fraction of the demand from center $i \in I$ served at site $j \in J$. The formulation of the *p*-median model is as follows:

(PM):

Minimize $$\sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \qquad (3.1)$$

Subject to: $$\sum_{j \in J} x_{ij} = 1,\ \forall i \in I \qquad (3.2)$$

$$x_{ij} \le y_j,\ \forall i \in I, j \in J \qquad (3.3)$$

$$\sum_{j \in J} y_j = p \qquad (3.4)$$

$$0 \le x_{ij} \le 1,\ \forall i \in I, j \in J \qquad (3.5)$$

$$y_j \in \{0,1\},\ \forall j \in J \qquad (3.6)$$

The objective function (3.1) of this mixed-integer optimization model expresses the minimization of travel costs, which can be seen as a proxy for accessibility maximization. Constraints (3.2) state that all centers have to be fully served. Constraints (3.3) link location and assignment decisions by stating that centers can only be assigned to an open facility. Constraint (3.4) sets the number of open facilities equal to parameter *p*. Finally, constraints (3.5) and (3.6) define decision variables.

Optimal solutions of (PM) have the so-called single assignment and closest assignment properties (Krarup and Pruzan, 1983), that is, centers are fully served by the closest open facility (or, if travel cost is not monotonically dependent on distance, the least travel cost facility). This happens because, as there are no capacity constraints, nothing is gained by splitting the demand from a given center across several facilities, and the objective forces centers to be assigned to the closest (least cost) facility.

The second basic model is derived from (PM) by adding constraints on the minimum and maximum capacity of facilities and deleting the constraint on the number of open facilities. Note that, with the capacity constraints, the number of open facilities becomes an output of the model rather than a parameter. Let b_j and B_j be the minimum and maximum capacity for a facility to be open at site *j*. This model, denoted capacitated median model, is formulated as follows:

(CM):

Minimize $$\sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \qquad (3.7)$$

Subject to: $$\sum_{j \in J} x_{ij} = 1, \ \forall i \in I \tag{3.8}$$

$$x_{ij} \le y_j, \ \forall i \in I, j \in J \tag{3.9}$$

$$\sum_{i \in I} u_i x_{ij} \ge b_j y_j, \ \forall j \in J \tag{3.10}$$

$$\sum_{i \in I} u_i x_{ij} \le B_j y_j, \ \forall j \in J \tag{3.11}$$

$$\sum_{k \in J | d_{ik} \le d_{ij}} x_{ik} \ge y_j, \ \forall i \in I, j \in J \tag{3.12}$$

$$x_{ij} \in \{0,1\}, \ \forall i \in I, j \in J \tag{3.13}$$

$$y_j \in \{0,1\}, \ \forall j \in J \tag{3.14}$$

Expressions (3.10) and (3.11) are, respectively, the minimum and maximum capacity constraints. As the closest assignment and single assignment properties of solutions do not hold for capacitated models, they are enforced explicitly with constraints (3.12) and (3.13), respectively. Closest assignment constraints (3.12) work as follows. For any center *i* and site *j*, if $y_j = 0$ then the constraint has no effect; if $y_j = 1$ then center *i* must be fully served from the facility located at site *j* or from a facility at the same or lower distance. Gerrard and Church (1996) thoroughly review formulations and applications of closest assignment constraints in several location models arising in the public and private sectors. Although the authors note that a different formulation, the so-called Rojeski-ReVelle (RR) constraints, is frequently used in the literature, we opted for constraints (3.12) for two reasons. First, unlike RR constraints, constraints (3.12) remain valid if a given center has two or more equidistant facilities that are the closest. Second, constraints (3.12) provide a tighter linear relaxation, as RR constraints are implied by (3.12) together with (3.9); thus, the model can be solved more efficiently with an integer optimization algorithm based on linear relaxations. Finally, constraints (3.13) force assignment variables to be binary. We remark that if all centers have a single closest facility then constraints (3.12) will force assignment variables to be integer, even if (3.13) is relaxed. Otherwise, if a center has two or more equidistant facilities that are the closest, constraints (3.12) allow demand to be freely distributed among those facilities and (3.13) is necessary to impose single assignment.

Unlike the *p*-median model, the capacitated median model has rarely been dealt with in the literature. Carreras and Serra (1999) use it without the maximum capacity constraints to represent a pharmacy location problem in a rural region, and solve it through a tabu search heuristic. Verter and Lapierre (2002) employ a similar model for locating preventive health care facilities with the objective of maximizing population coverage, and solve it with a commercial optimizer. Kalcsics et al. (2002) use model (CM) with constraint (3.4) for designing balanced and compact sales territories, and solve it through a variable neighborhood search heuristic.

3.3 Assignment constraints

In this section, we analyze the spatial pattern of user-to-facility assignments resulting from different assignment constraints. As noted previously, solutions to location models including capacity constraints do not have the single and closest assignment properties. This may happen because facilities have limited capacity, and thus users are diverted to other facilities, or users are "captured" to ensure the minimum capacity of a facility. In a public facility planning context, it should be prevented that users from the same center are split among different facilities; that users from neighboring centers are assigned to different facilities; that users are assigned to a distant facility when there are closer open facilities; that the path traveled by users to the facility they are assigned to crosses a center assigned to a different facility. If these conditions are violated, solutions are difficult to interpret by decision-makers and to explain to users, and will certainly be difficult to implement in practice. Gerrard and Church (1996) make a similar argument in their article, where they recommend the use of closest assignment constraints in capacitated location models for public facility planning because they are likely to improve public confidence in and acceptance of the corresponding solutions.

Our analysis focuses on three types of constraints: the closest assignment constraints (3.12); the single assignment constraints (3.13); and a new type of constraints designated as path assignment constraints (3.15). The latter are an alternative to closest assignment constraints. Although they do not require centers to be assigned to the closest facility, they guarantee that, if a center is assigned to a given facility, all centers "near" the "path" traveled by the users to reach the facility must also be assigned to it. The definition of "near" depends on the context of application of the model. The formulation of path assignment constraints is as follows:

$$\sum_{k \in P_{ij}} x_{kj} \geq \left| P_{ij} \right| \cdot x_{ij}, \ \forall i \in I, j \in J \qquad (3.15)$$

where P_{ij} is the subset of centers $k \in I$ that are "near" the shortest path from i to j, and $\left| P_{ij} \right|$ is the cardinality of this set. Expression (3.15) states that if $x_{ij} = 1$ then $x_{kj} = 1$ for all $k \in P_{ij}$.

Some constraints used in districting models are similar to the path assignment constraints. These models apply to the partitioning of a set of spatial units (i.e. city blocks, census tracts, or other geographic entities) into subsets, called districts, according to some objective. Like with location models, spatial units are represented with discrete centers connected through an underlying network. Desired properties of districts often include compactness and contiguity, i.e. they should be round shaped rather than spread out, and should be connected. Kalcsics et al. (2002) use model (CM) for sales districting, including closest assignment in order to produce connected districts. Zoltners and Sinha (1983) formulate a model for sales districting, where district centers (i.e. the seed units for the districts) are predefined. Considering binary decision variables $x_{ij} = 1$ if unit i is assigned to district j, and zero otherwise, contiguity is enforced with constraints

$$x_{ij} \leq \sum_{k \in A_{ij}} x_{kj} \, , \; \forall i \in I, j \in J \tag{3.16}$$

where A_{ij} is the set of units $k \in I$ that immediately precede unit i on a shortest path from district center j to unit i. Expression (3.16) states that unit i can be assigned to district j only if at least another unit, adjacent to i on a path connecting to the district center, is also assigned to j. In order to reduce the "rigidity" of assignments (as building districts along shortest paths guarantees contiguity but rules out some contiguous configurations), the authors propose augmenting sets A_{ij} by either considering the second, third, etc. shortest paths, or by manual modification by an expert user. A strict shortest-path approach is adopted by Mehrotra et al. (1998) to ensure contiguity in a political districting model. Caro et al. (2004) present a model for school districting (with predefined school locations) including constraints similar to (3.16), where sets A_{ij} are defined as the set of units $k \in I$ that are adjacent to unit i and are closer to district center j than to unit i (but not necessarily on the shortest path).

The impact of including the different assignment constraints in a location model will be illustrated with one of the random instances used to test model (CM).[9] The instances were built as follows. First, a set of $n = 20$ centers was generated uniformly in the unit square and their coordinates scaled by 1000. Sites were assumed coincident with centers. Then a planar network was created by computing the Delaunay triangulation (Weisstein, 1999), with edge length set equal to the Euclidean distance. Distances d_{ij} were computed by finding all shortest paths on the resulting network. Demands u_i were generated uniformly in the interval 5 to 95. Minimum capacities b_j were set to 200 for all sites. Maximum capacities B_j were set to the total demand (i.e. they are not binding).

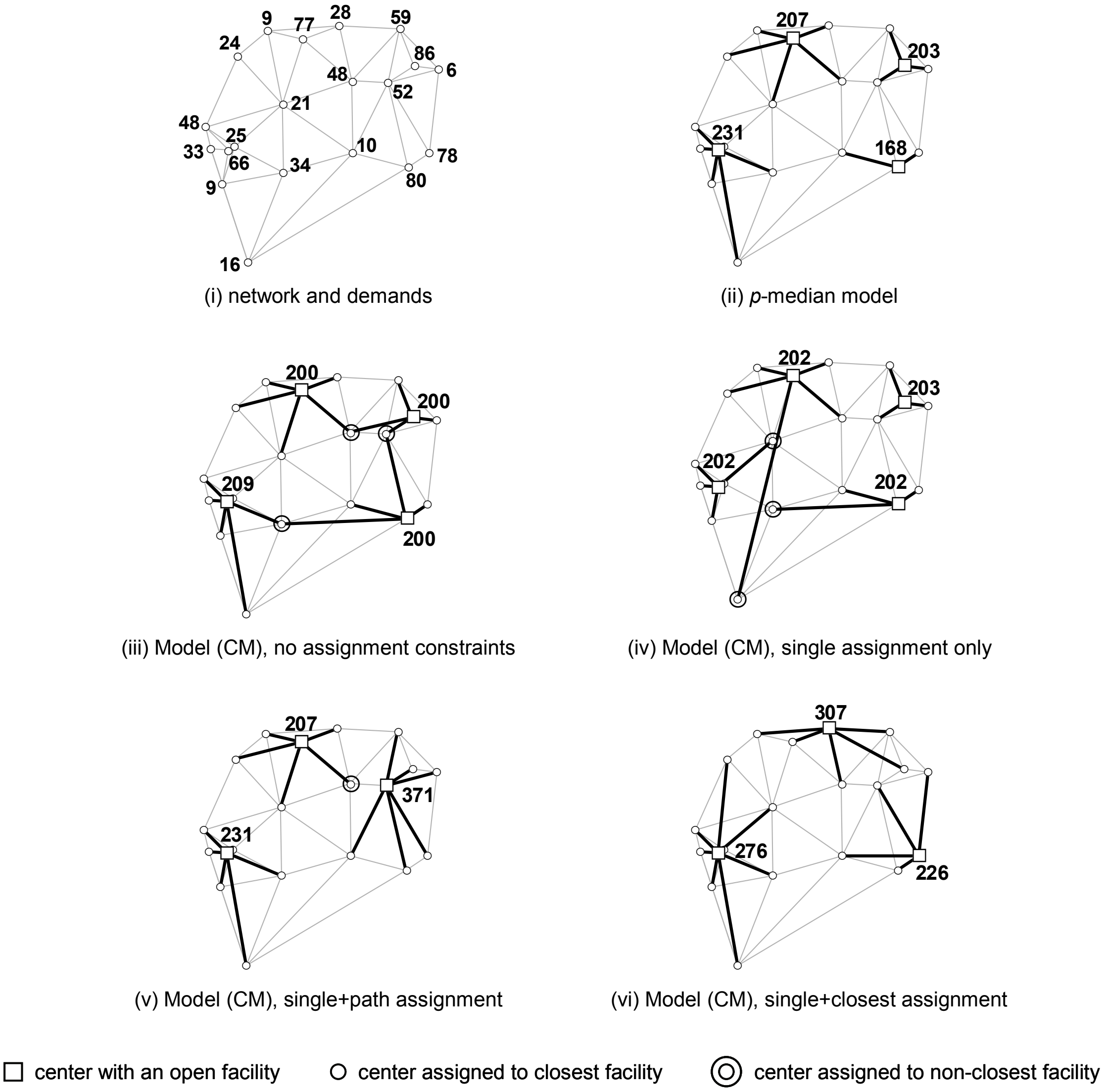
[9] This example partly repeats a similar one in chapter 2, but uses an instance built in a different way, more suitable to illustrate path assignment constraints.

Finally, path assignment sets P_{ij} were created by adding all centers in the shortest path from *i* to *j*, plus all centers within a radius of 100 from any node in this path. Panel (i) of Figure 3.1 shows the network and demands of the test instance. As the total demand is 809, at most four facilities can satisfy the minimum capacity of 200.

This instance was solved with four variants of model (CM): without assignment constraints, i.e. (3.12) removed and (3.13) relaxed; with single assignment only, i.e. (3.12) removed; with single+path assignment, i.e. (3.12) replaced by (3.15); and with single+closest assignment, i.e. the full model (CM). Solutions are shown in Figure 3.1 (where numbers in panels (ii)-(vi) refer to capacity). The *p*-median model (PM) solution is included for comparison, with *p*=4. In this solution, centers are fully assigned to the closest facility, but one of the facilities does not satisfy the minimum capacity constraints. On the other hand, all solutions of variants of model (CM) satisfy minimum capacities, but some undesirable user-to-facility assignments occur if path or closest assignment constraints are not imposed. Without any assignment constraints, the demand from some centers is split among two facilities. With single assignment only, split demands are eliminated but some centers are assigned to a facility much further than the closest. Adding path or closest assignment constraints eliminates these undesirable patterns (although with path assignment one of the centers is not assigned to the closest facility). As expected, adding more constraints degrades the optimal objective value (Table 3.1). In addition, in this example, solutions with path or closest assignment constraints have fewer open facilities. This can be expected to occur frequently because the number of feasible solutions diminishes. That is, path and closest assignment constraints prevent disadvantageous assignments of some small centers for the sake of the global optimum. The resulting solutions are easier to interpret and to explain in a planning context, therefore being more likely to be accepted by the users.

Table 3.1: Solutions for random instance

Model	Number of open facilities	Relative objective value
p-median	4	83%
(CM), no assignment constraints	4	91%
(CM), single assignment only	4	100%
(CM), single+path assignment	3	138%
(CM), single+closest assignment	3	146%



(i) network and demands

(ii) *p*-median model

(iii) Model (CM), no assignment constraints

(iv) Model (CM), single assignment only

(v) Model (CM), single+path assignment

(vi) Model (CM), single+closest assignment

□ center with an open facility ○ center assigned to closest facility ◎ center assigned to non-closest facility

Figure 3.1: Solutions for random instance

3.4 Hierarchical model

In this section we present a hierarchical version of the capacitated median problem of Section 2, considering several levels of demand and several types (or levels) of facilities. Specifically, the demand of centers is discriminated in levels, which must be separately assigned to facilities capable of providing them. We consider a nested (or successively inclusive) hierarchy of facilities where a level-*s* facility (*s*=1, 2, …, *ns*) can serve demands of level 1, …, *s*. Examples of nested hierarchical facilities arise in the context of public facility planning. A typical example is a health care network composed of local health care units, providing basic services, and central hospitals providing both basic and specialized services. Another example is an education network, composed of kindergartens, primary schools and secondary schools, in which a higher level of education can only be located at a site if all lower-levels are also located there. As noted by Weaver and Church (1991), the various levels of facilities may or may not be physically distinct (in the example of schools, these could be separate buildings located in the same community). Other examples of hierarchical facilities arising in postal and banking services are described by Daskin (1995).

Reviews and classifications of hierarchical models are provided by Narula (1986) and Church and Eaton (1987), focusing on models with distance minimization and with coverage maximization objectives, respectively. Before introducing our model, we will briefly describe some representative examples of hierarchical extensions of the *p*-median model. Weaver and Church (1991) formulate a model with a nested facility hierarchy with any number of levels. The model does not include capacity constraints, thus an optimal solution can always be found with the single and closest assignment properties. The model is solved with Lagrangian relaxation combined with an interchange heuristic, and results are reported for two test networks, with two and three levels. Galvão et al. (2006) formulate a model with a nested facility hierarchy with three-levels, addressing a real-world health care application. The model allows service referrals (that is, a fraction of demand served at a lower-level facility may be referred directly to a higher-level facility) and includes capacity constraints (however, single-assignment is not enforced and split assignments can occur). The authors propose a Lagrangian heuristic to solve the model. Eitan et al. (1991) formulate a model with nested or more general facility hierarchies (with any number of levels), capacity constraints, and service referrals. The model is applied to several problems appearing in the literature and is solved with a commercial integer optimizer.

Relatively to hierarchical models in the literature, the model we present here is the first to combine capacity constraints (minimum and maximum) with assignment constraints such as closest- or path-assignment (in addition to single-assignment). Consider the following additional or revised notation for data: $S = \{1,...,ns\}$ is the set of demand levels (and of facility types); u_{is} is level-*s* demand of center *i*; B_{js} and b_{js} are the maximum and minimum capacities of a type-*s* facility at site j; J_s^0 is the set of sites with existing type-*s* facilities; p_s is the maximum numbers of new type-*s* facilities to open; q_s is the maximum numbers of existing type-*s* facilities to close; D_s is the maximum user-to-facility distance for demand level *s*. Decision variables are defined as follows: x_{ijs} is the fraction of the level-*s* demand of center *i* satisfied by a facility located at site j; $y_{js} = 1$ if a type-*s* facility is located at j, and equals zero otherwise; z_{jst} is the capacity occupied with demand level *s* of a level-*t* facility located at *j*. The hierarchical capacitated median model is formulated as follows:

(HCM):

Minimize
$$\sum_{i \in I} \sum_{j \in J} \sum_{s \in S} d_{ij} u_{is} x_{ijs} \tag{3.17}$$

Subject to:
$$\sum_{j \in J} x_{ijs} = 1, \ \forall i \in I, s \in S \tag{3.18}$$

$$x_{ijs} \leq \sum_{t \in S | t \geq s} y_{jt}, \ \forall i \in I, j \in J, s \in S \tag{3.19}$$

$$\sum_{t \in S | t \geq s} z_{jst} = \sum_{i \in I} u_{is} x_{ijs}, \ \forall j \in J, s \in S \tag{3.20}$$

$$\sum_{s \in S | s \leq t} z_{jst} \geq b_{jt} y_{jt}, \ \forall j \in J, t \in S \tag{3.21}$$

$$\sum_{s \in S | s \leq t} z_{jst} \leq B_{jt} y_{jt}, \ \forall j \in J, t \in S \tag{3.22}$$

$$\sum_{k \in J | d_{ik} \leq d_{ij}} x_{iks} \geq y_{jt}, \ \forall i \in I, j \in J, s \in S, \forall t \in S \,|\, t \geq s \tag{3.23}$$

$$\sum_{j \in J \setminus J_s^0} y_{js} \leq p_s, \ \forall s \in S \tag{3.24}$$

$$\sum_{j \in J_s^0} y_{js} \geq \left| J_s^0 \right| - q_s, \ \forall s \in S \tag{3.25}$$

$$x_{ijs} = 0, \ \forall i \in I, j \in J, s \in S \,|\, d_{ij} > D_s \tag{3.26}$$

$$x_{ijs} \in \{0,1\}, \ y_{js} \in \{0,1\}, \ z_{jst} \geq 0, \ \forall i \in I, j \in J, s \in S, t \in S \tag{3.27}$$

Constraints (3.18) ensure that all demands of all levels from all centers are satisfied. Constraints (3.19) impose that a given level of demand can only be satisfied by a facility of equal or higher level. Constraints (3.20) define capacity variables z_{jst} by stating that the demand of each level assigned to a site has to be served by some facility of equal or higher level located there. Constraints (3.21) and (3.22) impose maximum and minimum limits on capacity, according to facility type. With this formulation, capacity is shared by all demand levels. Additional constraints could easily be added to impose separate capacity limits per demand level, as the model already includes capacity variables z_{jst} discriminating the demand levels. Note that constraints (3.19) are redundant for the integer formulation (that is, the same set of integer solutions is obtained if they are removed), as constraints (3.20) and (3.21) allow a variable x_{ijs} for a given $s \in S$ to be non-zero only if there is a variable y_{jt} equal to one for some $t \geq s$. However, (3.19) are kept in the formulation as they strengthen the linear relaxation. Closest assignment constraints (3.23) are written separately per demand level and state that each demand level must be assigned to the closest facility of equal or higher level. Constraints (3.24) and (3.25) limit the number of new facilities to open and existing facilities to close. Constraints (3.26) limit the maximum travel distance between centers and facilities. Finally, constraints (3.27) define decision variables and enforce single assignment.

Closest assignment constraints (3.23) may be replaced by path assignment constraints, stated separately per demand level:

$$\sum_{k \in P_{ij}} x_{kjs} \geq |P_{ij}| \cdot x_{ijs}, \ \forall i \in I, j \in J, s \in S \tag{3.28}$$

Note that in constraints (3.28) the assignments of different levels are independent, while in constraints (3.23) the location of higher-level facilities influences lower-level assignments.

The formulation of model (HCM) given above allows facility co-location, i.e. the location of different types of facilities at the same site. This can be advantageous to satisfy maximum capacity constraints. If co-location is not allowed then the following constraint should be added to the model:

$$\sum_{t \in S} y_{jt} \leq 1, \ \forall j \in J \tag{3.29}$$

In addition, if co-location is not allowed, capacity constraints (3.20)-(3.22) can be replaced with the following simpler constraints:

$$\sum_{t \in S} z_{jt} = \sum_{s \in S} \sum_{i \in I} u_{is} x_{ijs} \, , \, \forall j \in J \tag{3.30}$$

$$z_{jt} \geq b_{jt} y_{jt} \, , \, \forall j \in J, t \in S \tag{3.31}$$

$$z_{jt} \leq B_{jt} y_{jt} \, , \, \forall j \in J, t \in S \tag{3.32}$$

where constraints (3.30) define variables z_{jt} as the total demand of all levels served at site j by a type-t facility, and constraints (3.31) and (3.32) impose minimum and maximum capacities. If (3.30)-(3.32) are used, constraints (3.19) are not redundant for the integer formulation and are needed to impose the nested facility hierarchy.

3.5 Case study

In this section, we present a study on the redeployment of Coimbra's primary school network. Coimbra is a municipality located in the center-littoral region of Portugal with a population of 150,000 inhabitants (Figure 3.2). The primary school network of the municipality is composed of 96 schools (Table 3.2).

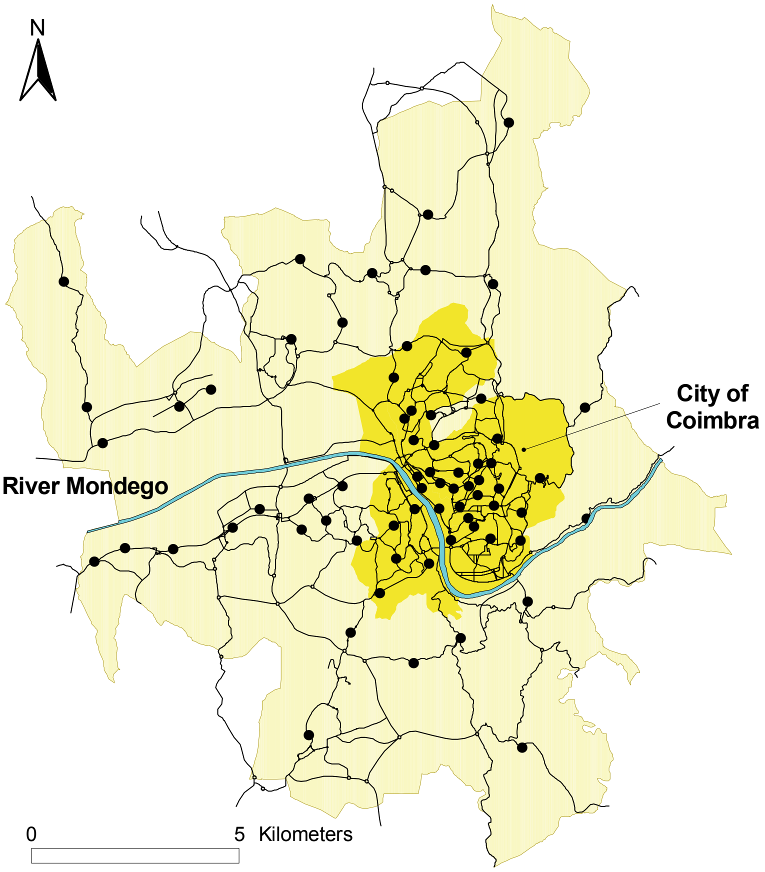


Figure 3.2: Municipality of Coimbra, demand centers and road network

Table 3.2: Coimbra's school network [10]

School Type	Number of schools	Number of classrooms	Fraction of capacity
Public EB1 (<4 classrooms)	43	90	12%
Public EB1 (4+ classrooms)	28	149	20%
Private EB1	7	49	6%
Public EB12	9	258	34%
Private EB12	9	210	28%
Total	96	756	100%

Most of these schools are public, but there is a significant number of private schools. Some of them are fully subsidized by the government because they are located in areas not covered by public schools. There are two types of primary schools: EB1, for the first cycle of education; and EB12, for the first and the second cycles. The first cycle of education comprises four years and is attended by children aged 6 to 9. The second cycle comprises two years and is attended by children aged 10 and 11. The EB1 school network consists of 78 schools with a total capacity of 288 classrooms (or 7200 students, assuming a maximum of 25 students per classroom). A large number of these schools has less than four classrooms, which means that students of different years must share the same classrooms. The EB12 network consists of 18 schools with a total capacity of 468 classrooms (or 11700 students). Following a period of fast decline in school age population, in 2004 the total number of enrollments in the primary schools of the municipality was 10659 students. Since the aggregate capacity of these schools is 18900 students, the aggregate occupation rate of the existing network was 56%.[11]

[10] Table 3.2 differs from Table 2.2 (chapter 2) in some of the data: (i) 15 public EB1 schools with only 1 classroom are excluded (in the next academic year they were to be closed or converted to kindergartens); (ii) private school capacity is measured differently: the number of classrooms was estimated by dividing occupation in 2003/04 by 20 students per room (this over-estimated capacity and was revised later in the study); for schools with both EB12 and ES12 types, each type was allocated 50% of the total number of classrooms.

[11] The EB1+EB12 capacity of 18900 differs from EB1+EB23 capacity in Table 2.2 (chapter 2) because the latter uses different private school capacities (as noted before) and includes public EB1 schools with 1 classroom. It also differs from EB1+EB12 capacity in Table 2.5 (chapter 2) because the latter uses different private school capacities and excludes all public EB1 schools with less than 4 classrooms.

The EB1+EB12 current occupation of 56% is not comparable to the EB1 and EB23 occupations higher than 80% in Table 2.3 (chapter 2) because the latter includes demand level B3 (and also uses different private school capacities).

The study was made using the hierarchical capacitated median (HCM) model introduced in the previous section. Applications of location models to school network planning are numerous and we cite only some representative, relatively recent examples. Church and Murray (1993) present a multi-objective model considering school openings and closures and capacity balancing between schools. Pizzolato and Silva (1997) and Pizzolato et al. (2004) use a *p*-median model for clustering population centers. The clusters thus found are then analyzed by confronting total existing school capacity and population. Antunes and Peeters (2001) develop a dynamic location model where schools can be opened or closed, and their capacity can be expanded or reduced over time, with the objective of minimizing total discounted costs. Related models, where location decisions are not involved, address the short-term school network management problem of assigning students to existing schools. Church and Schoepfle (1993) describe a multi-objective model considering student preferences for schools and balancing of school capacity occupation and racial mix across schools. Caro et al. (2004) present a model for school districting contemplating several desired properties of school districts, including capacity balancing and contiguity. As far as we know, no school network planning study (or other public facility planning study) reported in the literature relied on a hierarchical model with capacity and assignment constraints.

Three objectives were pursued by the education authorities. First, school capacity should be adjusted to education demand (as noted above, current aggregate occupation rate is just 56%). Second, accessibility of students to schools should be maximized. Third, changes to the existing network should be small, either because of scarce public budgets to build new schools or to avoid public reactions against school closure (particularly from parents and teachers). These objectives may of course be conflicting.

With regard to the first objective, following a decision by the education authorities, existing schools included in the study were current public schools with four classrooms or more and subsidized private schools. Other private schools, located in areas covered by public schools, which compete with public schools and are not subsidized, were left out of the study. Small public EB1 schools with less than four classrooms were also left out of the study, assuming these will be phased out in favor of larger, better equipped schools. In aggregate terms, the study included 25 EB1 schools with 4 to 10 classrooms and 14 EB12 schools with 15 to 36 classrooms, giving a total capacity of 149 and 367 classrooms, respectively.[12] As projected demand in 2015 for the two cycles of education

[12] The 25 EB1 schools with 4+ classrooms correspond to the 28 in Table 3.2, with 3 pairs of these having been aggregated into a single school in the model since they were located in the same discrete center.

is 6300 and 3150 students, the aggregate occupation rate of the resulting school network would be 73% [13].

The municipality of Coimbra was discretized in 68 population centers (Figure 3.2). Sites were assumed to be coincident with centers. For this level of aggregation, eight centers contain both EB12 and EB1 schools. Travel was assumed to be made along the main road network, which is fully served by public transportation.

Three scenarios for the redeployment of the school network were considered. In Scenario 1, the minimum occupation of EB1 and EB12 schools was set to 40 and 120 students, respectively, to meet the guidelines of the Ministry of Education for school capacity (MinEdu, 2000). In Scenario 2, the minimum occupation of schools was increased to 80 students for EB1 schools and to 75% of current maximum capacity for EB12 schools. Scenario 3 is the same as Scenario 2, but allowing a new EB12 school to be opened, with a capacity between 360 and 600 students. The purpose of Scenarios 2 and 3 was to find the best way of adjusting the existing capacity to forecast demand, while keeping schools with good occupation. The maximum student-to-school travel distance was set to 8 km in all scenarios, also to meet the guidelines of the Ministry of Education.

We first solved model (HCM) with closest assignment constraints (3.23). For this model, no feasible solutions could be found for any one of the three scenarios, as these constraints are too “rigid” given the spatial distribution of existing capacity versus forecast demand. Then model (HCM) was used with path assignment constraints (3.28) replacing (3.23). The computation of data for these constraints was carried out in two main steps (recall that the path-assignment set P_{ij} for a given center *i* and site *j* contains all centers “near” the travel path between *i* and *j*). First, buffers around each center were created with a radius of half the distance to the nearest neighbor, truncated to a maximum of 1 km, measured along the road network. Second, all centers whose buffer is intersected by the shortest path (on the road network) from *i* to *j* were added to set P_{ij}. In the example of Figure 3.3, $P_{38,8} = \{38, 36, 8\}$, which means that if center 38 is assigned to school 8, then center 36 must be assigned to the same school.

[13] The EB1+EB12 future occupation of 73% differs from similar data in Table 2.5 (chapter 2) because the latter includes non-subsidized private schools and uses different private school capacities.

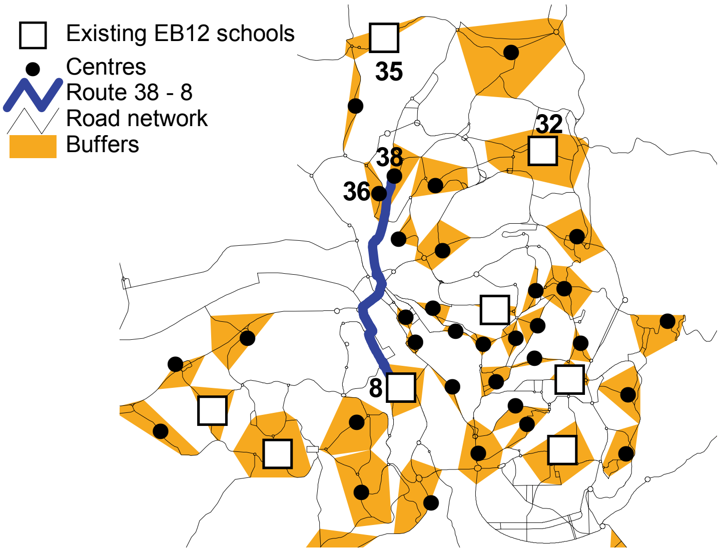


Figure 3.3: Detail of path assignment

The models were implemented with Dash Optimization's XPRESS-MP package. The modeling environment was XPRESS Mosel 1.4 (Dash, 2004) and the solver was XPRESS Optimizer version 15.30 (Dash, 2005), running under Windows XP on a computer with a Pentium-M 1.3 GHz CPU and 512 MB of memory. All model runs took less than three minutes, as in this study the number of schools to open or close is relatively small. Arcview GIS 3 (ESRI, 2000) was used in conjunction with the optimizer for data handling and result analysis. Three programs were developed in Arcview's scripting language for the following purposes: (i) computing the distance matrix between all centers using the road network; (ii) computing sets P_{ij} to be used with path assignment constraints; (iii) importing and displaying the solution output by the optimizer. The first two make use of Arcview's Network Analyst extension. In particular, the practical usefulness of path assignment constraints is closely tied to the ability to compute data with a real road network, and thus the use of a GIS is fundamental for this purpose. Church (2002) discusses the link between GIS and the development and application of location models.

The solutions obtained for the three scenarios are summarized in Table 3.3, where parameters q_1 and p_2 are the maximum number of EB1 schools to close and EB12 schools to open, respectively (for all scenarios, no new EB1 schools could be open and any number of EB12 schools could be closed). The existing school network is sufficient for the future needs of the municipality (Scenario 1), with around 30% of slack capacity. Three EB1 schools do not meet the minimum occupation requirements and are closed. A seemingly awkward assignment occurs for centers 36 and 38 (shown in Figure 3.4, expanded in detail in Figure 3.3), which are sent south to center 8. However, the existing schools at centers 35 and 32 do not have enough capacity, and center 8 is less than one kilometer further away than those schools. If the minimum occupation is augmented (Scenario 2), average occupation increases but total travel distance is degraded by 15%, mainly because of the closure of one EB12 school in the south-east of the municipality (Figure 3.5). In this scenario, 12 EB1 schools are closed. If a new school is allowed (Scenario 3), high average occupations are still guaranteed and travel distance is only 5% higher than in Scenario 1 (Figure 3.6). The main beneficiaries are centers 36 and 38, located in a fast growth area currently not covered satisfactorily.

Table 3.3: Summary of solutions for the three scenarios [14]

Scenario	Model parameters		Total travel distance		Aggregate school occupation			Number of schools					
								Total		New		Closed	
	q_1	p_2	km	Relative	EB1	EB12	Total	EB1	EB12	EB1	EB12	EB1	EB12
1	3	0	12527	100%	63%	79%	74%	22	14	0	0	3	0
2	Unlim.	0	14391	115%	78%	90%	88%	13	13	0	0	12	1
3	Unlim.	1	13183	105%	75%	87%	85%	12	14	0	1	13	1

[14] The model may have alternative optimal solutions with distinct assignments to co-located EB1 and EB12 schools. In scenarios 2 and 3, model solutions were post-processed to give priority to the occupation of EB12 schools: demand was transferred from EB1 schools to co-located EB12 schools while capacity allowed; EB1 schools were closed if all their demand was transferred.

We also note that, in Figure 3.4 and others below, when level-1 assignments are not visible, they coincide with superimposed level-2 assignments.

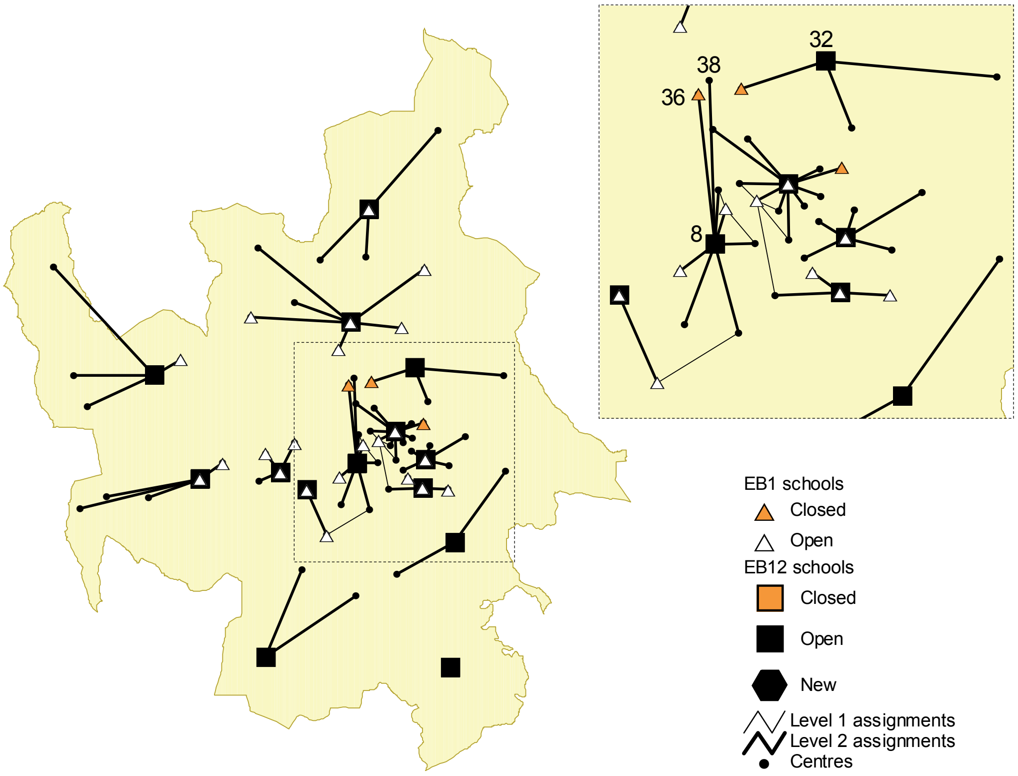


Figure 3.4: Solution for Scenario 1

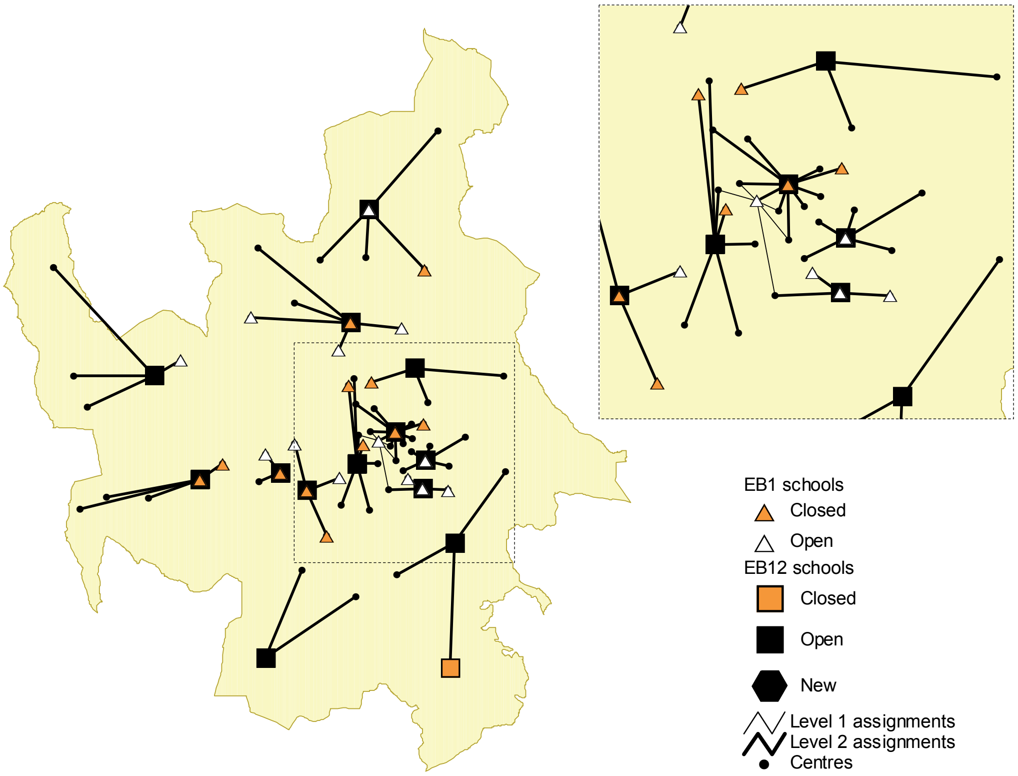


Figure 3.5: Solution for Scenario 2

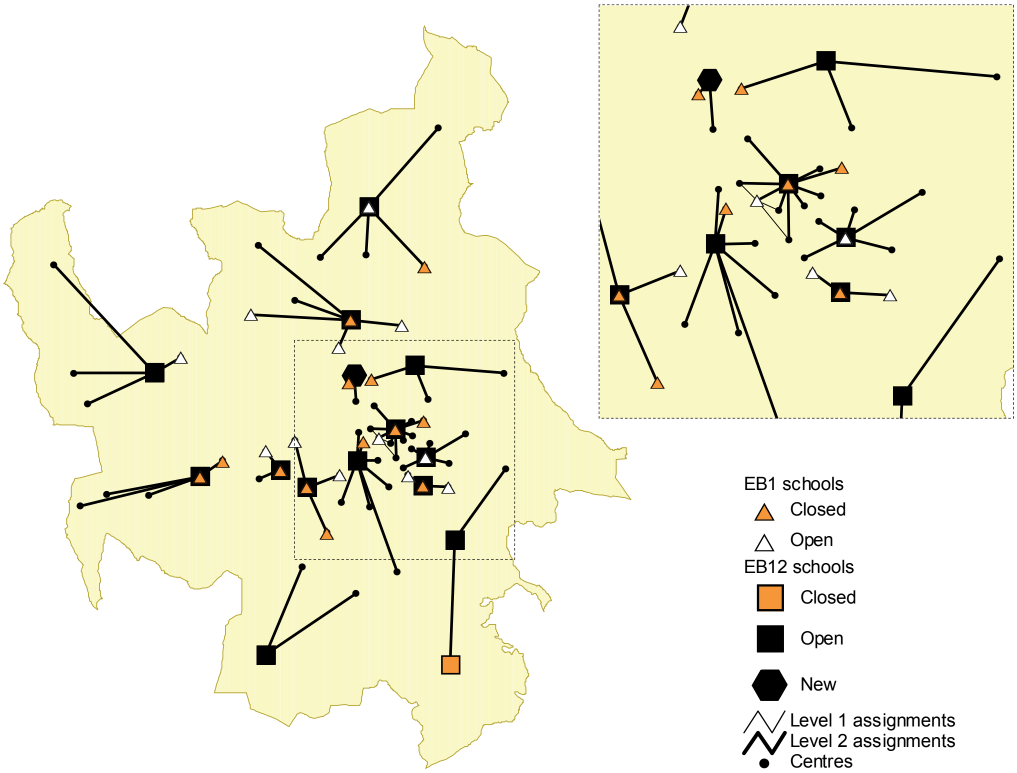


Figure 3.6: Solution for Scenario 3

3.6 Conclusion

In this chapter, we presented a discrete hierarchical location model for public facility planning. The main features of the model are: an accessibility-maximization objective; several levels of demand; several types (or levels) of facilities; a nested hierarchy of facilities (i.e., a facility of a given level can serve demand of equal and lower levels); maximum and minimum capacity (or occupation) constraints; and user-to-facility assignment constraints. The latter include single-assignment and closest-assignment constraints, as well as a new type of constraints called path-assignment constraints. They are used to enforce some desirable properties for the spatial pattern of assignments. The resulting solutions are easier to interpret and to explain in a public facility planning context, therefore being more likely to be accepted by the users. As far as the authors know, a model with this set of features has never been dealt with in the literature.

The hierarchical location model was developed to help in a real-world school network planning process conducted for the municipality of Coimbra, Portugal. At present, most public planning processes include complex participatory mechanisms. These mechanisms started to be introduced in the late 80s, as a reaction to the alleged failure of rational, model-based approaches to public planning (Chadwick, 1978). According to many planning theorists, a new type of approach was needed focusing on participation and debate rather than on rationality and modeling (Healey, 1992). This new type of approach was progressively adopted and now underlies many real-world planning processes. What we clearly realized from our involvement in the preparation of the Coimbra Educational Charter is that debate and modeling, instead of being substitutes, are close complements. Indeed, in the presence of stakeholders with different viewpoints and interests, like those represented in the education council of the municipality, objectives can be debated and agreed upon, but solutions that are not rational for the objectives retained are too fragile to prevail. When the number of possible solutions is very large, the only way of finding the rational solutions for a planning problem involves the application of optimization models. Without the hierarchical location model we developed, it would have been extremely difficult to arrive at planning solutions widely accepted by the Education Council.

The hierarchical location model was solved with a modern commercial optimizer rather easily. The use of a GIS package in conjunction with the optimization program was extremely valuable. In addition to simplify the analysis of results, the GIS was used for computing data for path assignment constraints, using features such as finding shortest paths and create buffers, or areas of influence, around centers with distances measured on the road network. The model was easy to solve because Coimbra, though being one of the largest municipalities in Portugal outside the metropolitan areas of Lisbon and Porto, still is a small-size municipality by European standards, and also because small changes in the number of schools were allowed. If any one of these conditions were not met, the optimization program would have been unable to do the job. Our future research will be directed towards the development of specialized solution procedures for solving large-scale hierarchical and (non-hierarchical) capacitated models with user-to-facility assignment constraints still out of reach of current modern commercial optimizers.

3.7 Appendix – Path assignment constraints

Introduction

In this appendix, we provide a better formulation of path assignment constraints, and cite references to previous work using such constraints that came to our attention only after publishing a journal article based on this chapter.

Alternative formulation of path assignment constraints

In this chapter, path assignment constraints were formulated as follows:

$$\sum_{k \in P_{ij}} x_{kj} \geq |P_{ij}| \cdot x_{ij}, \ \forall i \in I, j \in J \tag{3.15}$$

This expression states that if $x_{ij} = 1$ then $x_{kj} = 1$ for all $k \in P_{ij}$, where P_{ij} is the subset of centers $k \in I$ that are "near" the shortest path from *i* to *j* (including *i* and also *j* if $I = J$). An alternative formulation is as follows:

$$x_{ij} \leq x_{kj}, \ \forall i \in I, j \in J, k \in P_{ij} \tag{3.15b}$$

Formulation (3.15b) follows directly from the definition above and it is stronger than (3.15), which is obtained by summing (3.15b) over all *k*. Formulation (3.15b) was not used in the chapter since it apparently requires $O(n^3)$ constraints instead of $O(n^2)$ as (3.15). However, on closer analysis, (3.15b) may not actually increase formulation size, as discussed next.

In expression (3.15b), sets P_{ij} may be replaced by P'_{ij} defined as follows, in order to keep only non-dominated constraints:

$$P'_{ij} = P_{ij} \setminus \{i\} \setminus \{k \in P_{ij} : \exists s \in P_{ij} : P_{kj} \subset P_{sj} \subset P_{ij} \vee (P_{kj} \subset P_{sj} = P_{ij} \wedge s > i)\}$$

Example 1: If $I = J$ and sets P_{ij} are defined by shortest paths on a network, P'_{ij} contains a single node for $i \neq j$, which is the successor of node *i* on the path between *i* and *j*, and is empty for $i = j$. For each $j \in J$, formulation (3.15b) using sets P'_{ij} requires *n*-1 constraints, the same as (3.15) (the constraint for $i = j$ is redundant). Additionally, (3.15b) involves fewer non-zero elements.

Example 2: Consider $I = \{1,2,3,4\}$ and some $j \in J$ such that

$P_{1j} = P_{2j} = \{1,2,3,4\}$, $P_{3j} = \{3,4\}$, $P_{4j} = \{4\}$, and

$P'_{1j} = \{2\}$, $P'_{2j} = \{1,3\}$, $P'_{3j} = \{4\}$, $P'_{4j} = \varnothing$.

In this example, (3.15b) using sets P'_{ij} requires 4 constraints, while (3.15) requires 3 (the one for P_{4j} is redundant).

Additional simplifications would be possible with formulation (3.15b). If $P_{ij} = P_{kj}$ for some $i, k \in I$ and $j \in J$, then it results $x_{ij} = x_{kj}$; one of the variables is redundant and can be replaced and removed from the model (this occurs in example 2). When inequality $x_{ij} \leq x_{kj}$ is added, the variable upper bound constraint $x_{ij} \leq y_j$ becomes redundant and can be removed. Although we do not offer here a preprocessing procedure to produce a minimal formulation given the sets P_{ij}, we observe that even if (3.15b) is used with sets P_{ij} instead of P'_{ij}, all the simplifications above will likely be automatically performed by the presolve routines of a generic MIP optimizer.

To summarize, we expect that a location model formulated with constraints (3.15b) instead of (3.15) is likely to be solved faster with a generic MIP optimizer, since the formulation is stronger and its size may not increase significantly, or may even decrease. This is likely to apply even using sets P_{ij} instead of P'_{ij}, due to the automatic simplifications performed by presolve routines.

Previous work using path assignment constraints

It is stated in this chapter that path assignment constraints are a new type of assignment constraints, since we were unaware of previous work using them (although we cite related constraints to enforce contiguity in districting models). However, here we cite references to previous work using exactly the same constraints that came to our attention only after publishing a journal article based on this chapter.

Shulman and Vachani (1993) and Balakrishnan et al. (1995) study network design models for local-access telecommunication networks (copper and fiber-optic). These models consider a set of demand nodes, a tree network connecting nodes to a switching center, and binary decision variables for locating concentrator on nodes, assigning nodes to concentrators and expanding capacity on arcs.

Both models include contiguity requirements, stated as follows by Balakrishnan et al. (1995): "if a concentrator at node *j* serves node *i*, then it also serves all other nodes (including node *j*) on the path P_{ij} connecting nodes *i* and *j*". The path P_{ij} is unique in these models, since a tree network is assumed. Nevertheless, given a set of elements P_{ij}, this definition is equivalent to the definition of path assignment constraints used in this chapter.

Using the notation of this chapter, contiguity constraints were formulated by Shulman and Vachani (1993) as:

$$x_{ij} \leq x_{kj},\ \forall i, j, k \in I : k \in P_{ij}$$

and by Balakrishnan et al. (1995) as:

$$x_{ij} \leq x_{k_{ij}j},\ \forall i, j \in I$$

where k_{ij} is the node adjacent to node i on path P'_{ij}.

The first formulation is the same as (3.15b) given above, while the second formulation corresponds to the simplification using the sets P'_{ij} defined above when these sets contain at most one element. Thus, "path assignment constraints" as defined in this chapter had been proposed before by other authors. However, as far as we know, they had not been used before in a public facility location model with the paths P_{ij} being defined as in this chapter (and computed with a GIS) for the purposes discussed in section 3 of the chapter.

Chapter 4

Application of hierarchical models to the districting and location of courts of justice

4.1 Introduction

Throughout the years, location modeling techniques have been applied to an extremely wide variety of public facilities (see, for example, Current et al., 2002). However, their application to one of the most ubiquitous public facilities – courts of justice – is very rare (and, to the best of our knowledge, has never been described before in a refereed journal). A possible explanation for this situation is the principle of separation of powers, which leads governments to avoid interferences with judicial systems (in democratic countries). However, in Portugal, the judicial system has been going through a severe crisis and, under the pressure of the public opinion, the government was forced to take action, launching a vast judicial reform. One of the main constituents of this reform is a new judiciary map – that is, a new spatial organization for the judicial system. Indeed, the previously existing map was the result of an evolution with deep roots in the 19th century. It was based on very small territorial jurisdictions, called *comarcas*, which did not favor the specialization of justice. Moreover, in many cases, the *comarcas* were not consistent with the jurisdictions of institutions such as local administration, police, social security, and tax collection, thus contributing to efficiency losses in the judicial system.

In this chapter, we describe a study made in the University of Coimbra under contract with the Ministry of Justice of Portugal to define a proposal for the new judiciary map addressing the following main goals: (1) replace *comarcas* with larger districts, making judicial jurisdictions compatible with those of related institutions, and enabling more effective management of human and material resources; (2) promote the specialization of the judicial system (through the increase of the number of courts dedicated to specific litigation types); (3) promote a better balance of supply and demand (judicial litigation); (4) guarantee a good level of accessibility to courts. For tackling the problems involved in the accomplishment of these objectives, we developed two optimization models – a districting model, to determine the borders of new, large judicial districts; and a location model, to determine the location, type, size (measured in number of judges), and coverage area of the courts included in each new district. The study was carried out in close cooperation with a task force set up within the Ministry of Justice to accompany its preparation.

The chapter is organized as follows. We start by describing the context and objectives of the study, and by presenting the work plan developed to meet these objectives. Next, we explain how we have determined the reference values used within the study of judicial litigation and judicial productivity (that is, the number of cases entering and leaving the courts of justice, respectively). Then, we present the optimization models developed in the study and describe the results obtained through their application. In the conclusion, we describe events of the judiciary map reform occurred since the publication of the study, and summarize the contributions of this chapter.

4.2 Study context

The judicial system of Portugal is seen by many people as one of the country's greatest problems. This is the reason why justice was the object of the first and only pact ever made between the two parties that have alternated in government since the Carnation Revolution in 1974 – the center-left Socialist Party and the center-right Social Democratic Party. The pact was signed in September 2006 and involved several issues, including the reform of the judiciary map as one of the most important.

With respect to this reform, the provisions contained in the pact were the corollary of a process that started in 2002 when the Government commissioned the *Observatório Permanente da Justiça Portuguesa* (OPJP) – a research organization specialized in judicial affairs – with a study on the spatial organization of the Portuguese judicial system (OPJP, 2002). After that, many prominent personalities and institutions within the justice system emitted their opinions on the subject. These opinions were summarized in a new study of the OPJP released in 2006, where the main problems faced by the judicial system were plainly identified and the principles to be followed in the reform of the judiciary map were clearly established (OPJP, 2006). During this process, many pages of reflections were written. However, six months before the deadline established in the pact of justice there was still no map: the document specifying the type, location, size, and coverage area of the courts of justice of Portugal.

One problem identified by the OPJP as being related with the spatial organization of the justice system was the lack of specialization of courts. The basic courts in Portugal are the *comarca* courts, which are generic courts handling all types of cases except the ones for which there is a specialized court with jurisdiction in the area of the *comarca*. Before the ongoing reform there were 213 *comarcas* – and as many *comarca* courts (Figure 4.1). Some of the *comarca* courts (typically the largest ones) had separate civil and criminal sections, which intervened depending on the nature of the law violations to process. The specialized courts consisted of labor courts, family and juvenile courts, civil enforcement courts (dealing essentially with minor debt collection), and commerce courts. The area of jurisdiction of labor courts covered almost all the country, therefore *comarca* courts rarely had to take care of cases involving labor law. The situation was quite different with respect to the other specialized courts, whose area of jurisdiction is relatively small. This means that, in large parts of the country, judges in *comarca* courts had to deal with a wider variety of cases, and particularly with cases involving the complex family and commerce laws, with very negative implications for their productivity.

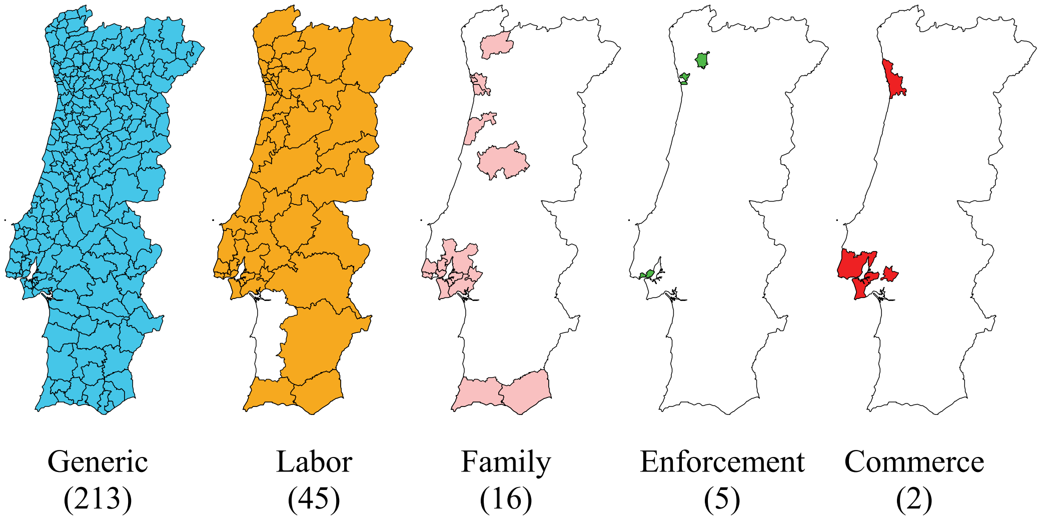


Figure 4.1: Number and territorial jurisdiction of generic and specialized courts (mainland Portugal)

A second problem pointed out by the OPJP was the lack of capacity of the courts for dealing with the increasing volume of litigation observed in the most developed areas of Portugal. Indeed, in some *comarcas* of the Littoral (the 50km wide stretch of land located between the metropolitan areas of Lisbon and Porto), the number of cases entered in courts was above 1,200 per judge in 2005 (the latest year for which this information was available when the study was made), when the expected productivity of a judge is 800 cases per year. The main implication of this has been a steady increase of the backlog of cases. In contrast, in a large number of *comarcas* of the Interior (the rest of the country with the exception of the southern province of The Algarve and the archipelagos of Madeira and Azores), the number of cases entered each year was below 800, and even below 500 (Figure 4.2). Since all *comarcas* had at least one judge, it is obvious that there was an excess of capacity in a large part of the country in parallel with the lack of capacity experienced in the Littoral.

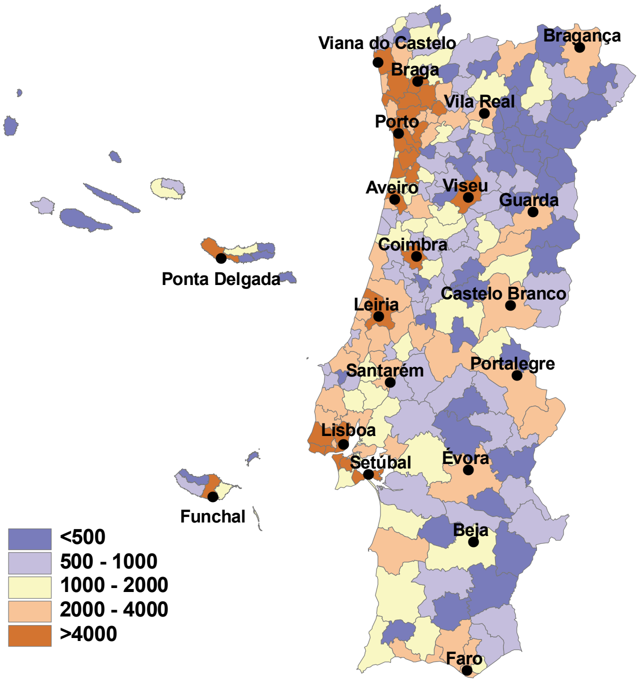


Figure 4.2: Judicial demand – Number of cases entered into generic courts in 2005

A third problem highlighted by the OPJP was the lack of coincidence between the territorial jurisdiction of courts and those of closely related services, such as local administration, police, social security, and tax collection. This problem was not felt in the northern part of the country but was severe in the southern part, with some *comarcas* being spread across as much as five municipalities, the geographic units in terms of which most other services are organized.

All these problems were addressed by the Pact of Justice. The main principles of the pact related with the reform of the judiciary map were as follows:

- The new jurisdictions will be organized in terms of the NUTS 3 regions (Figure 4.3), by promoting the aggregation of the existing *comarcas*, while trying to avoid splitting the territorial units within the existing *comarcas*.
- The new jurisdictions will be the geographic setting for the creation of new specialized courts, when justified, with a special emphasis on enforcement

courts (the specific reference to these courts reflects the belief that courts of this type, which had been introduced recently and were still rare in the country, would greatly help at fighting the slowness of justice).

- The new jurisdictions will be the setting for the joint management of human resources (judges, prosecutors, and other staff) and material resources. Each jurisdiction will be managed by a judge president and will have a support office providing technical legal assistance to judges within all courts.

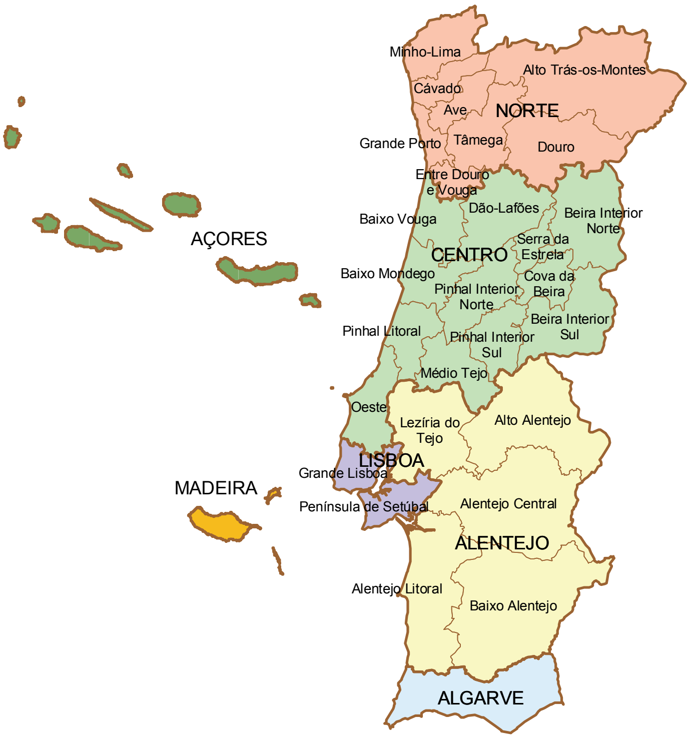


Figure 4.3: NUTS 2 and NUTS 3 regions of Portugal

The problems mentioned above refer to first instance courts. The judicial system of Portugal is a three-level system. No similar problems were identified with respect to higher-level courts (the five judicial courts of second instance, with jurisdiction over NUTS 2 regions, and the Supreme Court of Justice, with jurisdiction over the whole country), and they were left out of the study. Also out of the study were left the maritime court and the intellectual property court, because, despite being first instance courts, their jurisdiction covers the whole country.

4.3 Work plan

According to the contract signed with the Ministry of Justice, the development of the study would involve the following tasks:

1. Clarification of the problem(s) to be solved through the reform of the judiciary map.
2. Establishment of reference values for judicial litigation (civil, criminal, labor, etc.) in the reference year of 2015.
3. Establishment of reference values for judicial productivity.
4. Formulation of the optimization model(s) representing the problem(s) to solve.
5. Establishment of the proposal of the judiciary map – one and only one proposal – based on the solution(s) of the model(s).
6. Analysis of the sensitivity of the proposal to changes in problem definition.
7. Definition of implementation stages for the judiciary map.

The tasks defined are similar to those encountered in many public facility planning studies: the decision problem to address is properly stated; the demand for service is estimated; a proposal for the location and size of the facilities that meets the demand is established based on the solution obtained through an optimization model; a sensitivity analysis is performed, to better assess the proposal; finally, implementation stages of the proposal are defined, possibly starting with an experimental stage with limited regional scale.

The only aspect that we believe deserves a special mention is the fact that there should one and only one proposal. Our initial idea was that there could be several alternatives among which to choose. However, the Minister of Justice completely discarded this possibility saying that: "I do not want any alternatives; we really need to make this reform. If we propose alternatives, after some time no one really knows what exactly is being discussed."

4.4 Problem statement

The aim of the judiciary map proposal is to define the location, type, size (measured in number of judges), and coverage area (territorial jurisdiction) of first instance judicial courts in the whole territory of Portugal (mainland and archipelagos of Madeira and Azores).

The spatial unit adopted both for demand aggregation and for court location was the municipality, the basic local administration unit in Portugal, of which there are 278 in the mainland plus 30 in the archipelagos. The temporal unit of demand and court capacity was defined to be one year. The planning horizon was chosen to be the year 2015, which is sufficiently close for judicial litigation to be forecast with acceptable reliability, but also sufficiently distant for important planned road network expansions to be completed, according to the National Road Plan 2000, having direct influence on court accessibility.

The territory is to be partitioned into large districts, each aggregating the jurisdictions of several existing courts. Within each district, a municipality is selected to become the seat of the district, based on accessibility and current hierarchical level considerations (detailed below). The seat's court, or main court, is the headquarters of the judge president.

Courts were classified into the following types for the purposes of the study: generic and specialized of four types – family, labor, enforcement, commerce. Generic courts serve both civil and criminal cases, as well as other types of cases if a corresponding specialized court does not exist in the district. Specialized courts serve only cases of the corresponding type. A fundamental feature of specialization is that if a district has one or more specialized courts of a given type, then all demand of that type is served at those courts and not at generic courts.

The following goals were specified, in accordance with the principles defined in the Pact of Justice: (1) adopt NUTS 3 regions (28 in the mainland plus 2 in the archipelagos) as the reference for the new districts, keeping municipalities included in the same old *comarcas* together in the new districts; (2) create specialized courts in districts where demand is large enough; (3) promote a better balance of demand and supply, taking into account given maximum and minimum values of demand per court; (4) guarantee a good level of accessibility to courts, taking into account a given maximum travel time to courts.

These goals, together with additional rules for court location and assignment of municipalities to courts, were translated into a decision problem with the following decisions, objectives, and constraints.

Decisions:

- districting decisions: determine the new districts (municipality-based partitions of NUTS 3 regions) and select the seat of each district (municipality where the main court is located);

- location decisions: determine the location, size, and coverage area for generic and specialized courts consistent with districting decisions;

Objectives:

1. minimize the number of new districts in each NUTS 3 region;
2. maximize the number of types of specialized courts in the new districts;
3. minimize aggregate travel time to main court weighted by reference litigation;
4. minimize aggregate travel time to generic and specialized courts weighted by reference litigation;

General constraints:

- satisfy all demand (no backlog of cases);
- maximum travel time to the main court: 60 minutes;
- maximum total number of judges per district: 75, of average productivity;
- minimum number of judges in specialized courts: 1, with 80% of the reference productivity;
- minimum number of judges in generic courts: 1, with 50% of the reference productivity;

Constraints on the location of courts:

- main court: is located at a municipality with highest judicial and administrative hierarchical level, considering the following increasing levels – 0, no existing court; 1, existing generic court; 2, seat of old judicial circuit (group of old *comarcas*); 3, seat of administrative district (group of municipalities); 4, existing court of second instance;
- generic courts: can exist at any municipality with an existing generic court, that is, no new courts are allowed (however, this does not preclude capacity expansion of existing courts);
- specialized courts: can exist only in municipalities where a generic court is also located;
- enforcement and commerce courts: at most one court of each type can exist per district, located at the seat of the district;
- family and labor courts: one or more courts of each type can exist per district, located at the seat of the district or at a municipality with an existing specialized court of either type;

Constraints on the assignment of municipalities to courts:

- municipalities must be assigned to a single court of each type (i.e. the demand may not be split among courts located in different municipalities);
- municipalities must be assigned to the closest court of each type (in terms of travel time), with the possible exception of municipalities not having a generic court, for which the following rules prevail;
- municipalities assigned to the same generic court in an old *comarca* must be assigned to the same generic court in the future (this rule avoids disaggregating the old *comarcas*);
- municipalities assigned to the same generic court must also be assigned to the same specialized courts (this rule establishes coherency between generic and specialized assignments; coherency is automatically guaranteed for enforcement and commerce courts, since at most one of each type exists in a district);
- a municipality cannot be assigned to family and labor courts located in different municipalities (this implies family and labor courts must be co-located if both types exist in a district).

The four objectives above are ordered by priority, that is, each objective was assumed to be much more important than the next. This allowed the problem to be decomposed into two sequential sub-problems:

- a districting sub-problem, considering only the districting decisions and the first three objectives;
- a location sub-problem, considering only the location decisions and the fourth objective.

This decomposition into sub-problems is further discussed in section 4.7.3, after presenting model formulations.

Finally, we remark that location decisions considered that at most one generic court exists per municipality, and that specialized courts of each type are co-located with generic courts. In reality, depending on jurisdiction size and existing infrastructure, generic courts may either be a single unit or be subdivided into specialized civil and criminal sections, located in the same or distinct buildings; and specialized courts may be located in a single building together with a generic court or in distinct buildings. However, the subdivision of generic courts and the precise location of buildings where courts should be installed within a municipality were left outside the scope of the study.

4.5 Judicial litigation

Reference values for judicial litigation (measured in number of cases per year) were forecast to 2015 using the following method: (1) Stepwise regression of litigation rate against demographic, employment, and educational variables (using 2001 data); (2) Selection of the best regression equation; (3) Computation of forecasts for the litigation rate (from a forecast of the independent variables); (4) Application of litigation rate forecasts to population forecasts.

Regression models

The following regression models were tested:

$$L_n = \alpha + \beta_1 P_{A1} + \beta_2 P_{A2} + \beta_3 P_{A3}$$

$$L_n = \alpha + \beta_1 P_{E1} + \beta_2 P_{E2} + \beta_3 P_{E3}$$

$$L_n = \alpha + \beta_1 P_{H1} + \beta_2 P_{H2} + \beta_3 P_{H3}$$

L_n: litigation rate of type n (cases per thousand inhabitants); P_{A1}, P_{A2}, P_{A3}: difference from the national average of the percentage of active resident population (employed or unemployed) in the primary, secondary and tertiary sectors, respectively; P_{E1}, P_{E2}, P_{E3}: difference from the national average of the percentage of persons employed in the primary, secondary and tertiary sectors, respectively; P_{H1}, P_{H2}, P_{H3}: difference from the national average of the percentage of resident population per education level: first level of primary education or less, intermediate level, secondary education or higher, respectively.

The best regression models, obtained through forward stepwise regression (see e.g. Draper and Smith, 1998) considering parameters as statistically significant when absolute *t* values were higher than 2, were as follows:

$$L_{civil} = 30.52 - 0.82 \times P_{H1} \quad \left(R^2 = 0.249\right)$$

$$L_{criminal} = 11.25 - 0.18 \times P_{H1} + 0.19 \times P_{H3} \quad \left(R^2 = 0.291\right)$$

$$L_{family} = 3.54 + 0.17 \times P_{H3} \quad \left(R^2 = 0.216\right)$$

$$L_{labor} = 6.08 + 0.28 \times P_{A2} + 0.22 \times P_{A3} \quad \left(R^2 = 0.286\right)$$

We observe that the explanatory power of these models is low (R^2<0.30). Some attempts were made to improve them, such as an analysis of the spatial pattern of litigation rates (distinguishing north vs. south, littoral vs. interior, low vs. high administrative hierarchical level), but no patterns could be observed and thus no spatial dummy variables were included in the models. As we had anticipated, litigation is a complex phenomenon that would be difficult to model. In spite of this, the regression coefficients have the expected signs: litigation is higher for higher education levels (this reflects increased access to justice, since higher educated citizens tend to be better informed and have higher incomes, and also increased litigation in more complex economic activities, which tend to require higher educated workers); litigation is higher for higher weights of the secondary and tertiary sectors (this reflects increased litigation in more complex economic activities). Thus we decided to retain the regression models above for litigation forecasts, though aware that the forecast would not be very accurate.

Population forecasts

The reference value of population in 2015 was obtained with the following method. First, the population forecasts of the National Statistics Institute for NUTS 2 regions (INE, 2004) were used. Second, regional population growth was distributed by its constituent municipalities according to three scenarios: (1) proportionally to the population growth in 1991-2001 (the two last census years); (2) proportionally to the natural growth rate (births-deaths) in 2004 (the last year of available data); (3) proportionally to the population in 2001 (the last census year). The three scenarios favor municipalities with different demographics and dynamics: respectively, high recent growth, young population, large municipalities with recent population decrease. Finally, the population in each municipality was taken to be the maximum of all scenarios. This method produces an over-estimation of total population. However, even with this optimistic forecast a generalized decrease in the number of generic courts could be expected, provoking a negative reaction in the affected municipalities. The method chosen was expected to reduce claims that particular municipalities were treated unfairly due to the choice of a particular forecast method.

Regarding the projection of independent variables, the percentage of active population per economic sector was assumed to vary proportionally to the variation in 1991-2001 (a general increase in the tertiary sector was observed, accompanied by a general decrease in the primary sector and, in some cases, also in the secondary sector). The percentage of population per education level was also assumed to vary proportionally to the variation in 1991-2001 (a general decrease in the lower level and an increase in the higher levels was observed). According to the regression models adopted, these trends will produce an increase in litigation.

Litigation forecasts

In spite of the simplicity of the litigation forecast method, the results were judged to be credible when data for particular municipalities were analyzed for the years 2001 (the base year for the forecast), 2005 (the last year of available data, used for control) and 2015 (the forecast year).

A summary of the litigation forecasts is presented in Table 4.1, representing in 2015 a total of about 770,000 cases of the following types: civil 59% (including enforcement and commerce); criminal 24%; family 9%; labor 8%.

The significant increase in criminal and family litigation can be attributed to expected increases in education levels (as well as to the optimistic population forecast), while the decrease in labor litigation can be attributed to the significant decrease of weight of the secondary sector expected in a large proportion of municipalities.

Table 4.1: Results of litigation forecasts – number of cases per municipality

	Type of case				
	Civil	Criminal	Family	Labor	Total
Total 2001	414 838	104 768	42 875	67 316	629 797
Total 2015	453 963	185 151	70 309	62 582	772 005
Var. 2001-2015	9%	77%	64%	-7%	23%
Minimum 2015	22	7	5	5	39
Median 2015	594	227	75	85	995
Average 2015	1 474	601	228	203	2 507
Maximum 2015	48 075	21 262	8 807	5 852	83 996

Civil cases are divided into *declarative*, *commerce* and *enforcement* cases. The proportions of enforcement and commerce cases in 2005 were 60.0% and 1.4%, respectively, and were assumed to remain unchanged in the future and to apply to all municipalities. To account for recently introduced legislation that will tend to decrease litigation in courts (e.g. decriminalization of some infractions; possibility of extra-judicial settlements), litigation rates for all case types were multiplied by a factor of 0.9.

4.6 Judicial productivity

Reference values for judicial productivity (number of finished cases per judge in one year, applying to any judge) adopted in the study for 2015 were as follows:

- Global productivity (average for all courts in a district, combining all court types and case types): 800 cases;

- Generic court: 800 cases if the court receives enforcement cases; 550 cases otherwise;
- Enforcement court: 2750 cases;
- Family court: 800 cases;
- Labor court: 950 cases;
- Commerce court: 400 cases.

For specialized courts, the values above are approximately the national averages observed in 2005, and are within the reference productivity intervals used by the Supreme Judicial Council in the assessment of judges (according to information provided by the Ministry of Justice). For enforcement courts the middle point of this interval was adopted, rather than the observed average, since some of these courts had been installed only recently and the applicable law had been subject to recent changes.

For generic courts, the productivity of 800 is approximately the national average observed in 2005 (the observed global productivity is similar and the reference value was assumed to be equal). A lower productivity is considered if the generic court does not receive enforcement cases, which are generally simpler to judge, but no other discrimination is made regarding the mix and proportions of case types.

The latter option resulted from an analysis of productivity of judges in generic courts carried out within the study, using the following regression model:

$$F = \alpha + \beta_1 J_1 + \beta_2 J_2 + \beta_3 J_3 + \beta_4 T_{civil_d} + \beta_5 T_{civil_e} + \beta_6 T_{criminal} + \beta_7 T_{family} + \beta_8 T_{labor}$$

> F: number of finished cases per judge; J_1, J_2, J_3: difference from the national average of the percentage of judges with less than 3 years, between 3 and 6 years, and more than 6 years of experience, respectively; T_n: difference from the national average of the percentage of cases of type n (civil cases were divided into declarative, $civil_d$, and enforcement, $civil_e$, and commerce cases were excluded from the analysis).

The best regression model, obtained by forward stepwise regression and using 2005 data only from generic courts under demand pressure (where the number of new cases is not smaller than the number of finished cases), was the following:

$$F = 819.63 - 2.83 \times J_1 + 6.61 \times T_{civil_e} \quad \left(R^2 = 0.290\right)$$

As before, the explanatory power is low. Indeed, judicial productivity was expected to be difficult to capture in a model. Nevertheless, we considered the model to be useful, since the regression coefficients have the expected signs: productivity is lower in courts with higher proportion of younger judges and higher in courts with higher proportion of enforcement cases. According to this model, a court not receiving enforcement cases has a productivity of 543 cases per judge, while one in which all civil cases are enforcement cases has a productivity of 1205 (smaller than the 2750 assumed above for a specialized enforcement court). Coefficients of variables related to other case types were not retained in the model, i.e. other case types had no statistically significant effect on productivity.

4.7 Optimization models

4.7.1 Literature overview

We now briefly overview the relevant literature before introducing the formulations of the two optimization models developed in the study. The models presented here are discrete facility location models, which assume a discrete set of centers where demand is concentrated and a discrete set of sites where facilities can be located. ReVelle and Eiselt (2005) and Current et al. (2002) provide general reviews, and ReVelle (1987) and Marianov and Serra (2002) provide reviews focusing on public sector applications.

Basic models

The models we developed are extensions of a basic model that aims to locate facilities and assign demand centers to those facilities so that the total demand-weighted assignment distance is minimized, each center is fully assigned to the closest facility (or one of the closest, if several are equidistant), and the demand served by each facility satisfies given minimum and maximum capacities.

This model is called the *capacitated median model* (CM) by Teixeira and Antunes (2008), due to its relationship with the classic *p-median model* (PM), in which facility capacities are not imposed and the number of facilities is a parameter (*p*). In the PM model, solutions naturally have the so-called single and closest assignment properties, i.e. each center is fully assigned to the closest facility. In the CM model, facility capacities are imposed: a minimum capacity, representing a threshold to guarantee economic feasibility of individual facilities or to guarantee increased productivity through specialization of labor; and a maximum capacity, representing limited space availability or a threshold to avoid diseconomies of scale. The minimum and maximum capacities have two consequences: i) they imply, respectively, upper and lower bounds on the number of facilities, which becomes a model output rather than a parameter; ii) the single and closest assignment properties, if required, must be imposed through explicit constraints. The single and closest assignment properties are desirable in a public facility planning context where users are assumed to distinguish locations through their accessibility, by preferring the closest facility. Hanjoul and Peeters (1987) further discuss the representation of user preferences in a location model. Teixeira and Antunes (2008) compare spatial patterns of user-to-facility assignments between the PM and CM models.

The CM model, unlike the PM model, has rarely been dealt with in the literature. Carreras and Serra (1999) use it in a pharmacy location problem, and solve it through a tabu search heuristic. Kalcsics et al. (2002) use it together with a constraint on the number of facilities for designing balanced and compact sales territories, and solve it with a variable neighborhood search heuristic. Teixeira et al. (2007) apply it to the reorganization of a secondary school network, and solve it with a commercial optimizer. Bigotte and Antunes (2007) present several heuristics to solve the CM model. Related models have been proposed that do not require all demand centers to be served, but otherwise combine minimum capacity and closest assignment constraints. Verter and Lapierre (2002) present a model for locating preventive health care facilities with the objective of maximizing population coverage, and solve it with a commercial optimizer. Smith et al. (2009) present a model for locating primary health care facilities with the objective of maximizing the number of open facilities, and solve it with a commercial optimizer.

Hierarchical models

Teixeira and Antunes (2008) present a hierarchical version of the capacitated median model, considering demand for multiple services and multiple facility types, both classified into *n* levels, where facilities have a nested service hierarchy: a level *n* facility can serve level *n* demand and all lower levels. An application to the reorganization of a primary school network with two levels is presented, in which the model is solved with a commercial optimizer.

The two models presented here are also multiple-service hierarchical extensions of the capacitated median model. Both models consider two levels of facilities, generic (level 1) and specialized (level 2), organized into a nested hierarchy: level-2 facilities can only be installed at locations where level-1 facilities also exist. Hierarchical location models, dealing with the location of multiple facility types, are reviewed by Narula (1986), Church and Eaton (1987), and Sahin and Sural (2007).

Relatively to hierarchical models in the literature, the court location model presented here (the second model below) is the first to include so-called coherent assignment constraints together with capacity constraints and closest assignment constraints. Assignments are said to be coherent if all centers assigned to a given level-1 facility are assigned to the same level-2 facility. Explicit constraints enforcing coherency were first introduced by Serra and ReVelle (1993) for a model with two facility types and two service levels. Other models with the property of coherent assignment are cited in the survey by Sahin and Sural (2007).

Districting models

We also refer to the related literature on districting models – see e.g. the review by Kalcsics et al. (2005). Districting models have the purpose of partitioning of a set of spatial units (i.e. city blocks, census tracts, or other geographic entities) into subsets, called districts, according to some objective. As in discrete models for facility location, spatial units are represented with discrete centers connected through an underlying network. Desired properties of districts often include compactness and contiguity, i.e. they should be round shaped rather than spread out, and should be connected. The capacitated median model, when applied to districting problems, tends to produce districts with these properties, as noted by Kalcsics et al. (2002), who use this model for designing sales territories: compact districts are promoted by the objective of minimizing aggregate travel time together with the closest assignment constraints; also, closest assignment constraints tend to produce connected catchment areas. However, it should be noted that contiguity is not guaranteed and will depend on the data of particular instances. In the application described here, contiguity was always observed. In other cases, specific constraints enforcing contiguity may be required, such as those reviewed by Duque et al. (2011).

Applications to the justice sector

As far as we know, no previous applications of optimization models for the location of courts of justice were described or cited in refereed journals. However, there is at least one such application. Rømo and Sætermo (2000) report on a project of the research center SINTEF in which a discrete location model was developed for defining the location and districts of first instance courts in Norway, under contract with the Ministry of Justice. The model, which was implemented with XPRESS-MP and embedded into a decision support system, is an extension of the *p*-median model. The objective is to minimize total travel distance (optionally to minimize the total infrastructure and travel costs) subject to a number of courts to be located, a minimum workload per court, a maximum assignment distance, and optional configuration constraints (fixed location and assignment variables). Model data considers the municipality as the spatial unit. Relatively to the model presented here, their model does not include closest assignment constraints and is non-hierarchical, i.e. considers a single service representing all demand types and a single type of court.

4.7.2 Model formulations

We now present the formulations of the two models addressing the problem stated in section 4.4. The two models are applied sequentially:

- The districting model is applied to each NUTS 3 region considering all specialization types: family, labor, enforcement, and commerce. The solution defines one or more districts, the seats of district, and the specialization types existing in each district. Enforcement and commerce court locations are implied by this solution, since at most one court exists per district and is located at its seat.
- The court location model is then applied to each individual district, considering only family and labor specialization types if they exist (one or both). The solution defines generic and specialized court locations and the assignments of municipalities to those courts.

Districting model

Data:

M set of demand centers and facility sites – municipalities in the NUTS 3 region;

N set of demand types and court types – 1: generic, 2: family, 3: labor, 4: enforcement, 5: commerce;

Q_{jn} demand (number of cases) of center $j \in M$ of type $n \in N$;

$Q_n^{\min}$ minimum capacity (number of cases) to justify a court of type $n \in N$;

$Q_k^{\max}$ maximum capacity (number of cases) of a district with seat in $k \in M$;

D_{jk} travel time between centers $j, k \in M$;

$D^{\max}$ maximum travel time between a center and the seat of the district;

C_{jk} set of centers as close or closer to $j \in M$ than $k \in M$: $\{p \in M : D_{jp} \leq D_{jk}\}$;

H_j hierarchical level of municipality $j \in M$ – 0: no existing court; 1, 2, 3, 4: existing court and increasing hierarchical level (as defined in section 4.4);

M^T set of municipalities ($M^T \subset M$) currently without court and that are assigned to a generic court located in another municipality within the same NUTS 3 region;

T_j municipality ($T_j \in M$) with the generic court to which municipality $j \in M$ is currently assigned.

Variables (all are binary):

Z_{k1}^D =1 if center $k \in M$ has a generic court and is the seat of a district;

Z_{kn}^D =1 if the district with seat in $k \in M$ has one or more courts of type $n \in N \setminus \{1\}$;

X_{jk}^D =1 if center $j \in M$ is assigned to the district with seat in $k \in M$.

Formulation:

$$\text{Min } O_1 = \sum_{k \in M} Z_{k1}^D \tag{4.1}$$

$$\text{Max } O_2 = \sum_{k \in M} \sum_{n \in N} Z_{kn}^D \tag{4.2}$$

$$\text{Min } O_3 = \sum_{j \in M} \sum_{k \in M} \sum_{n \in N} D_{jk} Q_{jn} X_{jk}^D \tag{4.3}$$

$$\text{s.t.} \quad \sum_{k \in M} X_{jk}^D = 1 \qquad \forall j \in M \tag{4.4}$$

$$X_{jk}^D \le Z_{k1}^D \qquad \forall j \in M, k \in M \tag{4.5}$$

$$Z_{kn}^D \le Z_{k1}^D \qquad \forall k \in M, n \in N \setminus \{1\} \tag{4.6}$$

$$\sum_{j \in M} \sum_{n \in N} Q_{jn} X_{jk}^D \le Q_k^{\max} \cdot Z_{k1}^D \qquad \forall k \in M \tag{4.7}$$

$$\sum_{j \in J} Q_{jn} X_{jk}^D \ge Q_n^{\min} \cdot Z_{kn}^D \qquad \forall k \in M, n \in N \tag{4.8}$$

$$\sum_{p \in C_{jk}} X_{jp}^D \ge Z_{k1}^D \qquad \forall j \in M \setminus M^T, k \in M \tag{4.9}$$

$$X_{jk}^D = 0 \qquad \forall j \in M, k \in M : D_{jk} > D^{\max} \tag{4.10}$$

$$X_{jk}^D = 0 \qquad \forall j \in M, k \in M : H_j > H_k \tag{4.11}$$

$$X_{jk}^D = X_{T_j,k}^D \qquad \forall j \in M^T, k \in M \tag{4.12}$$

$$Z_{kn}^D \in \{0,1\} \qquad \forall k \in M, n \in N \tag{4.13}$$

$$X_{jk}^D \in \{0,1\} \qquad \forall j \in M, k \in M \tag{4.14}$$

The objectives (4.1), (4.2), (4.3) are respectively: to minimize the number of districts; to maximize the number of court types existing in districts; to minimize the total travel time between centers and seats of district, weighted by the demand of all types.

A lexicographic ordering of objectives was assumed, that is, (4.1) is infinitely more preferable than (4.2), which in turn is infinitely more preferable than (4.3). Since O_1 and O_2 are positive integers, the three objectives were replaced by the following single objective (where M_1 and M_2 are suitable large positive constants satisfying $M_2>O_3$ and $M_1>M_2 \cdot O_2+O_3$ for all values of parameters and variables):

$$\text{Min } M_1 \cdot O_1 - M_2 \cdot O_2 + O_3$$

The constraints (4.4)-(4.14) have the following meaning: (4.4) requires each center to be assigned to one district; (4.5) links the assignment and location variables through the standard strong formulation (it states that an assignment can only be made to a center defined as seat of district); (4.6) associates specialized courts ($n>1$) existing in a district with the center defined as seat of that district ($n=1$); (4.7) are maximum capacity constraints, stated for the total demand of all types in a district; (4.8) are minimum capacity constraints, stated separately for each demand and court type in a district; (4.9) are closest assignment constraints; they state that if center k is a seat of district, then any center j has to be assigned to a seat of district as close or closer than k; these constraints do not apply to centers in set M^T; (4.10) defines the maximum travel time to the seat of district; (4.11) forbids the assignment of a center to a seat of district with lower hierarchical level in the current organization (i.e. the seat is chosen among the municipalities with currently highest hierarchical level); (4.12) guarantees that centers belonging to the same old *comarca* have to be assigned to the same new district (i.e. old *comarcas* are not disaggregated); finally, (4.13) and (4.14) define all variables as binary.

We note that (4.11) and (4.12) are special purpose constraints imposing that the reorganization has to take into account the current organization of courts. These constraints, and also (4.10), lead to a reduction of the effective model size, performed automatically by the presolve routines of a MIP optimizer.

Location model

Note: The notation M, N, Q_{jn}, $Q_n^{\min}$ is retained from the previous model, but data is redefined, as described below and in the next section.

Data:

M set of demand centers and facility sites – municipalities in the district;

N set of demand types and court types – 1: generic, 2: family, 3: labor (types 2 and 3 are only included if they exist in the district);

Q_{jn} demand (number of cases) of center $j \in M$ of type $n \in N$;

$Q_n^{\min}$ minimum capacity (number of cases) to justify a court of type $n \in N$;

E_k =1 if a specialized court (of any type n>1) currently exists at center $k \in M$, or zero otherwise;

H_j, D_{jk}, $D^{\max}$, C_{jk}, M^T, T_j have the same definitions as in the previous model.

Variables (all are binary):

Z_{kn} =1 if a court of type $n \in N$ is installed in center $k \in M$;

X_{jkn} =1 if center $j \in M$ is assigned to a court located in center $k \in M$ for demand type $n \in N$.

Formulation:

$$\text{Min} \sum_{j \in M} \sum_{k \in M} \sum_{n \in N} D_{jk} Q_{jn} X_{jkn} \tag{4.15}$$

$$\text{s.t.} \quad \sum_{k \in C} X_{jkn} = 1 \qquad \forall j \in M, n \in N \tag{4.16}$$

$$X_{jkn} \le Z_{kn} \qquad \forall j \in M, k \in M, n \in N \tag{4.17}$$

$$Z_{kn} \le Z_{k1} \qquad \forall k \in M, n \in N \setminus \{1\} \tag{4.18}$$

$$\sum_{j \in M} Q_{jn} X_{jkn} \ge Q_n^{\min} Z_{kn} \qquad \forall k \in M, n \in N \tag{4.19}$$

$$\sum_{p \in C_{jk}} X_{jp1} \ge Z_{k1} \qquad \forall j \in M \setminus M^T, k \in M \tag{4.20}$$

$$\left(1 - Z_{j1}\right) + \sum_{p \in C_{jk}} X_{jpn} \ge Z_{kn} \qquad \forall j \in M, k \in M, n \in N \setminus \{1\} \tag{4.21}$$

$$X_{jk1} + X_{kpn} \le 1 + X_{jpn} \qquad \forall j \in M, k \in M, p \in M, n \in N \setminus \{1\} \tag{4.22}$$

$$X_{jkn} = 0 \qquad \forall j \in M, k \in M, n \in N : D_{jk} > D^{\max} \tag{4.23}$$

$$X_{jk2} = X_{jk3} \qquad \forall j \in M, k \in M, \text{ if } |N| = 3 \tag{4.24}$$

$$Z_{k2} = Z_{k3} \qquad \forall k \in M, \text{ if } |N| = 3 \tag{4.25}$$

$$X_{jk1} = 0 \qquad \forall j \in M, k \in M : H_j > H_k \tag{4.26}$$

$$X_{jkn} = X_{T_j,k,n} \qquad \forall j \in M \cap M^T, k \in M, n \in N \tag{4.27}$$

$$Z_{k1} = 0 \qquad \forall k \in M : H_k = 0 \tag{4.28}$$

$$Z_{kn} = 0 \qquad \forall k \in M, n \in N \setminus \{1\} : Z_{k1}^D = 0 \wedge E_k = 0 \tag{4.29}$$

$$Z_{kn} \in \{0,1\} \qquad \forall k \in M, n \in N \tag{4.30}$$

$$X_{jkn} \in \{0,1\} \qquad \forall j \in M, k \in M, n \in N \tag{4.31}$$

The objective (4.15) is to minimize the total travel time between centers and courts, weighted by demand. Travel times for generic demand (n=1) and for specialized demand (n>1) are assumed to be commensurable and are given the same weight.

The constraints (4.16)-(4.31) have the following meaning: (4.16) requires each demand type of each center to be assigned to one court of the corresponding type; (4.17) links the assignment and location variables of each type (they state that an assignment can only be made to a center where a court of the corresponding type is located); (4.18) states that specialized courts (n>1) can only be located in centers where a generic court (n=1) is also located (this constraint enforces the nested hierarchy); (4.19) are minimum capacity constraints for each court type; (4.20) are closest assignment constraints for generic courts; they do not apply to centers in set M^T, i.e. without an existing court; (4.21) are closest assignment constraints for each specialized type, applying only to centers j where a generic court exists (Z_{j1}=1); if no court exists (Z_{j1}=0) the constraint has no effect (the inequality is satisfied, irrespective of the values of all other variables) and the assignment will be determined only by coherency and other constraints; (4.22) are coherency constraints for each specialized type, stating that if a center j is assigned to a generic court at k (X_{jk1}=1) and center k is itself assigned to a specialized court at p (X_{kpn}=1), then center j must also be assigned to the specialized court at p (X_{jpn}=1); (4.23) defines the maximum travel time to a court; (4.24) implies that each center must be assigned to family and labor courts located in the same municipality, if both types exist (|N|=3); (4.25) implies that family and labor courts have to be co-located, if both types exist; (4.26) forbids the assignment of a center to a court in a municipality with lower hierarchical level in the current organization; (4.27) guarantees that centers belonging to the same old *comarca* have to be assigned to the same courts of each type; (4.28) forbids opening a new generic court in municipalities where no court currently exists; (4.29) states that specialized courts cannot be located in municipalities which are not the seat of the district and where no specialized court of any type currently exists; finally, (4.30) and (4.31) define all variables as binary.

Constraints on the maximum capacity of courts are not imposed explicitly since the individual courts are assumed to have the same capacity limit as the main court in the seat of the district, and this limit is guaranteed by the first model.

Regarding the combination of assignment constraints, centers where a level 1 facility is not installed are subject to coherent assignment but not to closest assignment constraints for level 2 assignments, through the term (1-Z_{j1}) in (4.21). This was a planning assumption in the present application. If that term was removed the model would remain consistent, but the feasible spatial configurations of solutions would be reduced.

We now comment on particular cases of the model, depending on the number of specialized court types that result from the districting model:

- If no specialized types exist (N={1}), a reduced model is obtained, dedicated to a single (generic) demand and court type; in particular, the following constraints are excluded: (4.18), (4.21), (4.22), (4.24), (4.25), (4.29).
- If both family and labor specializations exist (N={1,2,3}), the two types are not independent, due to the presence of constraints (4.24) and (4.25). These are included since, in the present application, centers cannot be assigned to family and labor courts in different municipalities, which in turn implies the co-location of family and labor courts (we note that it would suffice to add (4.24), because (4.25) are implied by the former together with (4.17), (4.19) and the binary constraints). These constraints effectively reduce assignment and location decisions to two types – generic (type 1) and specialized (type 2, representing both family and labor). The formulation could thus be simplified, retaining only the variables and constraints for a single specialized type, except for minimum capacity constraints (4.19), which must still be stated separately for each specialized subtype (that is, family and labor courts must be co-located, but each type must separately guarantee its minimum capacity). However, we opted to keep the general formulation above, which can address other applications where constraints (4.24) and (4.25) do not apply. Additionally, there is no computational burden, since the model will be automatically simplified as described above by the presolve routines of a MIP optimizer.

Similarly to the first model, (4.26)-(4.29) are special purpose constraints imposing that the reorganization has to take into account the current organization of courts. These constraints, together with (4.23) and, as noted above, (4.24)-(4.25), lead to a reduction of the effective model size, performed automatically by the presolve routines of a MIP optimizer.

Alternative formulations of closest assignment constraints have been proposed in the literature – see Gerrard and Church (1996) and Hansen et al. (2004). The latter authors show that formulation (4.20) provides a tighter linear programming relaxation than the alternatives discussed. The formulation of coherency constraints (4.22) is equivalent to the one proposed by Serra and ReVelle (1996).

Finally, we observe that both models above are hierarchical extensions of the basic, single-service capacitated median model. The first model (districting) is a straightforward extension, since it considers a single level of assignment (representing all demand types). Multiple facility types are considered because of the second objective (maximization of the number of facility types). The second model (court location) is a multiple-service hierarchical extension considering two levels of assignment and two facility levels. Level-1 demand and facilities are of a single type (n=1) while level-2 demand and facilities are of multiple types (n>1).

4.7.3 Model application and discussion

For applying the models within the study, data was prepared as follows:

- Demand of municipalities (Q_{jn}): is measured in number of cases of each type n and was estimated as described in section 4.5. For the generic type (Q_{j1}): in the first model, demand includes civil (except enforcement and commerce) and criminal cases; in the second model, it also includes other case types for which no specialized court exists in the district.

- Minimum capacity of courts ($Q_n^{\min}$): is defined as reference productivity per judge (section 4.6) multiplied by a minimum number of judges (section 4.4). For the generic type ($Q_1^{\min}$): in the first model, a minimum of 1 judge with reference productivity of 550 cases was adopted; in the second model, reference productivity was set to 550 cases if a specialized enforcement court exists in the district, or to 800 cases otherwise.

- Maximum capacity of a district ($Q_k^{\max}$): is defined as the global reference productivity for all case and court types (section 4.6) multiplied by a maximum number of judges (section 4.4); or is set equal to the total demand in the particular municipality k, if higher (in this case, a district composed of a single, large municipality will result).

- Travel time (D_{jk}): is measured in minutes and was computed with a representation of the main road network planned for 2015 (according to the National Road Plan – PRN 2000). Travel times refer to centroids of municipalities, and a time of zero is assumed if a municipality is assigned to a court located within it.

Models were implemented and solved with FICO's Xpress Optimization Suite version 2005B (Xpress Mosel 1.6 and Xpress MIP Optimizer 16.10) running on a personal computer with a Pentium M 755 2.0 GHz CPU, 1.0 GB of memory and Windows XP operating system. All instances were easily solved, within at most a couple of seconds, because their initial size was small (instances had 14 centers at most in both models) and the constraints limiting changes to the existing network reduced model size even further.

We now comment on the adoption of two separate, sequential models rather than a single integrated model involving both districting and location decisions. From the application perspective, the districting problem can indeed be seen as separated because it was assumed that its three objectives preempt the objective of maximizing accessibility to local generic and specialized courts in the second model. The two models will lead to a globally optimal solution, unless there is more than one optimal solution to the districting model and the one retained leads to a worse objective in the second model. However, this was deemed unlikely to occur in the present application, due to the third objective of the districting model, of minimizing aggregate travel time to the seat of district.

From the modeling perspective, separating the two models has two advantages. First, it makes the global problem easier to solve, as an integrated model would need to consider 3 levels of assignment decisions (to generic courts, to specialized courts, and to the district seat) and would become much larger. Second, once the first model is solved, the productivity per judge in generic courts, which underlies the minimum capacity of these courts and may depend on which specialized courts are installed in its district, becomes exogenous to the second model. On the other hand, an integrated model would have to determine an endogenous productivity in generic courts for each district, as a function of the combination of specializations installed. This may require a larger model, potentially making it much harder to solve.

4.8 Study results

4.8.1 Proposal of the new judiciary map

The proposal of the judiciary map is now briefly discussed, focusing on mainland Portugal (each of the archipelagos, Madeira and Azores, constitutes one additional district each). The map proposal is shown in Figure 4.4 to Figure 4.7. Assignments of municipalities to specialized courts are not represented explicitly as they are defined by district boundaries and, if they exist, by family and labor jurisdiction boundaries.

The proposal reduces the 213 existing *comarcas* to 38 districts. Of the 28 NUTS 3 regions, 20 correspond to a single district and 8 are partitioned into two or more districts to avoid excessive district size (occurs in the Lisbon and Porto regions) or excessive travel time to the seat of district. In 32 districts the seat is the most populous municipality, while in the other 6 districts a municipality with higher accessibility is chosen instead. Districts range in size from 1 to 16 courts and from 3 to 113 judges; there are two districts with only 1 court (both in the Alentejo region), created because of the imposed maximum travel time to the seat of district.

The number of courts (Table 4.2) is 284 in total. 27 existing generic courts would be closed (while no new generic courts would be open) and specialized courts would increase by 30 (the net effect of new courts to open and existing courts to close). The numbers of enforcement, family and commerce courts have large relative increases, while the number of labor courts decreases significantly. The latter is due to the significant decrease in labor litigation expected to occur (section 4.5).

The number of judges (Table 4.2) is 1060 in total, including judges assigned to main courts (in seats of district): 1 judge president and a number of assistant judges equal to 10% of the total number of judges assigned to courts in the district. The proposal implies 38 new judges would be necessary, while 97 existing judges would have to change to a court in a different district. Generic courts would lose 49 judges, while specialized courts would gain 87 judges in total (enforcement courts would gain the most, 77 judges, while labor courts would lose 23 judges).

Regarding accessibility to generic courts (Table 4.3), a generic court exists in about 2/3 of the municipalities (shown with a travel time of zero in Table 4.3). Among the other municipalities, only 7 require a travel time of more than 30 minutes to a generic court.

Table 4.2: Number of courts and judges (mainland Portugal)

	Type of court					
	Generic	Family	Labor	Enforcement	Commerce	Total
Number of Courts						
Current (2005)	213	16	45	5	2	281
Proposal (2015)	186	32	32	30	4	284
Difference	-27	16	-13	25	2	3
Closed	-27	-2	-17	0	0	-46
New	0	18	4	25	2	49
Number of Judges						
Current (2005)	866	48	85	16	7	1022
Proposal (2015)	817	82	62	93	6	1060
Difference	-49	34	-23	77	-1	38

Table 4.3: Travel time to generic courts (mainland Portugal), measured between centroids of municipalities

Travel time (min)	Number of municipalities	%
0	186	67%
>0 - 10	32	12%
>10 - 20	39	14%
>20 - 30	14	5%
>30 - 40	7	3%
>40	0	0%
Total	278	100%

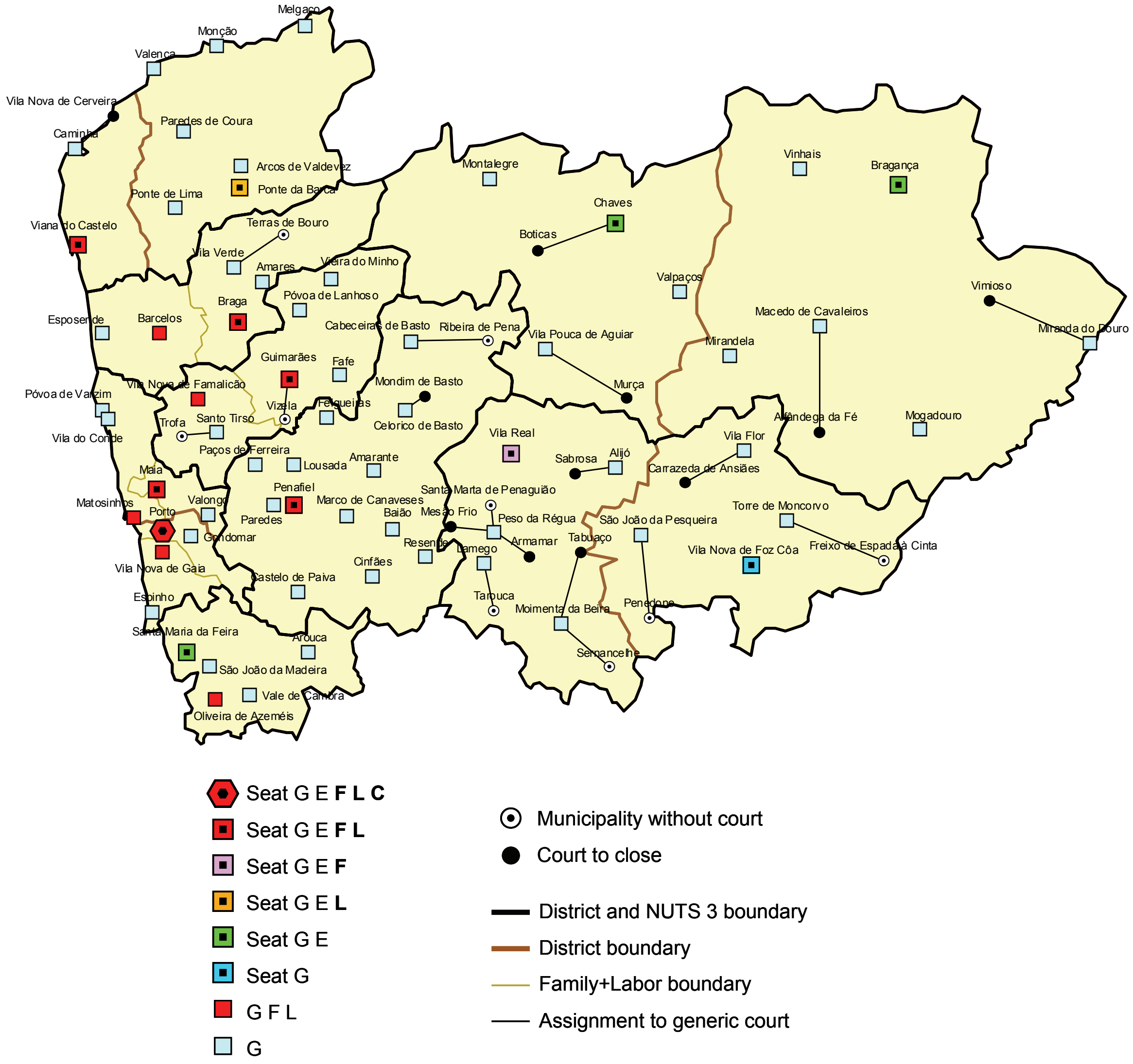


Figure 4.4: Judiciary map proposal – Norte region

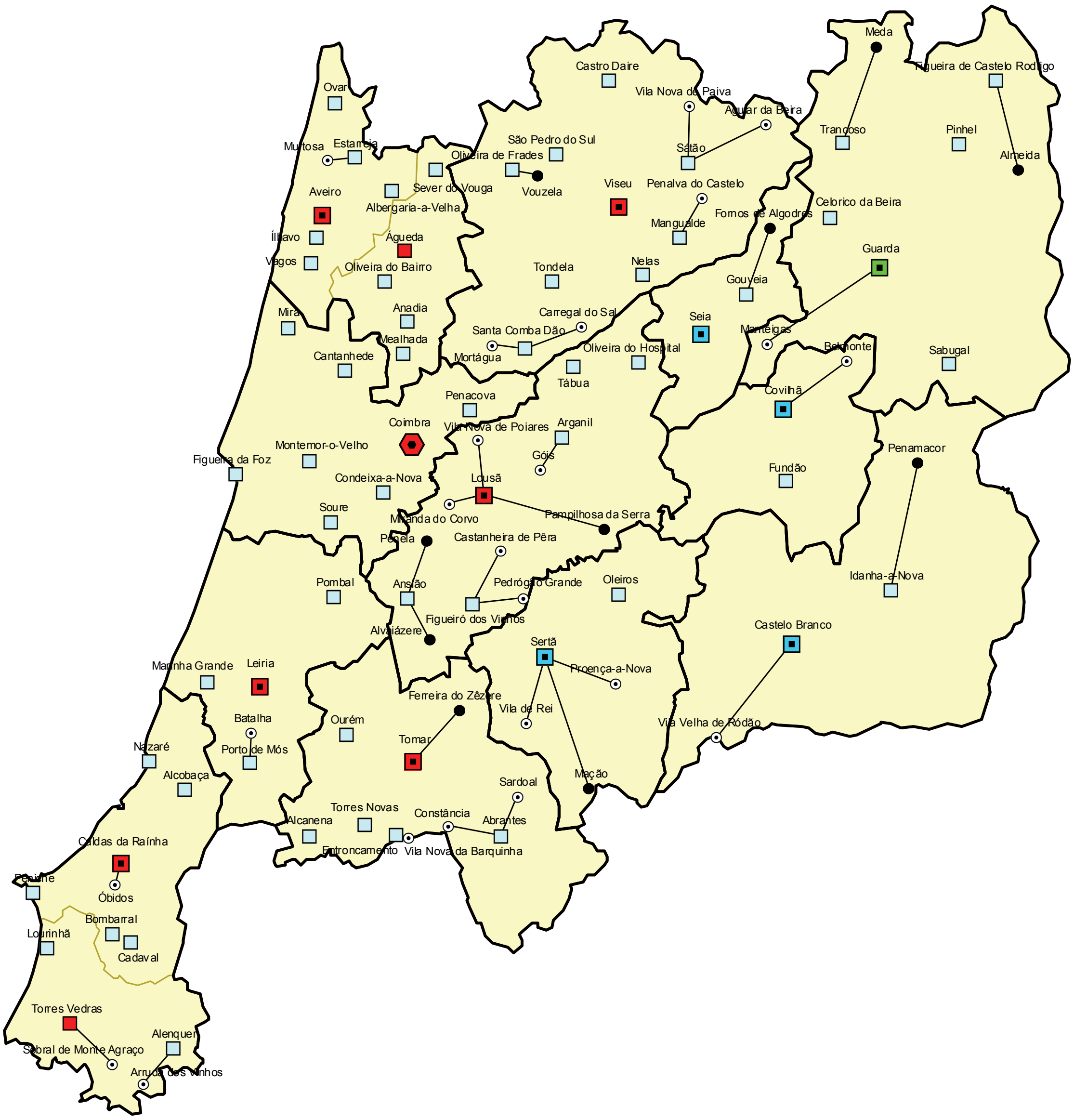


Figure 4.5: Judiciary map proposal – Centro region

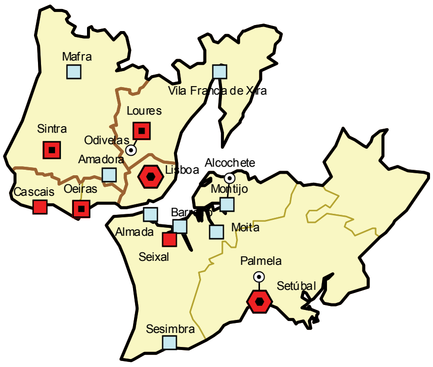


Figure 4.6: Judiciary map proposal – Lisbon region

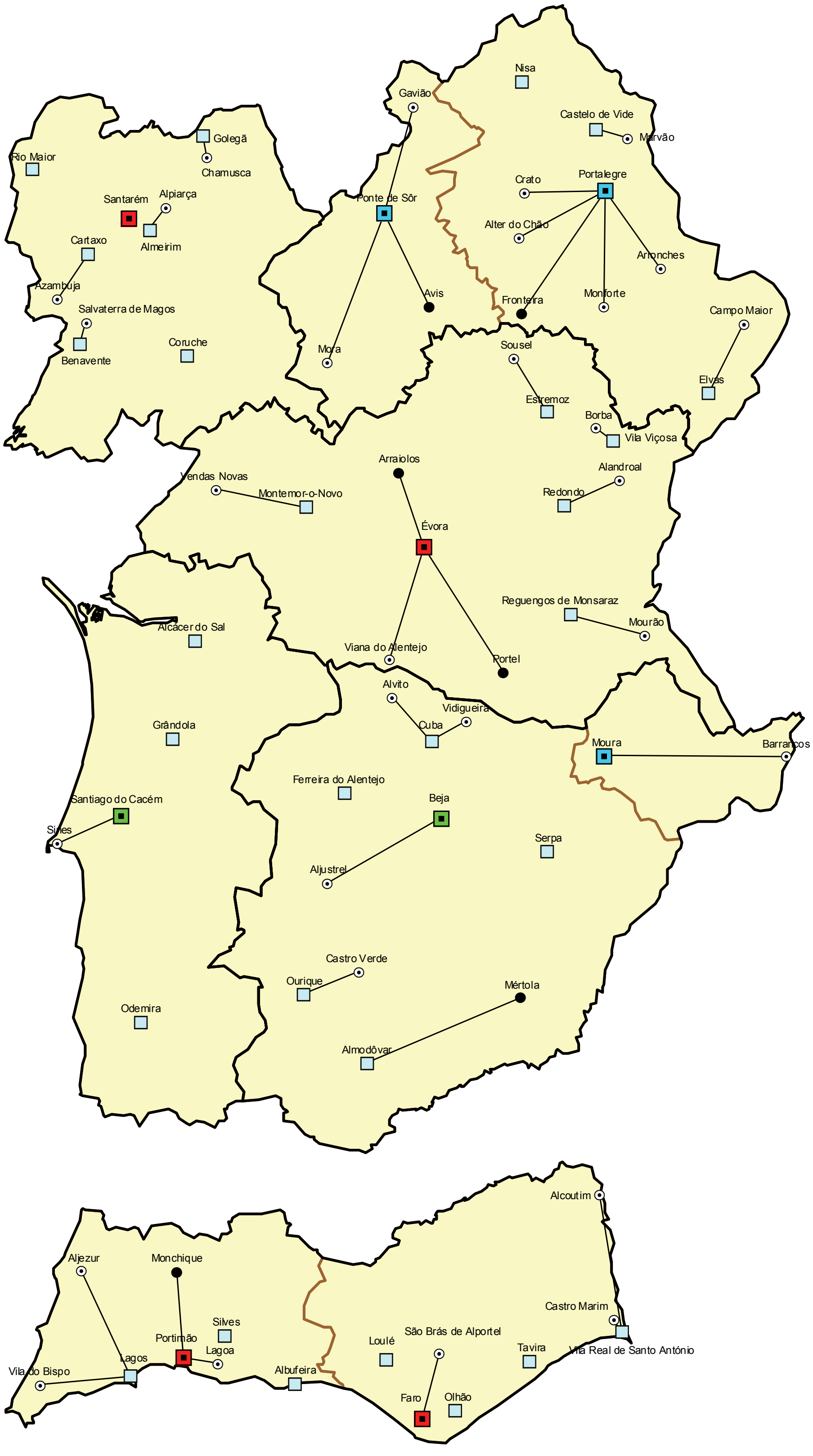


Figure 4.7: Judiciary map proposal – Alentejo region (top) and Algarve region (bottom)

4.8.2 Sensitivity analysis

A sensitivity analysis of the judiciary map proposal was carried out by changing individual parameters in the problem definition (Table 4.4). These changes can lead to significantly different outcomes, as measured by key criteria (Table 4.5).

Removing the travel time limit to the main court (Alternative I) would decrease the number of districts (from 38 to 32) and increase travel time to the main court (by 15%). It would produce only slight variations in the numbers of generic and specialized courts.

Decreasing the maximum number of judges per district (Alternative II) would significantly increase the number of districts (from 38 to 48). Consequently, the number of specialized courts would also increase (except commerce). While this contributes to reducing the number of judges (due to higher productivity), the total number would increase considerably (from 38 to 59), due to the additional president and assistant judges required in districts.

Requiring higher minimum workloads in generic courts (Alternatives III and IV) would significantly increase generic court closures and travel time to generic courts. However, only in Alternative IV the number of new judges would decrease significantly (from 38 to 10).

Requiring higher minimum workloads in specialized courts (Alternative V) would significantly reduce the number of specialized courts. However, the number of closed generic courts (which now receive additional specialized demand) would decrease only slightly, and consequently travel time to generic courts would only improve slightly.

In Alternative VI, districts are now based on the larger NUTS 2 regions (of which there are 5 in mainland Portugal: Norte, Centro, Lisboa, Alentejo, Algarve). This alternative would lead to fewer districts, better accessibility to general courts, and also fewer new judges. While the number of family and labor courts would remain unchanged and one additional commerce court would be open, fewer enforcement courts would be necessary (because there are fewer districts).

Overall, we observe that the selected planning parameters have significant impact on the judiciary map, in terms of the trade-off between judicial resources (number of districts, courts, and judges) and user benefits such as accessibility and availability of specialized courts. This highlights why the reform of the judiciary map can generate a long debate among decision makers and other stakeholders in the judicial system.

Table 4.4: Sensitivity analysis – parameters

Parameter	Proposal	Alternative					
		I	II	III	IV	V	VI
Maximum travel time to main court (min)	60	No limit	60	60	60	60	60
Maximum number of judges in a district	75	75	40	75	75	75	75
Minimum workload in a generic court relative to the reference productivity of one judge	50%	50%	50%	80%	180%	50%	50%
Idem, for specialized courts	80%	80%	80%	80%	80%	100%	80%
NUTS 3 based districts ?	Y	Y	Y	Y	Y	Y	N

Table 4.5: Sensitivity analysis – results

Criterion	Proposal	Alternative					
		I	II	III	IV	V	VI
Number of districts	38	32	48	38	38	38	31
Average travel time to main court (relative to the proposal)	1.00	1.15	0.82	1.00	1.00	1.00	1.14
Average travel time to generic court (relative to the proposal)	1.00	1.03	1.00	1.29	2.65	0.97	0.95
Number of generic courts to close	27	30	27	43	93	25	25
Number of family courts to open (net)	16	18	20	16	16	11	16
Number of labor courts to close (net)	13	12	11	13	13	20	13
Number of enforcement courts to open	25	23	34	25	24	19	19
Number of commerce courts to open	2	2	1	2	2	0	3
Number of new judges	38	27	59	36	10	34	26
Number of judges changing court	97	95	99	98	109	96	103

4.9 Conclusion

The study described here was released in March 2007, as a contribution to the discussion about the reform of the judiciary map of Portugal then being carried out by the Ministry of Justice and institutions within the judicial system.

The Government later prepared Proposal of Law 124/2008, specifying the reform of the organization and functioning of courts of justice, including the reform of the judiciary map, with a preamble where the contribution of our study was explicitly acknowledged. The Proposal of Law specifies the territories of districts, while the location of generic and specialized courts of justice of first instance and their territorial jurisdictions were left to subsequent legislation. The Proposal of Law followed the solutions proposed in the study with some changes. In particular, the two small districts with a single court (in the Alentejo region) were merged into neighboring districts (accepting larger maximum travel times), and two districts with large demand (with seats in Porto and Penafiel, in the Norte region) were divided into two districts each.

In May 2008, the Proposal of Law was voted and approved by the Parliament, becoming Law 52/2008. After that, the Government started preparing the law's implementation process, which will proceed in stages. The first stage started in April 2009, with the launching of three districts (with seats in Santiago do Cacém, Aveiro and Sintra, in the Alentejo, Centro and Lisbon regions, respectively), while the final stage is expected to finish in 2015. In those first three districts, the solution adopted for the location of specialized courts differs from that in the study, following the final set of planning criteria adopted after the discussion between the Ministry of Justice and other stakeholders. In particular, the implemented solution has a higher number and distribution of specialized courts among the municipalities composing a district, since the co-location of specialized courts and the preference given to the seat of district were de-emphasized.

At this point in time the true benefits of the reform are still uncertain, but we are certain that our study has given a contribution for making them possible, by helping to discuss relevant planning criteria and offering a complete and detailed proposal based on a particular set of plausible and clearly identified criteria.

The contributions of this chapter to the facility location literature can be outlined as follows. First, a multiple-service hierarchical version of the capacitated median model is formulated, combining features that appeared separately in previous models – capacitated facilities, closest assignment and coherent assignment. Second, an application of facility location models to courts of justice is described. To the best of our knowledge, no such application has been described before in a refereed journal, although we could find a research report on a similar application in Norway (Rømo and Sætermo, 2000).

Chapter 5

Solving the capacitated median model by a priori reformulation and branch-and-cut

5.1 Introduction

In this chapter we address a discrete facility location model, called the capacitated median (CM) model, which is related to the classic *p*-median (PM) model. In both models, we are given a set of demand centers with known demand quantities, a set of sites where facilities can be located, and travel distances (or costs) between centers and sites.

In the CM model, the aim is to locate facilities and assign demand centers to those facilities so that the total travel distance (or cost) is minimized, the demand served by each facility satisfies given minimum and maximum capacity bounds, and each center is fully served by the closest (or least cost) facility. This model is useful in the context of public facility location problems where facilities such as schools or hospitals should guarantee a minimum workload (e.g. a minimum number of users per year) in order to be economically feasible, and users minimize their costs by attending the closest facility.

The PM model is similar, except that the number of facilities is a given parameter (denoted *p*) and facility capacity is unrestricted. In comparison, in the CM model the number of open facilities is a model output, since the minimum and maximum capacity bounds impose implicit upper and lower bounds (respectively) on the number of facilities, with the model objective driving solutions towards the upper bound. Additionally, in the PM model solutions naturally have the so-called single assignment and closest assignment properties (Krarup and Pruzan, 1990), that is, centers are fully assigned to a single, closest facility. In the CM model, due to the presence of capacity constraints, these properties must be enforced through explicit constraints. Thus, we can say that capacity constraints and explicit single and closest assignment constraints are the defining features of the CM model relatively to the PM model.

The PM model has been studied extensively – see e.g. ReVelle and Eiselt (2005) and Marianov and Serra (2002) – but the CM model has been studied only rarely. Carreras and Serra (1999) use the CM model without the maximum capacity constraints for a pharmacy location problem, and solve it through a tabu search heuristic. Kalcsics et al. (2002) use it with a constraint on the number of facilities for designing balanced and compact sales territories, and solve it through a variable neighborhood search heuristic. Teixeira et al. (2007) use it to address a secondary school network redeployment problem and solve it with a commercial MIP solver. Bigotte and Antunes (2007) present several heuristics to solve the CM model without maximum capacity constraints, including construction and improvement heuristics, a genetic algorithm and a tabu search heuristic.

Thus, several heuristic methods have been proposed to solve the CM model, but no specialized exact method has been proposed, as far as we know. Such a method would be useful, as computational experiments with a generic MIP solver on a standard personal computer (detailed below) shown that instances with 50 and 70 centers can be solved to optimality on average within 1 minute and 10 minutes, respectively, but many instances with 100 centers cannot be solved within 1 hour.

Another relevant model related to the CM model is the uncapacitated facility location problem with clients' preference orderings (UFLPO), introduced by Hanjoul and Peeters (1987). The UFLPO is an extension of the classic uncapacitated facility location problem where each demand center has given preferences for facilities (independent of assignment costs) and must be assigned to the most preferred open facility (rather than to the least cost facility). The preferences are enforced by constraints equivalent to the closest assignment constraints in the CM model. The feasible set of the UFLPO is a relaxation of the feasible set of the CM model, so inequalities valid for the former are also valid for the latter. Cánovas et al. (2007) proposed an a priori reformulation procedure for the UFLPO that aims to strengthen the formulation and reduce its size (in terms of number of variables, constraints and non-zero elements) before applying a branch-and-cut algorithm. The procedure is implemented in a preprocessing algorithm and depends on the particular data of each instance.

The contributions of this chapter are the following. First, we extend the reformulation procedure of Cánovas et al. (2007) in order to address instances with preference ties between two or more locations (strict preference orderings were assumed in the original), and propose an additional step that further strengthens and reduces the formulation. Second, we propose new valid inequalities for the CM model. Third, we present computational experiments with an exact method to solve this model, composed of the reformulation procedure and cut generation procedures embedded within the branch-and-cut algorithm of a generic MIP solver, exploiting previously known valid inequalities for the UFLPO and the new valid inequalities for the CM model.

We start by presenting the formulation of the CM model. Next, we present valid inequalities and the reformulation procedure for the UFLPO, followed by new valid inequalities for the CM model and separation procedures for cut generation. We then present computational experiments and offer conclusions.

5.2 Model formulation

Consider a set of demand centers $I = \{1,...,n\}$ and a set of sites $J = \{1,...,m\}$ where facilities can be located. Each center i has associated a demand u_i and each site j has associated lower and upper capacity bounds, b_j and B_j, respectively. A cost $c_{ij} \geq 0$ is incurred if the demand of center i is satisfied from site j. We assume $c_{ij} = u_i d_{ij}$, where d_{ij} is a given travel distance, time or monetary cost per unit demand between center i and site j; in what follows, we use the terms “closest” and “least-cost” interchangeably. The problem is to locate facilities and assign centers to those facilities so that the total cost is minimized, facilities satisfy the capacity bounds, and centers are fully assigned to the least-cost facility.

Consider the following decision variables: $y_j = 1$ if a facility is installed or open at site j, and equals zero otherwise; x_{ij} is the fraction of the demand of center i satisfied from site j (if $x_{ij} = 1$, center i is said to be assigned to site j). The CM model can be formulated as follows:

Min
$$\sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \tag{5.1}$$

Subject to
$$\sum_{j \in J} x_{ij} = 1, \ \ \forall i \in I \tag{5.2}$$

$$x_{ij} \leq y_j, \ \ \forall i \in I, j \in J \tag{5.3}$$

$$\sum_{i \in I} u_i x_{ij} \geq b_j y_j, \ \ \forall j \in J \tag{5.4}$$

$$\sum_{i \in I} u_i x_{ij} \leq B_j y_j, \ \ \forall j \in J \tag{5.5}$$

$$\sum_{k \in N_{ij}} x_{ik} \geq y_j, \ \ \forall i \in I, j \in J, \ N_{ij} = \{k \in J \mid d_{ik} \leq d_{ij}\} \tag{5.6}$$

$$x_{ij} \in \{0,1\}, \ \ \forall i \in I, j \in J \tag{5.7}$$

$$y_j \in \{0,1\}, \ \forall j \in J \qquad (5.8)$$

Constraints (5.2) state that the demand of all centers should be fully served. Constraints (5.3) ensure that a center can only be served from a site where a facility is installed, and are termed variable upper bound (VUB) constraints. Inequalities (5.4) and (5.5) are, respectively, minimum and maximum capacity constraints, imposing lower and upper bounds on the demand served from open facilities. Inequalities (5.6) are the so-called closest assignment (CA) constraints, and state that if a facility is open at site *j* then center *i* must be assigned to it, to an equidistant or to a closer facility. Constraints (5.7) enforce single assignment, that is, the demand of a center must be fully satisfied from a single facility. Note that if there are no "distance ties" (i.e. $d_{ij} \neq d_{ik}$ for all $i \in I$, $j \in J$, $k \in J$ with $j \neq k$) then CA constraints (5.6) imply single assignment even if the integrality of the x_{ij} variables is relaxed. Note also that the CM model is obtained from the *p*-median model by replacing the constraint on the number of open facilities ($\sum_{j \in J} y_j = p$) with the capacity constraints (5.4) and (5.5), thus making the number of facilities a model output rather than a parameter, and by adding closest and single assignment constraints (5.6) and (5.7), which are redundant in the *p*-median model.

Regarding the choice of formulation of CA constraints, we adopted formulation (5.6) as it provides a tighter LP relaxation than other alternatives proposed in the literature – see the discussions by Hanjoul and Peeters (1987), Gerrard and Church (1996), and Espejo et al. (2012).

We now recall the definitions of two related problems that will be useful later. The uncapacitated facility location problem (UFLP) is the problem of minimizing $\sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \sum_{j \in J} f_j y_j$ subject to (5.2), (5.3), (5.7) and (5.8), where f_j is the fixed cost of installing a facility at site *j*. The UFLP with preference orderings (UFLPO) requires in addition the satisfaction of constraints (5.6), but these are now termed "preference constraints" and d_{ij} values represent given preferences, unrelated to costs c_{ij}, such that: $d_{ij} < d_{ik}$ means that users or clients at center *i* strictly prefer facility *j* over *k*, and $d_{ij} = d_{ik}$ means that facilities *j* and *k* are indifferent. Thus, the UFLPO is an extension of the UFLP where each demand center must be assigned to the most preferred open facility (rather than the least cost facility). Note that if c_{ij} is strictly increasing with d_{ij} for all centers *i*, then constraints (5.6) are redundant and the UFLPO reduces to the UFLP.

In the study of the CM model in this this chapter, we make the following assumptions about model data: minimum capacities are constant ($b_j = b$ for all sites *j*); maximum capacities are non-binding (B_j for all sites *j* is greater or equal to the total demand), and thus constraints (5.5) can be dropped from the formulation. Let X^{LSC} denote the feasible set of the CM model with those assumptions, i.e. the set of the location model with constant lower capacity bounds (L), single assignment (S) and closest assignment (C):

$$X^{LSC} = \left\{ x \in \{0,1\}^{n\times m}, y \in \{0,1\}^m : \text{(5.2), (5.3), (5.4), (5.6) are satisfied,} \right.$$
$$\left. \text{assuming } b_j = b \text{ for } j \in J \right\}$$

We will study valid inequalities for the following sets:

$$X^{UFLPO} = \left\{ x \in \{0,1\}^{n\times m}, y \in \{0,1\}^m : \text{(5.2), (5.3), (5.6) are satisfied} \right\}$$

$$X^{LS} = \left\{ x \in \{0,1\}^{n\times m}, y \in \{0,1\}^m : \text{(5.2), (5.3), (5.4) are satisfied,} \right.$$
$$\left. \text{assuming } b_j = b \text{ for } j \in J \right\}$$

$$X^{LSI} = \left\{ x \in \{0,1\}^{n\times m}, y \in \{0,1\}^m : \text{(5.2), (5.3), (5.4), } x_{ii} = y_i \text{ for } i \in I \text{ are satisfied,} \right.$$
$$\left. \text{assuming } I = J \text{, and } b_i = b \text{ for } i \in I \right\}$$

Sets X^{UFLPO} and X^{LS} are relaxations of (i.e. contain) the set X^{LSC}. Set X^{LSI} is a relaxation of a variant of X^{LSC} restricted to: (1) identical sets of centers and sites ($I = J$); (2) $x_{ii} = y_i$ for $i \in I$, i.e. facilities must serve the centers where they are located. Note that when $I = J$ and d_{ij} data is such that d_{ii} is the unique, minimum distance for $i \in I$, then the CA constraints (5.6) for $i \in I$ and $j = i$ reduce to $x_{ii} \geq y_i$, and combined with (5.3) give $x_{ii} = y_i$.

Before continuing, we note that the CM model is NP-hard, since the UFLP, which is NP-hard (Cornuejols et al., 1990), can be polynomially reduced to it. An instance of the UFLP can be transformed into an instance of the CM model by the following procedure, using closest assignment constraints to include fixed costs in the objective function through equivalent variable costs. (1) For $i \in I$, $j \in J$, set c_{ij} as in the original UFLP data and set $d_{ij} = 0$. (2) Create a set I' of fictitious centers corresponding to sites in J; for $i \in I'$, $j \in J$, set $c_{ij} = f_j$ if $i = j$ or equal to zero otherwise, and set $d_{ij} = 0$ if $i = j$ or equal to any positive value otherwise. (3) Augment the set I with set I' and set all other data (u_i, b_j, B_j) to zero. A similar procedure was cited by Hansen et al. (2004) to reduce the UFLPO to a restricted UFLPO without fixed costs ($f_j = 0$ for all sites j), thus showing that even this restricted problem is NP-hard.

5.3 Valid inequalities for X^{UFLPO}

We first introduce additional notation related to the preference (or closest assignment) constraints. For $i \in I$, $j \in J$:

$W_{ij} = \{k \in J \mid d_{ik} > d_{ij}\}$ (worse/farther sites)

$N_{ij} = J \setminus W_{ij} = \{k \in J \mid d_{ik} \le d_{ij}\}$ (not worse/farther sites)

$E_{ij} = \{k \in J \mid d_{ik} = d_{ij}\}$ (indifferent/equidistant sites)

$N'_{ij} = N_{ij} \setminus E_{ij} = \{k \in J \mid d_{ik} < d_{ij}\}$ (strictly better/closer sites)

In his PhD thesis, García (2006) studied several valid inequalities for X^{UFLPO} and proposed the reformulation procedure cited above, later also published by Cánovas et al. (2007). These authors assumed data with no preference ties, i.e. $E_{ij} = \{j\}$ holds for all i and j. Next we review relevant inequalities and discuss the case of data with preference ties, i.e. $|E_{ij}| \ge 1$.

Dominance inequalities – assuming no preference ties, $|E_{ij}| = 1$ (Cánovas et al., 2007):

$$x_{i_1 j} \le x_{i_2 j}, \text{ for } i_1, i_2 \in I, j \in J: W_{i_1 j} \subseteq W_{i_2 j} \text{ (or } N_{i_1 j} \supseteq N_{i_2 j}\text{)}$$

$$x_{i_1 j} = x_{i_2 j}, \text{ for } i_1, i_2 \in I, j \in J: W_{i_1 j} = W_{i_2 j} \text{ (or } N_{i_1 j} = N_{i_2 j}\text{)}$$

The first inequality states that assigning i_1 to j implies also assigning i_2 to j if all sites better than j for i_1 are also better than j for i_2. The equality results from combining two inequalities.

Inequality $x_{i_1 j} \le x_{i_2 j}$ dominates the VUB constraint $x_{i_1 j} \le y_j$, since $x_{i_2 j} \le y_j$ is also in the model, and can be used to strengthen the formulation. Thus, as noted by Cánovas et al. (2007), while all VUB constraints are facet defining for the UFLP polytope (Cornuejols et al., 1990), this is not the case for the UFLPO polytope, depending on d_{ij} data. Next we present a generalization for data with preference ties.

Dominance inequalities – generalized for preference ties, $|E_{ij}| \ge 1$:

$$x_{i_1 j} \le x_{i_2 j}, \text{ for } i_1, i_2 \in I, j \in J: N'_{i_1 j} \supseteq N_{i_2 j} \setminus \{j\} \qquad \text{(A)}$$

$$x_{i_1 j} = x_{i_2 j}, \text{ for } i_1, i_2 \in I, j \in J: N'_{i_1 j} = N'_{i_2 j} \text{ and } E_{i_1 j} = E_{i_2 j} = \{j\} \qquad \text{(A}_{\text{eq}}\text{)}$$

Proof of validity of (A): If $x_{i_1 j} = 0$ the inequality is trivially valid. If $x_{i_1 j} = 1$ then it must be $y_j = 1$ and $y_k = 0$ for $k \in N'_{i_1 j}$ in order to satisfy (5.3) and (5.6), respectively. Then, (5.3) implies $x_{i_2 k} = 0$ for $k \in N'_{i_1 j}$, and, given that $N'_{i_1 j} \supseteq N_{i_2 j} \setminus \{j\}$, (5.6) implies $x_{i_2 j} = 1$.

Proof of validity of (A$_{\text{eq}}$): As $N_{ij} = N'_{ij} \cup E_{ij}$, inequalities (A) hold for both (i_1, i_2) and (i_2, i_1) if $N'_{i_1 j} \supseteq N'_{i_2 j} \cup (E_{i_2 j} \setminus \{j\})$ and $N'_{i_2 j} \supseteq N'_{i_1 j} \cup (E_{i_1 j} \setminus \{j\})$, which leads to the conditions given.

Preference inequalities (Cánovas et al., 2007):

$$\sum_{k \in W_{i_1 j}} x_{i_1 k} \le 1 - y_j \text{, for } i_1 \in I, j \in J \tag{C1}$$

$$\sum_{k \in W_{i_1 j}} x_{i_1 k} + \sum_{k \in W_{i_2 j} \setminus W_{i_1 j}} x_{i_2 k} \le 1 - y_j \text{, for } i_1, i_2 \in I, j \in J \tag{C2}$$

$$\sum_{k \in W_{i_1 j}} x_{i_1 k} + \sum_{t=2}^{s} \sum_{k \in W_{i_t j} \setminus \bigcup_{r=1}^{t-1} W_{i_r j}} x_{i_t k} \le 1 - y_j \text{, for } i_1, \ldots, i_s \in I, j \in J \tag{Cs}$$

Inequalities (C1) are equivalent to (5.6), as demand constraints (5.2) imply that $\sum_{k \in N_{ij}} x_{ik} + \sum_{k \in W_{ij}} x_{ik} = 1$ for all $i \in I$, $j \in J$. The two formulations can be interpreted as follows: if facility j is open, (C1) forbids assignment to worse facilities, while (5.6) forces assignment to better or indifferent facilities. Note that formulation (5.6) implies that all centers must be assigned if at least one facility is open, while (C1) is compatible with models where not all demand requires satisfaction, i.e. with "=" replaced by "≤" in (5.2).

Inequalities (C2) and (Cs) are generalizations of (C1). Although inequalities (C1) can be dominated by the others, e.g. by (C2) for $i_1, i_2 \in I, j \in J$ such that $W_{i_2 j} \setminus W_{i_1 j} \ne \varnothing$, Cánovas et al. (2007) chose not to replace all such dominated inequalities (C1), since, according to their experiments, the improvement of LP bounds would not compensate the increase of the number of non-zero elements. However, a particular case is exploited in the reformulation: if $W_{i_1 j} \cap W_{i_2 j} = \varnothing$, inequality (C2) for (i_1, i_2, j) dominates the two (C1) inequalities for (i_1, j) and (i_2, j) while not increasing formulation size (this applies similarly to the generalized (Cs) inequalities). Finally, we note that the inequalities above remain valid when data has preference ties.

W inequalities (García, 2006):

$$\sum_{k \in W_{i_1 j}} x_{i_1 k} \le \sum_{k \in W_{i_1 j}} x_{i_2 k} \text{, for } i_1, i_2 \in I, j \in J. \tag{W}$$

$$\sum_{k \in S} x_{i_1 k} = \sum_{k \in S} x_{i_2 k} \text{, for } i_1, i_2 \in I, j, h \in J\text{: } S = W_{i_1 j} = W_{i_2 h}. \tag{W$_{\text{eq}}$}$$

Proof of validity of (W): If the left-hand side is zero, the inequality is trivially valid. If $\sum_{k \in W_{i_1 j}} x_{i_1 k} = 1$ then, for $k \in J \setminus W_{i_1 j}$, (5.2) implies $x_{i_1 k} = 0$, (5.6) requires $y_k = 0$, and (5.3) implies $x_{i_2 k} = 0$ for all $i_2 \in I$. Thus, for all $i_2 \in I$, $\sum_{k \in J \setminus W_{i_1 j}} x_{i_2 k} = 0$ and (5.2) requires $\sum_{k \in W_{i_1 j}} x_{i_2 k} = 1$.

Proof of validity of (W$_{\text{eq}}$): The equality follows from combining (W) for (i_1, i_2, j) and for (i_2, i_1, h).

Inequality (W) states that assigning i_1 to facilities it prefers less than j implies also assigning all other centers to those facilities (since all others must be closed). Equality (W$_{\text{eq}}$) states that if sites in set S are the worst for both i_1 and i_2 (but possibly ordered differently), then i_1 and i_2 must be either both assigned to facilities in S or both not assigned to facilities in S. The proof of (W) above uses the same arguments as the one by García (2006). We also note that (W) and (W$_{\text{eq}}$) remain valid when data has preference ties.

Inequality (W) for (i_1, i_2, j) dominates inequality (C1) $\sum_{k \in W_{i_1 j}} x_{i_1 k} \leq 1 - y_j$ if $W_{i_1 j} \subseteq W_{i_2 j}$, since $\sum_{k \in W_{i_1 j}} x_{i_1 k} \leq \sum_{k \in W_{i_1 j}} x_{i_2 k} \leq \sum_{k \in W_{i_2 j}} x_{i_2 k} \leq 1 - y_j$, following respectively from (W), $W_{i_1 j} \subseteq W_{i_2 j}$, and (C1) for (i_2, j). However, Cánovas et al. (2007) do not exploit this in the reformulation since replacing all such dominated (C1) inequalities would again not be compensated by the increase of the number of non-zero elements.

Inequalities (W) (actually, slightly less general variants) were independently introduced by Belotti et al. (2007) and Vasilyev et al. (2009), and were exploited to generate cuts in branch-and-cut algorithms by Belotti et al. (2007) and Vasilyev and Klimentova (2010). In this chapter, we exploit (W$_{\text{eq}}$) in an additional step of the reformulation procedure and exploit (W) to generate cuts in a branch-and-cut algorithm. Finally, we note that García (2006) presented additional new valid inequalities for X^{UFLPO} which are not useful for the a priori reformulation but may be useful to generate cuts. However, they were not tested by García (2006), nor in later work by any author, as far as we know.

5.4 Reformulation procedure for X^{UFLPO}

We next present an a priori reformulation procedure for the UFLPO that aims to strengthen the formulation and reduce its size (in terms of number of variables, constraints and non-zero elements) before applying a branch-and-cut algorithm. The procedure depends on the particular data of each instance, specifically on the d_{ij} data underlying CA constraints (5.6), and makes use of inequalities (A$_{\text{eq}}$), (A), (W$_{\text{eq}}$), and (Cs). This procedure is based on work by Cánovas et al. (2007), and is here extended for data with d_{ij} ties and with an additional step.

Step 1 (use (A$_{\text{eq}}$) to strengthen VUB constraints). For $j \in J$ and $i_1, i_2, ..., i_s \in I$, $i_1 < ... < i_s$, such that $N'_{i_1 j} = ... = N'_{i_s j}$ and $E_{i_1 j} = ... = E_{i_s j} = \{j\}$: replace constraint $x_{ij} \leq y_j$ with $x_{ij} = x_{i_1 j}$ for $i = i_2, ..., i_s$. The variables $x_{i_2 j}, ..., x_{i_s j}$ are termed *replaced* in the next step.

Step 2 (use (A) to strengthen VUB constraints). For $j \in J$ and $i_1, i_2 \in I$, $i_1 \neq i_2$, such that $N'_{i_1 j} \supseteq N_{i_2 j} \setminus \{j\}$: replace constraint $x_{i_1 j} \leq y_j$ with $x_{i_1 j} \leq x_{i_2 j}$ if (i) both variables $x_{i_1 j}$ and $x_{i_2 j}$ were not replaced in step 1, and (ii) $x_{i_1 j} \leq x_{i_2 j}$ is not dominated, i.e. no $i_3 \in I$, $i_3 \neq i_1, i_2$, exists such that $x_{i_1 j} \leq x_{i_3 j}$ and $x_{i_3 j} \leq x_{i_2 j}$ also hold. In addition, if more than one such $i_2 \in I$ exists, only one inequality is added so as not to enlarge the model. In this case, the minimum lexicographic index is chosen.

Step 3. New step, described further below.

Step 4 (remove trivially redundant CA constraints). For $i \in I$: (1) if a unique $j \in J$ exists such that $|N_{ij}| = 1$, remove the CA constraint for (i, j), which reduces to $x_{ij} \geq y_j$, and replace VUB constraint $x_{ij} \leq y_j$ with $x_{ij} = y_j$; (2) for $j \in J$ such that $|N_{ij}| = m$, remove the CA constraint for (i, j), which reduces to $1 \geq y_j$.

Step 5 (use (Cs) to strengthen CA constraints). For $j \in J$, let $T \subseteq I$ such that $|T| \geq 2$, $W_{ij} \neq \varnothing$ for all $i \in T$ and $W_{i_1 j} \cap W_{i_2 j} = \varnothing$ for all $i_1, i_2 \in T$. Replace CA constraints $\sum_{k \in N_{ij}} x_{ik} \geq y_j$ for $i \in T$ with $\sum_{i \in T} \sum_{k \in W_{ij}} x_{ik} \leq 1 - y_j$.

Cánovas et al. (2007) describe the following algorithm to find sets *T*. For each $j \in J$: (1) define $S = \{ i \in I : W_{ij} \neq \varnothing \wedge \exists t \in I : (W_{tj} \neq \varnothing \wedge W_{ij} \cap W_{tj} = \varnothing) \}$ and construct a graph (S, A) with set of arcs $A = \{ (i_1, i_2) \in S \times S : W_{i_1 j} \cap W_{i_2 j} = \varnothing \}$; (2) search a clique *T* in the graph and replace CA constraints for $i \in T$ as described above; (3) remove the nodes in *T* from the graph and return to the previous step until no edges remain in the graph. García (2006) includes the following further remarks. This procedure does not search for all possible cliques, in order to avoid increasing model size. In step (2), the clique is selected by adding elements remaining in *S* in lexicographic order (although alternative heuristic rules would be possible).

Step 6 (reduce the number of non-zero elements in CA constraints). For each CA constraint (original or strengthened in the previous step), if $|W_{ij}| < |N_{ij}|$ replace $\sum_{k \in N_{ij}} x_{ik}$ by $1 - \sum_{W_{ij}} x_{ik}$. This changes formulation (5.6) into (C1).

Notes:

- All steps above were part of the original procedure of Cánovas et al. (2007). The conditions of steps 1, 2 and 4 were modified to account for ties in d_{ij} data. The original steps 5 and 6 were valid with such ties, and remain unaltered in this respect. In steps 2 and 5, the original rules were retained to select among multiple alternative inequalities, when they exist. However, step 5 considers only the CA constraints which are not dropped in step 3, which is described below.

- The presolve routine of a MIP optimizer will automatically perform step 4, so this step could be dropped. It will also eliminate redundant variables after steps 1 and 4 even if they are kept in the initial formulation, which is useful to simplify the implementation.
- Church (2003) proposed a reformulation of the *p*-median model, called COBRA, including a step equivalent to step 1 above.

We next introduce an additional reformulation step exploiting equalities ($\mathrm{W_{eq}}$). The following additional notation is used:

CA(*i*, *j*) denotes CA constraints (5.6) for a particular pair of centers and sites (*i*,*j*), and $W_{eq}(i_1,i_2,j)$ is similar notation for ($\mathrm{W_{eq}}$);

$EQ(i_1,i_2,A) := \sum_{k\in A} x_{i_1k} = \sum_{k\in A} x_{i_2k}$ for $i_1,i_2 \in I$, $A \subseteq J$.

We start with some preliminary remarks. For $i_1,i_2 \in I$ and $j_1,j_2 \in J$:

Remark 1: If $A = N_{i_1j_1} = N_{i_2j_2}$, then $W_{eq}(i_1,i_2,j_1)$=$W_{eq}(i_1,i_2,j_2)$=$EQ(i_1,i_2,J\setminus A)$ and, given demand constraints (5.2), this equality is equivalent to $EQ(i_1,i_2,A)$.

Remark 2: If $A = N_{i_1j_1} = N_{i_2j_2}$ and $B = N_{i_1h_1} = N_{i_2h_2}$ with $h_1,h_2 \in J$ and $A \subset B$, then we can write the three equalities $EQ(i_1,i_2,A)$, $EQ(i_1,i_2,B\setminus A)$, and $EQ(i_1,i_2,B)$. Only two are independent and the first two have the least number of non-zero elements.

Remark 3: If $A = N_{i_1j_1} = N_{i_2j_2}$ and equality $EQ(i_1,i_2,A)$ is added to the formulation, then one or more CA constraints become redundant or dominated. In the following, we recall that $EQ(i_1,i_2,A)$:=$\sum_{k\in A} x_{i_1k} = \sum_{k\in A} x_{i_2k}$ and $CA(i,h)$:=$\sum_{k\in N_{ih}} x_{ik} \geq y_h$.

i) For $h \in E_{i_1j_1} \cap E_{i_2j_2}$, and thus $N_{i_1h} = N_{i_2h} = A$, $CA(i_1,h)$ and $CA(i_2,h)$ are equivalent and one can be dropped, e.g. the latter.

ii) For $h \in E_{i_1j_1} \setminus E_{i_2j_2}$, and thus $N_{i_2h} \subset N_{i_1h} = A$, $CA(i_1,h)$ is dominated by $CA(i_2,h)$ and can be dropped.

iii) For $h \in E_{i_2j_2} \setminus E_{i_1j_1}$, and thus $N_{i_1h} \subset N_{i_2h} = A$, $CA(i_2,h)$ is dominated by $CA(i_1,h)$ and can be dropped.

If no d_{ij} ties exist, i.e. $E_{i_1j_1} = \{j_1\}$ and $E_{i_2j_2} = \{j_2\}$, there are two cases: if $j_1 = j_2$, part (i) applies and either $CA(i_1,j_1)$ or $CA(i_2,j_2)$ can be dropped; if

$j_1 \neq j_2$, parts (ii) and (iii) apply and both $CA(i_1, j_1)$ and $CA(i_2, j_2)$ can be dropped.

Regarding the number of non-zero elements when $EQ(i_1, i_2, A)$ is added and redundant CA constraints are dropped: it increases if $|E_{i_1 j_1}| = |E_{i_2 j_2}| = 1$ and $j_1 = j_2$ and $|A| > 1$; otherwise, it does not increase or decreases.

The new step uses the remarks above and is as follows.

Step 3 (add equalities ($\mathbf{W_{eq}}$) and remove redundant CA constraints)

Step 3.1 (add EQ). For $i_1, i_2 \in I | i_2 > i_1$, find $j_h, j'_h \in J$ for $h = 1,...,s$ such that $S_h = N_{i_1 j_h} = N_{i_2 j'_h}$ and $S_0 = \varnothing \subset S_1 \subset ... \subset S_s \subseteq J$. For $h = 1,...,s$, set $T_h = S_h \setminus S_{h-1}$ and add $EQ(i_1, i_2, T_h)$ to the formulation if it is not redundant, i.e. if all of the following are false:

1. $|T_h| = J$. In this case, the only equality that can be written for the pair (i_1, i_2) is dominated by demand constraints (5.2).

2. $|T_h| = 1$. In this case, the equality was already added in step 1.

3. Equalities $x_{i_1 j} = x_{ij}$ for some $i \in I$ and all $j \in T_h$ were added in step 1, and $EQ(i, i_2, T_h)$ was already added in step 3.

4. Equalities $x_{i_2 j} = x_{ij}$ for some $i \in I$ and all $j \in T_h$ were added in step 1, and $EQ(i_1, i, T_h)$ was already added in step 3.

5. Equality $EQ(i, i_1, T_h)$ was already added for some $i \in I : i < i_1$. In this case, $EQ(i, i_2, T_h)$ was also already added or is implied by other added equalities.

6. $|T_h| = |T_m| : m = \arg\max \{k \in 1,...,s : |T_k|\}$. Since one equality will be redundant, given demand constraints (5.2), choose one with the maximum number of elements (if more than one exists, pick the lexicographic minimum m).

Step 3.2 (drop CA). For $i_1, i_2 \in I$, $j_1, j_2 \in J | A = N_{i_1 j_1} = N_{i_2 j_2}$: drop CA constraints according to Remark 3 above. In all cases, $EQ(i_1, i_2, A)$ is valid and is implied by the formulation after step 3.1. Includes the special case $|E_{i_1 j_1}| = |E_{i_2 j_2}| = 1$ and $j_1 = j_2$, corresponding to the equalities added in step 1.

The computational performance of the reformulated model is presented in the results section, including a comparison between the UFLPO and the CM model.

5.5 Valid inequalities for X^{LS}

We next present valid inequalities for X^{LS}. All inequalities are of the general form

$$\sum_{j\in S} y_j - m(S,T) \le \sum_{i\in I\setminus T} \sum_{j\in S} x_{ij}$$

where $S \subseteq J$ and $T \subseteq I$ are given subsets of sites and centers, respectively, and $m(S,T)$ is an upper bound on the number of facilities that can be open in S by being assigned all the demand centers in T. The inequality states that each additional open facility in S in excess of $m(S,T)$ requires the assignment of at least one center not in T. Different upper bounds $m(S,T)$ are derived from the minimum capacity and single assignment constraints.

Additional notation is used: $\lfloor a \rfloor$ and $\lceil a \rceil$ denote the integer round-down and round-up functions, respectively; $u(T) = \sum_{i\in T} u_i$ for $T \subseteq I$.

We start with a simple inequality setting an upper bound on the total number of open facilities implied by constant lower capacity bounds. This is a single inequality that can be added a priori to the formulation.

Proposition 1. The following inequality is valid for X^{LS}:

$$\sum_{j\in J} y_j \le \left\lfloor \frac{\sum_{i\in I} u_i}{b} \right\rfloor \tag{5.9}$$

Proof. Summing minimum capacity constraints (5.4) gives $\sum_{i\in I} u_i \sum_{j\in J} x_{ij} \ge \sum_{j\in J} b_j y_j$. Then using $\sum_{j\in J} x_{ij} = 1$ for $i \in I$ and $b_j = b$ for $j \in J$ gives $\sum_{j\in J} y_j \le \left(\sum_{i\in I} u_i\right)/b$. As the left-hand side is integral, the right-hand side can be rounded down, giving (5.9). ■

Next we introduce a general class of inequalities by using the *bin covering problem* (BCP), also called the *dual bin packing problem* (Labbé et al., 1995; Csirik et al., 2001). In the BCP, there is an unlimited number of bins and indivisible items of given weights, and the aim is to pack items into a maximum number of bins so that each bin receives at least a given minimum weight, equal for all bins.

In the context of this chapter, let $bc(T)$ be the maximum number of facilities that can be open by being assigned only centers in $T \subseteq I$, obtained by solving the following BCP:

$bc(T) = \quad \max \sum_{j\in J} y_j$

s.t. $\quad \sum_{j\in J} x_{ij} = 1, \;\; \forall i \in T$

$$\sum_{i \in T} u_i x_{ij} \geq b y_j, \ \ \forall j \in J$$

$$x_{ij} \in \{0,1\}, \ \ y_j \in \{0,1\}, \forall i \in T, j \in J$$

Proposition 2. Let $S \subseteq J$, $T \subseteq I$ and $bc(T)$ be the optimal value of the bin covering problem with demand set T and minimum capacity b. The following *bin covering inequality* is valid for X^{LS}:

$$\sum_{j \in S} y_j - bc(T) \leq \sum_{i \in I \setminus T} \sum_{j \in S} x_{ij} \tag{5.10}$$

Proof. Suppose that $\sum_{j \in S} y_j = bc(T) + k$. If $k \leq 0$, the inequality is trivially valid. So suppose that $k \geq 1$. By definition, at most $bc(T)$ open facilities can satisfy the minimum capacity constraints (5.4) with only the demand from T. Thus, at least k open facilities in S require additional demand from $I \setminus T$ to satisfy (5.4). Because of single assignment constraints (5.7), this implies $\sum_{i \in I \setminus T} \sum_{j \in S} x_{ij} \geq k$, and thus the inequality is valid. ■

We observe that bin covering inequalities are the analogue for constant minimum capacities of the *bin packing inequalities* studied by Deng and Simchi-Levi (1992) and by Labbé and Yaman (2006) for the polytope of the capacitated facility location problem (CFLP) with single assignment and constant maximum capacities.

Since there are an exponential number of inequalities (5.10), particular cases that can be added to the formulation a priori or as cuts may be useful in practice. To use (5.10) it is necessary to choose sets S and T and to compute $bc(T)$. We observe that $|S| > bc(T)$ must hold for the inequality not to be redundant, and T should be maximal for the inequality to be tight, i.e. $T' \subset T$ such that $bc(T') = bc(T)$ produces a dominated inequality.

If $bc(T)$ is replaced in (5.10) by an upper bound $U \geq bc(T)$, a weaker but still valid inequality is obtained. This is useful to avoid computing $bc(T)$ since the BCP is NP-hard, as noted by Labbé et al. (1995). The same authors present reduction criteria, upper bounds, lower bounds and a branch-and-bound algorithm for the BCP. They assume first that $u_i < b$, $\forall i \in T$. The four upper bounds provided are: $U_0 = \lfloor |T|/2 \rfloor$; $U_1 = \lfloor u(T)/b \rfloor$; and two other bounds U_2 and U_3 that can be computed in $O(n)$ time assuming items are sorted by weight. U_2 dominates the two first bounds, i.e. $U_2 \leq \min\{U_0, U_1\}$, and the best bound is $U = \min\{U_2, U_3\}$.

Note that by using $S = J$ and $T = I$ in (5.10) we get the following inequality dominating (5.9), whose right-hand side is the upper bound U_1:

$$\sum_{j\in J} y_j \le bc(I) \tag{5.11}$$

We now turn our attention to inequalities making use of upper bounds on $bc(T)$.

Proposition 3. Let $S \subseteq J$, $T \subseteq I$. The following inequality is valid for X^{LS}:

$$\sum_{j\in S} y_j - \lfloor u(T)/b \rfloor \le \sum_{i\in I\setminus T} \sum_{j\in S} x_{ij} \tag{LSa}$$

Proof. $\lfloor u(T)/b \rfloor$ is an upper bound on $bc(T)$, following the arguments used in the proof of proposition 1 with *I* replaced by *T*. Replacing $bc(T)$ by that upper bound, the arguments in the proof of proposition 2 remain valid. ■

To use (LSa) it is necessary to choose sets *S* and *T*. For the inequality not to be redundant, it must be $|S| > \lfloor u(T)/b \rfloor$. For the inequality to be tight, *T* should be maximal subject to $u(T) < (\lfloor u(T)/b \rfloor + 1)\cdot b$.

Particular cases of non-dominated inequalities (LSa) for a given *S* with $|S| \le 3$ are:

$$y_j \le \sum_{i\in I\setminus T} x_{ij}\,,\ \ T \subset I\ :\ u(T) < b \tag{LSa10}$$

$$y_{j_1} + y_{j_2} - 1 \le \sum_{i\in I\setminus T} \left(x_{ij_1} + x_{ij_2}\right),\ \ T \subset I\ :\ b \le u(T) < 2b \tag{LSa21}$$

$$y_{j_1} + y_{j_2} + y_{j_3} - 1 \le \sum_{i\in I\setminus T} \left(x_{ij_1} + x_{ij_2} + x_{ij_3}\right),\ \ T \subset I\ :\ b \le u(T) < 2b \tag{LSa31}$$

$$y_{j_1} + y_{j_2} + y_{j_3} - 2 \le \sum_{i\in I\setminus T} \left(x_{ij_1} + x_{ij_2} + x_{ij_3}\right),\ \ T \subset I\ :\ 2b \le u(T) < 3b \tag{LSa32}$$

Proposition 4. Let $S \subseteq J$, $T \subseteq I$, $u_i < b$, $\forall i \in T$. The following inequality is valid for X^{LS}:

$$\sum_{j\in S} y_j - \left\lfloor \frac{|T|}{2} \right\rfloor \le \sum_{i\in I\setminus T} \sum_{j\in S} x_{ij} \tag{LSb}$$

Proof. $\lfloor |T|/2 \rfloor$ is an upper bound on $bc(T)$ if $u_i < b$ for $i \in T$ (Labbé et al., 1995), and the arguments in the proof of proposition 2 remain valid with $bc(T)$ replaced by that bound. An alternative proof is as follows. Suppose that $\sum_{j\in S} y_j = \lfloor |T|/2 \rfloor + k$. If $k \le 0$, the inequality is trivially valid. So suppose that $k \ge 1$. To satisfy minimum capacity constraints (5.4) and single assignment constraints (5.7), and given that $u_i < b$ for $i \in T$, each open facility in *S* must be assigned either (i) at least two centers in *T*, or (ii) at most one center in *T* and at least one center not in *T*. The number of open facilities in *S* satisfying (i) is at most $\lfloor |T|/2 \rfloor$, and satisfying (ii) is at least *k*. This implies that $\sum_{i\in I\setminus T} \sum_{j\in S} x_{ij} \ge k$, and thus the inequality is valid. ■

To use (LSb) it is necessary to choose sets S and T. For the inequality not to be redundant, it must be $|T| < 2|S|$. For the inequality to be tight, $|T|$ should be odd (given S and T with $|T|$ even, adding an element to T produces a tighter inequality). Comparing (LSb) and (LSa), if $|S| \leq 2$ then (LSb) is dominated by (LSa), specifically by (LSa10) or (LSa21). If $|S| \geq 3$, no inequality always dominates the other.

Particular cases of inequalities (LSb) for a given S with $|S| = 3$ and for $|T| = 3$ are:

$$y_{j_1} + y_{j_2} + y_{j_3} - 1 \leq \sum_{i \in I \setminus T} \left(x_{ij_1} + x_{ij_2} + x_{ij_3} \right), \ T \subset I : |T| = 3, u_i < b, \forall i \in T \qquad \text{(LSb31)}$$

Inequality (LSb31) is only useful if $u(T) \geq 2b$, otherwise it is dominated by (LSa31) or by a combination of (LSa10) for all elements of S.

5.6 Valid inequalities for X^{LSI}

We now consider set X^{LSI}, i.e. the restriction of X^{LS} assuming $I = J$ and $x_{ii} = y_i$ for $i \in I$. These conditions allow some of the previous inequalities to be strengthened.

Proposition 5. Let $S \subseteq T \subseteq I$, $u_i < b$, $\forall i \in S$. The following inequality is valid for X^{LSI}:

$$2\sum_{j \in S} y_j - |T| \leq \sum_{i \in I \setminus T} \sum_{j \in S} x_{ij} \qquad \text{(LSc)}$$

or, equivalently,

$$2\left(\sum_{j \in S} y_j - \left\lfloor \frac{|T|}{2} \right\rfloor \right) \leq \sum_{i \in I \setminus T} \sum_{j \in S} x_{ij}, \text{ if } |T| \text{ is even}$$

$$2\left(\sum_{j \in S} y_j - \left\lfloor \frac{|T|}{2} \right\rfloor \right) - 1 \leq \sum_{i \in I \setminus T} \sum_{j \in S} x_{ij}, \text{ if } |T| \text{ is odd}$$

Proof. Suppose that $\sum_{j \in S} y_j = \lfloor |T|/2 \rfloor + k$. If $k \leq 0$, the inequality is trivially valid. So suppose that $k \geq 1$. Thus there are $A = \lfloor |T|/2 \rfloor + k$ centers in S with open facilities, assigned to their own facilities, as $x_{ii} = y_i$ for $i \in I$. Since by assumption $u_i < b$ for $i \in S$, in order to satisfy minimum capacity constraints (5.4) and single assignment constraints (5.7), each open facility in S must additionally be assigned either (i) at least one other center in T or (ii) at least one center in $I \setminus T$. The number of open facilities in S satisfying (i) is at most $B = |T| - A$, and satisfying (ii) is at least $A - B = 2A - |T| = 2\lfloor |T|/2 \rfloor + 2k - |T|$. This implies that $\sum_{i \in I \setminus T} \sum_{j \in S} x_{ij} \geq 2k$ if $|T|$ is even, or $\sum_{i \in I \setminus T} \sum_{j \in S} x_{ij} \geq 2k - 1$ if $|T|$ is odd, and thus the inequality is valid. ■

To use (LSc) it is necessary to choose sets S and T and it must be $|T| < 2|S|$ for the inequality not to be redundant. Comparing (LSc) and (LSb) for given S and T, if $|T|$ is even then (LSc) dominates (LSb). If $|T|$ is odd, the left-hand side of (LSc) is higher than the one of (LSb) when $\sum_{j\in S} y_j - \lfloor |T|/2 \rfloor > 1$; thus, (LSc) is dominated by (LSb) if $|S| - \lfloor |T|/2 \rfloor \le 1$ (e.g. $|S| = 2, |T| = 3$) and $u_i < b$ for $i \in T$ (this condition is required by (LSc) only for $i \in S$); otherwise, no inequality always dominates the other.

Particular cases of inequalities (LSc) for a given S with $|S|$ =2 or 3 and $T = S$ are:

$$2\left(y_{j_1} + y_{j_2} - 1\right) \le \sum_{i \in I\setminus T} \left(x_{ij_1} + x_{ij_2}\right),\ T = S \subset I:\ u_i < b,\ \forall i \in S \qquad \text{(LSc21)}$$

$$2\left(y_{j_1} + y_{j_2} + y_{j_3} - 1\right) - 1 \le \sum_{i \in I\setminus T} \left(x_{ij_1} + x_{ij_2} + x_{ij_3}\right),\ T = S \subset I:\ u_i < b,\ \forall i \in S \qquad \text{(LSc31)}$$

5.7 Separation procedures

In this section we present separation procedures for some of the valid inequalities from the previous sections in order to exploit them in a branch-and-cut algorithm. The separation problem associated with a given family of inequalities F for a formulation P is defined as follows. Given a fractional solution $(x^*, y^*) \in R_+^{n\times m} \times R_+^m$ obtained by solving the LP relaxation of P, find an inequality of family F that is violated by (x^*, y^*) or show that no such inequality exists.

For X^{LS} and X^{LSI}, we restricted our attention to inequalities with sets of small cardinality $|S| \le 3$, corresponding to particular cases presented before. It is also assumed that $I = J$ and that demand and capacity data is integer.

Separation of inequalities (LSa10). For each $j \in J$ such that $u_j < b$ and $y_j^* > 0$, find the set T that maximizes $\sum_{i\in T} x_{ij}^*$ subject to $u(T) \le b - 1$, which is a binary knapsack problem. If $y_j^* > \sum_{i\in I\setminus T} \sum_{j\in S} x_{ij}^*$, the inequality is violated and is added to the formulation. Separation run time is $O(n^2 \log n)$ if the knapsack problem is solved approximately by the standard greedy heuristic (Martello and Toth, 1990), requiring $O(n \log n)$ time.

Separation of inequalities (LSa21). For each $S \subset J$ such that $|S| = 2$, $u(S) < 2b$ and $\sum_{j\in S} y_j^* > 1$, find the set T that maximizes $\sum_{i\in T} \sum_{j\in S} x_{ij}^*$ subject to $u(T) \le 2b - 1$, which is a binary knapsack problem. If $\sum_{j\in S} y_j^* - 1 > \sum_{i\in I\setminus T} \sum_{j\in S} x_{ij}^*$, the inequality is violated and is added to the formulation. Separation run time is $O(n^3 \log n)$ using the greedy knapsack heuristic.

Separation of inequalities (LSa31). For each $S \subset J$ such that $|S| = 3$, $u(S) < 2b$ and $\sum_{j\in S} y_j^* > 1$, find the set T as for inequalities (LSa21). If the inequality is violated, add it to the formulation. Separation run time is $O(n^4 \log n)$ using the greedy knapsack heuristic.

Separation of inequalities (LSb31) and (LSc31). For each $S \subset J$ such that $|S| = 3$, $u_i < b$ for $i \in S$, and $\sum_{j \in S} y_j^* > 1$, check if inequality (LSb31) is violated for $T = S$. If so, add it to the formulation. If $\sum_{j \in S} y_j^* > 2$, then inequality (LSc31) is violated by a larger amount and is added instead. Separation run time is $O(n^3)$.

Separation of inequalities (W). The procedure of Belotti et al. (2007) was used. For each $i_1 \in I$, find $i_2 \in I$ and $j \in J$ that most violate inequality (W), and add all found violated inequalities to the formulation. Running time is $O(n^3)$ and up to n inequalities are added per iteration.

In the computational experiments performed, the greedy knapsack heuristic was replaced by the exact procedure MT1 of Martello and Toth (1990), since this was still very fast for the instance sizes tested.

5.8 Computational experiments

Computational experiments were carried out with the CM model. The objective was to compare three solution methods using a generic MIP optimizer implementing a branch-and-cut algorithm:

- S – standard formulation;
- R – a priori reformulation;
- RC – a priori reformulation and special purpose cuts (or *user cuts*) in addition to automatic cuts.

The software used was FICO's Xpress Optimization Suite version 2005B (released in Nov. 2005). The model formulation and code for the reformulation and separation (cut generation) procedures were implemented with Xpress Mosel 1.6, and the model was solved with Xpress MIP Optimizer 16.10. The computer used had a Pentium M 755 2.0 GHz CPU, 1.0 GB of memory, and Windows XP operating system.

Formulation (5.1)-(5.8) was used with the following assumptions: identical sets of centers and sites ($I = J$); constant minimum capacities ($b_j = b$ for all sites j); non-binding maximum capacities (B_j is greater or equal to total demand for all sites j), and thus constraints (5.5) were dropped from the formulation. Constraint (5.9) was added a priori to the formulation. The simple upper bound in (5.9) could have been replaced by the tightest bin covering upper bound of Labbé et al. (2005) mentioned in section 5.5, but this would make no difference in our tests since the two bounds were equal in the instances tested.

Regarding user cuts, after preliminary experiments the following procedure was adopted. Cuts are generated from inequalities (W) and (LSa10) only, as the others referred to in the previous section led to increased solution times in preliminary experiments. Cuts are generated at the top node only (after automatic cut generation). Only cuts violated more than a given threshold are added (0.01 was used). In each round, cuts of a single type are added. All added cuts remain in the formulation, including those that become inactive in later rounds.

Regarding solver parameters, branching priority was given to *y* variables (in preliminary tests this reduced solution time significantly, by 5-60%). A time limit of 1 hour was imposed. All other parameters were left at default values, except for method RC where presolve was disabled (to avoid interference of variable elimination with user cut generation).

Test instances were randomly generated as follows. First, for a given size *n* (number of centers), 9 data sets (c_{ij}, u_i) were created: points representing centers were uniformly generated in [0,100]×[0,100]; $u_i = 1000/n \times Uniform[0.1,1.9]$; $c_{ij} = u_i \times d_{ij}$, where d_{ij} is the Euclidean distance between centers *i* and *j*. Then, the capacity ratio *r*, equal to total demand divided by the minimum capacity (i.e. the expected maximum number of open facilities), was used as a control parameter to derive 4 instances from each data set (c_{ij}, u_i) by setting $b = \sum_{i \in I} u_i / r$ for $r = n \times 0.1$, 0.2, 0.3, and 0.4. All data was rounded to integer values.

Results are reported in the tables below, with the following definitions:

- n-r: instance group, defined by size *n* and capacity ratio *r*;
- inst.: number of the individual instance;
- IP: optimal value;
- nf: number of facilities in an optimal integer solution;
- finished: 1 if the instance was solved to optimality within the time limit, 0 otherwise;
- time: total solution time in seconds; includes the time for the reformulation procedure and user cut generation, if applicable (note: the reported time of unfinished instances may exceed the imposed limit of 3600, since this excludes time for the a priori reformulation procedure and the first LP relaxation);
- nodes: number of nodes in the branch-and-bound tree;

- gaps: gapLP=(IP-LP)/IP, gapXLP=(IP-XLP)/IP, gap closed=(XLP-LP)/(IP-LP), where LP is the value of the first LP relaxation, XLP is the value of the LP relaxation at the top node after addition of cuts, and IP is the optimal value;
- acuts and ucuts: number of cuts added at the top node by the automatic and user procedures, respectively; the number of generated automatic cuts may be higher, as cuts that become inactive in successive cut generation iterations are deleted;

Table 5.1 and Table 5.2 summarize results of individual instances in Table 5.3, where:

- "avg" denotes arithmetic averages of results of individual instances;
- gap closed with methods S, R and RC is relative to the LP value with method S;
- time ratio A/B is the time with method A divided by the time with method B, computed for individual instances.

First we observe that instances with smaller capacity ratios (*r/n*=0.1 and 0.2) are harder to solve than those with larger ratios (*r/n*=0.3 and 0.4). Method R relatively to S solves more instances within the time limit and reduces time by about 60% on average and by 20-90% for different instance groups (Table 5.1). LP and XLP gaps are also reduced (Table 5.2) and the number of nodes decreases significantly (Table 5.1), considering instance groups with comparable number of finished instances. Method RC relatively to R helps at further reducing XLP gaps and the number of nodes (but LP gaps with RC are larger, because presolve is disabled). Solution time decreases by about 30% relatively to method R for instances with *r/n*=0.2 (70-14 and 100-20), but for other instances time increases or is only slightly reduced. Thus we conclude that the reformulation procedure is very effective, while the user cuts are not generally effective.

The explanation for the poor performance of user cuts is mainly that these cuts are relatively weak, i.e. the additional LP gap closed relatively to automatic cuts is small. For most instance groups, this gap reduction is not compensated by the increase in number of rows and non-zero elements. It was verified that positive time differences between methods RC and R are not explained by presolve being disabled in RC (if it is also disabled in R, differences are still positive), nor by user cut generation time being excessive (it is lower than those differences). Also, if inactive cuts are deleted in user cut generation routines (instead of being managed automatically by the solver), the solution time is similar.

Table 5.1: Summary of results – nodes and time

n-r	total finished			avg nodes			avg time			avg time ratios		
	S	R	RC	S	R	RC	S	R	RC	R/S	RC/S	RC/R
70-7	9	9	9	2035	1529	1195	661	262	409	0.45	0.74	1.64
70-14	8	9	9	9423	3106	1080	1771	383	228	0.27	0.18	0.70
70-21	9	9	9	1430	279	170	241	47	51	0.29	0.32	1.14
70-28	9	9	9	309	13	8	64	17	22	0.41	0.51	1.26
70 total	35	36	36	3300	1232	613	685	178	178	0.36	0.44	1.18
100-10	1	3	2	1430	3728	1776	3495	2956	2951	0.83	0.83	0.96
100-20	1	6	8	2236	6022	1969	3344	2261	1323	0.64	0.38	0.67
100-30	4	9	9	3618	949	184	2819	292	192	0.11	0.09	0.90
100-40	8	9	9	4275	101	63	1723	80	95	0.10	0.12	1.22
100 total	14	27	28	2889	2700	998	2845	1397	1140	0.42	0.35	0.94

Table 5.2: Summary of results – gaps

n-r	avg gapLP (%)			avg gapXLP (%)			avg gap closed (%)		
	S	R	RC	S	R	RC	S	R	RC
70-7	5.6	5.6	5.6	4.8	4.5	3.6	19	25	39
70-14	9.9	9.7	9.7	6.4	4.9	3.3	36	52	68
70-21	13.7	12.7	13.5	4.4	2.5	1.7	70	83	88
70-28	15.5	13.5	20.6	3.1	1.0	0.8	80	94	95
70 total	11.2	10.3	12.3	4.7	3.2	2.4	51	64	73
100-10	6.9	6.7	6.7	6.2	5.5	4.3	12	21	38
100-20	11.4	10.9	10.9	7.0	4.4	2.8	39	62	77
100-30	16.3	14.0	16.1	6.0	3.0	2.0	64	82	88
100-40	18.6	14.5	23.5	5.7	1.6	1.3	70	92	93
100 total	13.3	11.5	14.3	6.2	3.6	2.6	46	64	74

Table 5.3: Detailed results

n-r	inst.	IP	nf	finished			time			nodes			gapLP (%)			gapXLP (%)			acuts			ucuts
				S	R	RC	S	R	RC	S	R	RC	S	R	RC	S	R	RC	S	R	RC	RC
70-7	1	15 389	7	1	1	1	29	23	55	21	21	182	1.5	1.4	1.4	0.8	0.6	0.5	25	30	24	65
	2	14 525	6	1	1	1	96	32	41	236	105	104	1.7	1.6	1.6	1.3	1.2	1.1	20	24	24	49
	3	15 298	6	1	1	1	743	184	544	2 418	1 081	2 053	7.8	7.8	7.8	6.9	6.5	5.4	30	69	78	233
	4	14 776	6	1	1	1	1 321	530	561	3 458	3 549	2 263	6.9	6.9	6.9	6.1	5.9	4.6	48	59	64	238
	5	15 146	6	1	1	1	945	434	814	3 139	2 044	2 163	9.1	9.0	9.0	8.0	6.9	5.3	51	104	116	354
	6	15 100	6	1	1	1	1 631	644	906	5 323	4 348	2 127	8.9	8.8	8.8	8.2	7.4	5.5	65	48	95	381
	7	16 337	6	1	1	1	85	35	41	278	175	71	3.3	3.2	3.2	2.5	2.3	1.8	15	23	26	86
	8	15 567	6	1	1	1	130	76	75	325	400	183	3.1	3.0	3.0	2.6	2.6	2.4	21	25	31	105
	9	14 896	6	1	1	1	971	400	645	3 121	2 034	1 606	8.1	8.1	8.1	7.1	6.8	5.6	54	83	84	281
70-14	1	10 377	14	1	1	1	1 037	316	130	5 855	2 651	280	9.9	8.9	8.9	6.5	4.7	2.2	107	138	162	231
	2	9 245	13	1	1	1	116	74	62	373	350	56	5.1	5.1	5.1	2.9	2.0	0.7	95	94	138	153
	3	9 792	13	1	1	1	225	44	41	1 030	101	15	6.2	5.9	5.9	3.5	2.6	1.4	95	115	143	179
	4	9 887	12	1	1	1	2 267	245	345	12 152	1 903	2 387	9.6	9.5	9.5	6.3	4.4	3.8	158	149	140	119
	5	9 579	13	1	1	1	1 755	345	186	8 916	2 793	649	12.3	12.2	12.2	8.1	6.7	5.0	108	146	175	173
	6	9 716	12	1	1	1	2 886	635	341	15 584	5 291	1 406	12.1	12.1	12.1	7.2	5.0	4.1	147	177	174	200
	7	10 162	11	0	1	1	3 601	535	429	17 645	4 066	2 282	13.3	13.0	13.0	9.2	7.3	5.0	113	154	168	213
	8	9 382	12	1	1	1	667	215	97	2 562	1 607	235	10.4	10.0	10.0	6.9	5.6	3.5	111	100	103	182
	9	10 023	12	1	1	1	3 386	1 037	424	20 694	9 190	2 407	10.3	10.2	10.2	7.0	5.5	4.2	114	143	138	191
70-21	1	7 137	21	1	1	1	26	18	21	36	23	13	10.4	9.6	10.7	1.3	0.7	0.7	110	78	102	27
	2	7 188	17	1	1	1	146	57	49	750	197	26	10.7	9.7	10.5	3.5	2.1	0.9	157	153	166	90
	3	7 933	18	1	1	1	178	44	42	1 017	93	17	17.4	15.4	17.3	4.5	1.8	0.8	153	129	151	102
	4	7 725	17	1	1	1	350	42	71	2 053	67	209	13.8	13.0	13.2	4.5	1.9	1.5	142	155	149	131
	5	6 624	18	1	1	1	132	21	25	741	42	7	13.9	13.2	13.7	5.4	1.9	1.1	88	71	81	54
	6	7 156	18	1	1	1	137	24	31	991	43	66	9.8	9.1	9.8	2.8	1.4	1.1	122	112	126	63
	7	7 456	17	1	1	1	372	98	68	2 561	1 000	259	17.9	17.8	18.1	8.3	6.5	5.6	118	107	111	124
	8	6 862	17	1	1	1	43	19	22	57	5	5	12.0	9.6	11.5	1.8	0.5	0.4	140	98	82	36
	9	7 655	18	1	1	1	789	100	129	4 667	1 037	927	17.1	16.6	17.1	7.2	5.4	3.7	130	116	121	119
70-28	1	5 999	25	1	1	1	22	13	13	23	25	7	13.4	11.8	23.3	1.5	0.9	0.6	83	56	97	15
	2	5 979	23	1	1	1	59	20	35	240	17	7	16.7	14.5	20.7	4.4	0.7	0.4	127	99	130	39
	3	6 090	24	1	1	1	38	18	20	97	1	1	15.9	12.9	19.9	3.1	0.1	0.1	113	76	110	29
	4	6 179	23	1	1	1	60	20	26	253	25	27	17.3	17.0	21.3	3.2	2.5	2.3	127	89	116	11
	5	5 448	22	1	1	1	21	13	17	33	13	3	14.2	11.3	17.8	3.7	0.7	0.9	64	55	58	25
	6	6 430	22	1	1	1	267	27	32	1 829	13	8	20.1	17.4	27.6	5.1	0.9	0.9	147	113	144	72
	7	5 815	21	1	1	1	36	13	18	83	5	1	14.6	13.1	19.7	2.0	0.4	0.7	119	54	96	40
	8	5 090	22	1	1	1	25	10	10	62	1	1	11.3	8.4	14.3	1.9	0.0	0.0	85	50	87	0
	9	5 953	22	1	1	1	51	19	25	160	19	19	16.1	15.5	20.8	3.4	2.4	1.7	119	93	100	48
100-10	1	25 848	10	0	0	0	3 609	3 637	3 634	1 326	4 578	1 920	8.8	8.8	8.8	7.9	7.6	6.3	41	69	92	298
	2	22 292	9	1	1	1	2 592	898	655	1 770	1 676	541	3.1	3.0	3.0	2.8	2.5	2.1	32	40	75	155
	3	23 295	9	0	0	0	3 610	3 636	3 634	1 469	4 449	2 042	7.6	7.3	7.3	6.9	5.9	5.0	59	106	101	272
	4	23 207	10	0	1	1	3 607	554	463	1 777	805	345	4.0	3.9	3.9	3.2	2.7	2.2	52	113	125	263
	5	22 356	9	0	0	0	3 607	3 637	3 635	1 035	2 570	1 365	9.5	9.3	9.3	9.1	8.4	5.4	55	111	108	430
	6	22 892	8	0	1	0	3 607	3 329	3 632	1 460	6 443	2 975	5.7	5.7	5.7	4.8	4.6	3.7	47	80	90	244
	7	25 621	10	0	0	0	3 607	3 637	3 633	1 317	4 947	2 646	8.2	8.1	8.1	6.9	5.9	4.3	76	108	132	335
	8	24 455	9	0	0	0	3 609	3 638	3 635	1 455	4 014	2 086	7.6	7.0	7.0	6.8	5.6	4.3	56	82	95	304
	9	24 949	9	0	0	0	3 608	3 638	3 635	1 259	4 072	2 068	8.0	7.7	7.7	7.4	6.6	5.6	75	101	115	358
100-20	1	15 492	18	0	0	1	3 604	3 634	1 181	2 281	10 848	1 920	11.8	11.3	11.3	7.4	5.3	3.8	198	177	203	177
	2	14 341	16	0	1	1	3 604	994	950	2 470	2 197	820	9.9	9.6	9.6	6.5	4.1	2.6	179	218	234	238
	3	14 687	17	0	1	1	3 604	2 136	875	2 037	5 785	1 018	11.9	11.5	11.5	7.1	4.6	2.5	263	236	257	234
	4	15 324	19	0	1	1	3 604	3 248	2 046	3 395	12 151	5 239	12.1	11.8	11.8	7.0	5.2	3.5	208	185	180	202
	5	14 076	16	0	1	1	3 605	1 629	601	2 092	2 348	177	10.2	9.8	9.8	7.2	3.6	0.9	208	280	294	381
	6	15 181	17	0	0	1	3 605	3 635	1 893	1 662	7 825	2 073	9.7	9.4	9.4	6.2	4.6	3.1	247	251	255	291
	7	16 812	19	0	0	0	3 604	3 634	3 631	2 466	8 682	5 870	19.0	18.0	18.0	12.8	7.5	5.7	249	230	234	264
	8	13 959	17	1	1	1	1 263	151	208	937	23	9	7.2	6.7	6.7	2.6	0.7	0.3	195	210	179	92
	9	14 506	18	0	1	1	3 604	1 288	520	2 780	4 341	592	10.3	10.2	10.3	6.5	4.3	3.0	193	172	243	117
100-30	1	11 750	26	0	1	1	3 604	250	225	4 943	818	349	16.4	13.8	16.7	5.6	3.4	2.5	181	147	175	76
	2	11 161	24	0	1	1	3 604	331	231	2 420	659	53	17.8	13.7	16.7	7.7	4.1	2.0	214	175	228	117
	3	10 432	24	1	1	1	1 667	74	75	2 924	25	15	15.0	12.8	15.0	3.8	1.0	1.0	206	125	132	27
	4	11 188	28	1	1	1	1 901	138	166	3 734	209	205	15.1	12.4	15.7	4.7	2.8	2.7	210	146	158	43
	5	10 481	22	0	1	1	3 603	1 054	307	2 568	4 568	245	19.5	18.5	19.5	10.1	4.6	2.9	237	171	209	131
	6	11 204	24	1	1	1	3 357	141	173	3 813	45	19	11.9	9.9	10.9	4.5	1.5	0.7	261	204	245	94
	7	11 619	26	0	1	1	3 604	344	259	8 268	1 995	721	19.9	17.9	19.7	8.3	5.2	3.5	173	126	176	95
	8	11 017	23	0	1	1	3 604	177	165	3 416	132	39	16.1	14.4	15.7	5.6	2.0	1.9	255	205	209	51
	9	10 998	25	1	1	1	427	116	129	474	93	11	15.1	12.2	14.9	3.9	2.3	0.8	186	110	173	55
100-40	1	9 160	33	1	1	1	629	110	95	1 807	328	135	19.3	16.7	21.6	4.8	3.7	3.3	145	65	94	30
	2	9 219	30	0	1	1	3 604	96	132	7 138	69	5	20.9	15.5	26.7	9.3	1.3	0.5	189	157	178	33
	3	8 942	32	1	1	1	2 300	77	87	6 329	82	44	19.2	16.0	24.2	7.9	1.6	0.7	144	97	131	19
	4	9 050	33	1	1	1	2 209	94	110	10 372	288	135	18.2	14.3	26.7	5.6	2.7	2.1	141	70	109	29
	5	7 992	29	1	1	1	2 380	65	88	4 814	9	7	19.3	15.1	22.0	7.2	0.8	0.7	144	80	127	0
	6	9 618	30	1	1	1	3 402	114	125	6 247	37	38	19.1	14.8	23.6	6.7	2.0	0.7	182	126	158	83
	7	9 170	32	1	1	1	282	55	83	617	92	201	17.6	13.1	24.2	3.3	2.2	3.0	123	48	90	8
	8	8 661	31	1	1	1	492	55	53	885	1	1	18.4	13.7	22.0	4.1	0.0	0.0	175	107	122	0
	9	8 736	31	1	1	1	211	51	77	264	3	5	15.7	10.8	20.4	2.5	0.2	0.3	164	70	122	11

We now turn to a more detailed discussion of the a priori reformulation procedure. Statistics are presented in Table 5.4, comparing our results for the CM model and the results of Cánovas et al. (2007) for the UFLPO (which refer to averages of sets of instances of sizes 50x50, 75x50 and 100x75). Statistics for the CM model are homogeneous for all capacity ratios (the reformulation does not depend on capacity data), except solution time, which is an average of all capacity ratios.

Recall that in the CM model the coefficients d_{ij} underlying the closest assignment constraints represent distances or costs (and objective function coefficients c_{ij} depend monotonically on them), while in the UFLPO they represent arbitrary preferences. In spite of this difference, for the instances tested, the two models have similar percentages of tightened VUB constraints (80-90%) and reductions of the number of non-zero elements (about 45%). The reformulation is very effective at reducing solution time for both models: 50-60% for the UFLPO, about 60% for the CM model.

Adding step 3 to the reformulation of the CM model further reduces the number of non-zero elements (while equalities are added in this step, many CA constraints are removed) and further reduces solution time (by 5-20% depending on capacity ratio and relatively to the reformulation, as shown in data further below).

The run time of the reformulation procedure for the CM model was 7 seconds for size n=70 and 28 seconds for n=100 (about 1% of the average total solution time with the standard formulation).

Table 5.4: Statistics of the reformulation procedure

	CM model (size n=70)	UFLPO (Cánovas et al., 2007)
Percentage of variable upper bound constraints (5.3) tightened in steps 1,2	93%	83-91%
Percentage of closest assignment constraints (5.6) tightened in step 5	7%	not reported
Number of equalities added in step 3 relative to the original number of closest assignment constraints	9%	not applicable
Percentage of closest assignment constraints (5.6) removed in steps 3, 4 and 5	9% without step 3, 28% with step 3	15-19% (without step 3)
Reduction of the number of non-zero elements (before LP presolve)	44% without step 3, 48% with step 3	43-45%
Reduction of total solution time	57% without step 3, 64% with step 3	49-58%

In Table 5.5 and Table 5.6 the following methods are compared:

- S: standard formulation;
- R12: reformulation with steps 1+2 only;
- R-35: reformulation with steps 1+2+4+6 (i.e. all steps except 3, 5);
- R-5: reformulation with steps 1-6 except 5;
- R-3: reformulation with steps 1-6 except 3;
- R: reformulation with all steps 1-6.

It can be seen that steps 1+2 (dedicated to VUB constraints) contribute the most to the overall performance of the reformulation. Step 6 in addition is also very effective at reducing time (R12 vs. R-35 in Table 5.5, assuming step 4 has negligible effect, since the MIP optimizer's presolve performs it automatically). Step 3 further reduces time by about 15% on average (R/R-3 in Table 5.5; about 20% for R-5/R-35).

Step 5 generally degrades the performance of the reformulation (R-3 vs. R-35 or R vs. R-5 in Table 5.5). This result was unexpected, but its cause was not analyzed. Thus, for the CM model and the type of data tested, step 5 should be excluded, i.e. method R-5 should be used. Unfortunately, the individual impact of step 5 was not assessed in our preliminary experiments (we had tested only R12, R-3 and R), and for this reason all the results with methods R and RC presented above included step 5 (however, this does not change any of the conclusions given before).

LP gaps with the reformulation are reduced only slightly, but XLP gaps (after adding cuts) are reduced more significantly. Step 3 helps to slightly reduce XLP gaps (R-3 vs. R in Table 5.6). An unexpected increase of LP gap is observed when step 3 is added for instance group 70-28. This is due to the operation of the automatic LP presolve (if disabled, LP gaps with step 3 are never higher than without it), but the precise cause was not further analyzed. Also, some unexpected increases of XLP gaps are observed when step 3 or step 5 are added; the increases are small and were not further analyzed.

Table 5.5: Reformulation steps – nodes and time

n-r	avg time						avg time ratios						
	S	R12	R-3	R	R-35	R-5	R12/S	R-3/S	R/S	R-35/S	R-5/S	R/R-3	R-5/R-35
70-7	661	446	302	262	263	249	0.70	0.50	0.45	0.43	0.40	0.95	0.95
70-14	1771	831	498	383	540	342	0.54	0.32	0.27	0.32	0.23	0.83	0.75
70-21	241	82	72	47	59	39	0.39	0.36	0.29	0.28	0.21	0.78	0.72
70-28	64	23	23	17	17	12	0.52	0.53	0.41	0.36	0.25	0.77	0.70
70 total	685	345	224	178	220	160	0.54	0.43	0.36	0.35	0.27	0.83	0.78

Table 5.6: Reformulation steps – gaps

n-r	avg gapLP (%)						avg gapXLP (%)					
	S	R12	R-3	R	R-35	R-5	S	R12	R-3	R	R-35	R-5
70-7	5.6	5.6	5.6	5.6	5.6	5.6	4.8	4.5	4.6	4.5	4.5	4.4
70-14	9.9	9.8	9.8	9.7	9.8	9.7	6.4	5.2	5.4	4.9	5.3	4.8
70-21	13.7	13.0	13.0	12.7	13.0	12.6	4.4	2.8	2.9	2.5	2.9	2.5
70-28	15.5	12.9	12.9	13.5	12.9	13.8	3.1	1.3	1.3	1.0	1.3	1.0
70 total	11.2	10.3	10.3	10.3	10.3	10.4	4.7	3.4	3.5	3.2	3.5	3.2

5.9 Conclusion

In this chapter we presented a specialized solution method for the CM model, composed of an a priori reformulation procedure and cut generation within a branch-and-cut algorithm using previously known and new valid inequalities. The reformulation procedure is very effective at reducing solution times relatively to a generic MIP optimizer (average reductions of 20-90% depending on capacity data). New valid inequalities were presented for the CM model, but these and the other known inequalities exploited to generate cuts turned out not to be effective at further reducing solution time in general, i.e. for all types of capacity data.

The reformulation procedure was based on previous work by Cánovas et al. (2007), but it was here extended in two ways: it was adapted to cope with distance or preference ties; a new step was added that further strengthens and reduces the formulation. The extended procedure may be applied to solve the UFLPO or any other model with a feasible set contained in that of the UFLPO.

The computational experiments reported in this chapter assumed a CM model variant with constant capacity lower bounds and no capacity upper bounds. We now comment on the generalization of these assumptions. The reformulation procedure is independent of capacity constraints and we expect it to remain effective for other capacity data, although this was not tested. Among the inequalities used to generate user cuts, (W) and

(LSa10) were found to be the most useful and both can be used with variable capacity lower bounds – the first is independent of capacities; the second relies on the capacity of a single facility. Regarding upper capacity bounds, if these are added all inequalities presented here remain valid and the solution method can be applied unaltered. If the upper capacity bounds are not tight, they may have little influence on computation time and on solutions, since the objective function generally improves with an increasing number of facilities (whose upper bound is determined by the lower capacity bounds). Otherwise, with tight upper capacity bounds, the branch-and-cut procedure may also benefit from the generation of cuts developed for the capacitated facility location problem – see e.g. Aardal (1998a), Labbé and Yaman (2006), Avella (2009).

We remark that there are location models related to CM not requiring all demand centers to be satisfied, i.e. "=" is replaced by "≤" in demand constraints (5.2), but that otherwise include constraints on minimum capacity, closest assignment (with formulation (C1) or an equivalent instead of (5.6)) and single assignment. Such is the case of the models of Verter and Lapierre (2002) and Smith et al. (2009), both applied to the location of health care facilities. The reformulation procedure for the UFLPO is not valid for those models since it relies on inequalities for which constraints (5.2) must hold with "=" (except step 5, which remains valid). On the other hand, inequalities for relaxation X^{LS} remain valid for such models and may be useful to generate cuts.

Possible future work includes developing other separation procedures for the inequalities proposed for relaxation X^{LS}, to better exploit them. Additional possible future work is to adapt the reformulation procedure and valid inequalities for a multiple-service hierarchical extension of the CM model, such as the one presented by Teixeira and Antunes (2008). In this model, centers have demands for multiple services, with each service type requiring independent assignment to facilities and being subject to closest assignment constraints. At least the adaptation of the reformulation procedure should be relatively straightforward, since valid inequalities for the UFLPO can be applied to each service type separately. However its computational effectiveness remains to be tested.

5.10 Appendix – Results with Xpress 7.2

The results reported before were obtained with Xpress suite version 2005B (released in Nov. 2005 and including Xpress MIP Optimizer 16.10), the latest available to us when the original experiments were carried out. Here we report results obtained with the more recent Xpress suite version 7.2 (released in May 2011 and including Xpress MIP Optimizer 22.01) in order to check if the previous conclusions on the relative performance of methods S, R and RC remain valid.

Tests with Xpress 7.2 were run on the same computer as before, thus solution times are comparable between solver versions. As before, all solver parameters were left at default values, except that branching priority was given to variables *y*. While method RC had been run with presolve disabled in version 2005B (to avoid interference of variable elimination with user cut generation), it was run with presolve enabled in version 7.2, as this version automatically presolves user cuts.

Table 5.7 to Table 5.9 below correspond to the previous Table 5.1 to Table 5.3.

Considering the standard formulation (method S), it can be seen that the performance of Xpress 7.2 has increased significantly relatively to 2005B: average solution time (Table 5.7 vs. Table 5.1) decreases by 20-90% for different instance groups (although several instances in groups 100-10 and 100-20 still cannot be solved to optimality within 1 hour). Automatic cuts are much more effective and certainly give a key contribution, as the average XLP gap is halved with Xpress 7.2 relatively to 2005B (Table 5.8 vs. Table 5.2), and is especially reduced for instances with large capacity ratios (r/n=0.3 and 0.4). Improvements in other solver components (presolve, heuristics, branching variable and node selection strategies, LP re-optimization) may also have contributed. By analyzing solver logs of particular instances, it is clear that MIP presolve at the top node (called "root presolve") is much more effective at reducing and tightening the formulation, and heuristics are much more effective at reducing the primal bound at the top node.

Method R, relatively to method S, reduces time (Table 5.7) by 20-30% on average (was 60% with Xpress 2005B) and by 0-50% for particular capacity ratios (was 20-90% with Xpress 2005B). Although reductions are smaller with Xpress 7.2 than with 2005B, they remain significant in absolute terms for the harder instances (r/n=0.1 and 0.2).

Method RC, relatively to method R, reduces time by about 10% for instances with r/n=0.2 (70-14 and 100-20), but for other instances solution time does not decrease or even increases (as happened with Xpress 2005B). On average, for all capacity ratios, method RC does not improve on method R.

Additional remarks:

- The additional gap closed at the top node with method RC relatively to R is now smaller with Xpress 7.2 than it was with 2005B (Table 5.8 vs. Table 5.6), possibly due to the much greater effectiveness of automatic cuts.
- Most instances with large capacity ratios (70-21, 70-28, 100-40) are now solved by Xpress 7.2 at the top node with method R (value 1 in column *nodes* of Table 5.9). Thus user cut generation in method RC is not invoked (value 0 in column *ucuts*).
- Comparing LP gaps between Xpress 7.2 and 2005B (Table 5.8 vs. Table 5.6), with method S gaps are equal or slightly lower in version 7.2, but with method R they are higher in version 7.2 for instances with r/n=0.3 and 0.4. This is due to differences in LP presolve (if disabled, LP gaps are exactly the same in the two versions), but the precise cause was not further analyzed.

Table 5.7: Summary of results – nodes and time (Xpress 7.2)

n-r	total finished			avg nodes			avg time			avg time ratios		
	S	R	RC	S	R	RC	S	R	RC	R/S	RC/S	RC/R
70-7	9	9	9	1472	1311	745	353	221	256	0.71	0.76	1.10
70-14	9	9	9	1563	925	468	346	129	106	0.52	0.48	0.91
70-21	9	9	9	74	46	29	38	30	30	0.81	0.82	1.01
70-28	9	9	9	6	2	2	20	20	20	1.06	1.04	1.00
70 total	36	36	36	779	571	311	189	100	103	0.78	0.78	1.00
100-10	2	4	3	4087	6822	2474	2912	2613	2676	0.88	0.83	0.98
100-20	8	8	9	3442	2399	1651	1341	803	727	0.55	0.51	0.90
100-30	9	9	9	1338	318	252	623	128	127	0.61	0.62	1.02
100-40	9	9	9	44	8	5	96	69	70	0.76	0.77	1.01
100 total	28	30	30	2228	2387	1095	1243	903	900	0.70	0.68	0.98

Table 5.8: Summary of results – gaps (Xpress 7.2)

n-r	avg gapLP (%)			avg gapXLP (%)			avg gap closed (%)		
	S	R	RC	S	R	RC	S	R	RC
70-7	5.6	5.6	5.6	4.0	3.8	3.3	38	40	49
70-14	9.9	9.7	9.7	3.4	2.8	2.4	68	74	78
70-21	13.6	13.2	13.2	1.0	0.5	0.5	94	97	97
70-28	15.3	14.8	14.8	0.4	0.3	0.2	98	98	99
70 total	11.1	10.8	10.8	2.2	1.9	1.6	74	77	81
100-10	6.9	6.7	6.7	5.3	5.2	4.3	25	26	38
100-20	11.4	10.9	10.9	3.7	3.3	2.6	69	73	79
100-30	16.3	15.5	15.5	2.6	2.0	1.7	85	89	90
100-40	18.5	17.3	17.3	2.0	0.0	0.0	89	100	100
100 total	13.3	12.6	12.6	3.4	2.6	2.2	67	72	77

Table 5.9: Detailed results (Xpress 7.2)

n-r	inst.	IP	nf	finished			time			nodes			gapLP (%)			gapXLP (%)			acuts			ucuts
				S	R	RC	S	R	RC	S	R	RC	S	R	RC	S	R	RC	S	R	RC	RC
70-7	1	15 389	7	1	1	1	40	27	28	1	1	1	1.5	1.4	1.4	0.0	0.0	0.0	17	224	224	0
	2	14 525	6	1	1	1	55	42	47	203	21	11	1.7	1.6	1.6	1.0	1.0	0.7	57	26	19	80
	3	15 298	6	1	1	1	449	286	237	1 812	2 398	1 503	7.8	7.8	7.8	6.0	5.7	5.2	213	57	30	205
	4	14 776	6	1	1	1	503	293	294	2 026	1 595	1 797	6.9	6.9	6.9	5.1	5.1	4.3	160	88	21	245
	5	15 146	6	1	1	1	932	336	338	3 348	1 687	1 093	9.1	9.0	9.0	6.1	5.5	4.5	84	95	40	357
	6	15 100	6	1	1	1	620	497	539	5 489	2 279	2 644	8.9	8.8	8.8	7.3	6.6	5.8	97	99	34	294
	7	16 337	6	1	1	1	46	49	50	127	153	31	3.3	3.2	3.2	1.8	1.7	1.6	43	31	31	70
	8	15 567	6	1	1	1	61	84	94	273	399	85	3.1	3.0	3.0	2.2	2.2	1.8	69	51	24	151
	9	14 896	6	1	1	1	609	342	384	2 411	1 453	1 573	8.1	8.1	8.1	6.1	6.2	5.7	80	106	30	236
70-14	1	10 377	14	1	1	1	205	76	76	1 022	207	55	9.9	8.9	8.9	2.8	2.3	1.9	189	165	165	196
	2	9 245	13	1	1	1	50	66	65	29	61	9	5.1	5.1	5.1	1.0	0.7	0.6	147	114	114	154
	3	9 792	13	1	1	1	50	19	19	17	1	1	6.2	5.9	5.9	0.9	0.0	0.0	160	73	73	0
	4	9 887	12	1	1	1	283	236	179	1 497	1 298	551	9.6	9.5	9.5	3.6	3.4	3.2	182	153	153	166
	5	9 579	13	1	1	1	432	196	175	873	1 051	466	12.3	12.2	12.2	5.6	4.0	3.8	216	160	160	167
	6	9 716	12	1	1	1	549	148	143	1 441	1 216	577	12.1	12.1	12.1	4.0	3.3	2.9	229	146	146	223
	7	10 162	11	1	1	1	539	257	175	3 693	1 618	383	13.3	13.0	13.0	5.2	5.0	4.4	217	167	167	186
	8	9 382	12	1	1	1	164	93	77	805	349	139	10.4	10.0	10.0	3.7	3.5	2.1	174	147	79	179
	9	10 023	12	1	1	1	707	251	186	2 079	1 359	706	10.3	10.2	10.2	3.8	3.3	3.2	243	142	142	154
70-21	1	7 137	21	1	1	1	40	16	15	1	1	1	10.3	10.1	10.1	0.0	0.0	0.0	66	79	79	0
	2	7 188	17	1	1	1	43	37	38	1	1	1	10.6	10.3	10.3	0.0	0.0	0.0	126	123	123	0
	3	7 933	18	1	1	1	48	44	45	1	1	1	16.9	16.3	16.3	0.0	0.0	0.0	101	169	169	0
	4	7 725	17	1	1	1	50	41	42	5	1	1	13.8	13.2	13.2	0.5	0.0	0.0	128	137	137	0
	5	6 624	18	1	1	1	49	19	19	1	1	1	13.9	13.7	13.7	0.0	0.0	0.0	207	0	0	0
	6	7 156	18	1	1	1	36	44	45	15	15	9	9.8	9.6	9.6	0.6	0.7	0.6	86	71	71	33
	7	7 456	17	1	1	1	99	51	53	227	35	41	17.9	17.8	17.8	4.3	1.6	1.5	192	105	105	44
	8	6 862	17	1	1	1	42	19	19	1	1	1	12.0	11.4	11.4	0.0	0.0	0.0	0	0	0	0
	9	7 655	18	1	1	1	90	61	64	495	159	103	17.0	16.8	16.8	3.6	2.5	2.3	187	127	127	108
70-28	1	5 999	25	1	1	1	12	19	16	1	1	1	12.9	13.2	13.2	0.0	0.0	0.0	94	78	78	0
	2	5 979	23	1	1	1	36	26	26	1	1	1	16.4	16.0	16.0	0.0	0.0	0.0	177	120	117	0
	3	6 090	24	1	1	1	29	19	20	1	1	1	15.9	14.8	14.8	0.0	0.0	0.0	130	80	80	0
	4	6 179	23	1	1	1	57	49	50	21	17	13	17.3	16.8	16.8	2.2	1.6	1.6	118	82	82	12
	5	5 448	22	1	1	1	16	13	13	1	1	1	14.0	11.5	11.5	0.0	0.0	0.0	104	0	0	0
	6	6 430	22	1	1	1	48	41	41	5	1	1	20.1	19.7	19.7	0.4	0.0	0.0	201	190	190	0
	7	5 815	21	1	1	1	39	19	19	1	1	1	14.0	14.0	14.0	0.0	0.0	0.0	133	74	74	0
	8	5 090	22	1	1	1	15	13	13	1	1	1	11.3	11.1	11.1	0.0	0.0	0.0	0	54	54	0
	9	5 953	22	1	1	1	59	30	33	5	7	13	16.1	15.9	15.9	1.2	0.7	0.6	89	69	69	34
100-10	1	25 848	10	0	0	0	3 600	3 627	3 627	2 346	4 893	3 279	8.8	8.8	8.8	7.0	7.1	6.3	128	68	22	328
	2	22 292	9	1	1	1	1 016	159	168	1 235	239	246	3.1	3.0	3.0	2.2	2.2	1.9	204	78	38	132
	3	23 295	9	0	0	0	3 600	3 627	3 627	2 467	4 780	2 639	7.6	7.3	7.3	5.9	5.2	4.4	598	77	77	349
	4	23 207	10	1	1	1	426	146	172	493	235	162	4.0	3.9	3.9	2.6	2.7	2.6	153	72	72	138
	5	22 356	9	0	1	0	3 600	3 081	3 627	1 740	4 241	5 138	9.5	9.3	9.3	7.5	7.3	5.1	124	91	19	455
	6	22 892	8	0	1	1	3 600	1 565	1 947	4 014	7 776	5 550	5.7	5.7	5.7	4.3	4.4	3.6	187	59	24	271
	7	25 621	10	0	0	0	3 600	3 628	3 628	2 446	5 811	3 008	8.2	8.1	8.1	5.5	6.0	4.8	121	93	93	347
	8	24 455	9	0	1	1	3 600	2 451	2 876	4 245	5 401	8 180	7.6	7.0	7.0	5.9	5.4	4.5	66	80	26	245
	9	24 949	9	0	1	0	3 600	3 558	3 628	2 592	7 368	4 565	8.0	7.7	7.7	6.5	6.6	5.7	120	85	82	331
100-20	1	15 492	18	1	1	1	2 044	440	298	2 351	1 615	709	11.8	11.3	11.3	4.1	3.9	3.4	292	277	277	157
	2	14 341	16	1	1	1	858	357	319	2 876	899	347	9.9	9.6	9.6	3.8	2.4	2.2	230	209	209	228
	3	14 687	17	1	1	1	1 023	292	200	2 900	732	559	11.9	11.5	11.5	3.8	4.1	2.1	285	175	124	255
	4	15 324	19	1	1	1	2 894	1 372	608	12 163	6 565	2 193	12.1	11.8	11.8	3.3	3.5	3.2	303	194	194	140
	5	14 076	16	1	1	1	2 590	165	147	2 039	147	1	10.2	9.8	9.8	3.4	2.1	0.0	337	196	165	352
	6	15 181	17	0	1	1	3 600	987	812	2 541	2 212	1 278	9.7	9.4	9.4	3.7	3.4	3.0	352	191	191	254
	7	16 812	19	0	0	1	3 600	3 628	2 964	1 608	5 951	5 290	19.0	18.0	18.0	7.3	7.0	6.4	350	200	200	254
	8	13 959	17	1	1	1	161	115	116	1	1	1	7.2	6.7	6.7	0.0	0.0	0.0	400	155	155	0
	9	14 506	18	1	1	1	1 370	295	299	4 517	1 166	741	10.3	10.3	10.3	3.9	3.4	3.3	262	176	176	115
100-30	1	11 750	26	1	1	1	128	111	120	179	7	1	16.3	15.7	15.7	2.0	0.8	0.0	250	101	101	14
	2	11 161	24	1	1	1	581	232	225	225	93	63	17.7	16.3	16.3	3.5	2.9	2.6	459	181	181	152
	3	10 432	24	1	1	1	140	92	94	51	123	45	15.0	14.1	14.1	1.4	0.8	0.8	198	124	124	9
	4	11 188	28	1	1	1	171	104	109	431	57	15	15.1	14.8	14.8	2.8	1.4	1.3	246	127	127	46
	5	10 481	22	1	1	1	627	383	229	2 955	2 643	955	19.5	19.2	19.2	4.6	4.4	3.9	303	160	160	118
	6	11 204	24	1	1	1	184	91	91	19	1	1	11.9	10.7	10.7	1.2	0.0	0.0	238	135	135	0
	7	11 619	26	1	1	1	951	263	263	4 796	813	583	19.9	19.3	19.3	5.3	4.4	3.9	234	164	164	95
	8	11 017	23	1	1	1	185	265	250	245	465	269	16.1	15.2	15.2	2.5	3.0	3.0	261	156	156	65
	9	10 998	25	1	1	1	149	113	115	1	1	1	14.8	13.9	13.9	0.0	0.0	0.0	241	191	191	0
100-40	1	9 160	33	1	1	1	88	98	99	63	1	1	18.8	17.8	17.8	2.1	0.0	0.0	146	31	31	0
	2	9 219	30	1	1	1	186	108	108	13	1	1	20.7	19.3	19.3	1.7	0.0	0.0	148	187	187	0
	3	8 942	32	1	1	1	150	81	80	243	1	1	19.2	18.6	18.6	3.9	0.0	0.0	261	151	151	0
	4	9 050	33	1	1	1	133	96	102	673	3	1	18.2	17.6	17.6	3.1	0.4	0.0	198	49	49	1
	5	7 992	29	1	1	1	116	81	82	53	1	1	19.3	17.5	17.5	3.0	0.0	0.0	228	114	114	0
	6	9 618	30	1	1	1	140	121	121	35	1	1	19.1	17.5	17.5	1.4	0.0	0.0	191	133	133	0
	7	9 170	32	1	1	1	62	72	72	71	1	1	17.6	16.4	16.4	3.1	0.0	0.0	101	48	48	0
	8	8 661	31	1	1	1	69	52	53	1	1	1	18.3	16.7	16.7	0.0	0.0	0.0	224	110	110	0
	9	8 736	31	1	1	1	61	103	103	1	1	1	15.5	14.5	14.5	0.0	0.0	0.0	175	142	142	0

Regarding the a priori reformulation procedure, step 6 is now performed automatically by the LP presolve procedure of Xpress 7.2 in most of the closest assignment constraints. Because of this, the reformulation procedure contributes much less to reducing the number of non-zero elements. The average reductions of the number of non-zero elements due to the reformulation procedure were as follows (for *n*=70):

- Before LP presolve with both versions: 44% without step 3, 48% with step 3;
- After LP presolve with Xpress 2005B: 45% without step 3, 49% with step 3;
- After LP presolve with Xpress 7.2: 8% without step 3, 16% with step 3.

Accordingly, the contribution of step 6 to reduce solution times with Xpress 7.2 is almost negligible, as seen in Table 5.10 by comparing R12 vs. R-35 (i.e. steps 1+2 vs. 1+2+4+6).

Adding step 3 to the reformulation procedure still reduces solution times with Xpress 7.2, by about 15% on average (R/R-3 or R-5/R-35 in Table 5.10), similar to the reduction with Xpress 2005B. Adding step 5 generally degrades performance, as had occurred with Xpress 2005B (R-3 vs. R-35 or R vs. R-5 in Table 5.10).

Table 5.10: Reformulation steps – nodes and time (Xpress 7.2)

n-r	avg time						avg time ratios						
	S	R12	R-3	R	R-35	R-5	R12/S	R-3/S	R/S	R-35/S	R-5/S	R/R-3	R-5/R-35
70-7	368	277	223	217	253	203	0.82	0.76	0.76	0.81	0.66	1.01	0.89
70-14	331	251	296	149	247	143	0.89	0.90	0.56	0.82	0.51	0.68	0.67
70-21	55	44	47	37	37	33	0.82	0.90	0.69	0.71	0.63	0.76	0.99
70-28	35	28	30	25	29	22	0.81	0.91	0.81	0.79	0.63	0.91	0.87
70 total	197	150	149	107	142	100	0.83	0.87	0.70	0.78	0.61	0.84	0.86

Table 5.11: Reformulation steps – gaps (Xpress 7.2)

n-r	avg gapLP (%)						avg gapXLP (%)					
	S	R12	R-3	R	R-35	R-5	S	R12	R-3	R	R-35	R-5
70-7	5.6	5.6	5.6	5.6	5.6	5.6	4.0	3.9	4.0	3.8	4.0	3.8
70-14	9.9	9.8	9.8	9.7	9.8	9.7	3.4	2.8	3.1	2.8	3.2	2.8
70-21	13.6	13.5	13.4	13.2	13.4	13.2	1.0	0.6	0.9	0.5	0.7	0.8
70-28	15.3	14.7	14.7	14.8	14.7	14.8	0.4	0.3	0.2	0.3	0.2	0.2
70 total	11.1	10.9	10.9	10.8	10.9	10.8	2.2	1.9	2.1	1.9	2.0	1.9

In conclusion, results with Xpress 7.2 corroborate those with 2005B, specifically the reformulation is generally effective at reducing solution times but user cuts are not generally effective. Time reductions relatively to the standard formulation are smaller with Xpress 7.2 than they were with 2005B, because the generic optimizer improved very significantly.

5.11 Appendix – Closest assignment constraints

1. Introduction

Some location models require closest assignment (CA) constraints or equivalent preference constraints – applications are discussed by Hanjoul and Peeters (1987) and by Gerrard and Church (1996).

In this text, we provide recommendations on selecting a formulation of CA constraints, among the several alternatives proposed in the literature, regarding expected computational performance with a generic MIP optimizer. For this purpose, we consider five commonly used alternative formulations, summarize previous results on their properties (including strength of LP relaxations), and report on computational experiments with the CM model.

Note: When a first version of this text was written, it contributed to previous literature with a comparison between the CA constraints denoted (WBR) below, which is adopted in some recent articles, with previous alternatives. However, Espejo et al. (2012) published a recent article providing a comprehensive comparison of CA constraints, including the contribution above and additional new results. Thus, this text loses some relevance, but it was still kept in the thesis (although revised to consider the results of that recent article) since it contains computational results and additional recommendations, complementing previous literature.

2. Formulations

The notation used here is the same as before: I and J are the sets of centers and sites, respectively, y_j are the location variables, x_{ij} are the assignment variables, d_{ij} are the distances between centers and sites. We also introduce the following notation for demand and variable upper bound constraints used in location models:

$$\sum_{j \in J} x_{ij} = 1, \ \forall i \in I \tag{D$^=$}$$

$$\sum_{j \in J} x_{ij} \le 1, \ \forall i \in I \tag{D$^\le$}$$

$$x_{ij} \le y_j, \ \forall i \in I, j \in J \tag{V}$$

We will consider the following two feasible sets of locations models:

$$X^{=} = \left\{ x \in \{0,1\}^{nm}, y \in \{0,1\}^{m} : (\mathrm{D}^{=}); (\mathrm{V}); (\mathrm{CA}) \right\}$$

$$X^{\leq} = \left\{ x \in \{0,1\}^{nm}, y \in \{0,1\}^{m} : (\mathrm{D}^{\leq}); (\mathrm{V}); (\mathrm{CA}) \right\}$$

where (CA) denotes a formulation of CA constraints stating that each center cannot be assigned to a facility farther than the closest open facility (determined by d_{ij} data). In set $X^{=}$, all centers must be fully served by a single, closest facility. In set $X^{\leq}$, centers may be either fully served from a single, closest facility or not served by any facility. The definition of CA constraints above is compatible with both sets, as it allows centers not to be served.

It is useful to distinguish the two sets because both arise in location problems studied in the literature. Note that CA constraints valid for these sets are also valid for restrictions of them, considering e.g. capacity constraints or a fixed number of facilities. We introduce notation for particular restrictions that will be useful later:

> X^{p*}: denotes a restriction of X^{*} with the additional constraint $\sum_{j \in J} y_j = p$, with p being a parameter.

In Table 5.12, we cite references studying models with different feasible sets. For set $X^{=}$ and restrictions of it, the list is not exhaustive of the literature. For set $X^{\leq}$ and restrictions of it, no additional references are known to the author, particularly none are known for set $X^{p\leq}$.

Table 5.12: References to location models with different feasible sets including CA constraints

	Without capacity constraints	With capacity constraints
$X^{=}$	Hanjoul and Peeters (1987)	Carreras and Serra (1999), Berman et al. (2006)
$X^{p=}$	Belotti et al. (2007), Scaparra and Church (2008)	Kalcsics et al. (2002), Berman et al. (2009)
$X^{\leq}$	Wagner and Falkson (1975)	Vedat and Verter (2002), Smith et al. (2009)

In formulations below, we retain the notation used before for sets of not farther sites, equidistant sites, and strictly closer sites, as follows: $N_{ij} = \left\{ k \in J \mid d_{ik} \leq d_{ij} \right\}$, $E_{ij} = \left\{ k \in J \mid d_{ik} = d_{ij} \right\}$, $N'_{ij} = N_{ij} \setminus E_{ij} = \left\{ k \in J \mid d_{ik} < d_{ij} \right\}$.

We now present alternative formulations of closest assignment constraints:

$$x_{ij} \geq y_j - \sum_{k \in N'_{ij}} y_k, \ \forall i \in I, j \in J \tag{RR}$$

$$\sum_{k \in N_{ij}} x_{ik} \geq y_j, \ \forall i \in I, j \in J \tag{CC}$$

$$\sum_{k \in J \setminus N_{ij}} x_{ik} \leq 1 - y_j, \ \forall i \in I, j \in J \tag{WF}$$

$$x_{ik} \leq 1 - y_j, \ \forall i \in I, j \in J, k \in J \setminus N_{ij} \tag{DK}$$

$$\sum_{k \in J} d_{ik} x_{ik} \leq d_{ij} y_j + M_i (1 - y_j), \ M_i = \max_{k \in J} \{d_{ik}\}, \ \forall i \in I, j \in J \tag{WBR}$$

Formulations (RR), (CC), (WF), and (DK) were introduced in the 70s and 80s, as discussed by Gerrard and Church (1996), and are denoted here by the initials of the original authors. Formulation (WBR) is here credited to Wang et al. (2002), the earliest reference known to us (Berman et al., 2006, used the same formulation in a related model developed independently).

The constraints above can be described as follows, for a given center $i \in I$ and a given site $j \in J$. If facility j is closed ($y_j = 0$), the constraints have no effect. If facility j is open ($y_j = 1$) then: (WF), (DK) and (WBR) state that i cannot be assigned to a facility k farther than j ($x_{ik} = 0$); (CC) states that i must be assigned to a facility k not farther than j ($x_{ik} = 1$); (RR) states that i must be assigned to j ($x_{ij} = 1$) if all facilities k closer than j are closed ($y_k = 0$).

Formulations (CC) and (WF) were adopted in the present chapter, and by Cánovas et al. (2007) in their reformulation of the UFLPO. All formulations were adopted or tested in recent work by other authors, for example: (CC) by Scaparra and Church (2008), (WF) by Belotti et al. (2007), (DK) by Smith et al. (2009), (WBR) by Berman et al. (2006) and Berman et al. (2009), (RR) by Berman et al. (2006) and Scaparra and Church (2008) (tested as an alternative formulation in both cases).

Previous work comparing alternative formulations of CA formulations is outlined next, with details being presented further below in this text.

Gerrard and Church (1996) discuss applications of models with CA constraints and provide a detailed comparison of properties of alternative formulations (CC), (WF), (RR), and (DK), although not covering the strength of LP relaxations and not including computational experiments. Regarding recommendations on computational performance, no formulation was singled out as being clearly better or worse than all others for all models. However, the authors emphasized that (RR) was the most widely cited and has a particular property (redundancy of VUB constraints) offering a potential computational advantage.

More recently, Espejo et al. (2012) provide an updated and thorough comparison of alternative CA formulations including all the commonly used ones: (CC), (WF), (RR), (DK), (WBR) (or *BDTW* in their notation), (C2) (or *CGLM* in their notation, the inequalities proposed by Cánovas et al. (2007) that generalize (WF) and were presented before in this chapter), and four additional formulations valid for set $X^{p=}$ but not for $X^{=}$. The authors provide a complete analysis of formulation strength, allowing any two CA formulations to be compared, and give formal proofs of properties previously noted by Gerrard and Church (1996). Regarding recommendations on computational performance, the authors favor two formulations leading to the strongest LP relaxations (as detailed further below). However, computational experiments are not reported.

Note: In the present text we do not analyze some of the CA formulations above: (C2) because it requires $O(n^3)$ constraints; the four formulations dedicated to set $X^{p=}$ because they are not valid for the more general set $X^{=}$.

Before the previous article, the strength of LP relaxations of the UFLPO with alternative CA formulations had been compared by Hanjoul and Peeters (1987) and by Hansen et al. (2004), as detailed below.

Computational experiments comparing alternative CA formulations have rarely been reported. In experiments with particular models and using a generic MIP optimizer, Scaparra and Church (2008) compared (CC) and (RR), and Berman et al. (2006) compared (WBR) and (RR). In both cases, (RR) was found to perform worse.

3. Properties

All the alternative formulations of CA constraints presented above lead to equivalent definitions of set $X^{=}$. However, they differ on suitability for different model and data assumptions, and on other properties influencing computational performance.

- **Suitability for models with different demand constraints:** For models requiring all centers to be assigned (set $X^{=}$), all CA formulations above are valid. For models allowing centers not to be assigned (set $X^{\leq}$), only (WF), (DK) and (WBR) are valid. Formulations (CC) and (RR), even if demand constraints are relaxed from "=" to "≤", force all centers to be assigned when at least one facility is open.

- **Suitability for data with distance ties**: All CA formulations above, except (RR), are valid when data has distance ties, i.e. $d_{ij} = d_{ih}$ for a center i and distinct sites j and h. If a center has two or more equidistant closest facilities, then it can be assigned to any one of them. On the other hand, (RR) is not generally valid with distance ties, as it rules out some feasible solutions: if two or more equidistant facilities were the closest, then (RR) would require full assignment to each of them; therefore at most one can be open. Generalizations of (RR) to deal with distance ties are presented in a section below.

- **Formulation size**: (DK) and (WBR) have disadvantages in terms of formulation size. All CA formulations above require $O(n^2)$ constraints, except (DK) which requires $O(n^3)$. Formulation (WBR) requires a larger number of non-zero elements than (RR), (CC), and (WF), as the first involves all sites $k \in J$ in all constraints, while the others involve only a subset.

- **Formulation strength**: The alternative CA formulations lead to models with different LP relaxations. This is discussed in the next section.

- **Property of single assignment**: All CA formulations above have the property that they suffice to guarantee single assignment in the definition of set $X^{=}$, that is, if all variables y take 0-1 binary values, variables x will also take 0-1 binary values, even if they are defined as continuous. However, this applies only to pairs (i,j) without distance ties ($E_{ij}=\{j\}$); for pairs (i,j) with distance ties ($|E_{ij}|>1$), all CA formulations (except (RR), as noted above) allow demand to be freely distributed among equidistant, closest facilities with fractional assignments. Thus, this property allows reducing the number of integer variables in the formulation of $X^{=}$.

- **Property of redundancy of constraints (V)**: Formulation (RR) has the property that constraints (V) are redundant in the definition of set $X^{=}$ restricted to at least one open facility ($\sum_{j\in J} y_j \geq 1$), e.g. to set $X^{p=}$. That is, if all variables y take binary values 0-1, constraints (V) are automatically satisfied. Thus, constraints (V) may be dropped, giving a smaller but still valid integer formulation of $X^{=}$, although leading to a weaker LP relaxation. The other formulations do not have this property.

All of the properties above were noted in the literature before, particularly in the thorough discussions by Gerrard and Church (1996) and Espejo et al. (2012). Here we provide only a compact summary and clarify the suitability for sets $X^{=}$ and $X^{\leq}$

(Gerrard and Church (1996) discussed both sets but did not include (WBR), Espejo et al. (2012) included (WBR) but focused on sets $X^{=}$ and $X^{p=}$).

4. Formulation strength

In the text below, given alternative formulations A and B of CA constraints valid for a set X formulated with integer variables, A is said to dominate B if the continuous relaxations of X defined with A and B, denoted P_A and P_B, are such that $P_A \subseteq P_B$. In this case, A leads to stronger LP relaxations of problems with feasible set X, i.e. providing bounds at least as tight.

The strength of LP relaxations of the UFLPO (which has feasible set $X^{=}$) with alternative CA formulations was discussed by Hanjoul and Peeters (1987), who note that (CC) dominates (RR), and by Hansen et al. (2004), who show that (CC) and (WF) are equivalent and dominate (DK) and (RR). Here we summarize these results and also analyze (WBR). We assume the feasible integer set is $X^{=}$ unless otherwise noted.

(WF) is equivalent to (CC): It is immediate to verify that one formulation can be converted into the other using demand constraints $\sum_{j \in J} x_{ij} = 1$ for $i \in I$.

(CC) dominates (RR): We assume there are no distance ties. For a given (i, j), (RR) can be re-written as

$$y_j \le x_{ij} + \sum_{k \in N'_{ij}} y_k .$$

Combining (CC) and (V), we get

$$y_j \le \sum_{k \in N_{ij}} x_{ik} = x_{ij} + \sum_{k \in N'_{ij}} x_{ik} \le x_{ij} + \sum_{k \in N'_{ij}} y_k .$$

Thus, (CC) implies (RR) if constraints (V) hold. The converse is not true, as shown by the following example. Given a center i and sites 1, 2, 3, 4 in increasing distance to i, the point y_1=y_2=0.5, y_3=y_4=1, x_{i1}=0.5, x_{i2}=x_{i3}=0 and x_{i4}=0.5 satisfies (RR) for j=1, 2, 3, 4, but violates (CC) for j=3.

(WF) dominates (DK): It is immediate to verify that (WF) implies (DK) in the forms given above, as the left-hand side of (WF) is greater than or equal to the one of (DK). The converse is not true, as shown by the following example. Given a center i and sites 1, 2, 3 in increasing distance to i, the point y_1=y_2=y_3=0.5, x_{i1}=0, x_{i2}=x_{i3}=0.5 satisfies (DK) but violates (WF) for j=1.

(WF) dominates (WBR): We assume $M_i \geq d_{ij} \geq 0$ for all i and j. For a given (i,j), (WBR) can be re-written as follows, by separating the sum over set J:

$$\sum_{k \in N_{ij}} d_{ik} x_{ik} + \sum_{k \in J \setminus N_{ij}} d_{ik} x_{ik} \leq d_{ij} y_j + M_i (1 - y_j).$$

Now considering that demand constraints imply

$$\sum_{k \in N_{ij}} x_{ik} + \sum_{k \in J \setminus N_{ij}} x_{ik} \leq 1$$

and that constraints (WF) are

$$\sum_{k \in J \setminus N_{ij}} x_{ik} \leq 1 - y_j,$$

summing the first multiplied by d_{ij} with the second multiplied by $M_i - d_{ij}$, we get:

$$\sum_{k \in N_{ij}} d_{ij} x_{ik} + \sum_{k \in J \setminus N_{ij}} M_i x_{ik} \leq d_{ij} y_j + M_i \left(1 - y_j\right).$$

This inequality dominates (WBR) as, in the left-hand sides, $d_{ik} \leq d_{ij}$ for $k \in N_{ij}$ and $d_{ik} \leq M_i$ for $k \in J \setminus N_{ij}$. Thus, (WF) implies (WBR) if demand constraints hold. The converse is not true, as shown by the following example. Given a center i and sites 1, 2, 3 such that d_{i1}=1, d_{i2}=2, d_{i3}=3, M_i=3, the point y_1=0.5, y_2=y_3=1, x_{i1}=0.5, x_{i2}=0 and x_{i3}=0.5 satisfies (WBR) for j=1, 2, 3, but violates (WF) for j=2.

The arguments above also show that (WF) dominates (DK) and (WBR) in the case of feasible set $X^{\leq}$.

Espejo et al. (2012) show in addition that (DK) dominates (WBR). They also show through particular examples that no domination relation exists between (RR) and (WBR), or between (RR) and (DK). They also analyze the strength of four additional CA formulations valid for set $X^{p=}$.

5. Generalizations of (RR) to deal with distance ties

Here we propose the following generalization of (RR) to deal with distance ties, which was also proposed by Espejo et al. (2012):

$$\sum_{k \in E_{ij}} x_{ik} \geq y_j - \sum_{k \in N'_{ij}} y_k, \quad \forall i \in I, j \in J \tag{RR2}$$

If $E_{ij}=\{j\}$, (RR2) reduces to (RR). If $|E_{ij}|$>1, (RR2) states that if facility j is open and all strictly closer facilities are closed then i must be assigned to j or to an equidistant site ($k \in E_{ij}$). These constraints now work properly if more than one facility in E_{ij} is open. We note that (CC) dominates (RR2), by a similar argument as used above for (RR).

Other generalizations of (RR) to deal with distance ties have been proposed:

$$\sum_{k\in S} x_{ik} \geq \frac{1}{|S|}\sum_{k\in S} y_k - \sum_{k\in N'_{ij}} y_k - \sum_{k\in E_{ij}\setminus S} y_k \text{ , } \forall i\in I, j\in J_i,\ S\subseteq E_{ij} \qquad \text{(RR3)}$$

$$\sum_{k\in E_{ij}} x_{ik} \geq \frac{1}{|E_{ij}|}\sum_{k\in E_{ij}} y_k - \sum_{k\in N'_{ij}} y_k \text{ , } \forall i\in I, j\in J_i \qquad \text{(RR4)}$$

Formulation (RR3) was proposed by Gerrard and Church (1996), who call it MAC1, and (RR4) was proposed by Berman et al. (2006). The set J_i contains only one element from each distinct set E_{ik} for $k\in J$. For a given (i, j) such that $|E_{ij}|>1$, (RR2) requires $|E_{ij}|$ constraints, while (RR3) requires a higher number (e.g. 3 constraints if $|E_{ij}|=2$, 7 constraints if $|E_{ij}|=3$), and (RR4) requires only one – actually, (RR4) corresponds to (RR3) written only for maximal sets $S = E_{ij}$. Gerrard and Church (1996) note that (RR3), like the original (RR), remains valid if (V) are dropped from the model. On the other hand, we note that (RR2) and (RR4) require (V) or equivalent constraints forbidding assignments to closed facilities for pairs (i, j) such that $|E_{ij}|>1$.

We note that (RR2) dominates (RR3): summing (RR2) for all $j\in S$ and dividing by $|S|$, an expression is obtained that dominates (RR3) if (V) holds. (RR2) also dominates (RR4): summing (RR2) for all $j\in E_{ij}$ and dividing by $|E_{ij}|$, (RR4) is obtained. Thus, (RR2) leads to stronger LP relaxations than the other generalizations, and in our computational experiments we considered only (RR2).

6. Computational experiments with the CM model

Four formulations of the CM model were tested: CC denotes our standard formulation (5.1)-(5.8) and (5.9), which uses CA formulation (CC); WBR, RR, DK denote formulations where (CC) was replaced by (WBR), (RR2) and (DK), respectively.

Note: In addition to the formulations above, we did preliminary experiments with a variant of RR without constraints (V), obtained by: using (RR2) instead of (CC), dropping (V) for all pairs (i, j) such that $|E_{ij}|=1$, and adding the constraint $\sum_{j\in J} y_j \geq 1$ (at least one open facility). This formulation had a very weak initial LP relaxation, violating many constraints (V) and even many of the weaker constraints $\sum_{i\in I} x_{ij} \leq n\cdot y_j$ for $j\in J$, and a very large time was required even for generating cuts at the top node. Thus, this formulation was dropped from consideration.

Instances of sizes n=50 and n=70 were tested, generated as described before. All instances have a significant number of distances ties.

Results were obtained for two versions of the Xpress solver: 2005B (MIP Optimizer 16.10) and 7.2 (MIP Optimizer 22.01). We show results with the two versions because some interesting observations can be made regarding the evolution of different solver components. Solution times are comparable between the two versions since the same computer was used, as described before. The time limit was set to 1 hour and solver parameters were set as before: branching priority was given to variables *y*; other parameters were left at default values.

The results reported below are the same as before and additionally:

- time XLP: time (in seconds) spent in cut generation and heuristics at the top node;
- htop gap 0%: number of instances where heuristics at the top node found an optimal solution (whether or not optimality was proven at the top node);
- htop gap 5%: number of instances where heuristics at the top node found a feasible solution with a value within 5% of the optimal value;

Note: Results for DK with n=70 are not reported since cut generation at the top node required more than the physical memory available in the computer used.

Note: Some of the results below, regarding solver components such as presolve, cut generation and heuristics, were surprising. Explaining such results is not always immediate, given that the operation of solver components is not fully documented.

Formulation size (Table 5.13)

Formulations CC and RR have the same number of non-zero elements before LP presolve, and WBR has many more (each CA constraint involves all sites, not only the closer or equidistant sites). LP presolve in Xpress 7.2 significantly reduces the number of non-zero elements of CC and WBR. With CC, it performs an equivalent simplification to step 6 of the reformulation procedure presented before. With WBR, it is even more effective than with CC. This latter result was surprising, but was not analyzed further.

DK is much larger than CC and RR (it has n^3 CA constraints instead of n^2). LP presolve in Xpress 2005B performs a large reduction of the model, while Xpress 7.2 reduces it only slightly (for reasons that are unclear).

Table 5.13: Formulation size (*n*=50, average for all capacity ratios *r*)

	LP presolve	CC	WBR	RR	DK
rows	before	5 101	5 101	5 101	63 190
	after - Xpress 2005B	4 947	5 047	5 001	21 092
	after - Xpress 7.2	4 947	5 047	5 001	60 639
non-zero	before	77 011	135 046	77 011	131 277
elements	after - Xpress 2005B	74 030	132 422	76 761	89 179
	after - Xpress 7.2	46 269	44 962	76 760	128 727

Solution time (Table 5.14)

Solution times are lowest with CC and WBR, and are much higher with RR and DK, by a factor of 5 or more (considering RR/CC for size *n*=70, DK/CC for size *n*=50, and any of the solver versions).

Comparing CC with WBR, using Xpress 2005B, CC is solved faster than WBR for all instance groups, except for *n*-*r* = 70-14 (note: the time ratio WBR/CC is 1.0 even though WBR has lower average time than CC; this is due to the fact that the best formulation varies for individual instances). Using Xpress 7.2, CC is solved faster than WBR for smaller capacity ratios and slower for larger capacity ratios; although the global average time ratio WBR/CC is 1.0 for *n*=70, CC can be considered more effective as it is solved faster for instances with lower capacity ratios, which are harder.

As noted before, Xpress 7.2 is much faster than 2005B. The improvement occurs for all formulations (except for DK, which performs worse, for reasons that are unclear).

LP and XLP gaps (Table 5.14)

The LP gaps of CC are the lowest, as expected. The difference relative to other formulations is small for smaller capacity ratios and increases for larger ones.

However, it turns out that the XLP gaps of CC are not the lowest: they are generally the highest among all formulations with Xpress 2005B; they are only the second lowest with Xpress 7.2. In both Xpress versions, WBR consistently has the lowest gaps across all capacity ratios. These results were surprising, but their causes were not analyzed further, particularly which cut types are responsible for closing the XLP gap of WBR (this task is made difficult since Xpress logs do not discriminate the types of added cuts).

Table 5.14: Main results (*n*=50 and 70)

		Xpress 2005B				Xpress 7.2			
total finished	n-r	CC	WBR	RR	DK	CC	WBR	RR	DK
	50-5	9	9	9	9	9	9	9	9
	50-10	9	9	9	9	9	9	9	9
	50-15	9	9	9	9	9	9	9	9
	50-20	9	9	9	9	9	9	9	9
	50 total	**36**	**36**	**36**	**36**	**36**	**36**	**36**	**36**
	70-7	9	9	6	-	9	9	8	-
	70-14	8	9	2	-	9	9	4	-
	70-21	9	9	6	-	9	9	9	-
	70-28	9	9	9	-	9	9	9	-
	70 total	**35**	**36**	**23**	**-**	**36**	**36**	**30**	**-**
avg time	n-r	CC	WBR	RR	DK	CC	WBR	RR	DK
	50-5	28	56	68	87	32	35	58	188
	50-10	52	67	183	146	33	26	116	242
	50-15	43	51	153	127	22	15	75	189
	50-20	9	19	31	65	10	7	10	98
	50 total	**33**	**48**	**109**	**106**	**24**	**21**	**65**	**179**
	70-7	661	988	1838	-	368	450	1135	-
	70-14	1771	1347	2996	-	331	360	2554	-
	70-21	241	250	1650	-	55	38	473	-
	70-28	64	98	768	-	35	23	116	-
	70 total	**685**	**671**	**1813**	**-**	**197**	**218**	**1070**	**-**
avg time ratios	n-r	CC	WBR	RR	DK	CC	WBR	RR	DK
	50-5	1.0	1.8	1.9	5.3	1.0	1.8	1.7	16.6
	50-10	1.0	1.6	3.3	3.5	1.0	0.8	3.3	8.4
	50-15	1.0	1.5	3.1	3.6	1.0	0.7	3.0	8.3
	50-20	1.0	2.3	3.5	9.3	1.0	1.0	1.2	12.0
	50 total	**1.0**	**1.8**	**2.9**	**5.4**	**1.0**	**1.1**	**2.3**	**11.3**
	70-7	1.0	2.3	2.5	-	1.0	1.2	2.6	-
	70-14	1.0	1.0	3.1	-	1.0	1.2	8.3	-
	70-21	1.0	1.6	6.9	-	1.0	0.7	6.3	-
	70-28	1.0	2.0	9.4	-	1.0	0.8	3.0	-
	70 total	**1.0**	**1.7**	**5.5**	**-**	**1.0**	**1.0**	**5.1**	**-**
avg gapLP (%)	n-r	CC	WBR	RR	DK	CC	WBR	RR	DK
	50-5	4.6	4.6	4.6	4.7	4.6	4.6	4.6	4.8
	50-10	10.2	10.2	10.3	10.3	10.2	10.2	10.3	10.3
	50-15	16.1	16.3	16.2	16.3	16.0	16.3	16.0	16.3
	50-20	14.5	20.9	14.9	20.7	14.1	20.8	14.4	20.8
	50 total	**11.3**	**13.0**	**11.5**	**13.0**	**11.2**	**13.0**	**11.3**	**13.0**
	70-7	5.6	5.6	5.6	-	5.6	5.6	5.6	-
	70-14	9.9	9.9	10.0	-	9.9	9.9	10.0	-
	70-21	13.7	13.9	13.8	-	13.6	13.8	13.8	-
	70-28	15.5	21.3	15.9	-	15.3	21.2	15.6	-
	70 total	**11.2**	**12.7**	**11.3**	**-**	**11.1**	**12.7**	**11.2**	**-**
avg gapXLP (%)	n-r	CC	WBR	RR	DK	CC	WBR	RR	DK
	50-5	3.9	2.8	3.5	3.1	3.0	2.8	3.4	3.1
	50-10	6.2	3.3	5.3	4.1	1.9	1.8	3.0	2.4
	50-15	6.1	2.9	4.9	3.9	0.9	0.7	2.1	0.7
	50-20	3.3	1.4	2.5	3.0	0.0	0.1	0.2	0.2
	50 total	**4.9**	**2.6**	**4.1**	**3.5**	**1.4**	**1.3**	**2.2**	**1.6**
	70-7	4.8	3.9	4.7	-	4.0	3.7	4.5	-
	70-14	6.4	3.9	5.9	-	3.4	3.1	4.2	-
	70-21	4.4	2.1	4.2	-	1.0	0.6	1.5	-
	70-28	3.1	1.4	4.0	-	0.4	0.2	0.6	-
	70 total	**4.7**	**2.8**	**4.7**	**-**	**2.2**	**1.9**	**2.7**	**-**

With Xpress 2005B, the small XLP gaps of WBR are obtained at the cost of a large number of cuts and a large cut generation time at the top node (Table 5.15) relatively to CC. However, with Xpress 7.2 both are similar relatively to CC.

As noted before, XLP gaps with Xpress 7.2 are much smaller than with 2005B. The improvement occurs for all formulations.

Heuristics (Table 5.15)

Formulation WBR also has the advantage that automatic heuristics find good solutions more frequently at the top node than with CC and RR, and this applies to both solver versions. In the same respect, CC is better than RR.

The lower XLP gaps and good heuristic solutions with formulation WBR help to explain the lower number of nodes relatively to CC (Table 5.15).

Table 5.15: Additional results (*n*=70)

		Xpress 2005B			Xpress 7.2		
avg nodes	n-r	CC	WBR	RR	CC	WBR	RR
	70-7	2035	1618	2721	1743	1103	2289
	70-14	9423	2644	5397	1273	1103	2840
	70-21	1430	279	3880	83	36	560
	70-28	309	39	2487	4	2	105
	70 total	**3300**	**1145**	**3621**	**776**	**561**	**1448**
avg time XLP	n-r	CC	WBR	RR	CC	WBR	RR
	70-7	18	192	27	41	52	44
	70-14	23	192	32	66	57	76
	70-21	18	147	29	47	34	67
	70-28	14	76	23	34	23	47
	70 total	**19**	**152**	**27**	**47**	**42**	**58**
avg acuts	n-r	CC	WBR	RR	CC	WBR	RR
	70-7	37	159	60	91	141	62
	70-14	116	264	174	195	203	210
	70-21	129	228	184	121	126	160
	70-28	109	217	152	116	122	143
	70 total	**98**	**217**	**142**	**131**	**148**	**144**
total htop gap 0%	n-r	CC	WBR	RR	CC	WBR	RR
	70-7	1	1	0	1	5	0
	70-14	0	1	0	1	3	0
	70-21	0	2	0	6	6	2
	70-28	2	3	0	7	9	5
	70 total	**3**	**7**	**0**	**15**	**23**	**7**
total htop gap 5%	n-r	CC	WBR	RR	CC	WBR	RR
	70-7	3	5	0	5	9	3
	70-14	3	5	0	8	6	0
	70-21	5	9	0	9	9	3
	70-28	2	8	0	9	9	5
	70 total	**13**	**27**	**0**	**31**	**33**	**11**

7. Conclusion

Our computational experiments with the CM model can be summarized as follows: formulation (CC) was the most efficient, (WBR) was competitive (with Xpress 7.2), while (RR) and (DK) had much higher solution times (by a factor of 5 or more).

In our experiments, we also observed that the relative performance of formulations can be influenced not only by the strength of the initial LP relaxation but also by the effect of the several components of a generic branch-and-cut solver, such as presolve, cut generation and heuristics. Indeed, our results with formulation (WBR) were surprising: the number of non-zero elements was the smallest after LP presolve (while it was the largest before) in Xpress 7.2; the LP gap after adding cuts at the top node was smaller than with (CC) (while it was larger before); heuristic solutions at the top node were better or found more frequently than with all other formulations. Thus, for models other than CM, or with other solvers, it may also be worthwhile trying (WBR).

The performance of alternative formulations of CA constraints may vary for different location models, depending e.g. on the type of objective function and on the presence of minimum and/or maximum capacity constraints. Nevertheless, on the basis of previous results in the literature and of our work in this chapter, we can offer some general observations and recommendations.

First, we observe that (DK) should not be used since it is clearly dominated by (WF). Both have the flexibility of being applicable to $X^{\leq}$ in addition to $X^{=}$, but (DK) leads to a weaker and much larger formulation. In our experiments with the CM model, (DK) had clearly the worst performance of all formulations.

We start with recommendations for solving a standard formulation with a generic MIP optimizer, without additional development effort. For models with feasible set $X^{=}$ (or a restriction of it):

- All the formulations (CC), (WF), (WBR), (RR), and (DK) are applicable.
- (CC) or its equivalent (WF) should be the first choice, as it leads to a stronger LP relaxation, while not requiring a larger formulation. It performed best in our experiments with the CM model.
- (WBR) may also be worthwhile to try. Even though (WBR) seems a poor choice relatively to (CC), because the initial formulation is weaker and has more non-zero elements, in practice it can perform well, as shown in our experiments on the CM model with Xpress 7.2.

- (RR) (and its generalization (RR2) to deal with distance ties) is less promising, as it leads to a weaker LP relaxation than (CC) and was found to perform worse in some particular models. In our experiments with the CM model, it performed significantly worse than (CC). In previous experiments with particular models, Scaparra and Church (2008) and Berman et al. (2006) found that (RR) performed worse than (CC) and (WBR), respectively.
- (DK) should not be used, as noted above.
- For set $X^{p=}$ (which is a restriction of $X^{=}$, so all the above applies), it may also be worthwhile to experiment with the new formulation *EMR* of CA constraints proposed by Espejo et al. (2012). This formulation also has $O(n^2)$ constraints and is not dominated by (CC) (nor dominates it). No computational results have been reported with this formulation.

For models with feasible set $X^{\leq}$ (or a restriction of it still allowing centers not to be assigned to any facility):

- Only formulations (WF), (WBR), (DK) are applicable.
- (WF) should be the first choice, as it leads to the strongest LP relaxation.
- (WBR) may also be worthwhile to try; however, as far as we know, no previous experiments have been reported.
- (DK) should not be used, as noted above.

We now comment on using the a priori reformulation procedure proposed in this chapter, based on a previous one by Cánovas et al. (2007). For models with feasible set $X^{=}$:

- This procedure is recommended since it requires only moderate implementation effort and reduces solution times significantly, according to our experiments and those of Cánovas et al. (2007).
- Steps 1-4 can be used with all CA formulations considered here. Note that steps 1, 2, 3.1, and 4 are valid with any constraints that enforce CA. Step 3.2 (drop CA constraints made redundant by equalities (W_{eq}) added in step 3.1) was shown to be valid for (WF). Since constraints (WF) dominate the alternative constraints considered here, step 3.2 can also be applied to the latter.
- For set $X^{p=}$ using formulation *EMR*, step 3.2 may no longer apply or may weaken the formulation, since (WF) does not dominate *EMR* (Espejo et al., 2012). We did not further analyze this issue.
- Steps 5-6 apply only to formulations (WF) and (CC).

For models with feasible set $X^{\leq}$:

- Only step 5 is valid for set $X^{\leq}$, and only if formulation (WF) is used. No computational results have been reported for this case. The other steps rely on inequalities that are valid only when all centers must be assigned (demand constraints hold with "=").

Finally, we offer some comments on other recommendations in the literature:

- **Relax the integrality of variables *x***: All CA formulations allow the integrality of variables x to be relaxed in the definition of set $X^{=}$ (for pairs (i, j) without distance ties). Gerrard and Church (1996) and Espejo et al. (2012) suggest that such reduction in the number of integer variables might improve the performance of a branch-and-bound algorithm, but no computational results are offered or cited.

  In our experiments, we chose to leave variables x defined as binary and to give branching priority to variables y (the rationale is that assignments are subsidiary to location decisions, especially when CA applies). This priority suffices to render variables x innocuous for the branch-and-bound tree (except when equidistant closest facilities occur), and indeed significantly reduced solution times, as noted before in this chapter. Additionally, leaving variables x defined as binary may be useful to help reduce or tighten the formulation through automatic presolve and cut generation procedures relying on that information. However, we did not perform experiments with continuous variables x to verify whether performance was improved or degraded.

- **Drop constraints (V) when using (RR)**: Formulation (RR) allows constraints (V) to be dropped from the definition of set $X^{=}$ (restricted to guarantee at least one open facility), although at the cost of a weaker LP relaxation. Gerrard and Church (1996) suggest that reducing model size by dropping all or a subset of (V) might improve computational performance, but no computational results are offered or cited.

  We observe that this approach does not seem very promising in general, since it is well known that constraints (V) are crucial to strengthen the formulations of many location models. That is, dropping (V) will likely only improve performance for particular models in which the objective function and constraints other than (V) and (RR) promote LP solutions that naturally tend to satisfy (V). In those cases, all CA formulations may also perform better if (V) is replaced by the weaker but smaller set of constraints $\sum_{i \in I} x_{ij} \leq n \cdot y_j$ for $j \in J$. In the case of the CM model, dropping (V) leads to very weak LP relaxations and degrades performance very significantly, as verified in experiments described above.

- **Other CA formulations**: In the conclusion of their recent article, Espejo et al. (2012) focus their recommendations on using formulations providing the tightest LP relaxations. The authors suggest using (C2) instead of (WF) but restricted to $O(n^2)$ constraints (by picking a single additional center for each pair of centers and sites), which would still provide a tighter formulation. For set $X^{p=}$, the authors suggest using (C2) as above, or the new formulation *EMR* proposed by the authors (which also requires $O(n^2)$ constraints, and is neither dominated by nor dominates (C2)), or both in conjunction with one set of constraints included a priori and the other set used to generate cuts. The authors leave implementation details and experiments to future work. Thus, while these approaches may be computationally effective, they have not been tested before.

Chapter 6

Solving facility location models with modern optimization software: the weak and strong formulations revisited

6.1 Introduction

The classic fixed-charge facility location problems – uncapacitated, capacitated, and capacitated with single-sourcing, denoted UFLP, CFLP and CFLPSS, respectively – consider decisions of locating facilities (such as plants or warehouses in supply chain networks or concentrators in telecommunications networks) in a discrete set of potential sites and assigning discrete demand centers to those facilities, with the objective of minimizing the total costs of satisfying all demand, composed of fixed costs of installing facilities and variable costs of operation and transportation. Formulations, model extensions, solution methods and applications are reviewed by Klose and Drexl (2005) and Gourdin et al. (2002).

Right from the first studies of these models, the so-called weak and strong variants of mixed-integer linear programming (MIP) formulations have been considered, differing in the constraints linking assignment and location variables, and involving a trade-off between model size and linear programming (LP) relaxation bounds. For the UFLP, it is well known that, for many instances of practical interest, the LP relaxation of the strong formulation gives integer or close to integer solutions, requiring no or very few additional nodes of branch-and-bound (Krarup and Pruzan, 1983). The same happens with other uncapacitated location models, such as the *p*-median problem (ReVelle, 1993). Capacitated models such as the CFLP are harder to solve, but still the strong formulation produces smaller LP gaps (Cornuejols et al., 1991). Although many early LP-based branch-and-bound algorithms were based on the weak formulation (Sridharan, 1995), the strong formulation of the CFLP is used frequently by several authors proposing different solution approaches. These include branch-and-cut (Aardal, 1998b; Avella and Boccia, 2009) and Lagrangian relaxation (Sridharan, 1995). Díaz and Fernández (2001) solve the strong formulation of the CFLPSS with a generic MIP optimizer to provide a benchmark for a specialized exact algorithm.

In this chapter we present computational experiments on solving the CFLP and CFLPSS (and also the UFLP for comparison) with a modern MIP optimizer implementing a generic branch-and-cut algorithm. The aim is to compare the computational performance of well-known formulation variants, combining the weak and strong variants, and additional constraints involving facility capacities that are redundant for the LP relaxation but help the optimizer recognize certain model relaxations and generate strong cutting planes, as recommended by Aardal (1998b).

Our empirical investigation is motivated by previous, more restricted computational experiments done by the authors with the CFLP, where surprisingly the weak formulation was found to be solved faster than the strong one with a generic MIP optimizer. Trick (2005) discusses a similar result with a particular instance of the CFLP. Here we aim to: (i) identify instance data types, including different levels of capacity and fixed costs, for which the weak formulation may be solved faster than the strong one; (ii) identify the most effective formulation, among the variants referred to above; (iii) check whether results for the CFLP also apply to the related CFLPSS, as well as to an unrelated facility location model, the capacitated median model, which does not involve fixed location costs and involves minimum rather than maximum capacity constraints.

We point out that using a generic MIP optimizer, even with the most effective formulation possible, may not be suitable for every application. There are efficient specialized algorithms that are able to solve large instances to optimality or near optimality. Exact algorithms include those of: Körkel (1989) for the UFLP; Avella and Boccia (2009), Görtz and Klose (2012) for the CFLP; Díaz and Fernández (2001), Avella et al. (2011) for the CFLPSS. Heuristic algorithms include those of: Barahona and Chudak (2005) for the UFLP and CFLP; Ahuja et al. (2004) for the CFLPSS.

Nevertheless, our results can be useful to practitioners wishing to solve models efficiently with minimal modeling effort, or to researchers using a generic MIP optimizer to provide a benchmark for specialized solution algorithms.

In our experiments we use FICO's Xpress MIP optimizer. General descriptions of MIP software implementing branch-and-cut algorithms are given by Atamturk and Savelsbergh (2005) and Lodi and Linderoth (2011). State-of-the-art commercial MIP software packages include: Xpress (Ashford, 2007; Laundy et al. 2009), CPLEX (Bixby et al., 2000; Bixby and Rothberg, 2007), and Gurobi (Bixby, 2011). The references cited describe the components of MIP solvers and their historical performance evolution.

The remainder of this chapter is organized as follows: section 2 addresses fixed-charge location problems; section 3 addresses the capacitated median model; section 4 offers overall conclusions.

6.2 Fixed-charge location problems

6.2.1 Formulations

We start by presenting mixed-integer linear programming formulations of the UFLP, CFLP and CFLPSS. We are given a set of demand centers $I=\{1,...,n\}$, a set of sites where facilities can be installed $J=\{1,...,m\}$, and the following data, for $i\in I$ and $j\in J$: d_i is the demand of center *i*; s_j is the maximum capacity of site *j*; f_j is the fixed cost of installing and operating a facility at site *j*; c_{ij} is the variable cost of serving all the demand of center *i* from site *j*. Decision variables are: $y_j=1$ if a facility is installed (or open) at site $j\in J$, and equals zero otherwise; x_{ij} is the fraction of the demand of center $i\in I$ served from site $j\in J$.

The uncapacitated facility location problem, UFLP, is the problem of finding which facilities to open, and assigning centers to those facilities, in order to minimize the total cost of serving all the demand. Capacities are assumed to be as large as necessary. The *weak formulation* of the UFLP is

(W-UFLP):

Min $\displaystyle\sum_{i\in I}\sum_{j\in J}c_{ij}x_{ij}+\sum_{j\in J}f_jy_j$

Subject to $\displaystyle\sum_{j\in J}x_{ij}=1,\ \forall i\in I$ (D)

$$\sum_{i\in I}x_{ij}\le n\cdot y_j,\ \forall j\in J \tag{U}$$

$$y_j\in\{0,1\},\ \forall j\in J \tag{I}$$

$$x_{ij}\ge 0,\ \forall i\in I, j\in J \tag{N}$$

In this formulation and those following, constraints are identified by letters using a notation similar to the one of Cornuejols et al. (1991). Constraints (D) state that all demand has to be served. Constraints (U) link *x* and *y* variables, and state that if a facility is open it can serve any number of centers, otherwise no center can be assigned to it. Expressions (I) and (N) are integrality and non-negativity constraints on variables. Note that (N) and (D) imply $0\le x_{ij}\le 1$ for all *i* and *j*.

The m constraints (U) can be replaced with the $n \times m$ variable upper-bound constraints

$$x_{ij} \leq y_j,\ \forall i \in I, j \in J \tag{B}$$

which state that center i can only be served by a facility at j if it is open. By doing this, we obtain the *strong formulation* of the UFLP:

(S-UFLP):

Min $\quad \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \sum_{j \in J} f_j y_j$

Subject to (D), (B), (I), (N)

The strong and weak integer formulations are equivalent, i.e. define the same set of integer solutions. However, their LP relaxations, obtained by replacing (I) with $0 \leq y_j \leq 1$ for $j \in J$, are not equivalent. Formulation (S-UFLP) is said to be stronger or tighter, as the set of solutions to its LP relaxation is strictly contained in the one of (W-UFLP). Moreover, constraints (B) have the property of being facet-defining for the UFLP polytope (Cornuejols et al., 1990). It is well known that, in many instances of practical interest, the LP relaxation of (S-UFLP) gives integer or close to integer solutions, requiring no or very few additional nodes of branch-and-bound (Krarup and Pruzan, 1983).

The capacitated facility location problem, CFLP, is obtained from the UFLP by imposing maximum capacities on facilities. Its weak formulation is

(W1-CFLP):

Min $\quad \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \sum_{j \in J} f_j y_j$

Subject to $\quad \sum_{j \in J} x_{ij} = 1,\ \forall i \in I \qquad$ (D)

$$\sum_{i \in I} d_i x_{ij} \leq s_j y_j,\ \forall j \in J \tag{C}$$

$$y_j \in \{0,1\},\ \forall j \in J \tag{I}$$

$$x_{ij} \geq 0,\ \forall i \in I, j \in J \tag{N}$$

where (C) are the capacity constraints. The strong formulation is obtained by adding constraints (B):

(S1-CFLP):

Min $\quad \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \sum_{j \in J} f_j y_j$

Subject to (D), (C), (B), (I), (N)

While constraints (B) are redundant for the integer formulation, they strengthen the LP relaxation (Cornuejols et al., 1991) and were shown (by Aardal, cited by Avella and Boccia, 2009) to be facet-defining for the CFLP polytope for all $i \in I$ and $j \in J$ such that $d_i < s_j$. Constraints (B) can also strengthen Lagrangian relaxations (Cornuejols et al., 1991; Sridharan, 1995). For these reasons, the strong formulation appears frequently in the literature, as was noted in the introduction.

Additional redundant constraints have been proposed in the literature, based on aggregate capacity and demand expressions. They are redundant even for the LP relaxation, but strengthen Lagrangian relaxations or help branch-and-cut solvers recognize particular model relaxations for which strong cutting planes can be generated.

First, we recall the total capacity constraint

$$\sum_{j \in J} s_j y_j \geq \sum_{i \in I} d_i \,, \tag{T}$$

which states that the total capacity of open facilities should cover total demand. Note that (T) is redundant both for the integer formulation and the LP relaxation of the CFLP, as it can be obtained from constraints (C) and (D). However, Cornuejols et al. (1991) show the theoretical and practical advantages of adding (T) to strengthen Lagrangian relaxations.

Second, we recall the constraint set

$$z_j = \sum_{i \in I} d_i x_{ij} \,, \ \forall j \in J \tag{Z}$$

$$z_j \leq s_j y_j \,, \ \forall j \in J \tag{C′}$$

$$\sum_{j \in J} z_j = \sum_{i \in I} d_i \tag{F}$$

$$z_j \geq 0 \,, \ \forall j \in J \tag{Nz}$$

where (Z) define auxiliary *occupied capacity* variables, (C′) replaces (C), and (F) establishes the *flow equilibrium* between total occupied capacity and total demand.

Aardal (1998b) proposes a formulation obtained from (S1-CFLP) and using the constraint set above. While not improving the LP relaxation bound, this formulation enables a MIP optimizer to recognize the surrogate knapsack set, X^K, and the single-node flow set, X^{SNF},

$$X^K = \left\{ y \in \{0,1\}^m : (\mathrm{T}) \right\}$$

$$X^{SNF} = \left\{ z \in R_+^m, y \in \{0,1\}^m : (\mathrm{F}), (\mathrm{C}') \right\}$$

and to generate lifted cover cuts and flow cover cuts, developed for X^K and X^{SNF}, respectively, which are facet-defining inequalities for the CFLP polytope under some conditions (Aardal, 1998b). In the experiments of Aardal, using the MINTO branch-and-cut solver, significant time reductions were obtained (on average 20-80% for groups of instances with varying capacity data).

Note that (T) was not explicitly added to the formulation proposed by Aardal, but presumably it was recognized by MINTO automatically by combining (C′) and (F). In our experiments using the Xpress solver (described below), we noticed that explicitly adding (T) as well led to some variation in cut generation (in terms of number of cuts and LP bound at the top node) and systematically reduced solution time (only slightly or by up to 30%, both for the weak and strong formulations of the CFLP and CFLPSS models). For this reason, (T) was also added to the formulation in our experiments.

To summarize, we consider the following CFLP formulation variants:

- (W1-CFLP) denotes Min $\sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \sum_{j \in J} f_j y_j$, s.t. (D), (C), (I), (N);
- (W2-CFLP) denotes (W1-CFLP) with (T) added;
- (W3-CFLP) denotes (W1-CFLP) with (C) removed, (Z), (C′), (F), (Nz) added, and (T) also added;
- (S1-CFLP) denotes (W1-CFLP) with (B) added;
- (S2-CFLP) denotes (W2-CFLP) with (B) added;
- (S3-CFLP) denotes (W3-CFLP) with (B) added;

We call W1 and S1 the *standard* formulations, and the others *aggregate* formulations.

Finally, the CFLPSS is the capacitated facility location problem with single sourcing (or single assignment), in which each center's demand cannot be split and must be served from a single facility. For the UFLP, an optimal solution can be found where variables x are either zero or one without explicitly requiring them to be integer (Krarup and Pruzan, 1983). This property does not hold for capacitated models and, if required, must be enforced with integrality constraints:

$$x_{ij} \in \{0,1\}, \ \forall i \in I, j \in J \qquad \text{(SS)}$$

We consider six variants of CFLPSS formulations, obtained by replacing (N) with (SS) in the CFLP formulations above.

6.2.2 Instances

The test instances were randomly generated using the procedure of Cornuejols et al. (1991), which was devised for the CFLP. Their main features include variable costs depending on Euclidean distances, fixed costs reflecting economies of scale, and varying capacity levels. We opted for this procedure as these features can be considered representative of real-world problems and it has been used by several other authors for benchmarking algorithms for the CFLP, e.g. Aardal (1998b), Barahona and Chudak (2005), Avella and Boccia (2009), and Görtz and Klose (2012).

The procedure is the following, for a given size $n \times m$:

- points representing centers and sites are randomly generated in $[0,1]\times[0,1]$;
- demands d_i are generated from a uniform distribution $U[5,35]$;
- variable costs are set to $c_{ij} = 10 \cdot d_i \cdot e_{ij}$, where e_{ij} is the Euclidean distance between center *i* and site *j*;
- capacities s_j are generated from $U[10,160]$;
- fixed costs are generated with the formula $f_j = U[0,90] + U[100,110]\sqrt{s_j}$;
- capacities s_j are then scaled to obtain instances with different capacity levels Cap = $\sum_{j\in J} s_j \big/ \sum_{i\in I} d_i$ (note that fixed costs are not affected by this scaling).

Cornuejols et al. (1991) used this procedure for varying sizes with *m*/*n* = 1, 2/3 and 1/3. Five instances of each size and capacity level were generated with Cap = 1.5, 2, 3, 5 and 10. Fixed costs were then multiplied by two for Cap = 1.5 and 2.

For our experiments, we opted for capacity levels spanning a wider range, and to generate instances with varying fixed costs, obtained by applying a multiplier, denoted Fix. We consider only one instance size for each of the CFLP and CFLPSS models, in order to reduce the amount of results to report and analyze here. We chose sizes that were neither too easy nor too hard to solve with the computer hardware used. We generated 10 instances of each of size, capacity level and fixed cost level as follows:

- Size 200x200, Cap = 2, 5, 10, 20, 50, and Fix = 0.2, 1, 5 for the CFLP;
- Size 50x50, Cap = 2, 3, 5, and Fix = 0.2, 1, 5 for the CFLPSS.

For the CFLPSS a smaller size was considered, as the model is much harder to solve for tight capacity levels, and a smaller maximum capacity level, as for higher levels the model becomes much easier to solve (it tends to reduce to the CFLP).

For the UFLP, we used the same instances as for the CFLP, although only the 10 basic instances for each fixed cost level, as capacities are not defined.

As shown in Table 6.1 and Table 6.2, the capacity and fixed cost levels considered cover wide variations of fixed cost weight and number of open facilities in optimal solutions.

Table 6.1: CFLP and UFLP instances – characteristics of optimal solutions (averages for 10 instances of size 200x200)

	Optimal value (IP)			% Fixed costs in IP			Open facilities			% Open facilities		
Cap \ Fix	0.2	1	5	0.2	1	5	0.2	1	5	0.2	1	5
2	18 610	81 233	391 685	85%	96%	99%	65	62	62	32%	31%	31%
5	9 443	33 649	151 740	67%	89%	97%	26	23	23	13%	12%	11%
10	7 253	19 711	78 119	50%	76%	93%	17	12	11	9%	6%	6%
20	6 321	14 050	43 987	45%	58%	84%	19	7	6	10%	4%	3%
50	5 623	11 265	26 101	36%	46%	63%	18	7	3	9%	3%	2%
Uncap.	5 519	9 390	16 166	33%	34%	38%	16	7	3	8%	3%	1%

Table 6.2: CFLPSS instances – characteristics of optimal solutions (averages for 10 instances of size 50x50)

	Optimal value (IP)			% Fixed costs in IP			Open facilities			% Open facilities		
Cap \ Fix	0.2	1	5	0.2	1	5	0.2	1	5	0.2	1	5
2	5 837	22 534	104 678	75%	91%	98%	18	17	17	36%	33%	33%
3	4 292	15 323	68 669	67%	88%	97%	12	11	11	24%	21%	21%
5	3 295	10 030	41 754	57%	80%	94%	8	6	6	17%	13%	13%

6.2.3 Software

All experiments were carried out with FICO's Xpress Optimization Suite version 7.2 (released in May 2011). Models were implemented with Xpress Mosel 3.2 and were solved with Xpress MIP Optimizer 22.01, running on a computer with a Pentium M 755 2.0 GHz CPU, 1.0 GB of memory, and Windows XP operating system.

A maximum computing time of 30 minutes was imposed for solving each instance. Optimizer parameters for presolve, cut generation, primal heuristics, and branch-and-bound were left at their default values, except the ones described next.

Cut generation parameters were set as follows: *cutfactor* = 100, *covercuts* = 50, *cutstrategy* = -1 (automatic). These parameters have the following meaning:

- *cutfactor*: sets a limit on the number of cuts and non-zero cut elements the optimizer is allowed to add to the formulation, relative to the size of the initial formulation (a value of 1 allows the number of rows and elements to double).

- *covercuts*: sets the maximum number of rounds of lifted cover cuts and other special purpose cut types (all types except Gomory and lift-and-project cuts).
- *cutstrategy*: specifies the aggressiveness of cut generation, from 1 to 3, with more cuts being generated with a higher level (no further details are provided in the documentation).

By default the solver sets these parameters automatically, depending on the model and instance being solved. In preliminary experiments with the UFLP, CFLP and CFLPSS, it was verified that automatic parameters were too conservative. The settings above enlarge the default limits and benefitted all formulations (strong and weak, standard and aggregate) of the three models and all data types (Cap, Fix), but particularly weak formulations with large capacity ratios (which have the largest LP gaps). Note: a *cutfactor* of 100 essentially removes the limit set by this parameter; *covercuts* by default was limited to 20, while the new limit of 50 was hit in less than 5% of CFLP instances tested and in less than 30% of CFLPSS instances tested; *cutstrategy* was left at the default, after verifying in preliminary tests that it is equivalent to setting 1 and that increasing it to 2 or 3 slightly increased solution times.

For the CFLPSS, branching priority was given to y variables over x variables. In preliminary experiments this reduced solution time significantly (total solution time decreased by 20-70% for formulations W2 and S2 and instances with Fix=1 and Cap=2 or 3).

Finally, we note that modern MIP solvers are able to automatically recognize the validity of variable upper bound constraints (B) when a weak formulation is used, e.g. using (U), and add them as cuts if they are violated. Such cuts are called implication cuts in Xpress (Ashford, 2007) and implied bound cuts in CPLEX (Bixby et al., 2000). In our experiments with Xpress 7.2, it was verified (by inspecting the formulation after the top node) that indeed all violated (B) constraints are added to the weak UFLP and CFLP formulations (as long as cut parameters limiting the number of cuts that can be added are suitably modified as indicated above; also not considering rounds of Gomory and lift-and-project cuts in Xpress, which are generated after all other cut types and may lead to a new fractional solution violating some of the (B) constraints). In the case of the CFLP, this means that the solver recognized (B) given (C) and (D). Thus, at the top node the solver can add to the weak formulation all (or most) of the (B) constraints that would be active in the strong formulation. However, the same will not occur at tree nodes, since cut generation is not invoked at all nodes.

6.2.4 Presentation of computational results

We present results disaggregated by capacity and fixed cost level, since results and the relative performance of formulations vary widely with these parameters. Results are not presented for individual instances, for the sake of brevity, and are aggregated for the 10 instances of each capacity and fixed cost level.

Arithmetic averages of the following results are reported:

- Time: computation time in seconds;
- Nodes: number of nodes in the branch-and-bound tree;
- Gap LP = (IP-LP)/IP: gap of the first LP relaxation value, LP, relative to the optimal value IP;
- Gap XLP = (IP-XLP)/IP: gap of the LP relaxation value at the top node after adding cuts, XLP, relative to the optimal value IP;
- Cuts: number of cuts added to the formulation at the top node (the total number of cuts generated may be higher, as cuts that become inactive in successive cut generation iterations are deleted);
- Gap H = (IP1-XLP)/XLP: gap of the heuristic solution value at the top node, IP1, relative to the best lower bound at that node (XLP);

In column “Optimal” we also report the number of instances solved to optimality within the time limit. For instances not solved to optimality, we considered the values of time and nodes observed at the time limit.

In addition, we compare solution times between pairs of formulations with two measures:

- Ratio of total time: ratio between the total solution times (or equivalently the average times) with the two formulations. This ratio assesses the relative performance to solve a batch of instances.
- Geometric mean of time ratios: geometric mean of the ratios between solution times of individual instances with the two formulations. This ratio assesses the relative performance to solve a single instance, and may differ from the previous ratio, which can be dominated by a large individual time. The geometric mean was chosen instead of the arithmetic mean since it is less sensitive and more conservative when large individual time ratios occur (it was also used e.g. by Bixby et al., 2000).

6.2.5 UFLP results

We start by presenting results for the UFLP (Table 6.3). The strong formulation is solved clearly faster. Although results of individual instances are not shown, the LP gap is positive in only 5 of the 30 instances tested, and in those cases it is closed at the top node with few cuts and no branching is required.

The weak formulation is solved more than 10 times slower, even though the solver is able to close the LP gap at the top node in most instances. It can also be observed that instances with higher fixed costs are harder to solve.

Note: For solving the weak formulation, cut generation parameters were set differently than indicated in section 6.2.3: only "implication cuts" enabled (*cutselect* = 4096), increased maximum number of cut rounds (*covercuts* = *m* = 200), and other parameters as before (*cutfactor* = 100, default for others). With this setting, only violated variable upper bound constraints (B) are added to the formulation, until no such violations remain at the top node. With the cut settings of section 6.2.3, with all cut types enabled by default, solution time was much higher (the solver wastes time generating cuts that are not as effective). We also tested the weak formulation with the default cut generation parameters of Xpress 7.2, which drastically limit the number of cut rounds and the number of cuts and cut elements that can be added. Branch-and-bound would be initiated with a large LP gap at the top node and instances could not be solved to optimality within the time limit.

In conclusion, the strong formulation is clearly better for the UFLP (assuming it fits the available computer memory), confirming well-known results from the literature.

Table 6.3: UFLP results (size 200x200)

Formul.	Fix	Optimal	Time	Nodes	Gap LP (%)	Gap XLP (%)	Cuts
S	0.2	10	1	1	0.01	0.00	1
	1	10	2	1	0.04	0.00	3
	5	10	4	1	0.07	0.00	2
W	0.2	10	10	1	70.57	0.00	2213
	1	10	70	1	75.08	0.01	4935
	5	10	246	8	67.40	0.05	9371

6.2.6 CFLP results

Results for the CFLP are presented in Table 6.4 and Table 6.5. We first note that solution times vary widely with capacity and fixed cost levels. Focusing first on formulation S1, instances are harder to solve for lower capacity levels (for higher levels the CFLP tends to reduce to the UFLP) and for higher fixed costs. Other formulations show similar trends, except that with aggregate formulations (strong or weak) instances become easier with higher fixed costs for lower capacity levels, particularly for the lowest level (Cap=2).

Regarding the relative performance of formulations:

- Aggregate formulations (W2, W3, S2, S3) can be solved much faster than the respective standard ones (W1, S1) for lower capacity levels and higher fixed cost levels. In the case of capacity levels, we observe that fewer cover inequalities are likely to be violated when facility capacity is less tight.

- Comparing formulations 3 and 2, although 3 would be expected to perform at least as well as 2, some variations are observed. S2 and S3 have similar performance, and no one dominates the other for all capacity and fixed cost levels. W2 is solved systematically faster than W3, particularly for higher capacity ratios (W3 can be solved slower by a factor of 2; although this could be perhaps changed with different solver cut parameters, this issue was not further analyzed as strong formulations are anyway solved faster in those cases, as discussed below).

- Weak formulations are solved faster than the respective strong ones for lower capacity levels and higher fixed costs, but are solved slower for other data levels. The precise transition capacity level depends on the fixed cost level; however, for the instances we tested, weak formulations were solved as fast (within 10%) or faster than strong ones for capacity levels up to 10 with all fixed cost levels.

- The good performance of weak formulations can be attributed to the following factors: (i) with lower capacity and higher fixed cost levels, fewer variable upper bound constraints (B) tend to be violated by LP relaxations (see Table 6.6); (ii) the smaller formulation size allows faster re-optimization of LP relaxations at branch-and-bound nodes (even if many more nodes are required than with strong formulations, as shown in Table 6.4).

- The summary in Table 6.5 (where W3 and S3 are omitted for simplicity) shows that, for Cap 10 or less and Fix 1 or more, formulation 2 relatively to 1 is solved faster by a factor of 2-10 or more, and weak formulations relative to strong ones are solved faster by a factor of 2 or more.

- The last panel of Table 6.5 (S1/W2) shows that if one adopts the standard strong formulation (S1) rather than the best formulation (W2) for lower capacity levels, solution times can be much higher, up to a factor of more than 100 for high fixed cost levels.

Other observations:

- Gaps LP and XLP: The weak formulation has much higher initial LP gap than the strong one (especially for higher capacity levels), but after adding cuts at the top node the gap is similar to the one of the strong formulation. It can also be observed that aggregate formulations are especially effective at reducing the top node LP gap for lower capacity levels and higher fixed costs (for which solution times also decrease the most, as noted above).

- Gap H: A high quality solution is found already at the top node by the solver's heuristic procedures. Such solution (which was found in all instances tested) had a small gap (Gap H) of less than 1% on average (when using an aggregate strong formulation, or an aggregate weak formulation for low capacity levels).

In conclusion, on the basis of our experiments, the following formulations are recommended for the CFLP: W2 or W3 for lower capacity ratios (Cap=10 or less); S2 or S3 for higher capacity ratios.

Note: In the original procedure of Cornuejols et al. (1991) to generate test instances, capacity levels varied up to 10, which is also the range for which we found the weak formulation to be competitive. However, we also note that those authors also generated instances with *m*<*n*, while here we test only *m*=*n*. If *m* is decreased (holding *n*, total demand and total capacity fixed) then fewer facilities will have to be open to satisfy all demand. If fixed costs per site remain at the same level (i.e. are not affected by capacity scaling, as is the case in this generation procedure), the importance of fixed costs in objective values will decrease. This tends to make the weak formulation less competitive, as noted above. Still, with *m*=*n*/2 or *m*=*n*/3 we expect the weak formulation to perform better than the strong one for a capacity level up to 10 and for a fixed cost level of 1 or more (in our test, the weak formulation was still competitive for *m*=*n* and Fix=0.2).

Table 6.4: CFLP results (size 200x200)

		Optimal						Nodes					
Fix	Cap	W1	W2	W3	S1	S2	S3	W1	W2	W3	S1	S2	S3
0.2	2	10	10	10	10	10	10	1960	956	1343	1761	1375	1253
	5	9	9	9	8	8	9	6224	4192	4517	1678	1749	1736
	10	10	10	10	10	10	10	1071	1166	1453	276	342	306
	20	10	10	9	10	10	10	542	623	789	150	147	127
	50	10	10	10	10	10	10	53	43	21	1	1	2
1	2	10	10	10	9	10	10	3841	43	52	2066	50	56
	5	8	10	10	6	10	10	10574	1673	1904	2791	1031	1266
	10	10	10	10	9	10	10	3102	726	1147	881	426	495
	20	10	10	8	10	10	10	1105	770	575	147	102	126
	50	10	10	9	10	10	10	213	240	188	26	28	25
5	2	6	10	10	5	10	10	12555	1	1	3317	4	2
	5	6	10	10	2	10	10	17511	129	113	779	122	178
	10	6	10	10	2	10	10	13865	118	93	464	73	65
	20	6	10	10	1	10	10	10815	249	319	312	59	49
	50	10	10	4	9	10	10	838	69	33	112	6	10

		Time						Ratio of total time (relative to W2)					
Fix	Cap	W1	W2	W3	S1	S2	S3	W1	W2	W3	S1	S2	S3
0.2	2	140	94	104	324	307	267	1.5	**1.0**	**1.1**	3.4	3.3	2.8
	5	665	520	603	937	1002	929	1.3	**1.0**	1.2	1.8	1.9	1.8
	10	247	272	477	232	273	264	**0.9**	**1.0**	1.8	**0.9**	**1.0**	**1.0**
	20	181	216	361	72	72	77	0.8	1.0	1.7	**0.3**	**0.3**	0.4
	50	60	55	53	5	7	9	1.1	1.0	1.0	**0.1**	**0.1**	0.2
1	2	222	10	10	494	30	26	22.0	**1.0**	**1.0**	48.9	3.0	2.6
	5	653	164	186	1382	554	547	4.0	**1.0**	**1.1**	8.4	3.4	3.3
	10	341	174	307	844	348	423	2.0	**1.0**	1.8	4.8	2.0	2.4
	20	400	442	884	394	236	265	0.9	1.0	2.0	0.9	**0.5**	0.6
	50	318	384	968	79	69	80	0.8	1.0	2.5	**0.2**	**0.2**	**0.2**
5	2	813	5	6	1159	27	19	162.6	**1.0**	1.2	231.8	5.4	3.8
	5	997	20	22	1551	108	94	50.3	**1.0**	**1.1**	78.3	5.5	4.7
	10	935	37	38	1571	87	95	25.5	**1.0**	**1.0**	42.9	2.4	2.6
	20	1026	134	241	1783	240	235	7.7	**1.0**	1.8	13.3	1.8	1.8
	50	715	370	1388	957	228	236	1.9	1.0	3.8	2.6	**0.6**	**0.6**

Bold – Fastest or within 10% of the fastest

Table 6.4 (continued)

		Gap LP (%)						Gap XLP (%)					
Fix	Cap	W1	W2	W3	S1	S2	S3	W1	W2	W3	S1	S2	S3
0.2	2	0.5	0.5	0.5	0.2	0.2	0.2	0.10	0.10	0.11	0.11	0.11	0.10
	5	9.2	9.2	9.2	0.6	0.6	0.6	0.49	0.50	0.50	0.48	0.49	0.48
	10	27.6	27.6	27.6	0.4	0.4	0.4	0.33	0.33	0.34	0.31	0.31	0.31
	20	44.9	44.9	44.9	0.4	0.4	0.4	0.29	0.30	0.30	0.22	0.20	0.21
	50	58.7	58.7	58.7	0.2	0.2	0.2	0.17	0.15	0.16	0.02	0.01	0.03
1	2	0.1	0.1	0.1	0.1	0.1	0.1	0.05	0.02	0.02	0.05	0.02	0.02
	5	1.6	1.6	1.6	0.3	0.3	0.3	0.29	0.24	0.25	0.30	0.25	0.25
	10	8.9	8.9	8.9	0.5	0.5	0.5	0.47	0.39	0.39	0.46	0.39	0.40
	20	27.6	27.6	27.6	0.4	0.4	0.4	0.40	0.36	0.36	0.39	0.35	0.35
	50	53.4	53.4	53.4	0.3	0.3	0.3	0.24	0.23	0.24	0.21	0.20	0.20
5	2	0.1	0.1	0.1	0.1	0.1	0.1	0.10	0.00	0.00	0.09	0.00	0.00
	5	0.3	0.3	0.3	0.2	0.2	0.2	0.21	0.07	0.06	0.21	0.07	0.07
	10	1.6	1.6	1.6	0.7	0.7	0.7	0.66	0.17	0.16	0.66	0.18	0.18
	20	8.1	8.1	8.1	1.8	1.8	1.8	1.74	0.30	0.31	1.73	0.31	0.31
	50	31.2	31.2	31.2	1.3	1.3	1.3	1.31	0.20	0.23	1.25	0.18	0.18

		Cuts						Gap H (%)					
Fix	Cap	W1	W2	W3	S1	S2	S3	W1	W2	W3	S1	S2	S3
0.2	2	92	98	82	31	29	32	0.2	0.1	0.2	0.2	0.2	0.2
	5	390	391	381	20	21	21	1.5	0.9	1.1	0.7	0.8	0.7
	10	1014	1011	941	15	15	18	1.3	1.1	0.9	0.4	0.4	0.4
	20	1536	1532	1280	20	19	20	1.2	1.7	0.9	0.2	0.2	0.2
	50	1499	1478	1243	12	12	15	2.3	1.5	0.3	0.0	0.0	0.0
1	2	37	59	52	18	26	23	0.3	0.0	0.0	0.2	0.0	0.0
	5	183	239	217	19	21	21	1.0	0.3	0.4	1.1	0.6	0.5
	10	572	691	617	11	14	14	1.3	0.9	0.9	0.9	0.5	0.6
	20	1848	1871	1482	13	12	11	2.3	3.8	1.8	0.5	0.5	0.6
	50	3982	3900	2207	12	12	9	2.5	2.6	1.3	0.2	0.2	0.2
5	2	12	54	47	11	25	20	0.5	0.0	0.0	0.6	0.0	0.0
	5	79	166	166	19	20	18	0.9	0.1	0.1	1.3	0.2	0.1
	10	191	362	348	12	17	16	2.2	0.5	0.2	2.6	0.2	0.2
	20	680	1119	929	9	11	11	3.7	0.8	0.7	4.3	0.7	0.4
	50	3972	4298	3099	11	15	18	4.2	1.9	1.5	2.9	0.2	0.2

Table 6.5: CFLP results (size 200x200) – summary of time ratios

Ratio of total time

S vs. W	S1/W1			S2/W2		
Cap \ Fix	0.2	1	5	0.2	1	5
2	2.3	2.2	1.4	3.3	3.0	5.4
5	1.4	2.1	1.6	1.9	3.4	5.5
10	0.9	2.5	1.7	1.0	2.0	2.4
20	0.4	1.0	1.7	0.3	0.5	1.8
50	0.1	0.2	1.3	0.1	0.2	0.6

1 vs. 2	W1/W2			S1/S2		
Cap \ Fix	0.2	1	5	0.2	1	5
2	1.5	22.0	162.6	1.1	16.3	43.3
5	1.3	4.0	50.3	0.9	2.5	14.3
10	0.9	2.0	25.5	0.8	2.4	18.2
20	0.8	0.9	7.7	1.0	1.7	7.4
50	1.1	0.8	1.9	0.7	1.1	4.2

Combined	S1/W2		
Cap \ Fix	0.2	1	5
2	3.4	48.9	231.8
5	1.8	8.4	78.3
10	0.9	4.8	42.9
20	0.3	0.9	13.3
50	0.1	0.2	2.6

Geometric mean of time ratios

S vs. W	S1/W1			S2/W2		
Cap \ Fix	0.2	1	5	0.2	1	5
2	2.8	4.3	4.3	3.0	3.1	5.4
5	1.6	3.3	3.1	2.2	3.4	5.2
10	1.0	3.0	3.9	1.0	1.9	2.3
20	0.4	1.0	2.5	0.3	0.5	1.9
50	0.1	0.3	1.3	0.1	0.2	0.6

1 vs. 2	W1/W2			S1/S2		
Cap \ Fix	0.2	1	5	0.2	1	5
2	1.2	5.9	42.5	1.2	8.2	33.9
5	1.3	3.1	25.2	0.9	3.0	14.8
10	0.9	1.5	12.1	0.9	2.4	20.6
20	0.8	0.9	5.8	1.0	1.8	7.8
50	1.0	0.8	2.0	0.8	1.1	4.2

Combined	S1/W2		
Cap \ Fix	0.2	1	5
2	3.5	25.7	182.5
5	2.1	10.2	77.0
10	0.9	4.5	47.2
20	0.3	0.9	14.6
50	0.1	0.2	2.6

Table 6.6: CFLP instances – average number of violated variable upper bound constraints (B) by the first LP relaxation of the weak formulation, relatively to the number of centers (*n*)

Cap \ Fix	0.2	1	5
2	25%	6%	2%
5	96%	69%	23%
10	100%	100%	73%
20	100%	100%	100%
50	100%	100%	100%

6.2.7 CFLPSS results

CFLPSS results are presented in Table 6.7 and Table 6.8. We first note that instances are harder to solve for lower capacity ratios and higher fixed costs, similarly to what was observed for the CFLP (however, for the CFLPSS this applies also to aggregate formulations with low capacity levels, which was not the case for the CFLP). For the instances tested, the CFLPSS becomes much easier to solve for capacity level 5 or higher. The explanation is that, with larger facility capacities, more centers satisfy single sourcing constraints even if these are omitted (solving a CFLP with the same instances we observed that the proportion of centers assigned to more than one facility in optimal solutions were 27%, 17%, 10% and 5% for capacity levels of 2, 3, 4 and 5, respectively, in the case of fixed cost level 1).

We find, like for the CFLP, that aggregate formulations are generally solved faster than the standard ones (except with fixed cost level 0.2, for which instances are relatively easy and standard formulations are slightly faster) and weak formulations are generally solved faster than the corresponding strong ones. Comparing aggregate formulations 3 and 2, the best one varies for different data, both for the strong and weak variants.

In Table 6.8 we summarize the relative performance of formulations (we omit Cap=5, as instances are easily solved in under 20 seconds on average with aggregate formulations; we consider formulation 3 instead of 2 like for the CFLP, since S3 is solved about twice as fast as S2 for the harder instances with Cap=2 and Fix=1 and 5). Weak formulations are solved faster than strong ones by a factor of up to about 1.5 for Fix=1. Aggregate formulations 3 are solved faster than the standard ones by a factor of about 1.5-3 times for Fix=1. The last panel (S1/W3) shows that if one adopts the standard strong formulation (S1) rather than the more effective W3, solution times can be higher by a factor of 4 or more for Fix=1 or 5.

Unlike for the CFLP, the relative performance of formulations for individual instances (tables show only aggregate results) is not systematic, i.e. one formulation with higher average time may be solved faster in particular instances. For this reason, in Table 6.8 the ratio of total time may be less than 1 but the geometric mean of time ratios may be higher than 1 (or vice versa).

In conclusion, on the basis of our experiments, the W2 or W3 formulations are recommended for the CFLPSS for capacity levels of up to 5 (and possibly higher). Although we did not test much higher capacity levels (say of more than 10), the

CFLPSS can be expected to reduce to the CFLP and formulations S2 or S3 are likely to perform better, as for the CFLP.

Table 6.7: CFLPSS results (size 50x50)

		Optimal						Nodes					
Fix	Cap	W1	W2	W3	S1	S2	S3	W1	W2	W3	S1	S2	S3
0.2	2	10	10	10	10	10	10	3664	3503	5118	5564	5135	7613
	3	10	10	10	10	10	10	1139	1527	1262	1146	1303	1140
	5	10	10	10	10	10	10	371	355	538	378	162	361
1	2	7	9	10	6	9	9	25963	20174	11181	22482	18420	12294
	3	9	10	10	10	10	10	98336	7788	8395	15889	9185	8600
	5	10	10	10	10	10	10	2641	1255	1224	3462	1231	1152
5	2	8	9	9	5	8	10	30249	16633	104725	50250	20939	9224
	3	10	10	10	7	10	10	74783	25966	11038	162461	16857	15548
	5	10	10	10	10	10	10	16449	3056	1440	24674	2551	2188

		Time						Ratio of total time (relative to W2)					
Fix	Cap	W1	W2	W3	S1	S2	S3	W1	W2	W3	S1	S2	S3
0.2	2	80	76	88	111	151	128	**1.1**	**1.0**	1.2	1.5	2.0	1.7
	3	18	28	25	23	28	25	**0.6**	1.0	0.9	0.8	1.0	0.9
	5	5	6	9	8	7	8	**0.8**	1.0	1.5	1.3	1.1	1.3
1	2	751	362	257	1025	610	357	2.1	1.0	**0.7**	2.8	1.7	1.0
	3	284	69	89	211	121	129	4.1	**1.0**	1.3	3.1	1.8	1.9
	5	18	12	16	29	14	13	1.5	**1.0**	1.3	2.4	1.2	**1.1**
5	2	852	364	473	1235	630	284	2.3	1.0	1.3	3.4	1.7	**0.8**
	3	289	164	93	915	170	240	1.8	1.0	**0.6**	5.6	1.0	1.5
	5	31	11	9	86	16	18	2.8	1.0	**0.8**	7.7	1.4	1.6

Bold – Fastest or within 10% of the fastest

		Gap LP (%)						Gap XLP (%)					
Fix	Cap	W1	W2	W3	S1	S2	S3	W1	W2	W3	S1	S2	S3
0.2	2	6.0	6.0	6.1	5.0	5.0	5.1	0.58	0.56	0.51	0.57	0.56	0.61
	3	7.5	7.5	7.5	3.6	3.6	3.6	0.75	0.78	0.83	0.70	0.70	0.78
	5	16.0	16.0	16.0	2.7	2.7	2.7	0.81	0.80	0.87	0.76	0.71	0.86
1	2	2.1	2.1	2.1	2.1	2.1	2.1	0.66	0.27	0.25	0.66	0.27	0.25
	3	2.4	2.4	2.4	1.8	1.8	1.8	0.82	0.55	0.55	0.82	0.54	0.54
	5	5.5	5.5	5.5	2.2	2.2	2.2	1.56	1.11	1.08	1.52	1.08	1.09
5	2	1.1	1.1	1.1	1.1	1.1	1.1	0.68	0.06	0.06	0.70	0.06	0.05
	3	1.1	1.1	1.1	1.0	1.0	1.0	0.82	0.09	0.08	0.77	0.10	0.09
	5	1.9	1.9	1.9	1.7	1.7	1.7	1.52	0.19	0.19	1.41	0.18	0.19

Table 6.7 (continued)

Fix	Cap	Cuts						Gap H (%)					
		W1	W2	W3	S1	S2	S3	W1	W2	W3	S1	S2	S3
0.2	2	182	235	148	249	166	148	3.7	2.5	2.8	1.9	2.3	5.3
	3	127	143	152	93	95	91	2.6	3.2	3.6	1.9	2.6	2.6
	5	150	153	169	52	52	46	1.7	2.1	3.2	2.5	1.0	3.6
1	2	147	225	216	168	206	188	4.1	5.2	7.3	3.9	5.0	8.2
	3	136	184	130	92	93	91	4.0	6.2	8.2	4.0	5.8	8.6
	5	97	140	152	36	32	38	4.9	6.3	8.0	5.0	6.1	7.4
5	2	172	236	238	176	190	192	4.4	5.2	7.6	5.1	4.9	7.5
	3	81	114	222	63	90	109	4.2	4.8	8.2	4.3	5.7	8.8
	5	58	119	104	27	36	37	5.7	4.5	6.0	4.2	4.1	3.5

Table 6.8: CFLPSS results (size 50x50) – summary of time ratios

Ratio of total time

S vs. W	S1/W1			S3/W3		
Cap \ Fix	0.2	1	5	0.2	1	5
2	1.4	1.4	1.4	1.5	1.4	0.6
3	1.3	0.7	3.2	1.0	1.4	2.6

1 vs. 3	W1/W3			S1/S3		
Cap \ Fix	0.2	1	5	0.2	1	5
2	0.9	2.9	1.8	0.9	2.9	4.3
3	0.7	3.2	3.1	0.9	1.6	3.8

Combined	S1/W3		
Cap \ Fix	0.2	1	5
2	1.3	4.0	2.6
3	0.9	2.4	9.8

Geometric mean of time ratios

S vs. W	S1/W1			S3/W3		
Cap \ Fix	0.2	1	5	0.2	1	5
2	1.2	1.6	1.8	1.6	1.1	1.0
3	1.2	1.2	2.8	1.2	1.4	2.5

1 vs. 3	W1/W3			S1/S3		
Cap \ Fix	0.2	1	5	0.2	1	5
2	1.1	2.7	2.9	0.9	4.0	5.3
3	0.8	1.5	2.3	0.8	1.3	2.6

Combined	S1/W3		
Cap \ Fix	0.2	1	5
2	1.4	4.4	5.2
3	0.9	1.8	6.4

6.3 Capacitated median model

6.3.1 Formulations

In the capacitated median (CM) model the aim is to locate facilities and assign all demand centers to those facilities so that the total travel distance (or time) is minimized, centers are assigned to the closest facility, and facilities satisfy minimum and maximum capacity bounds.

Fixed costs are not considered in this model but, like in the previous models, the number of facilities is a model output, resulting from the upper and lower bounds imposed by the minimum and maximum capacity constraints, respectively. The model is called the capacitated median model due to its relation with the classic *p*-median model – see the discussion by Teixeira and Antunes (2008), who also review applications, mostly dedicated to the location of health care and education facilities.

Here we consider a restricted version of this model with non-binding maximum capacity bounds, thus no maximum capacity constraints are included, and with constant minimum capacity bounds. We consider the same notation as in the previous models and additionally: $b_j = b$ is the minimum facility capacity for site $j \in J$; t_{ij} is the travel distance (or time) between center $i \in I$ and site $j \in J$; $c_{ij} = t_{ij} d_i$ for $i \in I, j \in J$.

The strong formulation of the CM model is as follows:

(S1-CM):

Min $\sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$

Subject to $\sum_{j \in J} x_{ij} = 1, \; \forall i \in I$ (D)

$x_{ij} \le y_j, \; \forall i \in I, j \in J$ (B)

$\sum_{i \in I} d_i x_{ij} \ge b_j y_j, \; \forall j \in J$ ($\underline{\text{C}}$)

$\sum_{k \in J : t_{ik} \le t_{ij}} x_{ij} \ge y_j, \; \forall i \in I, j \in J$ (CA)

$y_j \in \{0,1\}, \; \forall j \in J$ (I)

$x_{ij} \in \{0,1\}, \; \forall i \in I, j \in J$ (SS)

where ($\underline{\text{C}}$) are minimum capacity constraints and (CA) are closest assignment constraints, stating that if a facility is installed at *j*, then center *i* has to be assigned to a facility at a travel distance of t_{ij} or less.

The weak formulation of the CM model, (W1-CM), is obtained from (S1-CM) by replacing (B) with (U).

If the CM model is augmented with maximum capacity constraints (C), the aggregate constraints with redundant information discussed for the CFLP can also be added. We now discuss analogous constraints for the case of minimum capacities. The constraint analogous to (T) is

$$\sum_{j\in J} b_j y_j \le \sum_{i\in I} d_i \,. \qquad (\underline{\mathrm{T}})$$

Since $b_j = b$ for $j \in J$ is assumed, this constraint can be reduced to

$$\sum_{j\in J} y_j \le \left\lfloor \sum\nolimits_{i\in I} d_i \big/ b \right\rfloor \qquad (\underline{\mathrm{T}}')$$

which now is not redundant for the LP formulation since the integer round-down function was applied. The constraints analogous to (C′) are

$$z_j \ge b_j y_j,\ \forall j \in J \qquad (\underline{\mathrm{C}}')$$

Let (S2-CM) denote (S1-CM) with ($\underline{\mathrm{T}}'$) added.

Let (S3-CM) denote (S2-CM) with ($\underline{\mathrm{C}}$) replaced by ($\underline{\mathrm{C}}'$) and (Z), (F), (Nz) added.

Let (W2-CM) and (W3-CM) denote, respectively, (S2-CM) and (S3-CM) with (B) replaced by (U).

Note: Van Roy and Wolsey (1986) first introduced a variant of flow cover inequalities considering lower capacity bounds as well as upper bounds, which would apply to the single node flow structure in formulations S3 and W3 above. Even though Xpress can generate flow cover cuts, the documentation does not make clear if they include the variant for lower capacity bounds.

6.3.2 Instances

Test instances for the CM model were randomly generated and assumed identical sets of centers and sites ($I = J$). First, for a given size n (number of centers), 9 data sets (c_{ij}, u_i) were created: points representing centers were uniformly generated in [0,100]×[0,100]; $u_i = 1000/n\times \mathit{Uniform}[0.1,1.9]$; $c_{ij} = u_i \times d_{ij}$, where d_{ij} is the Euclidean distance between centers i and j. Then, the capacity ratio r, equal to total demand divided by the minimum capacity (i.e. the expected maximum number of open facilities), was used as a control parameter to derive 4 instances from each data set (c_{ij}, u_i) by setting $b = \sum_{i\in I} u_i \big/ r$ for

$r = n$×0.1, 0.2, 0.3, and 0.4. All data was rounded to integer values. Here we consider instances of a single size $n = 70$.

6.3.3 Software

The same MIP optimizer and computer were used as in the other models. All solver parameters were left at default values, except that branching priority was given to y variables. All instances were solved to optimality (in less than 30 minutes).

6.3.4 Results

Regarding weak formulations for the CM model, W2 was tested first and was verified not to be competitive with S2. Thus other weak formulations were not tested.

Results are shown in Table 6.9 and the following can be observed:

- S2 can be considered better than S1 and W2 as it is solved significantly faster for one or both of the harder, smaller capacity ratios r=7 and 14, while being similar or not much worse for other ratios r.
- S3 has an overall performance similar to S2 and no formulation seems to dominate the other.

Based on these results, we conclude that it is indifferent to choose either S2 or S3, and both are preferable to S1 and W2.

We also observe that the solver closes the XLP gaps of W2 to values similar to those of S2. Some surprising XLP gaps are observed (but were not further analyzed): the XLP gap of W2 is the smallest among all formulations for r=7; the XLP gap of S1 is smaller than the one of S2 for r=14 and 28.

Table 6.9: CM model results (size 70x70)

n-r	Optimal				Nodes			
	S1	S2	S3	W2	S1	S2	S3	W2
70-7	9	9	9	9	2509	1743	1259	2154
70-14	9	9	9	9	1055	1273	1451	1824
70-21	9	9	9	9	96	83	83	136
70-28	9	9	9	9	4	4	3	6
70 total	36	36	36	36	916	776	699	1030

n-r	Time				Ratio of total time			
	S1	S2	S3	W2	S1	S2	S3	W2
70-7	652	368	325	724	1.8	1.0	0.9	2.0
70-14	283	331	355	422	0.9	1.0	1.1	1.3
70-21	50	55	64	49	0.9	1.0	1.2	0.9
70-28	35	35	28	29	1.0	1.0	0.8	0.8
70 total	255	197	193	306	1.3	1.0	1.0	1.6

n-r	Gap LP (%)				Gap XLP (%)			
	S1	S2	S3	W2	S1	S2	S3	W2
70-7	8.3	5.6	5.6	49.1	6.3	4.0	4.2	3.8
70-14	10.7	9.9	9.9	30.8	3.2	3.4	3.8	3.3
70-21	13.9	13.6	13.8	23.1	1.0	1.0	0.7	1.0
70-28	15.3	15.3	21.1	26.6	0.2	0.4	0.2	0.4
70 total	12.1	11.1	12.6	32.4	2.7	2.2	2.2	2.1

n-r	Cuts				Gap H (%)			
	S1	S2	S3	W2	S1	S2	S3	W2
70-7	108	91	61	529	12.7	9.1	6.4	7.9
70-14	198	195	188	365	6.9	9.5	7.3	9.2
70-21	133	121	147	205	1.3	1.3	3.1	1.2
70-28	117	116	63	112	0.3	1.0	0.3	0.5
70 total	139	131	115	303	5.3	5.2	4.3	4.7

6.4 Conclusion

We presented an empirical investigation of solving the CFLP with a generic MIP optimizer, comparing several formulation variants. Test instances, generated with a procedure often used for benchmarking purposes, cover a wide range of capacity levels (total potential capacity divided by total demand) and fixed cost levels, since solution times and the relative performance of formulations strongly depend on these parameters.

Our results show that the CFLP can be solved faster with a generic MIP optimizer by considering: (i) a formulation with additional constraints to help the solver detect certain model relaxations, which may not be recognized automatically and for which strong cutting planes can be generated; (ii) the weak variant of the formulation instead of the strong one, for lower capacity levels (up to 10 in our tests and for a wide range of fixed cost levels).

The first recommendation follows the one by Aardal (1998b), who reported significant solution time reductions with the research-oriented MINTO optimizer. Here we confirm the same applies to a modern commercial optimizer, with time being reduced by a factor of 2-10 or more, depending on the capacity level.

Using the weak variant of the formulation reduces time further by a factor of 2 or more. Time reductions increase for higher fixed cost levels and lower capacity levels. The good performance of weak relative to strong formulations can be attributed to the fact that fewer variable upper bound constraints are required to strengthen the LP relaxation when capacities are tighter, and to faster re-optimization of the smaller formulations at branch-and-bound nodes (even though more nodes may be explored). For higher capacity levels, as the CFLP tends to reduce to the UFLP, the strong formulation becomes more effective, since variable upper bound constraints are essential to tighten LP bounds.

For the CFLPSS similar results were obtained, although with lower time reductions. For the capacitated median model, an unrelated model, similar results were not obtained and the strong formulation was more effective.

The results of our experiments can be useful for practitioners wishing to solve the CFLP or CFLPSS model with a generic MIP optimizer, with minimal effort without resorting to a specialized algorithm. They can also be useful for obtaining a benchmark for comparing with a specialized algorithm, by using the most effective formulation available in a generic MIP optimizer.

In out experiments we used the Xpress optimizer. We expect that similar results are likely to be obtained with other software, such as CPLEX or Gurobi, but this was not tested.

With the particular version of Xpress used, we verified that performance increased (for all formulations of all models) by changing default cut generation parameters to a less conservative one (in terms of the number of cuts the solver is allowed to add). However, we did not aim to optimize solver parameters for the location models tested (e.g. by switching off the generation of ineffective cut types or by tuning tree search strategies), and the conclusions above implicitly assume such optimization, if possible, would not change the relative performance of formulations significantly.

Chapter 7

Conclusion

7.1 Discussion of applications

Model applications in chapters 2, 3 and 4 were presented independently. This section offers a comparison of these applications, in order to further clarify the modeling assumptions adopted, and to further discuss the adequacy and limitations of the models.

Model assumptions

Inelastic demand: Both the school and court location applications assumed exogenous and inelastic demand with respect to travel costs (any other user-supported costs are assumed not to vary by location and to be already reflected in the demand forecasts). In the case of schools, this assumption is reasonable since primary and secondary education are mandatory (however, for extra-curricular education activities, such as music or sports, demand is likely to be elastic and the models used would not be representative). In the case of justice, it was deemed reasonable to assume most litigation demand is inelastic with respect to travel costs.

Minimum capacity: In applications to schools, minimum capacity bounds were set according to the guidelines on school size by the Ministry of Education, in the case of new schools, and were set for the purposes of the study in agreement with the Municipal Council of Coimbra, in the case of existing schools. These bounds reflect concerns both with student achievement in too small schools and with financial sustainability. In the application to courts, minimum capacity bounds were set for the purposes of the study in agreement with the Ministry of Justice and reflect concerns with financial sustainability, as well as with guaranteeing diversity of experience for judges in specialized courts.

Closest assignment: In all applications, assignment of demand centers to the closest facility was a planning assumption. In the case of courts, this assumption can be considered appropriate, since assignment to courts is mandated by the responsible public authority, taking into account proximity to the defendant's location. In this context, assuming closest assignment is likely to increase public confidence and acceptance of solutions.

In the case of schools, the appropriateness of the closest assignment assumption can be debated, since school assignment is not strictly mandated in Portugal: when applying their children to a new school, parents can specify a preference ordering of up to 5 schools (until 2011 primary schools had to be chosen within the home or work area of one of the parents; since 2012 any school can be chosen).

On the one hand, it is observed that proximity to home is, by far, the most important factor parents take into account when choosing a school for their children, as already stated in chapter 2. According to a survey carried out for Coimbra's Educational Charter (Canavarro et al., 2004), the two most frequent factors identified by parents were proximity to home (63% of respondents) and teacher quality (38%); proximity to workplace was much less frequent (18%). In addition, when the number of applicants for a school exceeds its capacity, proximity to home is by law one of the primary factors school directors take into account for approving applications. On the other hand, it should be acknowledged that proximity to home is not the only or the most important preference factor for all people.

To further discuss this issue, we outline two alternative approaches for a school network planning problem, differing in the treatment of user preferences (the two approaches were also discussed and exemplified in the literature review of chapter 2):

(1) Assume that accessibility is the main factor in school choice, and use a model that maximizes accessibility and assumes closest assignment. This was the approach followed in this thesis, with accessibility being defined in relation to the place of residence.

(2) Assume that preferences for schools include accessibility and other factors such as perceived teaching quality, and use a model that maximizes the preferences met. This approach requires describing and measuring the attractiveness of facilities and the preferences of users, and weighting preferences into a single measure.

It can be argued that the first approach is more appropriate for strategic planning problems that focus on defining the location and capacity of facilities in the long-term (e.g. in a horizon of 10 years). User preferences are considered only approximately by focusing on accessibility. The closest assignment assumption is also compatible with stated principles of education and spatial planning policies, of providing high education quality across all schools (according to the Basic Law of the Education System in

Portugal) and of providing education services in proximity to residential areas (as stated in Coimbra's Educational Charter).

In contrast, the second approach is more appropriate for operational problems of managing installed capacity in the short-term (e.g. on a yearly basis) that focus on defining assignments of users to facilities once user preferences are declared. If such an approach is used for a strategic planning problem, it requires forecasting the attractiveness of facilities and the preferences of users into the future (including in relation to potential new facilities), with results that are likely to be very inaccurate and to generate controversy among decision makers.

To conclude, assuming that closest assignment applies to all users is debatable in the case of schools. This was a planning assumption deemed reasonable in the applications in this thesis, but it may not be suitable for all school planning applications, e.g. if schools are expected in the future to have widely varying perceived education quality and parents can freely choose schools regardless of proximity to home.

Data preparation

Aggregation of population centers: In chapter 2 (secondary schools), the municipality of Coimbra was discretized into 43 population centers, while in chapter 3 (primary schools) the same region was discretized into 68 centers. The more detailed discretization is justified by the fact that proximity is a greater concern for primary schools, particularly for the first level schools (EB1), which exist in a larger number and have smaller catchment areas.

Travel distances or times: In all applications, travel distances or times were computed using a representation of the main road network. In the applications to schools, travel distances were used, which assumes that a constant average travel speed is representative of both private and public transportation under average congestion levels. This was considered acceptable since the spatial context is a municipality with moderate congestion levels and served by an extensive bus network. In the application to courts, travel times were used. Since the spatial context is a larger region, different average speeds were considered according to the hierarchical level of roads.

Model ingredients

Single vs. multiple services: in both secondary and primary school applications, education demand comprises two education levels. For secondary schools (chapter 2), a single service model is used, with demand for a single service representing two education levels (S1 and S2), being offered at a single facility type (ES12 schools). This could be assumed because both education levels should be assigned to the same school, and school capacity is shared by the two levels. For primary schools (chapter 3), a two-level service model is used, since two facility types are to be located – EB1 schools for B1 education and EB12 schools for both B1 and B2 education.

Co-location of existing and new facilities: In the secondary school model (chapter 2), the formulation allows the co-location of up to one existing facility and one new facility. This choice is justified because the spatial aggregation level is high and in some central population centers existing capacity is tight.

In the primary school model (chapter 3) the formulation allows the co-location of level-1 and level-2 facilities but not of new and existing facilities of the same type. In this application, the spatial aggregation level is smaller, but still level-1 and level-2 facilities exist in very close proximity in some centers. On the other hand, the existing capacity of level-2 facilities is generally not tight.

Path assignment: Path assignment (PA) constraints are introduced in chapter 3, where it is stated that they are an alternative to closest assignment (CA) constraints. Here we make two observations. First, in a model with PA replacing CA constraints, assignments may no longer be compatible with free choices of users that aim to minimize their individual travel costs. Thus, such model assumes that the public authority has the power to mandate assignments. In this context, PA constraints help by producing a spatial configuration of assignments that may be more easily accepted by users.

Second, PA and CA constraints may be used in combination if CA constraints are retained in the model but their data is modified so that they now forbid assignments further than a given tolerance relatively to the closest facility. As an example of such tolerance, model solutions in chapter 3 (primary schools) would remain the same if CA constraints were retained but given an absolute tolerance of 2.5 km.

Coherent assignment: This type of constraints was used in the application to courts (chapter 4), but not in the application to primary schools (chapter 3), where a two-level nested hierarchical model is also used. In the latter, the problem statement did not require coherent assignment constraints. However, if required, such constraints could be added to the model.

Hierarchical models: The applications to primary schools (chapter 3) and to courts (chapter 4) both consider models with a two-level nested hierarchy of facilities: low-level facilities serving only level-1 demand, and high-level facilities serving both level-

1 and level-2 demand. However, the formulation of the nested hierarchy is different in the two models, reflecting differences in problem statements. In the primary school location model, a level-2 facility corresponds to a single school that serves the two demand levels and capacity is shared by the two levels; the nested hierarchy is enforced by constraints (3.19) linking assignment and location variables. In the court location model, level-1 and level-2 facilities may be physically distinct, each one serves only the corresponding demand level and is subject to independent capacity constraints; the nested hierarchy is enforced by constraints (4.18) linking location variables.

7.2 Overall conclusion and contributions

An overall conclusion of the thesis is now presented, complementing the individual conclusions of previous chapters.

The discrete facility location models studied in this thesis provided a useful contribution to real-world public facility planning problems of reorganizing networks of schools and courts of justice. While the basic, single-service model had been previously presented in the literature, multiple-service (hierarchical) variants had not been presented before, combining hierarchical facilities, minimum capacity constraints and different types of constraints on the spatial pattern of user-to-facility assignments: closest assignment, path assignment, and coherent assignment.

Regarding solution methods, modern generic MIP optimization software proved to be robust and efficient to deal with the public facility planning applications in this thesis. Model instances in these applications turned out to be relatively easy to solve, since significant constraints were imposed on planning decisions in the context of reorganizing existing facility networks, such as limiting the numbers of new facilities to open and of existing facilities to close, or restricting assignments to existing administrative boundaries, thereby reducing the number of free decision variables.

The performance of generic MIP optimizers has been improving remarkably, as discussed in the references cited in section 1.3 and as illustrated by the results reported with Xpress versions 2005B and 7.2 in chapter 5. Nevertheless, a careful choice of model formulation is still recommended in order to take full advantage of modern MIP optimizers. For example, the alternative formulations of closest assignment constraints tested in chapter 5 (second appendix) lead to widely varying solution times. For the CFLP model, adding constraints with redundant information to help cut generation (following advice from the literature), and using the so-called weak formulation if capacity data is tight, can significantly reduce solution times, as tested in chapter 6.

The specialized solution method developed in chapter 5 for the single-service capacitated median model reduces the solution time of relatively large instances with 100 centers by 20-50% (with Xpress 7.2) relatively to the standard formulation. This reduction is mainly due to the a priori reformulation procedure adapted from previous work. Some new valid inequalities were presented, but they turned out not to be effective at further reducing solution times when used to generate cuts in a branch-and-cut algorithm.

Regarding the objectives outlined in the introduction, it can be said that they were generally accomplished, with some exceptions related to the development of specialized solution methods. First, for the single service model, the original work developed in this thesis gave only a modest improvement in solution times, as mentioned above. Second, while the development of specialized methods for multiple service models was an objective of the thesis, this work was not carried out and is left for future work.

The contributions of this thesis to the discrete facility location literature are the following:

- Formulation of optimization models combining multiple services, minimum and maximum capacity constraints, and constraints on the spatial pattern of assignments of users to facilities, extending previous hierarchical facility location models;
- Description of applications of models with single and multiple services to real-world problems of reorganizing networks of schools and courts of justice in Portugal;
- Development of new valid inequalities for the MIP formulation of the single service capacitated median model and proposal of an exact solution method, composed of a priori reformulation and branch-and-cut, that reduces solution times relatively to a generic MIP optimizer;
- Presentation of computational experiments on solving single service models with a modern generic MIP optimizer, including the fixed-charge capacitated facility location problem and the capacitated median model, in order to identify the most efficient formulation, among variants known from the literature, to solve these models to optimality without resorting to a specialized algorithm.

7.3 Future work

Possible future work arising from the work in this thesis can be outlined as follows:

- Adapt the reformulation procedure and specialized cuts of chapter 5 to the hierarchical models. Such an adaption should be relatively straightforward, since the valid inequalities can be applied to each service level separately. However its effectiveness to reduce solution times remains to be tested.
- Develop other separation procedures for the new valid inequalities proposed in chapter 5, to better exploit them in a branch-and-cut algorithm.
- Study the single and multiple-service models under uncertainty in the model data. Uncertainty in demand, rather than in travel costs, is deemed the most relevant for the types of applications studied in this thesis. The deterministic model can be used to solve independently a limited number of scenarios of data realizations (say, pessimistic, reference and optimistic scenarios of demand), but in this case comparing the solutions and selecting a final solution are left outside the scope of the model. In order to model uncertainty, and obtain a solution that "performs well" under all possible data realizations, a possible approach is scenario planning (Owen and Daskin, 1998; Current et al., 2002).
- Study alternative formulations of the hierarchical model with coherency constraints (the second model in chapter 4). A large number of coherency constraints is required (number of centers cubed) with the adopted formulation and its strength was not analyzed in this thesis (nor in previous work, as far as we know). Computational experiments with larger instances than those of the practical application in chapter 4 were also not performed. If such instances turn out to be hard to solve, an improved formulation would be useful.

References

Aardal, K. (1998a). Capacitated facility location: separation algorithms and computational experience. Mathematical Programming, 81 (2), 149-175.

Aardal, K. (1998b). Reformulation of capacitated facility location problems: how redundant information can help. Annals of Operations Research, 82 (0), 289-308.

Ahuja, R. K., Orlin, J. B., Pallottino, S., Scaparra, M. P., & Scutella, M. G. (2004). A multi-exchange heuristic for the single source capacitated facility location. Management Science, 50 (6), 749-760.

Antunes, A. (1994). De la Planification Optimale de l'Équipement Scolaire [Optimal planning of school infrastructure]. PhD thesis, Catholic University of Louvain, Belgium.

Antunes, A., & Peeters, D. (2001). On solving complex multi-period location models using simulated annealing. European Journal of Operational Research, 130 (1), 190-201.

Ashford, R. (2007), Mixed integer programming: a historical perspective with Xpress-MP. Annals of Operations Research, 149 (1), 5-17.

Atamturk, A., & Savelsbergh, M. W. P. (2005). Integer programming software systems. Annals of Operations Research, 140 (1), 67-124.

Avella, P., & Boccia, M. (2009). A cutting plane algorithm for capacitated facility location problems. Computational Optimization and Applications, 43 (1), 39-65.

Avella, P., Boccia, M., & Salerno, S. (2011). A computational study of dicut reformulation for the single source capacitated facility location problem. Studia Informatica Universalis, 9 (3), 21-42.

Balakrishnan, A., Magnanti, T. L., & Wong, R. T. (1995). A decomposition algorithm for local access telecommunications network expansion planning. Operations Research, 43 (1), 58-76.

Barahona, F., & Chudak, F. (2005). Near optimal solutions to large scale facility location problems. Discrete Optimization, 2 (1), 35-50.

Belotti, P., Labbé, M., Maffioli, F., & Ndiaye, M. M. (2007). A branch-and-cut method for the obnoxious p-median problem. 4OR, 5 (4), 299-314.

Berman, O., Drezner, Z., Tamir, A., & Wesolowsky, G. (2009). Optimal location with equitable loads. Annals of Operations Research, 167 (1), 307-325.

Berman, O., Krass, D., & Wang, J. (2006). Locating service facilities to reduce lost demand. IIE Transactions, 38 (11), 933-946.

Bigotte, J. F., & Antunes, A. P. (2007). Social infrastructure planning: a location model and solution methods. Computer-Aided Civil and Infrastructure Engineering, 22 (8), 570-583.

Bixby, R. (2011). The Gurobi optimizer. Presentation slides, Integer Programming Down Under (IPDU) workshop, University of Newcastle, Australia. Retrieved from http://carma.newcastle.edu.au/ nuor/ipdu/abstracts.html

Bixby, R., & Rothberg, E. (2007). Progress in computational mixed integer programming – a look back from the other side of the tipping point. Annals of Operations Research, 149 (1), 37-41.

Bixby, R., Fenelon, M., Gu, Z., Rothberg, E., & Wunderling, R. (2000). MIP: theory and practice – closing the gap. In Powell, M. J. D., & Scholtes, S. (Eds.), System modeling and optimization: methods, theory and applications (pp. 19-49). Kluwer.

Blum, C., & Roli, A. (2003), Metaheuristics in combinatorial optimization: overview and conceptual comparison. ACM Computing Surveys, 35(3), 268-308.

Brakman, S., Garretsen, H., & Van Marrewijk, C. (2001). An introduction to geographical economics. Cambridge University Press.

Canavarro, J. M., Pereira, M. D., David, R. M., Ramos, L. M., & Silva, P. (2004). Avaliação das expectativas da população sobre as escolas [Evaluation of the expectations of the population about schools]. Relatório para a Carta Educativa de Coimbra, Faculdade de Psicologia e Ciências da Educação, Universidade de Coimbra, Portugal.

Cánovas, L., García, S., Labbé, M., & Marín, A. (2007). A strengthened formulation for the simple plant location problem with order. Operations Research Letters, 35 (2), 141-150.

Caro, F., Shirabe, T., Guignard, M., & Weintraub, A. (2004). School redistricting: embedding GIS tools with integer programming. Journal of the Operational Research Society, 55 (8), 836-849.

Carreras, M., & Serra, D. (1999). On optimal location with threshold requirements. Socio-economic Planning Sciences, 33 (2), 91-103.

Chadwick, G. (1978). A systems view of planning. Pergamon Press.

Church, R. L., & Murray, T. (1993). Modeling school utilization and consolidation, Journal of Urban Planning and Development, 119 (1), 23-38.

Church, R. L. (2002). Geographical information systems and location science. Computers & Operations Research, 29 (6), 541-562.

Church, R. L. (2003). COBRA: a new formulation of the classic p-median location problem. Annals of Operations Research, 122 (1-4), 103-120.

Church, R. L., & Eaton, D. J. (1987). Hierarchical location analysis using covering objectives. In Ghosh, A., & Rushton, G. (Eds.), Spatial analysis and location-allocation models (pp. 163-185). Van Nostrand Reinhold.

Church, R. L., & Murray, A. T. (1993). Modeling school utilization and consolidation. Journal of Urban Planning and Development, 119 (1), 23-38.

Church, R. L., & Schoepfle, O. B. (1993). The choice alternative to school assignment. Environment and Planning B, 20 (4), 447-457.

Cordeau , J.-F., Pasin, F., & Solomon, M. M. (2006). An integrated model for logistics network design. Annals of Operations Research, 144 (1), 59-82.

Cornuejols, G., Nemhauser, G. L., & Wolsey, L. A. (1990). The uncapacitated facility location problem. In Mirchandani, P.B., & Francis, R.L. (Eds.), Discrete location theory (pp. 119-171). Wiley.

Cornuejols, G., Sridharan, R., & Thizy, J. M. (1991). A comparison of heuristics and relaxations for the capacitated facility location problem. European Journal of Operational Research, 50 (3), 280-297.

Csirik, J., Johnson, D. S., & Kenyon, C. (2001). Better approximation algorithms for bin covering. Proceedings of the 12th Annual ACM-SIAM Symposium on Discrete Algorithms, 557-566.

Current, J., Daskin, M. S., & Schilling, D. (2002). Discrete network location models. In Drezner, Z., & Hamacher, H. (Eds), Facility location: applications and theory (pp. 81-118). Springer.

Dash Optimization, Ltd (2002). Xpress-Optimizer reference guide – release 14. Blisworth, UK.

Dash Optimization, Ltd (2004), Xpress-Mosel language reference manual – release 1.4. Blisworth, UK.

Dash Optimization, Ltd (2005), Xpress-Optimizer reference manual – release 15. Blisworth, UK.

Daskin, M. S. (1995). Network and discrete location: models, algorithms, and applications. Wiley.

Deng, Q., & Simchi-Levi, D. (1992). Valid inequalities, facets and computational results for the capacitated concentrator location problem. Working paper, Department of Industrial Engineering and Operations Research, Columbia University, New York.

Densham, P. J., & Rushton, G. (1996). Providing spatial decision support for rural public service facilities that require a minimum workload. Environment and Planning B, 23 (5), 553-574.

DGS – Direcção-Geral da Saúde (2001). Rede de referenciação hospitalar de intervenção cardiológica [Hospital referral network of interventional cardiology]. Ministério da Saúde, Lisboa, Portugal.

Díaz, J. A., & Fernández, E. (2002). A branch-and-price algorithm for the single source capacitated plant location problem. Journal of the Operational Research Society, 53 (7), 728-740.

Draper, N. R., & Smith, H. (1998). Applied regression analysis (3rd ed.). Wiley.

Duque, J. C., Church, R. L., & Middleton, R. S. (2011). The p-regions problem. Geographical Analysis, 43 (1), 104-126.

Eiselt, H. A., & Laporte, G. (1995). Objectives in location problems. In: Drezner, Z. (Ed.), Facility location: a survey of application and methods (pp. 151-180). Springer.

Eitan, Y., Narula, S. C., & Tien, J. M. (1991). A generalized approach to modeling the hierarchical location-allocation problem. IEEE Transactions on Systems, Man, and Cybernetics, 21 (1), 39-46.

Erlenkotter, D. (1978). A dual-based procedure for uncapacitated facility location. Operations Research, 26 (6), 992-1009.

Espejo, I., Marín, A., & Rodríguez-Chía, A. M. (2012). Closest assignment constraints in discrete location problems. European Journal of Operational Research, 219 (1), 49-58.

ESRI – Environmental Systems Research Institute (2000). Using ArcView GIS. ESRI Press, Redlands, CA.

Galvão, R. D., Espejo, L. G. A., & Boffey, B. (2002), A hierarchical model for the location of perinatal facilities in the municipality of Rio de Janeiro. European Journal of Operational Research, 138 (3), 495-517.

Galvão, R. D., Espejo, L. G. A., Boffey, B., & Yates, D. (2006). Load balancing and capacity constraints in a hierarchical location model. European Journal of Operational Research, 172 (2), 631-646.

García, S. (2006). Advances in discrete location. PhD thesis, University of Murcia, Spain.

Geoffrion, A. M., & R. F. Powers (1980). Facility location analysis is just the beginning (if you do it right). Interfaces, 10 (2), 22-30.

Gerrard, R. A., & Church, R. L. (1996). Closest assignment constraints and location models: properties and structure. Location Science, 4 (4), 251-271.

Görtz, S., & Klose, A. (2012). A simple but usually fast branch-and-bound algorithm for the capacitated facility location problem. INFORMS Journal on Computing, published online before print July 21, 2011, doi:10.1287/ijoc.1110.0468.

Gourdin, E., Labbé, M., & Yaman, H. (2002). Telecommunication and location. In Drezner, Z., & Hamacher, H. (Eds.), Facility location: applications and theory (pp. 275-305). Springer.

Greenleaf, N., & Harrison, T. (1987), A mathematical programming approach to elementary school facility decisions. Socio-Economic Planning Sciences, 21 (6), 395-401.

Hanjoul, P., & Peeters, D. (1987). A facility location problem with clients' preference orderings. Regional Science and Economics, 17 (3), 451-473.

Hansen, P., Kochetov, Y., & Mladenovic, N. (2004). Lower bounds for the uncapacitated facility location problem with user preferences. Les Cahiers du GERAD G-2004-24 (discussion paper), HEC Montréal.

Hansen, P., Labbé, M., Peeters, D., & Thisse, J.-F. (1987). Facility location analysis. In Lesourne, J., & Sonnenschein, H. (Eds.), Systems of cities and facility location (pp. 1-70). Harwood.

Healey, P. (1992). Planning through debate: the communicative turn in planning theory. Town Planning Review, 63(2), 143-162.

INE – Instituto Nacional de Estatística (2003). Projecções de população residente [Resident population forecasts], Portugal, 2000-2050. Instituto Nacional de Estatística, Lisboa, Portugal.

INE – Instituto Nacional de Estatística (2004). Projecções de população residente [Resident population forecasts], Portugal e NUTS II, 2000-2050. Instituto Nacional de Estatística, Lisboa, Portugal.

Kalcsics, J., Melo, M. T., Nickel, S., & Gündra, H. (2002). Planning sales territories – a facility location approach. In Operations Research Proceedings 2001 (pp. 141-148). Springer.

Kalcsics, J., Nickel, S., & Schröder, M. (2005). Towards a unified territorial design approach – applications, algorithms and GIS integration. TOP, 13 (1), 1-74.

Klose, A., & Drexl, A. (2005). Facility location models for distribution system design. European Journal of Operational Research, 162 (1), 4-29.

Körkel, M. (1989). On the exact solution of large-scale simple plant location problems. European Journal of Operational Research, 39 (2), 157–173.

Krarup, J., & Pruzan, P. M. (1983). The simple facility location problem: survey and synthesis. European Journal of Operational Research, 12 (1), 36-81.

Krarup, J., & Pruzan, P. M. (1990). Ingredients of locational analysis. In Mirchandani, P. B., & Francis, R. L. (Eds.), Discrete location theory (pp. 1-54). Wiley.

Labbé, M., & Louveaux, F. V. (1997). Locations problems. In Dell'Amico, M., Maffioli, F., & Martello, S. (Eds.), Annotated bibliographies in combinatorial optimization (pp. 261-281). Wiley.

Labbé, M., & Yaman, H. (2006). Polyhedral analysis for concentrator location problems. Computational Optimization and Applications, 34 (3), 377-407.

Labbé, M., Laporte, G., & Martello, S. (1995). An exact algorithm for the dual bin packing problem. Operations Research Letters, 17 (1) 9-18 .

Laundy, R., Perregaard, M., Tavares, G., Tipi, J., & Vazacopoulos, A. (2009). Solving hard mixed-integer programming problems with Xpress-MP: a MIPLIB 2003 case study. INFORMS Journal on Computing, 21 (2), 304-313.

Lodi, A., & Linderoth, J. T. (2011). MILP software. In Cochran, J. J. (Ed.), Wiley Encyclopedia of Operations Research and Management Science (pp. 3239-3248). Wiley.

Marianov, V., & Serra, D. (2002). Location problems in the public sector. In Drezner, Z., & Hamacher, H. (Eds.). Facility location: applications and theory (pp. 119-150). Springer.

Martello, S., & Toth, P. (1990). Knapsack problems: algorithms and computer implementations. Wiley.

Mehrotra, A., Johnson, E. L., & Nemhauser, G. L. (1998). An optimization based heuristic for political districting. Management Science, 44 (8), 1100-1114.

Melo, M. T, Nickel, S., & Saldanha-da-Gama, F. (2009). Facility location and supply chain management - a review. European Journal of Operational Research, 196 (2), 401-412.

MinEdu – Ministério da Educação (2000). Critérios de Reordenamento da Rede Educativa [Criteria for the Redeployment of School Networks]. Ministério da Educação, Lisboa, Portugal.

Narula, S. C. (1986). Minisum hierarchical location-allocation problems on a network: a survey. Annals of Operations Research, 6 (8), 257-272.

Nemhauser, G. L., & Wolsey, L. A. (1988). Integer and combinatorial optimization. Wiley.

OPJP – Observatório Permanente da Justiça Portuguesa (2002). Os tribunais e o território: um contributo para o debate sobre a reforma da organização judiciária em Portugal [The courts and the territory: a contribution to the debate on the reform of the judicial organization in Portugal]. Centro de Estudos Sociais, Universidade de Coimbra, Portugal.

OPJP – Observatório Permanente da Justiça Portuguesa (2006). A geografia da justiça – para um novo mapa judiciário [The geography of justice – for a new judiciary map]. Centro de Estudos Sociais, Universidade de Coimbra, Portugal.

Owen, S. H., & Daskin, M. S. (1998), Strategic facility location: a review. European Journal of Operational Research, 111 (3), 423-447.

Peeters, D., Thisse, J.-F., & Thomas, I. (2002). Modèles opérationnels de localisation des équipements collectifs [Operational models for locating public facilities]. In Cliquet, G., & Josselin, J.-M. (Eds.), Stratégies de localisation des entreprises commerciales et industrielles. De Boeck, Bruxelles.

Pizzolato, N. D., & Silva, H. (1997). The location of public schools: evaluation of practical experiences. International Transactions in Operational Research, 4 (1), 13-22.

Pizzolato, N. D., Barcelos, F. B., & Lorena, L. A. N. (2004). School location methodology in urban areas of developing countries. International Transactions in Operational Research, 11 (6), 667-681.

ReVelle, C. S. (1987). Urban public facility location. In Mills, E. (Ed.), Handbook of regional and urban economics, Vol. II (pp. 1053-1096). Elsevier.

ReVelle, C. S. (1993). Facility siting and integer-friendly programming. European Journal of Operational Research, 65 (2), 147-158.

ReVelle, C. S., & Eiselt, H. A. (2005). Location analysis: a synthesis and survey, European Journal of Operational Research, 165 (1), 1-19.

ReVelle, C. S., & Swain, R. W. (1970). Central facilities location, Geographical Analysis, 2 (1), 30-42.

ReVelle, C. S., Marks, D., & Liebman, J. C. (1970). An analysis of private and public sector location models. Management Science, 16 (11), 692-707.

Rømo, F., & Sætermo, I.-A. F. (2000). New courts of first instance: a model for analysis of optimal location of courts of first instance in Norway. SINTEF report STF38 A00613 (technical report), Trondheim, Norway.

Sahin, G., & Sural, H. (2007). A review of hierarchical facility location models, Computers & Operations Research, 34 (8), 2310-2331.

São Pedro, M. E., Santos, M. F., Baptista, M. R., & Correia, R. (2000). Uma leitura quantitativa do sistema educativo. In Carneiro, R. (Coord.), O futuro da educação em Portugal: tendências e oportunidades [The future of education in Portugal: trends and opportunities]. Ministério da Educação, Lisboa, Portugal.

Scaparra, M. P., & Church, R. L. (2008). A bilevel mixed-integer program for critical infrastructure protection planning. Computers & Operations Research, 35 (6), 1905-1923.

Schoepfle, O. B., & Church, R. L. (1991), A new network representation of a classic school districting problem. Socio-Economic Planning Sciences, 25 (3), 189-197.

Serra, D., & ReVelle, C. S. (1993). The pq-median problem: location and districting of hierarchical facilities. Location Science, 1 (4), 299-312.

Shulman, A., & Vachani, R. (1993). A decomposition algorithm for capacity expansion of local access networks. IEEE Transactions on Communications, 41 (7), 1063-1073.

Smith, H. K., Harper, P. R., Potts, C. N., & Thyle, A. (2009). Planning sustainable community health schemes in rural areas of developing countries. European Journal of Operational Research, 193 (3), 768-777.

Snyder, L. V. (2006). Facility location under uncertainty: a review. IIE Transactions, 38 (7), 537-554.

Sridharan, R. (1995). The capacitated facility location problem. European Journal of Operational Research, 87 (2), 203-213.

Talen, E., & Anselin, L. (1998). Assessing spatial equity: an evaluation of measures of accessibility to public playgrounds. Environment and Planning A, 30 (4), 595-613.

Teixeira, J., & Antunes, A. (2008). A hierarchical location model for public facility planning. European Journal of Operational Research 185 (1), 92-104.

Teixeira, J., Antunes A., & Peeters, D. (2007). An optimization model for the redeployment of a secondary school network. Environment and Planning B, 34 (2), 296-315.

Trick, M. A. (2005). Formulations and reformulations in integer programming. In Barták, R., & Milano, M. (Eds.), Integration of AI and OR techniques in constraint programming for combinatorial optimization problems (CPAIOR 2005), Lecture Notes in Computer Science 3524 (pp. 366-379). Springer.

Van Roy, T. J., & Wolsey, L. A. (1986). Valid inequalities for mixed 0-1 programs, Discrete Applied Mathematics, 14 (2), 199-213.

Vasilyev, I. L., & Klimentova, K. B. (2010). The branch and cut method for the facility location problem with client's preferences. Journal of Applied and Industrial Mathematics, 4 (3), 441-454.

Vasilyev, I. L., Klimentova, K. B., & Kochetov, Y. (2009). New lower bounds for the facility location problem with clients' preferences. Computational Mathematics and Mathematical Physics, 49 (6), 1010-1020.

Verter, V., & Lapierre, S. (2002). Location of preventive health care facilities. Annals of Operations Research, 110 (1-4), 123-132.

Wagner, J., & Falkson, L. (1975). The optimal nodal location of public facilities with price-sensitive demand. Geographical Analysis, 7 (1), 69-83.

Wang, Q., Batta, R. J., & Rump, C. M. (2002). Algorithms for a facility location problem with stochastic customer demand and immobile servers. Annals of Operations Research 111 (1-4), 17-34.

Weaver, J. R., & Church R. L. (1991). The nested hierarchical median facility location model. INFOR, 29 (2), 100-115.

Weisstein, E. W. (1999). Delaunay triangulation. MathWorld – A Wolfram Web Resource. Retrieved from: http://mathworld.wolfram.com/DelaunayTriangulation.html

Williamson, D. P., & Shmoys, D. B. (2011). The design of approximation algorithms. Cambridge University Press.

Wolsey, L. A. (1998). Integer programming. Wiley.

Zipf, G. K. (1949). Human behavior and the principle of least effort. Addison-Wesley.

Zoltners, A. A., & Sinha, P. (1983). Sales territory alignment: a review and model. Management Science, 29 (11), 1237-1256.