SHAKEDOWN ANALYSIS:

CLASSICAL THEORY AND RECENT DEVELOPMENTS

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A report submitted in partial fulfilment of the requirements for the MSc. Degree in Engineering of the University of London and the Diploma of Imperial College of Science and Technology

JUNE 1979

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CONTENTS

1. INTRODUCTION

The objective of this work is to analyse and discuss the elasto-plastic behaviour of frames exposed to loads and/or other external actions which vary significantely and repeatedly in time.

Several theorems and methods were developed many years ago for predicting wether an elasto-plastic system will shakedown (sd) or otherwise it will undergo failure by incremental collapse or alternating plasticity. The limitations of the sd theory (in classical terms) lead to research activities lately in fields such as civil, mechanical and nuclear engineering.

Some of the developments were originally carried out by the continum approach; others by means of discrete structural models (fe) wich provide a basis for numerical solution.

It is also important to acess the order of magnitude of some displacements and/or intrinsic deformations even sd is acertained. Excessive deflections may mean unserviceability and excessive strain may imply local failure. Upper bounds on deformation parameters are the third subject of this work.

2. CLASSICAL THEORY BY L.P.

2.1 Historical Development

Bleich/1/ found a theorem for trusses generalized by Melan/2/ for trusses and continum. Symonds and Prager/3/ simplified the proof of the theorem and Neal/4/ adapted it for flexural framed structures considering moment-curvatures relatonship of linearly elastic and perfectly plastic. Also an unsafe method based on the combination of mechanisms Symonds and Neal/5/ and Neal/6/ may be implemented for lp. The problem of failure by unbounded plastic deformations was presented by Koiter/7/ but Horne showed/8/ that for most loading environments this type of failure is not significant. Maier/9/, Cedarini and Galvarini /10/ formulated the safe lp in mesh form and by dualization they obtained an unsafe lp corresponded to Neal's unsafe method. The connection between Koiter theorem and Neal's method was showed by Maier.

2.2 Main Assumptions(therefore, limitations)

Inviscid perfectly-plastic(non-hardening) laws govern local deformability, and involve convex yeld surfaces, associative flow rules and constant Young's modulus; the firt statement concerns structures subjected to variable thermal loading, with time dependent material properties.

Geometric changes do not affect signifficantely the equilibrium relations.

Material properties not influenced by temperature changes.

The loading acts so slowly that inertia and viscous forces are neglected: the system behaves in a quasi-static way.

Shakedown guarantees that failure willnot occur, not taking into account the development of excessive deformations or local failure.

2.3 Statics and Kinematics of elasto-plastic structures

When a mechanism develop prior to failure, at the limiting state of plastic collapse, increments of structural displacements $\Delta\delta$ are due to increase in plastic deformations at critical sections Δu. Since the mechanism must be compatible, the elastic deformation field remain unchanged and is associated with small displacements-original undeformed geometry of the structure to be considered. At pre-collapse state increments in displacements must be written in temporal rate $\dot{\delta}$ to distinguish from collapse mechanism due to either $\dot{u}$ increments in plastic deformations at critical sections or $\dot{u}_E$ increments in the elastic range.

The structure is considered to be a set of fe connected at the node points. $\dot{\theta}^1_m, \dot{\theta}^2_m$ represent the plastic rotation increments, the recoverable deformation associated with the curvature $\dot{\kappa}(x)$ represented by the deviation increment $\dot{\theta}^1_{Em}, \dot{\theta}^2_{Em}$ of the chord joining the nodes. $\dot{\Psi}^1_m, \dot{\Psi}^2_m$ are the total deformation increments. Both mesh and nodal description are applicable, but now an integration over the total duration of loading for statics and the deformation process for kinematics must be carried out.

MESH FORMAT $\dot{\underset{\sim}{\psi}} = \underset{\sim}{\dot{\theta}_E} + \underset{\sim}{\dot{\theta}}$ NODAL FORMAT

$\dot{\underset{\sim}{m}} = [B \;\; B_o] \begin{bmatrix} \dot{t} \\ \lambda \end{bmatrix}$ $\qquad \underset{\sim}{Q} = [A^T \;\; A_o^T] \left\{ \begin{matrix} \dot{M} \\ -\dot{\lambda} \end{matrix} \right\}$

$\left\{ \begin{matrix} \dot{c} \\ \dot{\underset{\sim}{\delta}} \end{matrix} \right\} = \begin{bmatrix} B^T \\ B_o^T \end{bmatrix} \dot{\underset{\sim}{\psi}}$ $\qquad \left\{ \begin{matrix} \dot{\psi} \\ \dot{\delta} \end{matrix} \right\} = \begin{bmatrix} A \\ A_o \end{bmatrix} \dot{\underset{\sim}{q}}$

2.4 Elasto-plastic Constitutive Relations

A material element of a flexural fe as the same loading-unloading linear elastic law as for initial loading. It also behaves as perfectly plastic: a plastic flow $\dot{k}$ may be induced whenever the applied moment is m_*^+ or $-m_*^-$ independently of total curvature of the element k.

Linear Elasticity $m(\wedge) = EI\; K(\wedge)$; $\dot{m}(\wedge) = EI\; \dot{k}(\wedge)$

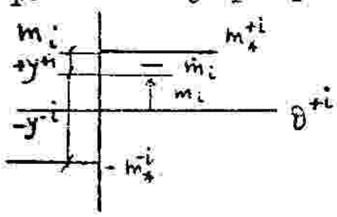
$\dot{\underset{\sim}{\theta}}_{Em} = \underset{\sim}{F}_m \dot{\underset{\sim}{m}}_m$ $\qquad \dot{\underset{\sim}{m}}_m = \underset{\sim}{K}_m \dot{\underset{\sim}{\theta}}_{Em}$

Flexibility Relations $\qquad$ Stiffness Relations

The plastic phase can only be reached at elements concentrated at critical sections. If elastic curvature subtracted, the plastic curvature resultant across the lumped point is the total ph rotation.

Considering the history of loading and unloading from initial conditions at critical section $\underline{i}$ with an applied bm m_i ,an increment of in the loading program will produce an increment of $\dot{\theta}_i$ in the plastic rotation. $\dot{\theta}_i$ depends not only on $\dot{m}_i$ but also on m_i

Let $\dot{\theta}_i = \dot{\theta}^{+i} - \dot{\theta}^{-i}$ pair of non-negative parameters which exhibit Complementary properties: $\dot{\theta}^{+i} \cdot \dot{\theta}^{-i} = 0 \quad ; \quad \dot{\theta}^{+i} , \dot{\theta}^{-i} \geq 0$



Thus: Activation condition $[y^{+i} \;\; y^{-i}] \times \begin{bmatrix} \dot{\theta}^{+i} \\ \dot{\theta}^{-i} \end{bmatrix} = 0$

Flow condition $[\dot{y}^{+i} \;\; \dot{y}^{-i}] \times \begin{bmatrix} \dot{\theta}^{+i} \\ \dot{\theta}^{-i} \end{bmatrix} = 0$

Increment of plastic energy dissipation

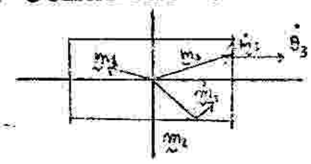
$\dot{\underset{\sim}{D}} = \{ m_*^{+i} \;\; m_*^{-i} \} \left\{ \begin{matrix} \dot{\theta}^{+i} \\ \dot{\theta}^{-i} \end{matrix} \right\}$

2.5 Governing Equations

Activation laws $\underset{\sim}{y}_* = [N^T \;\; m_*] \times \left\{ \begin{matrix} \underset{\sim}{m} \\ -1 \end{matrix} \right\}$; $\left\{ \begin{matrix} \dot{\theta} \\ \dot{D} \end{matrix} \right\} = \begin{bmatrix} N \\ m^T \end{bmatrix} \times \dot{\underset{\sim}{\theta}}_*$; $\underset{\sim}{y}_*^T \dot{\underset{\sim}{\theta}}_* = 0$

Flow laws $\dot{\underset{\sim}{y}}_* = \underset{\sim}{N}^T \dot{\underset{\sim}{m}}$; $\dot{\underset{\sim}{\theta}} = \underset{\sim}{N} \dot{\underset{\sim}{\theta}}_*$; $\dot{\underset{\sim}{y}}_*^T \dot{\underset{\sim}{\theta}}_* = 0$; $\underset{\sim}{y}_* \leq 0$; $\dot{\underset{\sim}{\theta}} \geq 0$

The 2c yeld hyperplanes form a rectangular box within which the the yeld conditions are satisfied by any moment set



$\longleftrightarrow$ Activation and neutral loading

$\longleftrightarrow$ Activation and unloading

The condition for unloading or neutral loading do not apply to all plastic potential rates, being related only to a subset for which the corresponding yeld hyperplanes have been activated.

Drucker's postulate alloud these plasticity relations to be extended to the case of stress-resultant interaction.

2.6 Geometrical Representation of the Safe Theorem /1/

Considering a loading domain in a load space based on n independently variable quasi-static loading, if a parameter $\mu \geq 0$ is applied to the domain, it will expands or contracts according to the variation on μ. For a guiven design of a structure, when the domain coincides with the instantaneous hypersurface of collapse a mechanism develops.

Points of the hypersurface are determined by pla and any convex combination of these points yelds a safe loading domain.

The moments at critical sections are found in the elastic phase applying a unit value of each loading separately in such a way that the structure does not yeld plastically. It is introduced a independent variable t which represents the loading hystory (temporal factor): $\underset{\sim}{m}(t) = \underset{\sim}{B}_e \cdot \underset{\sim}{\lambda}(t)$

If any part of the moment range is not contained in the yeld polytope, an admissible loading programme may cause plastic flow to occur at some critical sections. The linear nature of the parametric transformation of the domain alloud to restrict μt to the yeld polytope putting $\mu < 1$. However, a linear elastic response is produced for any load in the prescrbed domain even if a small amount of plastic flow takes place in the early stages of the process.

Continued loading in the above conditions produces a moment response $m(t)$ which differs from the entirely elastic moment response $m_e(t)$ That difference with respect to a guiven instant t is called residual moments. Since $m(t)$ and $m_e(t)$ are each in equilibrium with the loading $\lambda(t)$ it is therefore self-equilibrating.

This residual moments have the meaning of permanent plastic rotations induced during neutral loading, since they remain when the loading is remo ved, providing no further deformations occurs during unloading.

Any loading now produces the same elastic response $m_e(t)$ but these moments must be superimposed on the residual moments guiving a new elastic range, being changed the origin of the moment space.

If the plastic deformation associated with the loading programme may cause the elastic response to be wholly contained in the yeld polytope the structure is said to have adapted itself to its loading environment or to have shakedown. This adaptation is not possible if it exists any moment outside the elastic range.

The residual moments are not dependent on time and not related with the actual loading programme.

2.7 Safe Shakedown Theorem (Bleich-Melan)

SD occurs if there are constant self-equilibrated stresses (residual moments), such that the supperposition of them in the unlimited linear elastic response of the structure, to all loading programms of infinite duration contained in a prescribed domain, is statically admissible: $\underset{\sim}{\bar{y}}_* = \underset{\sim}{N}^T [\underset{\sim}{m}_e + \underset{\sim}{\bar{m}}_r] - \underset{\sim}{m}_* \leq \underset{\sim}{0}$ for all m_e in E

2.8 Determination of the Loading Parameter

It is a straightforward application of the safe theorem:

The elastic moment range is defined as the smallest set of M containing E bounded by the hyperplane corresponding to the yelding in the critical sections, which normals are parallel to those bounding the yeld polytope. If E is tranlated to E' which is contained in the yeld polytope, the envlope of bm is also contained in the hyperplane. The values of the bm at the critical sections are evaluated, and we find the maximum elastic stress: $\begin{Bmatrix} M^+ \\ M^- \end{Bmatrix} = \underset{\sim}{M}_* = \max_{m_e \in E} \{ \underset{\sim}{N}^T \underset{\sim}{m}_e \}$ $\underset{\sim}{N}$: Normal Matrix

If now the loading domain (D) is multiplied by a loading parameter μ, E becomes μE, M will be μM. For a single parameter loading domain, the envelope is defined by the perpendiculars μM_* which are the maximum elastic moments in both senses at each critical section due to any loading

The set of residual moments $\bar{m}_r$ must be such that the new set of bm, obtained by adding them to μM_* will be as large as possible but contained in the yeld polytope.

Depending on the mode of description used, two lp are obtained:

$$\underset{\sim}{M}_{t} = \max_{m_e \in E} [\underset{\sim}{N}^T \underset{\sim}{m}_e] \geq \underset{\sim}{N}^T \underset{\sim}{m}_e \rightarrow \left.\begin{array}{l} \underset{\sim}{y}_{\mu} = \underset{\sim}{M}_{*} + \underset{\sim}{N}^T \underset{\sim}{\bar{m}}_r - \underset{\sim}{m}_{*} \\ \underset{\sim}{\bar{y}}_{*} = \underset{\sim}{N}^T [\underset{\sim}{m}_e + \underset{\sim}{\bar{m}}_r] - \underset{\sim}{m}_{*} \end{array}\right\} \underset{\sim}{y} = \mu \underset{\sim}{M}_{*} + \underset{\sim}{N}^T \underset{\sim}{m}_r - \underset{\sim}{m}_{*} \leq 0$$

$$\underset{\sim}{m} = [B \;\; B_0] \times \begin{Bmatrix} p \\ \lambda \end{Bmatrix} \; ; \; \underset{\sim}{m}_r = \underset{\sim}{B} \, \underset{\sim}{p}_r \qquad 0 = [A^T \;\; A_0^T] \times \begin{Bmatrix} \underset{\sim}{m} \\ -\underset{\sim}{\lambda} \end{Bmatrix} \; ; \; A^T m_r = 0$$

MESH PRIMAL

$$\boxed{\begin{array}{l} \operatorname{Max} w = \{1 \;\; \underset{\sim}{0}^T\} \times \begin{Bmatrix} \mu \\ \underset{\sim}{p}_r \end{Bmatrix} \\ [\underset{\sim}{M}_{*} \;\; \underset{\sim}{N}^T \underset{\sim}{B}] \times \begin{Bmatrix} \mu \\ \underset{\sim}{p}_r \end{Bmatrix} \leq \underset{\sim}{m}_{*} \\ \mu \text{ unrestricted} ; p_r \text{ unrestricted} \end{array}}$$

NODAL PRIMAL

$$\boxed{\begin{array}{l} \operatorname{Max} w = [1 \;\; 0^T] \times \begin{Bmatrix} \mu \\ m_r \end{Bmatrix} \\ \begin{bmatrix} \underset{\sim}{M}_{*} & \underset{\sim}{N}^T \\ 0 & -\underset{\sim}{A}^T \end{bmatrix} \times \begin{Bmatrix} \mu \\ \underset{\sim}{m}_r \end{Bmatrix} \begin{matrix} \leq \\ = \end{matrix} \begin{Bmatrix} \underset{\sim}{m}_{*} \\ \underset{\sim}{0} \end{Bmatrix} \\ \mu \text{ unrestricted} ; m_r \text{ unrestricted} \end{array}}$$

When a loading domain D implies that the structure is unable to sd, plastic deformation will develop without bound under a loading programme to failure. If the mechanism is compatible, a temporal notation is adopted for both déformation and moment increments

$$\underset{\sim}{\dot{\psi}} = \underset{\sim}{\mathring{\theta}}_e + \underset{\sim}{\mathring{\theta}}_{er} + \underset{\sim}{\mathring{\theta}} \qquad . \qquad \underset{\sim}{\mathring{m}} = \underset{\sim}{\mathring{m}}_e + \underset{\sim}{\mathring{m}}_r$$

If there is no increment in plastic deformation $\underset{\sim}{\dot{\theta}} = 0; \underset{\sim}{\dot{m}}_r = 0; \underset{\sim}{\dot{\theta}}_{er} = 0$. Plastic hinge rotations $\underset{\sim}{\dot{\theta}}$ induce a unique set of residual bm $\underset{\sim}{\dot{m}}_r$ found by solving the unloaded elastic structure with the imposed rotation included. The uniqueness is assured: since $\underset{\sim}{\dot{\theta}}$ is compatible, $\underset{\sim}{\dot{\psi}}' = \underset{\sim}{\dot{\theta}}_{er} + \underset{\sim}{\dot{\theta}}$ is also compatible, being $\underset{\sim}{\dot{\theta}}_{er}$ and $\underset{\sim}{\dot{\theta}}$ not compatible. If $\underset{\sim}{\dot{\theta}}$ is compatible the deformation corresponds to an instantaneous plastic mechanism for which $\underset{\sim}{m}_r = \underset{\sim}{0}$ and hence $\underset{\sim}{\theta}_{er} = \underset{\sim}{0}$. Plastic deformations increase without bound if they are compatible to a load programme for an interval $(0,T)$ which can be repeated indefinitely.

When at least one section equal plastic rotations of opposite sense are induced during an interval $(0,T)$ the mean rotation becomes zero satisfying compatibility. As the loading programme is repeated both $\Delta\theta^{+i}$ and $\Delta\theta^{-i}$ increase without bound while $\Delta\theta_i$ remains zero. This mode of failure does not satisfy complementarity conditon $\underset{\sim}{\Delta\theta}^{+i} \underset{\sim}{\Delta\theta}^{-i} = 0$ and is called <u>alternating plasticity</u>. $\quad \Delta \underset{\sim}{\theta} = \underset{\sim}{N} \, \underset{\sim}{\Delta\theta}_{*} = \underset{\sim}{0} \quad ; \quad \Delta\theta_{*} \neq 0$

If the mode of failure satisfies complementarity and its components simcultaneously considered do not constitute an instantaneous collapse mechanism, the structure is subjected to an <u>incremental collapse</u> mechanism. The effect of the rotation during the interval $(0,T)$ is to produce a progressive collapse of the structure if the cycle is repeated indefinitely.

The dual programs of above can be written:

MESH DUAL
$$\boxed{\begin{gathered} \text{Min } z = m_*^T \Delta\theta_* \\ \begin{bmatrix} M_*^T \\ \hat{B}^T N \end{bmatrix} \times \Delta\theta_* = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \\ \Delta\theta_* \geq 0 \end{gathered}}$$

NODAL DUAL
$$\boxed{\begin{gathered} \text{Min } z = \begin{bmatrix} M_*^T & 0 \end{bmatrix} \times \begin{bmatrix} \Delta\theta_* \\ \Delta v \end{bmatrix} \\ \begin{bmatrix} M_*^T & 0 \\ N & -A \end{bmatrix} \times \begin{bmatrix} \Delta\theta_* \\ \Delta v \end{bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \\ \Delta\theta_* \geq 0 \;;\; \Delta v \text{ unrestricted} \end{gathered}}$$

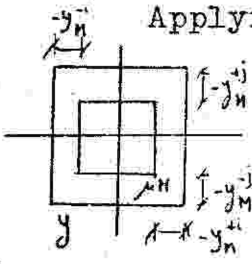
The dual variables have the meaning of displacements and plastic rotations in the nodal description and therefore the objective function represents dissipated energy. In both the normalization or scaling condition appears, being the compatibility condition for the admissible modes written for a mesh or nodal description of kinematics.

Both an unsafe method due to Neal and the application of Koiter theorem reach the same answer, although being founded in different principles: Neal's method is completely independent of sequential parameter, since it is based on the yelding moment envelope, while Koiter's approach is based on the loading domain and its sequential variation. /11/

2.9 Unsafe Shakedown Theorem (Koiter)

If for a prescribed loading domain D with parameter μ the quantity $\mu M_*^T \Delta\theta_*$ balances the energy dissipated in the ph on any kinematicaly admissible mode of failure $\Delta\theta$ to sd then μ is an upper bound to the limiting parameter of sd of the structure μ_c.

2.10 LP Duality



Applying the K-T conditions for the nodal description

KINEMATIC ADMISSABILITY $\quad 1 - M_*^T \Delta\theta_* = 0$ (Normalization)

$A \Delta v - N \Delta\theta_* = 0$ (Compatibility and Flow Rules)

STATIC ADMISSABILITY

(Modified Yeld Conditions) $y_M \leq 0$

(Modified Associative Rules) $y_M \Delta\theta_* = 0$

(Self-equilibrating Residual Moments) $A^T m_r = 0$

The non-negative loading parameter must be considered as unrestricted for the sake of duality.

2.11 General Comments

F.E. models of continum may be solved by these lp providing the symbols employed are guiven generalized meanings.

Beam-column problems may be solved either considering a piece wise linearization of the moment-rotaton relationship.Whithout doing this a nlp must be solved.

When the loading is reduced to a single point in the load space,both sd theorems reduce to the safe and unsafe theorems of pla, respectively.Since time-invariant external actions are assumed to be a single point of the loading space both elastic and residual strains are unimportant in the determinaton of loading parameter .

3.EXTENSIONS OF CLASSICAL SD THEORY

3.1 Recent Developments

The restrictions of sd theory mentioned in 2.2 were relaxed due to the work carried out mainly by the italian school,to meet structural needs in the various technological areas already mentioned.

Cedarini /12/ produced a safe theorem to cover dynamic loading, while Corradi and Maier /13/ have formulated the unsafe theorem.

To introduce the dynamic problem it is necessary to consider the introduction in the equilibrium equation (nodal description) of an additional term.

3.2 Hardening and Geometrical Effects

Two further assumptions must be made for the evaluation of these effects: a) It is considered a elastic workhardening law,changing the yeld limits linearly

b) During each cycle of loading the axial loading remains constant.

The former condition modifies the elasto-plastic constitutive relations ,and the activation laws can be rewritten:

$$y^{+i} = \underset{\sim}{N}^T \underset{\sim}{m} - (\underset{\sim}{m}_*^{+i} + H_{11}^i\, \theta^{+i} + H_{12}^i\, \theta^{-i})$$

$$y^{-i} = \underset{\sim}{N}^T \underset{\sim}{m} - (\underset{\sim}{m}_*^{-i} + H_{21}^i\, \theta^{+i} + H_{22}^i\, \theta^{-i})$$

The hardening modulus $\underset{\sim}{H}$ can describe several hardening rules:/19/

For non-interactive yeld modes the cross terms are null;for constant kinematic hardening with Bauschinger effect $H_{11}=H_{22}=-H_{12}=-H_{21}=H$

The equilibrium equation must also be added a term,since the displacements are supposed to be small:

$$-\underset{\sim}{A}^T \underset{\sim}{m}_r - \underset{\sim}{G}\,\Delta \underset{\sim}{q} = \underset{\sim}{0}$$

The compatibility matrix was already defined A ;Due to the last assumption,the geometric stiffness matrix G contains P linearly and is costant,producing a linear elastic moment response due to residual plastic strains:

Using the nodal-stiffness formulation

$$\underset{\sim}{m} = \underset{\sim}{k}\,\underset{\sim}{A}\,(\underset{\sim}{A}^T \underset{\sim}{k}\,\underset{\sim}{A})^{-1}\left[\underset{\sim}{A}_0^T \lambda - \underset{\sim}{A}^T(\underset{\sim}{m} - \underset{\sim}{k}\,\underset{\sim}{\theta}_E)\right] + \underset{\sim}{k}\,(\underset{\sim}{m} - \underset{\sim}{k}\,\underset{\sim}{\theta}_E)$$

where $\underset{\sim}{A}^T \underset{\sim}{k}\,\underset{\sim}{A} = S$; $\lambda = \underset{\sim}{0}$; $\underset{\sim}{m} = \underset{\sim}{0}$ $\therefore$ $\underset{\sim}{m}_r = \left[\underset{\sim}{k}\,\underset{\sim}{A}\,\underset{\sim}{S}^{-1}\underset{\sim}{A}^T\underset{\sim}{k} - \underset{\sim}{k}\right]\underset{\sim}{\theta}_{Er}$

Now,the stiffness matrix S is replaced by $S_G = S + G$ .Defining $Z = (k A S_G^{-1} A^T k - k)$ as the influence coefficient matrix,the following condition play an important role:

$\underset{\sim}{H} - \underset{\sim}{N}^T \underset{\sim}{Z}\,\underset{\sim}{N} = \underset{\sim}{C}$ Symmetric,positive definite with

It can be showed that the work produced by external loads is proportional to $\dot{\underset{\sim}{q}}_E\,\underset{\sim}{S}_G\,\dot{\underset{\sim}{q}}_E + \dot{\underset{\sim}{\theta}}^T \underset{\sim}{C}\,\dot{\underset{\sim}{\theta}}$,which for $\underset{\sim}{C}$ positive,guarantees the stability of all yelding modes.

The symmetry of $\underset{\sim}{C}$ require the symmetry of the hardening matrix $\underset{\sim}{H}$ and thus only a few hardening rules are to be considered.

If it does occur unstable behaviour in the local deformability,it is compensated by stabilizing geometrical effects and unstabilizing geometrical effects may be compensated by hardening behaviour.

The Safe SD theorem can be stated in this more general context:SD occurs if the overal stability condition holds and if there is a distribution of constant plastic strains such that the yeld conditions corresponding to these residual strains are satisfied by the sum of the stesses generated by them and the unlimited linear elastic response to all loading programs of infinite duration in a prescribed domain $\quad y = N^T m_e + N^T m_r - H\,\theta_{er} \leq m_*$

This can be expressed as $\quad y = N^T m_e - C\,\theta_{er} \leq m_*$

$C^T C = H - N^T Z N$ +ve definite is the other condition for SD to occur

Let $\quad M_* = \max\limits_{m_e \text{ in } E} [N^T m_e]$

LODAL PRIMAL
$$\boxed{\begin{array}{l} \text{Max } w = \begin{bmatrix} 1 & 0 \end{bmatrix} \times \begin{bmatrix} \mu \\ \theta_r \end{bmatrix} \\ \begin{bmatrix} M_* & -C^T \end{bmatrix} \begin{bmatrix} \mu \\ \theta_r \end{bmatrix} \leq m_* \end{array}}$$

NODAL DUAL
$$\boxed{\begin{array}{l} \text{Min } z = m_*^T \Delta\theta_* \\ \begin{bmatrix} M_*^T \\ -C \end{bmatrix} \times \Delta\theta_* = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \end{array}}$$

Now the static admissability is expressed in terms of kinematic constants (residual rotations).The dual program guives an upper bound to the loading parameter if the overal stability condition is not satisfied.

The feasibility of this program depends on the second constraint: If in both and are symmetric and positive semidefinite,a compatible set of plastic strains cannot alter the yeld polytope.When is positive definite the lp becomes infeasible,ie the system sd under any loading program. This occurs for an isotropic hardening without considering geometric effects or for stabilizing geometrical effects in absence of hardening.

A value of the loading parameter greater than 1 may lead to progressive failure even if sd predicted ,being a necessity to determine bounds on post-sd deformations.

The Unsafe sd theorem as the same statement as for the classical theory,but the overall stability condition must be included as necessary condition.

The Compatibility condition is affected by hardening laws and geometrical effects,while the stability condition is independent of hardening.

3.3 Rigid Workhardening Structures

When elastic strains are not to be considered and it occurs hardening effects in the structure due to external actions ,the elas to-plastic constitutive relations defined previously are modified by putting $\dot{\theta}_E = 0$ (or $K = \infty$). Thus: $\dot{\psi} = \dot{\theta}$

The main advantage of this assumption is to provide simplification in sd analysis. Prager introduced this approach /14/; further developments are due to Polizzoto /15/ and Konig and Maier /16/.

If it is defined a linear transformation on a set of strains (generally, not compatible) into ~~a~~ displacement set termed as consequent to, $\underset{\sim}{\theta}$ $\dot{\underset{\sim}{q}} = \underset{\sim}{W}\,\dot{\underset{\sim}{\theta}}$ where, $\underset{\sim}{W} = \underset{\sim}{B}_q^T$

The matrix W is positive definite. It is possible to write a equivalent condition for overall stability in this more restritive sense: $\underset{\sim}{F} = \underset{\sim}{H} + \underset{\sim}{W}^T \underset{\sim}{G}\, \underset{\sim}{W}$ symmetric, positive semi-definite

The Bleich-Melan theorem adapted to this formulation says that SD occurs if the overall stability condition holds and a constant set of plastic rotations such that the sum of the stresses generated by them and the stresses due to any loading program satisfy static admissability and yeld (in its deformed configuration contained in a prescribed domain and varying in time): $\underset{\sim}{N}^T \underset{\sim}{m}(t) - \underset{\sim}{F}\,\underset{\sim}{\theta}_{er}(t) \leq \underset{\sim}{m}_o$.

$$\underset{\sim}{m}_r = \underset{\sim}{B}\,\underset{\sim}{p}_r$$

These conditions can be used to implement a lp in the mesh format. Let $\underset{\sim}{M}_*$ be the envelope set of stresses due to each component of the loading domain at any time: $\underset{\sim}{M}_* = \max_{\lambda \text{ in } D} \left[\underset{\sim}{N}^T \underset{\sim}{B}_e \underset{\sim}{\lambda} \right]$

$$\text{Max} \quad w = \begin{bmatrix} 1 & 0^T & 0^T \end{bmatrix} \begin{bmatrix} \mu \\ p_r \\ \theta_r \end{bmatrix}$$

$$\begin{bmatrix} \underset{\sim}{M}_* & \underset{\sim}{N}^T \underset{\sim}{B} & -(\underset{\sim}{H} + \underset{\sim}{N}^T \underset{\sim}{B}_q \underset{\sim}{G}\,\underset{\sim}{W}) \end{bmatrix} \begin{bmatrix} \mu \\ p_r \\ \theta_r \end{bmatrix} \leq \underset{\sim}{m}_* \qquad \text{MESH PRIMAL}$$

The evaluation of the maximum value of the stress response is easier to perform than for the elastic statically indeterminate case.

As before, the loading parameter of SD to be accertained requires the overall stability condition to be satisfied; otherwise the maximization only guives an upper bound on .

The rigid hardening model theory could be developped as before, finding the Unsafe theorem by dualization and repeating the remarks concearning deformation boundings.

The lack of connection between plastic rotations and residual stresses are the main characteristic of this behaviour which also leads to underestimates in the evaluation of geometry changes in the structure.

3.4 Dynamic Effects

During the loading history the condition of slow variation of loading does not always hold; in this case, damping and inertia effects ought to be considered. Thus two further terms in the static admissability condition must be included: $-\underset{\sim}{A}^T \underset{\sim}{m}_r - \underset{\sim}{G} \Delta \underset{\sim}{q} - \underset{\sim}{I} \Delta \ddot{\underset{\sim}{q}} - \underset{\sim}{V} \Delta \dot{\underset{\sim}{q}} = 0$

A classical approach consists of having both the displacements and the velocities at t=0 and then follow a time-dependent loading history achieving a non-linear step by step analysis.

Predictions which guive accurate results with computer gains are described next, when considering periodic loads.

Let $\underset{\sim}{m}_e^*(t)$ represent the linear elastic fictitiousresponse of the structure to the load hystory and to initial conditions $\Delta \underset{\sim}{q}^*, \Delta \dot{\underset{\sim}{q}}^*$

The Safe theorem has a similar formulation as for the elasto-plastic material implemented of geometrical and hardening effects: The overall stability condition must be satisfied and a set of plastic rotations, displacements and velocities (all these unknowns being constant) such that the yeld conditions not contravyened for any $t \geq t^*$ (finite time) $\quad \underset{\sim}{N}^T \underset{\sim}{m}_e^*(t) - \underset{\sim}{G}\, \underset{\sim}{\theta}_{cr} \leq \underset{\sim}{m}_*$

The actual initial conditions do not influence the adaptation of the system.

The maximum elastic stress for each yeld mode will be calculated from the fictitious dynamic response $\quad \underset{\sim}{H}_*(t, \Delta q^*, \Delta \dot{q}^*) = \max_{\tau \geq t^*} \left[N^T m_e^*(t) \right]$

Two lp can be written once these values are known

NODAL PRIMAL

$$\text{Max } w = [1 \quad 0^T] \times \begin{bmatrix} \mu \\ \theta_r \end{bmatrix}$$

$$\begin{bmatrix} M_* & -C^T \end{bmatrix} \times \begin{bmatrix} \mu \\ \theta_r \end{bmatrix} \leq m_*$$

NODAL DUAL

$$\text{Min } z = m_*^T \times \Delta\theta_*$$

$$\begin{bmatrix} M_*^T \\ -C \end{bmatrix} \times \Delta\theta_* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Delta\theta_* \geq 0$$

These lp are identical to those derived in 3.2, but now since ω, z depend on the parameters $t^*, \Delta q_v^*, \Delta\dot{q}_v^*$ for each step.

Another maximization must be performed to find the maximum-maximorum of the loading parameter, which is extremely time-consuming.

Due to Gavarini /17/ and condidering periodic loads, such that $A_o(t) = A_o(t+\tau)$ for any t, τ a time-interval

As the time elapses, the fictitious elastic response of the structure is periodic with the forced vibrations only, since the free vibrations died out due to damping.

Since from linear analysis the forced motion is known, the displacements and velocities at an instant $t' = n\tau$ for n sufficientely large can be assumed as fictitious initial conditions. The linear elastic fictitious response $m_e(t)$ become periodic, being the maximum elastic stress determined for these conditions independentely of t.

The optimal value of μ is then accessed by linear combination of all components of M_*. $\quad M(0, \Delta q', \Delta\dot{q}') \leq M_*(t^*, \Delta q^*, \Delta\dot{q}^*)$

The same procedure can be applied for undamped systems.

For the statement of the Unsafe program, as before the plastic rotation rates are integrated during a cycle of loading; now, F_I^* and F_V^* are the inertia and damping forces due to the forced vibration $A_o(t)$ and to the fictitious initial conditions Δq^* and $\Delta\dot{q}^*$:

When the overall stability condition holds, SD does not occur if the quantity $m_*^T \Delta\theta_*$ balances the energy dissipated in the plastic hinge on any kinematic admissible mode of failure during a cycle over a time interval T.

3.5 Temperature Change Effects

Temperature changes cause thermal strains which are no more than imposed strains taken into account through the strain vector

Nodal Description $\mu \underset{\sim}{D} = \underset{\sim}{A} \, \Delta \underset{\sim}{q} - \underset{\sim}{\chi}$

Mesh Description $\mu \underset{\sim}{D} = \underset{\sim}{B}^T \underset{\sim}{\chi}$

Besides the straining effect, material properties (elasto-plastic constitutive relations) are sensible to certain ranges of temperature: The SD analysis was carried out by Konig /18/ but the program acertained is non-linear.

Another difficulty arises due to non-validity of elastoplastic model for the upper part of the temperature range where creep becomes signifficant, unless the cycles are of short duration.

3.6 Non-Associated Flow Rules

Until now the elastic moment range was considered to be bounded by the hyperplane corresponding to the yelding in the critical sections which normals are parallel to those bounding the yeld polytope.

For some materials the elasto-plastic behaviour must take into account some factors, eg internal friction in soils, not complying with the normality rule.

The outward normals previously defined as the matrix $\underset{\sim}{N}$ are now changed into the matrix $\underset{\sim}{U}$ which leads to a reduced moment range.

Maier / 9/ considered a loading parameter bracked by the classical calculation and the value determined considering deviations from normality.

Usually information about this behaviour is not available.

3.7 Multi-Parameter Loading Domain

The loading domain used with a single parameter is able to exclude impossible combinations of independent loads, assuming any shape.

It might be considered that each independent load can vary between independent limits $-\mu^- \leq \lambda(t) \leq \mu^+$. It is possible to define a set of single parameter SD problems by partitioning the loads in two atributed load parameters; one having fixed values, while the other varies whithin its range. The $\underset{\sim}{B}_c$ matrix is defined in terms of two non-negative and complementary matrices: $\underset{\sim}{B}_e = \underset{\sim}{B}_e^{+} - \underset{\sim}{B}_e^{-}$; $\underset{\sim}{B}_e^{+} \cdot \underset{\sim}{B}_e^{-} = \underset{\sim}{0}$; $B_e^{+}, B_e^{-} \geqslant 0$

Let $\underset{\sim}{M}_*$ be the maximum elastic moments in both senses

$$\underset{\sim}{M}_* = \begin{Bmatrix} M^+ \\ M^- \end{Bmatrix} = \begin{bmatrix} B_e^{+} & B_e^{-} \\ B_e^{-} & B_e^{+} \end{bmatrix} \begin{Bmatrix} \mu^+ \\ \mu^- \end{Bmatrix}$$

For compatibles modes of failure:

$$M_*^T \, \Delta\theta_{xu} \leq m_x^T \, \Delta\theta_{xu}$$

For any failure mode there is a relationship in the upper μ^+ and lower limits μ^- and n independent parameters must be considered. The corresponding geometrical representation will be a 2-n dimensional half-space.

The intersection of all possible modes of failure defines a region of parameter combination for which the structure will SD.

3.8 SD Design

Guiving a graph model of the structure, the problem is to find a moment-resistant distribution capable to ensure SD will occur for any loading program in a prescribed domain.

As for rigid plastic collapse the design vaiables are the yelding moments $\underset{\sim}{m}_*$ and it is assumed the objective function to be the total weight, a linear function of the design variables, whose coefficients are the lengths corresponding to the critical sections $\underset{\sim}{l}$. The whole design variables are dependent of some of them (independent), this linear technological constraint been represented by $\underset{\sim}{d}$

A Safe program can be written:

NODAL PRIMAL

$$\boxed{\begin{aligned} &\min z = \begin{bmatrix} \underset{\sim}{l}^T & 0 \end{bmatrix} \times \begin{bmatrix} \underset{\sim}{d} \\ \underset{\sim}{m_r} \end{bmatrix} \\ &\begin{bmatrix} \underset{\sim}{J} & \underset{\sim}{N}^T \\ 0 & \underset{\sim}{A}^T \end{bmatrix} \times \begin{bmatrix} \underset{\sim}{d} \\ \underset{\sim}{m_r} \end{bmatrix} \begin{matrix} \leq \\ = \end{matrix} \begin{bmatrix} -\underset{\sim}{M}_*(m_r) \\ 0 \end{bmatrix} \end{aligned}}$$

We obtain a very compact program, since it is the synthesis under variable loading inside a domain, excluding the hypothesis of instantaneous collapse. It might be simplified using the Foulkes simplifications for variable loading

Another comment is the introduction of the stiffness matrix for the elastic phase, depending on the 2nd moment of area which leads to a non-linear program.Cohn,Ghosh and Parimi /20/,used a iterative procedure assuming at each step the elastic stiffness corresponding to the optimal moment of the preciding step.

For reinforced concrete the stiffness is dependent on the depth being the elastic moment response easily acessed in the beggining of the program,and the preciding program becomes a lp.Also note the ultimate moments depend only on the ductility.

4.DEFORMATION BOUNDING TECHNIQUES

4.1 Introduction

It was already shown that even if plastic deformations can be bounded predicting adaptation of the structure,they may cause excessive displacements or exhaust the material ductility.

The acessement of the system's evolution is achieved by following the load history.When are defined the limits whithin loads can vary eg load domain or the loading history is not known it is usefull to bound some quantities such as dissipated energy and/or maximum displacements and rotations.

Three methods with pratical application are presented,belonging to two distinct philosophies.The structural model is an elasto-plastic discrete model in its virgin state in the beggining of the loading process.

The nodal description will be adopted,since these techniques are applied to dynamic loading,being the reduction to quasi-static case straightforward.Impulsive loading consisting of an initial velocity only,can be regarded as a special form of periodic loading wich has been already derived.

Geometrical effects are not considered in this first analysis, but since that unstabilizing geometrical effects compensated by hardening,whenever the overall stability condition holds,these methods can be extended to cover second order effects.

For dynamic loads the linear elastic static response depends on the instantaneous external actions as well as the inertia and damping forces, not coinciding with the unlimited linear elastic response unless dynamic effects neglected.

The self-equilibrating set of residual moments are the linear elastic response of the system due to residual plastic rotations.

Itis considered a fictitious process of loading with convenientely choosen initial conditions the SD analysis for periodic loads, where the linear elastic static response is the steady-state solution due to assumed initial conditions.

Considering impulsive loads, the steady-state response is assumed for homogeneous initial conditions and the program becomes infeasible eg the system will always SD, what is not true and may determine the application of these techniques.

Let χ be the sum of the potential and kinetic energy associated with the difference between the actual and fictious process in the beggining of the loading cycle:

$$\chi = \frac{1}{2}\left(\dot{\underset{\sim}{q}}_0^T - \dot{\underset{\sim}{q}}_0^{*T}\right) I$$

4.2 Indirect Bounding Methods

They provide bounds to the displacement vector q or its plastic $\underset{\sim}{q}_p$ component. If a constant dummy load vector $\underset{\sim}{\Delta}$ the total displacements developed in the structure during the interval $(0,\tau)$ are a linear combination of the displacement vector with the corresponding component of the load vector.

$$\underset{\sim}{\delta}_p(\tau) = \underset{\sim}{\Delta}^T \underset{\sim}{q}_p(\tau) \quad ; \quad \underset{\sim}{\delta}(\tau) = \underset{\sim}{\Delta}^T \underset{\sim}{q}(\tau)$$

Procedure A : A time independent vector $\hat{\underset{\sim}{\theta}}$ can be found for the fictious dynamic process, such that:

$$\underset{\sim}{N}^T[\underset{\sim}{m}^*(t) + \underset{\sim}{m}_e] - \underset{\sim}{C}\,\hat{\underset{\sim}{\theta}} - \underset{\sim}{m}_* \leq 0$$

And its respective upper bounds are:

$$\delta_p(\tau) \leq U_1^{\tau} = \chi + \frac{1}{2}\,\hat{\underset{\sim}{\theta}}^T \underset{\sim}{C}\,\hat{\underset{\sim}{\theta}}$$

$$\delta(\tau) \leq U^{\tau} = U_1^{\tau} + \frac{1}{2}\underset{\sim}{\Delta}^T \underset{\sim}{K}^{-1}\underset{\sim}{\Delta} + \underset{\sim}{\Delta}^T[q^*(\tau) - q_0]$$

Procedure B : If it is possible to find a load multipli er $\mu > 1$ two sets of time independent plastic strains and a fictitious dynamic $\bar{\theta} \geq 0$; $\hat{\theta} \geq 0$ $\quad 0 < t < \tau$

$$\mu N^T m^*(t) - C\,\bar{\theta} \leq m_*$$

then, $\quad N^T m_e - C\,\hat{\theta} \leq m_* \qquad \delta_p(\tau) \leq U^p_2(\tau) = \frac{\mu\chi}{\mu-1} + \frac{1}{2(\mu-1)}\bar{\theta}^T C\bar{\theta} + \frac{1}{2}\hat{\theta}^T C\hat{\theta}$

In this procedure upper bounds are found by an unlimited elastic response m_e and a fictitious dynamic solution $m^*(t)$

The maximum values of plastic or total displacements are determined by introducing the maximum value of $\bar{q}^*$ instead of $q^\circ(\tau)$.

The SD Safe theorem ensures adaptation after a certain length of time, but the loading history must be determined from the virgin state, what leads to the non-existance of a fictious process while the finite deformations are stabilized.

When external actions are periodic, only one cycle must be considered and the fictitious dynamic response is the steady state solution. If the loading factor greater than 1, the inequalities become strictly and a feasible value can be found.

In order to establish a method of solving let the dummy load be represented by a distribution of dummy loads of constant amplitude Δ' scaled by a non-negative factor η. Then a relative measure of displacements and their respective bounds is imediate: $\delta'_p = \delta_p/\eta$; $U'_p = U_p/\eta$

The best upper bounds will be those which guive lower values. A minimization must be carried out with respect to the independent variables, being obtained NLP. ProcedureA: $\quad \Delta = \eta\Delta'$; $m_e = \eta\, m'_e$

$$U'_p = \min_{\eta,\theta}\left[\frac{1}{\eta}\left(\chi + \frac{1}{2}\theta^T C\theta\right)\right] \quad ; \quad U' = \min_{\eta,\theta}\left[\frac{1}{\eta}\left(\chi + \frac{1}{2}\theta^T C\theta\right) + \eta/2\,\Delta'^T K^{-1}\Delta' + \Delta'^T(\bar{q}^* - q_0)\right]$$

$$\Pi_* + \eta N^T m'_e - C\theta \leq m_*$$

$$\theta \geq 0 \quad \eta \geq 0$$

Procedure B:

$$U'_p = \min_{\mu,\eta,\hat{\theta},\bar{\theta}}\left[\frac{1}{\eta}\left(\frac{\mu\chi}{\mu-1} + \frac{1}{2(\mu-1)}\bar{\theta}^T C\bar{\theta} + \frac{1}{2}\hat{\theta} C\hat{\theta}\right)\right]$$

$$\mu\,\Pi_* - C\,\bar{\theta} \leq m_* \qquad \mu > 1$$

$$\mu N^T m'^T_e - C\,\hat{\theta} \leq m_* \qquad \eta \geq 0,\ \hat{\theta} \geq 0,\ \bar{\theta} \geq 0$$

Impulse loading can be derived from above making $\underset{\sim}{M}=0, \underset{\sim}{\bar{\dot{v}}}_0$ in both and $\underset{\sim}{\bar{\theta}}=0$; $\mu=\infty$ in Procedure B. The relative plastic displacements remain the same. A general formulation for indirect bounding techniques can be stated in the form of NLP: $\quad \chi=\frac{1}{2}\dot{v}_0^T I \dot{v}_0$

$$U'=\min_{\eta,\underset{\sim}{\theta}}\left\{ g^P(\eta,\underset{\sim}{\theta})=\frac{1}{\eta}\left(\chi+\frac{1}{2}\underset{\sim}{\theta}^T\underset{\sim}{e}\,\underset{\sim}{\theta}\right);\ g(\eta,\underset{\sim}{\theta})=g^P(\eta,\underset{\sim}{\theta})+\frac{u}{2}\underset{\sim}{\Delta}'^T\underset{\sim}{K}'\underset{\sim}{\Delta}'\right\}$$

$$\eta\,\underset{\sim}{N}^T\underset{\sim}{m}'_0-\underset{\sim}{e}\,\underset{\sim}{\theta}\le m_* \qquad \eta\ge 0;\ \underset{\sim}{\theta}\ge 0$$

These NLP are very simple to solve, suitable for practical pur poses . The fictitious process, if convenientely choosen, different from steady state response may improve the results.

4.3 Direct Bounding Methods

This procedure is far more involved than the above formulations. The first step consists of determining the loading parameter μ and then find the residual plastic rotations. $\quad \underset{\sim}{M}-\underset{\sim}{e}\underset{\sim}{\theta}_r-\frac{1}{\mu}\underset{\sim}{m}_*\le 0$

The structure is acted by periodical external actions being the linear elastic dynamic response of the fictitious process the steady-state solution.

After SD has occured the bounds for total rotations $\underset{\sim}{\theta}$ are given by:

$$\underset{\sim}{M}-\underset{\sim}{e}\,\underset{\sim}{\theta}\le\underset{\sim}{m}_*$$

$$\left(1-\frac{1}{\mu}\right)\underset{\sim}{m}_*^T\underset{\sim}{\theta}\le\chi+\frac{1}{2}\underset{\sim}{\theta}_r^T\underset{\sim}{e}\,\underset{\sim}{\theta}_r-\frac{1}{2}\left(\underset{\sim}{\theta}^+-\underset{\sim}{\theta}_r^T\right)\underset{\sim}{e}\left(\underset{\sim}{\theta}-\underset{\sim}{\theta}_r\right)$$

The quantities which may be maximized are: a)particular components of b)particular components of the plastic rotations $\underset{\sim}{U}_i=\underset{\sim}{V}\underset{\sim}{\theta}_i$ c)particular components of the plasti displacements $\underset{\sim}{\delta}_P$ obtained from $\underset{\sim}{\delta}_P=\underset{\sim}{K}'\underset{\sim}{A}\underset{\sim}{N}\underset{\sim}{\theta}$ d)dissipated energy W_{iP} in each lumped point given by $W_{iP}=m_*^i\theta_i+\frac{1}{2}\underset{\sim}{\theta}_i^T\underset{\sim}{H}_i\underset{\sim}{\theta}_i$ (quadratic function).

The merrits of operating with each one of these quantities is acertained by making the objective function linear.

The best upper bound Ω_1 is obtained making the yeld poly tope as small as possible. The residual plastic rotation, time inde pendent, must be as small as possible.

If the quadratic term of W_{ip} dropped, the corresponding bound has a smaller value than before. The inequality becomes linear and the smallest yeld polytope is found making the quadratic form a minimum, minimizing the residual plastic rotation. The procedure consists of solving a QP and then a LP.

$$\psi = \min_{\underset{\sim}{\theta_r}} \left\{ \tfrac{1}{2} \underset{\sim}{\theta}_r^T \underset{\sim}{C} \underset{\sim}{\theta}_r \right\} \text{ subject to } \left[\underset{\sim}{M}_* - \underset{\sim}{C} \underset{\sim}{\theta}_r \leq \tfrac{1}{\mu} \underset{\sim}{m}_* \right] \text{ giving } \underset{\sim}{\theta}_r^*$$

$$\Omega_2 = \max_{\underset{\sim}{\theta}} f(\theta) : \left[\underset{\sim}{M}_* - \underset{\sim}{C} \underset{\sim}{\theta} \leq \underset{\sim}{m}_* \wedge \left(1 - \tfrac{1}{\mu}\right) \underset{\sim}{m}_*^T \underset{\sim}{\theta} \leq \chi + \psi \right]$$

If the quadratic term is not eliminated, after solving the QP a NLP must be satisfied:

$$\Omega_3 = \max_{\underset{\sim}{\theta}} f(\theta) : \left[\underset{\sim}{M}_* - \underset{\sim}{C} \underset{\sim}{\theta} \leq \underset{\sim}{m}_* \wedge \left\{ \left(1 - \tfrac{1}{\mu}\right) \underset{\sim}{m}_*^T - \theta_r^{*T} \right\} \underset{\sim}{\theta} + \tfrac{1}{2} \underset{\sim}{\theta}^T \underset{\sim}{C} \underset{\sim}{\theta} \leq \chi \right]$$

This bound is bracketed by the former results. For practical purposes the first program must be abandoned; since the values found for and are similar, the second formulation appears to be the best compromise between computational efficiency and usefulness of results.

For impulse loading, the residual rotations are zero and the quadratic form to be minimized vanishes $\tfrac{1}{2} \underset{\sim}{\theta}_r \underset{\sim}{C} \underset{\sim}{\theta}_r$. The bounds and are given by a LP and a NLP, respectively.

$$\Omega_2 = \max_{\theta} f(\theta); \left[-\underset{\sim}{C} \underset{\sim}{\theta} \leq \underset{\sim}{m}_* \wedge \underset{\sim}{m}^T \underset{\sim}{\theta} \leq \chi \right]$$

$$\Omega_3 = \max_{\theta} f(\theta); \left[-\underset{\sim}{C} \underset{\sim}{\theta} \leq \underset{\sim}{m}_* \wedge \underset{\sim}{m}^T \underset{\sim}{\theta} + \tfrac{1}{2} \underset{\sim}{\theta}^T \underset{\sim}{C} \underset{\sim}{\theta} \leq \chi \right]; \quad \underset{\sim}{\theta} \geq 0$$

4.4 General Comments

For structures subjected to periodic forces or to seismic loadings the actual plastic deformations are not known, since they depend on the loading history. Upper bounds must then be found.

Martin /21/, Hodge /22/ obtained a theorem in the sixties. A lot of work have been carried out by the italian school aiming to find reliable and good for engineering practice. Vitiello/23/, Maier /24/, Maier and Vitiello/25/ developed the direct bounding techniques, while Ponter/26/, Maier/27/, Corradi and Maier/28/ for procedure A and Capurso /29/ for procedure B, discussed /30/ some applications of the indirect methods.

5.CONCLUSIONS

1.SD analysis by LP establishes a link between elasto-plastic deformation analysis,too compreensive, and the general loading case for plastic limit analysis,which provides usually sufficient accurate results for loading parameter,but does not guive much more information about the internal behaviour.

2.Based on numerical experience /20/ SD analysis provides good results for a quasi-static varying loads,as well as for periodic dynamic case,when second order effects are not considered.

If the overall stability condition is not satisfied,the critical event is not inadaptation and bounding methods are needed to provide additonal information.Thus,inserting geometric effects,a check on the validity of SD analysis is made.

3.Other cases than periodic loading histories are of interest in the dynamic analysis.An extension to impulsive loads was performed sucessfully;for different pulse shapes,the transformation into a equivalent impulse or using a step function seems to be dependent on the structure and loading case to be considered not capable of generalization.

4.Most of SD problems can be dealt with on the basis of LP only,by piecewise linearization of elasto-plastic constitutive relations.The bounding techniques with practical application presented guive more complex formulation and must be analysed on the basis of their operativity:The direct method achieving Λ_2 is soved by a QP-LP sequence,while theNLP of the indirect procedure A is evaluated by minimizing the plastic deformatons for different values of $\hat{\underset{\sim}{\theta}}$;here the objective function is qudratic.The problem doubles its amplitude when dealing with procedure B,although some simplifying assumptions are possible.

5.It is also possible to reduce the number of variables(plastic deformations) by knowing a priori which hinges are not activated using the indirect methods,since a minimization is carried out This is not applicable in direct techniques where is maximized.

6.None of the methods is able to provide upper bounds on all the quantities needed. All of them achieve similar results for the residual deformations.The main advantage of the direct method is to provide in each discrete point information not only about the plastic rotations but also of the plastic dissipated energy wich may lead to alternating plasticity.No relation is established between single components and the total deformations,main feature of procedure A(indirect).

7.A big disavantage related to the direct method is its sensivity to the type of hardening law adopted,which is not usually easy to acess.A promissing field appears to be ductility problems in reinforced concrete,since its constitutive laws may vary considerably affecting the bounding method used.Indirect procedures are not much influenced by hardening laws,but they do not guive the local behaviour.

8.The application of this theory to continum and the errors involved in the discretization model adopted in this work,seems to be another interesting extension.

9.Prestressed concrete materials were not refered in this report since the elasto plastic constitutive relations apply to areas of plasticity and not only to discrete points; again the errors of a discritization with respect to planar frames could lead to a systematic procedure.

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