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RELIABILITY-BASED PLASTIC SYNTHESIS OF PORTAL FRAMES

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A mathematical programming technique is described which minimizes the total average volume of steel reinforcement of a reinforced concrete frame for a specified failure probability. The structural material is assumed to exhibit a perfectly-plastic behaviour so that plastic collapse is the only possible failure mode. It consists of solving alternatively a reliability assessment problem, which incorporates recent developments in large-scale constrained concave quadratic programming and an optimal sizing problem (convex minimization) until the best reliability-based design against collapse is found.

KEY WORDS: Frames, structural optimization, concave quadratic minimization, structural safety, reliability-based design.

1 INTRODUCTION

This paper describes a computer reliability-based optimization design procedure as a rational approach to the design of frames. Loads and strengths are treated as random variables and failure is defined as the event that the load effect exceeds the plastic resistance of the structure. The risk measure that corresponds to this event is the global probability of failure by plastic collapse which is obtained from a systematic analysis of the uncertainties in random variables and of their statistical inter-dependencies. The reliability-based optimization procedure involves an iterative process which is repeated until the best design against collapse occurs. Development of models for reliability-based optimization was apparently initiated by Forssell[1] who formulated the optimization as minimization of total cost. The term "total cost" implies the initial cost (sum of all costs associated with erection and operation of the structure during its projected design life without failure) and the expected cost of failure (sum of all costs associated with the probability of failure, e.g.: expected cost of repairs, expected cost of disruption of normal use, expected cost of loss of human life). This total cost criterion governs nearly all of the work that flourished during the 1950s and is recorded in the papers by Johnson[2], Ferry Borges[3], Freudenthal[4], Paez and Torroja[5], among others.

During the next decade (1960–1970) the importance of Bayesian decision theory in structural optimization was recognized, and several devices aimed at improving the value of the optimum reliability-based solutions were introduced by Benjamin[6], Cornell[7] and Turkstra[8]. In the same decade, the need to reduce structural weight without compromising structural reliability, particularly in aerospace applications, was also recognized. The first major efforts along this line were led by Hilton and Feigen[9] 30 years ago. Subsequently, other efforts in this category were carried out by Kalaba[10], Moses and Kinser[11] among others.

Nevertheless, by 1970 it had become apparent that the concepts and methods of probability are the proper bases for development of optimum criteria for structural design. In present design practice it is usually assumed that structural systems are as safe as their components. However, in recent years consideration of both ductile and brittle behaviour including load and strength correlation has permitted considerable improvement of solutions to system reliability problems (Ditlevsen[12], Ang and Ma[13], Cou, McIntosh and Corotis[14]). Developments in structural system reliability have again revived interest in methods for generating multiple failure modes. Each failure mode has a certain finite probability of occurring. Thus, it is necessary to generate the most significant failure modes to estimate the reliability of frames. Three basic procedures have been popular for generating failure modes: exhaustive enumeration, repeated structural analysis and optimization. Techniques for exhaustive enumeration have been discussed by Bjerager[15]. A number of recent procedures are based on the strategy of using successive elastic analysis. The incremental load method (Moses[16]) finds an initial failure mode by incrementing a load and following its path from the initial yield to the final failure criterion. Other modes are generated either by simulation or by applying variations successively. The method has also been used in conjunction with branching and bounding operations (Melchers and Tang[17], Murotsu[18]). Gorman[19] used simulation for generating a set of values for the basic variables which are then used with a traditional structural analysis procedure to determine the failure mode. Repeated application yields other failure modes. The β-unzipping method (Thoft-Christensen and Sorensen[20]) is another similar approach. Optimization techniques have also been used for failure mode generation. Simulation, along with sensitivity analysis procedures for a static LP model of plastic analysis have been proposed by Rashedi and Moses[21].

Alternatively, the problem was formulated as an unconstrained nonlinear programming model and solved by a trial-and-error search procedure (Ang and Ma[16]). These procedures are not very appropriate because they may overlook stochastic important modes. Nafday, Corotis and Cohon[22] proposed an extreme point ranking algorithm but in view of the NP-hard nature of the reliability assessment problem it is not efficient for large systems. There remains the need for reliable algorithmic procedures for the generation of failure modes of building frames. The solution method for the inner problem described in this paper, which incorporates recent developments in large-scale constrained concave quadratic programming seems ideally suited for this purpose. The computational technique is illustrated on a two-storey two-bay frame.

2 PROBLEM STATEMENT

2.1 Formulation

The general multicriteria reliability-based optimization problem with collapse and unserviceability as the failure criteria, may be stated as: find the vector of design variables d such that:

$$V = f(d) \leq V^o \quad (1a)$$

$$P_{fo} = P_{fo}(d) \leq P_{fo}^o \quad \text{or} \quad \beta_o = \beta_o(d) \geq \beta_o^o \quad (1b)$$

$$P_{fu} = P_{fu}(d) \leq P_{fu}^o \quad \text{or} \quad \beta_u = \beta_u(d) \geq \beta_u^o \quad (1c)$$

in which V is the total volume(weight) of the structure; P_{fo} is the probability of plastic collapse; P_{fu} is the probability of unserviceability (i.e.: probability of first plastic hinge occurrence); $\beta_0 = \Phi - 1(1 - P_{fo})$ is the reliability index against plastic collapse (in which Φ^{-1} is the inverse of the standard normal probability distribution); $\beta_u = \Phi^{-1}(1 - P_{fu})$ is the reliability index against unserviceability; and the superscript o denotes a prescribed value (e.g.: V^o = maximum allowable volume; P_{fo}^o is the maximum allowable probability of plastic collapse; β_u^o is minimum allowable reliability index against the occurrence of the first plastic hinge).

Nine possible formulations could be used for finding the optimum solution:

A) Minimize the total volume of a structure, V, subject to reliability constraints defined in Eqs. (1b) and (1c).

B) Minimize the probability of collapse P_{fo} (maximize the safety index β_0), when the maximum structural volume V^o and the allowable risk level against unserviceability P_{fu}^o are prescribed.

C) Minimize the probability of unserviceability P_{fu} (maximize the safety index β_u), when the maximum structural volume V^o and the allowable risk level against collapse P_{fu}^0 are prescribed.

D) Minimize the probability of collapse P_{fo} (maximize β_0), when the allowable risk level against unserviceability P_{fu}^o is prescribed (V is of no concern).

E) Minimize the probability of unserviceability P_{fu} (maximize β_u), when the allowable risk level against plastic collapse P_{fu}^o is prescribed (V is of no concern).

F) Minimize the total structural volume V, when the allowable risk level against plastic collapse P_{fu}^o is prescribed (P_{fu} is of no concern).

G) Minimize the total structural volume V, when the allowable risk level against unserviceability P_{fu}^o is prescribed (P_{fo} is of no concern).

H) Minimize the probability of collapse P_{fo} (maximize β_0), when the maximum structural volume V^o is prescribed (P_{fu} is of no concern).

I) Minimize the probability of unserviceability P_{fu} (maximize β_u), when the maximum structural volume V^o is prescribed (P_{fo} is of no concern).

The preceding formulations A, B, and C are multi-constrained optimization problems, while D–I are single-constraint problems. The formulations D and E are useful when weight is of no concern and the most reliable structure against the occurrence of one limit state is desired, given the specified reliability level against the occurrence of the other limit state. The formulations F and G are useful when the reliability level determines the minimum weight structure so that the reliability level against the occurrence of the other limit state will be equal to a preassigned value. These single-criterion formulations together with the formulations H and I should be used for comparison with constrained optimization solutions that for the case A can be written:

$$\text{Min } V = f(d) \tag{2a}$$

subject to

$$g_1(d, m, \lambda, \theta, u, \theta^*, u^*) = 0 \tag{2b}$$

$$g_2(d, m, \lambda, u, \theta^*, u^*) \leq 0 \tag{2c}$$

$$g_3(d) \leq 0 \tag{2d}$$

$$\Pr[Z_k \leq 0] \leq P_{fs} \qquad k = 1, \ldots, m \tag{2e}$$

$$\Pr\left[\bigcup_{k=1,m} Z_k \leq 0\right] \leq P_{fo} \tag{2f}$$

where $d, m, \lambda, \theta, u, \theta^*, u^*$ are the vectors of design variables, random bending moment resistances, random loads, critical section rotations, nodal displacements, total critical section rotations and total nodal displacements, respectively. The objective function and the constraints (2b)–(2d) are linear. The above formulation is different from plastic limit synthesis problems, owing to single mode failure probability constraints (2e) and the system failure probability constraint (2f). To check whether the probability constraint conditions are satisfied, an optimization problem called an inner problem must be solved.

2.2 Assumptions

The following assumptions are made in solving the constrained optimization problem (2):

1) The general structural configuration including the lengths of all prismatic and straight members is specified in a fixed (deterministic) manner.

2) Plastic collapse is the only possible failure mode. Safety against loss of serviceability, which is defined in this study by the formation of the first plastic hinge, is accessed by means of the simplified probabilistic procedure proposed by Parimi and Cohn[23].

3) The effects of axial forces, shear and torsion are not considered.

4) The magnitudes of static loads, $L_1, L_2, \ldots, L_r$ which form the load vector L are random but their positions are deterministic.

5) The statistical dependence between any pair of loads L_1, L_m is accounted for through the coefficient of correlation $\rho(L_1, L_m)$.

6) The ultimate (plastic) moment capacities in positive or negative bending for both beam and column critical sections are computed by the equation for flexural capacity of members with an ideal elastic-plastic moment-curvature relationship. Consequently, the actual plastic moments of the critical sections in positive or

negative bending, $m_{1+}, m_{1-}, m_{2+}, m_{2-}, \ldots, m_{s+}, m_{s-}$, are random variables which make up vector m.

7) The statistical dependence between any pair of plastic moments m_{j+}, m_{k-}, is accounted for through the coefficient of correlation $\rho(m_{j+}, m_{k-})$.

8) The components of plastic moment vector m are statistically independent of the components of the load vector L.

9) Consistent with a first-order second-moment reliability approach, the minimum statistical information required for the evaluation of the optimum solution is:

(a) The mean values of the loads $\mu_{L1}, \mu_{L2}, \ldots, \mu_{Lr}$, which make up the vector μ_L, the coefficients of variation of the loads $\Omega(L_1), \Omega(L_2), \ldots, \Omega(L_r)$ which make up the vector Ω_L, and the coefficients of correlation between pairs of loads which form a square symmetrical correlation matrix denoted C_u.

(b) The coefficients of variation of the plastic moments $\Omega(m_{1+}), \Omega(m_{1-}), \Omega(m_{2+}), \Omega(m_{2-}), \ldots, \Omega(m_{s+}), \Omega(m_{s-})$, which make up the vector Ω_m and the coefficients of correlation between pairs of plastic moments which form a square symmetrical matrix denoted C_θ.

2.3 *Single mode failure probability constraint*

The probability of failure via the kth individual collapse mode p_k can be obtained from the probability that a certain performance function Z_k:

$$Z_k = U_k - E_k = m^{+t}\theta^{*+} + m^{-t}\theta^{*-} - L^t u^* \tag{3}$$

is negative. In Eq. (3) U_k and E_k are the internal and external random work associated with the kth collapse mode. Consistent with a first-order second-moment reliability analysis, the failure probability may be measured entirely with a function of the first and second moments of random parameters. It is assumed that safety with regard to plastic collapse via the failure mode k depends only on reliability index β_k, that is defined as the shortest distance from the origin to a failure surface in the reduced random variables coordinate system.

$$\beta_k = \mu_{Z_k}/\sigma_{Z_k} \tag{4}$$

It is important to note that the standard deviation of the safety margin of an individual collapse mode and the probability of occurrence of this mode increases as the statistical positive dependence between plastic moments and/or between loads that are active in producing the mechanism increases and vice-versa.

2.4 *Multi-mode failure probability constraint*

Evaluation of the failure probability appearing in constraints (2f) is one of the major concerns in the solution of reliability-based optimization problems. Since multiple integration with respect to random parameters must be executed and correlations of each failure mode must be known *a priori*, in a multi-mode failure probability, an exact calculation of probability Pr[ ] is practically impossible, without resorting to an approximation method. In general, the admissible failure probability for structural design is very low. Hence, approximation by Cornell's first-order upper bound is suitable for calculating multi-mode failure probability and it can be used for the designer's benefit, since it is conservative,

$$\operatorname*{Max}_{\text{all}\,k}\,[\Pr(Z_k)] \leq P_{fo} \leq 1 - \prod_{k=1,m} [1 - \Pr(Z_k)] \tag{5}$$

The lower bound, which represents the probability of occurrence of the most critical mode (dominant mode) is obtained by assuming the mode failure events Z_k to be perfectly dependent, and the upper bound is derived by assuming independence between mode failure events. Since this multi-mode failure probability is a summation of single mode failure probabilities, the constraint (2f) can be stated as,

$$\sum_{k=1,m} \Pr[Z_k \leq 0] = \sum_{k=1,m} P_{fk} \leq P_{fo} \tag{6}$$

For optimization purposes and since the failure probabilities are very small compared with other quantities, the multi-mode probability constraint can be replaced by the convex approximation,

$$\ln \sum_{k=1,m} c_1 e^{-2.3c_2\beta_k^{c_3}} \leq \ln P_{fo} \tag{7}$$

where P_{fo} denotes an admissible system failure probability, c_1, c_2 and c_3 are constants. Cornell has shown that in the 10^{-4} regions the value of c_2 varies from about 0.5 to 1.5 for several common distributions. The value of c_2 depends primarily on the skewness of the probability density function. For the normal distribution c_3 would be about 2.

The bound (6) can be improved by taking into account the probabilities of joint failure events such as $P(F_i \cap F_j)$ which means the probability that both events F_i and F_j will simultaneously occur. The resulting closed-form solutions for the lower and upper bounds are as follows:

$$p_f \geq P(F_1) + \sum_{i=2}^{m} \operatorname{Max}\left\{\left[P(F_i) - \sum_{j=1}^{i-1} P(F_i \cap F_j)\right]; 0\right\} \tag{8}$$

$$p_f \leq \sum_{i=1}^{m} P(F_i) - \sum_{i=2}^{m} \operatorname*{Max}_{j<1} P(F_i \cap F_j) \tag{9}$$

The above bounds can be further approximated using Ditlevsen's method of conditional bounding[12]. This is accomplished by using a Gaussian distribution space in which it is always possible to determine three numbers β_i, β_j and ρ_{ij} for each pair of collapse modes F_i and F_j such that if $\rho_{ij} > 0$ (i.e., if F_i and F_j are positively dependent):

$$P(F_i \cap F_j) \geq \operatorname{Max}\left\{\Phi(-\beta_j)\Phi\left(-\frac{\beta_i - \beta_j\rho_{ij}}{\sqrt{1-\rho_{ij}^2}}\right);\quad \Phi(-\beta_i)\Phi\left(-\frac{\beta_j - \beta_i\rho_{ij}}{\sqrt{1-\rho_{ij}^2}}\right)\right\} \tag{10}$$

$$P(F_i \cap F_j) \leq \Phi(-\beta_j)\cdot\Phi\left(-\frac{\beta_i - \beta_j\rho_{ij}}{\sqrt{1-\rho_{ij}^2}}\right) + \Phi(-\beta_i)\cdot\Phi\left(-\frac{\beta_j - \beta_i\rho_{ij}}{\sqrt{1-\rho_{ij}^2}}\right) \tag{11}$$

in which β_i and β_j are the safety indices of the ith and the jth failure mode, ρ_{ij} is the correlation coefficient between the ith and the jth failure mode, and $\Phi(\ )$ is the standardized normal probability distribution function.

The probabilities of the joint events $P(F_i \cap F_j)$ in (8) and (9) are then approximated with the appropriate sides of (10) and (11). For example, if F_i and F_j are positively dependent, for the lower (8) and upper (9) bounds it is necessary to use the approximations given by the upper (11) and lower (10) bounds, respectively.

Moses and Kinser[10] have shown that the overall probability of collapse of a system, Eq. (6), can be expressed in the following way:

$$p_f = P(F_1) + \sum_{i=2}^{m} a_i P(F_i) \tag{12}$$

where,

$$a_i = P(S_1 \cap S_2 \cap \cdots \cap S_{i-1} | F_i) \tag{13}$$

is the conditional probability that the first $i-1$ modes survive given that mode i occurs. Note that the failure modes are arranged so that $P(F_1) \geq P(F_2) \geq \cdots \geq P(F_i) \geq \cdots \geq P(F_m)$ because the value of the conditional probability (13) depends on the ordering of failure modes.

The method introduced by Vanmarcke[24] reduces in relation (13) the number of survival events to one, such that:

$$a_i \leq \operatorname*{Min}_{j=1}^{i-1} P(S_j|F_i) = a_i^* \tag{14}$$

Therefore, $a_i^* = a_1 = 1$, $a_2^* = a_2$ and,

$$P(F_1) + \sum_{i=2}^{m} a_i P(F_i) \tag{15}$$

is an upper bound to the overall probability of collapse (6). Using a first-order approach, Vanmarcke introduced a useful approximation of the conditional probability $P(S_j|F_i)$ in terms of the safety indices β_i and β_j and of the coefficient of correlation ρ_{ij} between the failure modes F_i and F_j, as follows:

$$P(S_j|F_i) \cong 1 - \frac{\Phi[\mathrm{Max}(\beta_j/|\rho_{ij}|, \beta_i)]}{\Phi[\beta_i]} \tag{16}$$

in which it is assumed that the probability of occurrence of the ith mode $P(F_i) = \Phi(\beta_i)$ depends on β_i only.

A different approximate method which avoids calculating conditional probabilities resulting from conditions leading to failure via pairs of failure modes is the PNET method[13]. This method requires the determination of the coefficients of correlation between any two failure modes i and j and is based on the notion of a demarcating correlation coefficient ρ_0 assuming those failure modes with high correlation ($\rho_{i,j} \geq \rho_0$) to be perfectly correlated and those with low correlation ($\rho_{i,j} < \rho_0$) to be statistically independent.

On the basis of one mechanism, Klingmüller[25] evaluated the failure probabilities of large systems performing the theorem of total probability. It is stated that:

$$P(A) = \sum_{i=1,m} P(A|B_i)P(B_i) \tag{17}$$

where the event A represents the system failure and B_i is the occurrence of mechanism i respectively. From this theorem the following approximation may be derived:

$$P(A) = P_F \cong \frac{\max_i P(A|B_i)}{P(B_i)} \tag{18}$$

with $\max_i P(A|B_i)$ being the probability of failure of the deterministic most relevant mechanism and $P(B_i)$ the probability that this is the relevant mechanism. This method neglects the fact that the deterministic mechanism may not yield the highest failure probability and therefore is not necessarily identical with the stochastic most relevant mechanism. The mechanical model also implies that all loads are mutually perfectly correlated, since only a common load factor can be introduced as a random variable. However, if one replaces the deterministic by the stochastic most important mechanism the solution given by this method can be a useful approach for large systems.

3 SOLUTION METHOD

The solution method can be divided into two alternating subprocedures:

a) an optimization procedure for the nonconvex inner problem, that finds the stochastic most important mechanism and enumerates other relevant collapse modes for a given value of the design variables (plastic moments of resistance).

b) an optimization of the convex outer problem on the design variables, that is the solution with least cost satisfying serviceability and technological requirements and the reliability constraints (2e) and (2f). In order to satisfy these, the number of reliability constraints is expanded to include several stochastic dominant modes.

The procedure is repeated until the vector of design variables converges. Since these two procedures are themselves mathematical programmes, any suitable techniques can be applied. The probability constraints are checked by using the results of the inner problem.

Bracken and McGill[26] considered a mathematical program with optimization problems in the constraints. They developed a criteria under which the problem is convex, differing from the problem treated here. This form of reliability-based optimization is similar to the parametric optimization problem treated by Kwak and Haug[27] except that the domains of subproblems are functions not only of random parameters but also of design variables. This problem was overcome by Lee and Kwak[28] for elastic trusses, but it was assumed that the solution of each inner problem is unique. This is not true for structures with plastic behaviour, because each collapse mechanism is a local solution of the inner problem.

4 INNER PROBLEM

4.1 Structural relations

A frame can be reduced to a determinate form in a variety of different ways. The bending moments at the critical sections can be expressed in terms of the coordinates of the influence diagrams associated with unit magnitudes of the loads L and the indeterminate forces p.

$$m = Bp + B_0 L \tag{19}$$

If a mechanism is created by allowing flexural deformations, the kinematic relations for the mesh description to ensure compatibility are:

$$0 = B^t\theta \tag{20}$$

The rates of the displacements u which correspond to the loads L can be evaluated in terms of the flexural deformation rates:

$$u = B_0 t\theta \tag{21}$$

Since the angular rotation at the critical sections are the independent kinematic variables, scaling can be performed through the following normalization condition:

$$b_0^t = 1 \Rightarrow b_0^t\theta^+ - b_0^t\theta^- = 1 \tag{22}$$

where the elements of b_0 are linear combinations of the rows of B_0.

The familiarity of structural engineers with respect to a determinate basis with physical release systems may be responsible for the limited use that has been made of mesh procedures in structural applications. With more complex frames, the derivation of the basic matrices (B, B_0) becomes very tedious. Therefore, the selection of the most suitable release system becomes more important and also more difficult. The most convenient basis for the frame of Figure 1 is that of the regional meshes shown (Munro[29]).

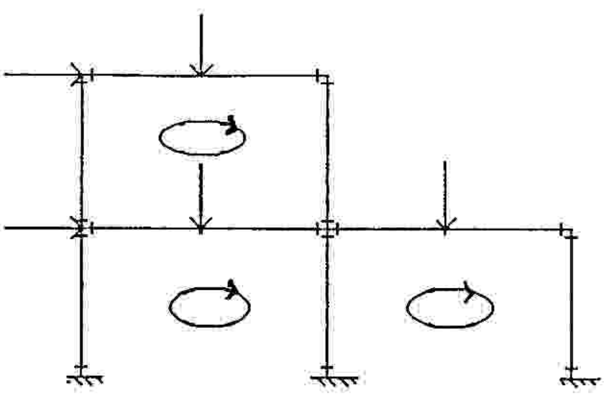

Figure 1

The B matrix can be assembled in terms of four submatrices and the elements of B_0 can be obtained from any set of bending moments which are in equilibrium with the unit loads.

4.2 *Constitutive relations*

The plastification in frames can be considered to be restricted to pre-located sections and the yield criterion imposes bounds upon the values of the moments in all critical sections. For example, if the plastic resisting moments, negative and positive, in the critical section i are m^*_{i-}, and m^*_{i+}, respectively, it must be:

$$-m^*_{i-} \leq m_i \leq m^*_{i+} \tag{23}$$

Since the variables θ are real and the standard algorithms of mathematical programming work with nonnegative variables, the rotation in the critical section i, θ_i is decomposed in the pair of nonnegative variables θ_{i+} and θ_{i-}:

$$\theta_i = \theta_{i+} - \theta_{i-} \tag{24}$$

Therefore, the section i can be:

1) A positive plastic hinge participating in the failure mechanism,

$$\theta_{i+} > 0; \quad \theta_i = \theta_{i+} > 0; \quad m_i = m^*_{i+} \tag{25a}$$

2) A negative plastic hinge participating in the failure mechanism,

$$\theta_{i-} > 0; \quad \theta_i = -\theta_{i-} < 0; \quad m_i = -m^*_{i-} \tag{25b}$$

3) An elastic point or a plastic hinge not participating in the mechanism,

$$\theta_{i+} = 0 = \theta_{i-}; \quad \theta_i = 0; \quad -m^*_{i-} \leq m_i \leq m^*_{i+} \tag{25c}$$

4.3 *Second moment approach*

To represent explicitly the uncertainty in the strength of the structure and the loading acting on it random variables need to be considered. The problem arises to evaluate the conditional failure probability of the structural system given a certain load event, and the program seeks the stochastic most relevant mechanism. For the uncorrelated random variables m, L and positive $[\mu^t_m\theta^* - \mu^t_L u^*]$, the identification of the stochastic most relevant mechanism consists of minimizing the reliability index β,

$$\min \beta = \frac{[\mu^t_m\theta^* - \mu^t_L u^*]}{\sqrt{(\sigma_m)^{2t}(\theta^*)^2 + (\sigma_L)^{2t}(u^*)^2}} \tag{26}$$

subject to the compatibility relations (20), the linear incidence equations,

$$\theta^* = J_{\theta+}\theta^+ + J_{\theta-}\theta^-; \quad u^* = J_u u \tag{27}$$

where the incidence matrices $J_{\theta+}$, $J_{\theta-}$ and J_u are obtained by associating the rotations of the element sides with the rotations of the members represented by the same random variables θ^* and the displacements of the point loads linked by the same random variables u^*. Sign constraints on the variables need also to be considered:

$$\theta^+ \geq 0,\ \theta^- \geq 0,\ u \geq 0,\ \theta^* \geq 0,\ u^* \geq 0 \tag{28}$$

If matrices C_θ and C_u represent the correlations between the bending moments of resistance and between the loads, respectively, the reliability index is given by,

$$\beta = \frac{[\mu_m^t \theta^* - \mu_L^t u^*]}{\sqrt{(\theta^{*t} \sigma_m^t C_\theta \sigma_m \theta^*) + u^{*t} \sigma_L^t C_u \sigma_L u^*}} \tag{29}$$

For random variables that are correlated, the original variates may be transformed to a set of uncorrelated variables. The required set of uncorrelated transformed variates can be obtained through an orthogonal transformation. If the probability distribution functions of the random variables are not Gaussian, the Rosenblatt[30] transformation may be used. The identification of all the significant collapse modes of a ductile structural system is necessary in the analysis and evaluation of the system reliability, including the evaluation of the corresponding bounds.

These mathematical programs belong to the class of fractional programming problems. The minimization of β shares its solutions with the quadratic concave minimization:

$$\max - 1/\beta^2 = -(\sigma_m)^{2t}(\theta^*)^2 - (\sigma_L)^{2t}(u^*)^2 \tag{30a}$$

subject to,

$$\mu_m^t \theta^* - \mu_L^t u^* = 1 \tag{30b}$$

$$\theta^* = J_{\theta^+}\theta^+ + J_{\theta^-}\theta^-;\ u^* = J_u u \tag{30c}$$

$$\theta = Cu \tag{30d}$$

$$\theta^+ \geq 0,\ \theta^- \geq 0,\ u \geq 0,\ \theta^* \geq 0,\ u^* \geq 0 \tag{30e}$$

This problem cannot be solved by convex programming techniques because of the possibility of nonglobal local minima. The global optimum of these programs gives the plastic deformations for the stochastic most important mechanism and the reduced random variables can be evaluated by using,

$$m^{*\prime} = -\sigma_m \theta^* \beta^2 \tag{31a}$$

$$\lambda_{L'} = \sigma_L u^* \beta^2 \tag{31b}$$

5 SOLUTION OF A CONCAVE QUADRATIC MINIMIZATION PROBLEM

This constrained optimization problem is equivalent to other combinatorial optimization problems such as 0–1 integer programming and remains an NP-hard problem. From a computational complexity point of view, this means that in the worst case the computer time will grow exponentially with the number of nonlinear variables. The most general methods for global optimization can be divided in two classes:

deterministic and stochastic. Here we are going to consider only deterministic methods. The most important approaches for concave quadratic programming are enumerative techniques, cutting-plane methods, branch and bound, bilinear programming methods or different combinations of these techniques.

5.1 Enumerative methods by ranking extreme points

An important property of concave functions is that every local and global solution is achieved at some extreme point of the feasible domain. This property makes the problem more tractable since the search for an optimal solution can be restricted to the set of extreme points. Cabot and Francis[31] presented the following procedure for solving the quadratic concave programming problem: First a linear program is solved where the objective function is an underestimator to the original objective function. Then the extreme point ranking approach developed by Murty[32] is applied using this underestimating function to obtain an optimal solution to the original problem. Although this algorithm in the worst case will degenerate to complete inspection of all vertices, this approach is computationally infeasible for large problems.

5.2 Cutting plane methods and partition of feasible domain techniques

Tuy's method[33] is based on the use of cuts to exclude parts of the feasible domain and a cone splitting procedure. The feasible set can be visualized as contained in a cone generated by the edges coincident with a vertex. The method solves a sequence of subproblems associated with subcones of this initial cone. The cone splitting algorithm described by Tuy has been shown by Zwart[34] to be nonconvergent. In the same paper, Zwart gives a counterexample for a method developed by Ritter[35] in which an infinite sequence of cutting planes is needed to reduce the feasible domain. Tohai and Tuy[36] proposed a class of algorithms which are based upon a combination of branch and bound techniques with Tuy's original cutting plane method.

5.3 Branch and bound methods using approximations of the objective function

Quadratic concave minimization belongs to the more general class of nonconvex separable programming. Critical to the algorithm proposed by Falk and Soland[37] for nonconvex separable programming is the use of convex envelopes. The algorithm is of the branch and bound type and solves a sequence of subproblems in each of which the objective function is linear (or convex). These problems correspond to successive partitions of the feasible domain. two different refining rules lead to convergence of the algorithm under different requirements on the problem functions. Horst[38] proposed a different approach for separable nonconvex programming: Instead of considering convex envelopes and minimizing several convex subproblems he only minimizes the objective function over suitable intervals. The efficiency of these methods depend upon the tightness of the underestimating functions.

5.4 *Bilinear programming approaches*

A mathematical program with a bilinear objective function over a linear domain may possess many local optima. Konno[39] used the equivalence of the quadratic concave minimization problem to the associated bilinear program. The method searches for an optimal solution among those basic solutions with equilibrium points of the problem. The bilinear programming algorithm for quadratic programming is greatly simplified as a consequence of the problem structure and because of the fact that the verification of necessary and sufficient conditions for the existence of an optimal solution is reduced to the solution of a linear program.

5.5 *Methods for large scale concave quadratic programming*

The motivation for considering this type of problem is similar to that for problems of 0–1 integer linear programming. Large-scale 0–1 mixed integer programming can be solved in a reasonable time (Marsten[40]), providing most of the variables are continuous. The computational method presented by Rosen[41] for finding the global minimum of a quadratic concave function over a polyhedral set takes advantage of the ellipsoid-like level surfaces of the objective function to find a good initial vertex and to eliminate a rectangular domain (enclosed in a level surface) from further consideration. The basic step used is to initially determine a rectangular domain which contains the projection of the domain on the space of the nonlinear variables. This can be done by a multiple-cost-row LP with n objective functions. Then a linear underestimating function is computed and a linear underestimating problem is solved to give lower and upper bounds for the global optimum. This solution also gives a bound on the relative error in the function value of this incumbent vertex. If the incumbent is not a satisfactory approximation to the global optimum, a guaranteed ε-approximate solution is obtained by solving a single 0–1 mixed integer programming problem. This integer problem is formulated by a piecewise linear underestimation of the separable problem.

5.5.1 Reduction to separable form For simplicity of notation, the concave quadratic program involving correlated random variables can be written in the following form,

$$\text{Min } \psi(z, y) = -1/2z^tQz \tag{32a}$$

$$\text{over } \Omega = \{(z, y): A_1z + A_2y = b, z \geq 0, y \geq 0\} \tag{32b}$$

with Q a positive definite symmetric matrix and A_1 and A_2 have m rows. z, y are n and m vectors corresponding to the random variables and the rotations of the critical sections, respectively. To carry out the reduction to separable form it is necessary to compute the real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of Q and the corresponding eigenvectors $u_1, u_2, \ldots, u_n$. Then $Q = U\,D\,U^t$ where $U = [u_1, \ldots, u_n]$, $D = \text{diag}[\lambda_1, \ldots, \lambda_n]$.

The multiple-cost-row linear program must be solved first,

$$\max u_i^t z \tag{33a}$$

$$\text{subject to: } (z, y) \in \Omega, \qquad i = 1, 2, \ldots, n \tag{33b}$$

Denote by β_i the corresponding optimal values. The concave quadratic programming can be formulated as a separable programming in terms of the new variables x_i,

$$\min \phi(x) = \sum_{i=1,n} -1/2\lambda_i x_i^2 \tag{34a}$$

$$\text{subject to: } A_3 x + A_2 y = b, \qquad 0 \leq x_i \leq \beta_i, \qquad y \geq 0 \tag{34b}$$

where $A_3 = A_1 U$.

5.5.2 Linear underestimator and error bounds The smallest rectangular domain R_x in x-space can be constructed by using β_i. A linear function $\Gamma(x)$ which interpolates $\phi(x)$ at every vertex of R_x and underestimates $\phi(x)$ on R_x is given by,

$$\Gamma(x) = \sum_{i=1,n} -1/2\lambda_i \beta_i x_i \tag{35}$$

and

$$R_X = \{x: 0 \leq x_i \leq \beta_i; i = 1, \ldots, n\}$$

The following linear underestimating program, which differs from the multiple-cost-row only in its objective function, must be solved,

$$\min \Gamma(x) \text{ over } (x, y) \in \Omega \tag{36}$$

The solution to this problem will give a vertex $\underline{v} = (\underline{x}, y)$ which is a candidate for the global minimum ψ^* of the original problem and,

$$\Gamma(\underline{x}) \leq \psi^* \leq \phi(\underline{x}) \tag{37}$$

Therefore, the error at $(\underline{x}, \underline{y})$ is given by $\phi(\underline{x}) - \psi^*$ and this error is bounded by,

$$E(\underline{x}) = \phi(\underline{x}) - \Gamma(\underline{x}) \tag{38}$$

If $E(\underline{x})$ is sufficiently small, $\psi(\underline{x}, \underline{y})$ is an acceptable approximation to the global optimum ψ^*. It is necessary to obtain bounds on $E(\underline{x})$ relative to the range of $\phi(x)$ over R_x. The quantity,

$$\Delta\phi = -\min_{x \in R_x} \phi(x) \tag{39}$$

is used as a scaling factor to measure $E(x)$ on Ω.

Assuming, without loss of generality that,

$$\lambda_1 \beta_1^2 \geq \lambda_i \beta_i^2, \qquad i = 2, \ldots, n \tag{40}$$

and defining the ratios,

$$\rho_i = \lambda_i \beta_i^2 / \lambda_1 \beta_1^2, \qquad i = 2, \ldots, n \tag{41}$$

the lower bound on $\Delta\phi$ is given by

$$\Delta\phi \geq 1/2\lambda_1\beta_1 \sum_{i=1,n} \rho_i \tag{42}$$

$E(x)$ attains its maximum at $x_i = \beta_i/2$, $i = 1, \ldots, n$, so that for any $x \in R_x$,

$$E(x) = \phi(x) - \Gamma(x) = 1/2 \sum_{i=1,n} \lambda_i(\beta_i - x_i)x_i \leq 1/8\lambda_1\beta_1 \sum_{i=1,n} \rho_i$$

An *a priori* bound on the relative error is given by:

$$(\psi(\underline{x}, \underline{y}) - \psi^*)/\Delta\phi \leq 0.25 \tag{43}$$

5.5.3 Piecewise linear approximation and zero-one integer formulation Using a piecewise linear approximation to each function $-1/2\lambda_i x_i^2$ a mixed integer zero-one LP can be formulated such that the finding of a solution $(\underline{x}, \underline{y})$ for which,

$$(\psi(\underline{x}, \underline{y}) - \psi^*)/\Delta\phi \leq \varepsilon \tag{44}$$

can be guaranteed for any specified tolerance ε.

Each interval $[0, \beta_i]$ is partitioned into k_i equal subintervals of length $h_i = \beta_i/k_i$ and the new variables w_{ij} are introduced, such that the variable x_i is represented uniquely by w_{ij},

$$x_i = h_i \sum_{j=1,k_i} w_{ij} \tag{45}$$

where the variables w_{ij} are restricted to the range $[0, 1]$ and the vector $(w_{i1}, w_{i2}, \ldots, w_{ik_i})$ is restricted to have the form $(1, \ldots, 1, w_{il}, 0, \ldots, 0)$.

Assuming, without loss of generality, that:

$$0 < p_n \leq p_{n-1} \leq \cdots \leq p_1 = 1 \tag{46}$$

k_i are the smallest integers chosen such that,

$$k_i \geq \left[n\rho_i \Big/ \left(4\varepsilon \sum_{i=1,n} \rho_i \right) \right]^{1/2} \tag{47a}$$

where,

$$1 \leq k_n \leq k_{n-1} \leq \cdots \leq k_1 \tag{47b}$$

By defining,

$$\Delta_{ij} = -1/2\lambda_i j^2 h_i^2 + 1/2\lambda_i (j-1)^2 h_i^2, \qquad i = 1, \ldots, n, j = 1, \ldots, k_i \tag{48}$$

It follows that the linear function,

$$\Gamma_i(x_i) = \sum_{i=1,k_i} \Delta_{ij} w_{ij} \tag{49}$$

interpolates $-1/2\lambda_i x_i^2$ at the points $x_i = jh_i$, $j = 0, 1, \ldots, k_i$ and since $-1/2\lambda_i x_i^2$ is concave, it satisfies $\Gamma_i(x_i) \leq -1/2\lambda_i x_i^2$ for $x_i \in [0, \beta_i]$. That is $\Gamma_i(x_i)$ is a piecewise linear underestimating function for $-1/2\lambda_i x_i^2$.

Therefore, if the objective function $\psi(x, y)$ is approximated by the piecewise linear underestimating functions $\sum_{i=1,n} \Gamma_i(x_i)$, the separable quadratic concave minimization can be approximated by the following 0–1 mixed integer problem in the continuous variables w_{ij} and the binary variables z_{ij}:

$$\text{Min} \sum_{i=1,n} \sum_{j=1,k_i} \Delta_{ij} w_{ij} \tag{50a}$$

$$\text{subject to} \sum_{i=1,n} h_i a_i \sum_{j=1,k_i} w_{ij} + A_2 y \tag{50b}$$

$$0 \le w_{ij} \le 1,\ y \ge 0 \tag{50c}$$

$$w_{i,j+1} \le z_{ij} \le w_{ij},\ z_{ij} \in \{0, 1\},\ j = 1, \ldots, k_i - 1,\ i = 1, \ldots, n \tag{50d}$$

where a_i is the ith column of A_3.

5.6 *Parallel branch and bound algorithm*

An algorithm for concave quadratic minimization which is designed to be efficient for problems with many design variables and can take full advantage of parallel processing was recently presented (Phillips and Rosen[42]). It considers linear underestimating functions and upper and lower bounds on the global minimum in the way described above. Branch and bound techniques are then applied to reduce the feasible region under consideration and decrease the difference between the upper and lower bounds. The average computational performance for problems with 25 nonlinear and 400 linear variables and a maximum error bound of $\varepsilon = 0.001$ using a four processor Cray2 was 15 seconds. Results with problems with up to 50 nonlinear and 400 linear variables have shown that an approximate solution with a minimum of computation but with a relative large bound ($\varepsilon = 0.1$) can be obtained in a computational time depending linearly on the number of nonlinear variables.

6 ENUMERATION OF OTHER STOCHASTIC IMPORTANT MECHANISMS

An appraisal of the current procedures for generating the stochastic most representative failure modes indicate that they are variously dependent on simulation, trial-and-error, perturbation, human judgement, complex heuristic strategies, or approximations, either for choosing the appropriate starting points or for continuing the method at different stages. Some of the methods generate the modes in random order and thus many of the important modes may be missed without ever knowing about them. The techniques described above to find the stochastic most representative mechanism either by mixed integer linear programming or by the parallel processing algorithm are associated with branch and bound strategies that reduce the feasible region to decrease the differences between upper and lower bounds. Both methods can be employed to enumerate other stochastic important modes by assigning the incumbent solution at a desired level of significance, larger than the global solution.

Since the domain is partitioned with respect to the nonlinear variables only (random loads and resisting moments), it remains the possibility that within the same range of bounds other mechanisms may exist (and are not identified). Moreover, mechanisms with plastic hinges at different locations but associated with the same values of the random variables might be overlooked. For this reason, after finding the stochastic dominant mode over each of the subregions, one of the two procedures described next must be employed to enumerate the remaining mechanisms.

6.1 *Branch and bound tree*

Once some of the critical sections participating in the most representative mechanisms over each subregion earlier detected are ruled out of the basis, other modes can be identified by a branch and bound based strategy. A strong branching rule is employed: the number of nodes created at each stage from an intermediate node is equal to the number of critical sections participating in the mechanism associated with the intermediate node. The result obtained at any node is a lower bound on those obtained by branching from it and if the reliability index associated with that node is larger than a prespecified value or is infeasible, then that branch of the combinatorial tree can be terminated. Since a large number of problems created by branching at intermediate nodes have no feasible solution or have a large lower bound, the procedure is reasonably efficient.

6.2 *Vertex enumeration and ranking*

Murty's method[32] can be used for ordering the extreme points of a linear domain. It is based on a theorem which states that if $x^1, \ldots, x^r$ are r best points, x^{r+1} will be an adjacent point of one of the first extreme points. The new point is distinct from the first r and maximizes the objective function giving $1/\beta^2$ among all the remaining extreme points. All the adjacent extreme points are found from the canonical tableau corresponding to the linear domain $A\,x = b$ by bringing one by one all the nonbasic variables—only those corresponding to rotations in the critical sections and that do not participate in the mechanism—into the basis. Denoting the entries in any canonical tableau by the usual simplex notation a_{il}, b_i in which l is the column index for any nonbasic column, define:

$$\Delta_l = \min b_i/a_{il} \qquad \forall i, l \text{ such that } a_{il} > 0 \tag{51a}$$

$$\Delta_l = +\infty, \text{ if } a_{il} \leq 0 \qquad \forall i, l \tag{51b}$$

Variables corresponding to index i yielding the minimum value of Δ_l indicate the basic variable that will leave the basis when the variable corresponding to l enters the basis.

For each l with finite Δ_l, an adjacent point of x is given by,

$$x^l(\Delta_l) = [x_1^l(\Delta_l) \cdots x_n^l(\Delta_l)]^t \tag{52}$$

whose basis vector is found from,

$$x^l_{s_i} = b_i - a_{il}\Delta_l \tag{53a}$$

$$x^l_1 = \Delta_l \tag{53b}$$

where s denotes a basis. Note that at least one of the $x^l_{s_i}$ will always be zero. The procedure is repeated until either a prespecified number of extreme points is found or all the extreme points in the ranked sequence whose objective value gives a reliability index which is less than a given value of β_{max} are obtained. On the basis of plastic limit analysis, failure modes are generated in Ref. [22] by Murty's method, although they end up with a much larger number of degenerate mechanisms caused by the larger number of state variables that correspond to the nodal description. The smaller number of variables used in the mesh description reduces the likelihood of such solutions. Moreover, since this procedure is carried out over the subintervals, the number of adjacent feasible vertices is not very large.

7 OUTER PROBLEM

7.1 *Objective function*

The aim of reliability-based plastic optimum design is to determine the average values of the bending moments at the critical sections for a specified failure probability of the structure.

$$\text{Min } V = c^t d = l^t \mu_m \tag{54}$$

where l is the vector of member lengths. In the case of reinforced concrete frames the object is to find the concrete cross-sectional dimensions and the corresponding amount of reinforcing steel so that a specified reliability level against plastic collapse is provided and an adopted objective function is minimized. The relevance of the adopted objective function is an open subject with important consequences on the optimum solution (Surahman and Rojiani[43]). If the concrete section sizes are assigned, the design objective is simpler: the minimization of the longitudinal steel volume. The reinforcement for a critical section is assumed to be extended over a definite length called equivalent length for the calculation of the volume of reinforcement. The equivalent length of reinforcement adopted for beam and column sections of the examples presented here are shown in Figure 2.

For under-reinforced sections of known lever arm, the area of reinforcement has a linear relation to the plastic moment of resistance. The total volume of reinforcement can therefore be expressed as the linear function,

$$\text{Min } V = c^T d = a^t_+ \mu_{m^+} + a^t_- \mu_{m^-} \tag{55}$$

where a_+ and a_- are vectors of known constants whilst μ_{m^+} and μ_{m^-} are the vectors of design variables: means of the plastic moments of resistance with respect to positive and negative bending, respectively.

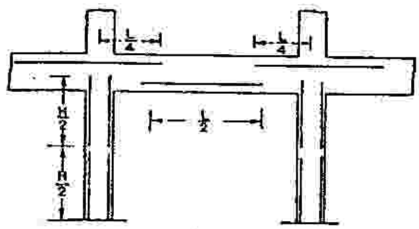

Figure 2

7.2 *Technological and serviceability constraints*

Frequently, it is convenient to establish relationships between the design variables of various members in order to satisfy requirements and simplify the calculations. These are termed technological constraints.

Any design of a reinforced concrete frame based on the behaviour under ultimate load must satisfy the serviceability requirements under working load. These requirements are that: (i) no yielding of a section occurs under working load; (ii) the deflections are within allowable limits; (iii) the width and spacing of cracks are within desired limits. Investigations show that the serviceability requirements for members subjected predominantly to bending depend mainly on the yielding of the critical sections and hence depend on the yield moment factor. Safety against loss of serviceability, which is defined by the formation of the first yield line can be checked by means of a simplified probabilistic procedure. Thus, for practical design, additional constraints will be required and these serviceability constraints may be most readily included in the form of lower bounds on the design variables.

7.3 *Reliability constraints*

By fixing the design variables, the inner problem gives the yield line rotations θ^* and nodal displacements u^* associated with the stochastic most important mechanism and other relevant modes. Clearly, the reliability analysis for another set of design variables (but the same mechanism) would give proportional yield line rotations and nodal displacements. For a prespecified reliability index β_s, the single mode probability constraints defined by m collapse modes will be satisfied if,

$$\mu_m^{t}\theta_k^* - \mu_L^{t}\mu_k^* \geq \beta_s\left[\frac{2D}{\sqrt{\sigma_m^{2t}\theta_k^{*2} + \sigma_F^{2t}v_k^{*2}}}\right] \tag{56}$$

where $k = 1, \ldots, m$. It can be shown that these constraints are convex with respect to the design variables. Since the convex approximation of the multi-mode constraint (7) is convex, the outer problem can be solved by any convex programming technique. It has a linear objective function (19) and the domain is defined by linear (20) and nonlinear constraints (7),(21). These nonlinear constraints could be linearized by creating a first-order Taylor series expansion and the linearized problem could be

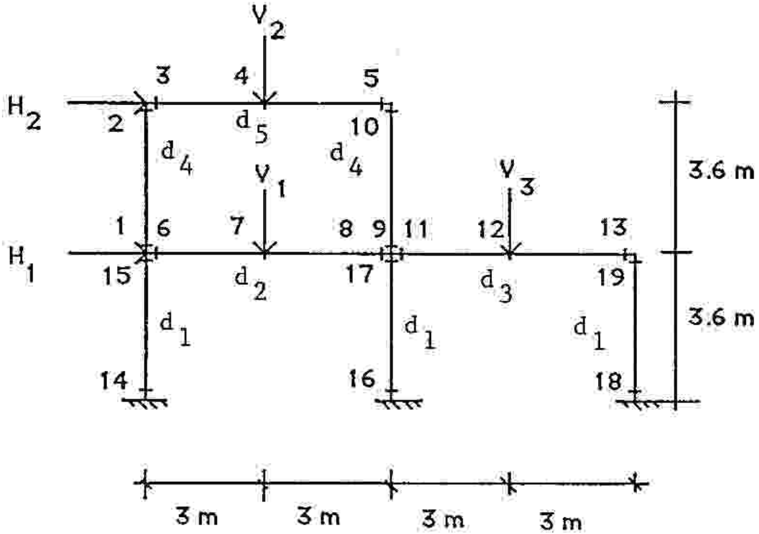

Figure 3

solved by using linear programming. The design is updated and a new iteration performed repeating the process until convergence is achieved. Since these move limits are critical, the procedure can be made much more efficient by using either Sequential Quadratic Programming or Sequential Convex Programming (Vanderplaats[44]).

8 NUMERICAL EXAMPLE

It is intended to find the minimum volume of the rigid portal frame with fixed geometry represented in Figure 3 which satisfies given reliability requirements. 5 design variables corresponding to average bending moments of resistance of the columns and beams are considered. The random loads possess the following characteristics:

$$\mu_{V1} = 169 \text{ kN}; \quad \Omega_{V1} = 0.15; \quad \mu_{V2} = 89 \text{ kN}; \quad \Omega_{V2} = 0.25$$

$$\mu_{V3} = 116 \text{ kN}; \quad \Omega_{V3} = 0.25$$

$$\mu_{H1} = 62 \text{ kN}; \Omega_{H1} = 0.25; \mu_{H2} = 31 \text{ kN}; \Omega_{H2} = 0.25$$

Several types of correlation between loads and between the design variables are considered:

Case I—Perfect correlation within members and column–column correlation. Statistically independent decision variables. Statistically independent random loads except the horizontal loads which are perfectly correlated.

Case II—Perfect correlation among all plastic moments. Statistically independent random loads except the horizontal loads which are perfectly correlated.

Case III—Perfect correlation within members and column–column correlation. Statistically independent decision variables. All loads are perfectly correlated.

Case IV—Perfect correlation among all plastic moments. All loads are perfectly correlated.

As an illustrative example of the method consider an upper limit of 10^{-5} in the individual modes of collapse probabilities (Cornell's lower bound) for Case I. Take as initial average plastic moment capacities the design variables given by the plastic limit synthesis LP multiplied by 2.5.

d_1	d_2	d_3	d_4	d_5	Vol
140.5	515.8	243.0	139.5	194.3	8245

At least 5 mechanisms are needed to avoid move limits becoming critical. The inner problems pick up 9 modes with higher probabilities of failure, the highest of which is 4×10^{-4}. These mechanisms are used by the outer problem to produce a new set of design variables:

d_1	d_2	d_3	d_4	d_5	Vol
182.8	323.3	293.3	69.1	291.0	7918

By solving the reliability assessment inner problem, another mechanism is added to those chosen in the first iteration. The corresponding outer problem leads to,

d_1	d_2	d_3	d_4	d_5	Vol
191.8	380.8	299.0	129.4	204.8	8312

The stochastic most important modes for this design are those given previously and the convex outer problem has for solution,

d_1	d_2	d_3	d_4	d_5	Vol
191.5	385.8	298.6	122.6	228.7	8429

which is a good approximation of the final design. This design is 2% more costly than the starting point, but the probability of failure of the stochastic most relevant mode is only 1/40 of the latter. It is also important to analyze the contribution of the dominant modes to the overall probability of failure.

Number of dominant modes	Cornell upper bound	Varmarcke upper bound	Ditlevsen upper bound
	(24 mechanisms)		
5	59%	68%	73%
7	83%	91%	96%

The same problem is solved by imposing an upper limit of 10^{-2} on the maximum probability of collapse of the individual modes. The starting point is obtained by multiplying the moment capacities of the design variables given by the plastic limit synthesis LP by 1.5.

d_1	d_2	d_3	d_4	d_5	Vol
84.5	309.5	145.8	83.7	116.5	4947

After solving the inner problem 10 mechanisms that exceeded the prescribed probability of failure—the highest of which is 6.4×10^{-2}—where chosen.

The outer problem gives,

d_1	d_2	d_3	d_4	d_5	Vol
115.4	163.8	166.7	103.6	125.4	4595

The reliability assessment of this design lead to consider another 8 mechanisms with high probabilities of collapse. The solution of the convex outer problem involving 18 mechanisms is,

d_1	d_2	d_3	d_4	d_5	Vol
104.8	207.7	172.5	77.1	154.7	4896

The stochastic most important modes for this design are those given previously and the convex outer problem has for solution,

d_1	d_2	d_3	d_4	d_5	Vol
104.8	213.9	172.4	74.5	157.8	4930

which is close to the optimum sizing. For a solution with approximately the same volume, the reliability of the most important mode has been increased more than six times.

The contribution of the dominant modes to the overall probability of failure is:

Number of dominant modes	Cornell upper bound	Varmarcke upper bound	Ditlevsen upper bound (24 mechanisms)
5	35%	48%	49%
7	52%	61%	65%

Therefore, the contribution of the dominant modes becomes more important by increasing the reliability requirements.

Figures 4 to 12 show the influence of several parameters on the variation of the optimum volume of the frame of Figure 3. In Figure 4 is represented the variation of the minimum volume with respect to the overall collapse probability given by the upper Ditlevsen bound. Cases I and II are considered for the coefficients of variation (c.o.v) of the bending moments,

$$\Omega_m = \Omega_{d_1} = \Omega_{d_2} = \Omega_{d_3} = \Omega_{d_4} = \Omega_{d_5} = 0.15$$

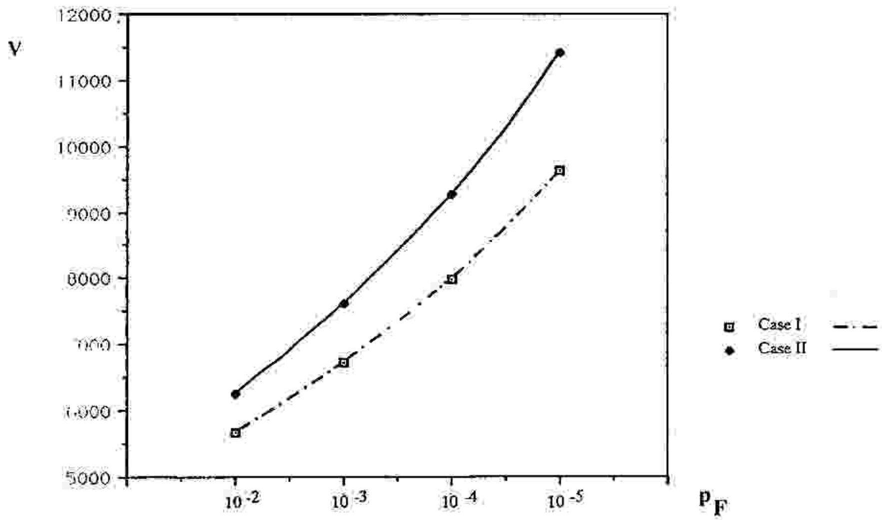

Figure 4

In Figure 5 is represented the variation of the minimum volume with respect to the overall collapse probability (given by the upper Ditlevsen bound) for Case I supposing the coefficients of variation of strength Ω_m are 0.10, 0.15 and 0.20.

Figure 6 shows the variation of the minimum volume required to satisfy the overall probability of collapse given by Cornell (upper bound), Vanmarcke (upper bound), Ditlevsen (upper bound), Ditlevsen (lower bound) and Cornell (lower bound) for Case I and a c.o.v. of the resisting bending moments of 0.15.

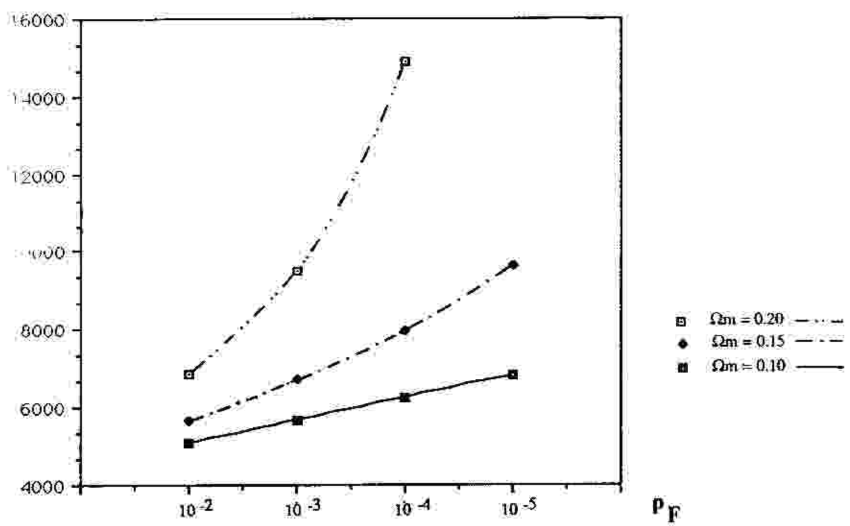

Figure 5

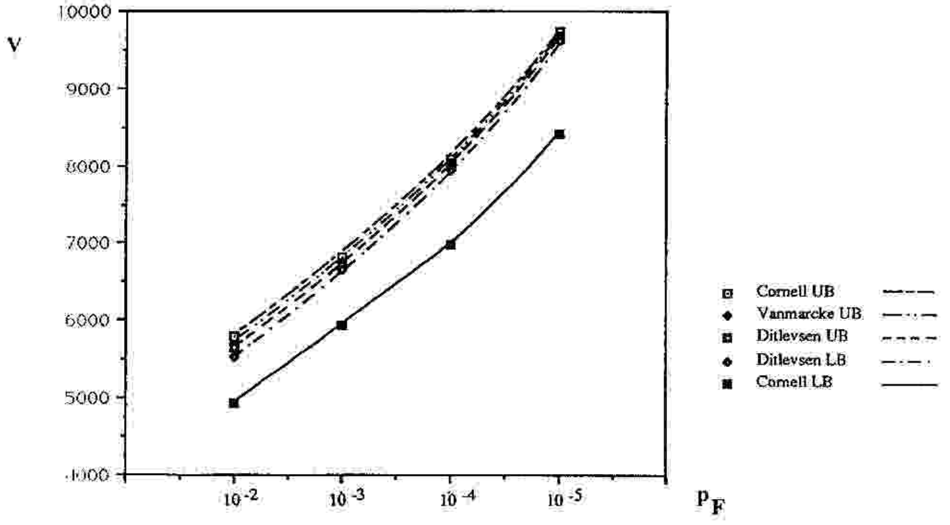

Figure 6

For a specified correlation between plastic moments C_θ, bounds on the optimum volume range can be obtained by means of upper and lower bounds on the system collapse. All methods for estimating global collapse probabilities yield results within the optimum volume range whose bounds are obtained by assuming perfect correlation among failure modes (Cornell's lower bound) and mutual independence (Cornell's upper bound), although they are much closer to the latter.

Figure 7 shows how the minimum volume varies with the probability of failure

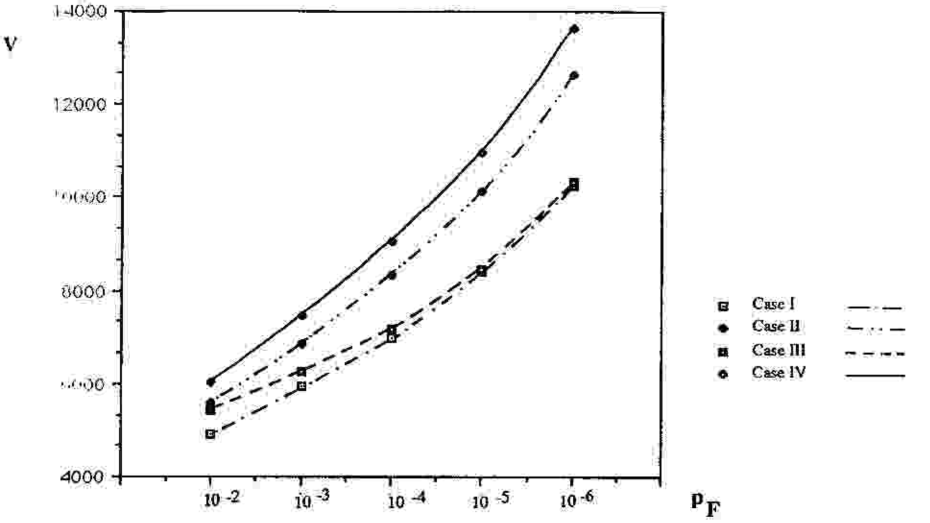

Figure 7

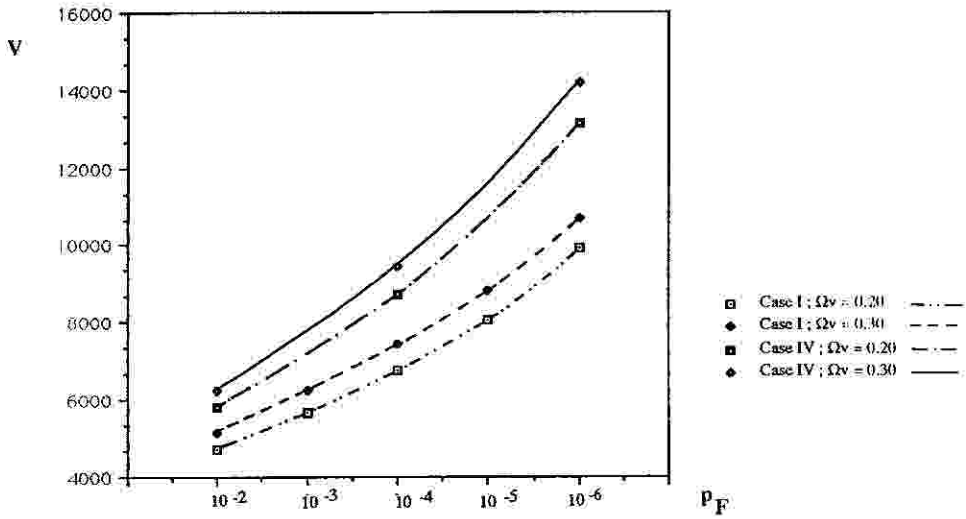

Figure 8

when the constraints associated with individual modes of collapse only (Cornell lower bound) are considered for Cases I, II, III and IV and $\Omega_m = 0.15$.

The minimum volume increases with increasing correlation between loads and between moment capacities. So an assumption of independent beam-column moment capacities may be very inappropriate and on the unconservative side.

By considering the constraints associated with individual modes of collapse only, Figure 8 shows how the c.o.v. of the gravity loads changes the minimum sizing.

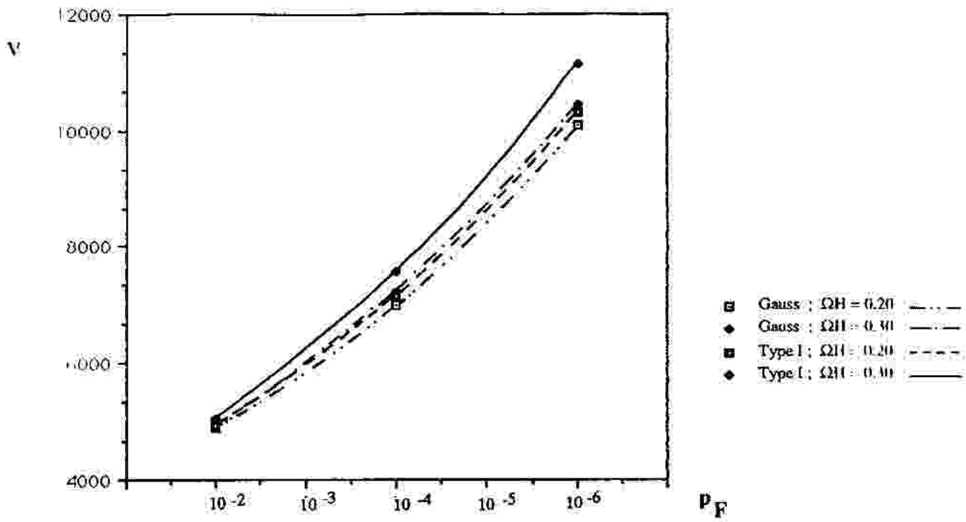

Figure 9

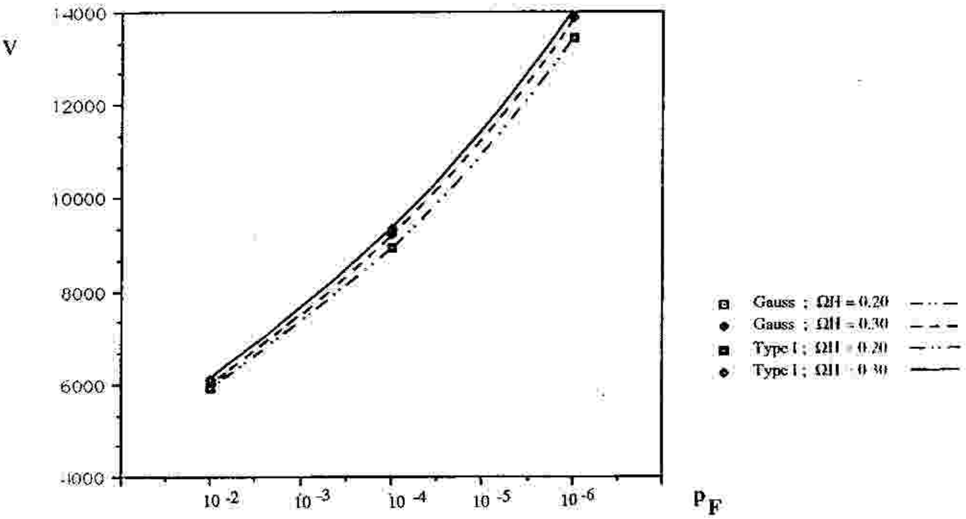

Figure 10

Only plastic moment capacities and loads are assumed to be random variables. Since the types of their distributions may have significant influence on the resulting failure probabilities, the question of how to model their distributions is very important. For example, a lognormal distribution shows the best fit to observed data on ultimate stress of the most commonly used steel. Moreover available data on wind loading are often given through maximum wind speeds which can only be realistically described by an extreme value distribution. Figures 9 and 10 show how the type of probabilistic distribution function of the horizontal loads (Gaussian and Extreme

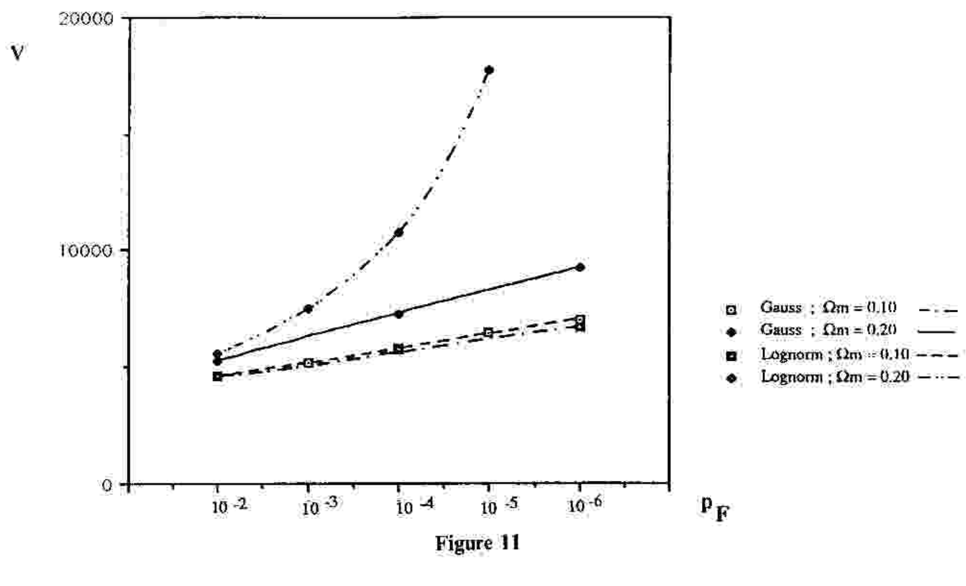

Figure 11

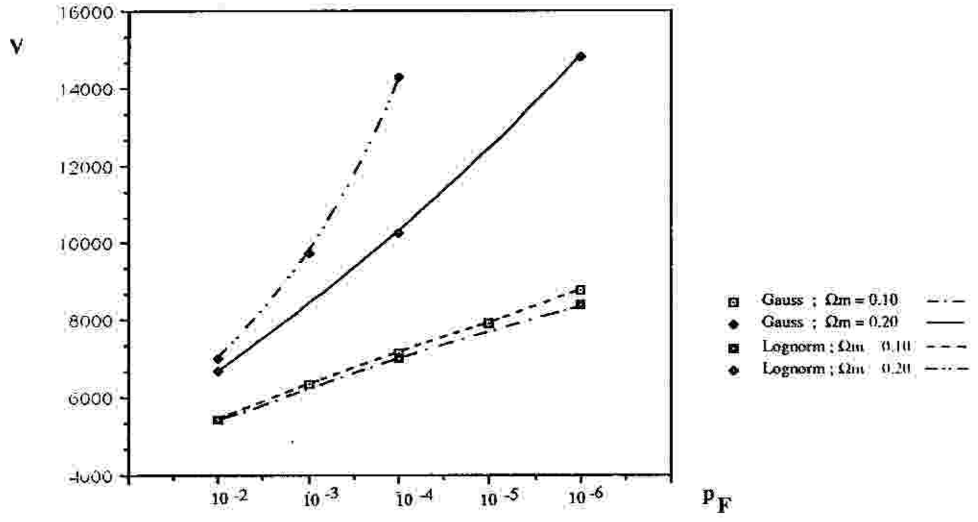

Figure 12

type I) may affect the minimum sizing for Cases I and IV, respectively, considering individual collapse modes only.

In this particular example, the optimum solution is not very sensitive to the distribution functions of the horizontal loads.

The sensitivity of the optimum volume solution with respect to changes in the c.o.v. of the strength are represented in Figures 11 and 12 for Cases I and IV, respectively, considering individual collapse modes only. Gaussian and lognormal distribution functions for the resisting moments are considered.

They indicate that the lognormal distribution results in upper values for the minimum volume solutions. The influence of the type of distribution decreases with increasing values of correlation between loads and between plastic moments. The influence of the type of distribution highly increases with the prescribed value of the probability of failure.

9 CONCLUSIONS

The solution method for the reliability-based design of frames can be divided into two alternating subprocedures: a) an optimization of the nonconvex fractional program giving the reliability index of the stochastic most important mechanism (and also enumerating the remaining relevant local solutions); b) an optimization of the convex outer problem that includes the cost function.

The proposed technique is illustrated on a two-storey two-bay steel frame. The following remarks may be made regarding the influence of the parameters mentioned in the optimum solution. As expected, the optimum volume increases with increasing values of Ω_m, Ω_L and/or increasing the correlation between plastic moments or between loads. The sensitivity of the optimum volume solution increases more with increasing values of Ω_m than with increasing values of Ω_L. For a specified correlation structure between plastic moments, the rate of increase in the optimum volume range is larger at high than at low values of the c.o.v. Ω_m and/or Ω_L. In addition it is interesting to note that the optimum solution is not very sensitive to either the c.o.v. of the lateral load Ω_H or the statistical variation between loads C_u. Preliminary results on larger frames seem to agree with these conclusions.

The method presented in this paper provides a powerful tool to obtain a practical optimum solution for large frames, offering the potential for application within a reliability-based code context which is an important goal in structural design.

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