Structural Safety, 5 (1988) 95–108
Elsevier Science Publishers B.V., Amsterdam – Printed in The Netherlands

A BRANCH AND BOUND STRATEGY FOR FINDING THE RELIABILITY INDEX WITH NON-CONVEX PERFORMANCE FUNCTIONS

L.M.C. Simões

Departamento de Engenharia Civil, Faculdade de Ciencias e Tecnologia, Universidade de Coimbra, 3049 Coimbra (Portugal)

(Received April 2, 1987; accepted December 15, 1987)

Key words: safety; reliability index; branch and bound; quadratic programming; non-convex limit-state equations.

ABSTRACT

When faced with non-convex limit-state equations, the Lagrangean-based algorithms that find the most probable failure point (Rackwitz, Direct Method) are unable to guarantee that the solution found is indeed the minimum distance to the origin of the variates. Some illustrative examples of this undersirable behaviour are given. A branch and bound approach associated with suitable convex underestimating functions is presented in this paper. This technique can be accepted as competitive, since each resulting subproblem is solved by using an algorithm for quadratic programming.

1. INTRODUCTION

The evaluation of the exact probability of safety will involve the integration of the joint probability density function over the non-linear safe region. To avoid a multiple numerical quadrature it is sufficient to find an approximation to the exact probability, i.e. the point on the failure surface with minimum distance to the origin of the variates is the most probable failure point. With the second-moment approach the reliability may be measured entirely with a function of the first and second moments of the design variables, namely, the reliability index, β, when there is no information on the probability distributions. If the appropriate forms of the distributions are prescribed, the corresponding probability may be evaluated on the basis of equivalent normal distributions.

For performance functions that are non-linear, the evaluation of the exact probability of safety will generally be involved. Unlike the linear case, there may be no unique distance from

0167-4730/88/$03.50

the failure surface to the origin of the reduced variables. If the failure surface is approximated by its tangent plane, depending on whether the exact failure surface is convex or concave towards the origin, this approximation will be on the safe side or unsafe side, respectively.

When the limit-state equation is non-convex, there may be several points on the failure surface that locally minimize the distance to the origin of the variates. Some simple examples of this non-convex behaviour are given. This justifies the need for the devopment of methods that lead to the global solution, i.e.: the overall minimum distance.

The branch and bound (B&B) approach described here is a globally convergent method that can be applied to the solution of the general separable problems such as functions of one variable and their products. This procedure can be accepted as competitive, since each problem can be solved by using a quadratic programming (QP) technique.

2. RELIABILITY ASSESSMENT

If the random variables are uncorrelated normal variates, the probality of the safe state is:

$$p_s = P\left(a_0 + \sum_i a_i x_i > 0\right)$$

$$= 1 - \Phi\left\{\frac{-\left(a_0 + \sum_i a_i \mu_{x_i}\right)}{\sqrt{\sum_i (a_i \sigma_{x_i})^2}}\right\} = \Phi\left\{\frac{a_0 + \sum_i a_i \mu_{x_i}}{\sqrt{\sum_i (a_i \sigma_{x_i})^2}}\right\} = \Phi(\beta) \tag{1}$$

Introducing the set of uncorrelated variables $\boldsymbol{X}'$ [1]:

$$x_i' = (x_i - \mu_{x_i})/\sigma_{xi}; \quad i = 1,2,\ldots,n \tag{2}$$

The safe and the failure state may be portrayed in the space of the reduced variates by the appropriate limit-state equation. In terms of the reduced variates $\boldsymbol{X}_i'$ the limit state equation would be:

$$g(\sigma_{x_1} x_1' + \mu_{x_1}, \ldots, \sigma_{x_n} x_n' + \mu_{x_n}) = 0 \tag{3}$$

Observe from Fig. 1 that as the limit-state surface $g(\boldsymbol{x}) = 0$ moves closer to the origin, the safe region $g(\boldsymbol{x}) > 0$ decreases. Therefore the position of the failure surface may be represented by the minimum distance from the surface $g(\boldsymbol{x}) = 0$ to the origin of the reduced variates [2]. Shinozuka [3] has shown that the point on the failure surface with the minimum distance to the origin is the most probable failure point and can be taken as a measure of reliability.

For random variables that are correlated, the original variates may be transformed to a set of uncorrelated variables [4]. The required transformation is necessarily dependent on the covariance matrix $\boldsymbol{C}$ of the original variates $x_1, x_2, \ldots, x_n$, whose elements $\text{cov}(x_i, x_j)$ are the covariances between the pairs of variables x_i and x_j. The covariance between a pair of reduced variates x_i' and x_j' is equal to the correlation coefficient ρ_{ij} between the pair of the original variates x_i and x_j. The required set of uncorrelated transformed variates can be obtained from $\boldsymbol{X}'$ through the following orthogonal transformation:

$$\boldsymbol{Y} = \boldsymbol{T}^{\mathrm{T}} \boldsymbol{X}' \tag{4}$$

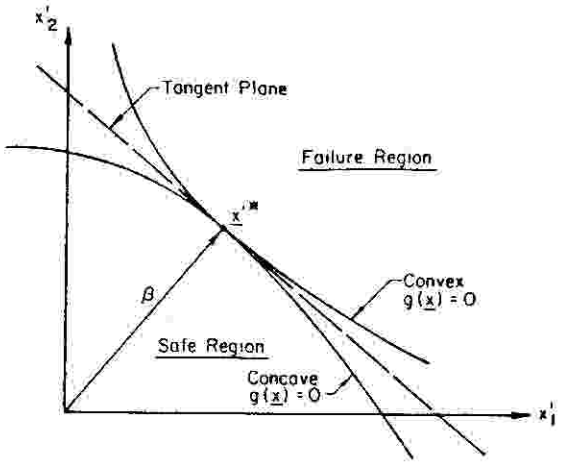

Fig. 1. Limit-state surface $g(x) = 0$.

$\boldsymbol{T}$ will be an orthogonal matrix if it is composed by the eigenvectors corresponding to the eigenvalues of the correlation matrix $\boldsymbol{C}'$:

$$\boldsymbol{T}^{\mathrm{T}}\boldsymbol{C}'\boldsymbol{T} = [\lambda] \tag{5}$$

in which $[\lambda]$ is a diagonal matrix of the eigenvalues of the real symmetric matrix of the reduced variates $\boldsymbol{C}'$. This transformation represents a rotation of the coordinates from $\boldsymbol{X}'$ to $\boldsymbol{Y}$; for the two-variable case, this transformation is illustrated in Fig. 2.

As the eigenvectors are mutually orthogonal,

$$\boldsymbol{T}^{\mathrm{T}}\boldsymbol{C}'^{-1}\boldsymbol{T} = [1/\lambda] \tag{6}$$

Since $\boldsymbol{T}$ is orthogonal, $\boldsymbol{T}^{-1} = \boldsymbol{T}^{\mathrm{T}}$. The reduced variates $\boldsymbol{X}'$ and the original variates $\boldsymbol{X}$ are related to $\boldsymbol{Y}$ as follows:

$$\boldsymbol{X}' = \boldsymbol{T}\boldsymbol{Y} \tag{7}$$

and

$$\boldsymbol{X} = [\sigma_{\boldsymbol{X}}]\,\boldsymbol{X}' + \{\mu_{\boldsymbol{X}}\} \tag{8}$$

The distance D is given by,

$$D = \boldsymbol{X}'^{\mathrm{T}}\boldsymbol{C}^{-1}\boldsymbol{X}' = \boldsymbol{Y}^{\mathrm{T}}\boldsymbol{T}^{\mathrm{T}}\boldsymbol{C}'^{-1}\boldsymbol{T}\boldsymbol{Y} = [1/\lambda]\,\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{Y} \tag{9}$$

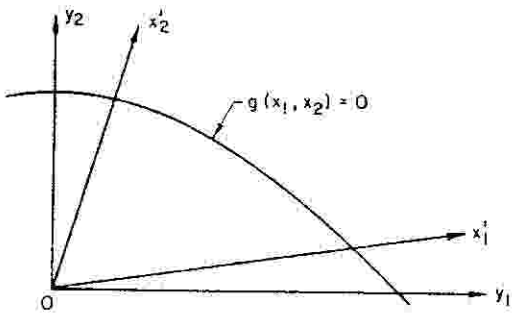

Fig. 2. Transformation of coordinates from $\boldsymbol{X}'$ to $\boldsymbol{Y}$.

If the distributions of the original random variables are non-normal, the corresponding probability of safety may be evaluated using equivalent normal distributions [5,6]; in such a case, the mean values and standard deviations of the equivalent normal distributions $\mu_{x_i}^N$ and $\sigma_{x_i}^N$ must be used in place of μ_{x_i} and σ_{x_i}.

The distance from a point $\boldsymbol{x}' = (x_1', x_2', \ldots, x_n')$ on the failure surface to the origin of the uncorrelated variates $\boldsymbol{X}'$ is,

$$D = \sqrt{x_1'^2 + \ldots + x_n'^2} = (\boldsymbol{x}'^{\mathrm{T}} \boldsymbol{x}')^{1/2} \tag{10}$$

If the variables are correlated, the point on the failure surface $(x_1'^*, x_2'^*, \ldots x_n'^*)$ having the minimum distance to the origin may be determined by minimizing the function D subject to the constraint $g(\boldsymbol{x}) = 0$, that is:

$$\begin{aligned} &\min \quad (\boldsymbol{x}'^{\mathrm{T}} \boldsymbol{C}'^{-1} \boldsymbol{x}')^{1/2} = D \\ &\text{s.t.} \quad g(\boldsymbol{x}) = 0 \end{aligned} \tag{11}$$

The solution of $g(\boldsymbol{x}) = 0$ under the norm is equivalent to the mathematical programming problem:

$$\begin{aligned} &\min \quad 1/2\ \boldsymbol{x}'^{\mathrm{T}}\ \boldsymbol{C}'^{-1}\ \boldsymbol{x}' \\ &\text{s.t.} \quad g(\boldsymbol{x}) = 0 \end{aligned} \tag{12}$$

since both satisfy the first-order necessary conditions.

3. EXAMPLES OF NON-CONVEX PERFORMANCE FUNCTIONS

The multiplicity of the points that locally minimize the distance between the failure surface and the origin of the variates may even arise for simple performance functions such as bilinear expressions. Consider the random normal variables X and Y uncorrelated and let:

$\overline{X} = 15.0; \quad \Omega_X = 0.10; \quad \sigma_X = 1.5$

$\overline{Y} = 10.0; \quad \Omega_Y = 0.15; \quad \sigma_Y = 1.5$

Let the performance function be defined as,

$$-13X - 16.5Y + XY + 221.25 = 0 \tag{13}$$

The failure surface represented in Fig. 3 has a discontinuity for $X' = 1$ (i.e. $X = 16.5$). Depending on the starting point, the Lagrangean-based algorithms (Rackwitz [7], Direct Method [8]) converge to one of the following solutions:

$\beta = 3.392; \quad X = 12.7; \quad Y = 14.8$

$\beta = 2.219; \quad X = 18.1; \quad Y = 8.8$

The non-convex limit-state equation, represented by the performance function (eqn. 13), consists of the two disjoint regions shown in Fig. 4. There are two local solutions that are obtained when using Rackwitz's algorithm. The direct method produced an oscillatory behaviour being unable to converge in some instances:

$\beta = 3.392; \quad X = 12.7; \quad Y = 14.8$

$\beta = 2.219; \quad X = 18.1; \quad Y = 8.8$

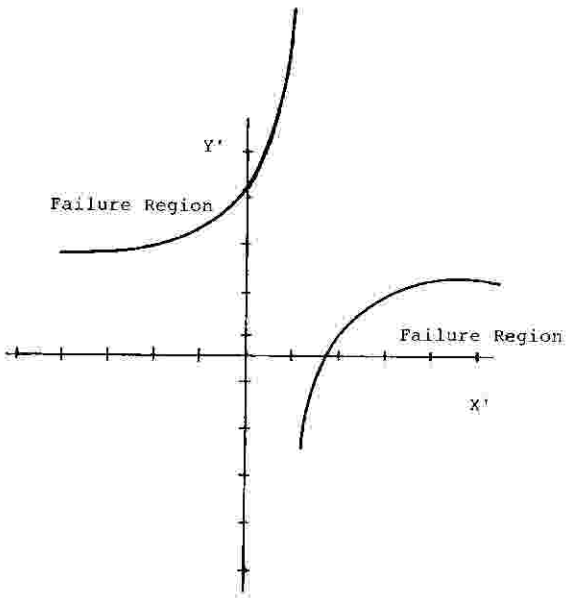

Fig. 3. Failure surface of the example.

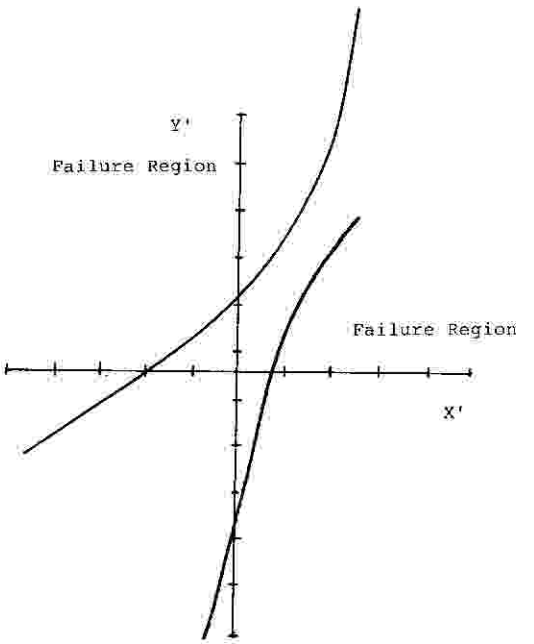

Fig. 4. Limit-state equation of the example.

It is also possible to have limit-state functions defining non-convex connected domains. In any case, it is not possible to guarantee that Lagrangean-based methods yield the global minimum distance to the failure surface, as opposed to any solution that locally minimizes the distance.

4. BRANCH AND BOUND METHODS

One method that avoids this local behaviour is outlined next, followed by a discussion of the definition of the factorable function (to which class the problem functions are assumed to belong) and the procedure for obtaining underestimating convex and overestimating concave functions. Algorithmic details on how to implement this strategy are contained in Section 4.3. Section 4.4 contains a rule for successively dividing up the regions over which the convex underestimating problems are computed.

4.1 General methodology

The general non-convex domain is transformed in the B&B strategy into a sequence of intersecting convex domains by the use of underestimating convex functions. It is well known that a local solution to a problem having a convex objective function and being restricted by a convex domain is also its global solution.

The two main ingredients are a combinatorial tree with appropriately defined nodes and some upper and lower bounds to the final solution associated with each node of the tree. It is then possible to eliminate a large number of possible solutions without evaluating them. As the implicit enumeration program relies on an upper bound, its efficiency can be greatly improved by providing a good feasible initial solution.

A partial solution is said to be fathomed if the best feasible completion of the solution can be found, or if can be determined that, no matter how the design variables are choosen, it will be impossible to find a feasible completion of smaller distance than that previously found. If a partial solution is fathomed, this means that all possible completions of this partial solution have been enumerated implicitly and therefore need not be enumerated explicitly. When the last node is fathomed the algorithm terminates with the optimum solution. Back-tracking in the tree is performed so that no solution is repeated or omitted from consideration.

4.2 Transformation of factorable functions into convex underestimates

In this section details are presented for the generation of convex underestimates. Basic to this is the notion of a convex underestimating function. Suppose the function has the form:

$$T[t(\boldsymbol{x})] + U[u(\boldsymbol{x})] \cdot V[v(\boldsymbol{x})] \tag{14}$$

where $t(\boldsymbol{x})$, $u(\boldsymbol{x})$ and $v(\boldsymbol{x})$ are continuous functions of n variables and $T(\cdot)$, $U(\cdot)$ and $V(\cdot)$ are continuous functions of a single variable; it is desired to find a convex function which bounds the original factorable function from below for all $\boldsymbol{x}$ in some convex set $\boldsymbol{S}$ when $t(\boldsymbol{x})$, $u(\boldsymbol{x})$ and $v(\boldsymbol{x})$ are restricted in range. Assume that there are convex functions $c_t(\boldsymbol{x})$, $c_u(\boldsymbol{x})$ and $c_v(\boldsymbol{x})$, concave functions $C_t(\boldsymbol{x})$, $C_u(\boldsymbol{x})$ and $C_v(\boldsymbol{x})$ and numbers a_t, a_u, a_v, b_t, b_u and b_v for which it is known that for $\boldsymbol{x}$ in $\boldsymbol{S}$

$$c_t(\boldsymbol{x}) \leqslant t(\boldsymbol{x}) \leqslant C_t(\boldsymbol{x}) \tag{15}$$

$$c_u(\boldsymbol{x}) \leqslant u(\boldsymbol{x}) \leqslant C_u(\boldsymbol{x}) \tag{16}$$

$$c_v(\boldsymbol{x}) \leqslant v(\boldsymbol{x}) \leqslant C_v(\boldsymbol{x}) \tag{17}$$

and for which it is required that,

$$a_t \leqslant t(\boldsymbol{x}) \leqslant b_t \tag{18}$$

$$a_u \leqslant u(\boldsymbol{x}) \leqslant b_u \tag{19}$$

$$a_v \leqslant v(\boldsymbol{x}) \leqslant b_v \tag{20}$$

The convex envelope of a function over a closed convex set is the highest convex function which underestimates the function everywhere and the concave envelope is the lowest concave

TABLE 1

Examples of convex and concave envelopes and the corresponding minimizing and maximizing points

Function	Interval	Convex envelope minimizing point	Concave envelope maximizing point
z^2	$[a,b]$	$e(z) = z^2$ $z_{\min} = a$ if $a \geqslant 0$ $z_{\min} = b$ if $b \leqslant 0$ $z_{\min} = 0$ otherwise	$E(z) = (a+b)z - ab$ $z_{\max} = a$ if $a^2 \geqslant b^2$ $z_{\max} = b$ if $b^2 > a^2$
e^z	$[a,b]$	$e(z) = e^z$ $z_{\min} = a$	$E(z) = [(e^b - e^a)/(b-a)]z + (be^a - ae^b)/(b-a)$ $z_{\max} = b$

function which overestimates the function everywhere. The convex envelope of $T(\cdot)$ on the interval $[a_t, b_t]$ is denoted by $e_T(\cdot)$ and the concave envelope by $E_T(\cdot)$. Compute a point at which each function achieves its minimum on its interval domain and a point at which each function achieves its maximum. That is, let $z^T_{\min}$, $z^T_{\max}$, A_T, B_T be the values where,

$$A_T = T(z^T_{\min}) = \min\ T(z), \quad z \in [a_t, b_t] \tag{21}$$

$$B_T = T(z^T_{\max}) = \max\ T(z), \quad z \in [a_t, b_t] \tag{22}$$

The lower-bounding convex function of $T[t(\boldsymbol{x})]$ for $\boldsymbol{x} \in \boldsymbol{S} \cap \{\boldsymbol{x} \mid a_t \leqslant t(\boldsymbol{x}) \leqslant b_t\}$ is

$$T[t(\boldsymbol{x})] \geqslant e_T[\mathrm{mid}\{c_t(\boldsymbol{x}),\ C_t(\boldsymbol{x}),\ z^T_{\min}\}] \tag{23}$$

where $\mathrm{mid}(z_1, z_2, z_3)$ is the function which selects the middle value of three scalars. The concave overestimating function for $T[t(\boldsymbol{x})]$ is

$$T[t(\boldsymbol{x})] \geqslant E_T[\mathrm{mid}\{c_t(\boldsymbol{x}),\ C_t(\boldsymbol{x}), z^T_{\max}\}] \tag{24}$$

For two functions of a single variable, Table 1 gives the convex and concave envelopes and their minimizing and maximizing points.

Finding the underestimating function for the product term is more complicated. First note that for the convex envelope of the function $U \cdot V$ in the rectangle $A_U \leqslant U \leqslant B_U$, $A_V \leqslant V \leqslant B_V$, the function values at the corners must coincide with the underestimate taken. Since three points are enough to define a plane in 3-D, the convex underestimate will be taken as the coordinate on the highest of the two planes defining a ridge through the two intermediate function values.

$$e_{U,V} = \max[B_V U + B_U V - B_U B_V, A_V U + A_U V - A_U A_V] \tag{25}$$

assume without loss of generality that $A_V \geqslant 0, A_U < 0$ and $B_U \geqslant 0$. It then follows that for $\boldsymbol{x} \in \boldsymbol{S} \cap \{\boldsymbol{x} \mid a_u \leqslant u(\boldsymbol{x}) \leqslant b_u,\ a_v \leqslant v(\boldsymbol{x}) \leqslant b_v\}$,

$$\begin{aligned} U[u(\boldsymbol{x})] \cdot V[v(\boldsymbol{x})] \geqslant e_{U,V} = \max\{ & B_V e_U[\mathrm{mid}\{c_u(\boldsymbol{x}),\ C_u(\boldsymbol{x}), z^U_{\min}\}] \\ & + B_U e_V[\mathrm{mid}\{c_v(\boldsymbol{x}), C_v(\boldsymbol{x}), z^V_{\min}\}] - B_U B_V, \\ & A_V e_U[\mathrm{mid}\{c_u(\boldsymbol{x}), C_u(\boldsymbol{x}), z^U_{\min}\}] \\ & + A_U e_V[\mathrm{mid}\{c_v(\boldsymbol{x}), C_v(\boldsymbol{x}), z^V_{\min}\}] - A_U A_V\} \end{aligned} \tag{26}$$

Since the maximum of two convex functions is convex, the complete convex underestimating function for the product term has been obtained.

The concave overestimating functions are accordingly defined as the coordinate at the lowest of the two planes defining a ridge through the corners having the higher and lower function values:

$$E_{U,V} = \min[B_V U + A_U V - A_U B_V, A_V U + B_U V - B_U A_V] \tag{27}$$

When these estimating functions are switched around to another quadrant, the approximations are suitably modified. Both the underestimating and overestimating functions are not differentiable everywhere. If the solution is sought of a quadratic programming (QP) problem where some of these constraints are present, there are several ways of handling this by altering the problem to

create an equivalent QP. The simplest way involves the addition of some extra inequality constraints, i.e.:

$$\begin{aligned} \min \quad & \boldsymbol{x}^T\boldsymbol{C}\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geqslant b \\ & \boldsymbol{f}^{\mathrm{T}}\boldsymbol{x} + \max\left\{\boldsymbol{g}_1^{\mathrm{T}}\boldsymbol{x},\ \boldsymbol{g}_2^{\mathrm{T}}\boldsymbol{x}\right\} \leqslant h \\ & \boldsymbol{p}^{\mathrm{T}}\boldsymbol{x} + \min\left\{\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{x},\ \boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{x}\right\} \geqslant r \end{aligned} \tag{28}$$

is equivalent to the QP

$$\begin{aligned} \min \quad & \boldsymbol{x}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{D}\boldsymbol{x} \geqslant e \\ & \boldsymbol{f}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{g}_1^{\mathrm{T}}\boldsymbol{x} \leqslant h \\ & \boldsymbol{f}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{g}_2^{\mathrm{T}}\boldsymbol{x} \leqslant h \\ & \boldsymbol{p}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{x} \geqslant r \\ & \boldsymbol{p}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{x} \geqslant r \end{aligned} \tag{29}$$

Other formulations may be employed. When the constraints are equalities, they should be replaced by a pair of inequalities.

4.3 Outside-in approach

In the following the algorithm originally presented by Soland [9] for separable functions is outlined and it is shown that it can easily be extended to cover the convex underestimating functions described above. Let $\boldsymbol{x}^p$ be the solution of the linear underestimating subproblem (P_p), with $k = 1$

$$\begin{aligned} \min \quad & \boldsymbol{x}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geqslant b \\ & \boldsymbol{l}_p \leqslant \boldsymbol{x} \leqslant L_p \end{aligned} \tag{30}$$

If $\boldsymbol{x}^p$ is not a feasible solution of the original problem, one may try to strengthen the constraints or to restrict the domain of the subproblem (P_p) in order to make the solution $\boldsymbol{x}^p$ infeasible. (P_p) is replaced by a set of problems that bound the original problem in the sense that there exist one optimal solution $\boldsymbol{x}^*$ for at least one problem $j \in W^p$. Suppose an optimal solution to each such problem is obtained and let

$$\boldsymbol{x}^s = \min_{j \in W^p} \boldsymbol{x}^{j\mathrm{T}}\boldsymbol{C}\boldsymbol{x}^j \tag{31}$$

If $\boldsymbol{x}^s$ is not a feasible solution of the original problem one of the problems of the bounding set is replaced by a set of new problems. Make $p = p + 1$. The problem s is replaced by a set W^p, such that $W^{\mathrm{p}} = (W^{p-1} - \{s\}) \cup W^s$ contains an optimal solution of the original problem feasible for at least one problem W^p.

For each problem $j \in W^{\mathrm{p}}$ either $\boldsymbol{x}^j$ is infeasible for j, or $\boldsymbol{x}^{j\mathrm{T}}\boldsymbol{C}\boldsymbol{x}^j > \boldsymbol{x}^{*T}\boldsymbol{C}\boldsymbol{x}^*$. This is a condition ensuring that some progress towards the final solution is made.

The combinatorial tree has each node identified with a subproblem j. The problems that replace j in the bounding set W^p are pointed to by the branches directed outward from that

node. At any intermediate point in the calculations, the set W^p of the current bounding problems is identified with the set of nodes that are the leaves of the tree.

Each note of the tree is associated with an incumbent bound v. Any leaf node of the tree whose bound is strictly less than v is active. Otherwise it is designated as terminated and need not be considered in any further computation. The B&B tree will be developped until every leaf can be terminated.

4.4 Rules for splitting the intervals

In order for the method of computing underestimating convex programmes to be implementable, it must be possible to compute the convex envelopes of functions of a single variable. The general approach is to decide on the number of different intervals on which the representation of the convex envelopes takes a different form. When the function has convex portions, sometimes the convex envelope is the original function itself for an interval. It is a line for an adjacent one and may revert to the function itself for the next. The next step is to determine the exact point at which the convex envelope changes representation from a line to follow the curve of the function (and vice-versa).

Presumably bounds on the interval in which this occurs are given by the first step. This is simply the problem of finding a point where the slope of the curve equals the slope of the line connecting the point with a given one. If $T(z)$ is a function of a single variable and a is a given point at the start of the interval, then the problem is to find a point $t \in [a,b]$ such that

$$\frac{\mathrm{T}(t)-\mathrm{T}(a)}{t-a} = \frac{\mathrm{d}T(t)}{\mathrm{d}t} = T'(t) \tag{32}$$

which is equivalent to solving the equation

$$T(t)-T(a)-(t-a)T'(t)=0 \tag{33}$$

When the underestimate does not coincide with the true function value, it is also required to define (in a heuristic way) a refining rule for splitting the bounds on the selected variable: Choose the index i of the variable that maximizes the difference between the factorable form and its convex underestimate out of the more violated constraint by x^s in the original problem. The corresponding interval is divided into two new intervals $[l_i,x_i]$ and $[x_i,L_i]$. Therefore as soon as a node is selected to be branched, the partition of its interval is only dependent on its solution value and is not related to other partitions at the same level of the tree. This corresponds to a weaker form of the convergence theorem not requiring the completion of the intervals partitioned.

These rules for splitting reduce to those proposed by Soland when the problem functions are separable. The intuitive motivation for them is that in some sense the convex envelopes (and concave envelopes) computed for the new feasible regions will be brought up closer to the value of the function at x^p using this splitting rule than using any other.

4.5 Application

The first example given in Section 3 was choosen in order to illustrate how this algorithm works. In this problem the correlation matrix C' is equal to the identity matrix, because the

variables X and Y are uncorrelated. Introducing the set of uncorrelated variates X' and Y' such that,

$$X' = (X - \bar{X})/\sigma_X = (X - 15.0)/1.5 \tag{34}$$

$$Y' = (Y - \bar{Y})/\sigma_Y = (Y - 10.0)/1.5 \tag{35}$$

the limit-state function becomes:

$$-13.0X - 16.5Y + XY + 221.25 = 0 \Leftrightarrow -2X' - Y' + X'Y' + 5.0 = 0 \tag{36}$$

It is necessary to define the intervals where X and Y can vary,

$$9.0 \leqslant X \leqslant 21.0; \quad 4.0 \leqslant Y \leqslant 16.0 \tag{37}$$

i.e.

$$-4.0 \leqslant X' \leqslant 4.0; -4.0 \leqslant Y' \leqslant 4.0 \tag{38}$$

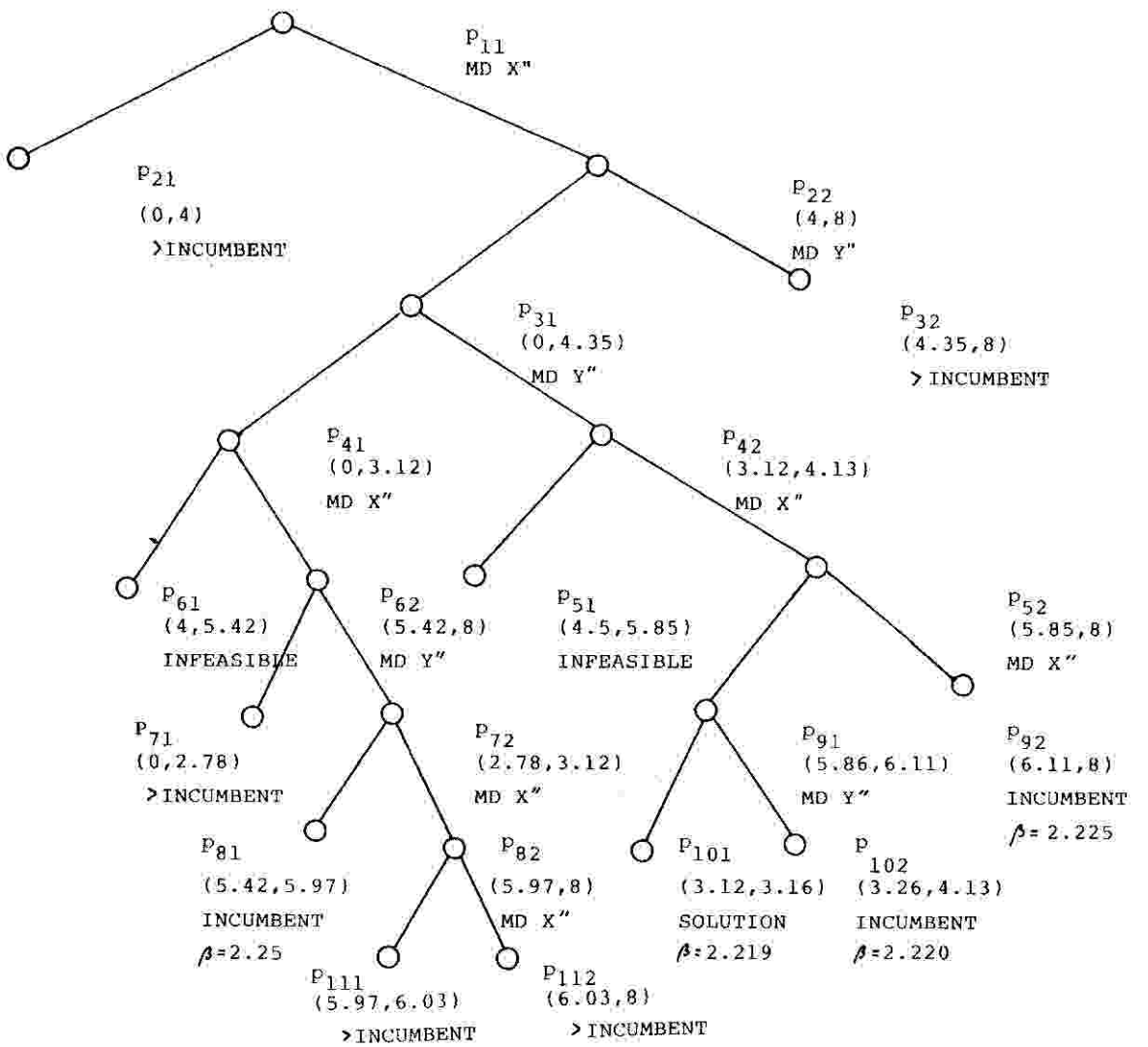

Fig. 5. Branch and bound tree.

The reliability can be measured if the solution X'^* and Y'^* of the mathematical programming problem (MP):

$$\begin{aligned} \min \quad & 1/2(X'^2 + Y'^2) \\ \text{s.t.} \quad & -2.0X' - Y' + X'Y' + 5.0 = 0 \\ & -4.0 \leqslant X' \leqslant 4.0; -4.0 \leqslant Y' \leqslant 4.0 \end{aligned} \tag{39}$$

is known. The domain of this problem consists of factorable functions and is non-convex. Before proceeding with the replacement of the limit-state function by its convex underestimates, it is neceassary to transform the programme (39) into a standard form (non-negative variables). Let,

$$X'' = X' + 4.0; \quad Y'' = Y' + 4.0 \tag{40}$$

The following MP is obtained:

$$\begin{aligned} \min \quad & 1/2(X''^2 + Y''^2) - 4.0X'' - 4.0Y'' \\ \text{s.t.} \quad & -6.0X'' - 5.0Y'' + X''Y'' + 33.0 = 0 \\ & 0.0 \leqslant X'' \leqslant 8.0; \quad 0.0 \leqslant Y'' \leqslant 8.0 \end{aligned} \tag{41}$$

Replacing the factorable functions by their convex underestimates, a QP in 2 variables and 4 inequality constraints that is the root of the combinatorial tree (corresponding to the bounds specified "ab-initio") is obtained:

$$\begin{aligned} \min \quad & 1/2(X''^2 + Y''^2) - 4.0X'' - 4.0Y'' \\ \text{s.t.} \quad & (8.0 - 6.0)X'' + (8.0 - 5.0)Y'' - 64.0 + 33.0 \leqslant 0 \\ & (0.0 - 6.0)X'' + (0.0 - 5.0)Y'' - 0.0 + 33.0 \leqslant 0 \\ & (-8.0 + 6.0)X'' + (0.0 + 5.0)Y'' + 0.0 - 33.0 \leqslant 0 \\ & (0.0 + 6.0)X'' + (-8.0 + 5.0)Y'' + 0.0 - 33.0 \leqslant 0 \\ & 0.0 \leqslant X'' \leqslant 8.0; \; 0.0 \leqslant Y'' \leqslant 8.0 \end{aligned} \tag{42}$$

Any node of the B&B tree is defined for each set of bounds on the variables X and Y. The results of the branching strategy known as breadth first (choose the node with lower bound) are represented in the combinatorial tree of Fig. 5.

4.6 Inside-out approach

An alternative B&B algorithm originally presented by Reeves [10] and intended for all-quadratic programming, can also be adapted to programmes in which the constraints can be reduced to factorable forms. It relies on performing each iteration over a subinterval of the original domain given by bounds on the variables. It consists of three basic steps:

The first step of each iteration is to determine a base point from which to branch and bound. Highly desirable points are local optima, although it is possible to start the algorithm with a point that may be not a local optimum (or even feasible). If this algorithm is used as a verification procedure then the local solution obtained by convex programming techniques is an ideal starting point.

Once a base point is obtained the second step consists of eliminating an interval surrounding it. For a feasible point x^V, where V represents the iteration number of the algorithm, an interval is eliminated for which x^V is the global solution of the optimization problem. It is composed of three basic substeps. First the interval under consideration is divided into subintervals around its

base point. Next a region of each subinterval is defined over which the base point is global to the original problem. Finally the total elimination interval is formed from the union of the regions eliminated over the individual subintervals. For unfeasible x^V, an interval for which the QP derived using convex underestimates (QCE) is not feasible is found and eliminated.

In the last step the B&B section is entered. Uneliminated regions are partitioned in subintervals and a QCE problem is solved for each of them. These lower bounds are compared to the value of the best upper bound (incumbent solution). All previously uneliminated intervals with bounds which equal or exceed the incumbent are eliminated from further consideration. If it remains any subset of the original domain, a new iteration is initiated over the uneliminated subinterval with the lower bound around a new base point.

The termination criteria for verifying the global minimum for a fixed set of bounds requires that the solution point y^V given by the QCE on the same interval should also be feasible to the original problem. Considering the convex underestimates taken, this implies that all design variables are endpoints.

4.7 Discussion of the two B&B strategies

In many instances the first found minimum will be global. This approach tends to accelerate the remainder of the algorithm in which it is only verified whether the local is global or determine a better local and repeat the process. Obtaining a local minimum and eliminating the region around it increases the likelihood that the QCE will be either unfeasible or have a solution that will exceed the incumbent. Theoretically in verifying an optimum, the number of elimination subintervals determined in Step 2 grows exponentially with the size of the problem. This appears to be a major drawback to application of this approach. A factor which tends to lessen the importance of the growth rate in the number of elimination subintervals is that as the problem size increases, it is increasingly unlikely that all subintervals need to be investigated. Another criticism is related to the trial and error procedure for determining the endpoints for the elimination interval. An exponentially increasing number of QP may need to be solved a number of times.

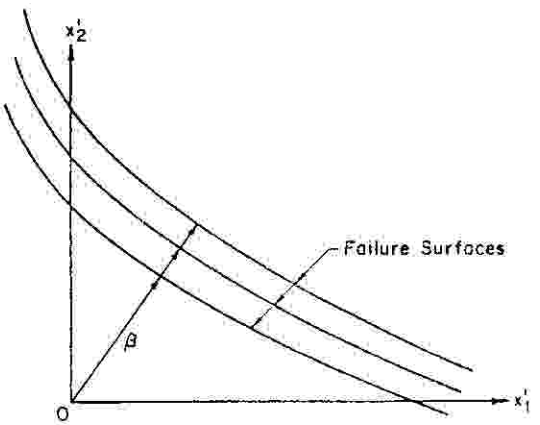

Fig. 6. Different failure surfaces are characterized by different distances from the origin, β.

5. PROBABILITY-BASED DESIGN CRITERIA

The probability-based design criteria are developed on the basis of the second moment approach, i.e., using the information provided by the first and second moments of the design variables. In the space of the reduced variates, designs at different levels of safety may be viewed as corresponding to satisfying different failure surfaces represented by varying distances β to the origin as shown in Fig. 6. The design criterion consists of the determination of the design factors that will result in designs having failure surfaces that comply with a required safety index. The distance from the failure surface to the origin of the reduced variates satisfies some target value of p_s.

Applying a design factor on the mean value of each of the basic design variables yields

$$g(\bar{\gamma}_1\mu_{x_1}, \bar{\gamma}_2\mu_{x_2}, \ldots, \bar{\gamma}_n\mu_{x_n}) = 0 \tag{43}$$

$\bar{\gamma}_i\mu_{x_i}$ should be on the failure surface and at the most probable failure point. Then, the required partial design factors are

$$\gamma_i = x_i^*/\mu_{x_i} \tag{44}$$

Therefore, the determination of the required design factor is also a problem of determining the most probable failure point x_i^*. In the space of the reduced variates, the most probable failure point is,

$$x_i'^* = -\alpha_i^*\beta \tag{45}$$

where,

$$\alpha_i^* = \frac{(\partial g/\partial x_i)^*}{\sqrt{\sum_i (\partial g/\partial x_i)^{*2}}} \tag{46}$$

From this, the original variates are obtained,

$$x_i = \mu_{x_i} - \alpha_i^*\beta\ \sigma_{x_i} = \mu_{x_i}(1 - \alpha_i^*\beta\ \Omega_{x_i}) \tag{47}$$

Therefore the required design factors are,

$$\gamma_i = 1 - \alpha_i^*\beta\ \Omega_{x_i} \tag{48}$$

The direction cosines α_i^* must be evaluated at the most probable failure point x_i^*. The design problem then consists of finding the uncorrelated variates X_j' and the mean value of one of the variables (e.g.: $\bar{X}_1$) such as to achieve the reliability β. This is equivalent to the mathematical programming problem:

$$\begin{aligned} &\min \quad 1/2\ \boldsymbol{x}'^{\mathrm{T}}\boldsymbol{C}'^{-1}\boldsymbol{x}' \\ &\text{s.t.} \quad g(\boldsymbol{x}, \bar{x}_1) = 0 \\ &\qquad \boldsymbol{x}'^{\mathrm{T}}\boldsymbol{C}'^{-1}\boldsymbol{x}' - \beta^2 \geqslant 0 \Rightarrow \beta^2 - \boldsymbol{x}'^{\mathrm{T}}\boldsymbol{C}'^{-1}\boldsymbol{x}' \leqslant 0 \end{aligned} \tag{49}$$

Inherent to the inclusion of the variable $\bar{X}_1$ is the increase in the non-linearity of the limit-state function, that needs to be transformed into a factorable form. The second set of constraints requires the convex underestimates of quadric and product terms in $\boldsymbol{X}'$ that are easily derived.

CONCLUSIONS

It is well known that the usual methods may fail in searching the global minimum distance (point P^*). That occurs especially for irregular geometries of the limit-state (LS) surface, i.e. when it shows convex and concave parts. The algorithm described in this paper gives a systematic strategy to find the global optimum P^*, or at least to approach it as closely as possible; the algorithm can actually be applied for any geometry of the LS surface. The B&B methodology described in this paper is therefore recommended when assessing the safety or designing nonconvex limit-state functions for a fixed reliability. The original non-convex domain is transformed by the use of underestimating functions; a sequence of QP is then solved, terminating with the absolute minimum distance to the failure surface. Its extensions may constitute checking methods allowing the assessment of upper and lower bounds of the probability of failure associated with the nodes of the combinatorial tree.

REFERENCES

1 A.M. Freudenthal, Safety and the probability of structural failure, ASCE Trans., 121 (1956) 1337–1397.
2 O. Ditlevsen, Generalized second moment reliability index, J. Struct. Mech., 7 (1979) 453–472.
3 M. Shinozuka, Basic analysis of structural safety, J. Struct. Div. ASCE, 109 (1983) 721–740.
4 A.H.S. Ang and W.H. Tang, Probability Concepts in Engineering Planning and Design, Wiley, New York, NY, 1984.
5 E. Paloheimo and H. Hannus, Structural design based on weighted fractiles, J. Struct. Div. ASCE, 100 (1974) 1367–1378.
6 M. Rosenblatt, Remarks on a multivariate transformation, Ann. Math. Stat, 23 (1952) 470–472.
7 R. Rackwitz, Practical Probabilistic Approach to Design, Bulletin 112, Comité Européen du Beton, Paris, France, 1976.
8 A.H.S. Ang, Private communication, Short Course on Structural Reliability, Lausanne, 1985.
9 R.M. Soland, Algorithm for separable non-convex programming problem: Nonconvex constraints, Manage. Sci., 17 (1971) 759–773.
10 G.R. Reeves, Global optimization in nonconvex all quadratic programming, Manage. Sci., 22 (1975) 76–86.
11 L.M.C. Simões, Search for the global optimum of least volume trusses, Eng. Opt., 11 (1987) 49–67.