Stochastically Dominant Modes of Frames by Mathematical Programming

By Luis Miguel da Cruz Simões[1]

Abstract: Most important in calculating the failure probability of structural systems is the search for the stochastically most relevant failure mechanism of ductile structural systems—particularly framed structures. For this reason an automatic method for finding all modes can be employed, but for very large systems these procedures may become intractable. Mesh and nodal methods for frames are formulated with particular attention to simple data preparation. The generalized mesh description leads to the simpler and more efficient formulation of the mechanism compatibility equations. Two nonconvex problems, one with bilinear constraints and the other with a fractional objective function, have been derived. The latter formulation can be solved quite effectively when cast in the form of the maximization of a quadratic convex function over a linear domain. This is followed by mathematical programming techniques that are more appropriate for nonconvex optimization problems. The paper concludes with a discussion of the methodologies for finding other important modes.

Introduction

Assessment of the reliability of a structure has to take into account that, during its design life, the structure is generally subjected to a number of varying loads and their combinations and that its resistance may deteriorate with time. In this context, there arises the problem of evaluating the conditional failure probability of the structural system given a certain load event. Variation of the parameters of the probability distributions, the respective types of distributions, and the correlation among the variables involved have a significant effect on the results. Most important in calculating the failure probability of structural systems—as the number of failure modes for larger systems is extraordinarily high—is the search for the stochastically most relevant failure mechanism. Three basic procedures have been popular for generating failure modes: exhaustive enumeration, repeated structural analysis, and optimization. The dominant modes are easily selected if the failure probabilities of all possible modes of ductile structural systems—in particular, framed structures, which are of primary interest for engineering application—can be evaluated. The modes of a small system can be found by simple investigation, but this may be very tedious, or even impossible, for large systems. For this reason an automatic method to find all modes can be employed, but for very large systems these procedures may become intractable. Ditlevsen and Bjerager (1984) have identified modes from the consideration of lower-bound reliability analysis. Except for special cases, the programming problem becomes nonlinear in the coefficients of the linear combinations. Ang and Ma (1982) developed a method for finding the stochastically relevant mode directly by solving a nonlinear optimization prob-

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lem. The optimization procedure is performed to find the minimal safety index β that can be interpreted as the shortest distance from the limit state function. The constraints are formulated in terms of positive external work and kinematic admissibility. Because of computational difficulties due mainly to the nonconvexity of the problem, the method is limited to small systems and is solved by a trial-and-error search procedure. Nafday et al. (1987) identified this problem as NP-hard and proposed a method based on vertex enumeration and ranking of a plastic limit analysis problem. However, since an approach based on such algorithms has a tendency to grow exponentially with the number of vertices, it does not appear very practical for large structures. Simulation together with a sensitivity analysis procedure for plastic limit analysis has been proposed by Rashedi and Moses (1983). The nodal description, extensively used in the literature (Watwood 1979), is much less efficient than its mesh counterpart, formulated here with particular attention t e simple preparation of data. This is followed by mathematical programming strategies that are more appropriate for the bilinearly constrained optimization problem, consisting of finding the stochastically most relevant failure mechanism. This problem is also identified as a fractional program. The latter formulation can be solved quite effectively when cast in the form of the maximization of a quadratic convex function over a linear domain. Some examples show the relative efficiency of the various algorithmic procedures. The paper concludes with a discussion of the methodologies for finding other important modes and presents examples using one of these techniques.

Mesh and Nodal Descriptions

Nodal Description of Kinematics

The nodal description for the present case may be considered to have its origins in the concept of mechanisms. The elementary mechanisms for rectangular frames are the collection of all the beam, sway, and joint mechanisms. They are associated with parameters that play the role of nodal displacements, **q**. Any set of compatible deformation rates, $\boldsymbol{\Theta}$ can be written in the following form:

$$\boldsymbol{\Theta} = \mathbf{C}\mathbf{q} \quad (1)$$

...ere $\mathbf{C}$ = a c_r by b matrix; c_r, $b = c_r - \alpha$, α = the number of critical ...tions, the number of independent mechanisms in the basis, and the static indeterminacy number, respectively. The displacement rates u can be expressed in terms of the nodal displacements q

$$\mathbf{u} = \mathbf{C}_o\mathbf{q} \quad (2)$$

The nodal displacements, which are independent kinematic variables, can be scaled by the following normalization condition:

$$\mathbf{c}_o\mathbf{q} = 1 \quad (3)$$

where the matrix $\mathbf{C}_o$ is reduced to the vector-column $\mathbf{c}_o$.

Mesh Description of Statics and Kinematics

A frame can be reduced to a determinate form in a variety of ways. The bending moments at the critical sections can be expressed in terms of the

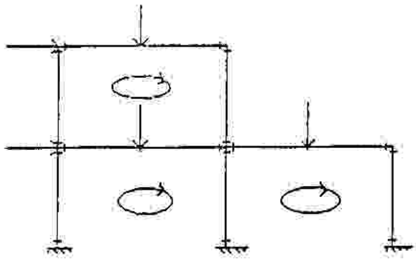

FIG. 1. Two-Story Two Bay Frame

coordinates of the influence diagrams associated with unit magnitudes of the loads $\mathbf{F}$ (matrix $\mathbf{B}_o$) and the indeterminate forces $\mathbf{p}$ (matrix $\mathbf{B}$).

$$\mathbf{m} = (\mathbf{B}\ \ \mathbf{B}_o)\begin{vmatrix}\mathbf{p}\\ \mathbf{F}\end{vmatrix} \qquad (4)$$

If a mechanism is created by allowing flexural deformations, the kinematic relations for the mesh description to ensure compatibility are

$$\mathbf{0} = \mathbf{B}^t\mathbf{\Theta} \qquad (5)$$

The rates of the displacements $\mathbf{u}$, which correspond vectorially to the loads $\mathbf{F}$, can be evaluated in terms of the flexural deformation rates:

$$\mathbf{u} = \mathbf{B}_o^t\mathbf{\Theta} \qquad (6)$$

Since the angular rotation at the critical sections are the independent kinematic variables, scaling can be performed through the following normalization condition:

$$\mathbf{b}_o^t\mathbf{\Theta} = 1 \ \ \Rightarrow \ \ \mathbf{b}_o^t\mathbf{\Theta}^+ - \mathbf{b}_o^t\mathbf{\Theta}^- = 1 \qquad (7)$$

where the elements of $\mathbf{b}_o$ are linear combinations of the rows of $\mathbf{B}_o$.

The familiarity of structural engineers with respect to a determinate basis with physical release systems may be responsible for the limited use that has been made of mesh procedures in structural applications. With more complex frames, the derivation of the basic matrices ($\mathbf{B}$, $\mathbf{B}_o$) becomes very tedious. Therefore, the selection of the most suitable release system becomes more important and also more difficult. The most convenient basis for the frame of Fig. 1 is that of the regional meshes (Munro 1965).

If the unit diagrams (b_1, b_2, b_3) of Fig. 2 are applied to each of the regional meshes in turn, the $\mathbf{B}$ matrix then takes a particularly simple form. First, with the critical section numbering in this particular example, the sign convention for bending moments associated with the member orientation are shown in Fig. 3. The $\mathbf{B}$ matrix is then assembled in terms of four submatrices $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, and $\mathbf{B}_4$.

These submatrices are as follows:

$$\mathbf{B}_1 = \begin{vmatrix}-1 & -1 & 0\\ 0 & -1 & -1\end{vmatrix}; \qquad \mathbf{B}_2 = \begin{vmatrix}0 & -1 & -1\\ 0 & -1/2 & -1\\ 0 & 0 & -1\end{vmatrix}$$

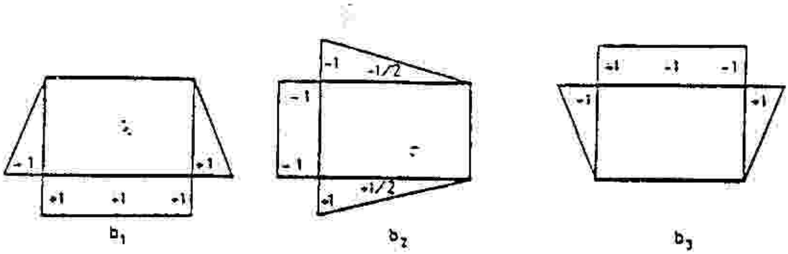

FIG. 2. Unit Bending-Moment Diagrams

$$\mathbf{B}_3 = \begin{vmatrix} +1 & +1 & 0 \\ +1 & +1/2 & 0 \\ +1 & 0 & 0 \end{vmatrix}; \qquad \mathbf{B}_4 = \begin{vmatrix} +1 & 0 & 0 \\ 0 & 0 & +1 \end{vmatrix}$$

and the mesh matrix is:

$$\mathbf{B} = \begin{vmatrix} B_1 & 0 & 0 \\ B_2 & 0 & 0 \\ B_3 & B_2 & 0 \\ B_4 & 0 & 0 \\ 0 & 0 & B_2 \\ 0 & B_1 & 0 \\ 0 & B_4 & B_1 \\ 0 & 0 & B_4 \end{vmatrix}$$

The elements of the **B** matrix are independent of the dimensions of the rectangular meshes, provided that the critical sections are maintained in the same relative positions. The elements of $\mathbf{B}_0$ can be obtained from any set of bending moments that are in equilibrium with the unit loads.

Constitutive Relations

The plastification in frames can be considered restricted to prelocated sections, and the yield criterion imposes bounds to the values of the moments

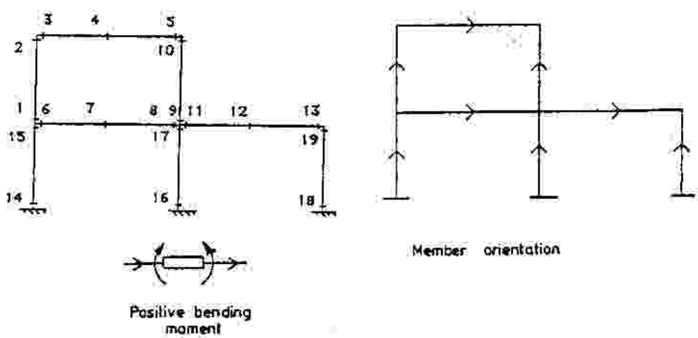

FIG. 3. Positive Bending Moment; Member Orientation

in all critical sections. For example, if the negative and positive plastic resisting moments in the critical section i are $m*_i^-$ and $m*_i^+$, respectively, then

$$-m*_i^- \leq m_i \leq m*_i^+ \quad (8)$$

Since the variables Θ are real and the standard algorithm of mathematical programming work with nonnegative variables, the rotation in the critical section i, Θ_i is decomposed in the pair of nonnegative variables Θ_i^+ and Θ_i^-.

$$\Theta_i = \Theta_i^+ - \Theta_i^- \quad (9)$$

Therefore, the section i can be

1. A positive plastic hinge participating in the failure mechanism:

$$\Theta_i^+ > 0; \quad \Theta_i = \Theta_i^+ > 0; \quad m_i = m*_i^+ \quad (10)$$

2. A negative plastic hinge participating in the failure mechanism:

$$\Theta_i^- > 0; \quad \Theta_i = -\Theta_i^- < 0; \quad m_i = -m*_i^- \quad (11)$$

3. An elastic point or a plastic hinge not participating in the mechanism:

$$\Theta_i^+ = 0 = \Theta_i^-; \quad \Theta_i = 0; \quad -m*_i^- \leq m_i \leq m*_i^+ \quad (12)$$

Mathematical Programming Formulations

Quadratic Programming Bilinearly Constrained

The simplest problem is concerned with a rectangular plane frame whose topology and geometry are known a priori, whose members are straight and prismatic, and whose material properties contain a terminal phase that is perfectly plastic. The member sections will be taken as symmetric with respect to the bending axes and will have the same yield stress for both senses. It will also be assumed that the loading can be considered a single predominant load state. In the case of structures composed of ductile members, such as components with elastic perfectly plastic behavior, the structural strength would be independent of the failure sequences of the component. Hinge moment capacities and loads are the random variables considered to represent the uncertainty in the strength of the structure and the loading acting on it. If they are Gaussian, they may be characterized through their means and standard deviations. The identification of the stochastically most important mechanism consists of finding the minimum distance β from the origin of the reduced normal variables $m'*$, λ'_F to a point in the failure surface (Shinozuka 1983):

$$\min \beta = \sqrt{(\mathbf{m}'*)^2 + (\boldsymbol{\lambda}'_F)^2} \quad (13)$$

such that the performance function $g(\mathbf{m}*, \boldsymbol{\lambda})$ is a bilinear equation expressing the equality between the external and internal work produced by the mechanism:

$$g(\mathbf{m}*, \boldsymbol{\lambda}) = \mathbf{m}*^t\Theta* - \boldsymbol{\lambda}_F^t\mathbf{u} = 0 \quad (14)$$

Using the mesh description of kinematics, the set of equations that guarantees mechanism compatibility and the existence of a nontrivial solution, respectively, are

$$\mathbf{B}^t\Theta = 0 \;\Rightarrow\; (\mathbf{B}^t - \mathbf{B}^t) \begin{vmatrix} \Theta^+ \\ \Theta^- \end{vmatrix} = 0 \qquad (15)$$

$$\mathbf{b}_o^t\Theta = 1 \;\Rightarrow\; (\mathbf{b}_o^t - \mathbf{b}_o^t) \begin{vmatrix} \Theta^+ \\ \Theta^- \end{vmatrix} = 1 \qquad (16)$$

and:

$$\mathbf{u} = \mathbf{B}_o^t\Theta = (\mathbf{B}_o^t - \mathbf{B}_o^t) \begin{vmatrix} \Theta^+ \\ \Theta^- \end{vmatrix} \qquad (17)$$

$$\Theta* = [\mathbf{J}\ \mathbf{J}] \begin{vmatrix} \Theta^+ \\ \Theta^- \end{vmatrix} \qquad (18)$$

$$\Theta^+ \geq 0, \quad \Theta^- \geq 0, \quad \Theta* \geq 0, \quad \mathbf{u} \geq 0 \qquad (19)$$

where $\mathbf{J}$ = the incidence matrix that relates the rotations of the critical sections with the rotations of the members represented by the same random variable.

Assuming that the random normal variables are statistically independent, the relationship linking the reduced normal variables $\mathbf{m}'*$, λ'_F to the normal variables $\mathbf{m}*$, λ_F are

$$\mathbf{m}* = \mu_M + \sigma_M\mathbf{m}'* \qquad (20)$$

$$\lambda_F = \mu_F + \sigma_F\lambda'_F \qquad (21)$$

where μ_M, σ_M and μ_F, σ_F = the mean and standard deviation of the random variable $\mathbf{m}*$ and λ_F, respectively. Substituting $\mathbf{m}*$ and λ in the limit-state equation:

$$(\sigma_M\mathbf{m}'*)'\Theta* - (\sigma_F\lambda'_F)'\mathbf{u} + \mu'_M\Theta* - \mu'_F\mathbf{u} = 0 \qquad (22)$$

which is a bilinear equation.

If the nodal description of kinematics could be employed

$$\Theta^+ - \Theta^- = \mathbf{C}\mathbf{q} \qquad (23)$$

$$\mathbf{q} = 1 \qquad (24)$$

$$\mathbf{u} = \mathbf{C}_o\mathbf{q} \qquad (25)$$

replace Eqs. 15–17. As α cannot exceed c_r, the latter formulation always has a larger number of constraints and variables and, since for the same type of programming problems the computational effort varies with the product of the number of variables and the cube of the number of constraints, the use of the mesh description of kinematics is always more effective. Another important question concerns the ease of data preparation. The regional meshes discussed earlier are the more convenient basis, largely eliminating this problem.

Finding the stochastically most important mechanism consists of minimizing β subject to Eqs. 15–19 and 22. This mathematical programming problem shares its solutions with the minimization of the quadratic function:

$$\min \beta^2 = (\mathbf{m}'*)^2 + (\boldsymbol{\lambda}_F')^2 \quad (26)$$

subject to the same constraints. This is a bilinear programming problem named problem A.

Solution Algorithm for Bilinearly Constrained Minimization—Problem A

The existence of multiple solutions in the optimization of problem A caused by the bilinear constraint Eq. 22, leads to the consideration of solution methods that overcome this nonconvex behavior. The general nonconvex domain is transformed in the branch and bound (B & B) strategy into a sequence of intersecting convex domains by the use of underestimating convex functions. It is well known that a local solution to a problem possessing a convex objective function and a convex domain is also its global solution. The two main ingredients are a combinatorial tree with appropriately defined nodes and some upper and lower bounds to the final solution associated with each node of the tree. It is then possible to eliminate a large number of possible solutions without evaluating them.

Since the convex underestimate of the linear part of Eq. 22 is trivial, the envelope of the product terms is needed. Let the function $f(x,y) = xy$, representative of a bilinear term, be defined in the rectangle of bounds:

$$a \le x \le b; \qquad c \le y \le d \quad (27)$$

The convex underestimate will be taken as the z-coordinate on the highest of the two planes:

$$z_1 = cx + ay - ac; \qquad z_2 = dx + by - bd \quad (28)$$

$$z = \max(z_1, z_2) \le f(x,y) = xy \quad (29)$$

The concave overestimates of product functions are also required and can be obtained in a similar way. However, the underestimating functions z that are used to build up the convex envelope of Eq. 22 are not differentiable everywhere. There are several ways of handling this by altering the problem in order to create an equivalent QP. The simplest way involves the addition of some extra linear inequality constraints and variables. Considering the problem

$$\min \beta^2 = \mathbf{x}^T\mathbf{x} \quad (30a)$$

subject to

$$\mathbf{A}\mathbf{x} \ge \mathbf{b} \quad (30b)$$

$$\mathbf{f}^T\mathbf{x} + \max(\mathbf{g}_1^T\mathbf{x}, \mathbf{g}_2^T\mathbf{x}) \le \mathbf{h} \quad (30c)$$

where Eq. 30*a* is equal to Eq. 26, Eq. 30*b* represents Eqs. 15–19, and Eq. 30*c* is the convex underestimate of Eq. 22. This mathematical program is equivalent to the QP:

$$\min \beta^2 = \mathbf{x}^T\mathbf{x} \quad (31a)$$

subject to

$$\mathbf{A}\mathbf{x} \ge \mathbf{b} \quad (31b)$$

$$\mathbf{f}^T\mathbf{x} + \mathbf{u} \leq \mathbf{h} \quad (31c)$$

$$\mathbf{u} \geq \mathbf{g}_1^T\mathbf{x} \qquad \mathbf{u} \geq \mathbf{g}_2^T\mathbf{x} \quad (31d)$$

Therefore, the underestimate of each factorable term requires the addition of one variable and a pair of constraints. Each node of the tree associated with the B & B strategy is a quadratic programming problem QP. In the first iteration P_p is a QP constrained by Eqs. 15–29 and the convex underestimate of Eq. 22 defined in the range (l_p, L_p). The solution x_P of P_p is a lower bound to the optimum solution. If x_P is not a feasible solution of the original problem, one may restrict the domain of the subproblem P_p in order to make the solution xP feasible. P_p is replaced by a set of problems that bound the original problem in the sense that there exists one optimal solution x^* for at least one of these problems.

Each node of the tree is associated with an incumbent bound v. Any leaf node of the tree whose bound is strictly less than v is active. Otherwise it is designated as terminated and need not be considered in any further computation. Since the solution of each QP is a lower bound to the optimal solution required, the domain is subdivided until a partial solution is found that also satisfies Eq. 22. A partial solution is said to be fathomed if one of the following situations occurs: (1) The solution does not belong to the domain of the original problem; (2) the subproblem has no feasible solution; or (3) it is not possible to find a solution whose lower bound is less than the incumbent. If a partial solution is fathomed, then all possible completions of this partial solution have been implicitly enumerated and therefore need not be explicitly enumerated. When the last node is fathomed the algorithm terminates with the optimal solution. Backtracking in the tree is performed so that no solution is repeated or omitted from consideration.

An alternative branch-and-bound algorithm originally presented by Reeves (1975) and intended for all-quadratic programming can also be adapted to programs in which the constraints can be reduced to factorable forms. It consists of obtaining a local minimum and eliminating a region surrounding it in which the minimum is global and is called the inside-out approach as opposed to the outside-in strategy previously described. Computational experience shows that the latter is more efficient. Full details of these algorithms can be found in Simões (1987a).

Fractional Programming

In limit-state Eq. 22, the plastic deformations of the mechanism are present as state variables. Since it is necessary to find the minimum distance from the origin of the reduced normal variates to the yield surface, the values of $\boldsymbol{\lambda}_F'$ and $\mathbf{m}'^*$ are given for the minimum norm solution of Eq. 22. The pseudo-inverse (Simões 1987b) that gives the minimum norm solution of the undetermined system of equations:

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad (32)$$

is the product:

$$\mathbf{A}_{\#} = \mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1} \quad (33)$$

By making $\mathbf{x}^t = [\mathbf{m}'^* \ \boldsymbol{\lambda}_F']$, the identification of the stochastically most relevant mechanism consists in finding the minimum of the fractional programming (Schaibel and Ibaraki (1983):

$$\min \beta = \frac{\mu_M^t \Theta * - \mu_F^t \mathbf{u}}{\sqrt{(\sigma_M)^{2t}(\Theta *)^2 + (\sigma_F)^{2t}(\mathbf{u})^2}} \quad (34a)$$

for positive ($\mu_M^t \Theta * - \mu_F^t \mathbf{u}$), subject to the linear constraints:

$$(\mathbf{B}^t - \mathbf{B}^t) \begin{vmatrix} \Theta^+ \\ \Theta^- \end{vmatrix} = 0 \quad (34b)$$

$$\mathbf{u} = \mathbf{B}_o^t \Theta = (\mathbf{B}_o^t - \mathbf{B}_o^t) \begin{vmatrix} \Theta^+ \\ \Theta^- \end{vmatrix} \quad (34c)$$

$$\Theta * = (\mathbf{J}\ \mathbf{J}) \begin{vmatrix} \Theta^+ \\ \Theta^- \end{vmatrix} \quad (34d)$$

and sign constraints:

$$\Theta^+ \geq 0, \quad \Theta^- \geq 0, \quad \Theta * \geq 0, \quad u \geq 0 \quad (34e)$$

This mathematical formulation is equivalent to the bilinear programming, Eq. 13, subject to Eqs. 15–19 and 22, consequently sharing its solutions with Eq. 26, subject to the same constraints. The global minimum of the fractional programming Eq. 34 gives the plastic deformations for the stochastically most important mechanism.

Solution Algorithm for Fractional Programming

A class of algorithms is based on the following parametric problem associated with Eq. 34:

$$\max F(\gamma) = (\sigma_M)^{2t}(\Theta *)^2 + (\sigma_F)^{2t}(\mathbf{u})^2 - \gamma(\mu_M^t \Theta * - \mu_F^t \mathbf{u})^2 \quad (35)$$

subject to Eqs. 16 and 34*b–e*, where γ = a parameter. This parametric convex quadratic maximization is more tractable than Eq. 34 because of its simpler structure. Thus solving Eq. 35 subject to Eqs. 16 and 34*b–e* consists of finding the root of the quadratic equation $F(\gamma) = 0$ with one variable γ. This procedure, called the Dinkelbach method (Dinkelbach 1984), is essentially similar to the convex quadratic maximization (problem C) described later, although it is less efficient because of the larger number of nonlinear terms involved.

All-Quadratic Programming—Problem C

Another way of solving the quadratic fractional programming problem consists of replacing it by another whose objective function is linear and nonnegative (problem B):

$$\min \beta = \mu_M^t \Theta * - \mu_F^t \mathbf{u} \quad (36)$$

and its domain defined by the linear constraints (Eqs. 34*b–e*) and the quadratic equation:

$$(\sigma_M)^{2t}(\Theta *)^2 + (\sigma_F)^{2t}(\mathbf{u})^2 = 1 \quad (37)$$

To solve this nonconvex problem the enumerative technique just described may be employed. The convex underestimate of Eq. 37 is required. This equality will be replaced by the pair of inequalities:

$$(\sigma_M)^{2T}(\Theta*)^2 + (\sigma_F)^{2T}(\mathbf{u})^2 \leq 1 \qquad (38a)$$

- and

$$-(\sigma_M)^{2T}(\Theta*)^2 - (\sigma_F)^{2T}(\mathbf{u})^2 \leq -1 \qquad (38b)$$

A quadratic term, for example, x^2, can be linearized using a first-order Taylor series expansion around a point $(x)_o$ belonging to the domain:

$$x^2 \cong -(x)_o^2 + 2(x)_o u \qquad (39)$$

Since the linear constraint derived using this approximation is not binding at the solution, Eq. 38*a* need not be considered.

For the concave function $-x^2$ defined in the interval (Eqs. 38*a* and *b*) the convex underestimate is

$$+ b)x + ab \leq -x^2 \qquad (40)$$

Therefore, the convex underestimate of Eq. 37 is a linear inequality, not involving the addition of extra variables and constraints. Each node of the combinatorial tree is now associated with a linear program: The linear objective function (Eq. 36) is considered instead of Eqs. 31*a* and 34*b–e* to define the linear constraints (Eq. 31*b*), and a single linear inequality replaces Eqs. 31*c* and *d*.

Problem B is clearly much more effective than problem A. It has a smaller number of variables because the reduced normal variables $m'*$, λ'_F are expressed in terms of the rotations $\Theta*$ and displacements u. Moreover, it no longer requires the factorable functions' underestimates that are associated with a large increase in the number of constraints and variables. Unfortunately, it leads to an increase in the number of iterations required to give a feasible solution because the convex underestimates of the quadratic functions are not very tight.

Convex Quadratic Maximization

An alternative formulation of the minimization of Eq. 34 consists in maximizing the quadratic convex function (problem C):

$$\max \frac{1}{\beta^2} = (\sigma_M)^{2t}(\Theta*)^2 + (\sigma_F)^{2t}(\mathbf{u})^2 \qquad (41)$$

subject to the linear constraints (Eqs. 34*b–e*) and

$$\mu'_M\Theta* - \mu'_F\mathbf{u} = 1 \qquad (42)$$

The scaling condition in formulations B and C becomes Eqs. 37 and 42, respectively, and Eq. 16 is no longer required.

Branch-and-Bound Strategy for Solving Quadratic Convex Maximization

The finding of the global optimum of mathematical programs with nonconvex objective function and/or constraints is normally considered an NP-hard problem. One of the few cases for which there are algorithms that work well is the maximization of a quadratic convex objective function subject to linear constraints. Since the nonconvexity is restricted to the objective function, it is possible to solve this problem through a technique that explores its special structure: The local solutions of Eq. 41 subject to Eqs. 42 and 34*b*–*e* are vertices of the domain. This is not valid in the alternative problem formulations A and B. Since vertex enumeration techniques are expensive, other methods were considered. The branch-and-bound technique can be employed to solve this optimization problem in the form proposed by Falk and Soland (1969). It only requires the concave overestimate of the quadratic terms in the objective function (Eq. 41), since the domain defined by the expressions Eqs. 34*b*–*e* and 42 is now linear.

Cutting Plane Based Method for Solving Quadratic Convex Maximization

Cutting plane based methods can also find the global solution. In Konno's method (1976), it is defined as an associated equivalent program with bilinear objective function (BLP). The algorithm operates by keeping one set of variables fixed and partially dualizing the BLP using linear duality theory. A local maximum is computed and a cutting plane generated by exploiting the symmetric nature of BLP. In the next iteration, this procedure either generates a point that is strictly better than the last local maximum or generates a cut that is deeper, until the convergence criteria to an ϵ-solution are met.

Tui's cut (1964), which is devised using local information only, becomes more shallow as the dimension increases and the results of numerical experiments reported in Zwart (1974) were quite disappointing. Konno combined the Ritter (1966) and Tui methods, thus producing a much stronger cut. It is a common contention that cutting plane based methods do not work well when the number of constraints and variables is very large. Computational results quoted in the literature seem limited to the case where the number of nonlinear variables is ≤ 25. Nevertheless, the special nature of this problem, in which the nonlinearities are restricted to the variables associated with the random variables, means that the possibility of obtaining very deep cuts and generating suboptimal solutions corresponding to other dominant modes at a small cost makes this algorithm attractive from a computational point of view.

Falk's Linear Max-Min Problem

This nonconvex problem is shown in Falk (1973) to be equivalent to the maximization of a convex function over a linear domain, and a method of solution based on the branch and bound philosophy is proposed. The algorithm will implicitly search all possible solutions that are vertices of the polyhedron and select the best of these as the global solution. The simplex method is used to obtain the upper bound required in the branch-and-bound algorithm and to select candidate solutions. Branching takes place on individual variables and is affected by holding variables out of the basis. Although better than explicit vertex enumeration, the procedure is not as efficient as those described here because it makes no use of the explicit representation of the objective and constraining functions.

NUMERICAL EXAMPLES

Example 1

For the portal frame represented in Fig. 4, the random variables are the loads H, V, and the member resistances M_b, M_c. Assuming they are not

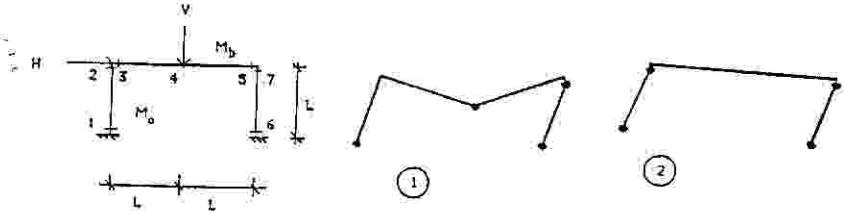

FIG. 4. One-Story One-Bay Frame—Example 1; Collapse Mechanisms

correlated and Gaussian, it is intended to find the stochastically most relevant mechanism

$$= 50 \text{ kN}; \quad \mu_V = 40 \text{ kN}; \quad \Omega_H = \Omega_V = 0.30 \quad (43a)$$

$$\mu_{Mc} = 115.5 \text{ kNm}; \quad \mu_{Mb} = 161.6 \text{ kNm}; \quad \Omega_{Mc} = \Omega_{Mb} = 0.05 \quad (43b)$$

Two failure modes of this simple frame are compared. The combined mechanism is more relevant from a deterministic point of view because of its minimum load factor $\lambda_1 = 1.74$, whereas its probability is $p_{F1} \cong 3 \times 10^{-4}$ ($\beta = 3.3487$). However, stochastically most important is the sway mechanism with $p_{F2} \cong 3 \times 10^{-3}$ ($\beta = 2.7014$) and a load factor $\lambda_2 = 1.85$.

The **J** matrix is:

$$\mathbf{J} = \begin{vmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{vmatrix} \quad (44)$$

and

$$\begin{bmatrix} \mathbf{\Theta}_b \\ \mathbf{\Theta}_c \end{bmatrix} = (\mathbf{J}\ \mathbf{J}) \begin{vmatrix} \mathbf{\Theta}^+ \\ \mathbf{\Theta}^- \end{vmatrix} \quad (45)$$

The generalized mesh matrix for this problem is

$$\mathbf{B}' = \begin{vmatrix} B_1 \\ B_2 \\ B_4 \end{vmatrix}^t = \begin{vmatrix} -1 & 0 & 0 & 0 & 0 & +1 & 0 \\ -1 & -1 & -1 & -1/2 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 & +1 \end{vmatrix} \quad (46)$$

and

$$(\mathbf{B}' - \mathbf{B}') \begin{vmatrix} \mathbf{\Theta}^+ \\ \mathbf{\Theta}^- \end{vmatrix} = 0 \quad (47)$$

The generalized static mesh matrix for the applied load is

$$\mathbf{B}'_o = \begin{vmatrix} -L & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L/2 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.5 & 0 & 0 & 0 \end{vmatrix} \quad (48a)$$

$$\mathbf{b}'_o = |-\mathrm{L}\ 0\ 0\ \mathrm{L}/2\ 0\ 0\ 0| = |-5\ 0\ 0\ 2.5\ 0\ 0\ 0| \quad (48b)$$

and

$$\begin{bmatrix} \mathbf{u}_H \\ \mathbf{u}_V \end{bmatrix} = (\mathbf{B}'_o - \mathbf{B}'_o) \begin{vmatrix} \mathbf{\Theta}^+ \\ \mathbf{\Theta}^- \end{vmatrix} \qquad (\mathbf{b}'_o - \mathbf{b}'_o) \begin{bmatrix} \mathbf{\Theta}^+ \\ \mathbf{\Theta}^- \end{bmatrix} = 1$$

[Continued]

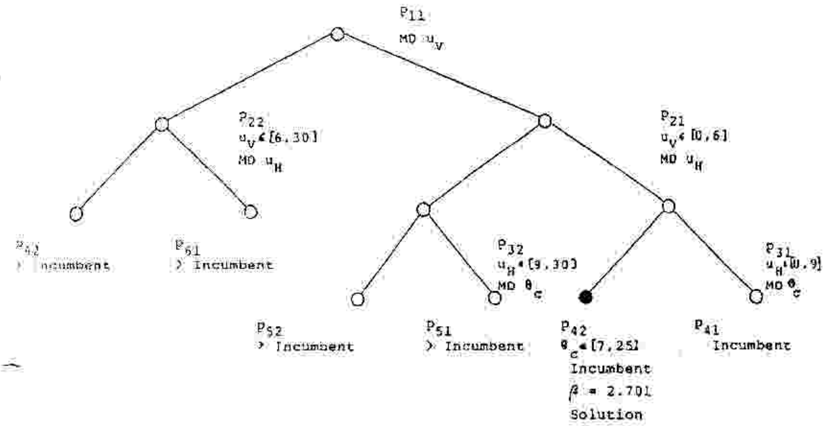

FIG. 5. Branch-and-Bound Tree—Example 1

$$\Theta^+ \geq 0, \quad \Theta^- \geq 0, \quad \Theta_b \geq 0, \quad \Theta_c \geq 0, \quad \mathbf{u}_H \geq 0, \quad \mathbf{u}_V \geq 0 \quad \text{.......... (49)}$$

The number of variables in problem A (nodal) are the $2c_r = 14$ plastic rotations Θ^+ and Θ^-, $b = c_r - \alpha = 4$ independent displacements q, the four random variables Θ_b, Θ_c, u_H, u_V, and eight variables giving convex underestimates of the bilinear terms. The constraints are the $c_r = 7$ compatibility equations, the scaling condition, four incidence relations corresponding to the random variables, eight inequalities giving the underestimates, and the linear inequality that replaces the bilinear equation. The number of variables in problem C (mesh) are 14 plastic rotations Θ^+ and Θ^+ and the four random variables Θ_b, Θ_c, u_H, u_V. The constraints are the $c_r = 3$ compatibility equations, four incidence relations corresponding to the random variables, and the linear equality (Eq. 42).

The branch-and-bound outside-in strategy was adopted to solve all these problems. In problem A (mesh) and problem A (nodal), each node of the combinatorial tree was associated with a quadratic program, whereas problem B and problem C required a linear program routine at each node. Branching was done by splitting the domain in which the random variables may vary. The selection of the domain to be split was based on the constraint violation or objective function difference with respect to the exact value (in the problem C setup). The collapse mechanism with lower reliability index is obtained, regardless of the formulation employed, and the random variables

TABLE 1. Results of Reliability Analysis—Example 1

Problem (1)	Number of variables (2)	Number of constraints (3)	Iterations required (4)	CPU time (sec) (5)
A (nodal)	30	21	35	58.4
A (mesh)	26	17	35	29.6
B (mesh)	18	8	90	10.7
C (mesh)	18	8	12	1.7

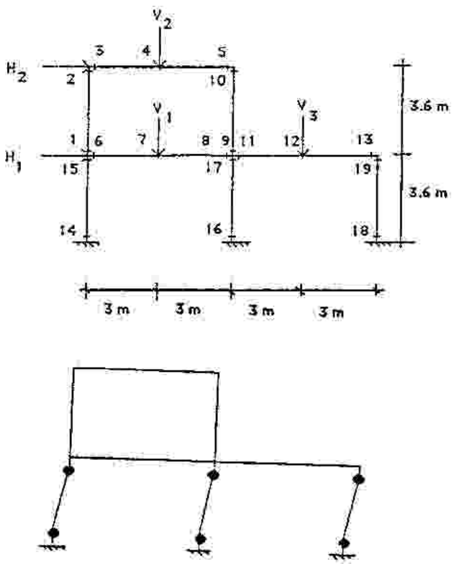

FIG. 6. Two-Story Two-Bay Frame—Example 2; Collapse Most Important Mechanism

Θ_b, Θ_c, u_H, and u_V are directly available. The reduced random variables $\mathbf{M}'_b$, $\mathbf{M}'_c$, $\mathbf{H}'$, and $\mathbf{V}'$ will be obtained from

$$\mathbf{M}'_b = \sigma_{Mb}\Theta_b\beta^2; \quad \mathbf{M}'_c = \sigma_{Mc}\Theta_c\beta^2 \quad (50a)$$

$$\mathbf{H}' = -\sigma_H \mathbf{u}_H\beta^2; \quad \mathbf{V}' = -\sigma_V \mathbf{u}_V\beta^2 \quad (50b)$$

The combinatorial tree corresponding to the problem C (mesh) is represented in Fig. 5.

The results of the reliability analysis are summarized in Table 1. Clearly, problem C is much more efficient with respect to computer effort. In the

TABLE 2. Data for Frame in Fig. 6—Example 2

Variable (1)	Mean (2)	Coefficient of variation (3)	Coefficient of correlation (4)
M_i (i = 1,2,11,12,18,19)	95 kN m	0.15	1.0
M_j (j = 3,4,13,14)	95	0.15	1.0
M_k (k = 5,6,7)	204	0.15	1.0
M_l (l = 8,9,10)	122	0.15	1.0
M_m (i = 15,16,17)	163 kN m	0.15	1.0
V_1	169 kN	0.15	—
V_2	89	0.25	—
V_3	116	0.25	—
H_1	62	0.25	—
H_2	31	0.25	—

TABLE 3. Results of Reliability Analysis—Example 2

Problem (1)	Number of variables (2)	Number of constraints (3)	Iterations required (4)	CPU time (sec) (5)
A (nodal)	75	38	52	420.0
C (mesh)	47	19	13	9.8

first place, it leads to smaller problems. This explains why problem B (mesh) is more efficient than Problem A (mesh), although it is associated with a larger number of nodes. The number of nodes to be solved is closely related to the tightness of the underestimates. Moreover, the number of iterations (nodes) is substantially reduced in problem C because the underestimates are close to the exact value.

Example 2

This example consists of an unsymmetrical two-story two-bay frame shown in Fig. 6 is analyzed with the data given in Table 2. M_i, M_j, M_k, M_l, and M_m are independent random variables. The loads are also independent random variables, except $\rho_{H_1,H_2} = 1.0$.

The same branch-and-bound strategy was employed to solve this problem, but only two formulations were considered: problem A (nodal) and problem C (mesh). The same conclusions as in example 1 can be drawn. The number of underestimating subproblems does not grow exponentially with the number of independent random variables, thus making this algorithm competitive. The results of the optimization algorithm summarized in Table 3 lead to the mechanism of Fig. 6 that has a reliability index $\beta = 1.966$.

Enumeration of Remaining Stochastically Important Mechanisms

An appraisal of the current procedures for generating the stochastically most representative failure modes indicates that they are variously dependent on simulation, trial-and-error, perturbation, human judgment, complex heuristic strategies, or approximations, either in the choice of appropriate starting points or for continuing the method at different stages. Some of the methods generate the modes in random order, thus many of the important modes may be missed without ever knowing them. Three alternative procedures that can be used in a mathematical programming setup to enumerate the remaining mechanisms are next described. Formulation C (convex maximization) was seen to be more efficient in finding the most important mechanism, and all these procedures are based on it.

Vertex Enumeration and Ranking

Murty's method (1969) can be used for ordering the extreme points of a linear domain. It is based on a theorem that states that if $x^1, \ldots x^r$ are r best points, x^{r+1} will be an adjacent point of one of the first extreme points. The new point is distinct from the first r and maximizes the objective function giving $1/\beta^2$ among all the remaining extreme points. All the adjacent extreme points are found from the canonical tableau corresponding to the linear

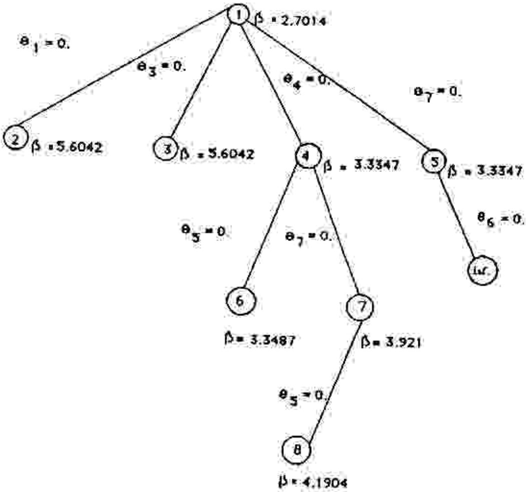

FIG. 7. Enumeration of Suboptimal Solutions—Example 1

domain $Ax = b$, that is, Eqs. 42 and 34*b–e*, by bringing, one by one, all the nonbasic variables—only those corresponding to rotations in the critical sections—into the basis. The procedure is repeated until either a prespecified number of extreme points is found, or all the extreme points in the ranked sequence whose objective value gives a reliability index, which is less than a prespecified value of β_{max}, are obtained. This method amounts to an implicit enumeration of the basis corresponding to different collapse mechanisms and is similar to the strategy used in Falk's linear max-min problem for holding variables out of the basis. On the basis of plastic limit analysis, failure modes are generated in Nafday et al. (1987) by Murty's method, although they end up with a much larger number of degenerate mechanisms caused by the larger number of state variables that correspond to the nodal description. Moreover, the efficiency of the method decreases as the number of kinematically admissible mechanisms increases.

Branch-and-Bound Tree

When formulation C is employed, the resulting convex maximization can be solved efficiently. This leads to the consideration of a B & B strategy that gives the stochastically most relevant mechanism, once some of the critical sections participating in the mechanism earlier detected are ruled out of the basis. This strategy is an adaptation of the algorithm for separable piecewise convex programming problems of Soland (1973), where a strong branching rule is employed: The number of nodes created at each stage from an intermediate node is equal to the number of critical sections participating in the mechanism associated with the intermediate node. The result obtained at any node is a lower bound on those obtained by branching from it, and if the reliability index associated with that node is larger than a prespecified value, then the leaf of the combinatorial tree can be terminated. Nothing prevents the same mode from being obtained by branching at different nodes.

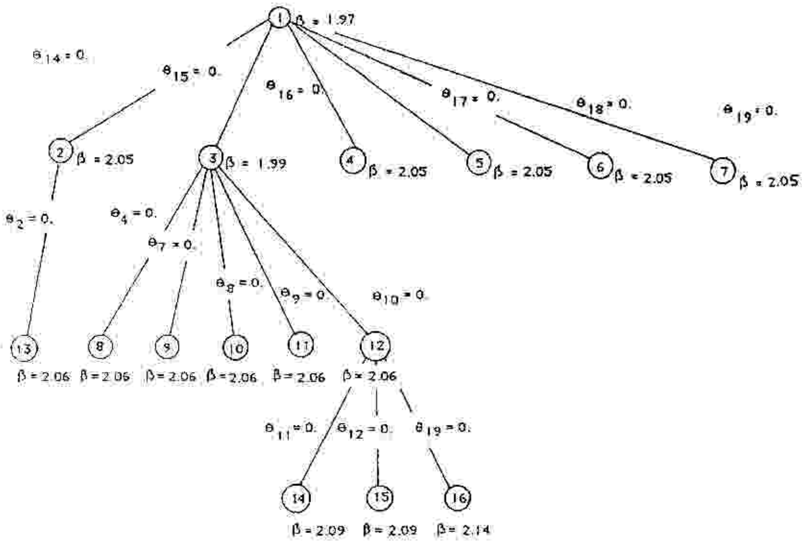

FIG. 8. Enumeration of Suboptimal Solutions—Example 2

TABLE 4. Stochastically Most Important Failure Modes—Example 1

Mechanism (1)	Reliability index β (2)	Critical sections participating in mechanism (3)
1	2.7014	1, 3, 4, 7
4	3.3347	1, 3, 5, 7
5	3.3347	1, 3, 4, 6
6	3.3487	1, 2, 3, 7
7	3.9210	1, 2, 5, 6
8	4.1904	1, 2, 3, 6
2 = 3	5.6042	2, 4, 7

TABLE 5. Stochastically Most Important Failure Modes—Example 1

Mechanism (1)	Reliability index β (2)	Critical sections participating in mechanism (3)
1	1.97	14, 15, 16, 17, 18, 19
3	1.99	4, 7, 8, 9, 10, 11, 14, 16, 17, 18, 19
2 = 4 = 5 = 6 = 7	2.05	2, 4, 10
11	2.06	4, 7, 8, 10, 12, 14, 16, 18, 19
8 = 9 = 10 = 12 = 13	2.06	11, 12, 19
14 = 15	2.09	1, 7, 8, 14, 16, 17, 18, 19
16	2.14	5, 7, 8

TABLE 6. Computational Effort Required to Enumerate Stochastically Most Important Failure Modes

Example (1)	Number of problems solved (2)	Number of different mechanisms found (3)	CPU time (sec) (4)
1	9	8	22
2	16	7	176

Nevertheless, the data available concerning mechanisms already obtained can be used to predict the solutions of a large number of nonconvex problems created by branching at intermediate nodes, which makes the procedure reasonably efficient.

BLP-Based Procedure

Konno's method for bilinear programming partitions the linear domain into basic and nonbasic vectors. The selection of the incoming variable into the basis is based on the information available from the dual linear problem. It is potentially more efficient than enumerating all possible bases without taking into account the explicit form of the objective function, which is used to define the domain of BLP's dual problem.

Numerical Examples

The branch-and-bound tree is used in this section to find other dominant modes of examples 1 and 2. Each node of the tree consists of a convex quadratic maximization (problem C). The initial node corresponds to the stochastically dominant mode. The branching operation from this node forces all the critical sections participating in the collapse mechanism out of the basis, one at a time. Each of the nodes created in this way is associated with a relevant mechanism obtained by convex quadratic programming. The branch-and-bound trees giving the stochastically most important modes for examples 1 and 2 are given in Tables 4 and 5 and drawn in Fig. 7 and 8, respectively.

In the latter problem some branches are not represented because they lead already-known solutions and can be pruned. The computational time (Table 6) is spent almost entirely solving the nonconvex problems associated with the nodes of the combinatorial tree.

Conclusions

Most of the methods used to find system-failure probabilities require the evaluation of the probabilities of failure associated with the stochastically most representative mechanisms. This paper has dealt with the application of various mathematical programming algorithms that can be used for finding stochastically dominant modes of ductile frames. It has started with a nonlinear model and solved it by algorithms more appropriate for nonconvex programming. The mesh and nodal descriptions have been briefly summarized and applied to the problem of finding the stochastically most relevant mechanism of ductile structural frames. Some attention has been given to simplifying the data preparation for this mathematical program. Two non-convex problems, one bilinearly constrained, the other a fractional program, have been derived. Examples have been given, showing that the latter formulation can be solved quite effectively when cast in the form of the maximization of a quadratic convex function over a linear domain. The results have shown that the number of iterations required in this method to find the more important mechanism grows polynomially with the number of random variables and the problem size, and not exponentially, as expected, in view of the NP-hard nature of this problem. The CPU time also depends on the particular choice of nonlinear variables' range. The methods for large-scale quadratic concave minimization described in Simões (1989) are more efficient, because they treat the nonlinear variables in a different manner than these, appearing linearly, and use a computational procedure for finding nonlinear variables' narrow bounds. The main advantage of these techniques is that they ensure that the global solution is obtained (the stochastically most important mechanism), as opposed to a merely local solution (any kinematically admissible mechanism). On the basis of the most efficient formulation, algorithms that enumerate other dominant modes have been described and examples of one of these strategies have been solved in order to illustrate its applicability.

APPENDIX I. REFERENCES

Ang, A. H. S., and Ma, H. F. (1982). "On the reliability of structural systems." *3rd Intl. Conf. Struct. Safety and Reliability*, Elsevier Scientific Publishing Co., Throndeim, Holland.

Ang, A. H. S., and Tang, W. H. (1984). *Probability concepts in engineering planning and design*. John Wiley and Sons, Inc., New York, N.Y.

Dinkelbach, W. (1967). "On nonlinear fractional programming." *Mgmt. Sci.*, 13(7), 492–498.

Ditlevsen, O. (1979). "Narrow reliability bounds for structural systems." *J. Struct. Mech.*, 7(4), 453–472.

Ditlevsen, O., and Bjerager, P. (1984). "Reliability of highly redundant plastic structures." *J. Engrg. Mech. Div.*, ASCE, 110(5), 671–693.

Falk, J. E., and Soland, R. M. (1969). "An algorithm for separable nonconvex programming problems." *Mgmt. Sci.*, 15(9), 551–568.

Falk, J. E. (1973). "A linear max-min problem." *Math. Program.*, 5(2), 169–188.

Konno, H. (1976). "Maximization of a convex quadratic function under linear constraints." *Math. Program.*, (1965). 11(2), 117–127.

Munro, J. "The stress analysis and topology of skeletal structures." *Res. Report CSTR 14*, Dept. of Civ. Engrg., Imperial College, London, England.

Murotsu, Y. (1983). "Reliability analysis of frame structure through automatic generation of failure modes." *Reliability theory and its application in structural and soil mechanics*, P. Thoft Christensen, ed. NATO ASI series, 525–540.

Murty, K. G. (1969). "An algorithm for ranking all assignments in increasing order of costs." *Oper. Res.*, 16(3), 682–687.

Nafday, A. M., Corotis, R. B., and Cohon, J. L. (1987). "Failure mode identification for structural frames." *J. Struct. Engrg.*, ASCE, 113(7), 1415–1432.

Rashedi, M., and Moses, F. (1983). "Studies on reliability of structural systems." *Report No. R83-3*, Case Western Univ., Cleveland, Ohio.

Reeves, G. R. (1975). "Global optimization in nonconvex all-quadratic programming." *Mgmt. Sci.*, 22(1), 76–86.

Ritter, K. (1966). "A method for solving maximum problems with a nonconcave quadratic objective function." *Zeitschraft für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 4(4), 340–351.

Schaible, S., and Ibaraki, T. (1983). "Fractional programming." *Eur. J. Oper. Res.*, 12(4), 325–338.
Shinozuka, M. (1983). "Basic analysis of structural safety." *J. Struct. Div.*, ASCE, 109(3), 721–740.
Simões, L. M. C. (1987a). "Search for the global optimum of least volume trusses." *Engrg. Optim.*, 11(1–2), 49–67.
Simões, L. M. C. (1987b). "Approximate design of structures using pseudo-inverses." *Comput. Struct.*, 27(2), 311–316.
Simões, L. M. C. (1988). "A branch and bound strategy for finding the reliability index with nonconvex performance functions." *Struct. Safety*, 5(2), 95–108.
Soland, R. M. (1973). "An algorithm for separable piecewise convex programming problems." *Nav. Res. Logist. Q.*, 2, 325–340.
Tui, H. (1964). "Concave programming under linear constraints." *Sov. Math.*, 5(6), 1437–1440.
Watwood, V. B. (1979). "Mechanism generation for limit analysis of frames." *J. Struct. Div.* ASCE 109(1), 1–15.
.wart, P. (1973). "Nonlinear programming: Counterexamples to two global optimization algorithms," *Oper. Res.*, 21(6), 1260–1266.

Appendix II. Notation

The following symbols are used in this paper:

$\mathbf{A}_{\#}$ = matrix giving pseudo-inverse of matrix **A**;
$\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \mathbf{B}_4$ = submatrices derived from unit diagrams for regional meshes used to built up mesh matrix **B**;
b = number of independent elementary mechanisms;
b_{ij} = element of static matrix **B** giving bending moment at section i caused by unit indeterminate force p_j;
$\mathbf{b}_{oi}$ = element of static vector $\mathbf{b}_o$ giving bending moment at critical section i caused by sum of applied loads F;
$\mathbf{b}_{oij}$ = element of static matrix $\mathbf{B}_o$ giving bending moment at section i caused by applied load F_j;
c_{ij} = element of nodal kinematic matrix **C** giving deformation at section i caused by nodal displacement q_j;
c_{oij} = element of nodal kinematic matrix $\mathbf{C}_o$ giving displacement u_i caused by q_j;
c_r = number of critical sections;
F_i = applied loading;
$g(X)$ = limit state function;
$\mathbf{j}_{ij}$ = element of incidence matrix **J** that is one if critical section j has random plastic moment capacity i or else, 0;
$\mathbf{m}*, \lambda_{\mathbf{F}}$ = vector of random plastic moment capacities and applied loads, respectively;
$\mathbf{m}'*, \lambda'_{\mathbf{F}}$ = vectors of reduced normal variables corresponding to $\mathbf{m}*$ and $\lambda_{\mathbf{F}}$, respectively;
$m*_i^+, m*_i^-$ = positive and negative plastic moments of resistance at section i;
m_i = bending moment at section i;
PF = probability of failure;
Pi = indeterminate forces, mesh forces;
q = nodal displacements;

u_i = displacement corresponding to applied load F_i;
v_j = discontinuity at release j of reduced structure;
α = indeterminacy number;
β = reliability index;
$\Theta*$ = vector of sums of plastic rotations associated with random variables $m*$;
Θ_B, Θ_C = sum of plastic rotations at critical section of beam and columns, respectively;
Θ_i = plastic rotation at critical section i;
Θ_i^+, Θ_i^- = positive and negative plastic rotations at section i;
λ = collapse load parameter;
$\mu_{\mathbf{M}}, \mu_{\mathbf{F}}$ = vector of mean values of $\mathbf{m}*$ and λ_F, respectively;
μ_{Mb}, μ_{Mc} = mean plastic moment capacity of beam and columns, respectively;
μ_H, μ_V = mean horizontal and vertical applied loads, respectively;
$\sigma_{\mathbf{M}}, \sigma_{\mathbf{F}}$ = vector of standard deviations of $\mathbf{m}*$ and λ_F, respectively; and
$\Omega_{Mb}, \Omega_{Mc}, \Omega_H, \Omega_V$ = coefficient of variation of random variables m_b, m_c, F_H and F_V, respectively.