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ENTROPY-BASED SYNTHESIS OF PRETENSIONED CABLE NET STRUCTURES

L. M. C. SIMÕES

Departamento de Engenharia Civil,
Universidade de Coimbra,
3049 Coimbra, Portugal

and

A. B. TEMPLEMAN

Department of Civil Engineering,
University of Liverpool,
P.O. Box 147, Liverpool, L69 3BX, England

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Cable nets structures exhibit highly non-linear behaviour under applied static loads. Non-linearities are caused by the changes in configuration necessary to equilibrate applied loads and by the slackening and/or yielding of the cables. In order to stiffen such cable net structures and reduce displacements under applied loads the cables are often pretensioned. Such pretensioning, however, requires more substantial cables, stronger connections and stronger supporting structures. It is therefore desirable to be able to design a cable net structure which satisfies all displacement and stress performance criteria with as small a level of pretensioning as possible.

This paper describes a method which sets the above design problem in a multicriteria optimization context with goals of minimum prestressing force, displacement and stress. A minimax solution is found by means of an entropy-based optimization algorithm. Illustrative examples are solved showing the applicability of the method.

KEY WORDS: Structural optimization, cable net structures, prestressing, entropy, multicriteria optimization, minimax optimization.

1 INTRODUCTION

This paper is concerned with aspects of the optimum design of prestressed cable net structures subjected to multiple independent quasi-static loads. The behaviour of cable net structures is characterized by a combination of geometrical and physical non-linearities. Geometric non-linearities arise from the fact that such structures respond to changes in applied loading by large changes in shape (but small strains). Physical non-linearities are caused by the fact that cables can only carry tension forces and become slack under compressive loads. Additional physical non-linearity may be present if yielding occurs in the tension cables. Consequently, the behaviour of cable net structures may be described as grossly non-linear. This makes the analysis of such structures particularly difficult and very few attempts to optimize the design of cable nets have been reported in the literature.

Two recent books[1,2] provide a comprehensive conspectus of cable net structures. In respect of their structural analysis the gross non-linearities have been handled by two main approaches. Buchholdt[1] places particular emphasis on methods which directly minimize the total potential energy of the structure. In an optimization context this is an attractive approach and consists of an almost unconstrained minimization of the total potential function expressed in terms of the nodal coordinates of the cable net (loads are assumed to be applied only at the nodes). The minimization is not completely unconstrained because checks on the cable strains during the optimization process must be incorporated to identify cable slackening or yielding and the energy function must be altered accordingly. The other approach, described in Krishna[2], consists of solving the governing non-linear equations iteratively by the Newton-Raphson method or variants. This is the method used in the present work.

In practice cable nets are usually prestressed in order to reduce the magnitudes of displacements under applied loading. This presents added difficulties in analysis and design. The designer has to solve two conceptually distinct problems. Firstly, he must address the so-called form-finding problem: starting with a design consisting of individual cable segment lengths which correspond to some desired net shape, called the reference configuration, when prestressing is applied to this net its shape will change from that desired in order to equilibrate the prestressing forces. Consequently, the designer must first calculate the configuration of the prestressed but otherwise unloaded net; this is called the zero configuration. Secondly, he must analyse what happens to this zero configuration when various design load cases are applied to the net, calculating cable stresses, shape changes and nodal displacements which can be compared with performance requirements for the structure.

As was noted above, the magnitudes of nodal displacements can usually be reduced by increasing the levels of prestressing forces. However, this implies the use of larger diameter cables, more robust connections and much stiffer supporting structures. Consequently, highly prestressed cable nets are much more expensive than lightly prestressed nets. As a design goal it is desirable to satisfy performance requirements for a cable net with as low a level of prestressing as possible. The aspect of optimum design which is studied in this paper is that of attempting to find the minimum level and optimum distribution of prestress which can be used in a cable net whilst satisfying prescribed limits upon cable stresses and nodal displacements from the reference configuration. Cinquini and Contro[3] have studied this prestress minimization problem and used linear programming to solve iteratively the optimality conditions corresponding to a non-linear and non-convex mathematical programming problem. In the present work it has been found that the linearization of the compatibility relationships in Ref. [3] can introduce significant errors into the analysis process for such grossly non-linear cable nets. Moreover, the present authors have found that the structural behaviour of the optimized cable nets can sometimes be very sensitive to small variations of some of the design variables. The results described in this paper for the example problems solved by Cinquini and Contro are quite different from those reported in Ref. [3].

In this paper the prestressed cable net design problem is posed in a vector (multicriteria) optimization format and a minimax solution is sought. An entropy-based technique[4] is used to determine the minimax solution via the minimization of a convex non-linear scalar function. The results of several test problems are presented and show that considerable savings can be made by using an optimized distribution of prestressing forces rather than a uniform distribution.

2 PROBLEM FORMULATION

2.1 Element Equilibrium Equations

Figure 1 shows a node q in a general space cable network connected to adjoining nodes p by cable elements i. The cable net is pretensioned but otherwise unloaded. Let F_i and s_i be the pretension force and length of a particular cable element i. The coordinates of the nodes p and q are x_p, y_p, z_p and x_q, y_q, z_q respectively. If a system of forces P_{xq}, p_{yq}, P_{zq}, is now applied at node q, the nodes p and q will displace through u_p, v_p, w_p and u_q, v_q, w_q respectively in the x, y and z directions. The force F_i and the length s_i will change by amounts ΔF_i and Δs_i, becoming F_i' and s_i' respectively. For a linearly elastic cable material

$$\Delta F_i = EA_i\,([s_i'/s_i] - 1) \tag{1}$$

where EA_i is the extensional stiffness of the ith cable element.

The equations of equilibrium at node q before the application of the nodal loads can be written as

$$\sum_{i=1}^{m} \frac{F_i}{s_i}(x_p - x_q) = 0 \tag{2a}$$

$$\sum_{i=1}^{m} \frac{F_i}{s_i}(y_p - y_q) = 0 \tag{2b}$$

$$\sum_{i=1}^{m} \frac{F_i}{s_i}(z_p - z_q) = 0 \tag{2c}$$

where m is the number of cables connected at node q. This set of equations (2) implicitly assumes that the cable net is in equilibrium with the pretension forces. On

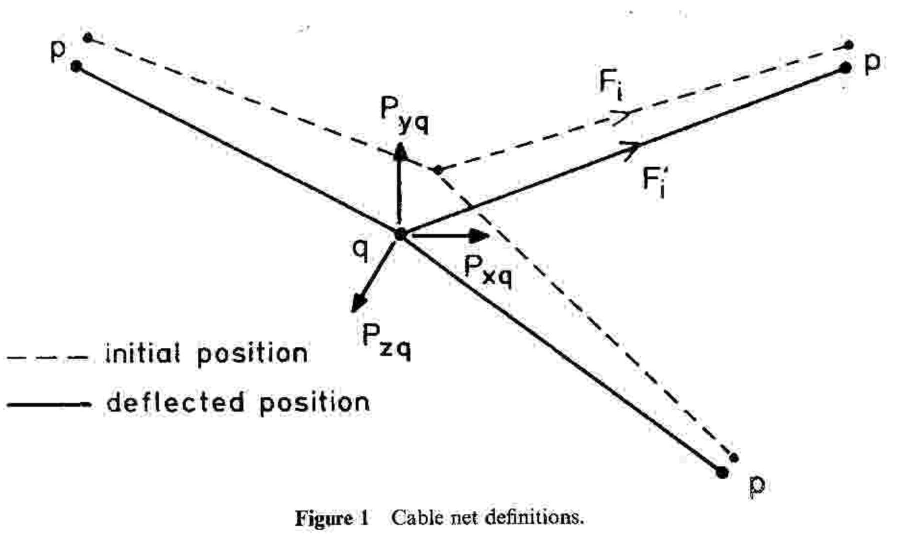

Figure 1 Cable net definitions.

the application of the nodal loads, Eqs. (2) will change to

$$\sum_{i=1}^{m} \frac{F_i'}{s_i'}(x_p + u_p - x_q - u_q) + P_{xq} = 0 \tag{3a}$$

$$\sum_{i=1}^{m} \frac{F_i'}{s_i'}(y_p + v_p - y_q - v_q) + P_{yq} = 0 \tag{3b}$$

$$\sum_{i=1}^{m} \frac{F_i'}{s_i'}(z_p + w_p - z_q - w_q) + P_{zq} = 0 \tag{3c}$$

By substituting the original cable length s_i, given by

$$s_i = [(x_p - x_q)^2 + (y_p - y_q)^2 + (z_p - z_q)^2]^{1/2} \tag{4}$$

into the final cable length s_i', given by

$$s_i' = [(x_p + u_p - x_q - u_q)^2 + (y_p + v_p - y_q - v_q)^2 + (z_p + w_p - z_q - w_q)^2]^{1/2} \tag{5}$$

it can be shown that

$$s_i' = s_i[1 + 2a_i + b_i]^{1/2} \tag{6}$$

where

$$a_i = \frac{1}{s_i^2}[(x_p - x_q)(u_p - u_q) + (y_p - y_q)(v_p - v_q) + (z_p - z_q)(w_p - w_q)] \tag{7}$$

$$b_i = \frac{1}{s_i^2}[(u_p - u_q)^2 + (v_p - v_q)^2 + (w_p - w_q)^2] \tag{8}$$

Expanding Eq. (6) by the binomial theorem and substituting into Eq. (1) gives

$$\Delta F_i = EA_i\left(a_i + \frac{b_i}{2} - \frac{a_i b_i}{2} - \frac{a_i^2}{2} + \frac{a_i^3}{2} + \ldots\right) \tag{9}$$

Similarly, expressing Eq. (6) as

$$\frac{1}{s_i'} = \frac{1}{s_i}[1 + 2a_i + b_i]^{-1/2} \tag{10}$$

and expanding by the binomial theorem gives

$$\frac{1}{s_i'} = \frac{1}{s_i}\left(1 - a_i - \frac{b_i}{2} + \frac{3a_i b_i}{2} + \frac{3a_i^2}{2} - \frac{5a_i^3}{2} + \ldots\right) \tag{11}$$

Noting that $F_i' = F_i + \Delta F_i$, Eqs. (9) and (11) may be substituted into Eqs. (3) to express the equilibrium of the nodally loaded and pretensioned net in terms of the unloaded but pretensioned forces and lengths. If these equations are then rearranged so that only the first order terms in u, v and w are retained on the left-hand side they become

$$\sum_{i=1}^{m}\left[\frac{F_i}{s_i}(u_p-u_q)+(EA_i-F_i)(x_p-x_q)a_i\right]=-P_{xq}+R_{xq} \tag{12a}$$

$$\sum_{i=1}^{m}\left[\frac{F_i}{s_i}(v_p-v_q)+(EA_i-F_i)(y_p-y_q)a_i\right]=-P_{yq}+R_{yq} \tag{12b}$$

$$\sum_{i=1}^{m}\left[\frac{F_i}{s_i}(w_p-w_q)+(EA_i-F_i)(z_p-z_q)a_i\right]=-\mathrm{P}_{zq}+\mathrm{R}_{zq} \tag{12c}$$

where

$$R_{xq}=-\sum_{i=1}^{m}\frac{(EA_i-F_i)}{s_i}[(u_p-u_q)c_i+(x_p-x_q)d_i] \tag{13a}$$

$$R_{yq}=-\sum_{i=1}^{m}\frac{(EA_i-F_i)}{s_i}[(v_p-v_q)c_i+(y_p-y_q)d_i] \tag{13b}$$

$$R_{zq}=-\sum_{i=1}^{m}\frac{(EA_i-F_i)}{s_i}[(z_p-z_q)c_i+(w_p-w_q)d_i] \tag{13c}$$

$$c_i=(2a_i+b_i-3a_i^2)/2 \tag{14}$$

$$d_i=(b_i-3a_ib_i-3a_i^2+5a_i^3)/2 \tag{15}$$

The expressions for R_{xq}, R_{yq} and R_{zq} are obtained by retaining terms up to the third order of u, v and w and neglecting terms of higher order. The R terms may be considered as residual forces corresponding to the difference between this non-linear analysis and the results which would have been obtained by a strictly linear analysis.

2.2 Structural Analysis

Consider a cable structure in service conditions and a set of l loading cases. If the number of nodal degrees of freedom is denoted by n, the nodal displacements and nodal loads for the jth load condition can be represented by the n vectors $u_j=[u_j\ v_j\ w_j]^t$ and $P_j=[P_{xj}\ P_{yj}\ P_{zj}]^t$. The problem of analysing the cable structure reduces to the solution of a system of non-linear equilibrium equations that, when assembled for the whole structure, can be represented in matrix form as:

$$Ku_j=-P_j+R_j \tag{16}$$

where K is the stiffness matrix consisting of the coefficients of the unknowns u, v, w and R_j is a column vector containing terms R_{xq}, R_{yq} and R_{zq}. For a space cable structure three equations will be required per joint and this will lead to a set of $3n$ simultaneous equations where n is the total number of free joints, the supporting structure being treated as rigid. The solution of these equations by a suitable iterative method will give values for u, v, w. The substitution of these values into Eq. (9) will give the changes in the forces.

2.3 *Newton-Raphson Method*

There can be no direct solution of Eq. (16) since the right-hand side contains the terms R which are functions of the unknowns. An iterative numerical procedure has therefore to be adopted. The computations in the Newton-Raphson method are based on the instantaneous stiffnesses of the structure derived anew in each iterative cycle. Although a fast convergence rate can be obtained by the use of this method, it entails a considerable amount of computational effort because the stiffness matrix has to be computed in every cycle of the iteration. On the other hand, any changes in the elastic or physical properties of the structure can be easily introduced to the stiffness matrix. The steps involved in this scheme are:

1) Initialization: Iteration $k = 0$. Assume $u_j^{(0)} = R_j^{(0)} = 0$
2) $k = 1$. Solve $K_j^{(0)} \cdot u_j^{(1)} = -P_j$ to evaluate $u_j^{(1)}$
3) Compute $R_j^{(k)}$ using $u^{(k)}$
   a) If $R_j^{(k)} < \varepsilon_1$, Stop.
   b) Otherwise, go to step 4.
4) Solve $K_j^{(k)} \cdot \Delta u_j^{(k)} = R^{(k)}$ to obtain $\Delta u_j^{(k)}$
   a) If $\Delta u_j^{(k)} < \varepsilon_2$, Stop.
   b) Otherwise, iteration $k = k + 1$. Evaluate:
   $$u_j^{(k)} = \Delta u_j^{(k-1)} + u_j^{(k-1)}$$
   and go to step 3.

2.4 *Determination of the Zero Configuration*

When the reference geometry of the structure is not in equilibrium with the applied forces, a zero configuration needs to be computed. The problem can be tackled by a suitable modification of the equations given in Section 2.1. Assuming that the geometry of the supporting structure as well as the pretensioning forces in the cables are specified, the nodal coordinates in the network have to be set and suitably modified by a process of successive iterations to be finally in equilibrium with the forces. In the beginning of this procedure, Eqs. (2) are assumed not to hold good since it is not assumed that the given geometry is in equilibrium. Cable pretension is given a constant value and so,

$$\Delta F_i = 0 \quad \text{and} \quad F_i' = F_i \tag{17}$$

Equations (12) then become:

$$\sum_{i=1}^{m} \frac{F_i}{s_i} [(u_p - u_q) - (x_p - x_q)a_i] = R_{xq} \tag{18a}$$

$$\sum_{i=1}^{m} \frac{F_i}{s_i} [(v_p - v_q) - (y_p - y_q)a_i] = R_{yq} \tag{18b}$$

$$\sum_{i=1}^{m} \frac{F_i}{s_i} [(w_p - w_q) - (z_p - z_q)a_i] = R_{zq} \tag{18c}$$

where

$$R_{xq} = \sum_{i=1}^{m} \frac{F_i}{s_i} [(u_p - u_q)c_i - (x_p - x_q)(1 - d_i)] \tag{19a}$$

$$R_{yq} = \sum_{i=1}^{m} \frac{F_i}{s_i} [(v_p - v_q)c_i - (y_p - y_q)(1 - d_i)] \tag{19b}$$

$$R_{zq} = \sum_{i=1}^{m} \frac{F_i}{s_i} [(w_p - w_q)c_i - (z_p - z_q)(1 - d_i)] \tag{19c}$$

The Newton–Raphson method can be used to determine a solution of the non-linear system of Eq. (18). If the initial geometry before pretensioning is used to start the iterations the risk of converging to a completely different equilibrium configuration will be small, and a nearby configuration of stable equilibrium will be found.

3 OPTIMIZATION

3.1 Minimax Formulation

Assume that the zero configuration has been found and is in equilibrium with some known initial prestress. Reducing the value of this initial pretensioning would be desirable since this would permit the use of less substantial supporting structures and anchorages. Let the cable net have a total of N anchorage points and let D_i denote the horizontal component of the pretensioning force in a cable where it meets an anchorage point. Then D is a vector of cable-end horizontal pretensioning force components. D_i, $i = 1, \ldots, N$. These are design variables for the optimization. Non-negativity must be imposed upon each of these variables since cables cannot carry compressive forces. One goal for the optimization is that some measure representing the total amount of prestressing force on the anchorages should be as small as possible. The measure chosen in this work is the sum of the cable-end pretensioning forces at the anchorage points, each weighted by the horizontal component of the appropriate cable length, i.e.

$$W = c^t D \tag{20}$$

A second set of goals arises from the requirement that under both pretensioning and applied loads the displacement of some specified nodes should all be as small as possible. This raises the question of which configuration the displacements should be measured relative to: the initial reference surface or the zero configuration. In this work it was decided that a maximum desirable value of the displacement $u_{\max}$ of any node under both pretensioning and applied loads from the reference surface should be imposed. If u_{pj} is the displacement of node j relative to the reference surface caused by pretensioning and u_{sj} is the displacement of node j relative to the zero configuration caused by the addition of the applied loads then the displacement goal for node j can be written as:

$$u_j \equiv u_{pj} + u_{sj} \leqslant u_{\max} \tag{21}$$

Further goals come from the imposition of a lower limit on the total force in a cable which ensures that cable elements do not become slack under any form of loading. If

F_{min} is the value of this minimum desirable tensile force in any cable element this goal can be written in the form:

$$D + \Delta\hat{F} \geqslant F_{min} \tag{22}$$

where $\Delta\hat{F}$ is the difference between the total force in a particular cable element and the pretensioning force in the anchorage end of that cable. Similarly, the maximum force in cable element may be limited to some desirable maximum value F_{max} by means of the goal

$$D + \Delta\hat{F} \leqslant F_{max} \tag{23}$$

The optimization method used in this work and described in the next section requires that all these goals should be cast in a normalized form. If some reference cost $\bar{W}$ is specified the goals (20) to (23) may be written in the form

$$g_1(D) \equiv \frac{c^t D}{\bar{W}} - 1 \leqslant 0 \tag{24}$$

$$g_j(D) \equiv \frac{u_j}{u_{max}} - 1 \leqslant 0 \qquad j = 2, \ldots, \bar{J} + 1 \tag{25}$$

$$g_k(D) \equiv \frac{F_{min}}{D + \Delta\hat{F}_k} - 1 \leqslant 0 \qquad k = \bar{J} + 2, \ldots, \bar{J} + \bar{K} + 1 \tag{26}$$

$$g_1(D) \equiv \frac{D + \Delta\hat{F}_k}{F_{max}} - 1 \leqslant 0 \qquad 1 = \bar{J} + 2\bar{K} + 1 \tag{27}$$

where $\bar{J}$ is the total number of nodal displacement restrictions and $\bar{K}$ is the total number of cable segments. The objective of this work is to minimize all of these goals over variables D. This is achieved by the minimax optimization problem:

$$\min_{D} \max_{i} \langle g_1, \ldots, g_i, \ldots, g_{J+2k+1} \rangle \tag{28}$$

3.2 Minimax and Multicriteria Optimization

The method used to solve the minimax optimization problem (28) with goals defined by (24) to (27) is a recently developed entropy-based approach[4]. The minimax problem (28) is discontinuous and non-differentiable, both of which attributes make its numerical solution by direct means difficult. In Ref. [4] it is shown that the minimax solution to problems such as (24) to (28) may be found indirectly by the unconstrained minimization of a scalar function which is both continuous and differentiable, and is thus considerably easier to solve. In this section some of the theory behind this entropy-based approach to minimax and multicriteria optimization is briefly described and a solution algorithm for problem (28) is outlined.

For any set of real, positive numbers U_j, $j = 1, \ldots, J$ and real $p \geqslant q \geqslant 1$, Jensen's inequality ($p$th norm inequality) states that,

$$\left(\sum_{j=1}^{J} U_j^p\right)^{1/p} \leqslant \left(\sum_{j=1}^{J} U_j^q\right)^{1/q} \tag{29}$$

Inequality (29) states that the pth norm[5] of the set U decreases monotonically as its order, p, increases. Another important property of the pth norm is its limit as p tends towards infinity:

$$\lim_{p\to\infty} \left(\sum_{j=1}^{J} U_j^p\right)^{1/p} = \operatorname*{Max}_{j\in J}\langle U_j\rangle \tag{30}$$

Consider the minimax optimization problem,

$$\operatorname*{Min}_{x\in X}\operatorname*{Max}_{j\in J}\langle g_j(x)\rangle \tag{31}$$

and Jensen's inequality. Let $U_j = \exp[g_j(x)]$, $j = 1, \ldots, J$ thus ensuring that $U_j > 0$ for all positive, zero or negative $g_j(x)$. Then

$$\left\{\sum_{j=1}^{J} U_j^p\right\}^{1/p} = \left\{\sum_{j=1}^{J} \exp[pg_j(x)]\right\}^{1/p} \tag{32}$$

And from (30),

$$\lim_{p\to\infty}\left\{\sum_{j=1}^{J} \exp[pg(x)]\right\}^{1/p} = \operatorname*{Max}_{j\in J}\langle\exp[g_j(x)]\rangle \tag{33}$$

Taking natural logarithms of both sides and noting that,

$$\log\lim(f) = \lim\log(f) \quad \text{and} \quad \log\operatorname{Max}(f) = \operatorname{Max}\log(f) \tag{34}$$

Equation (33) becomes,

$$\lim_{p\to\infty}(l/p)\log\left\{\sum_{j=1}^{J}\exp[pg_j(x)]\right\} = \operatorname*{Max}_{j\in J}\langle g_j(x)\rangle \tag{35}$$

Result (35) holds for any set of vectors $g(x)$, including that set which results from minimizing both sides of (35) over $x \in X$. Thus (35) can be extended to:

$$\operatorname*{Min}_{x\in X}\operatorname*{Max}_{j\in J}\langle g_j(x)\rangle = \operatorname*{Min}_{x\in X}(1/p)\log\left\{\sum_{j=1}^{J}\exp[pg_j(x)]\right\} \tag{36}$$

with increasing p in the range $1 \leqslant p \leqslant \infty$.

Result (36) shows that the minimax optimization problem may be solved by the minimization of the scalar function,

$$(1/p)\log\left\{\sum_{j=1}^{J}\exp[pg_j(x)]\right\} \tag{37}$$

over variables $x \in X$ with a sequence of values of increasingly large positive $p \geq 1$.

Reference [4] explores further the relationships between the minimax optimization problem (31) and the scalar optimization function (37), and extends the equivalences to general multicriteria optimization. It is shown in Ref. [4] that the Shannon/Jaynes maximum entropy principle[6,7] plays a key role in these classes of problems, hence the characterization of these methods as entropy-based.

For the purposes of this paper these entropy considerations are incidental so this aspect is not pursued further. In this work the minimax optimization problem (28) for the cable net structure with goals defined by Eqs. (24) to (27) was solved by the scalar minimization of function (37) in the form:

$$\underset{D}{\mathrm{Min}}\ (1/p)\log\left\{\sum_{j=1}^{J+2K+1} \exp[pg_j(\mathrm{D})]\right\} \tag{38}$$

over an increasing positive sequence of values of p.

3.3 *Scalar Optimization*

Problem (38) is unconstrained and differentiable which, in theory, gives a wide choice of possible numerical solution methods. However the goal functions $g_j(D)$ in (38) do not have explicit algebraic forms in most cases. In particular u_j and ΔF_k in goals (25), (26) and (27) are only calculable numerically and are found from the analysis results of a particular design. This presents a considerable impediment to the optimization process.

The strategy adopted was to solve the implicit optimization problem (38) by means of an iterative sequence of explicit approximation models. An explicit approximation to problem (38) can be formulated by taking Taylor series expansions of all the goal functions $g_i^{(D)}$ in problem (38), truncated after the linear terms. This gives:

$$\underset{D}{\mathrm{Min}}\ (1/p)\log\left\{\sum_{j=1}^{J+2K+1} \exp p\left[g_j(D^o) + \sum_{i=1}^{N}\left(\frac{\partial g_j}{\partial D_i}\right)_{D^o}(D - D_o)\right]\right\} \tag{39}$$

where N is the total number of design prestressing force variables D, and D^0 is the current vector of those variables at which the Taylor series expansion is made. Problem (39) is now an explicit approximation to problem (38) if values of all the $g_j(D^0)$ and $(\partial g_j/\partial D_i)_{D^0}$ are known numerically. Given such values (and Section 4 addresses this aspect), problem (39) can be solved directly by any standard unconstrained optimization method. In the examples presented later, a quasi-Newton algorithm (the NAG Library routine EO4JAF) was used to solve (39).

Solving (39) for particular numerical values of $g_j(D^0)$ and $(\partial g_j/\partial D_i)_{D^0}$ forms only one iteration of the complete solution of problem (38), however. The solution vector D^1 of such an iteration represents a new design which must be analysed, checked for feasibility, altered if necessary, and gives new values for $g_j(D^1)$ and $(\partial g_j/\partial D_i)_{D^1}$ to replace those corresponding to D^0 in (39). Iterations continue until changes in the design variables D become small. Also, during these iterations the parameter p must be increased in value to ensure that a minimax optimum solution is found.

Section 5 describes the optimum design algorithm in more detail, but first the determination of numerical values for the goal functions and their derivatives in problem (39) is considered.

4 SENSITIVITY ANALYSIS

To formulate and solve the explicit approximation problem (39) numerical values are required for all the goal functions $g_j(D)$ and all their first derivatives with respect to the design variables D. Goal (25) is explicit and need not be considered further. Goals (26), however, contain all the nodal displacement under all loading cases and these are implicit functions of D. Given some design variable values a full analysis of the cable net will yield numerical values for all the u_j. The first derivative of each element of u_j with respect to each design variable is also required, and their calculation is a considerable task. One way of evaluating these derivatives is to calculate them from analytical expresssions, as follows.

The cable net equilibrium equations (16) may be differentiated with respect to a particular design variable D_i as below:

$$\frac{\partial K}{\partial D_i} u_j + K \frac{\partial u_j}{\partial D_i} = -\frac{\partial P_j}{\partial D_i} + \frac{\partial R_j}{\partial D_i}$$

Hence

$$\frac{\partial u_j}{\partial D_i} = K^{-1}\left\{ -\frac{\partial P_j}{\partial D_i} + \frac{\partial R_j}{\partial D_i} - \frac{\partial K}{\partial D_i} u_j \right\} \tag{40}$$

the $\partial P_j/\partial D_i$ terms in (40) are all zero as the P_i are transverse applied loads. Analytical expressions for the $\partial R_j/\partial D_i$ terms in (40) can be written by direct differentiation of Eqs. (13) or (19), and the terms $\partial K/\partial D_i$ come from an analytical differentiation of the left-hand sides of Eqs. (12) or (18). Care must be taken, particularly with the stiffness matrix derivatives, that the appropriate equations from (12) or (18) are differentiated and combined together according to the structure of the stiffness matrix itself. Note also that design sensitivities must be calculated for both the zero configuration and for the zero configuration plus each applied load case.

The evaluation of design sensitivities for cable net structures by the above analytical approach is computationally efficient, but requires great care, programming skill and time to write the necessary coding. An alternative means of calculating sensitivities which is much simpler to implement but uses much more CPU time is to use finite difference approximations for the derivatives. This was the method used in this work since the number of design variables in the examples ($N = 8$) is not very large. The scheme adopted used backward finite differences in order to give a check on the possibility of slackening of a cable.

For finite difference sensitivity analysis the current design is first analysed in full and then a further N complete analyses are performed, reducing each design variable in D in turn by a small amount, ΔD. Approximate values for the derivatives of nodal displacements, for example, are then given by

$$\frac{\partial u_j}{\partial D_i} \approx \frac{u_j(D) - u_j(D - \Delta D_i)}{\Delta D_i} \tag{41}$$

Goals (26) and (27) contain quantities $\Delta \hat{F}_k$, and represent the forces in all individual segments of the cable net. Values of these for any particular design are given by a full analysis but derivative values are also needed by problem (39). In the present work they were obtained through the backward finite difference scheme but they may also be calculated analytically by differentiation of Eq. (9).

5 THE OPTIMUM PRETENSION DESIGN ALGORITHM

The previous sections have examined the major elements of the design method—the non-linear analysis of the net, the minimax optimum design formulation and its solution, and the sensitivity analysis of the structure. All three major elements are complicated and time-consuming to perform. Each of the elements is iterative. To combine these elements together into an automatic optimum design program of general applicability to cable net structures would be a major task and the use of such a program on a large cable net structure would require a large amount of CPU time. This was not attempted in this work.

Instead, an interactive strategy was used in combining the elements together, with the operator carefully monitoring iterations, guiding the design process towards a feasible and pseudo optimum solution, and terminating the process when satisfactory results were obtained. No guarantees of optimality can be given for such a strategy. Indeed the strategy did not aim at finding an accurate, unique solution: the overall aim was to take an initial design for a cable net structure and to make successive improvements to that design by the methods described until the rate of improvement became too small to warrant further computational effort. Viewed in the context of this strategy the algorithm outlined below was very successful.

5.1 *Initialization*

For the given cable net structure a uniform pretensioning force distribution in all cables was assumed for initial design purposes. Complete prestressing plus applied load analyses of the cable net for different values of this uniform pretension force enabled an approximate lowest value to be found which gave a just-feasible design. The cost c^tD of this prestressing formed the reference cost $\bar{W}$ against which improvements in cost were measured.

The minimax optimization algorithm requires a sequence of positive values of p increasing towards infinity. Many different schemes are possible. One way of estimating a value for p is to iterate to that value which makes the objective function of problem (38) stationary with respect to p; i.e. to iteratively solve

$$p = \frac{\left\{\sum_{j=1}^{J+2K+1} \exp[pg_j(D)]\right\}\left\{\log \sum_{j=1}^{J+2K+1} \exp[pg_j(D)]\right\}}{\sum_{j=1}^{J+2K+1} g_j(D)\exp[pg_j(D)]} \tag{42}$$

From this expression it can be seen that p increases as the infeasibility of the current design decreases, i.e. increasing p tends to enforce feasibility. Also, expression (42) requires the starting point to be infeasible in order to generate $p > 0$. In practice it was found that the optimization algorithm works well even when the starting point is feasible though is improved by the use of an infeasible starting point. In the present work a value of p in the range $30 \leqslant p \leqslant 50$ was used for the first iteration of problem (39). For subsequent iterations p was increased to 100 then 200 by the operator.

5.2 *The Algorithm*

1) Find the zero configuration for the structure loaded by the pretension forces only, and the deviations of the zero configuration from the reference configuration.

2) Analyse the structure for pretension plus each applied load system using the zero configuration as the initial geometry.

3) Perform the sensitivity analysis of the structure by backward finite differences for all types of loading. This gives numerical values of all goal functions and their first derivatives for use in problem (39).

4) Choose a value for p and solve problem (39) by any unconstrained optimization method (such as NAG EO4JAF). This gives new estimates for all the pretension forces.

5) Repeat steps 1 and 2 for the new pretension forces.

At this point the results of the iteration are assessed and a choice is made from several alternative strategies for further iterations, as follows.

A) If the new design is feasible with reduced cost, retain it as an incumbent optimum design. Use it as a starting point for a further iteration of problem (39) with the same value of p, starting with a new sensitivity analysis, step 3.

B) If the new design is feasible with only a small reduction in cost, retain it as an incumbent design. Use it as a starting point for a further iteration of problem (39) with an increased value of p, starting with a new sensitivity analysis, step 3.

C) If the starting design is feasible and the new design leads to a cable structure design of higher cost, increase the value of p and repeat the iteration of problem (39) with the same data.

D) If the starting design is feasible but the new design is infeasible, scale up the pretension forces in order to restore feasibility. Use this as a starting point for a further iteration of problem (39) with the same value of p, starting with step 1.

E) If the starting design is infeasible and the new design is less infeasible, reduce the value of p and repeat the iteration of problem (39) with the same data. If the new design is more infeasible stop the algorithm and use the incumbent design as the optimum.

6 NUMERICAL EXAMPLES

The algorithm described in Section 5 has been used to find the optimum distribution of prestressing forces in the hyperbolic paraboloid cable net first described by Krishna[2] who used it as an analysis example, and which was subsequently used as a design example by Cinquini and Contro[3]. The geometry of the reference surface is shown in Figure 2. The extensional stiffnesses (EA) of the sagging and hogging cables are 293.6 MN and 197.5 MN respectively. Three loading cases were chosen from Ref. [2] for design purposes and are shown in Figure 3.

Symmetry of the cable layout implies that this net may have eight different pretension forces in the different cables so the design problem has eight design variables D_1 to D_8. Four different design problems were studied corresponding to four different combinations of minimum permissible cable force and maximum permissible nodal displacement. Target values used were 190 kN and 335 kN for minimum cable force F_{min} and 0.8 m and 1.0 m for maximum nodal displacement u_{max} from the reference surface. As a starting point for all designs a uniform prestressing force of 670 kN in all cables was used. This represents an infeasible design for three of the four examples. The cost coefficients used were $c^t = (146.4.\ 219.6\ 146.4\ 73.2\ 146.4\ 219.6\ 146.4\ 73.2)$.

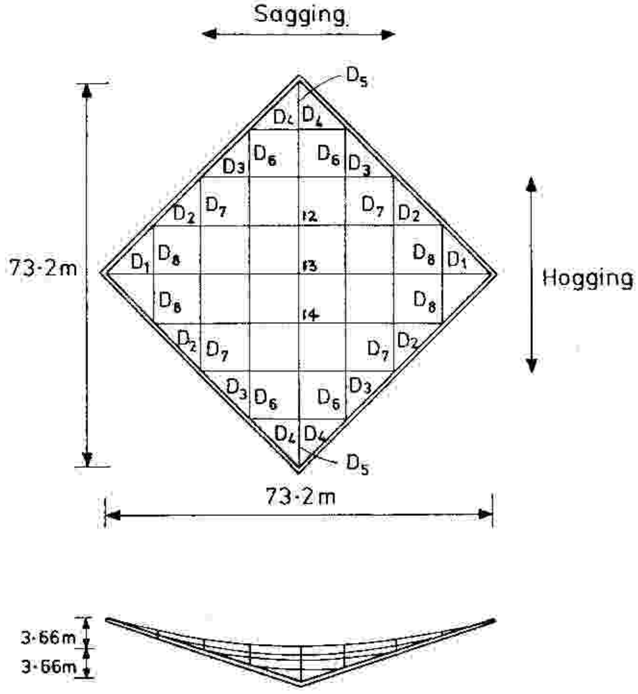

Figure 2 Hyperbolic paraboloidal cable net example.

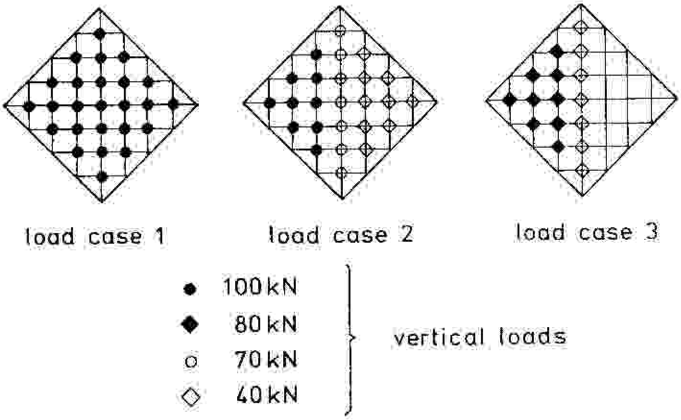

Figure 3 Loading cases for the example.

Table 1 Optimization results for the four examples.

Example No.	*1*	*2*	*3*	*4*
D_1 (kN)	906	838	945	890
D_2	567	711	607	821
D_3	364	360	368	485
D_4	407	106	408	184
D_5	722	868	712	840
D_6	677	775	643	756
D_7	460	592	466	591
D_8	278	387	264	399
F_{min} (kN)	199	335	206	335
u_{max} (m)	0.89	0.99	0.77	0.77
W(kNm × 10^3)	686	751	688	800

Example 1

For this example target values of F_{min} and u_{max} were 190 kN and 1.0 m respectively. The solution for a uniform prestressing force distribution has a cost of 776 × 10^3 kNm units corresponding to a horizontal component of $D = 663$ kN prestressing in each cable. The final design is tabulated in Table 1 and its cost is approximately 12% less than that of the uniform design. Figure 4 shows the reference, zero and loaded configurations of the sagging and hogging cables under the uniform load case 1. Such configurations are typical for all the examples. Figure 5 shows diagrammatically those cable segments in the final design in which the axial force is between F_{min} and $1.1 \times F_{min}$ (indicated by heavy lines) under each load case and those cable segments in which the lower bound is exactly attained (indicated by stars).

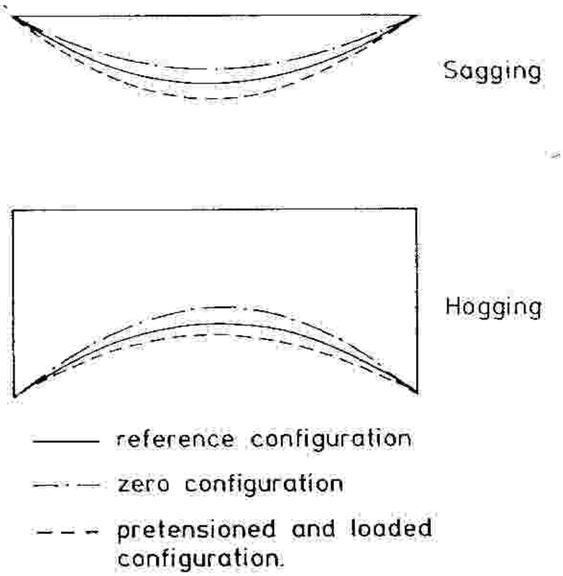

Figure 4 Typical cable configurations.

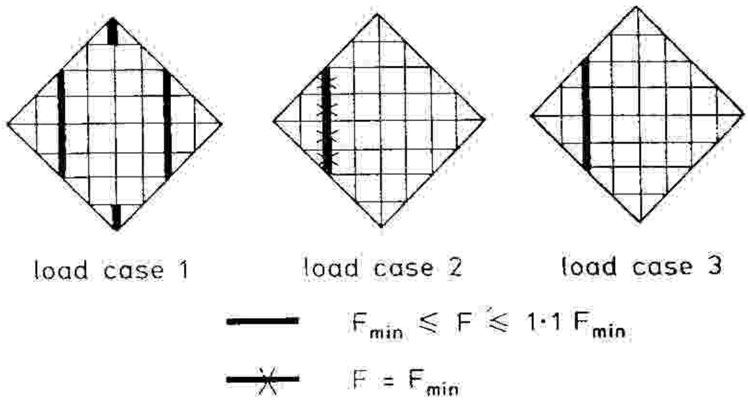

Figure 5 Cable minimum force maps. Example 1.

This example had a feasible starting point and the solution of Table 1 was obtained after two iterations of problem (39) with $p = 50$. Further iterations produced no further improvements.

Example 2

In this example target values of F_{min} and u_{max} were 335 kN and 1.0 m respectively, i.e. cable elements may not be as lightly loaded as in the first example. The uniform prestressing solution for this example has $D = 790$ kN in all cables and a cost of 925×10^3 kNm units. The final design given in Table 1 is 19% cheaper than the uniform design. Figure 6 shows maps of cable segments which are near or at their minimum force limits under each applied load case.

In this example two iterations of problem (39) were carried out with $p = 50$, two with $p = 100$ and two with $p = 200$. The starting point is infeasible for this example, but the first iteration leads to a feasible design which is 14% cheaper than the uniform design. The subsequent iterations raise this saving to 19%. It was noted that the prestressing force distribution changed very little after the second iteration. Thus the

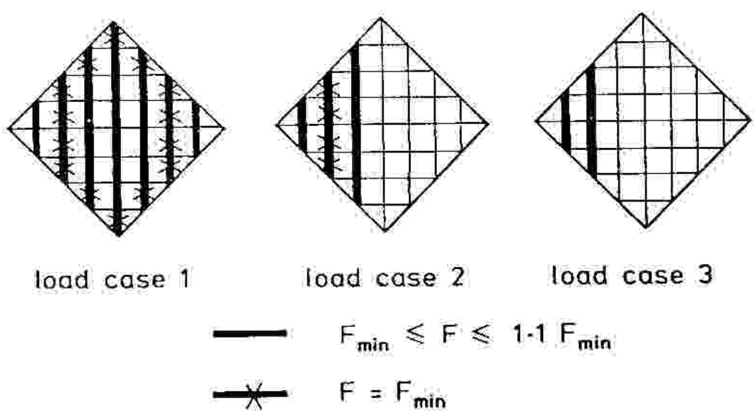

Figure 6 Cable minimum force maps. Example 2.

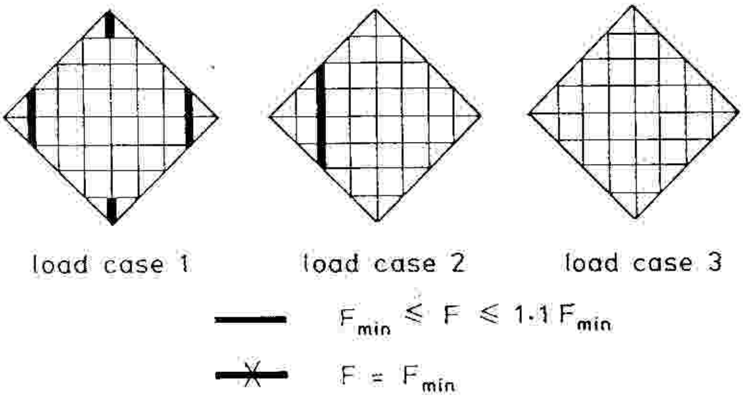

Figure 7 Cable minimum force maps. Example 3.

sensitivity information used for the second iteration could also have been used for subsequent iterations with little effect upon accuracy but with very large savings of CPU time.

Example 3

Target values of F_{min} and u_{max} were 190 kN and 0.8 m respectively, i.e. more stringent nodal displacement limits than in the first example. The uniform prestressing force solution has $D = 978$ kN and a cost of 1145×10^3 kNm units. Table 1 gives the final design results and cable minimum force maps are shown in Figure 7.

The Table 1 solution was obtained after two iterations of problem (39) with $p = 50$. From an infeasible starting point the first iteration resulted in a feasible design with a saving of 35% over the uniform design. The second iteration raised this cost saving figure to 40% but very little further savings were made in subsequent iterations.

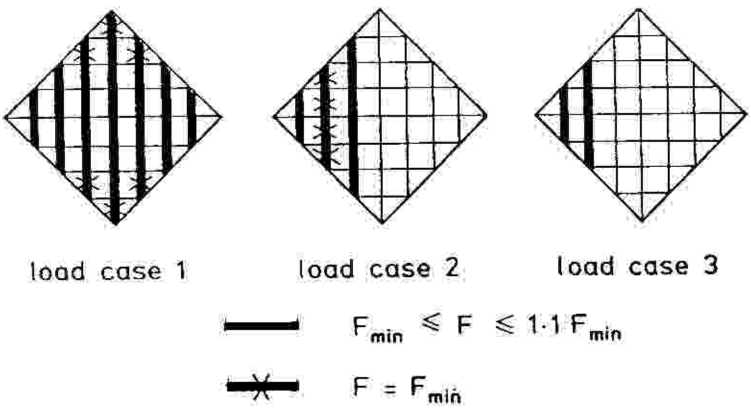

Figure 8 Cable minimum force maps. Example 4.

Example 4

This example had the most stringent set of limits among the examples studied: F_{min} was 335 kN and u_{max} was 0.8 m. The uniform prestressing force solution is as for Example 3. The final design is tabulated in Table 1 and Figure 8 shows maps of cable segment minimum forces.

In this example the usual starting point of $p = 50$ had to be reduced to $p = 30$ in order to reduce the infeasibilities in the starting design. Eventually the Table 1 result was obtained after two iterations at $p = 100$. The final design represents a saving of 30% over the uniform design.

In all four examples load case 1 dominated the final optimum design in the sense that a design optimized for load case 1 only also satisfied load cases 2 and 3. It is noted that the greatest savings were made in the last two examples, both of which had stringent displacement limits. More examples would be needed to confirm whether this is a significant result but it may indicate that prestressing force optimization is particularly valuable in cable nets in which small displacements are required. Cinquini and Contro[3] also give solutions for all these four examples but direct comparison with the results presented here is not possible. Their results for examples 1 and 3 seem to be in error since a full non-linear analysis of their optimum designs reveals that some cable segments not only violate the minimum force limit but are actually slack. In examples 2 and 4 similar full analyses reveal that although the nodal displacement limit is satisfied at node 13 (where a maximum displacement might intuitively be thought to occur), at nodes 12 and 14 the displacement is actually greater than at node 13 and the limit is exceeded by a considerable amount.

7 DISCUSSION AND CONCLUSIONS

This paper has shown that the optimization of cable net structures must be classed as a difficult problem. The prestress optimization process has three major elements: analysis, sensitivity analysis and optimization. Each of these elements is significantly difficult and requires considerable computational effort. A completely automatic optimum design program which combines all three is currently not an achievable goal but the use of optimization in a user-interactive mode is feasible and can result in considerable reductions of prestress levels over a uniformly prestressed design.

The non-linear analysis of pretensioned cable net structures involves considerable computation because the shape-finding problem must be done iteratively. With pretensioned nets at least two shape-finding problems must be solved in each analysis: one to find the shape of the zero configuration under pretension forces only, and one more for each applied load case. The Newton Raphson method used in the present work proved efficient and accurate.

Because the non-linear analysis can only be done numerically it is not possible to obtain closed form algebraic expressions for most of the functions needed in the optimization model. Consequently the optimization must be based upon approximate models which employ numerical function values and first derivative values calculated for particular designs. The non-linear and iterative nature of the analysis process means that the calculation of the necessary sensitivity information must inevitably be expensive in programming effort and CPU time by whatever means it is done. Analytical sensitivities were not used in this work although they might be preferable in terms of accuracy and CPU time to the backward finite difference method used here.

The minimax formulation adopted in this work is satisfying in that it allows the simultaneous optimization and control of many different engineering goals. The entropy-based approach to solving minimax optimization formulations also proved very successful in transforming the problem to a scalar optimization with just one control parameter, p. In another application[8] it proved fairly easy to develop rules for choosing a suitable increasing sequence of values for p which caused smooth convergence to an optimum solution. In this cable net optimization it proved more difficult to provide clear-cut rules for up-dating p. This is probably because of the highly non-linear nature of cable net structures. However, with care and experiment suitable values for p can be found. It is worth noting in this context that all the examples solved gave very substantial reductions in total cost of pretensioning after only the first iteration with a preset value of p between 30 and 50. In all cases the first iteration was responsible for approximately 80% of the total savings made.

The examples solved in the course of this work provided considerable insights into the behaviour of prestressed cable net structures. Nodal displacements and individual cable segment forces were sometimes very sensitive to comparatively small changes in the initial pretensioning forces. Structural intuition was very poor at predicting which cable segments might become slack or which node would have the largest displacement. Cable net structures appear to resist very similar loading cases by very different structural actions. This is disturbing from a design point of view and needs further investigation. It also suggests that many local optima may exist in this design optimization problem. This would not be surprising but has not been investigated. However, experience with the current algorithm in respect of its convergence shows that the initial iterations reach the general area of an optimum quickly and smoothly but further iterations do not home in on a solution as would be expected if only a single solution were present: the final convergence of the algorithm may be affected by the presence of several different local optima.

Finally, this work has only touched the surface of the general area of optimizing the design of cable net structures. It is possible to criticise many aspects of this work. Nevertheless it has shown that there are considerable potential savings to be made through the use of optimization in the design of pretensioned cable systems and has raised more questions than it has answered about the behaviour of such structures under load.

Acknowledgements

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