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STABILITY OF FINITE DIFFERENCE SCHEMES FOR COMPLEX DIFFUSION PROCESSES

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ABSTRACT: Complex diffusion is a common and broadly used denoising procedure in image processing. The method is based on an explicit finite difference scheme applied to a diffusion equation with a proper complex diffusion parameter in order to preserve edges and the main features of the image, while eliminating noise. In this paper we present a rigorous proof for the stability condition of complex diffusion finite difference schemes.

KEYWORDS: finite differences, complex diffusion, stability.

1. Introduction

Diffusion processes are commonly used in image processing in order to remove noise. The main idea is that if one pixel is affected by noise, than the noise should be diffused among the neighboring pixels in order to smooth the region. In this way proper diffusion partial differential equations have been considered to achieve this end. Taking the heat equation

$$\frac{\partial u(x,t)}{\partial t} = \nabla \cdot (D(x,t,u)\nabla u(x,t))$$

where $u(x,t)$ represents the denoised image at time t with the initial noisy image $u(x,0)$, the choice of the diffusion parameter D plays a very important role for the purpose of denoising. Roughly speaking, one wants D to allow diffusion on homogeneous areas affected only by noise and to forbid diffusion on edges to preserve features of the original denoised image. In this way, several expressions for D have been suggested. The first approaches indicated that D should depend on the gradient of u with an inverse proportion [5, 9, 10]. However, this approach had a few handicaps. For instance, within a ramp edge the diffusion coefficient is similar along all the edge delaying the diffusion process, not distinguishing between the end points and interior points of the ramp edge where diffusion should differ. Therefore, the use of the laplacian was suggested as being more appropriate since it has a higher

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amplitude near the end points and low magnitude elsewhere, namely by considering

$$D = \frac{1}{1+|\Delta u|^2}.$$

The drawback is that the computation of a higher order derivative is needed leading to higher ill-posedness in the first steps while the image is strongly affected by noise. To overcome this problem, Gilboa et al. [3] suggested to use complex diffusion. The use of a complex diffusion coefficient turns the partial differential equation into some sort of combination between the heat and the Schrödinger equation. Having our goal of preserving image features in view, Gilboa et al. [3] suggested

$$D = \frac{e^{i\vartheta}}{1+\left(\frac{\mathrm{Im}(u)}{\kappa\vartheta}\right)^2}, \tag{1}$$

with $\vartheta \approx 0$ and $\kappa > 0$ a normalization constant, involving an approximation to the laplacian of the image, namely by using the identity

$$\lim_{\vartheta\to 0} \frac{\mathrm{Im}(u(.,t))}{t\vartheta} = G * \Delta u(.,0)$$

where G is a gaussian and therefore the convolution with G represents a low pass filter. We note that the right hand side is a low-pass filter of the structure of edges of the initial image $u(.,0)$ given by its laplacian, and therefore the choice of D as in (1) penalizes diffusion across edges.

Complex diffusion proved to be much better conditioned numerically and successfully applied in medical imaging despeckling and denoising [2, 6]. More recently an improvement of D by making κ adaptive was also strongly suggested [1]. However, to the best of our knowledge a theoretical gap as been open since, having in mind that explicit finite difference schemes for the discretization of the equation are usually considered in order to obtain fast computational methods: a stability condition for complex diffusion. It is known that for a real positive diffusion parameter D in the one-dimensional case, the method is stable if

$$\Delta t \leq \frac{h^2}{2\max D},$$

where Δt and h are, respectively, the temporal and spatial discretization steps. This upper bound has generally been used by engineers when implementing complex diffusion, though no rigorous proof of stability condition was presented in this case. Moreover, the generalization seems not to be straightforward, since the von Neumann stability analysis approach seems to fail due to a non-constant D, as well as typical approaches using the Gershgorin theorem ([8]).

The main result of this paper is the proof of a stability condition for complex diffusion. We only present in detail the proof for the Neumann case. Remark 2 in section 4 is devoted to the generalization of the proof for the easier Dirichlet case. The papers is organized as follows. In section 2 the complex diffusion equation is presented. In section 3 the details on the considered numerical scheme are also given. In section 4 the stability condition is proven for Neumann boundary conditions. Finally we present numerical examples in section 5 to illustrate the stability condition.

2. Complex diffusion equation

Let Ω be a bounded open set in $\mathbb{R}^d$, $d \geq 1$, with boundary $\Gamma = \partial\Omega$. Typically Ω is the cartesian product of open intervals in $\mathbb{R}$, i.e.,

$$\Omega = \prod_{j=1}^{d}]a_j, b_j[,$$

with $a_j, b_j \in \mathbb{R}$. We consider a diffusion process with a non-constant complex coefficient $D = D(x, t, u)$, where

$$0 \leq |\mathrm{Im}(D(x,t,u))| \leq \mathrm{Re}(D(x,t,u)), \qquad x \in \Omega, t \in [0,T], u \in \mathbb{C}, \tag{2}$$

$T > 0$. Let $Q = \Omega \times]0, T]$. We define the initial boundary value problem for the unknown function $u : \bar{Q} = \bar{\Omega} \times [0,T] \longrightarrow \mathbb{C}$

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \nabla \cdot (D(x,t,u)\nabla u(x,t)), \quad (x,t) \in Q \\ \\ u(x,0) = u^0(x), \quad x \in \overline{\Omega}, \\ \\ \alpha u(x,t) + \beta \frac{\partial u}{\partial \nu}(x,t) = 0, \quad x \in \Gamma, \quad t \in [0,T], \end{cases} \tag{3}$$

where $\frac{\partial u}{\partial \nu}$ denotes the derivative in the direction of the exterior normal to Γ. With a complex parameter D, the equation can be seen as a combination of the Schrödinger equation with the second Fick's Law or the Heat Equation, depending on the science field.

For the boundary conditions we consider that

$$\alpha\beta = 0 \quad \text{and} \quad \alpha + \beta \neq 0. \tag{4}$$

3. Numerical method

Let us construct a mesh on $\overline{Q}$. Let $h = (h_1, \ldots, h_d)$, where h_k denotes the mesh-size in the kth spatial coordinate direction, such that $h_k = (b_k - a_k)/N_k$, $k = 1, \ldots, d$, with $N_k \geq 2$ an integer. Let $\Delta t = T/M$ be the mesh-size in the t-direction, where $M \geq 1$ is an integer. We define a mesh in $\overline{Q}$ by

$$\begin{aligned}\overline{Q}_h^{\Delta t} \;\; = \;\; & \{(x_j, t^m) : x_j = (a_1 + j_1 h_1, \ldots, a_d + j_d h_d), 0 \leq j_k \leq N_k, k = 1, ..., d, \\ & t^m = m\Delta t, 0 \leq m \leq M\} .\end{aligned}$$

Note that $\overline{Q}_h^{\Delta t}$ could be viewed as the cartesian product of a space grid $\overline{\Omega}_h$ and a grid in the temporal domain. Let $Q_h^{\Delta t} = \overline{Q}_h^{\Delta t} \cap Q$ and $\Gamma_h^{\Delta t} = \overline{Q}_h^{\Delta t} \cap \Gamma \times [0, T]$.

With the coordinate $(j, m) = (j_1, \ldots, j_d, m)$ is associated the point $(x_j, t^m) \in \overline{Q}_h^{\Delta t}$ of the form

$$(x_j, t^m) = (a_1 + j_1 h_1, \ldots, a_d + j_d h_d, m\Delta t).$$

We denote by V_j^m the value of a mesh function V, defined on $\overline{Q}_h^{\Delta t}$, at the point (x_j, t^m). We define the forward and backward finite differences with respect to (x_j, t^m) in the kth spatial direction by

$$\delta_k^+ V_j^m = \frac{V_{j+e_k}^m - V_j^m}{h_k}, \quad \delta_k^- V_j^m = \frac{V_j^m - V_{j-e_k}^m}{h_k},$$

where e_k denotes the kth element of the natural basis in $\mathbb{R}^d$.

On $\overline{Q}_h^{\Delta t}$ we approximate (3) by the one-parameter family of finite difference schemes

$$\begin{cases} \dfrac{U_j^{m+1} - U_j^m}{\Delta t} = \displaystyle\sum_{k=1}^{d} \delta_k^+ (D_{j-(1/2)e_k}^m \delta_k^- U_j^{m+\theta}) \quad \text{in} \quad \tilde{Q}_h^{\Delta t}, \\[2ex] U_j^0 = u_0(x_j) \quad \text{in} \quad \overline{\Omega}_h, \\[2ex] \alpha U_j^m + \dfrac{\beta}{2} \displaystyle\sum_{k=1}^{d} \left(\delta_k^+ U_j^m + \delta_k^- U_j^m\right) \cdot \nu_k = 0 \quad \text{in} \quad \Gamma_h^{\Delta t}, \end{cases} \tag{5}$$

where $U_j^{m+\theta} = \theta U_j^{m+1} + (1-\theta)U_j^m$, $\theta \in [0,1]$, U_j^m representes the approximation of $u(x_j, t^m)$,

$$D^m_{j-(1/2)e_k} = \frac{D(x_j, t^m, U_j^m) + D(x_{j-e_k}, t^m, U^m_{j-e_k})}{2},$$

we use the notation $\tilde{Q}_h^{\Delta t}$ for the set $Q_h^{\Delta t}$ or $\overline{Q}_h^{\Delta t}$, respectively, in the case of Dirichlet or Neumann boundary conditions, and ν_k represents the kth component of the normal vector ν.

Assuming enough regularity for the solution of (3), it is not difficult to prove that this method is consistent with the differential problem of order $(h^2, \Delta t)$, if $\theta \neq \frac{1}{2}$, and of order $(h^2, (\Delta t)^2)$, if $\theta = \frac{1}{2}$.

Remark 1. *If the diffusion coefficient $D = D(x,t)$ does not depend on the field u, than the differential equation is linear and Lax equivalence theorem* [4] *tells us that to prove convergence one only needs to prove the stability of the method. However if $D = D(x,t,u)$ - as in our case of interest - the proof of stability might not imply convergence, since the fact that the differential equation is non-linear does not satisfy the hypothesis of Lax equivalence theorem. However, the proof of a stability condition is of importance in order to guarantee that the solution stays bounded.*

4. Stability of the numerical scheme

In this section we investigate the stability of the finite difference scheme (5). The approach we use here has some analogies with strategies more commonly used in the context of the finite element method but it has already been used in the context of the finite difference method (see e.g. [7]).

Theorem 1. *Suppose that (2) and (4) hold. Then, if $\theta \in [\frac{1}{2}, 1]$ the method (5) is unconditionally stable and if $\theta \in [0, \frac{1}{2}[$ the method (5) is stable under the condition*

$$\Delta t \leq \frac{(\min\{h_1, \ldots, h_d\})^2}{d(2(1-2\theta)) \max{(D_R + |D_I|)}}.$$

To prove this result we will consider the unidimensional case and considering Neumann boundary conditions ($\alpha = 0$). For $d \geq 2$ or Dirichlet boundary conditions the proof follows the same steps.

We rewrite (5) as a system by separating the real and imaginary parts of the main variable $U = U_R + iU_I$ and the diffusion coefficient $D = D_R + iD_I$.

We shall then study the stability of the family of finite difference schemes: find $U_j^m \approx u(x_j, t^m)$, $j = 0, \ldots, N$, $m = 0, \ldots, M$, such that

$$\begin{cases} \dfrac{U_{Rj}^{m+1} - U_{Rj}^m}{\Delta t} = \delta_x^+ (D_{Rj-}^m \delta_x^- U_{Rj}^{m+\theta}) - \delta_x^+ (D_{Ij-}^m \delta_x^- U_{Ij}^{m+\theta}), & \begin{array}{l} j = 0, \ldots, N, \\ m = 0, \ldots, M-1, \end{array} \\ \\ \dfrac{U_{Ij}^{m+1} - U_{Ij}^m}{\Delta t} = \delta_x^+ (D_{Ij-}^m \delta_x^- U_{Rj}^{m+\theta}) + \delta_x^+ (D_{Rj-}^m \delta_x^- U_{Ij}^{m+\theta}), & \begin{array}{l} j = 0, \ldots, N, \\ m = 0, \ldots, M-1, \end{array} \\ \\ U_{Rj}^0 = u_R^0(x_j), U_{Ij}^0 = u_I^0(x_j), j = 0, \ldots, N, \\ \\ \delta_x^+ U_{R0}^m + \delta_x^- U_{R0}^m = 0,\ \delta_x^+ U_{RN}^m + \delta_x^- U_{RN}^m = 0, m = 0, \ldots, M, \\ \\ \delta_x^+ U_{I0}^m + \delta_x^- U_{I0}^m = 0,\ \delta_x^+ U_{IN}^m + \delta_x^- U_{IN}^m = 0, m = 0, \ldots, M, \end{cases} \tag{6}$$

where

$$D_{-j}^m = \frac{D(x_{j-1}, t^m, U_{j-1}^m) + D(x_j, t^m, U_j^m)}{2},\ j = 1, \ldots, N,\ m = 0, \ldots, M.$$

In (6) we need the extra points $x_{-1} = x_0 - h$ and $x_{N+1} = x_N + h$ and we define $D_{-0}^m = D_{-1}^m$, $D_{-(N+1)}^m = D_{-N}^m$.

We consider the discrete L^2 inner products

$$(U, V)_h = \sum_{j=0}^{N}{}'' h U_j \overline{V}_j = \frac{h}{2} U_0 \overline{V}_0 + \sum_{j=1}^{N-1} h U_j \overline{V}_j + \frac{h}{2} U_N \overline{V}_N$$

and

$$(U, V)_{h^*} = \sum_{j=1}^{N} h U_j \overline{V}_j,$$

and their corresponding norms

$$\|U\|_h = (U, U)_h^{1/2} \quad \text{and} \quad \|U\|_{h^*} = (U, U)_{h^*}^{1/2}.$$

Multiplying both members of the first and second equations of (6) by, respectively, $U_R^{m+\theta}$ and $U_I^{m+\theta}$, according to the discrete inner product $(\cdot, \cdot)_h$ and using summation by parts we obtain

$$\left(\frac{U_R^{m+1} - U_R^m}{\Delta t}, U_R^{m+\theta} \right)_h + \|(D_{R-}^m)^{1/2} \delta_x^- U_R^{m+\theta}\|_{h^*}^2 = \left(D_{I-}^m \delta_x^- U_I^{m+\theta}, \delta_x^- U_R^{m+\theta} \right)_{h^*}$$

and

$$\left(\frac{U_I^{m+1}-U_I^m}{\Delta t}, U_I^{m+\theta}\right)_h + \|(D_{R-}^m)^{1/2}\delta_x^- U_I^{m+\theta}\|_{h^*}^2 = -\left(D_{I-}^m \delta_x^- U_R^{m+\theta}, \delta_x^- U_I^{m+\theta}\right)_{h^*}.$$

Then

$$\left(\frac{U^{m+1}-U^m}{\Delta t}, U^{m+\theta}\right)_h + \|(D_{R-}^m)^{1/2}\delta_x^- U^{m+\theta}\|_{h^*}^2 = 0.$$

Since we can write

$$U^{m+\theta} = \Delta t\left(\theta - \frac{1}{2}\right)\frac{U^{m+1}-U^m}{\Delta t} + \frac{U^{m+1}+U^m}{2},$$

we obtain

$$\Delta t\left(\theta - \frac{1}{2}\right)\left\|\frac{U^{m+1}-U^m}{\Delta t}\right\|_h^2 + \frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t} + \|(D_{Rj-}^m)^{1/2}\delta_x^- U^{m+\theta}\|_{h^*}^2 = 0. \tag{7}$$

Let us now consider two different cases. First, we suppose that $\theta \in [\frac{1}{2}, 1]$. In this case $\theta - \frac{1}{2} \geq 0$ and from (7) we immediately obtain that

$$\|U^{m+1}\|_h \leq \|U^m\|_h, \quad m = 0, \ldots, M-1,$$

which proves that the scheme is stable without any limitation in the time step.

Now, let us consider that $\theta \in [0, \frac{1}{2}[$. In this case,

$$\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t} + \|(D_{R-}^m)^{1/2}\delta_x^- U^{m+\theta}\|_{h^*}^2 = \Delta t\left(\frac{1}{2} - \theta\right)\left\|\frac{U^{m+1}-U^m}{\Delta t}\right\|_h^2. \tag{8}$$

We recall that

$$\left\|\frac{U^{m+1}-U^m}{\Delta t}\right\|_h^2 = \left\|\frac{U_R^{m+1}-U_R^m}{\Delta t}\right\|_h^2 + \left\|\frac{U_I^{m+1}-U_I^m}{\Delta t}\right\|_h^2. \tag{9}$$

Using the right hand side of the difference equations and the boundary conditions in (6), and applying the iniquality $(a-b)^2 \leq 2a^2+2b^2$, we obtain

$$\begin{aligned}\left\|\frac{U_R^{m+1}-U_R^m}{\Delta t}\right\|_h^2 &\leq \frac{4}{h^2}\|D_{R^-}^m\delta_x^- U_R^{m+\theta} - D_{I^-}^m\delta_x^- U_I^{m+\theta}\|_{h^*}^2\\ &= \frac{4}{h^2}\left(\|D_{R^-}^m\delta_x^- U_R^{m+\theta}\|_{h^*}^2 + \|D_{I^-}^m\delta_x^- U_I^{m+\theta}\|_{h^*}^2\right)\\ &\quad -\frac{8}{h}\sum_{j=1}^{N} D_{R^-}^m\delta_x^- U_R^{m+\theta} D_{I^-}^m\delta_x^- U_I^{m+\theta}. \qquad (10)\end{aligned}$$

Likewise,

$$\begin{aligned}\left\|\frac{U_I^{m+1}-U_I^m}{\Delta t}\right\|_h^2 &\leq \frac{4}{h^2}\|D_{I^-}^m\delta_x^- U_R^{m+\theta} + D_{R^-}^m\delta_x^- U_I^{m+\theta}\|_{h^*}^2\\ &= \frac{4}{h^2}\left(\|D_{I^-}^m\delta_x^- U_R^{m+\theta}\|_{h^*}^2 + \|D_{R^-}^m\delta_x^- U_I^{m+\theta}\|_{h^*}^2\right)\\ &\quad +\frac{8}{h}\sum_{j=1}^{N} D_{R^-}^m\delta_x^- U_R^{m+\theta} D_{I^-}^m\delta_x^- U_I^{m+\theta}. \qquad (11)\end{aligned}$$

From (9), (10) and (11), using the assumption (2), we obtain

$$\left\|\frac{U^{m+1}-U^m}{\Delta t}\right\|_h^2 \leq \frac{4}{h^2}\max\left(D_R+|D_I|\right)\|(D_{R^-}^m)^{1/2}\delta_x^- U^{m+\theta}\|_{h^*}^2.$$

Then, (8) implies that

$$\begin{aligned}&\frac{\|U^{m+1}\|_h^2-\|U^m\|_h^2}{2\Delta t}\\ &+\left(1-\frac{2\Delta t(1-2\theta)\max\left(D_R+|D_I|\right)}{h^2}\right)\|(D_{R^-}^m)^{1/2}\delta_x^- U^{m+\theta}\|_{h^*}^2 \leq 0.\end{aligned}$$

If we assume that

$$\Delta t \leq \frac{h^2}{2(1-2\theta)\max\left(D_R+|D_I|\right)} \qquad (12)$$

then

$$\|U^{m+1}\|_h^2 \leq \|U^m\|_h^2, \quad m=0,\ldots,M-1.$$

We conclude that, in the case $\theta \in [0,\frac{1}{2}[$, the scheme (6) is stable provided that the condition (12) holds.

Remark 2. *In the case of Dirichlet boundary conditions ($\beta = 0$), we obtain a similar stability result defining the following discrete L^2 inner product*

$$(U, V)_h = \sum_{j=1}^{N-1} hU_j\overline{V}_j.$$

5. Numerical results

We have implemented the explicit scheme (5) in 1D and 2D to illustrate the stability condition in Theorem 1, considering homogeneous Dirichlet boundary conditions. We considered $\vartheta = \pi/180$, $\kappa = 10$ for the complex diffusion filter (1), which is commonly used to remove noise while preserving edges in image processing.

5.1. Examples in 1D. To this end, we considered a characteristic function scaled to the image color scale (that is, 255) as initial condition

$$U(x, 0) = \begin{cases} 255, & 8 \leq x \leq 15 \\ 0, & \text{otherwise.} \end{cases}$$

We considered the discretization points $x_j = j$, $j = 1, 2, \ldots, 25$, that is, $h = 1$. From Theorem 1, the stability condition is achieved if $\Delta t \leq 0.50865$.

In figure 1 we present reconstructions with several time steps Δt in order to illustrate the stability condition in Theorem 1. It is clear that the reconstructions with time steps larger than the stabily condition allow (dotted lines) present instability, while the ones that satisfy the stability condition (full lines) do not. In this case, it is the first iteration that presents the higher value for the right hand side of the stability condition (1) therefore the instability is attenuated as the diffusion time increases.

To illustrate this, we plot the first two iterations with time steps around the cut-off for stability $\Delta t = 0.50865$.

We also considered a profile with ramp edges, namely considering a initial condition of

$$U(x, 0) = \begin{cases} 255, & 10 \leq x \leq 15 \\ 125, & 15 < x \leq 20 \wedge 30 \leq x \leq 40 \\ \frac{255*(30-x)+125*(x-20)}{10}, & 20 < x < 30 \\ 0, & \text{otherwise.} \end{cases}$$

and discretization points $x_j = j$, $j = 1, 2, \ldots, 51$, again with $h = 1$. The stability condition stays the same in this case. It is again clear in figures 3

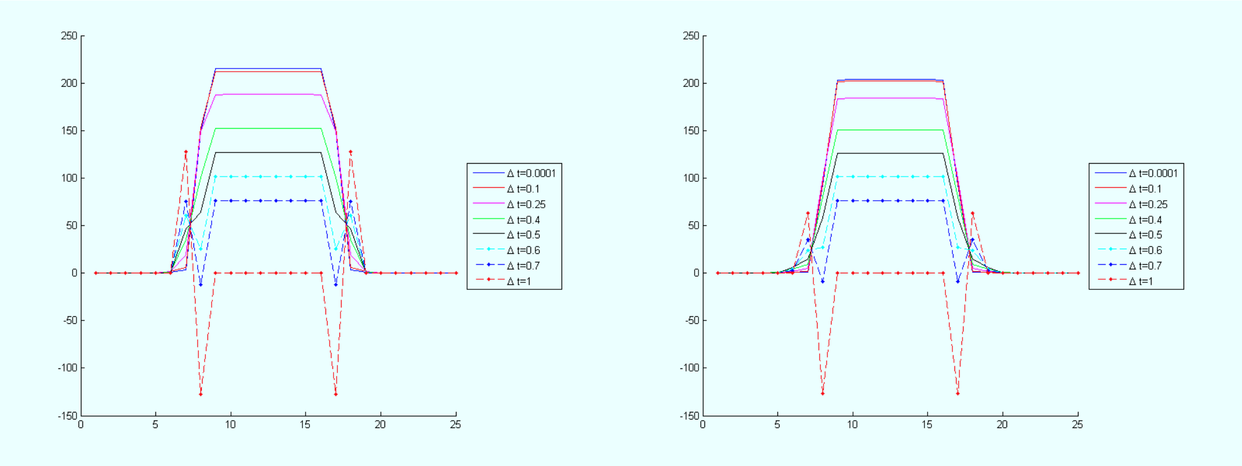

FIGURE 1. Numerical solution for the explicit method with different time step and diffusion times of $T = 1$ (left) and $T = 2$ (right).

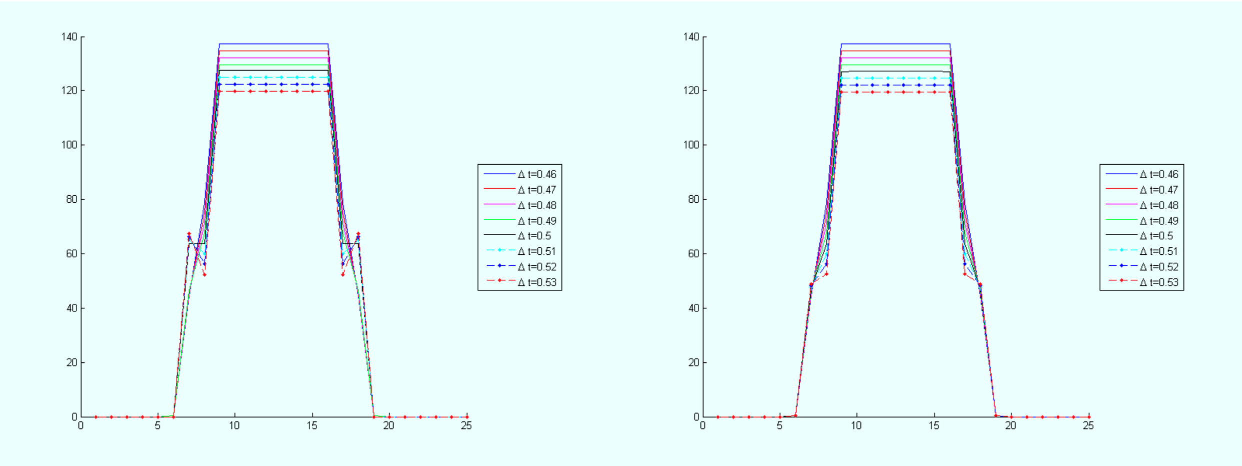

FIGURE 2. First (left) and second (right) iteration for the explicit method with time steps close to the cut-off value for stability $\Delta t = 0.50865$.

and 4 that the instability arise again in areas with strong edges whenever the stability condition is not fulfilled, while the scheme is stable if the condition is satisfied.

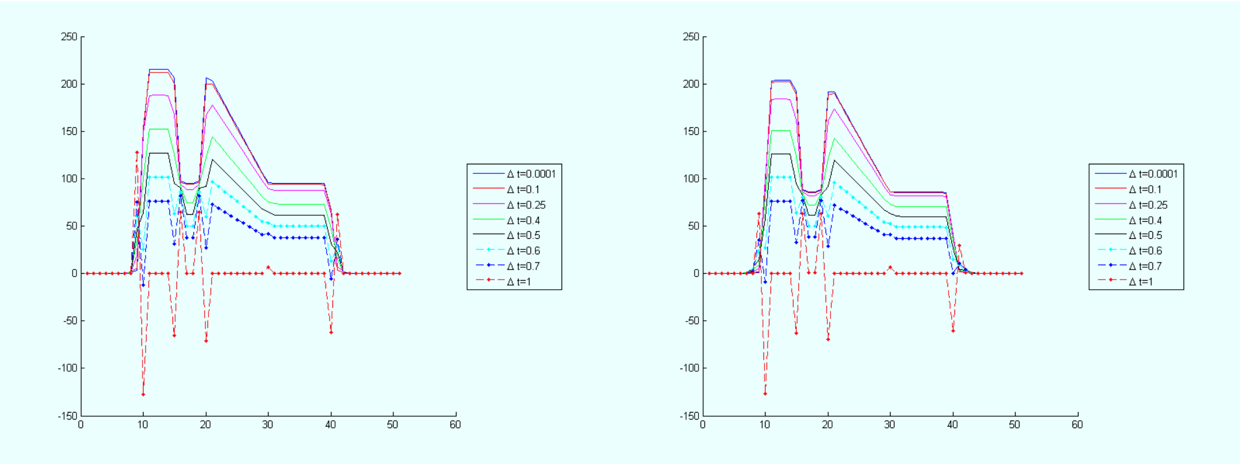

FIGURE 3. Numerical solution for the explicit method with different time step and diffusion times of $T = 1$ (left) and $T = 2$ (right).

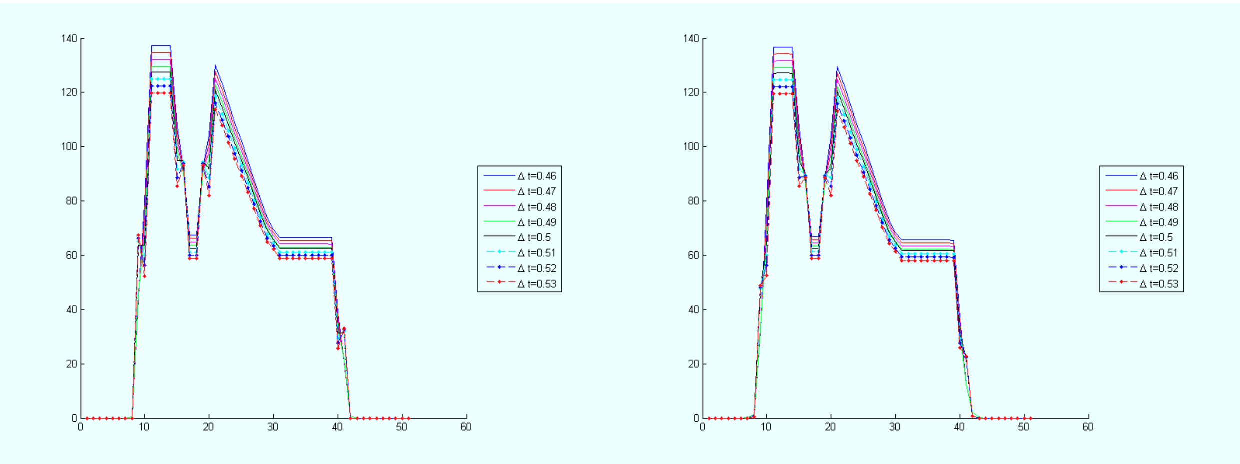

FIGURE 4. First (left) and second (right) iteration for the explicit method with time steps close to the cut-off value for stability $\Delta t = 0.50865$.

5.2. Example in 2D. We considered

$$U(x,0) = \begin{cases} 240, & x_1 = x_2 = 10 \\ 150, & 13 < x_1 \leq 14 \wedge 13 \leq x_2 \leq 14 \\ 0, & \text{otherwise.} \end{cases}$$

and discretization points $x = (i,j)$, $i,j = 1,2,\ldots,21$, again with unitary spatial spacement $h_1 = h_2 = 1$, that give rise to an image that we affected by additive noise. Reconstructions using the explicit finite difference scheme are

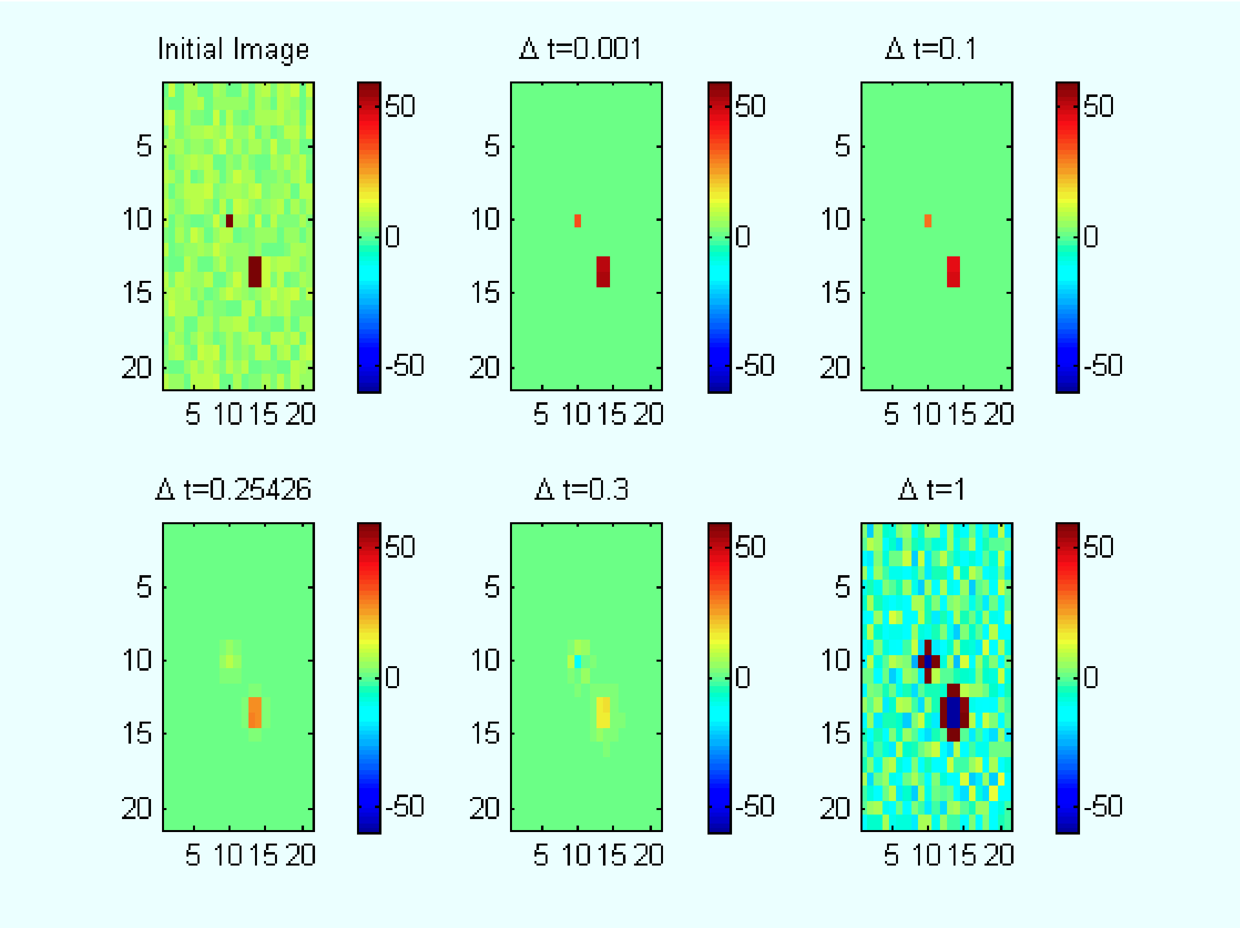

FIGURE 5. Approximation with the explicit method with different time steps for a diffusion time of $T = 1$.

presented in figures 5 and 6, illustrating the stability condition $\Delta t \leq 0.25432$ given by Theorem 1 for this case.

The figures clearly illustrate the stability condition. In fact, when a time step greater than the cut-off is used, the image has pixels with negative values, which do not occur in the original noisy image. These examples in 1D and 2D might even suggest that the stability condition obtained in theorem 1 additionally to be sufficient is actually also necessary, considering that the spatial step is equal in all directions.

Two more remarks should be made in what concerns the numerical application of this scheme to image denoising. The first is that the time step used should not be close to the cut-off value for stability. The examples illustrate that in this case, though stability is assured in the sense that the L^2-norm of the image does not explode, the image features are not preserved. In this

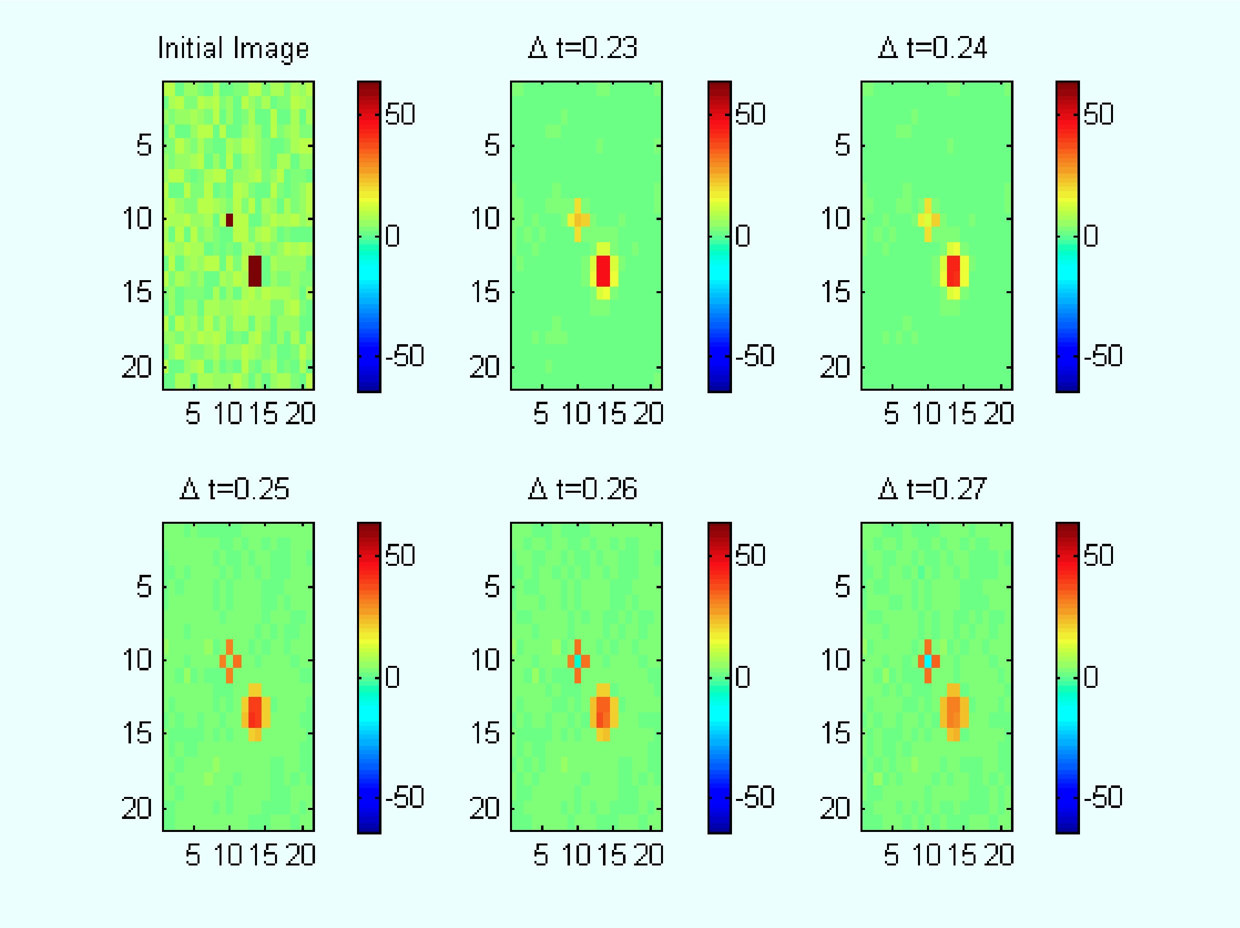

FIGURE 6. First iteration for the explicit method with time steps close to the cut-off value for stability $\Delta t = 0.25432$.

way, a time step around half of the cut-off value seems to be a better choice to this purpose. The second remark is that it seems also profitable to use an adaptative time step along iterations, since the cut-off value for stability seems to be smaller at early iterations and may allow the numerical scheme to take higher steps in time as iterations go by. In this way, the time step should be computed at each iteration having in mind the stability condition in Theorem 1.

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