Relating Lagrangian and Hamiltonian Formalisms of *LC* Circuits

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***Abstract*—The Lagrangian formalism defined by Scherpen *et al.*for (switching) electrical circuits, is adapted to the Lagrangian formalism defined on Lie algebroids. This allows us to define regular Lagrangians and consequently, well-defined Hamiltonian descriptions of arbitrary *LC* networks. The relation with other Hamiltonian approaches is also analyzed and interpreted.**

***Index Terms*—Hamiltonian formulation, Lagrangian formulation, *LC* circuits, Lie algebroids.**

I. Introduction

In the last few years, an evident interest for the Lagrangian and Hamiltonian description of electrical circuits has arisen in the literature [2], [7]–[9], [14], [16]. A recent Lagrangian description [14]–[16] leads to a successful picture of *LC* dynamics and provides a step-by-step construction for the description of the components, the definition of the Lagrangian, and the corresponding Euler–Lagrange dynamics. Kirchhoff's current law defines a set of constraints for the corresponding Lagrangian system while the corresponding voltage law defines the Euler–Lagrange equations for the system.

Regarding the Hamiltonian description of the dynamics of electrical circuits, a recent and successful approach is based on the concept of Dirac structures and port-controlled Hamiltonian systems [7], [18]. This approach also provides a suitable description of the dynamics of the system.

It seems quite natural to compare both approaches and to try to relate the solutions of both methods for electrical circuits. Since dissipative elements can be viewed as external elements, we only consider electrical *LC* circuits here. The formulation of both frameworks is done in $\mathbb{R}^n$ spaces and, hence, the canonical procedure would suggest to use the Legendre transform to go from dynamics given by the Lagrangian formalism into dynamics given by the Hamiltonian formalism, and vice versa. The problem in this case is that the Lagrangian formalism proposed in [14]–[16], yields a singular Lagrangian description, which makes the Legendre transform ill defined and, thus, no straightforward Hamiltonian formulation can be related. In this brief, we intend to complement the original Lagrangian picture proposed in [14]–[16], with a procedure that transforms the singular Lagrangian system into a regular Lagrangian system. Then, the Lagrangian system can be related with a Hamiltonian system by using a well-defined Legendre transform. The main new ingredient of the approach is the use of Lie algebroids in the description. A Lie algebroid is a geometrical object which generalizes the concept of tangent bundles (which is the natural framework of usual Lagrangian mechanics) such that a Lagrangian formulation on them is still possible [6], [19]. Essentially, we just need one of the simplest examples of the Lie algebroid, namely an integrable subbundle of a tangent bundle, which in the case of electrical *LC* circuits is even a vector space.

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The structure of the paper is as follows. Section II introduces,very briefly, the Lagrangian formulation presented in [14]–[16]. In Section III, we present the main theoretical aspects of our paper and we conclude with an example.

II. Lagrangian Formulation

In this section, we present the Lagrangian formalism as introduced in [14]–[16] for the topologically complete class (e.g., [20]) of (switching) electrical networks. The basic construction is as follows.

- The system is described as a dynamical system defined on $\mathcal{T}\mathbb{R}^n \sim \mathbb{R}^{2n}$, where $n-p$ is the number of inductances of the network and p the number of capacitors.
- Each element has two variables associated, a charge (generalized position) q_L or q_C (for inductances and capacitors, respectively), and a current (generalized velocity) $\dot{q}_L$ and $\dot{q}_C$.
- Kirchhoff's current law is represented by a set of linear algebraic equations in the generalized velocities, what can be considered as a set of holonomic constraints on $\mathcal{T}\mathbb{R}^n \sim \mathbb{R}^{2n}$, i.e., the set of velocities that satisfy these constraints form a subalgebra of the algebra of vector fields on $\mathbb{R}^n$. The dynamics are thus not defined on the whole space $\mathbb{R}^{2n}$ but in the subspace $\mathcal{A} \subset \mathcal{T}\mathbb{R}^n$ defined by the velocities which satisfy the set of constraints.
- Dynamics are introduced in the Lagrangian formalism by defining the Lagrangian function which includes the electrical and the magnetic (co)energy, in the form

$$L(q,\, \dot{q}) = \sum_{i=1}^{n-p} \frac{1}{2}\, L_i \dot{q}_{L_i}^2 - \sum_{j=n-p+1}^{n} \frac{1}{2C_j}\, q_{C_j}^2. \tag{1}$$

  From the Lagrangian, it can be directly seen that a singularity occurs in the system, namely the kinetic term $T = \sum_{i=1}^{n-p}(1/2)L_i\dot{q}_{L_i}^2$ does not contain any term involving the velocities (currents) on the capacitors, what implies that thinking in mechanical terms, the mass matrix of the system is singular. This yields that the Lagrangian is not regular and that the Legendre transformation connecting it with the Hamiltonian formalism is not well defined.

The corresponding Euler–Lagrange equations provide the suitable dynamical description, by including voltage sources as external forces and, eventually, Rayleigh dissipation terms to include resistive elements. The Euler–Lagrange equations are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\,(q,\dot{q})\right) - \frac{\partial L}{\partial q}\,(q,\dot{q}) = \mathcal{F}_e + A(q)\lambda + \frac{\partial \mathcal{D}}{\partial \dot{q}}\,(\dot{q}) \tag{2}$$

$$A(q)\dot{q} = 0 \tag{3}$$

where $\mathcal{F}_e$ denotes the external voltage sources, $A(q)$ the constraint forces and $\mathcal{D}(\dot{q})$ the Rayleigh dissipation function. In the above formulation we can recognize the Kirchhoff laws for the circuit, namely the dynamical equation corresponds to the voltage law and the constraint equation to the current law.

Note 2.1: Please note that this method (or the Hamiltonian discussed later) does not yield other dynamical behavior than the classical network theoretic methods, e.g., mesh current and node voltage analysis, e.g., [1]. For a more extensive treatment of the relations (also with methods like bond graphs), we refer to [15]. The main motivation of our energy based formulations is given by the explicit passive form, which is beneficial for analysis, design, and control purposes.

III. From Lagrangian to Hamiltonian Formulation of *LC* Circuits via Algebroids

The description proposed in Section II is very useful if we want a purely Lagrangian description of the dynamics. But if we are trying to relate this construction with other approaches based on a Hamiltonian formulation (for example, the approach based on Dirac structures and port controlled Hamiltonian systems as in [17]), it has a clear disadvantage: the Lagrangian (1) is singular when formulated on the full tangent bundle $\mathcal{T}\mathbb{R}^n$, as we saw above. Hence, the Legendre transform can not be directly computed. Then the definition of the Hamiltonian is, in general, involved. The use of Lie algebroids solves this problem: since we remove the singular variables when quotienting the constraints, the transition to the Hamiltonian formalism is canonical and straightforward.

A. What is a Lie Algebroid?

The intuitive image of a Lie algebroid is a geometric object which intertwines between tangent bundles (in mechanical terms, the space of coordinates and velocities of a system) and Lie algebras. The concept of Lie algebroids has been used in the last 50 years in the algebraic geometric framework, under different names (see [3]–[5][10], [11], and [13]); but, the first proper definition, from the point of view of differential geometry, is due to Pradines [12].

We now summarize the main properties of these objects. First of all, a Lie algebroid is a vector bundle, i.e., a manifold $\mathcal{A}$ which, locally, can be seen as a product of a manifold of smaller dimension (called the base which we denote as M) and a vector space (which is called the fiber and in the following assumed to be $\mathbb{R}^n$). The best known example of a vector bundle (which also happens to be a Lie algebroid) is the tangent bundle, i.e., the geometric object defined as the ensemble of all the coordinates and velocities of a mechanical system. If we assign a point of the fiber to each point of the base manifold, we are defining a so-called section of $\mathcal{A}$. The natural image of this is a set of vector fields defined on a manifold (they define a velocity vector at each point of the manifold).

Consider now the set of all the sections of the vector bundle $\mathcal{A}$ (in our example above, the set of all vector fields of a manifold). Assume that we can endow that set with a Lie algebra structure, i.e., a skew-symmetric operation which satisfies the Jacobi identity, as it happens with the commutator of vector fields.

Assume that we can realize the sections of $\mathcal{A}$ as true vector fields on the base manifold M (in the case of the tangent bundle this is a trivial issue, but it is not for more general situations). This is done by means of a so-called anchor mapping ρ: $A \rightarrow TM$ that satisfies a couple of conditions. The first condition requires that the Lie algebra structure of the sections of $\mathcal{A}$ and the natural Lie algebra structure of vector fields defined by the commutator are *homomorphic*, i.e., the image by ρ of a bracket of any two sections σ_1, σ_2 is equal to the commutator of the images $\rho(\sigma_1), \rho(\sigma_2)$, or in more algebraic terms $\rho([\sigma_1, \sigma_2]) = [\rho(\sigma_1), \rho(\sigma_2)]$. The second condition requires that the Lie algebra structure of the sections of $\mathcal{A}$ satisfies $[\eta, f\xi] = [\eta, \xi] + (\rho(\eta)f)\xi \, \forall \eta, \xi \in \Gamma(\mathcal{A}), \forall f \in C^\infty(M)$.

Any $\mathcal{A}$ satisfying the conditions above is called a Lie algebroid. The vector bundle A, the Lie algebra structure of the space of sections $[\cdot, \cdot]$ and the anchor mapping ρ characterizes the Lie algebroid we need. These three elements identify the structure, and hence, when using coordinates, we need the set of coordinates of $\mathcal{A}$, $\{x^i, \lambda^\alpha\}$, i.e., coordinates for the base points and for the fiber points, the structure functions for the Lie algebra structure $\{C^\gamma_{\alpha,\beta}\}$, and the coordinate expression of the anchor mapping $\{\rho^i_\alpha\}$.

B. Mechanics on Lie Algebroids

An interesting property of Lie algebroids is that the dual bundle $\mathcal{A}^*$ (i.e., the vector bundle defined with the same base and with the dual vector space as fiber and, therefore, exhibits a natural duality product with $\mathcal{A}$) admits a canonical Poisson tensor. The corresponding coordinate expression, with the coordinates $\{x^i, \mu_\alpha\}$ for $\mathcal{A}^*$ are

$$\{x^i, x^j\} = 0 \quad \{x^i, \mu_\alpha\} = \rho^i_\alpha \quad \{\mu_\alpha, \mu_\beta\} = C^\gamma_{\alpha\beta}\mu_\gamma. \tag{4}$$

The reason for its importance is the following. Being an object fairly similar to a tangent bundle, it is reasonable to try to extend the usual Lagrangian formalism (which is naturally defined on them) to the case of Lie algebroids. Indeed, such construction is possible and has been introduced in two different approaches (see [6] and [19]). Referring to the references for the details, we will extract only the expression of the Euler–Lagrange equations corresponding to a regular Lagrangian function $L \in C^\infty(A)$

$$\begin{aligned} \frac{dq^i}{dt} &= \{q^i, E_L\} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \lambda^\alpha}\right) &= \left\{\frac{\partial L}{\partial \lambda^\alpha}, E_L\right\} \end{aligned} \tag{5}$$

where $E_L = \sum_\alpha \lambda^\alpha(\partial L/\partial\lambda^\alpha) - L$ and the bracket $\{\cdot, \cdot\}$ corresponds to the Poisson bracket (4).

C. The Case of Electrical Circuits: The Algebroid

The Lie algebroids we are interested in now are defined by the set of coordinates and admissible velocities of a constrained system motivated by Section II, i.e., electrical circuits with the natural set of generalized coordinates and velocities and the constraints defined by Kirchoff's current law. We can define a new vector bundle by taking the generalized coordinates and the set of admissible velocities only. Being velocities (currents), the sections heritage the natural Lie algebraic structure of the total case. When the set of admissible generalized velocities is closed under this Lie bracket, the constraints are called holonomic and the bundle is a Lie algebroid (see [19]).

In Section II, we saw that the singularities of the Lagrangian (1) are the source of our problem with the description of the circuit. As we are still able to define a Lagrangian formalism on a Lie algebroid, it is natural to ask whether it is possible to obtain a Lie algebroid which describes the circuit such that the corresponding Lagrangian description is regular. Roughly speaking, the idea is to use the constraints to remove the generalized velocities corresponding to the capacitors. We thus define the coordinates λ^α of the fiber of the algebroid as

$$\lambda^\alpha = \dot{q}^L_\alpha - \sum_{j=n+p+1}^{n}\left(\dot{q}^C_\alpha \cdot \hat{\phi}_j(\dot{q})\right)\hat{\phi}_j(\dot{q})$$

where $\{\hat{\phi}_j\}$ are the expressions of the hyperplanes which define the constraints (they are simply the expression of Kirchoff's current laws, i.e., a linear combination of currents equal to zero).

With such definition, the anchor mapping is

$$\rho^\alpha_i = \left(\dot{q}^A_\alpha - \sum_{j=n+p+1}^{n}\left(\dot{q}^A_\alpha \cdot \hat{\phi}_j(\dot{q})\right)\hat{\phi}_j(\dot{q})\right)_i. \tag{6}$$

Finally, the structure functions fulfill $\{C^\gamma_{\alpha\beta} = 0\}$, since the algebra of the constraints is commutative.

Note 3.1: In the construction of this basis, we are assuming that the projection of the elements $\dot{q}^L_j$ on the algebroid can be defined. In order to do that, we need a Riemannian metric on the manifold. We can take either the natural one (which defines the kinetic energy of the Lagrangian) or any other metric (a Euclidean metric is the simplest choice). It is convenient to remark that since the Lie algebroid and its dynamics is a well-defined intrinsic object, this choice of the metric would only affect the particular coordinate expression of the equations but not its geometrical meaning. Any two choices would define equivalent systems of equations.

D. The Case of Electrical Circuits: The Dynamics

The Lagrangian for the Lie algebroid is defined by the restriction of the original Lagrangian to this subbundle of admissible velocities. The condition for the Lagrangian defined on the algebroid to be regular is easily expressed in terms of the number of capacitors and the number of constraints of our system.

Theorem: The Lagrangian defined on the Lie algebroid is regular if the number of capacitors equals the number of independent constraints involving the currents through the capacitive branches.

Proof: The singularities of the original Lagrangian arise from the lack of kinetic terms for the capacitors, and we can remove one degree of freedom with each constraint function. Hence, we need as many constraints as capacitor velocities we must remove. This condition is naturally fulfilled by the actual electrical networks described in this framework. The variable which represents the current intensity through the capacitor is meaningless from a physical point of view and, hence, if the original description is physically well defined, it is possible to remove them from the equations by a corresponding Kirchhoff current law. Therefore, there must be as many independent constraints involving capacitors as the number of capacitors of the network. □

In the following, we assume that the condition of the theorem above is fulfilled. In such a case, we obtain, by restricting the original Lagrangian (1) to the Lie algebroid, a regular Lagrangian

$$L^A(q,\,\lambda) = \sum_{\alpha,\,\beta=1}^{n-p} \frac{1}{2}(\lambda^\alpha)^T M_{\alpha\beta}\lambda^\beta - \sum_{j=n-p+1}^{n} \frac{1}{2C_j}\, q_{C_j}^2 \qquad (7)$$

where M is the mass matrix restricted to the subbundle of admissible velocities.

Euler–Lagrange equations for the Lie algebroid provide the desired dynamics, treated now as an unconstrained system

$$\frac{dq^i}{dt} = \rho_\alpha^i(\lambda^\alpha) \qquad M_{\alpha\beta}\dot{\lambda}^\beta = \rho_j^\alpha\, \frac{q_{C_j}}{2C_j} \qquad (8)$$

where the constraints are now encoded in the anchor mapping ρ_j^α, which is given by (6).

The definition of the Hamiltonian formalism is now straightforward, since the Legendre transform is well defined. The Poisson structure takes the form

$$\begin{aligned} \{x^i, x^j\} &= 0 \\ \{\mu_\alpha,\, \mu_\beta\} &= 0 \\ \{x^i, \mu_\alpha\} &= \left(\dot{q}_\alpha^L - \sum_{j=n+p+1}^{n} \left(\dot{q}_\alpha^L \cdot \phi_j(\dot{q})\right)\phi_j(\dot{q})\right)_i. \end{aligned} \qquad (9)$$

Hence, the Lagrangian formalism on Lie algebroids provides a direct solution for any *LC* circuit, and a direct Hamiltonian implementation. A problem with the approach above is that, though it is very natural from the geometrical point of view, it loses the physical interpretation of the original singular model. This is due to the definition of the dynamics directly on the constrained manifold. The differential geometric dynamics takes place there, but then, the physical components (inductances, capacitors, etc.) are mixed; that is the reason why the mass matrix of the Lagrangian restricted to the constrained manifold is no longer diagonal. In case of Lagrangian modeling, the subset of the state space with a physical meaning is denoted by $\mathcal{F}$. $\mathcal{F}$ contains the base coordinates (charges corresponding to inductive and capacitive elements) and the inductance velocities. Nonetheless, since the solutions corresponding to the regular Lagrangian are defined on the algebroid, which is nothing but a hyperplane on the original $\mathbb{R}^{2n}$ manifold, we can define a transformation which maps the curves that are

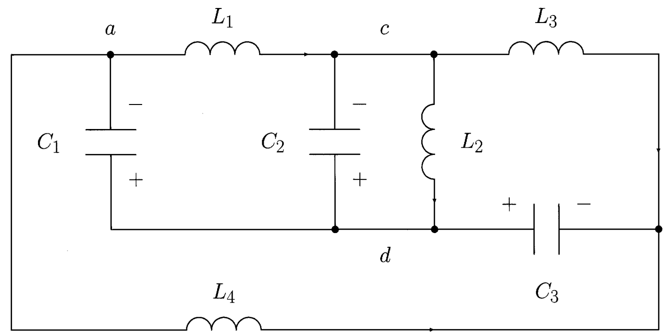

Fig. 1. Electrical circuit of the example.

solutions of the dynamics on the constrained manifold to curves defined on $\mathcal{F}$. It is trivial to check that $\mathcal{F}$ is also a Lie algebroid. To connect both algebroids, we define the transformation

$$\Psi\colon \mathcal{A} \to \mathcal{F}, \qquad \Psi(q,\,\lambda^1,\,\ldots\lambda^p) = (q,\,\dot{q}^1\cdots\dot{q}^p) \qquad (10)$$

where we denote by $\dot{q}^i$ the elements of $\mathcal{F}$ (considered as a subset of $\mathbb{R}^{2n}$). The transformation is the inverse of the anchor mapping (6) restricted to $\mathcal{F} \subset \mathcal{T}\mathbb{R}^n$.

Hamiltonian dynamics arise straightforwardly from this framework. Again Hamiltonian dynamics, which incorporate the constraints in the Poisson tensor, have a clear geometrical meaning, but are not meaningful from the physical point of view. In the Hamiltonian formalism, the variables with physical meaning are the charge on the capacitors and the magnetic flux of the inductances. This space is precisely the dual of the space $\mathcal{F}$ and we denote it by $\mathcal{F}^*$ (for simplicity, we are including the charge of the inductances which is also physically meaningless but much easier to handle). We can define the restriction of the Lagrangian to $\mathcal{F}$, that we denote by $L^{\mathcal{F}}$. This Lagrangian is regular on $\mathcal{F}$, but we cannot write the dynamics of the circuit as its Euler–Lagrange equations because they do not include the constraints. And it is not possible to include them because the constraints can not be written on $\mathcal{F}$.

Similar to the Lagrangian formalism, we cannot define the Hamiltonian dynamics on $\mathcal{F}^*$ by using the natural Hamiltonian function, since the constraints can not be written on it. Again, the Hamiltonian formulation on the Lie algebroid has the advantage of encoding the constraints in the Poisson structure itself (because it depends of the anchor mapping ρ which is defined by them) and, hence, it provides directly the differential geometric dynamics of the system, treated as an unconstrained system on $\mathcal{A}^*$. In order to relate this dynamics with the physically meaningful dynamics defined on $\mathcal{F}^*$, we use the transformation

$$\begin{aligned} \Phi &= (\Psi^{-1})^*\colon \mathcal{A}^* \to \mathcal{F}^* \\ \Phi(q,\,\mu_1,\,\ldots\mu_p) &= (q,\,\mu_{L_1}\cdots\mu_{L_p}). \end{aligned} \qquad (11)$$

E. Example

Consider the circuit shown in Fig. 1.

The Lagrangian is given by

$$L(q,\,\dot{q}) = \frac{1}{2}\sum_{i=1}^{4} L_i\dot{q}_{L_i}^2 - \frac{1}{2}\sum_{i=1}^{3}\frac{1}{C_i}\, q_{C_i}^2.$$

The set of independent constraints can be written as

$$\begin{aligned} \dot{q}_{C_1} - \dot{q}_{L_1} - \dot{q}_{L_4} &= 0 \\ \dot{q}_{C_3} + \dot{q}_{L_3} + \dot{q}_{L_4} &= 0 \\ \dot{q}_{C_2} + \dot{q}_{L_1} - \dot{q}_{L_2} - \dot{q}_{L_3} &= 0. \end{aligned}$$

Since we have more than one constraint, we proceed to orthonormalize the system of vectors. Following the Gram–Schmidt method we find

$$\phi_1 = \left(\frac{1}{\sqrt{3}}, 0, 0, -\frac{1}{\sqrt{3}}, 0, 0, -\frac{1}{\sqrt{3}}\right)$$
$$\phi_2 = \left(\frac{1}{2\sqrt{6}}, 0, \sqrt{\frac{3}{8}}, -\frac{1}{2\sqrt{6}}, 0, \sqrt{\frac{3}{8}}, \frac{1}{\sqrt{6}}\right)$$
$$\phi_3 = \left(\frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{2\sqrt{3}}, 0\right)$$

for the normal vectors of the hyperplanes written in the basis $\{\dot{q}_{C_1}, \dot{q}_{C_2}, \dot{q}_{C_3}, \dot{q}_{L_1}, \dot{q}_{L_2}, \dot{q}_{L_3}, \dot{q}_{L_4}\}$. The projection of the inductance currents defines a basis for the algebroid and, according to it, we obtain the expression for the anchor mapping

$$\rho = \begin{pmatrix} \frac{7}{24} & -\frac{1}{6} & \frac{1}{24} & \frac{13}{24} & \frac{1}{6} & \frac{5}{24} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} & -\frac{1}{6} & 0 \\ -\frac{1}{24} & \frac{1}{6} & -\frac{7}{24} & \frac{5}{24} & -\frac{1}{6} & \frac{13}{24} & -\frac{1}{4} \\ \frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

If we restrict the Lagrangian to this subspace of velocities, the mass matrix becomes $M = \{m_{ij}\}_{i,j=1,\ldots,4}$, where

$$m_{11} = \frac{169\, L_1}{576} + \frac{L_2}{36} + \frac{25\, L_3}{576} + \frac{L_4}{16}$$
$$m_{12} = \frac{1}{2}\left(\frac{13\, L_1}{72} + \frac{2\, L_2}{9} - \frac{5\, L_3}{72}\right)$$
$$m_{13} = \frac{1}{2}\left(\frac{65\, L_1}{288} - \frac{L_2}{18} + \frac{65\, L_3}{288} + \frac{L_4}{8}\right)$$
$$m_{14} = \frac{1}{2}\left(\frac{-13\, L_1}{48} - \frac{5\, L_3}{48} - \frac{L_4}{4}\right)$$
$$m_{21} = \frac{1}{2}\left(\frac{13\, L_1}{72} + \frac{2\, L_2}{9} - \frac{5\, L_3}{72}\right)$$
$$m_{22} = \frac{L_1}{36} + \frac{4\, L_2}{9} + \frac{L_3}{36}$$
$$m_{23} = \frac{1}{2}\left(\frac{5\, L_1}{72} - \frac{2\, L_2}{9} - \frac{13\, L_3}{72}\right)$$
$$m_{24} = \frac{1}{2}\left(\frac{-L_1}{12} + \frac{L_3}{12}\right)$$
$$m_{31} = \frac{1}{2}\left(\frac{65\, L_1}{288} - \frac{L_2}{18} + \frac{65\, L_3}{288} + \frac{L_4}{8}\right)$$
$$m_{32} = \frac{1}{2}\left(\frac{5\, L_1}{72} - \frac{2\, L_2}{9} - \frac{13\, L_3}{72}\right)$$
$$m_{33} = \frac{25\, L_1}{576} + \frac{L_2}{36} + \frac{169\, L_3}{576} + \frac{L_4}{16}$$
$$m_{34} = \frac{1}{2}\left(\frac{-5\, L_1}{48} - \frac{13\, L_3}{48} - \frac{L_4}{4}\right)$$
$$m_{41} = \frac{1}{2}\left(\frac{-13\, L_1}{48} - \frac{5\, L_3}{48} - \frac{L_4}{4}\right)$$
$$m_{42} = \frac{1}{2}\left(\frac{-L_1}{12} + \frac{L_3}{12}\right)$$
$$m_{43} = \frac{1}{2}\left(\frac{-5\, L_1}{48} - \frac{13\, L_3}{48} - \frac{L_4}{4}\right)$$
$$m_{44} = \frac{L_1}{16} + \frac{L_3}{16} + \frac{L_4}{4}.$$

The projection onto the physical space $\mathcal{F}$ whose fiber is defined by the set $\{\dot{q}_{L_1}, \dot{q}_{L_2}, \dot{q}_{L_3}, \dot{q}_{L_4}\}$

$$\Psi = \begin{pmatrix} 13/24 & 1/6 & 5/24 & -1/4 \\ 1/6 & 2/3 & -1/6 & 0 \\ 5/24 & -1/6 & 13/24 & -1/4 \\ -1/4 & 0 & -1/4 & 1/2 \end{pmatrix}.$$

The Euler Lagrange equations on the algebroid provide a set of equations for the base coordinates $\{q\}$ and for the fiber variables $\{\lambda^\alpha\}$, but as we saw, it is not physically meaningful. Nonetheless, if we map these dynamics on $\mathcal{F}$ we obtain

$$\ddot{q}_{L_1} = \frac{q_{C_2}}{L_1 C_2} - \frac{q_{C_1}}{L_1 C_1}$$
$$\ddot{q}_{L_2} = \frac{q_{C_2}}{L_2 C_2}$$
$$\ddot{q}_{L_3} = -\frac{q_{C_2}}{L_3 C_2} + \frac{q_{C_3}}{L_3 C_3}$$
$$\ddot{q}_{L_4} = \frac{q_{C_3}}{L_4 C_3} - \frac{q_{C_1}}{L_4 C_1}$$

for the fibers and

$$\frac{dq_{C_1}}{dt} = \dot{q}_{L_1} + \dot{q}_{L_4}$$
$$\frac{dq_{C_3}}{dt} = -\dot{q}_{L_3} - \dot{q}_{L_4}$$
$$\frac{dq_{C_2}}{dt} = -\dot{q}_{L_1} + \dot{q}_{L_2} + \dot{q}_{L_3}$$

for the base coordinates. (we omit the trivial relations in the form $dq_{L_i}/dt = \dot{q}_{L_i}$).

The corresponding Hamiltonian dynamics can now be obtained by performing the Legendre transformation for the algebroid. But again, the dynamics are not meaningful and must be transferred to $\mathcal{F}^*$ by the mapping Ψ^*.

The result

$$\dot{\varphi}_{L_1} = \frac{q_{C_2}}{C_2} - \frac{q_{C_1}}{C_1}$$
$$\dot{\varphi}_{L_2} = \frac{q_{C_2}}{L_2 C_2}$$
$$\dot{\varphi}_{L_3} = -\frac{q_{C_2}}{C_2} + \frac{q_{C_3}}{C_3}$$
$$\dot{\varphi}_{L_4} = \frac{q_{C_3}}{C_3} - \frac{q_{C_1}}{C_1}$$
$$\dot{q}_{C_1} = \frac{1}{L_1}\varphi_{L_1} - \frac{1}{L_4}\varphi_{L_4}$$
$$\dot{q}_{C_3} = -\frac{1}{L_3}\varphi_{L_3} - \frac{1}{L_4}\varphi_{L_4}$$
$$\dot{q}_{C_2} = -\frac{1}{L_1}\varphi_{L_1} + \frac{1}{L_2}\varphi_{L_2} + \frac{1}{L_3}\varphi_{L_3}$$

coincides (excepting the trivial equations for the currents of the inductances) with the solutions of [8].

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Periodic Solution in Multimachine Power Systems Affected by Perturbation of Nonlinear Loads

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***Abstract*—Some results of higher dimensional space periodic solution of differential and algebraic systems are presented. The structure preserving models are considered for power systems with loads influenced by periodic disturbance. The existence and uniqueness of periodic solutions for multimachine power systems with nonlinear loads are given. The existent conditions of periodic solution for multimachine power systems with constraints are obtained. Some examples are given to illustrate the feasibility by higher dimensional space periodic solution theory.**

***Index Terms*—Differential and algebraic systems (DAS), existence, periodic perturbation, periodic solution, uniqueness.**

I. Introduction

The dynamics of a large class of power systems can be modeled by using nonautonomous differential and algebraic systems (DAS). As a result of the extension of power systems in scale and the variety of loads, in some cases, occurrence of power oscillation increases, the perturbation is aroused by periodic disturbance of loads.

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In power systems, the dynamical analysis on nonlinear model are mainly the assessments of transient stability and voltage stability [1]–[4]. Problems associated with steady-state stability and voltage collapse become more significant in recent years and consequently attracted the attention of many investigators. Low voltage can result in loss of stability, voltage collapse and ultimately to cascading power outages. Electric power systems operate under the influence of numerous parameters that vary with time and circumstances. As these parameters change, the power system may lose stability in one of two ways: either with the abrupt appearance of self-sustained oscillations or with the disappearance of the equilibrium point. Usually, the latter is considered to be a static bifurcation.

In studying power-system stability, it is used to analyze by classical generators model. For such models, there is a well-defined energy function consisting of a kinetic-energy part and a potential-energy part [5]–[7]. The major difficulty to the integral to the energy is that the transmission system is not lossless, i.e., if there are transfer conductances between busses, then, the potential-energy integral is not path independent. In classical stability analysis the loads are considered as constant impedances to ground. Hill and Iven [1] use the constant real power-constant voltage loads, i.e., power voltage (PV) type busses. The instability of the system is detected by the state variables, for instance, frequency, angles, and voltages, by an oscillatory behavior or a continuous change (voltage decrease, i.e., collapse, and frequency and angle increase, i.e., loss of synchronism). The region of stability around the current operating point (stable equilibrium point) is small; hence, the system is not able to withstand load perturbations and becomes unstable [2], [8]–[11].

In this brief, some fundamental concepts of the general spatial periodic solution theory are applied to multimachine systems with reserved structure in Section II, in other words, the spatial periodic solution theory of DAS, are introduced. In Section III, the model of multimachine power systems affected by perturbation of nonlinear loads is presented. In Section IV, the model of multimachine power systems with DAS structure that yield periodic oscillatory solution as system parameters satisfying particular conditions is analyzed, which provide powerful analytical methods for further studying the disturbed problems with nonlinear loads. In Section V, the method is applied to some practice examples.

II. Periodic Solution of Nonlinear System

In this section, the theoretical results of periodic solution of nonlinear systems without constraints is summarized [12], [13]. Then, some results of periodic solution of DAS are presented.

Consider the linear homogeneous periodic system

$$\dot{x} = A(t)x \tag{2.1}$$

and linear nonhomogeneous periodic system

$$\dot{x} = A(t)x + p(t) \tag{2.2}$$

where $x = (x_1, x_2, \ldots, x_n)^T \in \mathcal{R}^n, A(t) \in C(\mathcal{R}^n \times \mathcal{R}^n)$ is the $n \times n$ matrix $A(t + \mathcal{T}) = A(t); p(t) \in C(\mathcal{R}^n), p(t + \mathcal{T}) = p(t)$. Under the following conditions, (2.2) has a unique $\mathcal{T}$ periodic solution. Using the notations, $M(t)$ is the basic matrix solution of (2.1) and $M(0) = I, I$ is identity matrix; then, the general solution of (2.1) is $x(t) = M(t)x_0, x(0) = x_0$. With a nontrivial initial condition, $x(t)$ is the nonzero $\mathcal{T}$ periodic solution $x(0) = x(\mathcal{T}) \neq 0$; then, the necessary and sufficient conditions that (2.1) has nonzero $\mathcal{T}$ periodic solution are