# $L$-Splines - a Manifestation of Optimal Control

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**Abstract**

We show how to generate a class of Euclidean splines, called $L$-splines, as solutions of a high-order variational problem. We also show connections between $L$-splines and optimal control theory, leading to the conclusion that $L$-splines are manifestations of an optimal behavior.

**Keywords:** $L$-splines, variational problems, optimal control.

## 1 Introduction

In recent years there has been an effort to combine ideas of splines and control theory, having in mind engineering applications such as path-planning trajectories of mechanical systems. Most of these applications require generalizations of splines in Euclidean spaces to Lie groups and other Riemannian manifolds. Related to that we mention the work of Crouch and Jackson [8], Crouch and Silva Leite [5], [4], Crouch, Kun and Silva Leite [7], [6], Ge and Ravani [9], [10], [11], Noakes, Heinzinger and Paden [14] and Park and Ravani [15]. In some of these papers the connection between splines and optimal control is explored, leaving behind the

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idea that splines and control theory are manifestations of the same phenomena. Following previous work on generalized splines and optimal control [17], which in turn was inspired by the work of Martin et al [13], we now show that another class of Euclidean splines, called $L$-splines, can be realized as output functions associated with the optimal solution of a linear optimal control problem with interpolation conditions.

An interesting property of this optimal control problem is that it allows uneven conditions at the interpolating points. Another important feature is that the optimal control problem associated with an $L$-spline may not have a unique solution. This is particularly appealing from an engineering point of view, since there is some degree of freedom in the choice of the optimal control that gives rise to an output function with the properties of an $L$-spline. This will be explained in detail in section 3. Before that, in section 2, we define $L$-splines as solutions of the Euler-Lagrange equations associated with a particular variational problem. In the last section we show that this general case gives rise to the generalized splines studied in [13] and [17]. We also illustrate the theory with examples.

# 2 $L$-spline functions - a variational approach

Now, we formulate a variational problem and provide a solution based on results from calculus of variation. As a consequence, we will be able to show that its solution belongs to a well know class of spline functions. This approach will be crucial to make clear what splines and optimal control have in common.

Before formulating the variational problem, we give some definitions which will be used throughout this paper. For each positive integer $m$, let $K^m[t_0, T]$ denote the collection of all real-valued functions $f$ defined on the real time interval $[t_0, T]$, such that $f \in \mathcal{C}^{m-1}[t_0, T]$, $D^{m-1}f$ is absolutely continuous on $[t_0, T]$ and $D^m f \in L_2[t_0, T]$. Here $D^m f$ is used to denote the derivative $\frac{d^m f}{dt^m}$. Later on, it will be convenient to use the notation $f^{(m)}$ with the same meaning. For $\Delta : t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ a partition of $[t_0, T]$, let $\mathcal{Z} = (z_1, z_2, \ldots, z_{n-1})$, be an *incidence vector associated with* $\Delta$. That is, $\mathcal{Z}$ is an $(n-1)$-vector with positive integer components $z_k$ where $1 \leq z_k \leq m$ for $k = 1, 2, \ldots, n-1$. Finally, let $L$ be the linear differential operator of order $m$ with constant coefficients

$$L \equiv D^m + b_m D^{m-1} + \cdots + b_2 D + b_1 D^0,$$

and $L^*$ its formal adjoint

$$L^* \equiv (-1)^m D^m + (-1)^{m-1} b_m D^{m-1} + \cdots - b_2 D + b_1 D^0.$$

Consider the following variational problem $(\mathcal{P}_1)$:

Given:

the differential operator $L \equiv D^m - a_m D^{m-1} - \cdots - a_2 D - a_1 D^0$;

a partition $\Delta : t_0 < t_1 < \cdots < t_{n-1} < t_n = T$, of the time interval $[t_0, T]$;

$(z_1, z_2, \ldots, z_{n-1})$ the incidence vector associated with $\Delta$;

real constants $\alpha_0^i$, $\alpha_n^i$, $i = 0, 1, \ldots, m-1$,

and $\alpha_k^i$, $k = 1, 2, \ldots, n-1$, $i = 0, 1, \ldots, z_k - 1$,

find $y \ : \ y\,|_{[t_{k-1}, t_k]} \in \mathcal{C}^{2m}[t_{k-1}, t_k]$, that minimizes the functional

$$J(y) = \int_{t_0}^{T} (Ly(t))^2 \, dt$$

and satisfies the boundary conditions

$$D^i y(t_0) = \alpha_0^i, \quad D^i y(T) = \alpha_n^i, \quad i = 0, 1, \ldots, m-1,$$

the interpolation conditions

$$D^i y(t_k) = \alpha_k^i, \quad k = 1, 2, \ldots, n-1, \ i = 0, 1, \ldots, z_k - 1,$$

and the continuity conditions

$$D^i y(t_k^-) = D^i y(t_k^+), \quad k = 1, 2, \ldots, n-1, \ i = 0, 1, \ldots, 2m - 1 - z_k.$$

The theorem 2.2 below gives a necessary condition for $y$ to be a solution of problem $(\mathcal{P}_1)$. The proof of this theorem requires the next intermediate result.

**Lemma 2.1** *Let $\Delta : t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ be a partition of the time interval $[t_0, T]$, $(z_1, z_2, \ldots, z_{n-1})$ an incidence vector associated with $\Delta$ and $f, g \in \mathcal{C}^q\,[t_{k-1}, t_k]$, for $k = 1, 2, \ldots, n$ and some $(q \in \mathbb{N})$. If $f$ satisfies the following conditions*

$$D^i f(t_0) = D^i f(T) = 0, \quad i = 0, 1, \ldots, q-1; \tag{1}$$

$$D^i f(t_k) = 0, \quad k = 1, 2, \ldots, n-1, i = 0, 1, \ldots, z_k - 1; \tag{2}$$

$$D^i f(t_k^-) = D^i f(t_k^+), \quad k = 1, 2, \ldots, n-1, i = 0, 1, \ldots, 2q - 1 - z_k, \tag{3}$$

*and $g$ satisfies*

$$D^i g(t_k^-) = D^i g(t_k^+), \quad k = 1, 2, \ldots, n-1, i = 0, 1, \ldots, q - z_k, \tag{4}$$

*then the following holds*

$$\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} D^q f(t)\, g(t)\, dt = (-1)^q \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f(t)\, D^q g(t)\, dt.$$

**Proof** It is clear that

$$g(t)\,D^q f(t) + (-1)^{q-1} D^q g(t) f(t) = \frac{d}{dt}\left(\sum_{i=0}^{q-1} (-1)^i D^i g(t)\, D^{q-i-1} f(t)\right).$$

Then

$$\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} g(t)\, D^q f(t)\, dt + \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (-1)^{q-1} D^q g(t)\, f(t)\, dt$$

$$= \sum_{k=1}^{n} \left(\sum_{i=0}^{q-1} (-1)^i D^i g(t)\, D^{q-i-1} f(t)\right|_{t_{k-1}}^{t_k}, \quad k = 1, 2, \ldots, n.$$

It is now easy to show, that the right hand side of the previous equation, vanishes whenever $f$ and $g$ satisfy (1), (2), (3) and (4). □

**Theorem 2.2** *A necessary condition for $y$ to be a solution of the variational problem $(\mathcal{P}_1)$ is that $L^* L y(t) = 0$ in each subinterval $[t_{k-1}, t_k]$, $k = 1, 2, \ldots, n$.*

**Proof** It is enough to show that the Euler-Lagrange equation associated with the Lagrangean $H(t, y, y^{(1)}, \ldots, y^{(m)}) = (Ly(t))^2$ is given by $L^* L y(t) = 0$, in each subinterval. First of all we note that

$$H \in \mathcal{C}^m[t_{k-1}, t_k], \quad k = 1, 2, \ldots, n;$$

$$D^i H(t_k^-, y, y^{(1)}, \ldots, y^{(m)}) = D^i H(t_k^+, y, y^{(1)}, \ldots, y^{(m)}),$$
$$k = 1, 2, \ldots, n-1, i = 0, 1, \ldots, m-1-z_k \ \ (1 \leq z_k \leq m),$$

and all first order partial derivatives of $H$ are continuous in each subinterval $[t_{k-1}, t_k]$, $k = 1, 2, \ldots, n$, at least up to order $m+1$. If $y$ is an extremal of the functional $J$, then, according to the fundamental theorem of the calculus of variations, we must find conditions on $y$ so that $\delta J(y, \delta y)$ vanishes, for all admissible variations $\delta y$ of $y$. $\delta y$ is an admissible variation of $y$ if

$$\delta y \in \mathcal{C}^{2m}[t_{k-1}, t_k] \quad k = 1, 2, \ldots, n;$$

$$D^i \delta y(t_0) = D^i \delta y(T) = 0, \quad i = 0, 1, \ldots, m-1;$$
$$D^i \delta y(t_k) = 0, \quad k = 1, 2, \ldots, n-1, i = 0, 1, \ldots, z_k - 1;$$

and

$$D^i \delta y(t_k^-) = D^i \delta y(t_k^+), \quad k = 1, 2, \ldots, n-1, i = 0, 1, \ldots, 2m-1-z_k.$$

Since the variation $\delta J(y,\delta y)$ is the linear part of $\Delta J(y,\delta y) = J(y+\delta y) - J(y)$, we first compute the function $\Delta J$.

$$
\begin{aligned}
\Delta J(y,\delta y) &= \int_{t_0}^{T} H(t, y+\delta y, \ldots, y^{(m)} + (\delta y)^{(m)})\, dt - \int_{t_0}^{T} H(t, y, \ldots, y^{(m)})\, dt \\
&= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} H(t, y+\delta y, \ldots, y^{(m)} + (\delta y)^{(m)})\, dt \\
&\quad - \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} H(t, y, \ldots, y^{(m)})\}\, dt \\
&= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} H(t, y+\delta y, \ldots, y^{(m)} + (\delta y)^{(m)}) - H(t, y, \ldots, y^{(m)})\, dt.
\end{aligned}
$$

Using Taylor's formula, we may write

$$
\begin{aligned}
\Delta J(y,\delta y) &= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \{\, (\delta y)\frac{\partial H}{\partial y} + (\delta y)^{(1)} \frac{\partial H}{\partial y^{(1)}} + \cdots + (\delta y)^{(m)} \frac{\partial H}{\partial y^{(m)}} \\
&\quad + \frac{1}{2} [\, (\delta y)^2 \frac{\partial^2 H}{\partial y^2} + 2(\delta y)(\delta y)^{(1)} \frac{\partial^2 H}{\partial y^{(1)} \partial y} + \cdots \\
&\quad + 2(\delta y)(\delta y)^{(m)} \frac{\partial^2 H}{\partial y^{(m)} \partial y} + \cdots + ((\delta y)^{(m)})^2 \frac{\partial^2 H}{\partial (y^{(m)})^2}\, ] + R_2 \,\}\, dt
\end{aligned}
$$

and, consequently,

$$
\begin{aligned}
\delta J(y,\delta y) &= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (\delta y) \frac{\partial H}{\partial y}\, dt + \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (\delta y)^{(1)} \frac{\partial H}{\partial y^{(1)}}\, dt + \cdots \\
&\quad + \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (\delta y)^{(m)} \frac{\partial H}{\partial y^{(m)}}\, dt.
\end{aligned}
$$

Using lemma 2.1, with $f = \delta y$, in each term of the previous formula, we obtain the following:

$$
\begin{aligned}
\delta J(y,\delta y) &= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (\delta y) \frac{\partial H}{\partial y}\, dt - \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (\delta y) \frac{d}{dt}\left(\frac{\partial H}{\partial y^{(1)}}\right) dt + \cdots \\
&\quad + (-1)^m \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (\delta y) \frac{d^m}{dt^m}\left(\frac{\partial H}{\partial y^{(m)}}\right) dt \\
&= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left[\frac{\partial H}{\partial y} - \frac{d}{dt}\left(\frac{\partial H}{\partial y^{(1)}}\right) + \cdots + (-1)^m \frac{d^m}{dt^m}\left(\frac{\partial H}{\partial y^{(m)}}\right)\right] (\delta y)\, dt.
\end{aligned}
$$

So, we must have

$$\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left[\frac{\partial H}{\partial y}-\frac{d}{dt}\left(\frac{\partial H}{\partial y^{(1)}}\right)+\cdots+(-1)^m\frac{d^m}{dt^m}\left(\frac{\partial H}{\partial y^{(m)}}\right)\right](\delta y)\,dt=0, \qquad (5)$$

for all admissible variations $\delta y$ of $y$.
We now show that the last equation implies that

$$\frac{\partial H}{\partial y}-\frac{d}{dt}\left(\frac{\partial H}{\partial y^{(1)}}\right)+\cdots+(-1)^m\frac{d^m}{dt^m}\left(\frac{\partial H}{\partial y^{(m)}}\right)=0,\ \forall t\in[t_{k-1},t_k],\ k=1,2,\ldots,n. \qquad (6)$$

Assume that there exists a $j\in\{1,2,\ldots,n\}$ such that (6) does not hold in $[t_{j-1},t_j]$. Taking into consideration that the left hand side of (6) is a continuous function, there exists $[c,d]\subset[t_{j-1},t_j]$ where that function has constant sign. (Assume, without loss of generality, that the sign is positive). Consider the following admissible variation

$$\delta y(t)=\begin{cases}[(t-c)(d-t)]^{2m+1}, & t\in[c,d]\\ 0, & t\in[t_0,T]\setminus[c,d]\end{cases}$$

from which it follows that

$$\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left[\frac{\partial H}{\partial y}-\frac{d}{dt}\left(\frac{\partial H}{\partial y^{(1)}}\right)+\cdots+(-1)^m\frac{d^m}{dt^m}\left(\frac{\partial H}{\partial y^{(m)}}\right)\right](\delta y)\,dt$$

$$=\int_{c}^{d}\left[\frac{\partial H}{\partial y}-\frac{d}{dt}\left(\frac{\partial H}{\partial y^{(1)}}\right)+\cdots+(-1)^m\frac{d^m}{dt^m}\left(\frac{\partial H}{\partial y^{(m)}}\right)\right](\delta y)\,dt>0,$$

which contradicts (5) and so proves that (6) holds.
Finally, replacing $H(t,y,y^{(1)},\ldots,y^{(m)})$ by $(Ly(t))^2$ in (6) and computing the necessary derivatives, we conclude that $y$ satisfies the following Euler-Lagrange equation

$$L^*Ly(t)=0,\quad \forall t\in[t_{k-1},t_k],\quad \forall k\in\{1,2,\ldots,n\}. \qquad (7)$$

□

The function $y$, solution of the Euler-Lagrange equation associated with $(\mathcal{P}_1)$ is a spline function of a particular type, called an $L$-spline of type I. These functions were first introduced, in 1967, by Schultz and Varga in [18]. In the following we limit ourselves to a brief outline of their results.

**Definition 2.3** *A real function $s$, defined on $[t_0,T]$, is said to be an $L$-spline for the differential operator $L$, the partition $\Delta$ and the incidence vector $\mathcal{Z}$, if the following holds simultaneously:*

$$s\in K^{2m}[t_{k-1},t_k],\ k=1,2,\ldots,n;$$

$$L^*Ls(t)=0,\ \ for\ \ t\in[t_{k-1},t_k],\ \ k=1,2,\ldots,n;$$

$$D^is(t_k^-)=D^is(t_k^+),\ k=1,2,\ldots,n-1,\ i=0,1,\ldots,2m-1-z_k.$$

The conclusion that $y \in K^{2m}[t_{k-1}, t_k]$, $k = 1, 2, \ldots, n$, follows from the fact that $y\,|_{[t_{k-1},t_k]} \in \mathcal{C}^{2m}[t_{k-1}, t_k]$.

**Definition 2.4** *An L-spline s is said to be of type I if it satisfies the following boundary and interpolation conditions:*

$$\begin{aligned} D^i s(t_k) &= \beta_k^i,\ k \in \{0, n\},\ i = 0, 1, \ldots, m-1; \\ D^i s(t_k) &= \alpha_k^i,\ k = 1, 2, \ldots, n-1,\ i = 0, 1, \ldots, z_k - 1, \end{aligned} \tag{8}$$

*for given real numbers $\beta_k^i$ and $\alpha_k^i$.*

Schultz and Varga also proved the existence and uniqueness of such spline function and its already noticed extremal property. We sumarize their conclusion in the next theorem.

**Theorem 2.5** *Given the differential operator L, the partition $\Delta$ of the time interval $[t_0, T]$, and the incidence vector $\mathcal{Z}$, there exists a unique L-spline of type I, for each set of boundary and interpolation conditions of type (8). This L-spline (of type I) also minimizes the functional $J(f) = \int_{t_0}^{T} (Lf(t))^2\, dt$, among all functions belonging to $K^m[t_0, T]$ and satisfying the same boundary and interpolation conditions.*

For the sake of completeness we include here a proof of the fact that all solutions of the Euler-Lagrange equation associated with $(\mathcal{P}_1)$ also minimize the functional $J$. For that we need the following lemma.

**Lemma 2.6** *If y is a solution of the Euler-Lagrange equation associated with $(\mathcal{P}_1)$ and if $f \in K^m[t_0, T]$ and satisfies the same boundary and interpolation conditions, then*

$$\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} Ly(t) L(f(t) - y(t))\, dt = 0. \tag{9}$$

**Proof** We use the following relation that connects $L$ and $L^*$ and which is known as the *Lagrange Identity*,

$$v\, Lu - u\, L^* v = \frac{d}{dt} B(u, v), \tag{10}$$

where $u$ and $v$ are any two real functions defined and possesing at least $m$ derivatives on a closed interval and

$$B(u, v) = \sum_{j=0}^{m-1} D^{m-j-1} u(t) \sum_{i=0}^{j} (-1)^{i+1} D^i (a_{m+1-j+i} v(t)).$$

Choosing $u = f - y$, $v = Ly$ and integrating (10) in $[t_{k-1}, t_k]$, we get for $k = 1, 2, \ldots, n$

$$\int_{t_{k-1}}^{t_k} Ly(t)L(f(t) - y(t))\,dt$$

$$= \left(\sum_{j=0}^{m-1} D^{m-j-1}(f(t) - y(t)) \sum_{i=0}^{j} (-1)^{i+1} D^i(a_{m+1-j+i} Ly(t))\right|_{t_{k-1}}^{t_k}.$$

Then

$$\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} Ly(t)L(f(t) - y(t))\,dt$$

$$= \sum_{k=1}^{n} \left(\sum_{j=0}^{m-1} D^{m-j-1}(f(t) - y(t)) \sum_{i=0}^{j} (-1)^{i+1} D^i(a_{m+1-j+i} Ly(t))\right|_{t_{k-1}}^{t_k}.$$

The right hand side of the previous equation gives rise to the following identical expression for $t_0$ and $t_n$,

$$\left(\sum_{j=0}^{m-1} D^{m-j-1}(f(t) - y(t)) \sum_{i=0}^{j} (-1)^{i+1} D^i(a_{m+1-j+i} Ly(t))\right|_{t=t_0, t_n},$$

that vanishes, since $f$ and $y$ satisfy the same boundary conditions. For all other $t_k$, $k = 1, 2, \ldots, n-1$, we have

$$\sum_{j=0}^{m-1} D^{m-j-1}(f(t_k) - y_k(t_k) \sum_{i=0}^{j} (-1)^{i+1} D^i(a_{m+1-j+i} Ly_k(t_k))$$

$$- \sum_{j=0}^{m-1} D^{m-j-1}(f(t_k) - y_{k+1}(t_k) \sum_{i=0}^{j} (-1)^{i+1} D^i(a_{m+1-j+i} Ly_{k+1}(t_k)),$$

where $y_k$ is the expression of $y$ in $[t_{k-1}, t_k]$ and $y_{k+1}$ the expression of $y$ in $[t_k, t_{k+1}]$. The last expression also vanish for each $k$, since $f$ and $y$ satisfy the same interpolation conditions and $y$ satisfies continuity conditions. $\square$

**Theorem 2.7** *The solutions of the Euler-Lagrange equation associated with ($\mathcal{P}_1$), minimize the functional $J$ among all functions $f \in K^m[t_0, T]$ that satisfy the same boundary and interpolation conditions.*

**Proof**

$$\int_{t_0}^{T} [L(f(t)-y(t))]^2\,dt$$

$$= \int_{t_0}^{T} (Lf(t))^2\,dt - 2\int_{t_0}^{T} Lf(t)Ly(t)\,dt + \int_{t_0}^{T} (Ly(t))^2\,dt$$

$$= \int_{t_0}^{T} (Lf(t))^2\,dt - 2\int_{t_0}^{T} L(f(t)-y(t))Ly(t)\,dt - \int_{t_0}^{T} (Ly(t))^2\,dt.$$

Using (9) we get

$$\int_{t_0}^{T} (Lf(t))^2\,dt = \int_{t_0}^{T} (Ly(t))^2\,dt + \int_{t_0}^{T} [L(f(t)-y(t))]^2\,dt.$$

The conclusion that $y$ minimizes $J$ follows. □

To find this unique spline it is enough to determine $2mn$ unknowns, corresponding to $2m$ arbitrary constants in the general solution of the differential equation $L^*Ls(t) = 0$, for each subinterval of the partition $\Delta$. The required interpolation conditions provide $\sum_{k=1}^{n-1} z_k$ equations, the boundary conditions generate $2m$ equations and the continuity requirements give rise to $2m(n-1) - \sum_{k=1}^{n-1} z_k$ equations, leading to a total of $2mn$ linear algebraic equations in the $2mn$ unknowns. Solving this system is all that one needs to determine the corresponding $L$-spline.

# 3 $L$-splines and optimal control

This section provides a direct connection between a linear optimal control problem and the spline functions of the previous section. We begin by showing that the variational problem $(\mathcal{P}_1)$ is equivalent to an optimal control problem with interpolation conditions, where the function $y$ plays the role of the output function. We refer to Brockett [2] and Luenberger [12] for a general treatment of linear control systems.

Consider a time invariant single input/single output linear control system defined on $[t_0, T]$ and evolving in $\mathbb{R}^m$. Let $\Delta : t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ be a partition of $[t_0, T]$, and assume that the system is controllable so that it may be written in canonical form:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_m \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \tag{11}$$

$$y(t) = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{m-1} \end{bmatrix} x(t),$$

where $a_i,\ c_{i-1} \in \mathbb{R},\ i = 1, 2, \ldots, m$. First of all notice that the output function can be expressed in terms of the first coordinate of the state vector, which will be denoted by $x^1$, and its derivatives as

$$y(t) = c_0 x^1(t) + c_1 D x^1(t) + \cdots + c_\sigma D^\sigma x^1(t), \quad t \in [t_0, T], \tag{12}$$

where $\sigma$ is the following integer $\sigma = \max\{ i : \ c_i \neq 0\}$, $(0 \leq \sigma \leq m-1)$. It also follows from the first equation in (11) that $Lx^1(t) = u(t)$, where $L$ is the linear differential operator $L \equiv D^m - a_m D^{m-1} - \cdots - a_1 D^0$. As a consequence, the input/output equation for the sistem (11) may be written as the differential equation

$$Ly(t) = c_0 u(t) + c_1 D u(t) + \cdots + c_\sigma D^\sigma u(t), \quad t \in [t_0, T]. \tag{13}$$

Finally, if $y \mid_{[t_{k-1}, t_k]} \in \mathcal{C}^{2m}[t_{k-1}, t_k]$, $k = 1, 2, \ldots, n$, then from (13), one can deduce that $u \mid_{[t_{k-1}, t_k]} \in \mathcal{C}^{m+\sigma}[t_{k-1}, t_k]$, $k = 1, 2, \ldots, n$, where $\sigma$ $(0 \leq \sigma \leq m-1)$ is the order of the higher derivative of $u$ present in the right hand side of (13).

It's now clear that the variational problem $(\mathcal{P}_1)$ may be formulated as the following equivalent *optimal control problem* $(\mathcal{P}_2)$

Given:

a partition $\Delta : t_0 < t_1 < \cdots < t_{n-1} < t_n = T$, of the time interval $[t_0, T]$;

$(z_1, z_2, \ldots, z_{n-1})$ the incidence vector associated with $\Delta$;

real constants $\alpha_0^i,\ \alpha_n^i,\ i = 0, 1, \ldots, m-1$,

and $\alpha_k^i,\ k = 1, 2, \ldots, n-1,\ i = 0, 1, \ldots, z_k - 1$;

find $u \ : \ u \mid_{[t_{k-1}, t_k]} \in \mathcal{C}^{m+\sigma}[t_{k-1}, t_k]$, that minimizes the functional

$$J(u) = \int_{t_0}^{T} \left(c_0 u(t) + c_1 D u(t) + \cdots + c_\sigma D^\sigma u(t)\right)^2 \, dt \tag{14}$$

subject to:

$$\begin{cases} \dot{x} = Ax + bu & , \quad (A, b) \text{ in controllability canonical form} \\ y = Cx & , \quad C = [\ c_0 \quad c_1 \quad \cdots \quad c_{m-1}\ ]; \end{cases} \tag{15}$$

$$D^i y(t_0) = \alpha_0^i, \quad D^i y(T) = \alpha_n^i, \quad i = 0, 1, \ldots, m-1; \tag{16}$$

$$D^i y(t_k) = \alpha_k^i, \quad k = 1, 2, \ldots, n-1,\ i = 0, 1, \ldots, z_k - 1; \tag{17}$$

$$D^i y(t_k^-) = D^i y(t_k^+), \quad k = 1, 2, \ldots, n-1,\ i = 0, 1, \ldots, 2m-1-z_k. \tag{18}$$

Notice that there are precisely $2m$ conditions in every interior point of $\Delta$.

## 3.1 Solution of the optimal control problem with interpolation conditions

The next result, which is a consequence of theorem 2.2 from the last section, gives a necessary and sufficient condition for $u$ to be a solution of the optimal control problem $(\mathcal{P}_2)$.

**Theorem 3.1** *The optimal solution of $(\mathcal{P}_2)$ exists, and satisfies in each subinterval $[t_{k-1}, t_k]$, $k = 1, 2, \ldots, n$, the linear homogeneous differential equation of order $m + \sigma$ with constant coefficients*

$$L^* R\, u(t) = 0,$$

*where $R$ is the linear differential operator $R = c_\sigma D^\sigma + \cdots + c_1 D + c_0 D^0$.*

**Proof** It was shown in the last section, that there exists a unique output function $y$ such that $y|_{[t_{k-1},t_k]} \in K^{2m}[t_{k-1}, t_k]$, which fullfils the conditions (16)-(17)-(18) in the problem $(\mathcal{P}_2)$. $y$ satisfies, in each subinterval $[t_{k-1}, t_k]$, $k = 1, 2, \ldots, n$, the differential equation $L^* L y(t) = 0$, and on the other hand, according to (13), $Ly = Ru$. So, the conclusion follows.

$\square$

This theorem shows that the optimal solution $u$ of the optimal control problem $\mathcal{P}_2$ is piecewisely defined as the solution of a linear homogeneous differential equation, of order $m + \sigma$, with constant coefficients. It's clear that the optimal input has a rather simple expression in each subinterval of $\Delta$. Nevertheless, the theorem doesn't give any answer about the uniqueness of the optimal $u$. In fact, for each set of boundary conditions (16), interpolation conditions (17) and continuity conditions (18), there exists an $n\sigma$-parameter family of optimal controls, which gives rise to the same minimum value of the functional $J$ and generate the same $L$-spline function. Indeed, since $u$ must satisfy a linear homogeneous differential equation of order $m + \sigma$, we must necessarily find $m + \sigma$ real constants to characterize $u$ in each interval $[t_{k-1}, t_k]$, $k = 1, 2, \ldots, n$. The conditions needed are obtained using equation (13)

$$L\, y(t) = R\, u(t).$$

For instance, to find $u_k$, the expression of $u$ in the interval $[t_{k-1}, t_k]$, and since $u_k \in \mathcal{C}^{m+\sigma}[t_{k-1}, t_k]$ ($y_k \in \mathcal{C}^{2m}[t_{k-1}, t_k]$, where $y_k$ is the expression of the spline function $y$ in $[t_{k-1}, t_k]$) we get, choosing for example $t = t_k$, the following set of $m + 1$ equations

$$\begin{aligned} L\, y_k(t_k) &= R\, u_k(t_k) \\ DL\, y_k(t_k) &= DR\, u_k(t_k) \\ &\vdots \\ D^m L\, y_k(t_k) &= D^m R\, u_k(t_k). \end{aligned}$$

However, each side of the last equation can be obtained, as a linear combination of the corresponding sides of the previous equations. This conclusion is simply due to the fact that $L^*L\,y_k(t_k) = 0$ and $L^*R\,u_k(t_k) = 0$. The previous argument can also be used to show that the same applies to any other equation of the form $D^pL\,y_k(t_k) = D^pR\,u_k(t_k)$, with $p > m$. Notice that any other $t \in [t_{k-1}, t_k[$ could have been chosen. The m equations remaining define a linear system of $m$ algebraic equations in $m + \sigma$ unknowns. It can also be easily shown that such system of algebraic equations has always a solution, which is unique only when $\sigma = 0$. For general $\sigma$, $u_k$ is not defined uniquely. Indeed, there is a number of $\sigma$ free parameters to define $u_k$ (a total of $n\sigma$ to define $u$). This freedom in the choice of an optimal control has advantages in practical applications.

## 3.2 Computation of the optimal input and the optimal output

In order to write explicitely the optimal output function associated with the problem $(\mathcal{P}_2)$, in each interval $[t_{k-1}, t_k]$, it is sufficient to know the spectrum of the coefficient matrix $A$ in system (11). Indeed, if $\lambda \in \mathrm{Sp}(A)$, or equivalently, if $\lambda$ is a root of the characteristic polynomial associated with the operator $L$, then $-\lambda$ is a root of the characteristic polynomial associated with the operator $L^*$ and, consequently, all the solutions of $L^*Ly(t) = 0$ may be written as linear combinations of $2m$ linearly independent solutions, associated with $\pm\lambda$, for each $\lambda \in \mathrm{Sp}(A)$. Polynomial $L$-splines correspond to the extremal situation when $\mathrm{Sp}(A) = \{0\}$. In particular, the cubic $L$-spline is associated with the case $m = 2$. To compute the $L$-spline we need to find the value of $2mn$ parameters which form the solution of a linear algebraic system generated by the interpolation, boundary and continuity conditions, as explained at the end of section 2.

Since $u$ satisfies the equation $L^*R\,u(t) = 0$ in each interval $[t_{k-1}, t_k]$, the same reasoning applies to find an explicit formula for the optimal control.

Once we have computed the $L$-spline, to find an optimal control we only need to choose $\sigma$ parameters.

## 3.3 Examples

The figures presented at the end of this section show the graphs of the optimal input function and corresponding output function, associated with the optimal control problem $(\mathcal{P}_2)$, for a bidimensional system in the controllability form (11), with $c_0 = 1$ and $c_1 = 2$ $(\sigma = 1)$. We consider three cases only. Other cases may be treated similarly, as described before.

*Case* 1 - Sp(A) $= \{0\}$,
*Case* 2 - Sp(A) $= \{1, -10\}$,

*Case* 3 - Sp(A) $= \{\pm 6i\}$.

For all cases we assume the time interval $[0,3]$ and the partition

$$t_0 = 0 < 1/2 < 1 < 2 < 9/4 < 3 = t_5,$$

with the incidence vector $\mathcal{Z} = (2,1,2,1)$, boundary conditions

$$y(0) = 3, \quad Dy(0) = -1$$

$$y(3) = 0, \quad Dy(3) = 0,$$

interpolation conditions on $y$

$$y(1/2) = 3/2,\ y(1) = 1,\ y(2) = 1/2,\ y(9/4) = 1/6,$$

interpolation conditions on $Dy$

$$Dy(1/2) = -1,\ Dy(2) = -1/2,$$

and continuity conditions

$$y(1/2^-) = y(1/2^+), \quad Dy(1/2^-) = Dy(1/2^+);$$

$$y(1^-) = y(1^+), \qquad Dy(1^-) = Dy(1^+), \qquad D^2y(1^-) = D^2y(1^+);$$

$$y(2^-) = y(2^+), \qquad Dy(2^-) = Dy(2^+);$$

$$y(9/4^-) = y(9/4^+), \quad Dy(9/4^-) = Dy(9/4^+), \quad D^2y(9/4^-) = D^2y(9/4^+).$$

In all cases the linear system gives rise to a differential equation of the form

$$D^2y(t) - a_2Dy(t) - a_1y(t) = u(t) + 2Du(t)$$

while the differential equations $L^*Ly(t) = 0$ and $L^*R\,u(t) = 0$ may be written as

$$D^4y(t) - ({a_2}^2 + 2a_1)D^2y(t) + {a_1}^2y(t) = 0$$

and

$$2D^3u(t) + (2a_2 + 1)D^2u(t) + (a_2 - 2a_1)Du(t) - a_1u(t) = 0$$

respectively.

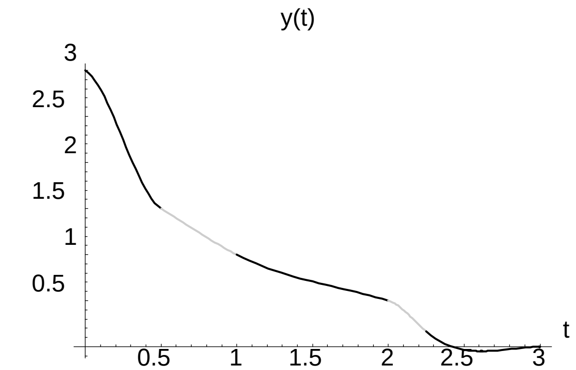


Figure 1: cubic $L$-spline

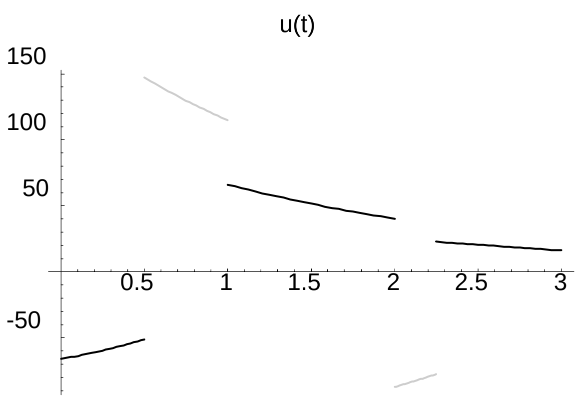


Figure 2: input - cubic $L$-spline

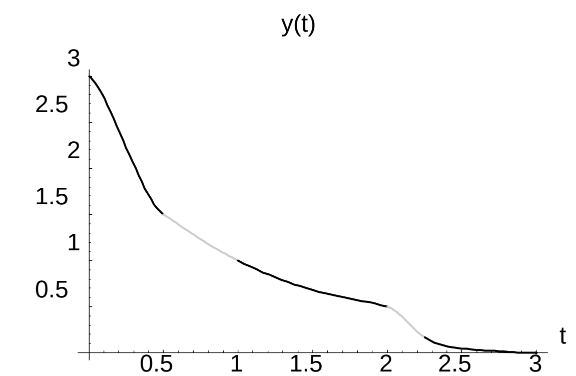


Figure 3: exponential $L$-spline

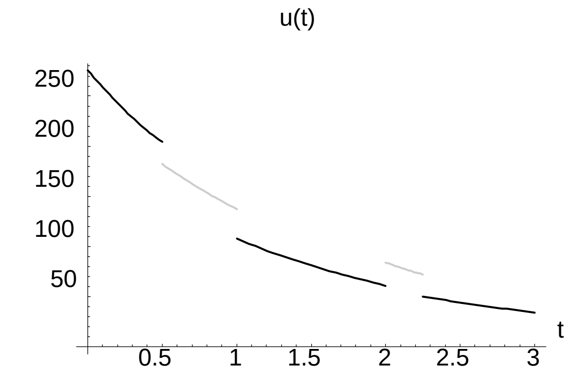


Figure 4: input - exponential $L$-spline

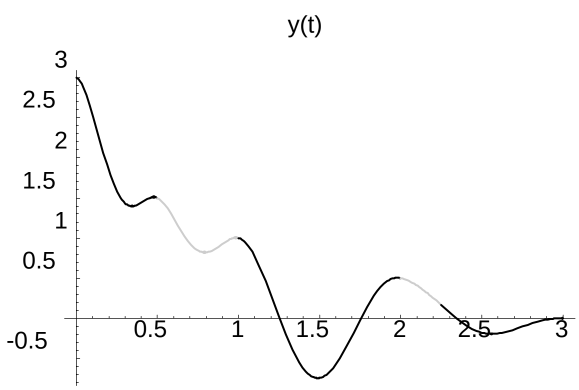


Figure 5: trigonometric $L$-spline

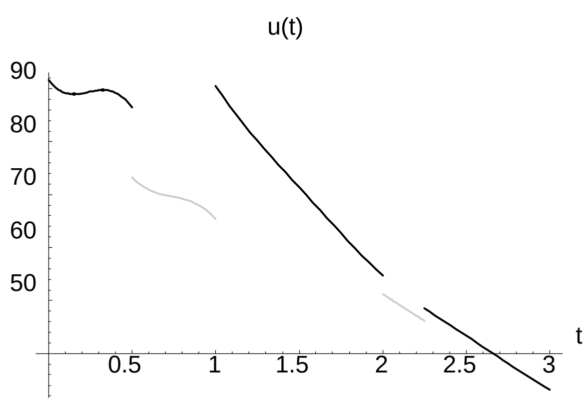


Figure 6: input - trigonometric $L$-spline

# 4 A particular case - generalized splines

Choosing $\sigma = 0$ with $c_0 = 1$ and $\mathcal{Z} = (1, 1, \ldots, 1)$, the optimal control problem $(\mathcal{P}_2)$ simplifies and can be restated as follows

Given:

a partition $\Delta : t_0 < t_1 < \cdots < t_{n-1} < t_n = T$, of the time interval $[t_0, T]$;

real constants $\alpha_0^i$, $\alpha_n^i$, $i = 0, 1, \ldots, m-1$,

and $\alpha_k$, $k = 1, 2, \ldots, n-1$;

$(\mathcal{P}_{\mathbf{2}}^*)$ find $u \, : \, u\,|_{[t_{k-1}, t_k]} \in \mathcal{C}^m[t_{k-1}, t_k]$, that minimizes the functional

$$J(u) = \int_{t_0}^{T} u^2(t)\, dt$$

subject to:

$$\begin{cases} \dot{x} = Ax + bu & , \quad (A, b) \text{ in controllability canonical form} \\ y = Cx & , \quad C = [\begin{array}{cccc} 1 & 0 & \cdots & 0 \end{array}]; \end{cases} \tag{19}$$

$$D^i y(t_0) = \alpha_0^i, \quad D^i y(T) = \alpha_n^i, \quad i = 0, 1, \ldots, m-1;$$

$$y(t_k) = \alpha_k, \quad k = 1, 2, \ldots, n-1;$$

$$D^i y(t_k^-) = D^i y(t_k^+), \quad k = 1, 2, \ldots, n-1, \; i = 0, 1, \ldots, 2m-2. \tag{20}$$

Note that the differentiability conditions (20) mean, in this context, that $y \in \mathcal{C}^{2m-2}[t_0, T]$ and that this condition, together with the constraint (19), implies that $u \in \mathcal{C}^{m-2}[t_0, T]$. This particular case was studied in [13] and [17] and gives rise to another special class of spline functions, called *generalized splines*. Generalized splines were first defined and studied by "Ahlberg et al. [1]. The connection with optimal control appeared more recently in the work of Martin et al. [13] and Rodrigues et al. [17]. As the following definition shows, generalized splines are particular cases of $L$-splines, for which the incidence vector is chosen to be equal to $(1, \cdots, 1)$.

**Definition 4.1** *A real function $s$, defined on $[t_0, T]$, is said to be a generalized spline, for the partition $\Delta$ and the operator $L$, if the following holds simultaneously:*

$$s \in K^{2m}[t_{k-1}, t_k], \; k = 1, 2, \ldots, n;$$

$$L^* Ls(t) = 0, \;\; \textit{for almost all} \;\; t \in [t_{k-1}, t_k], \;\; k = 1, 2, \ldots, n;$$

$$s \in \mathcal{C}^{2m-2}[t_0, T].$$

**Definition 4.2** *A generalized spline $s$ is said to be of type I if it satisfies the following boundary and interpolation conditions:*

$$\begin{aligned} D^i s(t_k) &= \beta_k^i, \; k \in \{0, n\}, \; i = 0, 1, \ldots, m-1; \\ s(t_k) &= \alpha_k, \; k = 1, 2, \ldots, n-1, \end{aligned}$$

*for given real numbers $\beta_k^i$ and $\alpha_k$.*

Notice that the choice of $\mathcal{Z} = (1, \cdots, 1)$ means that there is the same number of continuity conditions and the same number of interpolation conditions, at each interior point of the partition $\Delta$. As already mentioned in section 3.1, since $\sigma = 0$, the optimal control problem $(\mathcal{P}_2^*)$ has a unique solution. As shown in [17], this optimal control is, on each subinterval $[t_{k-1}, t_k]$, the solution of $L^* u(t) = 0$ that satisfies conditions which are derived from the boundary, interpolation and differentiability conditions of the output.

If no interpolation conditions are considered in $(\mathcal{P}_2^*)$, our results reduce to those of a classical optimal control problem, which may be found, for instance, in [2].

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