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ORIGINAL PAPER

The algebraic and geometric classification of nilpotent Lie triple systems up to dimension four

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Abstract

In this paper we generalize the Skjelbred–Sund method, used to classify nilpotent Lie algebras, in order to classify triple systems with non-zero annihilator. We develop this method with the purpose of classifying nilpotent Lie triple systems, obtaining from it the algebraic classification of the nilpotent Lie triple systems up to dimension four. Additionally, we obtain the geometric classification of the variety of nilpotent Lie triple systems up to dimension four.

Keywords Nilpotent Lie triple systems · Algebraic classification · Geometric classification · Annihilator extension · Non-associative triple systems

Mathematics Subject Classification 17A30 · 17A40 · 17B30 · 17D99 · 14D06 · 14L30

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1 Introduction

In recent years, numerous works have been published on the algebraic [2, 5, 11, 13–17, 19, 27, 29, 31, 33] and geometric [3–5, 7, 9, 11, 20–23, 30, 34–37, 41] classification of different varieties of low-dimensional algebras. Meanwhile, although there are a few works on the subject [8, 18], the classification of triple systems is a matter still to be explored. In this work, we give the algebraic and geometric classification of nilpotent Lie triple systems, including the graph of primary degenerations and the inclusion graph of orbit closures of the variety. Lie triple systems were introduced by Jacobson [39] as a generalization of the Lie algebras to the ternary case. These structures arise on the study of the curvature tensor of the space of continuous groups of transformations with affine connection and without torsion [43].

The algebraic classification of the nilpotent Lie triple systems is obtained by calculating the annihilator extensions of the same variety but with smaller dimension. To compute these extensions, we develop a generalization of the classic Skjelbred–Sund method, adapted to Lie triple systems. This method was first used for the classification of nilpotent Lie algebras [42] and has been used in the last years for the classification of different varieties of algebras. For example, non-Lie central extensions of 4-dimensional Malcev algebras [26], non-associative central extensions of 3-dimensional Jordan algebras [25], anticommutative central extensions of 3-dimensional anticommutative algebras [10], central extensions of 2-dimensional algebras [12] and some others were described. There is also an algebraic classification of nilpotent associative algebras up to dimension 4 [16], nilpotent Novikov algebras up to dimension 4 [33], nilpotent restricted Lie algebras up to dimension 5 [14], nilpotent Jordan algebras up to dimension 5 [24], nilpotent Lie algebras up to dimension 6 [13, 15], nilpotent Malcev algebras up to dimension 6 [27], nilpotent binary Lie algebras up to dimension 6 [1, 2], nilpotent algebras up to dimension 3 [12, 19] nilpotent commutative algebras up to dimension 4 and nilpotent anticommutative up to dimension 5 [19].

The geometric classification is an interesting matter. The problem of determining the irreducible components of a variety of low-dimensional algebras (with respect to the Zariski topology) has been studied in various works. This allows identifying the so-called rigid algebras of the variety, which correspond to those algebras whose orbit by the action of the generalized linear group is an irreducible component. This subject was studied for 2-dimensional pre-Lie algebras in [6], 2-dimensional terminal algebras in [11], 3-dimensional Novikov algebras in [7], 3-dimensional Jordan algebras in [21], 3-dimensional Jordan superalgebras in [4], 3-dimensional Leibniz and 3-dimensional anticommutative algebras in [31], 4-dimensional Lie algebras in [9], 4-dimensional Zinbiel and 4-dimensional nilpotent Leibniz algebras in [34], 3-dimensional nilpotent algebras, 4-dimensional nilpotent commutative algebras and 5-dimensional nilpotent anticommutative algebras [19], 6-dimensional nilpotent Lie algebras in [22, 41], 6-dimensional nilpotent Malcev algebras in [35] and 2-dimensional algebras in [36]. To obtain the geometric classification of the variety of nilpotent Lie triple systems, we used a similar approach as the one used in these cited papers.

This work is organized as follows. In the first section, we will introduce the basic definitions related to Lie triple systems. In the second section, we will develop a method to classify the annihilator extensions of a Lie triple system and we will prove that every nilpotent Lie triple system is an annihilator extension of another lower-dimensional nilpotent Lie triple system. In the third section, we will apply this method and we will obtain the algebraic classification of nilpotent Lie triple systems up to dimension four. In the fourth section, we will study the geometric classification of the variety of nilpotent Lie triple systems of dimension three and four.

Throughout this paper, unless stated otherwise, $\mathbb{F}$ denotes an arbitrary ground field of characteristic not two. All vector spaces are assumed to be finite dimensional and defined over $\mathbb{F}$.

Definition 1.1 [32] A vector space $\mathcal{T}$ together with a trilinear map $[-,-,-]:\mathcal{T}\times\mathcal{T}\times\mathcal{T}\longrightarrow\mathcal{T}$ is a *Lie triple system* if the following identities are satisfied:

(A1) $[x,y,z]+[y,x,z]=0$,
(A2) $[x,y,z]+[y,z,x]+[z,x,y]=0$,
(A3) $[u,v,[x,y,z]]=[[u,v,x],y,z]+[x,[u,v,y],z]+[x,y,[u,v,z]]$,

for all x,y,z,u,v in $\mathcal{T}$. The *dimension* of the triple system $\mathcal{T}$ is the dimension of $\mathcal{T}$ as a vector space.

Example 1.1 Any Lie algebra with product $[x,y]$ can be viewed as a Lie triple system with respect to the operation defined as $[x,y,z]:=[[x,y],z]$. A Malcev algebra with product $[x,y]$ gives rise to a Lie triple system with respect to the operation defined as $[x,y,z]:=2[[x,y],z]-[[y,z],x]-[[z,x],y]$ (see [40]). Every Jordan algebra with product $x\circ y$ gives rise to a Lie triple system with respect to the operation $[x,y,z]:=(y\circ z)\circ x-y\circ(z\circ x)$ (cf. [38]).

Let $\mathcal{T}$ be a Lie triple system. A subspace $\mathcal{I}$ of $\mathcal{T}$ is an *ideal* if $[\mathcal{I},\mathcal{T},\mathcal{T}]\subset\mathcal{I}$. If $\mathcal{I}$ is an ideal of $\mathcal{T}$, then the quotient vector space $\overline{\mathcal{T}}=\mathcal{L}/\mathcal{I}$ together with the trilinear map

$$[x+\mathcal{I},y+\mathcal{I},z+\mathcal{I}]:=[x,y,z]+\mathcal{I},$$

for all $x,y,z,\in\mathcal{T}$, is also a Lie triple system.

Definition 1.2 [32] The *annihilator* of a Lie triple system $\mathcal{T}$ is the ideal $\operatorname{Ann}(\mathcal{T}):=\{x\in\mathcal{T}:[x,\mathcal{T},\mathcal{T}]=0\}$. Obviously, $[x,\mathcal{T},\mathcal{T}]=0$ if and only if $[\mathcal{T},x,\mathcal{T}]=[\mathcal{T},\mathcal{T},x]=0$. A Lie triple system $\mathcal{T}$ is called *abelian* if $\mathcal{T}=\operatorname{Ann}(\mathcal{T})$.

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two Lie triple systems. A linear map $\phi:\mathcal{T}_1\longrightarrow\mathcal{T}_2$ is a *homomorphism* of Lie triple systems if $\phi\left([x,y,z]_{\mathcal{T}_1}\right)=[\phi(x),\phi(y),\phi(z)]_{\mathcal{T}_2}$ for all x,y,z in $\mathcal{T}_1$. If $\phi:\mathcal{T}_1\longrightarrow\mathcal{T}_2$ is a homomorphism of Lie triple systems, then $\ker\phi$ is an ideal of $\mathcal{T}_1$ and $\phi(\mathcal{T}_1)\cong\mathcal{T}_1/\ker\phi$.

Given an ideal $\mathcal{I}$ of a Lie triple system $\mathcal{T}$, we may introduce the following sequence of subspaces:

$$\mathcal{I}^{(0)}:=\mathcal{I},\quad \mathcal{I}^{(n+1)}:=\left[\mathcal{I}^{(n)},\mathcal{T},\mathcal{I}\right]+\left[\mathcal{I},\mathcal{T},\mathcal{I}^{(n)}\right],$$

for $n\geq 0$.

Definition 1.3 (See [28]) An ideal $\mathcal{I}$ of a Lie triple system $\mathcal{T}$ is $\mathcal{T}$-*nilpotent* if $\mathcal{I}^{(m)}=0$, for some $m\geq 0$. A Lie triple system $\mathcal{T}$ is *nilpotent* if itself is $\mathcal{T}$- nilpotent.

Note that if $\mathcal{I}=\mathcal{T}$, then $\mathcal{T}^{(n+1)}=\left[\mathcal{T}^{(n)},\mathcal{T},\mathcal{T}\right]$, for all $n\geq 0$. So if $\mathcal{T}$ is nilpotent and $\mathcal{T}\neq 0$, then $\operatorname{Ann}(\mathcal{T})\neq 0$. Moreover, $\mathcal{T}$ is nilpotent if and only if $\mathcal{T}/\operatorname{Ann}(\mathcal{T})$ is nilpotent (see [28]).

2 The algebraic classification of nilpotent Lie triple systems

An algebraic classification of the class of Lie triple systems of dimension n is a classification, up to isomorphisms, of this class. Throughout this work, unless stated otherwise, $\mathcal{T}$ denotes a Lie triple system and $\mathbb{V}$ a vector space.

Definition 2.1 Let $\mathcal{T}$ be a Lie triple system and let $\mathbb{V}$ be a vector space. The vector space $Z^3(\mathcal{T}, \mathbb{V})$ is defined as the set of all trilinear maps $\theta : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathbb{V}$ such that:

(B1) $\theta(x, y, z) + \theta(y, x, z) = 0$,
(B2) $\theta(x, y, z) + \theta(y, z, x) + \theta(z, x, y) = 0$,
(B3) $\theta(u, v, [x, y, z]) + \theta([v, u, x], y, z) + \theta(x, [v, u, y], z) + \theta(x, y, [v, u, z]) = 0$,

for all x, y, z, u, v in $\mathcal{T}$. The elements of $Z^3(\mathcal{T}, \mathbb{V})$ are called *cocycles*.

Let $\mathcal{T}$ be a Lie triple system and $\mathbb{V}$ be a vector space. Given a trilinear map $\theta : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathbb{V}$, consider the vector space $\mathcal{T}_\theta := \mathcal{T} \oplus \mathbb{V}$ endowed with the trilinear map $[-,-,-]_{\mathcal{T}_\theta} : \mathcal{T}_\theta{}^3 \to \mathcal{T}_\theta$ given by $[x + u, y + v, z + w]_{\mathcal{T}_\theta} := [x, y, z]_{\mathcal{T}} + \theta(x, y, z)$ for all $x, y, z \in \mathcal{T}$ and $u, v, w \in \mathbb{V}$. The following result shows this construction $\mathcal{T}_\theta$ is a Lie triple system if and only if $\theta \in Z^3(\mathcal{T}, \mathbb{V})$.

Lemma 2.1 *Let $\mathcal{T}$ be a Lie triple system, $\mathbb{V}$ be a vector space and $\theta : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \longrightarrow \mathbb{V}$ be a trilinear map. The pair $(\mathcal{T}_\theta, [-,-,-]_{\mathcal{T}_\theta})$ is a Lie triple system if and only if $\theta \in Z^3(\mathcal{T}, \mathbb{V})$.*

Proof Since $\mathcal{T}$ is a Lie triple system, from (A1)–(A3) we have the following identities:

$$
\begin{aligned}
&[x, y, z]_{\mathcal{T}_\theta} + [y, x, z]_{\mathcal{T}_\theta} = \theta(x, y, z) + \theta(y, x, z),\\
&[x, y, z]_{\mathcal{T}_\theta} + [y, z, x]_{\mathcal{T}_\theta} + [z, x, y]_{\mathcal{T}_\theta} = \theta(x, y, z) + \theta(y, z, x) + \theta(z, x, y),\\
&\left[u, v, [x, y, z]_{\mathcal{T}_\theta}\right]_{\mathcal{T}_\theta} + \left[[v, u, x]_{\mathcal{T}_\theta}, y, z\right]_{\mathcal{T}_\theta} + \left[x, [v, u, y]_{\mathcal{T}_\theta}, z\right]_{\mathcal{T}_\theta} + \left[x, y, [v, u, z]_{\mathcal{T}_\theta}\right]_{\mathcal{T}_\theta}\\
&\quad = \theta(u, v, [x, y, z]) + \theta([v, u, x], y, z) + \theta(x, [v, u, y], z) + \theta(x, y, [v, u, z]),
\end{aligned}
$$

for any x, y, z, u, v in $\mathcal{T}_\theta$. So, $\mathcal{T}_\theta$ is a Lie triple system if and only if $\theta \in Z^3(\mathcal{T}, \mathbb{V})$. □

If $\dim \mathbb{V} = s$, the Lie triple system $\mathcal{T}_\theta$ is called an s- *dimensional annihilator extension* of $\mathcal{T}$ by $\mathbb{V}$.

Definition 2.2 Let $\mathcal{T}$ be a Lie triple system and let $\mathbb{V}$ be a vector space. The *radical* of a trilinear map $\theta \in Z^3(\mathcal{T}, \mathbb{V})$ is the set

$$\mathrm{Rad}(\theta) = \{x \in \mathcal{T} : \theta(x, \mathcal{T}, \mathcal{T}) = 0\}.$$

Clearly, $\theta(x, \mathcal{T}, \mathcal{T}) = 0$ if and only if $\theta(\mathcal{T}, x, \mathcal{T}) = \theta(\mathcal{T}, \mathcal{T}, x) = 0$.

Lemma 2.2 *Let $\mathcal{T}$ be a Lie triple system, let $\mathbb{V}$ be a vector space and $\theta \in Z^3(\mathcal{T}, \mathbb{V})$. Then*

$$\mathrm{Ann}(\mathcal{T}_\theta) = (\mathrm{Rad}(\theta) \cap \mathrm{Ann}(\mathcal{T})) \oplus \mathbb{V}.$$

Proof Since $\mathbb{V} \subset \mathrm{Ann}(\mathcal{T}_\theta)$, we may write $\mathrm{Ann}(\mathcal{T}_\theta) = \mathbb{W} \oplus \mathbb{V}$ where $\mathbb{W}$ is a subspace of $\mathcal{T}$. For any $x \in \mathcal{T}$, we have $[x, \mathcal{T}_\theta, \mathcal{T}_\theta]_{\mathcal{T}_\theta} = [x, \mathcal{T}, \mathcal{T}]_{\mathcal{T}} + \theta(x, \mathcal{T}, \mathcal{T})$. Thus, $x \in \mathbb{W}$ if and only if $x \in \mathrm{Rad}(\theta) \cap \mathrm{Ann}(\mathcal{T})$. So $\mathbb{W} = \mathrm{Rad}(\theta) \cap \mathrm{Ann}(\mathcal{T})$. □

Lemma 2.3 *Let $\mathcal{T}$ be a Lie triple system, let $\mathbb{V}$ be a vector space and $\theta \in Z^3(\mathcal{T}, \mathbb{V})$. In this condition, $\mathcal{T}_\theta$ is nilpotent if and only if $\mathcal{T}$ is nilpotent.*

Proof Since $\left[\mathcal{T}_{\theta}^{(n)}, x+u, y+v\right]_{\mathcal{T}_{\theta}} = \left[\mathcal{T}^{(n)}, x, y\right] + \theta(\mathcal{T}^{(n)}, x, y)$, for all $x, y \in \mathcal{T}$ and $u, v \in \mathbb{V}$, we conclude that $\mathcal{T}_{\theta}$ is nilpotent if and only if $\mathcal{T}$ is nilpotent. $\square$

The following result shows that every Lie triple system with a non-zero annihilator is an annihilator extension of a smaller-dimensional Lie triple system.

Lemma 2.4 *Let $\mathcal{T}$ be a n-dimensional Lie triple system with* $\dim(\mathrm{Ann}(\mathcal{T})) = m \neq 0$. *Then there exists, up to isomorphism, an unique $(n-m)$-dimensional Lie triple system $\mathcal{T}'$ and a trilinear map $\theta \in \mathrm{Z}^3\left(\mathcal{T}', \mathrm{Ann}\left(\mathcal{T}\right)\right)$ with* $\mathrm{Rad}\,(\theta) \cap \mathrm{Ann}\left(\mathcal{T}'\right) = 0$ *such that $\mathcal{T} \cong \mathcal{T}'_{\theta}$ and $\mathcal{T}/\mathrm{Ann}\,(\mathcal{T}) \cong \mathcal{T}'$.*

Proof Let $\mathcal{T}'$ be a linear complement of $\mathrm{Ann}\,(\mathcal{T})$ in $\mathcal{T}$. Define a linear map $P : \mathcal{T} \longrightarrow \mathcal{T}'$ by $P(x+v) := x$ for $x \in \mathcal{T}'$ and $v \in \mathrm{Ann}\,(\mathcal{T})$, and define a trilinear map on $\mathcal{T}'$ by $[x, y, z]_{\mathcal{T}'} := P([x, y, z]_{\mathcal{T}})$ for $x, y, z \in \mathcal{T}'$. Then we get

$$
\begin{aligned}
P([x, y, z]_{\mathcal{T}}) &= P([x - P(x) + P(x), y - P(y) + P(y), z - P(z) + P(z)]_{\mathcal{T}}) \\
&= P\left([P(x), P(y), P(z)]_{\mathcal{T}}\right) \\
&= [P(x), P(y), P(z)]_{\mathcal{T}'},
\end{aligned}
$$

for any $x, y, z \in \mathcal{T}$, thus P is a homomorphism of Lie triple systems. So $P(\mathcal{T}) = \mathcal{T}'$ is a Lie triple system and $\mathcal{T}/\mathrm{Ann}\,(\mathcal{T}) \cong \mathcal{T}'$, which give us the uniqueness. Now, define the trilinear map $\theta : \mathcal{T}' \times \mathcal{T}' \times \mathcal{T}' \longrightarrow \mathrm{Ann}\,(\mathcal{T})$ by $\theta(x, y, z) := [x, y, z]_{\mathcal{T}} - [x, y, z]_{\mathcal{T}'}$. Consequently, $\mathcal{T}'_{\theta}$ is $\mathcal{T}$, therefore, $\theta \in Z^3\left(\mathcal{T}', \mathrm{Ann}\,(\mathcal{T})\right)$ by Lemma 2.1, as well $\mathrm{Rad}\,(\theta) \cap \mathrm{Ann}\left(\mathcal{T}'\right) = 0$ by Lemma 2.2. $\square$

Let $\mathcal{T}$ be a Lie triple system and let $\mathbb{V}$ be a vector space. We denote as usually the set of all linear maps from $\mathcal{T}$ to $\mathbb{V}$ by $\mathrm{Hom}\,(\mathcal{T}, \mathbb{V})$. For $f \in \mathrm{Hom}\,(\mathcal{T}, \mathbb{V})$, consider the map $\delta f : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \longrightarrow \mathbb{V}$ defined by $\delta f(x, y, z) := f([x, y, z])$, for any $x, y, z \in \mathcal{T}$, then $\delta f \in Z^3\,(\mathcal{T}, \mathbb{V})$. The following definition arises:

Definition 2.3 Let $\mathcal{T}$ be a Lie triple system and let $\mathbb{V}$ be a vector space. Define the vector subspace of $Z^3\,(\mathcal{T}, \mathbb{V})$ by

$$
B^3\,(\mathcal{T}, \mathbb{V}) = \{\delta f : f \in \mathrm{Hom}\,(\mathcal{T}, \mathbb{V})\}\,.
$$

The elements of $B^3\,(\mathcal{T}, \mathbb{V})$ are called *coboundaries*.

Lemma 2.5 *Let $\mathcal{T}$ be an n-dimensional Lie triple system and let $\{e_1, e_2, \ldots, e_m\}$ be a basis of $\mathcal{T}^{(1)} = [\mathcal{T}, \mathcal{T}, \mathcal{T}]$. Then the set $\left\{\delta e_1^*, \delta e_2^*, \ldots, \delta e_m^*\right\}$, where $e_i^*(e_j) := \delta_{ij}$ and δ_{ij} is the Kronecker's symbol for $i, j \in \{1, 2, \ldots, m\}$, is a basis of $B^3\,(\mathcal{T}, \mathbb{F})$.*

Proof Let $\{e_1, \ldots, e_m, e_{m+1}, \ldots, e_n\}$ be a basis of $\mathcal{T}$, by extending the basis of $[\mathcal{T}, \mathcal{T}, \mathcal{T}]$. Then $\left\{e_1^*, e_2^*, \ldots, e_n^*\right\}$ is a basis of the dual space $\mathcal{T}^* = \mathrm{Hom}\,(\mathcal{T}, \mathbb{F})$. For any $\delta f \in B^3\,(\mathcal{T}, \mathbb{F})$, with $f = \sum_{l=1}^{n} \alpha_l e_l^*$, we have

$$
\begin{aligned}
\delta f(e_i, e_j, e_k) &= \sum_{l=1}^{n} \alpha_l e_l^*\left([e_i, e_j, e_k]\right) = \sum_{l=1}^{n} \alpha_l e_l^* \left(\sum_{p=1}^{m} \tau_{ijk}^{p} e_p\right) \\
&= \sum_{l=1}^{m} \alpha_l e_l^*\left([e_i, e_j, e_k]\right) = \sum_{l=1}^{m} \alpha_l \delta e_l^*(e_i, e_j, e_k).
\end{aligned}
$$

Therefore $\delta f = \sum_{l=1}^{m} \alpha_l \delta e_l^*$, proving that $\delta e_1^*, \delta e_2^*, \ldots, \delta e_m^*$ spans $B^3(\mathcal{T}, \mathbb{F})$. Moreover, let $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$ be such that $\sum_{l=1}^{m} \alpha_l \delta e_l^* = 0$, then

$$\left(\sum_{l=1}^{m} \alpha_l e_l^*\right)([\mathcal{T}, \mathcal{T}, \mathcal{T}]) = \left(\sum_{l=1}^{m} \alpha_l \delta e_l^*\right)(\mathcal{T}, \mathcal{T}, \mathcal{T}) = 0.$$

This implies that $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$ and consequently $\delta e_1^*, \delta e_2^*, \ldots, \delta e_m^*$ are linearly independent. □

Let $\mathcal{T}$ be a Lie triple system and let $\mathbb{V}$ be a vector space. We define $H^3(\mathcal{T}, \mathbb{V})$ as the quotient space $Z^3(\mathcal{T}, \mathbb{V}) \big/ B^3(\mathcal{T}, \mathbb{V})$. The equivalence class of $\theta \in Z^3(\mathcal{T}, \mathbb{V})$ will be denoted $[\theta] \in H^3(\mathcal{T}, \mathbb{V})$.

Lemma 2.6 *Let $\mathcal{T}$ be a Lie triple system and let $\mathbb{V}$ be a vector space. Given two cocycles $\theta, \vartheta \in Z^3(\mathcal{T}, \mathbb{V})$ such that $[\theta] = [\vartheta]$, then* $\operatorname{Ann}(\mathcal{T}_\theta) = \operatorname{Ann}(\mathcal{T}_\vartheta)$. *Moreover, $\mathcal{T}_\theta \cong \mathcal{T}_\vartheta$.*

Proof Suppose $[\theta] = [\vartheta]$, then $\vartheta = \theta + \delta f$ for some $f \in \operatorname{Hom}(\mathcal{T}, \mathbb{V})$. So for all $x, y, z \in \mathcal{T}$, we have

$$\vartheta(x, y, z) = \theta(x, y, z) + f([x, y, z]).$$

Hence, $\theta(x, y, y) = [x, y, z] = 0$ if and only if $\vartheta(x, y, z) = [x, y, z] = 0$. Therefore, $\operatorname{Rad}(\theta) \cap \operatorname{Ann}(\mathcal{T}) = \operatorname{Rad}(\vartheta) \cap \operatorname{Ann}(\mathcal{T})$, so $\operatorname{Ann}(\mathcal{T}_\theta) = \operatorname{Ann}(\mathcal{T}_\vartheta)$ by Lemma 2.2. Further, we define a linear map $\varphi : \mathcal{L}_\theta \to \mathcal{T}_\vartheta$ by $\varphi(x + v) := x + f(x) + v$ for $x \in \mathcal{T}$ and $v \in \mathbb{V}$. Clearly φ is bijective. Moreover, if $x, y, z \in \mathcal{T}$ and $u, v, w \in \mathbb{V}$, then we have

$$\begin{aligned}
\varphi\left([x + u, y + v, z + w]_{\mathcal{T}_\theta}\right) &= \varphi\left([x, y, z]_{\mathcal{T}} + \theta(x, y, z)\right) \\
&= [x, y, z]_{\mathcal{T}} + f\left([x, y, z]_{\mathcal{T}}\right) + \theta(x, y, z) \\
&= [x, y, z]_{\mathcal{T}} + \delta f(x, y, z) + \theta(x, y, z) \\
&= [x, y, z]_{\mathcal{T}} + (\delta f + \theta)(x, y, z) \\
&= [x, y, z]_{\mathcal{T}} + \vartheta(x, y, z) \\
&= [x + f(x) + u, y + f(y) + v, z + f(z) + w]_{\mathcal{T}_\vartheta} \\
&= [\varphi(x + u), \varphi(y + v), \varphi(z + w)]_{\mathcal{T}_\vartheta},
\end{aligned}$$

hence, $\mathcal{T}_\theta$ and $\mathcal{T}_\vartheta$ are isomorphic. □

Let $\operatorname{Aut}(\mathcal{T})$ be the automorphism group of the Lie triple system $\mathcal{T}$ and $\mathbb{V}$ be a vector space. For an automorphism $\phi \in \operatorname{Aut}(\mathcal{T})$ and a cocycle $\theta \in Z^3(\mathcal{T}, \mathbb{V})$, define $\phi\theta : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \longrightarrow \mathbb{V}$ by $\phi\theta(x, y, z) := \theta(\phi(x), \phi(y), \phi(z))$, for all $x, y, z \in \mathcal{T}$. Then $\phi\theta \in Z^3(\mathcal{T}, \mathbb{V})$. This is an action of the group $\operatorname{Aut}(\mathcal{T})$ on $Z^3(\mathcal{T}, \mathbb{V})$.

Lemma 2.7 *Let $\mathcal{T}$ be a Lie triple system and let $\mathbb{V}$ be a vector space. For an automorphism $\phi \in \operatorname{Aut}(\mathcal{T})$ and a cocycle $\theta \in Z^3(\mathcal{T}, \mathbb{V})$, the map $\phi\theta \in B^3(\mathcal{T}, \mathbb{V})$ if and only if $\theta \in B^3(\mathcal{T}, \mathbb{V})$.*

Proof Let $\theta \in B^3(\mathcal{T}, \mathbb{V})$, so $\theta = \delta f$ for some $f \in \operatorname{Hom}(\mathcal{T}, \mathbb{V})$. We have

$$\begin{aligned}
\phi\theta(x, y, z) = \theta(\phi(x), \phi(y), \phi(z)) &= \delta f(\phi(x), \phi(y), \phi(z)) = f([\phi(x), \phi(y), \phi(z)]) \\
&= f(\phi([x, y, z])) = \delta(f \circ \phi)(x, y, z),
\end{aligned}$$

for any $x, y, z \in \mathcal{T}$, hence $\phi\theta = \delta(f \circ \phi) \in B^3(\mathcal{L}, \mathbb{V})$. Conversely, suppose $\phi\theta \in B^3(\mathcal{T}, \mathbb{V})$, then by the direct implication $\theta = \phi^{-1}(\phi\theta) \in B^3(\mathcal{L}, \mathbb{V})$. □

Since $B^3(\mathcal{T},\mathbb{V})$ is invariant under the action of $\mathrm{Aut}(\mathcal{T})$, we have an induced action of $\mathrm{Aut}(\mathcal{T})$ on $H^3(\mathcal{T},\mathbb{V})$. Our goal is to find all annihilator extensions of a Lie triple system $\mathcal{T}$ by a vector space $\mathbb{V}$. In order to solve the isomorphism problem, we need to study the action of $\mathrm{Aut}(\mathcal{T})$ on $H^3(\mathcal{T},\mathbb{V})$.

Lemma 2.8 *Let $\mathcal{T}_1, \mathcal{T}_2$ be two Lie triple systems and let $\mathbb{V}$ be a vector space. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are isomorphic, then it is verified the following equivalences:*

(1) $Z^3(\mathcal{T}_1,\mathbb{V}) \cong Z^3(\mathcal{T}_2,\mathbb{V})$.
(2) $B^3(\mathcal{T}_1,\mathbb{V}) \cong B^3(\mathcal{T}_2,\mathbb{V})$.
(3) $H^3(\mathcal{T}_1,\mathbb{V}) \cong H^3(\mathcal{T}_2,\mathbb{V})$.

Proof (1) Consider an isomorphism $\phi : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$. We can prove that $\theta \in Z^3(\mathcal{T}_1,\mathbb{V})$ if and only if $\phi^{-1}\theta \in Z^3(\mathcal{T}_2,\mathbb{V})$, where $\phi^{-1}\theta : \mathcal{T}_2 \times \mathcal{T}_2 \times \mathcal{T}_2 \longrightarrow \mathbb{V}$ is defined by $\phi^{-1}\theta(x,y,z) := \theta\left(\phi^{-1}(x), \phi^{-1}(y), \phi^{-1}(z)\right)$, for $x,y,z \in \mathcal{T}_2$. This defines an isomorphism $\tau : Z^3(\mathcal{T}_1,\mathbb{V}) \longrightarrow Z^3(\mathcal{T}_2,\mathbb{V})$ by $\tau(\theta) := \phi^{-1}\theta$.
(2) Take $\theta \in B^3(\mathcal{T}_1,\mathbb{V})$, so $\theta = \delta f$ for some $f \in \mathrm{Hom}(\mathcal{T}_1,\mathbb{V})$. Then $\tau(\theta) = \tau(\delta f) = \phi^{-1}(\delta f) = \delta(f \circ \phi^{-1}) \in \mathcal{B}^3(\mathcal{T}_2,\mathbb{V})$, since $f \circ \phi^{-1} \in \mathrm{Hom}(\mathcal{T}_2,\mathbb{V})$. On the other hand, take $\vartheta \in B^3(\mathcal{T}_2,\mathbb{V})$, so $\vartheta = \delta f \in B^3(\mathcal{T}_2,\mathbb{V})$ for some $f \in \mathrm{Hom}(\mathcal{T}_2,\mathbb{V})$. Then $\tau(\theta) = \vartheta$, where $\theta = \delta(f \circ \phi) \in B^3(\mathcal{T}_1,\mathbb{V})$. So $\tau : B^3(\mathcal{T}_1,\mathbb{V}) \longrightarrow B^3(\mathcal{T}_2,\mathbb{V})$ is an isomorphism.
(3) It follows from (1) and (2). □

Lemma 2.9 *Let $\mathcal{T}$ be a Lie triple system and let $\mathbb{V}$ be a vector space. If $\theta \in Z^3(\mathcal{T},\mathbb{V})$ and $\phi \in \mathrm{Aut}(\mathcal{T})$, then $\mathcal{T}_\theta \cong \mathcal{T}_{\phi\theta}$ and* $\mathrm{Rad}(\theta) \cong \mathrm{Rad}(\phi\theta)$.

Proof Consider the linear map $\varphi : \mathcal{T}_{\phi\theta} \longrightarrow \mathcal{T}_\theta$ defined by $\varphi(x+v) := \phi(x) + v$, for $x \in \mathcal{T}$ and $v \in \mathbb{V}$. Then φ is bijective. For any $x,y,z \in \mathcal{T}$ and $u,v,w \in \mathbb{V}$, we have

$$\begin{aligned}
\varphi\left([x+u, y+v, z+w]_{\mathcal{T}_{\phi\theta}}\right) &= \varphi\left([x,y,z]_{\mathcal{T}} + \phi\theta(x,y,z)\right) \\
&= \phi([x,y,z]_{\mathcal{T}}) + \phi\theta(x,y,z) \\
&= [\phi(x), \phi(y), \phi(z)]_{\mathcal{T}} + \theta(\phi(x), \phi(y), \phi(z)) \\
&= [\varphi(x+u), \varphi(y+v), \varphi(z+w)]_{\mathcal{T}_\theta},
\end{aligned}$$

hence φ is an isomorphism of Lie triple systems. With respect to the radical, for $x \in \mathcal{T}$ we obtain $\phi\theta(x,\mathcal{T},\mathcal{T}) = \theta(\phi(x),\mathcal{T},\mathcal{T})$. So $x \in \mathrm{Rad}(\phi\theta)$ if and only if $\phi(x) \in \mathrm{Rad}(\theta)$, therefore $\varphi|_{\mathrm{Rad}(\phi\theta)} : \mathrm{Rad}(\phi\theta) \longrightarrow \mathrm{Rad}(\theta)$ is bijective. □

Denote as usually the set of linear isomorphisms from $\mathbb{V}$ to itself by $GL(\mathbb{V})$. Given an automorphism $\psi \in GL(\mathbb{V})$ and a cocycle $\theta \in Z^3(\mathcal{T},\mathbb{V})$, define $\psi\theta(x,y,z) := \psi(\theta(x,y,z))$, for any $x,y,z \in \mathcal{T}$. Then $\psi\theta \in Z^3(\mathcal{T},\mathbb{V})$. So it is an action of the group $GL(\mathbb{V})$ on $Z^3(\mathcal{T},\mathbb{V})$.

Lemma 2.10 *Let $\mathcal{T}$ be a Lie triple system and let $\mathbb{V}$ be a vector space. For $\theta \in Z^3(\mathcal{T},\mathbb{V})$ and $\psi \in GL(\mathbb{V})$, we have $\psi\theta \in B^3(\mathcal{T},\mathbb{V})$ if and only if $\theta \in B^3(\mathcal{T},\mathbb{V})$.*

Proof Suppose that $\psi\theta \in B^3(\mathcal{T},\mathbb{V})$. Let $\psi\theta = \delta f$ for some $f \in \mathrm{Hom}(\mathcal{T},\mathbb{V})$. Then

$$\begin{aligned}
\theta(x,y,z) = \psi^{-1}\psi(\theta(x,y,z)) = \psi^{-1}(\delta f(x,y,z)) &= (\psi^{-1} \circ f)([x,y,z]) \\
= \delta(\psi^{-1} \circ f)(x,y,z)&,
\end{aligned}$$

for any $x,y,z \in \mathcal{T}$. Hence, $\theta = \delta(\psi^{-1} \circ f) \in B^3(\mathcal{T},\mathbb{V})$.

Conversely, assume that $\theta \in B^3(\mathcal{T}, \mathbb{V})$ and let $\theta = \delta f$ for some $f \in \operatorname{Hom}(\mathcal{T}, \mathbb{V})$. Then

$$\psi\theta(x, y, z) = \psi(\delta f)(x, y, z) = (\psi \circ f)([x, y, z]) = \delta(\psi \circ f)(x, y, z),$$

for any $x, y, z \in \mathcal{T}$. So $\psi\theta = \delta(\psi \circ f) \in B^3(\mathcal{T}, \mathbb{V})$. □

Since $B^3(\mathcal{T}, \mathbb{V})$ is invariant under the action of $GL(\mathbb{V})$ in the space $Z^3(\mathcal{T}, \mathbb{V})$, we have an induced action of $GL(\mathbb{V})$ on $H^3(\mathcal{T}, \mathbb{V})$.

Lemma 2.11 *Let $\mathcal{T}$ be a Lie triple system and let $\mathbb{V}$ be a vector space. If $\theta \in B^3(\mathcal{T}, \mathbb{V})$ and $\psi \in GL(\mathbb{V})$, then $\mathcal{T}_\theta \cong \mathcal{T}_{\psi\theta}$ and* $\operatorname{Rad}(\theta) = \operatorname{Rad}(\psi\theta)$.

Proof Define a linear map $\varphi : \mathcal{T}_\theta \longrightarrow \mathcal{T}_{\psi\theta}$ by $\varphi(x + v) := x + \psi(v)$ for $x \in \mathcal{T}$ and $v \in \mathbb{V}$. Then φ is a bijective map. Also, for any $x, y, z \in \mathcal{T}$ and $u, v, w \in \mathbb{V}$, we have

$$\begin{aligned}\varphi\left([x+u, y+v, z+w]_{\mathcal{T}_\theta}\right) &= \varphi\left([x, y, z]_{\mathcal{T}} + \theta(x, y, z)\right) = [x, y, z]_{\mathcal{T}} + (\psi\theta)(x, y, z)\\ &= [x+u, y+v, z+w]_{\mathcal{T}_{\psi\theta}},\end{aligned}$$

hence, $\mathcal{T}_\theta \cong \mathcal{T}_{\psi\theta}$. With respect to the radical, since $\theta(x, \mathcal{T}, \mathcal{T}) = \psi^{-1}\psi(\theta(x, \mathcal{T}, \mathcal{T})) = \psi^{-1}((\psi\theta)(x, \mathcal{T}, \mathcal{T}))$ we conclude that $x \in \operatorname{Rad}(\psi\theta)$ if and only if $x \in \operatorname{Rad}(\theta)$. □

Lemma 2.12 *Let $\mathcal{T}$ be a Lie triple system, let $\mathbb{V}$ be a vector space and let $\theta, \vartheta \in Z^3(\mathcal{T}, \mathbb{V})$. If there exist a map $\phi \in Aut(\mathcal{T})$ and a map $\psi \in GL(\mathbb{V})$ such that $[\phi\theta] = [\psi\vartheta]$, then $\mathcal{T}_\theta \cong \mathcal{T}_\vartheta$.*

Proof Let $\phi \in Aut(\mathcal{T})$ and $\psi \in GL(\mathbb{V})$ such that $[\phi\theta] = [\psi\vartheta]$. Then by Lemma 2.6 we have $\mathcal{T}_{\phi\theta} \cong \mathcal{T}_{\psi\vartheta}$, by Lemma 2.9 we get $\mathcal{T}_\theta \cong \mathcal{T}_{\phi\theta}$, and by Lemma 2.11 we obtain $\mathcal{T}_{\psi\vartheta} \cong \mathcal{T}_\vartheta$. □

Now, using the two actions on $Z^3(\mathcal{T}, \mathbb{V})$ that we have established before, we can state the following result.

Lemma 2.13 *Let $\mathcal{T}$ be a Lie triple system and let $\mathbb{V}$ be a vector space. Let θ, ϑ be two cocycles in $Z^3(\mathcal{T}, \mathbb{V})$ such that* $\operatorname{Ann}(\mathcal{T}_\theta) = \operatorname{Ann}(\mathcal{T}_\vartheta) = \mathbb{V}$. *In this conditions, $\mathcal{T}_\theta \cong \mathcal{T}_\vartheta$ if and only if there exist an automorphism $\phi \in Aut(\mathcal{T})$ and a linear isomorphism $\psi \in GL(\mathbb{V})$ such that* $[\phi\theta] = [\psi\vartheta]$.

Proof If there exist a map $\phi \in Aut(\mathcal{T})$ and a map $\psi \in GL(\mathbb{V})$ such that $[\phi\theta] = [\psi\vartheta]$ then $\mathcal{T}_\theta \cong \mathcal{T}_\vartheta$ by Lemma 2.12. Now let us show the converse. Consider $\theta, \vartheta \in Z^3(\mathcal{T}, \mathbb{V})$ such that $\operatorname{Ann}(\mathcal{T}_\theta) = \operatorname{Ann}(\mathcal{T}_\vartheta) = \mathbb{V}$. Suppose that $\mathcal{T}_\theta \cong \mathcal{T}_\vartheta$, there exists an isomorphism $\Phi : \mathcal{T}_\theta \longrightarrow \mathcal{T}_\vartheta$. Since $\Phi(\mathbb{V}) = \Phi(\operatorname{Ann}(\mathcal{T}_\theta)) = \operatorname{Ann}(\mathcal{T}_\vartheta) = \mathbb{V}$, we define $\psi := \Phi|_{\mathbb{V}} \in GL(\mathbb{V})$. Let $e_1, e_2, \ldots, e_n$ be a basis of $\mathcal{T}$, and let $\Phi(e_i) = e_i' + v_i$, where $e_i' \in \mathcal{T}$ and $v_i \in V$, for every $i \in \{1, \ldots, n\}$. Then Φ induces an automorphism $\phi : \mathcal{T} \longrightarrow \mathcal{T}$ defined by $\phi(e_i) := e_i'$, and a linear map $\varphi : \mathcal{T} \longrightarrow \mathbb{V}$ defined by $\varphi(e_i) := v_i$. So we can realize Φ as a matrix of the form

$$\Phi = \begin{pmatrix} \phi & 0 \\ \varphi & \psi \end{pmatrix},$$

where $\phi \in Aut(\mathcal{T})$, $\psi = \Phi|_{\mathbb{V}} \in GL(\mathbb{V})$ and $\varphi \in \operatorname{Hom}(\mathcal{T}, \mathbb{V})$. Furthermore, for any $x, y, z \in \mathcal{T}$ we have

$$\begin{aligned}\Phi\left([x, y, z]_{\mathcal{T}_\theta}\right) = \Phi\left([x, y, z]_{\mathcal{T}} + \theta(x, y, z)\right) &= \phi([x, y, z]_{\mathcal{T}})\\ &+\varphi([x, y, z]_{\mathcal{T}}) + \psi(\theta(x, y, z)),\end{aligned}$$

and

$$\begin{aligned}[\Phi(x), \Phi(y), \Phi(z)]_{\mathcal{T}_\vartheta} &= [\phi(x)+\varphi(x), \phi(y)+\varphi(y), \phi(z)+\varphi(z)]_{\mathcal{T}_\vartheta}\\ &= [\phi(x), \phi(y), \phi(z)]_{\mathcal{T}} + \vartheta(\phi(x), \phi(y), \phi(z))\\ &= \phi([x, y, z]_{\mathcal{T}}) + \vartheta(\phi(x), \phi(y), \phi(z)).\end{aligned}$$

Since Φ is an isomorphism, it follows that

$$\vartheta(\phi(x), \phi(y), \phi(z)) = \varphi\left([x, y, z]_{\mathcal{T}}\right) + \psi(\theta(x, y, z)), \tag{1}$$

for all $x, y, z \in \mathcal{T}$. Hence we have $\phi\vartheta = \delta\varphi + \psi\theta$ and $[\phi\vartheta] = [\psi\theta]$. □

In case of $\theta = \vartheta$, we obtain from Condition (1) the following description of $Aut(\mathcal{T}_\theta)$.

Corollary 2.1 *Let $\mathcal{T}$ be a Lie triple system, let $\mathbb{V}$ be a vector space and $\theta \in Z^3(\mathcal{T}, \mathbb{V})$ such that* $\mathrm{Rad}(\theta) \cap \mathrm{Ann}(\mathcal{T}) = 0$. *Then the automorphism group $Aut(\mathcal{T}_\theta)$ consists of all linear maps of the matrix form*

$$\Phi = \begin{pmatrix} \phi & 0 \\ \varphi & \psi \end{pmatrix},$$

where $\phi \in Aut(\mathcal{T})$, $\psi \in GL(\mathbb{V})$ and $\varphi \in \mathrm{Hom}(\mathcal{T}, \mathbb{V})$, such that $\theta(\phi(x), \phi(y), \phi(z)) = \varphi\left([x, y, z]_{\mathcal{T}}\right) + \psi(\theta(x, y, z))$ for any $x, y, z \in \mathcal{T}$.

Definition 2.4 Let $\mathcal{T}$ be a Lie triple system. If $x \in \mathcal{T}$ satisfies $x \notin \mathcal{T}^{(1)} = [\mathcal{T}, \mathcal{T}, \mathcal{T}]$ and $x \in \mathrm{Ann}(\mathcal{T})$, we call $\mathbb{F}x$ an *annihilator component* of $\mathcal{T}$.

Remark 2.1 If $\phi : \mathcal{T}_1 \to \mathcal{T}_2$ is an isomorphism of Lie triple systems, then $\mathcal{T}_1$ has a annihilator component if and only if so has $\mathcal{T}_2$. In fact, consider the vectors $x \in \mathcal{T}_1$ and $y \in \mathcal{T}_2$ such that $\phi(x) = y$, it follows that $\mathbb{F}x$ is an annihilator component of $\mathcal{T}_1$ if and only if $\mathbb{F}y$ is an annihilator component of $\mathcal{T}_2$.

Let $\mathbb{V}$ be an s-dimensional vector space and let $e_1, e_2, \ldots, e_s$ be a fixed basis of $\mathbb{V}$. Then $\theta \in Z^3(\mathcal{T}, \mathbb{V})$ can be uniquely written as $\theta(x, y, z) = \sum_{i=1}^{s} \theta_i(x, y, z) e_i$, where $\theta_i \in Z^3(\mathcal{T}, \mathbb{F})$. Moreover, $\mathrm{Rad}(\theta) = \cap_{i=1}^{s} \mathrm{Rad}(\theta_i)$.

Lemma 2.14 *Let $\theta(x, y, z) = \sum_{i=1}^{s} \theta_i(x, y, z) e_i \in Z^3(\mathcal{T}, \mathbb{V})$ and* $\mathrm{Rad}(\theta) \cap \mathrm{Ann}(\mathcal{T}) = 0$. *Then $\mathcal{T}_\theta$ has an annihilator component if and only if $[\theta_1], [\theta_2], \ldots, [\theta_s]$ are linearly dependent in $H^3(\mathcal{T}, \mathbb{F})$.*

Proof Suppose $\mathcal{T}_\theta$ has an annihilator component and fix an annihilator component $\mathbb{F}v_1 \subseteq \mathbb{V}$. Enlarge the set $\{v_1\}$ to a set $\{v_1, v_2, \ldots, v_s\}$ to form a basis of $\mathbb{V}$. Then there exists an invertible matrix $\left(a_{ij}\right)$ change of basis such that $e_i = \sum_{j=1}^{s} a_{ij} v_j$, for any $i = 1, \ldots, s$. So

$$\theta(x, y, z) = \sum_{j=1}^{s} \left(\sum_{i=1}^{s} a_{ij} \theta_i(x, y, z) \right) v_j.$$

Since $v_1 \notin \mathcal{T}_\theta^{(1)}$, it follows that $\sum_{i=1}^{s} a_{i1} \theta_i(x, y, z) = 0$ for all $x, y, z \in \mathcal{T}$. Then $\sum_{i=1}^{s} a_{i1} \theta_i = 0$, and therefore $\sum_{i=1}^{s} a_{i1} [\theta_i] = 0$. Since $\det\left(a_{ij}\right) \neq 0$, then $[\theta_1], [\theta_2], \ldots, [\theta_s]$ are linearly dependent in $H^3(\mathcal{T}, \mathbb{F})$.
Conversely, suppose that $[\theta_1], [\theta_2], \ldots, [\theta_s]$ are linearly dependent and $[\theta_s] = \sum_{i=1}^{s-1} \alpha_i [\theta_i]$. Now, define a new cocycle $\vartheta(x, y, z) = \sum_{i=1}^{s} \vartheta_i(x, y, z) e_i$ by setting $\vartheta_i = \theta_i$ for $i = 1, \ldots, s-1$ and $\vartheta_s = \sum_{i=1}^{s-1} \alpha_i \theta_i$. Then $[\theta] = [\vartheta]$ and by Lemma 2.6 we get $\mathcal{T}_\theta \cong \mathcal{T}_\vartheta$.

Further, we have $\vartheta(x,y,z) = \sum_{i=1}^{s-1}\theta_i(x,y,z)(e_i+\alpha_i e_s)$. For $i = 1,\ldots,s-1$, set $w_i := e_i + \alpha_i e_s$. Then $\vartheta(x,y,z) = \sum_{i=1}^{s-1}\theta_i(x,y,z)\,w_i$. Hence, $\mathcal{T}_\vartheta^{(1)} = [\mathcal{T}_\vartheta,\mathcal{T}_\vartheta,\mathcal{T}_\vartheta] \subset \mathcal{T}\oplus\langle w_1, w_2,\ldots,w_{s-1}\rangle$, so that $\mathcal{T}_\vartheta$, and therefore also $\mathcal{T}_\theta$, has a annihilator component. □

The statement in Lemma 2.13 can be rephrased as follows.

Lemma 2.15 *Given two cocycles* $\theta(x,y,z) = \sum_{i=1}^{s}\theta_i(x,y,z)e_i$ *and* $\vartheta(x,y,z) = \sum_{i=1}^{s}\vartheta_i(x,y,z)e_i$ *in* $Z^3(\mathcal{T},\mathbb{V})$. *Suppose that* $\mathcal{T}_\theta$ *has no annihilator component and* $\mathrm{Rad}(\theta)\cap\mathrm{Ann}(\mathcal{T}) = \mathrm{Rad}(\vartheta)\cap\mathrm{Ann}(\mathcal{T}) = 0$. *In these conditions, the Lie triple systems* $\mathcal{T}_\theta$ *and* $\mathcal{T}_\vartheta$ *are isomorphic if and only if there exists an automorphism* $\phi\in Aut(\mathcal{T})$ *such that the set* $\{[\phi\vartheta_i] : i = 1,\ldots,s\}$ *spans the same subspace of* $H^3(\mathcal{T},\mathbb{F})$ *as the set* $\{[\theta_i] : i = 1,\ldots,s\}$.

Proof Suppose first that $\mathcal{T}_\theta \cong \mathcal{T}_\vartheta$. Then, by Lemma 2.13, there exist a map $\phi\in Aut(\mathcal{T})$ and a map $\psi\in GL(\mathbb{V})$ such that $[\phi\vartheta] = [\psi\theta]$. Let $\psi(e_i) = \sum_{j=1}^{s}a_{ij}e_j$. Then

$$(\phi\vartheta-\psi\theta)(x,y,z) = \sum_{j=1}^{s}\left(\phi\vartheta_j - \sum_{i=1}^{s}a_{ij}\theta_i\right)(x,y,z)\,e_j. \tag{2}$$

Since $\phi\vartheta_j - \sum_{i=1}^{s}a_{ij}\theta_i \in B^3(\mathcal{T},\mathbb{F})$, then $\left[\phi\vartheta_j\right] = \sum_{i=1}^{s}a_{ij}[\theta_i]$. Hence, $\{[\phi\vartheta_i] : i = 1,\ldots,s\}$ spans the same subspace of $H^3(\mathcal{T},\mathbb{F})$ as the set $\{[\theta_i] : i = 1,\ldots,s\}$.

Conversely, suppose $\{[\phi\vartheta_i] : i = 1,\ldots,s\}$ spans the same vector space as $\{[\theta_i] : i = 1,\ldots,s\}$. Then there exists an invertible matrix (a_{ij}) such that $\left[\phi\vartheta_j\right] = \sum_{i=1}^{s}a_{ij}[\theta_i]$. Define a linear map $\psi : \mathbb{V}\longrightarrow\mathbb{V}$ by $\psi(e_i) := \sum_{j=1}^{s}a_{ij}e_j$. Then $\psi\theta(x,y,z) = \sum_{i=1}^{s}\sum_{j=1}^{s}a_{ij}\theta_i(x,y,z)\,e_j$ and we have Eq. (2). Hence, $[\phi\vartheta] = [\psi\theta]$, and therefore $\mathcal{T}_\theta\cong\mathcal{T}_\vartheta$ by Lemma 2.13. □

Let $G_s\left(H^3(\mathcal{T},\mathbb{F})\right)$ be the Grassmannian of subspaces of dimension s in $H^3(\mathcal{T},\mathbb{F})$. The automorphism group $Aut(\mathcal{T})$ acts on $G_s\left(H^3(\mathcal{T},\mathbb{F})\right)$ as follows: if $\phi\in Aut(\mathcal{T})$ and vector space $\mathbb{W} = \langle[\theta_1],[\theta_2],\ldots,[\theta_s]\rangle \in G_s\left(H^3(\mathcal{T},\mathbb{F})\right)$, then $\phi\mathbb{W} := \langle[\phi\theta_1],[\phi\theta_2],\ldots,[\phi\theta_s]\rangle \in G_s\left(H^3(\mathcal{T},\mathbb{F})\right)$. We denote the orbit of $\mathbb{W}\in G_s\left(H^3(\mathcal{T},\mathbb{F})\right)$ under the action of $Aut(\mathcal{T})$ by $\mathcal{O}(\mathbb{W})$.

Lemma 2.16 *Given two vector spaces* $\mathbb{W}_1 = \langle[\theta_1],[\theta_2],\ldots,[\theta_s]\rangle$ *and* $\mathbb{W}_2 = \langle[\vartheta_1],[\vartheta_2],\ldots,[\vartheta_s]\rangle$ *in* $G_s\left(H^3(\mathcal{T},\mathbb{F})\right)$. *If* $\mathbb{W}_1 = \mathbb{W}_2$, *then* $\cap_{i=1}^{s}\mathrm{Rad}(\theta_i)\cap\mathrm{Ann}(\mathcal{T}) = \cap_{i=1}^{s}\mathrm{Rad}(\vartheta_i)\cap\mathrm{Ann}(\mathcal{T})$.

Proof Assume $\mathbb{W}_1 = \mathbb{W}_2$. Then there exists an invertible matrix (a_{ij}) such that $[\theta_i] = \sum_{j=1}^{s}a_{ij}\left[\vartheta_j\right]$. Therefore, we have $\theta_i = \sum_{j=1}^{s}a_{ij}\vartheta_j + \delta f_i$ for some $f_i\in\mathrm{Hom}(\mathcal{T},\mathbb{F})$. Then,

$$\theta_i(x,y,z) = \sum_{j=1}^{s}a_{ij}\vartheta_j(x,y,z) + f_i\left([x,y,z]_{\mathcal{T}}\right).$$

Hence, $\theta_1(x,y,z) = \cdots = \theta_s(x,y,z) = [x,y,z]_{\mathcal{T}} = 0$ if and only if $\vartheta_1(x,y,z) = \cdots = \vartheta_s(x,y,z) = [x,y,z]_{\mathcal{T}} = 0$. Therefore $\cap_{i=1}^{s}\mathrm{Rad}(\theta_i)\cap\mathrm{Ann}(\mathcal{T}) = \cap_{i=1}^{s}\mathrm{Rad}(\vartheta_i)\cap\mathrm{Ann}(\mathcal{T})$. □

From this result, we can define the set

$$\mathcal{T}_s(\mathcal{T}) := \left\{\mathbb{W} = \langle[\theta_1],[\theta_2],\ldots,[\theta_s]\rangle \in G_s\left(H^3(\mathcal{T},\mathbb{F})\right) : \bigcap_{i=1}^{s}\mathrm{Rad}(\theta_i)\cap\mathrm{Ann}(\mathcal{T}) = 0\right\}.$$

Lemma 2.17 *The set $\mathcal{T}_s(\mathcal{T})$ is stable under the action of $Aut(\mathcal{T})$.*

Proof Let $\phi \in Aut(\mathcal{T})$ and $\mathbb{W} = \langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle \in G_s\left(H^3(\mathcal{T}, \mathbb{F})\right)$. Then, $x \in \cap_{i=1}^s \mathrm{Rad}(\phi\theta_i) \cap \mathrm{Ann}(\mathcal{T})$ if and only if $\phi(x) \in \cap_{i=1}^s \mathrm{Rad}(\theta_i) \cap \mathrm{Ann}(\mathcal{T})$. So $\cap_{i=1}^s \mathrm{Rad}(\phi\theta_i) \cap \mathrm{Ann}(\mathcal{T}) = 0$ if and only if $\cap_{i=1}^s \mathrm{Rad}(\theta_i) \cap \mathrm{Ann}(\mathcal{T}) = 0$. Consequently, $\phi\mathbb{W} \in \mathcal{T}_s(\mathcal{T})$ if and only if $\mathbb{W} \in \mathcal{T}_s(\mathcal{T})$. □

Let $\mathcal{T}$ be a Lie triple system, let $\mathbb{V}$ be an s-dimensional vector space and let $\{e_1, e_2, \ldots, e_s\}$ be a basis of $\mathbb{V}$. Consider the set $\mathcal{E}(\mathcal{T}, \mathbb{V})$ of all Lie triple systems without annihilator components, which are s-dimensional annihilator extensions of $\mathcal{T}$ by $\mathbb{V}$ and have s-dimensional annihilator. Then

$$\mathcal{E}(\mathcal{T}, \mathbb{V}) = \left\{ \mathcal{T}_\theta : \theta(x, y, z) = \sum_{i=1}^{s} \theta_i(x, y, z) e_i \text{ and } \langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle \in \mathcal{T}_s(\mathcal{T}) \right\}.$$

Given $\mathcal{T}_\theta \in \mathcal{E}(\mathcal{T}, \mathbb{V})$, we denote by $[\mathcal{T}_\theta]$ the isomorphism class of $\mathcal{T}_\theta$. Using this new notation, we can rephrase Lemma 2.15 as follows.

Lemma 2.18 *Let $\mathcal{T}_\theta, \mathcal{T}_\vartheta \in \mathcal{E}(\mathcal{T}, \mathbb{V})$. Suppose that $\theta(x, y, z) = \sum_{i=1}^s \theta_i(x, y, z) e_i$ and $\vartheta(x, y, z) = \sum_{i=1}^s \vartheta_i(x, y, z) e_i$. In this conditions, $[\mathcal{T}_\theta] = [\mathcal{T}_\vartheta]$ if and only if $\mathcal{O}\langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle = \mathcal{O}\langle [\vartheta_1], [\vartheta_2], \ldots, [\vartheta_s] \rangle$.*

Proof Consider two Lie triple systems $\mathcal{T}_\theta, \mathcal{T}_\vartheta \in \mathcal{E}(\mathcal{T}, \mathbb{V})$. By Lemma 2.15, $\mathcal{T}_\theta \cong \mathcal{T}_\vartheta$ if and only if there exists an automorphism $\phi \in Aut(\mathcal{T})$ such that $\langle [\phi\vartheta_1], [\phi\vartheta_2], \ldots, [\phi\vartheta_s] \rangle = \langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle$. Hence, $[\mathcal{T}_\theta] = [\mathcal{T}_\vartheta]$ if and only if $\mathcal{O}\langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle = \mathcal{O}\langle [\vartheta_1], [\vartheta_2], \ldots, [\vartheta_s] \rangle$. □

Thus, each orbit of $Aut(\mathcal{T})$ on $\mathcal{T}_s(\mathcal{T})$ corresponds uniquely to an isomorphism class of $\mathcal{E}(\mathcal{T}, \mathbb{V})$, and vice-versa. This correspondence is defined by

$$\mathcal{O}\langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle \in \{\mathcal{O}(\mathbb{W}) : \mathbb{W} \in \mathcal{T}_s(\mathcal{T})\} \longleftrightarrow [\mathcal{T}_\theta] \in \{[\mathcal{T}_\vartheta] : \mathcal{T}_\vartheta \in \mathcal{E}(\mathcal{T}, \mathbb{V})\},$$

where $\theta(x, y, z) = \sum_{i=1}^s \theta_i(x, y, z) e_i$. We call $\mathcal{T}_\theta$ the Lie triple system corresponding to the representative $\langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle$. Finally, we have the following correspondence theorem.

Theorem 2.1 *There exists a one-to-one correspondence between the set of $Aut(\mathcal{T})$-orbits on $\mathcal{T}_s(\mathcal{T})$ and the set of isomorphism classes of $\mathcal{E}(\mathcal{T}, \mathbb{V})$.*

By this theorem, we can construct all Lie triple systems of dimension n with s-dimensional annihilator, given those algebras of dimension $n - s$, following this steps:

(1) For a Lie triple system $\mathcal{T}$ of dimension $n-s$, determine $H^3(\mathcal{T}, \mathbb{F})$, $\mathrm{Ann}(\mathcal{T})$ and $Aut(\mathcal{T})$.
(2) Determine the set of $Aut(\mathcal{T})$-orbits on $\mathcal{T}_s(\mathcal{T})$.
(3) For each orbit, construct the Lie triple system corresponding to a representative of it.

3 The algebraic classification of nilpotent Lie triple systems up to dimension four

Let us introduce the following notations. Let $\mathcal{T}$ be a Lie triple system with a basis $\{e_1, e_2, \ldots, e_n\}$. Let us now define a trilinear form $\Delta_{i,j,k} : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \longrightarrow \mathbb{F}$ by

$$\Delta_{i,j,k}(e_l, e_m, e_t) = \begin{cases} 1 & \text{if } (i, j, k) = (l, m, t), \\ -1 & \text{if } (j, i, k) = (l, m, t), \\ 0 & \text{otherwise.} \end{cases}$$

Then $Z^3(\mathcal{T},\mathbb{F})$ is a subspace of $\left\langle \Delta_{i,j,k} : 1 \le i < j \le n, 1 \le k \le n \right\rangle$. In particular, we have $Z^3(\mathcal{T},\mathbb{F}) = 0$ if $\dim\mathcal{T} = 1$ and $\dim Z^3(\mathcal{T},\mathbb{F}) \le 2$ if $\dim\mathcal{T} = 2$. Further, if $\dim\mathcal{T} \ge 3$, then $\dim Z^3(\mathcal{T},\mathbb{F}) \le 2\binom{n+1}{3}$.

A cocycle $\theta \in Z^3(\mathcal{T},\mathbb{F})$ is uniquely determined by its values $\theta(e_i, e_j, e_k) = \alpha_{i,j,k}$ and it may be represented by a block matrix $\mathcal{C} = \left(\mathcal{C}_1\ \mathcal{C}_2\ \cdots\ \mathcal{C}_n\right)$ where for $t \in \{1,2,\ldots,n\}$ the $\mathcal{C}_t$ is the matrix representing the skew symmetric bilinear form $\mathcal{T}\times\mathcal{T} \longrightarrow \mathbb{F}$ defined by $(x,y) \longmapsto \theta(x,y,e_t)$ (i.e., if $\mathcal{C}_t = \left(c_{ij}\right)$ then $c_{ij} = \alpha_{i,j,t}$). We call $\mathcal{C}$ the *matrix form* of θ.

Let $\phi = \left(a_{ij}\right) \in \operatorname{Aut}(\mathcal{T})$, $\theta \in Z^3(\mathcal{T},\mathbb{F})$, $\mathcal{C} = \left(\mathcal{C}_1\ \mathcal{C}_2\ \cdots\ \mathcal{C}_n\right)$ be the matrix form of θ and $\mathcal{C}' = \left(\mathcal{C}_1'\ \mathcal{C}_2'\ \cdots\ \mathcal{C}_n'\right)$ be the matrix form of $\phi\theta$. Then

$$\begin{aligned}\phi\theta(x,y,e_k) = \theta(\phi(x),\phi(y),\phi(e_k)) &= \theta\left(\phi(x),\phi(y),\sum_{i=1}^{n} a_{ik}e_i\right)\\ &= \sum_{i=1}^{n} a_{ik}\theta(\phi(x),\phi(y),e_i),\end{aligned}$$

thus, if $\mathcal{B} = \sum_{i=1}^{n} a_{ik}\mathcal{C}_i$, therefore $\mathcal{C}_k' = \phi^t\mathcal{B}\phi$.

Example 3.1 Consider the abelian 2 -dimensional Lie triple system $\mathcal{T} := \langle e_1, e_2\rangle$. Then $Z^3(\mathcal{T},\mathbb{F}) = \left\langle \Delta_{1,2,1}, \Delta_{1,2,2}\right\rangle$ and $\operatorname{Aut}(\mathcal{T}) = \{\phi = \left(a_{ij}\right) \in \mathcal{M}_{2\times 2}(\mathbb{F}) : \det\phi \ne 0\}$. Let $\theta = \alpha_{1,2,1}\Delta_{1,2,1} + \alpha_{1,2,2}\Delta_{1,2,2}$. Then the matrix form of θ is $\mathcal{C} = \left(\mathcal{C}_1\ \mathcal{C}_2\right)$ where $\mathcal{C}_1 = \begin{pmatrix} 0 & \alpha_{1,2,1}\\ -\alpha_{1,2,1} & 0\end{pmatrix}$, $\mathcal{C}_2 = \begin{pmatrix} 0 & \alpha_{1,2,2}\\ -\alpha_{1,2,2} & 0\end{pmatrix}$. If $\mathcal{C}' = \left(\mathcal{C}_1'\ \mathcal{C}_2'\right)$ is the matrix form of $\phi\theta$, then

$$\begin{aligned}\mathcal{C}_1' &= \phi^t\left(a_{11}\begin{pmatrix} 0 & \alpha_{1,2,1}\\ -\alpha_{1,2,1} & 0\end{pmatrix} + a_{21}\begin{pmatrix} 0 & \alpha_{1,2,2}\\ -\alpha_{1,2,2} & 0\end{pmatrix}\right)\phi,\\ \mathcal{C}_2' &= \phi^t\left(a_{12}\begin{pmatrix} 0 & \alpha_{1,2,1}\\ -\alpha_{1,2,1} & 0\end{pmatrix} + a_{22}\begin{pmatrix} 0 & \alpha_{1,2,2}\\ -\alpha_{1,2,2} & 0\end{pmatrix}\right)\phi.\end{aligned}$$

Proposition 3.1 *Let $\mathcal{T}$ be a Lie triple system and* $\dim\mathcal{T} \le 2$. *If $\mathcal{T}$ is nilpotent, then $\mathcal{T}$ is abelian.*

Proof Since $\mathcal{T}$ is nilpotent, $\operatorname{Ann}(\mathcal{T}) \ne 0$. So if $\dim\mathcal{T} = 1$, then $\mathcal{T} = \operatorname{Ann}(\mathcal{T})$. If $\dim\mathcal{T} = 2$, then $\mathcal{T}$ is a annihilator extension of 1-dimensional Lie triple system $\mathcal{T}'$. But $Z^3\left(\mathcal{T}',\mathbb{F}\right) = 0$. So, any annihilator extension of $\mathcal{T}'$ is trivial. Thus, $\mathcal{T}$ is abelian. □

We denote by $\mathcal{T}_{1,1}$ the abelian Lie triple system of dimension 1 and by $\mathcal{T}_{2,1}$ the abelian Lie triple system of dimension 2.

Remark 3.1 Let $X = \left(\alpha\ \beta\right) \in \mathcal{M}_{1\times 2}(\mathbb{F})$ and $X \ne 0$. Then there exists an invertible matrix $A \in \mathcal{M}_{2\times 2}(\mathbb{F})$ such that $XA = \left(1\ 0\right)$. To see this, suppose first that $\alpha \ne 0$. Then $\left(\alpha\ \beta\right)\begin{pmatrix}\alpha^{-1} & -\beta\\ 0 & \alpha\end{pmatrix} = \left(1\ 0\right)$. Assume now that $\alpha = 0$. Then $\left(0\ \beta\right)\begin{pmatrix} 0 & 1\\ \beta^{-1} & 0\end{pmatrix} = \left(1\ 0\right)$.

Proposition 3.2 *Let $\mathcal{T}$ be a nilpotent Lie triple system of dimension* 3. *Then $\mathcal{T}$ is isomorphic to one of the following:*

- $\mathcal{T}_{3,1} = \mathcal{T}_{2,1} \oplus \mathcal{T}_{1,1}$ *(abelian Lie triple system).*
- $\mathcal{T}_{3,2} : [e_1, e_2, e_1] = e_3$.

Proof Suppose that $\mathcal{T}$ is not abelian. Then $\mathcal{T}$ is a 1 -dimensional annihilator extension of $\mathcal{T}_{2,1}$. Since, we have $Z^3\left(\mathcal{T}_{2,1},\mathbb{F}\right)=\left\langle\Delta_{1,2,1},\Delta_{1,2,2}\right\rangle$ and $B^3\left(\mathcal{T}_{2,1},\mathbb{F}\right)=0$, then $Z^3\left(\mathcal{T}_{2,1},\mathbb{F}\right)=H^3\left(\mathcal{T}_{2,1},\mathbb{F}\right)$. Furthermore, we have $\operatorname{Aut}\left(\mathcal{T}_{2,1}\right)=GL(\mathcal{T}_{2,1})$.

Now, choose an arbitrary subspace $\mathbb{W}\in\mathcal{T}_1\left(\mathcal{T}_{2,1}\right)$. Then $\mathbb{W}$ is spanned by $\theta=\alpha_{1,2,1}\Delta_{1,2,1}+\alpha_{1,2,2}\Delta_{1,2,2}$ such that $\left(\alpha_{1,2,1},\alpha_{1,2,2}\right)\neq(0,0)$. Let $\phi=\left(a_{ij}\right)\in\operatorname{Aut}\left(\mathcal{T}_{2,1}\right)$. Write $\phi\theta=\beta_{1,2,1}\Delta_{1,2,1}+\beta_{1,2,2}\Delta_{1,2,2}$. Then

$$\beta_{1,2,1}=(a_{11}a_{22}-a_{12}a_{21})\left(a_{11}\alpha_{1,2,1}+a_{21}\alpha_{1,2,2}\right),$$
$$\beta_{1,2,2}=(a_{11}a_{22}-a_{12}a_{21})\left(a_{12}\alpha_{1,2,1}+a_{22}\alpha_{1,2,2}\right).$$

Set $\alpha'_{1,2,1}=\det(\phi)\,\alpha_{1,2,1}$ and $\alpha'_{1,2,2}=\det(\phi)\,\alpha_{1,2,2}$. Then $\beta_{1,2,1}=a_{11}\alpha'_{1,2,1}+a_{21}\alpha'_{1,2,2}$, $\beta_{1,2,2}=a_{12}\alpha'_{1,2,1}+a_{22}\alpha'_{1,2,2}$ and $\left(\alpha'_{1,2,1},\alpha'_{1,2,2}\right)\neq(0,0)$. By Remark 3.1, there exists a $\phi\in\operatorname{Aut}\left(\mathcal{T}_{2,1}\right)$ such that $\phi\theta=\Delta_{1,2,1}$. So we get the Lie triple system $\mathcal{T}_{3,2}$. □

Theorem 3.1 *Let $\mathcal{T}$ be a nilpotent Lie triple system of dimension 4 over an algebraically closed field $\mathbb{F}$ of characteristic $\neq 2,3$. Then $\mathcal{T}$ is isomorphic to one of the following:*

- $\mathcal{T}_{4,1}=\mathcal{T}_{3,1}\oplus\mathcal{T}_{1,1}$ *(abelian Lie triple system).*
- $\mathcal{T}_{4,2}=\mathcal{T}_{3,2}\oplus\mathcal{T}_{1,1}:[e_1,e_2,e_1]=e_3.$
- $\mathcal{T}_{4,3}:[e_1,e_2,e_1]=e_3,[e_1,e_2,e_2]=e_4.$
- $\mathcal{T}_{4,4}:[e_2,e_3,e_2]=e_4,[e_3,e_1,e_3]=e_4.$
- $\mathcal{T}_{4,5}:[e_2,e_3,e_1]=e_4,[e_3,e_1,e_2]=e_4,[e_2,e_1,e_3]=2e_4,[e_2,e_3,e_2]=e_4.$
- $\mathcal{T}_{4,6}^{\lambda}:[e_1,e_2,e_3]=-(\lambda+1)\,e_4,[e_2,e_3,e_1]=\lambda e_4,[e_3,e_1,e_2]=e_4.$
- $\mathcal{T}_{4,7}:[e_1,e_2,e_1]=e_3,[e_1,e_2,e_3]=e_4,[e_1,e_3,e_2]=e_4.$
- $\mathcal{T}_{4,8}:[e_1,e_2,e_1]=e_3,[e_1,e_3,e_1]=e_4,[e_1,e_2,e_2]=e_4.$
- $\mathcal{T}_{4,9}:[e_1,e_2,e_1]=e_3,[e_1,e_3,e_1]=e_4.$

Among these Lie triple systems, there are precisely the following isomorphisms:

- $\mathcal{T}_{4,6}^{0}\cong\mathcal{T}_{4,6}^{-1}.$
- $\mathcal{T}_{4,6}^{\alpha}\left(\alpha^2+\alpha\neq 0\right)\cong\mathcal{T}_{4,6}^{\beta}\left(\beta^2+\beta\neq 0\right)$ *if and only if* $\xi(\alpha)=\xi(\beta)$ *where*

$$\xi(\lambda)=\frac{\left(\lambda^2+\lambda+1\right)^3}{\lambda^2(\lambda+1)^2}:\lambda^2+\lambda\neq 0.$$

Proof Suppose that $\mathcal{T}$ has no annihilator components. On the one hand, if $\dim\operatorname{Ann}(\mathcal{T})=2$, then $\mathcal{T}$ is a 2-dimensional annihilator extension of $\mathcal{T}_{2,1}$. Since $H^3\left(\mathcal{T}_{2,1},\mathbb{F}\right)=\left\langle\Delta_{1,2,1},\Delta_{1,2,2}\right\rangle$, $\mathcal{T}_1(\mathcal{T})=\{\mathbb{W}=\left\langle\Delta_{1,2,1},\Delta_{1,2,2}\right\rangle\}$. So we get the Lie triple system $\mathcal{T}_{4,3}$. On the other hand, if $\dim\operatorname{Ann}(\mathcal{T})=1$, then $\mathcal{T}$ is a 1-dimensional annihilator extension of $\mathcal{T}_{3,1}$ or $\mathcal{T}_{3,2}$.

Assume first that $\mathcal{T}$ is a 1-dimensional annihilator extension of $\mathcal{T}_{3,1}$. In this case, we have

$$Z^3\left(\mathcal{T}_{3,1},\mathbb{F}\right)=\left\langle\Delta_{1,2,1},\Delta_{1,2,2},\Delta_{1,3,1},\Delta_{1,3,3},\Delta_{2,3,2},\Delta_{2,3,3},\Delta_{1,2,3}+\Delta_{1,3,2},\Delta_{2,3,1}\right.$$
$$\left.+\Delta_{1,3,2}\right\rangle,$$

and $B^3\left(\mathcal{T}_{3,2},\mathbb{F}\right)=0$. So $H^3\left(\mathcal{T}_{3,1},\mathbb{F}\right)=Z^3\left(\mathcal{T}_{3,1},\mathbb{F}\right)$. Moreover, the automorphism group $\operatorname{Aut}\left(\mathcal{T}_{3,1}\right)=GL(\mathcal{T}_{3,1})$.

Now, choose an arbitrary subspace $\mathbb{W}\in\mathcal{T}_1\left(\mathcal{T}_{3,1}\right)$. Then $\mathbb{W}$ is spanned by

$$\theta=\alpha_{1,2,1}\Delta_{1,2,1}+\alpha_{1,3,1}\Delta_{1,3,1}+\alpha_{2,3,1}\Delta_{2,3,1}+\alpha_{1,2,2}\Delta_{1,2,2}+\alpha_{1,3,2}\Delta_{1,3,2}$$
$$+\alpha_{2,3,2}\Delta_{2,3,2}+\alpha_{1,2,3}\Delta_{1,2,3}+\alpha_{1,3,3}\Delta_{1,3,3}+\alpha_{2,3,3}\Delta_{2,3,3}$$

such that $\alpha_{1,3,2} = \alpha_{1,2,3} + \alpha_{2,3,1}$ and $\mathrm{Rad}\,(\theta) \cap \mathcal{T}_{3,1} = 0$. Further, $\mathrm{Rad}\,(\theta) \cap \mathcal{T}_{3,1} = 0$ if and only if the matrix Ω^t has rank 3 where

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & \alpha_{1,2,1} & \alpha_{1,2,2} & \alpha_{1,2,3} & \alpha_{1,3,1} & \alpha_{1,3,2} & \alpha_{1,3,3} \\ -\alpha_{1,2,1} & -\alpha_{1,2,2} & \alpha_{2,1,3} & 0 & 0 & 0 & \alpha_{2,3,1} & \alpha_{2,3,2} & \alpha_{2,3,3} \\ -\alpha_{1,3,1} & -\alpha_{1,3,2} & -\alpha_{1,3,3} & -\alpha_{2,3,1} & -\alpha_{2,3,2} & -\alpha_{2,3,3} & 0 & 0 & 0 \end{pmatrix}.$$

Let $\phi = \left(a_{ij}\right) \in \mathrm{Aut}\left(\mathcal{T}_{3,1}\right)$ and write

$$\begin{aligned}\phi\theta = {} & \beta_{1,2,1}\Delta_{1,2,1} + \beta_{1,3,1}\Delta_{1,3,1} + \beta_{2,3,1}\Delta_{2,3,1} + \beta_{1,2,2}\Delta_{1,2,2} + \beta_{1,3,2}\Delta_{1,3,2} \\ & + \beta_{2,3,2}\Delta_{2,3,2} + \beta_{1,2,3}\Delta_{1,2,3} + \beta_{1,3,3}\Delta_{1,3,3} + \beta_{2,3,3}\Delta_{2,3,3}.\end{aligned}$$

Then

$$\begin{aligned}
\beta_{1,2,1} = {} & (a_{11}a_{22} - a_{12}a_{21})\left(a_{11}\alpha_{1,2,1} + a_{21}\alpha_{1,2,2} + a_{31}\alpha_{1,2,3}\right) \\
& + (a_{11}a_{32} - a_{12}a_{31})\left(a_{11}\alpha_{1,3,1} + a_{21}\alpha_{1,3,2} + a_{31}\alpha_{1,3,3}\right) \\
& + (a_{21}a_{32} - a_{22}a_{31})\left(a_{11}\alpha_{2,3,1} + a_{21}\alpha_{2,3,2} + a_{31}\alpha_{2,3,3}\right), \\
\beta_{1,3,1} = {} & (a_{11}a_{23} - a_{21}a_{13})\left(a_{11}\alpha_{1,2,1} + a_{21}\alpha_{1,2,2} + a_{31}\alpha_{1,2,3}\right) \\
& + (a_{11}a_{33} - a_{13}a_{31})\left(a_{11}\alpha_{1,3,1} + a_{21}\alpha_{1,3,2} + a_{31}\alpha_{1,3,3}\right) \\
& + (a_{21}a_{33} - a_{31}a_{23})\left(a_{11}\alpha_{2,3,1} + a_{21}\alpha_{2,3,2} + a_{31}\alpha_{2,3,3}\right), \\
\beta_{2,3,1} = {} & (a_{12}a_{23} - a_{13}a_{22})\left(a_{11}\alpha_{1,2,1} + a_{21}\alpha_{1,2,2} + a_{31}\alpha_{1,2,3}\right) \\
& + (a_{12}a_{33} - a_{13}a_{32})\left(a_{11}\alpha_{1,3,1} + a_{21}\alpha_{1,3,2} + a_{31}\alpha_{1,3,3}\right) \\
& + (a_{22}a_{33} - a_{23}a_{32})\left(a_{11}\alpha_{2,3,1} + a_{21}\alpha_{2,3,2} + a_{31}\alpha_{2,3,3}\right), \\
\beta_{1,2,2} = {} & (a_{11}a_{22} - a_{12}a_{21})\left(a_{12}\alpha_{1,2,1} + a_{22}\alpha_{1,2,2} + a_{32}\alpha_{1,2,3}\right) \\
& + (a_{11}a_{32} - a_{12}a_{31})\left(a_{12}\alpha_{1,3,1} + a_{22}\alpha_{1,3,2} + a_{32}\alpha_{1,3,3}\right) \\
& + (a_{21}a_{32} - a_{22}a_{31})\left(a_{12}\alpha_{2,3,1} + a_{22}\alpha_{2,3,2} + a_{32}\alpha_{2,3,3}\right), \\
\beta_{1,3,2} = {} & (a_{11}a_{23} - a_{21}a_{13})\left(a_{12}\alpha_{1,2,1} + a_{22}\alpha_{1,2,2} + a_{32}\alpha_{1,2,3}\right) \\
& + (a_{11}a_{33} - a_{13}a_{31})\left(a_{12}\alpha_{1,3,1} + a_{22}\alpha_{1,3,2} + a_{32}\alpha_{1,3,3}\right) \\
& + (a_{21}a_{33} - a_{31}a_{23})\left(a_{12}\alpha_{2,3,1} + a_{22}\alpha_{2,3,2} + a_{32}\alpha_{2,3,3}\right), \\
\beta_{2,3,2} = {} & (a_{12}a_{23} - a_{13}a_{22})\left(a_{12}\alpha_{1,2,1} + a_{22}\alpha_{1,2,2} + a_{32}\alpha_{1,2,3}\right) \\
& + (a_{12}a_{33} - a_{13}a_{32})\left(a_{12}\alpha_{1,3,1} + a_{22}\alpha_{1,3,2} + a_{32}\alpha_{1,3,3}\right) \\
& + (a_{22}a_{33} - a_{23}a_{32})\left(a_{12}\alpha_{2,3,1} + a_{22}\alpha_{2,3,2} + a_{32}\alpha_{2,3,3}\right), \\
\beta_{1,2,3} = {} & (a_{11}a_{22} - a_{12}a_{21})\left(a_{13}\alpha_{1,2,1} + a_{23}\alpha_{1,2,2} + a_{33}\alpha_{1,2,3}\right) \\
& + (a_{11}a_{32} - a_{12}a_{31})\left(a_{13}\alpha_{1,3,1} + a_{23}\alpha_{1,3,2} + a_{33}\alpha_{1,3,3}\right) \\
& + (a_{21}a_{32} - a_{22}a_{31})\left(a_{13}\alpha_{2,3,1} + a_{23}\alpha_{2,3,2} + a_{33}\alpha_{2,3,3}\right), \\
\beta_{1,3,3} = {} & (a_{11}a_{23} - a_{21}a_{13})\left(a_{13}\alpha_{1,2,1} + a_{23}\alpha_{1,2,2} + a_{33}\alpha_{1,2,3}\right) \\
& + (a_{11}a_{33} - a_{13}a_{31})\left(a_{13}\alpha_{1,3,1} + a_{23}\alpha_{1,3,2} + a_{33}\alpha_{1,3,3}\right) \\
& + (a_{21}a_{33} - a_{31}a_{23})\left(a_{13}\alpha_{2,3,1} + a_{23}\alpha_{2,3,2} + a_{33}\alpha_{2,3,3}\right), \\
\beta_{2,3,3} = {} & (a_{12}a_{23} - a_{13}a_{22})\left(a_{13}\alpha_{1,2,1} + a_{23}\alpha_{1,2,2} + a_{33}\alpha_{1,2,3}\right) \\
& + (a_{12}a_{33} - a_{13}a_{32})\left(a_{13}\alpha_{1,3,1} + a_{23}\alpha_{1,3,2} + a_{33}\alpha_{1,3,3}\right) \\
& + (a_{22}a_{33} - a_{23}a_{32})\left(a_{13}\alpha_{2,3,1} + a_{23}\alpha_{2,3,2} + a_{33}\alpha_{2,3,3}\right).
\end{aligned}$$

Let us now associate θ with another matrix $\mathcal{A}_\theta$ and $\phi\theta$ with another matrix $\mathcal{A}_{\phi\theta}$ in the following way:

$$\mathcal{A}_\theta = \begin{pmatrix} \alpha_{2,3,1} & \alpha_{2,3,2} & \alpha_{2,3,3} \\ -\alpha_{1,3,1} & -\alpha_{1,3,2} & -\alpha_{1,3,3} \\ \alpha_{1,2,1} & \alpha_{1,2,2} & \alpha_{1,2,3} \end{pmatrix}, \mathcal{A}_{\phi\theta} = \begin{pmatrix} \beta_{2,3,1} & \beta_{2,3,2} & \beta_{2,3,3} \\ -\beta_{1,3,1} & -\beta_{1,3,2} & -\beta_{1,3,3} \\ \beta_{1,2,1} & \beta_{1,2,2} & \beta_{1,2,3} \end{pmatrix}.$$

Then $\operatorname{tr}(\mathcal{A}_\theta) = \operatorname{tr}(\mathcal{A}_{\phi\theta}) = 0$. Moreover, we have $\mathcal{A}_{\phi\theta} = \det(\phi)\,\phi^{-1}\mathcal{A}_\theta\phi$. Define

$$\mathcal{S} = \left\{\mathcal{A}_\theta : \theta \in Z^3\left(\mathcal{T}_{3,1}, \mathbb{F}\right) \text{ such that } \operatorname{Rad}(\theta) \cap \mathcal{T}_{3,1} = 0\right\}.$$

Then $\operatorname{Aut}\left(\mathcal{T}_{3,1}\right)$ acts on $\mathcal{S}$ by $\phi \bullet \mathcal{A}_\theta = \mathcal{A}_{\phi\theta}$. It follows that any element in $\mathcal{S}$ is in the same orbit as one of the following:

(i) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. (ii) $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix} : \lambda \neq 0$.

(iii) $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix} : \lambda \neq 0$. (iv) $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & -(\lambda+\mu) \end{pmatrix} : \lambda\mu(\lambda+\mu) \neq 0$.

We will study every case individually.

- If $\mathcal{A}_\theta$ has the form (i), then $\theta = \Delta_{2,3,2} - \Delta_{1,3,3}$. So we get the Lie triple system $\mathcal{T}_{4,4}$.
- If $\mathcal{A}_\theta$ has the form (ii), then

$$\vartheta = \lambda^{-1}\theta = \Delta_{2,3,1} - \Delta_{1,3,2} - 2\Delta_{1,2,3} + \lambda\Delta_{2,3,2}.$$

Since

$$\mathcal{A}_\vartheta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the matrix $\mathcal{A}_\vartheta$ is similar to:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

So we may assume that $\vartheta = \Delta_{2,3,1} - \Delta_{1,3,2} - 2\Delta_{1,2,3} + \Delta_{2,3,2}$. Since $\mathbb{W} = \langle\theta\rangle = \langle\vartheta\rangle$, we get the triple system $\mathcal{T}_{4,5}$.
- If $\mathcal{A}_\theta$ has the form (iii), we may assume that $\theta = -\Delta_{1,3,2} - \Delta_{1,2,3}$. Then we get the Lie triple system $\mathcal{T}_{4,6}^0$.
- If $\mathcal{A}_\theta$ has the form (iv), then we may assume that $\theta = \lambda\Delta_{2,3,1} - \Delta_{1,3,2} - (\lambda+1)\,\Delta_{1,2,3}$ such that $\lambda(\lambda+1) \neq 0$. So we get the Lie triple systems $\mathcal{T}_{4,6}^\lambda\left(\lambda^2+\lambda \neq 0\right)$. Further, if $\mathcal{T}_{4,6}^\lambda \cong \mathcal{T}_{4,6}^{\lambda'}$, then there exists a non-zero element $\mu \in \mathbb{F}$ such that the following matrices have the same eigenvalues:

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -(\lambda+1) \end{pmatrix}, \mu\begin{pmatrix} \lambda' & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\left(\lambda'+1\right) \end{pmatrix}.$$

So we have $\{1, \lambda, -(\lambda+1)\} = \left\{\mu, \mu\lambda', -\mu\left(\lambda'+1\right)\right\}$ and hence, $\lambda' \in \{\lambda, -(\lambda+1), \frac{1}{\lambda}, -\frac{\lambda+1}{\lambda}, -\frac{1}{\lambda+1}, -\frac{\lambda}{\lambda+1}\big\}$. Conversely, if $\lambda' \in \left\{\lambda, -(\lambda+1), \frac{1}{\lambda}, -\frac{\lambda+1}{\lambda}, -\frac{1}{\lambda+1}, -\frac{\lambda}{\lambda+1}\right\}$, we have the following isomorphisms:

$$
\begin{aligned}
&\sigma_1 : \mathcal{T}_{4,6}^{\lambda} \longrightarrow \mathcal{T}_{4,6}^{\lambda} : \sigma_1(e_1) = e_1, \sigma_1(e_2) = e_2, \sigma_1(e_3) = e_3, \sigma_1(e_4) = e_4,\\
&\sigma_2 : \mathcal{T}_{4,6}^{\lambda} \longrightarrow \mathcal{T}_{4,6}^{-(\lambda+1)} : \sigma_2(e_1) = e_3, \sigma_2(e_2) = e_2, \sigma_2(e_3) = e_1, \sigma_2(e_4) = -e_4,\\
&\sigma_3 : \mathcal{T}_{4,6}^{\lambda} \longrightarrow \mathcal{T}_{4,6}^{\frac{1}{\lambda}} : \sigma_3(e_1) = e_2, \sigma_3(e_2) = e_1, \sigma_3(e_3) = e_3, \sigma_1(e_4) = -\frac{1}{\lambda}e_4,\\
&\sigma_4 : \mathcal{T}_{4,6}^{\lambda} \longrightarrow \mathcal{T}_{4,6}^{-\frac{\lambda+1}{\lambda}} : \sigma_4(e_1) = e_2, \sigma_4(e_2) = e_3, \sigma_4(e_3) = e_1, \sigma_4(e_4) = \frac{1}{\lambda}e_4,\\
&\sigma_5 : \mathcal{T}_{4,6}^{\lambda} \longrightarrow \mathcal{T}_{4,6}^{-\frac{1}{\lambda+1}} : \sigma_5(e_1) = e_3, \sigma_5(e_2) = e_1, \sigma_5(e_3) = e_2, \sigma_5(e_4) = -\frac{1}{\lambda+1}e_4,\\
&\sigma_6 : \mathcal{T}_{4,6}^{\lambda} \longrightarrow \mathcal{T}_{4,6}^{-\frac{\lambda}{\lambda+1}} : \sigma_6(e_1) = e_1, \sigma_6(e_2) = e_3, \sigma_6(e_3) = e_2, \sigma_6(e_4) = \frac{1}{\lambda+1}e_4.
\end{aligned}
$$

Moreover, from here we have that $\mathcal{T}_{4,6}^{0} \cong \mathcal{T}_{4,6}^{-1}$. Now, for any non-zero elements $\alpha, \beta, \gamma \in \mathbb{F}$, we define $\kappa(\alpha, \beta, \gamma) = \frac{(\alpha^2+\beta^2+\gamma^2)^3}{8\alpha^2\beta^2\gamma^2}$. If $\{\alpha, \beta, \gamma\} = \{\mu\alpha', \mu\beta', \mu\gamma'\}$, then $\kappa(\alpha, \beta, \gamma) = \kappa(\alpha', \beta', \gamma')$. Thus,

$$
\xi(\lambda) = \kappa(1, \lambda, -(\lambda+1)) = \frac{(\lambda^2+\lambda+1)^3}{\lambda^2(\lambda+1)^2} : \lambda \neq -1, 0
$$

is an invariant for $\mathcal{T}_{4,6}^{\lambda}$. If $\xi(\lambda) = \xi(\lambda')$, then $\lambda' \in \left\{\lambda, -\frac{1}{\lambda+1}, \frac{1}{\lambda}, -\lambda-1, -\frac{\lambda}{\lambda+1}, -\frac{1}{\lambda}(\lambda+1)\right\}$. Hence, $\mathcal{T}_{4,6}^{\lambda} \cong \mathcal{T}_{4,6}^{\lambda'}$ if and only if $\xi(\lambda) = \xi(\lambda')$.

Now, let us assume that $\mathcal{T}$ is a 1-dimensional annihilator extension of $\mathcal{T}_{3,2}$. In this case, we have

$$
Z^3(\mathcal{T}_{3,2}, \mathbb{F}) = \langle \Delta_{1,2,1}, \Delta_{1,2,2}, \Delta_{1,3,1}, \Delta_{1,2,3} + \Delta_{1,3,2} \rangle
$$

and $B^3(\mathcal{T}_{3,2}, \mathbb{F}) = \langle \Delta_{1,2,1} \rangle$. Therefore, $H^3(\mathcal{T}_{3,2}, \mathbb{F}) = \langle [\Delta_{1,2,2}], [\Delta_{1,3,1}], [\Delta_{1,2,3}] + [\Delta_{1,3,2}] \rangle$. Moreover, by Corollary 2.1, the automorphism group $\mathrm{Aut}(\mathcal{T}_{3,2})$ consists of invertible matrices of the form

$$
\phi = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{11}^2 a_{22} \end{pmatrix}.
$$

Choose an arbitrary subspace $\mathbb{W} \in \mathcal{T}_1(\mathcal{T}_{3,2})$. Then $\mathbb{W}$ is spanned by

$$
[\theta] = \alpha_{1,3,1}[\Delta_{1,3,1}] + \alpha_{1,2,2}[\Delta_{1,2,2}] + \alpha_{1,2,3}([\Delta_{1,2,3}] + [\Delta_{1,3,2}])
$$

such that $\mathrm{Rad}(\theta) \cap \langle e_3 \rangle = 0$. Let $\phi = (a_{ij}) \in \mathrm{Aut}(\mathcal{T}_{3,2})$ and write $\phi\theta = \beta_{1,3,1}[\Delta_{1,3,1}] + \beta_{1,2,2}[\Delta_{1,2,2}] + \beta_{1,2,3}([\Delta_{1,2,3}] + [\Delta_{1,3,2}])$. Then

$$
\begin{aligned}
\beta_{1,3,1} &= a_{11}^3 a_{22}(a_{11}\alpha_{1,3,1} + a_{21}\alpha_{1,2,3}),\\
\beta_{1,2,2} &= a_{11}a_{22}(a_{22}\alpha_{1,2,2} + 2a_{32}\alpha_{1,2,3}),\\
\beta_{1,2,3} &= \det(\phi)\,\alpha_{1,2,3}.
\end{aligned}
$$

Since $\beta_{1,2,3} \neq 0$ if and only if $\alpha_{1,2,3} \neq 0$, we have $\mathcal{O}\langle[\theta] : \alpha_{1,2,3} \neq 0\rangle \cap \mathcal{O}\langle[\theta] : \alpha_{1,2,3} = 0\rangle = \emptyset$ and therefore $\mathrm{Aut}(\mathcal{T}_{3,2})$ has at least two orbits on $\mathcal{T}_1(\mathcal{T}_{3,2})$. We distinguish two cases:

- $\alpha_{1,2,3} \neq 0$. So we may assume $\alpha_{1,2,3} = 1$. Let ϕ be the following automorphism:

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ -\alpha_{1,3,1} & 1 & 0 \\ 0 & -\frac{1}{2}\alpha_{1,2,2} & 1 \end{pmatrix}.$$

Then $\phi\mathbb{W} = \langle [\Delta_{1,2,3}] + [\Delta_{1,3,2}] \rangle$. So we get the Lie triple system $\mathcal{T}_{4,7}$.
- $\alpha_{1,2,3} = 0$. Then $\mathrm{Rad}\,(\theta) \cap \langle e_3 \rangle = 0$ implies that $\alpha_{1,3,1} \neq 0$ and so we may assume $\alpha_{1,3,1} = 1$. From here, we have

$$\beta_{1,3,1} = a_{11}^4 a_{22},\ \beta_{1,2,2} = a_{11} a_{22}^2 \alpha_{1,2,2},\ \beta_{1,2,3} = 0.$$

This in turn shows that if $\alpha_{1,2,3} = 0$ then $\mathcal{O}\langle [\theta] : \alpha_{1,2,2} \neq 0 \rangle \cap \mathcal{O}\langle [\theta] : \alpha_{1,2,2} = 0 \rangle = \emptyset$. We distinguish two subcases:

  - $\mu = \alpha_{1,2,2} \neq 0$. Let ϕ be the following automorphism:

$$\phi = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu^2 & 0 \\ 0 & 0 & \mu^4 \end{pmatrix}.$$

  Then $[\phi\theta] = \mu^6 \left([\Delta_{1,3,1}] + [\Delta_{1,2,2}] \right)$. Hence, $\phi\mathbb{W} = \langle [\Delta_{1,3,1}] + [\Delta_{1,2,2}] \rangle$ and we get the Lie triple system $\mathcal{T}_{4,8}$.
  - $\alpha_{1,2,2} = 0$. Then $\mathbb{W} = \langle [\Delta_{1,3,1}] \rangle$ and we get the Lie triple system $\mathcal{T}_{4,9}$.

□

4 The geometric classification of nilpotent Lie triple systems up to dimension four

Given a vector space $\mathbb{V}$ of dimension n, the set of trilinear maps $\mathrm{Tri}(\mathbb{V}, \mathbb{V}) \cong \mathrm{Hom}(\mathbb{V}^{\otimes 3}, \mathbb{V}) \cong (\mathbb{V}^*)^{\otimes 3} \otimes \mathbb{V}$ is a vector space of dimension n^4. This vector space has the structure of the affine space $\mathbb{F}^{n^4}$ in the following sense: fixed a basis $e_1, \ldots, e_n$ of $\mathbb{V}$, then any triple system with multiplication $\mu \in \mathrm{Tri}(\mathbb{V}, \mathbb{V})$, is determined by some parameters $c_{ijk}^p \in \mathbb{F}$, called *structural constants*, such that

$$\mu(e_i, e_j, e_k) = \sum_{p=1}^{n} c_{ijk}^p e_p$$

which corresponds to a point in the affine space $\mathbb{F}^{n^4}$. Then a set of triple systems $\mathcal{S}$ corresponds to an algebraic variety, i.e., a Zariski closed set, if there are some polynomial equations in variables c_{ijk}^p with zero locus equal to the set of structural constants of the triple systems in $\mathcal{S}$. Since given the identities defining Lie triple systems we can obtain a set of polynomial equations in variables c_{ijk}^p, the class of n-dimensional Lie triple systems is a variety. Moreover, the class of n-dimensional nilpotent Lie triple systems, denoted $\mathcal{T}_n$, is a subvariety of the variety of n-dimensional Lie triple systems.

Now, consider the action of $\mathrm{GL}(\mathbb{V})$ on $\mathcal{T}_n$ by conjugation:

$$(g * \mu)(x, y, z) = g\mu(g^{-1}x, g^{-1}y, g^{-1}z)$$

for $g \in \mathrm{GL}(\mathbb{V})$, $\mu \in \mathcal{T}_n$ and for any $x, y, z \in \mathbb{V}$. Observe that the $\mathrm{GL}(\mathbb{V})$-orbit of μ, denoted $O(\mu)$, contains all the structural constants of the triple systems isomorphic to the Lie triple system with structural constants μ.

In the previous section, we gave a decomposition of $\mathcal{T}_n$, for $n \leq 4$, into $\mathrm{GL}(\mathbb{V})$-orbits (acting by conjugation), i.e., an algebraic classification of the nilpotent Lie triple systems up to dimension four. In this section, we will describe the closures of orbits of $\mu \in \mathcal{T}_n$, denoted by $\overline{O(\mu)}$, and we will give a geometric classification of $\mathcal{T}_n$, which consist in describing its irreducible components. Recall that any affine variety can be represented as a finite union of its irreducible components in a unique way.

Additionally, describing the irreducible components of a variety, such as $\mathcal{T}_n$, gives us the rigid triple systems of the variety, which are those triple systems with an open $\mathrm{GL}(\mathbb{V})$-orbit. This is due to the fact that a triple system is rigid in variety if and only if the closure of its orbit is an irreducible component of the variety.

Definition 4.1 Let $\mathcal{T}$ and $\mathcal{T}'$ be two n-dimensional Lie triple systems and $\mu, \lambda \in \mathcal{T}_n$ be their representatives in the affine space, respectively. We say $\mathcal{T}$ *degenerates* to $\mathcal{T}'$, and write $\mathcal{T} \to \mathcal{T}'$, if $\lambda \in \overline{O(\mu)}$. If $\mathcal{T} \not\cong \mathcal{T}'$, then we call it a *proper degeneration*.

Conversely, if $\lambda \notin \overline{O(\mu)}$ then we call it a *non-degeneration* and we write $\mathcal{T} \not\to \mathcal{T}'$.

Note that the definition of a degeneration does not depend on the choice of μ and λ. Also, due to the transitivity of the notion of degeneration (that is, if $\mathcal{T} \to \mathcal{T}''$ and $\mathcal{T}'' \to \mathcal{T}'$ then $\mathcal{T} \to \mathcal{T}'$) we have the following definitions.

Definition 4.2 Let $\mathcal{T}$ and $\mathcal{T}'$ be two n-dimensional Lie triple systems such that $\mathcal{T} \to \mathcal{T}'$. If there is no $\mathcal{T}''$ such that $\mathcal{T} \to \mathcal{T}''$ and $\mathcal{T}'' \to \mathcal{T}'$ are proper degenerations, then $\mathcal{T} \to \mathcal{T}'$ is called a *primary degeneration*. Analogously, let $\mathcal{T}$ and $\mathcal{T}'$ be two n-dimensional Lie triple systems such that $\mathcal{T} \not\to \mathcal{T}'$, if there are no $\mathcal{T}''$ and $\mathcal{T}'''$ such that $\mathcal{T}'' \to \mathcal{T}$, $\mathcal{T}' \to \mathcal{T}'''$, $\mathcal{T}'' \not\to \mathcal{T}'''$ and one of the assertions $\mathcal{T}'' \to \mathcal{T}$ and $\mathcal{T}' \to \mathcal{T}'''$ is a proper degeneration, then $\mathcal{T} \not\to \mathcal{T}'$ is called a *primary non-degeneration*.

Note that it suffices to prove primary degenerations and non-degenerations to fully describe the geometry of a variety. Therefore, in this work we will focus on proving the primary degenerations and non-degenerations, and for this we will use the results applied to Lie algebras in [9, 22, 23, 41]. These results are summarized below.

Firstly, since $\dim O(\mu) = n^2 - \dim \mathfrak{Der}(\mu)$, then if $\mathcal{T} \to \mathcal{T}'$ and $\mathcal{T} \not\cong \mathcal{T}'$, we have that $\dim \mathfrak{Der}(\mathcal{T}) < \dim \mathfrak{Der}(\mathcal{T}')$, where $\mathfrak{Der}(\mathcal{T})$ denotes the Lie algebra of derivations of $\mathcal{T}$. Therefore, we will check the assertion $\mathcal{T} \to \mathcal{T}'$ only for $\mathcal{T}$ and $\mathcal{T}'$ such that $\dim \mathfrak{Der}(\mathcal{T}) < \dim \mathfrak{Der}(\mathcal{T}')$.

Secondly, to prove primary degenerations, let $\mathcal{T}$ and $\mathcal{T}'$ be two Lie triple systems represented by the structures μ and λ from $\mathcal{T}_n$, respectively. Let c_{ijk}^p be the structure constants of λ in a basis $e_1, \ldots, e_n$ of $\mathbb{V}$. If there exist n^2 maps $a_i^j(t) : \mathbb{F}^* \to \mathbb{F}$ such that $E_i(t) = \sum_{j=1}^n a_i^j(t) e_j$ $(1 \leq i \leq n)$ form a basis of $\mathbb{V}$ for any $t \in \mathbb{F}^*$ and the structure constants $c_{ijk}^p(t)$ of μ in the basis $E_1(t), \ldots, E_n(t)$ satisfy $\lim_{t \to 0} c_{ijk}^p(t) = c_{ijk}^p$, then $\mathcal{T} \to \mathcal{T}'$. In this case, $E_1(t), \ldots, E_n(t)$ is called a parametrized basis for $\mathcal{T} \to \mathcal{T}'$.

Thirdly, to prove primary non-degenerations we will use the following lemma (see [22]).

Lemma 4.1 *Let $\mathcal{B}$ be a Borel subgroup of* $\mathrm{GL}(\mathbb{V})$ *and $\mathcal{R} \subset \mathcal{T}_n$ be a $\mathcal{B}$-stable closed subset. If $\mathcal{T} \to \mathcal{T}'$ and $\mathcal{T}$ can be represented by $\mu \in \mathcal{R}$ then there is $\lambda \in \mathcal{R}$ that represents $\mathcal{T}'$.*

In particular, it follows:

Corollary 4.1 *Let $\mathcal{T} \to \mathcal{T}'$ be a proper degeneration, then*

(1) $\dim \operatorname{Ann}(\mathcal{T}) \leq \dim \operatorname{Ann}(\mathcal{T}')$,
(2) $\dim\,[\mathcal{T}, \mathcal{T}, \mathcal{T}] \geq \dim\,\left[\mathcal{T}', \mathcal{T}', \mathcal{T}'\right]$.

Defining $\mathcal{R}$ by a set of polynomial equations in variables c_{ijk}^{p} and in the conditions of the previous result, such that $\mu \in \mathcal{R}$ and $O(\lambda) \cap \mathcal{R} = \emptyset$, give us the non-degeneration $\mathcal{T} \not\to \mathcal{T}'$. In this case, we call $\mathcal{R}$ a separating set for $\mathcal{T} \not\to \mathcal{T}'$. To prove non-degenerations, we will present the corresponding separating set and we will omit the verification of the fact that $\mathcal{R}$ is stable under the action of the Borel subgroup of lower triangular matrices and of the fact that $O(\lambda) \cap \mathcal{R} = \emptyset$, which can be obtained by straightforward calculations.

Finally, if the algebraic classification of the class under consideration is finite, then the graph of primary degenerations gives the whole geometric classification: the description of irreducible components can be easily obtained. This is the case of the variety $\mathcal{T}_3$ of 3-dimensional nilpotent Lie triple systems. However, the variety $\mathcal{T}_4$ of 4-dimensional nilpotent Lie triple systems contains infinitely many non-isomorphic triple systems, since we have the family $\mathcal{T}_{4,6}^{*}$. Therefore, we have to fulfil some additional work.

Definition 4.3 Let $\mathcal{T}(*) = \{\mathcal{T}(\alpha) : \alpha \in I\}$ be a family of n-dimensional Lie triple systems and let $\mathcal{T}'$ be another Lie triple system. Suppose that $\mathcal{T}(\alpha)$ is represented by the structure $\mu(\alpha) \in \mathcal{T}_n$ for $\alpha \in I$ and $\mathcal{T}'$ is represented by the structure $\lambda \in \mathcal{T}_n$. We say the family $\mathcal{T}(*)$ *degenerates* to $\mathcal{T}'$, and write $\mathcal{T}(*) \to \mathcal{T}'$, if $\lambda \in \overline{\{O(\mu(\alpha))\}_{\alpha \in I}}$.

Conversely, if $\lambda \notin \overline{\{O(\mu(\alpha))\}_{\alpha \in I}}$ then we call it a *non-degeneration*, and we write $\mathcal{T}(*) \not\to \mathcal{T}'$.

To prove $\mathcal{T}(*) \to \mathcal{T}'$, suppose that $\mathcal{T}(\alpha)$ is represented by the structure $\mu(\alpha) \in \mathcal{T}_n$ for $\alpha \in I$ and $\mathcal{T}'$ is represented by the structure $\lambda \in \mathcal{T}_n$. Let c_{ijk}^{p} be the structure constants of λ in a basis $e_1, \ldots, e_n$ of $\mathbb{V}$. If there is a pair of maps $(f, (a_i^j))$, where $f : \mathbb{F}^* \to I$ and $a_i^j : \mathbb{F}^* \to \mathbb{F}$ are such that $E_i(t) = \sum_{j=1}^{n} a_i^j(t) e_j$ $(1 \leq i \leq n)$ form a basis of $\mathbb{V}$ for any $t \in \mathbb{F}^*$ and the structure constants $c_{ijk}^{p}(t)$ of $\mu\big(f(t)\big)$ in the basis $E_1(t), \ldots, E_n(t)$ satisfy $\lim\limits_{t \to 0} c_{ijk}^{p}(t) = c_{ijk}^{p}$, then $\mathcal{T}(*) \to \mathcal{T}'$. In this case $E_1(t), \ldots, E_n(t)$ and $f(t)$ are called a parametrized basis and a parametrized index for $\mathcal{T}(*) \to \mathcal{T}'$, respectively.

To prove $\mathcal{T}(*) \not\to \mathcal{T}'$, we will use an analogue of the Lemma 4.1 for families of triple systems.

Lemma 4.2 *Let $\mathcal{B}$ be a Borel subgroup of* $\mathrm{GL}(\mathbb{V})$ *and $\mathcal{R} \subset \mathcal{T}_n$ be a $\mathcal{B}$-stable closed subset. If $\mathcal{T}(*) \to \mathcal{T}'$ and $\mathcal{T}(\alpha)$ can be represented by $\mu(\alpha) \in \mathcal{R}$ for $\alpha \in I$, then there is $\lambda \in \mathcal{R}$ that represents $\mathcal{T}'$.*

Similarly, constructing a set $\mathcal{R}$ in the conditions of the previous result, such that $\mu(\alpha) \in \mathcal{R}$ for any $\alpha \in I$ and $O(\lambda) \cap \mathcal{R} = \varnothing$, gives us the non-degeneration $\mathcal{T}(*) \not\to \mathcal{T}'$.

4.1 Geometric classification of 3-dimensional nilpotent Lie triple systems

Since any triple system degenerates to the trivial triple system, we have the following result.

Theorem 4.1 *The variety of* 3*-dimensional nilpotent Lie triple systems has only one irreducible component, corresponding to the rigid Lie triple system $\mathcal{T}_{3,2}$. Moreover,* $\dim O(\mathcal{T}_{3,2}) = 4$.

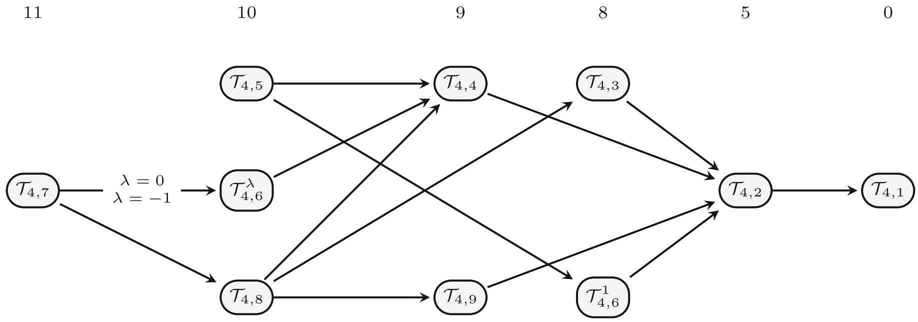

Fig. 1 Graph of primary degenerations and non-degenerations

4.2 Geometric classification of 4-dimensional nilpotent Lie triple systems

In Theorem 3.1 we present the classification, up to isomorphism, of the nilpotent Lie triple systems of dimension 4 over an algebraically closed field $\mathbb{F}$ of characteristic different from two and three. To obtain the irreducible components of this variety, we have to study the primary degenerations and non-degenerations first.

Lemma 4.3 *The graph of primary degenerations and non-degenerations for the variety of* 4*-dimensional nilpotent Lie triple systems is given in* Fig. 1*, where the numbers above are the dimensions of the corresponding orbits.*

Proof From Table 1 we deduce the dimensions of the orbits for each nilpotent Lie triple system of dimension 4. Every primary degeneration and non-degeneration can be proven using the parametrized bases and separating sets included in Tables 2 and 3 below, respectively. □

at this point, only the description of the closure of the orbit of the parametric family $\mathcal{T}_{4,6}^{*}$ is missing.

Lemma 4.4 *The closure of the orbit of the parametric family $\mathcal{T}_{4,6}^{*}$ in the variety $\mathcal{T}_4$ contains the closures of the orbits of the nilpotent Lie triple systems $\mathcal{T}_{4,5}$, $\mathcal{T}_{4,4}$, $\mathcal{T}_{4,2}$ and $\mathcal{T}_{4,1}$.*

Proof The primary degenerations and non-degenerations that do not follow from the previous results are included in Tables 4 and 5 below, respectively. □

From Lemmas 4.3 and 4.4, we have the following result that summarizes the geometric classification of $\mathcal{T}_4$.

Theorem 4.2 *The variety of* 4*-dimensional nilpotent Lie triple systems $\mathcal{T}_4$ has two irreducible components corresponding to the rigid Lie triple system $\mathcal{T}_{4,7}$ and the family of Lie triple systems $\mathcal{T}_{4,6}^{*}$. The lattice of subsets for the orbit closures is given in Fig. 2, where the numbers above are the dimensions of the corresponding orbits.*

Table 1 The dimension of derivations of 4-dimensional nilpotent Lie triple systems

$\mathcal{T}$	Multiplication table				$\dim \mathfrak{Der}$
$\mathcal{T}_{4,1}$					16
$\mathcal{T}_{4,2}$	$[e_1, e_2, e_1] = e_3$				9
$\mathcal{T}_{4,3}$	$[e_1, e_2, e_1] = e_3$	$[e_1, e_2, e_2] = e_4$			8
$\mathcal{T}_{4,4}$	$[e_2, e_3, e_2] = e_4$	$[e_3, e_1, e_3] = e_4$			7
$\mathcal{T}_{4,5}$	$[e_2, e_3, e_1] = e_4$	$[e_3, e_1, e_2] = e_4$	$[e_2, e_1, e_3] = 2e_4$	$[e_2, e_3, e_2] = e_4$	6
$\mathcal{T}_{4,6}^{\lambda}$	$[e_1, e_2, e_3] = -(\lambda + 1)\, e_4$	$[e_2, e_3, e_1] = \lambda e_4$	$[e_3, e_1, e_2] = e_4$		8 $\lambda = 1, -2, -1/2$ 6 Otherwise
$\mathcal{T}_{4,7}$	$[e_1, e_2, e_1] = e_3$	$[e_1, e_2, e_3] = e_4$	$[e_1, e_3, e_2] = e_4$		5
$\mathcal{T}_{4,8}$	$[e_1, e_2, e_1] = e_3$	$[e_1, e_3, e_1] = e_4$	$[e_1, e_2, e_2] = e_4$		6
$\mathcal{T}_{4,9}$	$[e_1, e_2, e_1] = e_3$	$[e_1, e_3, e_1] = e_4$			7

Table 2 Degenerations of 4-dimensional nilpotent Lie triple systems, except of the family $\mathcal{T}_{4,6}^{*}$

Degeneration			Parametrized basis			
$\mathcal{T}_{4,7}$	$\to$	$\mathcal{T}_{4,6}^{0}$	$E_1(t)=e_1,$	$E_2(t)=e_2,$	$E_3(t)=t^{-1}e_3,$	$E_4(t)=-t^{-1}e_4$
$\mathcal{T}_{4,5}$	$\to$	$\mathcal{T}_{4,6}^{1}$	$E_1(t)=e_1,$	$E_2(t)=te_2,$	$E_3(t)=e_3,$	$E_4(t)=te_4$
$\mathcal{T}_{4,8}$	$\to$	$\mathcal{T}_{4,3}$	$E_1(t)=te_1,$	$E_2(t)=e_2,$	$E_3(t)=t^2e_3,$	$E_4(t)=te_4$
$\mathcal{T}_{4,8}$	$\to$	$\mathcal{T}_{4,9}$	$E_1(t)=e_1,$	$E_2(t)=te_2,$	$E_3(t)=te_3,$	$E_4(t)=te_4$
$\mathcal{T}_{4,4}$	$\to$	$\mathcal{T}_{4,2}$	$E_1(t)=e_2,$	$E_2(t)=te_3,$	$E_3(t)=te_4,$	$E_4(t)=te_1$
$\mathcal{T}_{4,9}$	$\to$	$\mathcal{T}_{4,2}$	$E_1(t)=e_1,$	$E_2(t)=te_2,$	$E_3(t)=te_3,$	$E_4(t)=e_4$
$\mathcal{T}_{4,3}$	$\to$	$\mathcal{T}_{4,2}$	$E_1(t)=e_1,$	$E_2(t)=te_2,$	$E_3(t)=te_3,$	$E_4(t)=e_4$
$\mathcal{T}_{4,2}$	$\to$	$\mathcal{T}_{4,1}$	$E_1(t)=te_1,$	$E_2(t)=te_2,$	$E_3(t)=te_3,$	$E_4(t)=te_4$
$\mathcal{T}_{4,8}$	$\to$	$\mathcal{T}_{4,4}$	$E_1(t)=t^{-1}e_3,$	$E_2(t)=-ie_2,$	$E_3(t)=te_1,$	$E_4(t)=te_4$
$\mathcal{T}_{4,6}^{1}$	$\to$	$\mathcal{T}_{4,2}$	$E_1(t)=te_1-\frac{1}{3t}e_3,$	$E_2(t)=e_2,$	$E_3(t)=e_4,$	$E_4(t)=e_3$
$\mathcal{T}_{4,7}$	$\to$	$\mathcal{T}_{4,8}$	$E_1(t)=e_1-\frac{1}{2t^3}e_2-\frac{1}{4t^5}e_3,$	$E_2(t)=\frac{1}{2t}e_2-\frac{1}{4t^3}e_3,$	$E_3(t)=\frac{1}{2t}e_3,$	$E_4(t)=-\frac{1}{4t^4}e_4$
$\mathcal{T}_{4,5}$	$\to$	$\mathcal{T}_{4,4}$	$E_1(t)=\frac{t}{3}e_1,$	$E_2(t)=e_2,$	$E_3(t)=-\frac{1}{3t}e_1+t^{-1}e_2+e_3,$	$E_4(t)=e_4$
$\mathcal{T}_{4,6}^{\lambda\neq 1}$	$\to$	$\mathcal{T}_{4,4}$	$E_1(t)=e_2,$	$E_2(t)=e_1+\frac{t^{-1}}{(\lambda-1)}e_2,$	$E_3(t)=-\frac{t^{-1}}{(2\lambda+1)}e_1-\frac{t^{-2}}{(\lambda^2+\lambda-2)}e_2+e_3,$	$E_4(t)=t^{-1}e_4$

Table 3 Non-degenerations of 4-dimensional nilpotent Lie triple systems, except of the family $\mathcal{T}_{4,6}^{*}$

Non-degeneration			Arguments
$\mathcal{T}_{4,7}$	$\not\to$	$\mathcal{T}_{4,5}, \mathcal{T}_{4,6}^{\lambda\neq 0,-1}$	$\mathcal{R} = \left\{ \begin{array}{l} c_{1,2,1}^{3} = -c_{2,1,1}^{3}, c_{1,2,1}^{4} = -c_{2,1,1}^{4}, c_{1,2,2}^{4} = -c_{2,1,2}^{4}, c_{1,2,3}^{4} = -c_{2,1,3}^{4}, \\ c_{1,3,1}^{4} = -c_{3,1,1}^{4}, c_{1,3,2}^{4} = -c_{3,1,2}^{4} = c_{1,2,3}^{4} \text{ and } c_{ijk}^{p} = 0 \text{ otherwise.} \end{array} \right\}$
$\mathcal{T}_{4,6}^{\lambda}$	$\not\to$	$\mathcal{T}_{4,6}^{1}$	$\mathcal{R} = \left\{ \begin{array}{l} c_{1,2,1}^{4} = -c_{2,1,1}^{4}, c_{1,2,2}^{4} = -c_{2,1,2}^{4}, c_{1,2,3}^{4} = -c_{2,1,3}^{4} = (1+\lambda)c_{1,3,2}^{4}, c_{1,3,1}^{4} = -c_{3,1,1}^{4}, \\ c_{1,3,2}^{4} = -c_{3,1,2}^{4}, c_{2,3,1}^{4} = -c_{3,2,1}^{4} = -\lambda c_{1,3,2}^{4} \text{ and } c_{ijk}^{p} = 0 \text{ otherwise.} \end{array} \right\}$
$\mathcal{T}_{4,9}$	$\not\to$	$\mathcal{T}_{4,3}$	$\mathcal{R} = \left\{ c_{1,2,1}^{3} = -c_{2,1,1}^{3}, c_{1,2,1}^{4} = -c_{2,1,1}^{4}, c_{1,3,1}^{4} = -c_{3,1,1}^{4} \text{ and } c_{ijk}^{p} = 0 \text{ otherwise.} \right\}$
$\mathcal{T}_{4,5}$	$\not\to$	$\mathcal{T}_{4,9}, \mathcal{T}_{4,3}$	Corollary 4.1 (2)
$\mathcal{T}_{4,6}^{\lambda}$	$\not\to$	$\mathcal{T}_{4,9}, \mathcal{T}_{4,3}$	Corollary 4.1 (2)

Table 4 Degenerations of the the 4-dimensional nilpotent Lie triple systems for the family $\mathcal{T}^*_{4,6}$

Degeneration			Parametrized basis				Parametrized index
$\mathcal{T}^*_{4,6}$	$\to$	$\mathcal{T}_{4,5}$	$E_1(t) = \frac{1}{2}e_1 + \frac{1}{2t}e_2,$	$E_2(t) = -\frac{1}{2t}e_1 + \frac{1}{2t^2}e_2,$	$E_3(t) = e_3,$	$E_4(t) = \frac{1}{2t^2}$	$f(t) = \frac{2}{1+t} - 1$

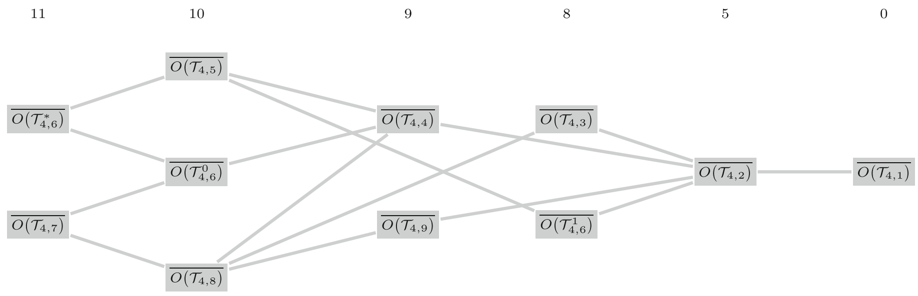

Fig. 2 Inclusion graph for orbit closures and dimension

Table 5 Non-degenerations of the 4-dimensional nilpotent Lie triple systems for the family $\mathcal{T}^*_{4,6}$

Non-degeneration	Arguments
$\mathcal{T}^*_{4,6} \not\to \mathcal{T}_{4,9}, \mathcal{T}_{4,3}$	$\mathcal{R} = \left\{ \begin{array}{l} c^4_{1,2,1} = -c^4_{2,1,1}, c^4_{1,2,2} = -c^4_{2,1,2}, c^4_{1,2,3} = -c^4_{2,1,3}, c^4_{1,3,1} = -c^4_{3,1,1}, \\ c^4_{1,3,2} = -c^4_{1,3,2}, c^4_{2,3,1} = -c^4_{3,2,1} \text{ and } c^p_{ijk} = 0 \text{ otherwise.} \end{array} \right\}$

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References

1. Abdelwahab, H., Calderón, M.A., Fernández, O.A.: Central extensions of 4-dimensional binary Lie algebras. Rocky Mt. J. Math. **50**(5), 1541–1559 (2020)
2. Abdelwahab, H., Calderón, A.J., Kaygorodov, I.: The algebraic and geometric classification of nilpotent binary Lie algebras. Int. J. Algebra Comput. **29**(6), 1113–1129 (2019)
3. Alvarez, M.A.: On rigid 2-step nilpotent Lie algebras. Algebra Colloq. **25**(2), 349–360 (2018)
4. Alvarez, M.A., Hernández, I., Kaygorodov, I.: Degenerations of Jordan superalgebras. Bull. Malays. Math. Sci. Soc. **42**(6), 3289–3301 (2019)
5. Ancochea Bermúdez, J.M., Fresán, J., Margalef, B.J.: Contractions of low-dimensional nilpotent Jordan algebras. Comm. Algebra **39**(3), 1139–1151 (2011)
6. Beneš T., Burde D.: Degenerations of pre-Lie algebras. J. Math. Phys. **50**(11), 112102 (2009)
7. Beneš T., Burde D.: Classification of orbit closures in the variety of three-dimensional Novikov algebras. J. Algebra Appl. **13**(2), 1350081 (2014)
8. Bouetou, T.: Classification of solvable 3-dimensional Lie triple systems. Non-associative algebra and its applications, Lecture Notes Pure Application Mathematics, vol. **246**. Chapman & Hall/CRC, Boca Raton, FL, pp. 41–54 (2006)
9. Burde, D., Steinhoff, C.: Classification of orbit closures of 4-dimensional complex Lie algebras. J. Algebra **214**(2), 729–739 (1999)
10. Calderón, M.A., Fernández, O.A., Kaygorodov, I.: The classification of n-dimensional anticommutative algebras with $(n-3)$-dimensional annihilator. Comm. Algebra **47**(1), 173–181 (2019)
11. Calderón, M.A., Fernández, O.A., Kaygorodov, I.: The classification of 2-dimensional rigid algebras. Linear Multilinear Algebra **68**(4), 828–844 (2020)
12. Calderón, M.A., Fernández, O.A., Kaygorodov, I.: On the classification of bilinear maps with radical of a fixed codimension. Linear Multilinear Algebra (2020) https://doi.org/10.1080/03081087.2020.1849001
13. Cicalò, S., De Graaf, W., Schneider, C.: Six-dimensional nilpotent Lie algebras. Linear Algebra Appl. **436**(1), 163–189 (2012)
14. Darijani, I., Usefi, H.: The classification of 5-dimensional p-nilpotent restricted Lie algebras over perfect fields. I. J. Algebra **464**, 97–140 (2016)

15. De Graaf, W.: Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2. J. Algebra **309**(2), 640–653 (2007)
16. De Graaf, W.: Classification of nilpotent associative algebras of small dimension. Int. J. Algebra Comput. **28**(1), 133–161 (2018)
17. Demir, I., Misra, K., Stitzinger, E.: On classification of four-dimensional nilpotent Leibniz algebras. Comm. Algebra **45**(3), 1012–1018 (2017)
18. Neher, E.: On the classification of Lie and Jordan triple systems. Comm. Algebra **13**(12), 2615–2667 (1985)
19. Fernández, O.A., Kaygorodov, I., Khrypchenko, M., Volkov, Y.: Degenerations of nilpotent algebras. J. Pure Appl. Algebra **226**(3), 106850 (2022)
20. Gorshkov, I., Kaygorodov, I., Khrypchenko, M.: The geometric classification of nilpotent Tortkara algebras. Comm. Algebra **48**(1), 204–209 (2020)
21. Gorshkov, I., Kaygorodov, I., Popov, Yu.: Degenerations of Jordan algebras and "marginal" algebras. Algebra Colloq. **28**(2), 281–294 (2021)
22. Grunewald, F., O'Halloran, J.: Varieties of nilpotent Lie algebras of dimension less than six. J. Algebra **112**(2), 315–325 (1988)
23. Grunewald, F., O'Halloran, J.: A characterization of orbit closure and applications. J. Algebra **116**(1), 163–175 (1988)
24. Hegazi, A., Abdelwahab, H.: Classification of five-dimensional nilpotent Jordan algebras. Linear Algebra Appl. **494**, 165–218 (2016)
25. Hegazi, A., Abdelwahab, H.: The classification of N-dimensional non-associative Jordan algebras with $(N-3)$-dimensional annihilator. Comm. Algebra **46**(2), 629–643 (2018)
26. Hegazi, A., Abdelwahab, H., Calderón, M.A.: The classification of n-dimensional non-Lie Malcev algebras with $(n-4)$-dimensional annihilator. Linear Algebra Appl. **505**, 32–56 (2016)
27. Hegazi, A., Abdelwahab, H., Calderón, M.A.: Classification of nilpotent Malcev algebras of small dimensions over arbitrary fields of characteristic not 2. Algebr. Represent. Theory **21**(1), 19–45 (2018)
28. Hopkins, N.C.: Nilpotent ideals in Lie and anti-Lie triple systems. J. Algebra **178**(2), 480–492 (1995)
29. Ismailov, N., Kaygorodov, I., Mashurov, F.: The algebraic and geometric classification of nilpotent assosymmetric algebras. Algebr. Represent. Theory **24**(1), 135–148 (2021)
30. Ismailov N., Kaygorodov I., Volkov Yu.: The geometric classification of Leibniz algebras. Int. J. Math. **29**(5), 1850035 (2018)
31. Ismailov, N., Kaygorodov, I., Volkov, Yu.: Degenerations of Leibniz and anticommutative algebras. Can. Math. Bull. **62**(3), 539–549 (2019)
32. Lister, W.G.: A structure theory of Lie triple systems. Trans. Am. Math. Soc. **72**, 217–242 (1952)
33. Karimjanov, I., Kaygorodov, I., Khudoyberdiyev, A.: The algebraic and geometric classification of nilpotent Novikov algebras. J. Geom. Phys. **143**, 11–21 (2019)
34. Kaygorodov, I., Popov, Yu., Pozhidaev, A., Volkov, Yu.: Degenerations of Zinbiel and nilpotent Leibniz algebras. Linear Multilinear Algebra **66**(4), 704–716 (2018)
35. Kaygorodov, I., Popov, Yu., Volkov, Yu.: Degenerations of binary Lie and nilpotent Malcev algebras. Comm. Algebra **46**(11), 4928–4940 (2018)
36. Kaygorodov, I., Volkov, Yu.: The variety of two-dimensional algebras over an algebraically closed field. Can. J. Math. **71**(4), 819–842 (2019)
37. Kaygorodov, I., Volkov, Yu.: Complete classification of algebras of level two. Mosc. Math. J. **19**(3), 485–521 (2019)
38. Jacobson, N.: General representation theory of Jordan algebras. Trans. Amer. Math. Soc. **70**, 509–530 (1951)
39. Jacobson, N.: Lie and Jordan triple systems. Am. J. Math. **71**, 149–170 (1949)
40. Loos, O.: Über eine Beziehung zwischen Malcev-Algebren und Lietripelsystemen. Pac. J. Math. **18**, 553–562 (1966)
41. Seeley, C.: Degenerations of 6-dimensional nilpotent Lie algebras over $\mathbb{C}$. Comm. Algebra **18**(10), 3493–3505 (1990)
42. Skjelbred, T., Sund, T.: Sur la classification des algèbres de Lie nilpotentes. C. R. Acad. Sci. Paris Sér. A-B **286**(5), A241–A242 (1978)
43. Yamaguti, K.: On algebras of totally geodesic spaces (Lie triple systems). J. Sci. Hiroshima Univ. Ser. A **21**, 107–113 (1957/58)