Research Article

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Representing the Stirling polynomials $\sigma_n(x)$ in dependence of n and an application to polynomial zero identities

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Abstract: Denote by σ_n the n-th Stirling polynomial in the sense of the well-known book Concrete Mathematics by Graham, Knuth and Patashnik. We show that there exist developments $x\sigma_n(x) = \sum_{j=0}^{n}(2^j j!)^{-1}q_{n-j}(j)x^j$ with polynomials q_j of degree j. We deduce from this the polynomial identities

$$\sum_{a+b+c+d=n}(-1)^d\frac{(x-2a-2b)^{3n-s-a-c}}{a!b!c!d!(3n-s-a-c)!} = 0, \quad \text{for } s \in \mathbb{Z}_{\geq 1},$$

found in an attempt to find a simpler formula for the density function in a five-dimensional random flight problem. We point out a probable connection to Riordan arrays.

Keywords: Stirling polynomials, polynomial identities, difference equations, random flights, Riordan arrays

MSC 2020: 05A19, 11B37, 11B73, 39A06

1 Introduction

In the context of research on a problem of random flights in dimension 5 – which we discuss briefly in Section 4 – the second author conjectured the identities in the abstract, for which the authors could not find many hints in the literature. The work on its proof led us to a (for us at least) surprising result about the behaviour of the coefficients of sequences of Stirling polynomials. Let $\sigma_n(x)$ be the nth Stirling polynomial in the sense of [6]; the precise definition is given in Section 2, but see Table 1 for the coefficients of the first few Stirling polynomials. The table tells us, for example, that $x\sigma_3(x) = 0 + 0x - \frac{1}{48}x^2 + \frac{1}{48}x^3$.

We agree to begin row and column indices both with 0 and then multiply the entries in column j of Table 1 with $2^j j!$. We then get Table 2.

The first main result proven in the current article can be expressed as saying that the jth diagonal, i.e., the sequence of numbers in positions $(j, 0), (j + 1, 1), (j + 2, 2), \ldots$ of Table 2, gives the values of a polynomial of degree j on the nonnegative integers.

Accept this for the moment and denote the sequence by $(q_j(n))_{n\geq 0}$. As is well known, see almost any text on numerical analysis, e.g. [2, p. 95ff], using $j + 1$ interpolation points of distinct abscissae, in our case $n = 0, 1, 2, \ldots$, one can compute a unique polynomial of degree $\leq j$ whose graph passes through these points. From Table 2, one finds, for example,

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Table 1: Coefficients of the first few polynomials $x\sigma_n(x)$

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8
$x\sigma_0(x)$	1								
$x\sigma_1(x)$	0	$\frac{1}{2}$							
$x\sigma_2(x)$	0	$-\frac{1}{24}$	$\frac{1}{8}$						
$x\sigma_3(x)$	0	0	$-\frac{1}{48}$	$\frac{1}{48}$					
$x\sigma_4(x)$	0	$\frac{1}{2880}$	$\frac{1}{1152}$	$-\frac{1}{192}$	$\frac{1}{384}$				
$x\sigma_5(x)$	0	0	$\frac{1}{5760}$	$\frac{1}{2304}$	$-\frac{1}{1152}$	$\frac{1}{3840}$			
$x\sigma_6(x)$	0	$-\frac{1}{181440}$	$-\frac{1}{69120}$	$\frac{13}{414720}$	$\frac{1}{9216}$	$-\frac{1}{9216}$	$\frac{1}{46080}$		
$x\sigma_7(x)$	0	0	$-\frac{1}{362880}$	$-\frac{1}{138240}$	$\frac{1}{829440}$	$\frac{1}{55296}$	$-\frac{1}{92160}$	$\frac{1}{645120}$	
$x\sigma_8(x)$	0	$\frac{1}{9676800}$	$\frac{101}{348364800}$	$-\frac{1}{2580480}$	$-\frac{67}{39813120}$	$-\frac{1}{1658880}$	$\frac{1}{442368}$	$-\frac{1}{1105920}$	$\frac{1}{10321920}$

Table 2: Result of multiplying column j of Table 1 with $2^j\, j!$

	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	0	$-\frac{1}{12}$	1						
3	0	0	$-\frac{1}{6}$	1					
4	0	$\frac{1}{1440}$	$\frac{1}{144}$	$-\frac{1}{4}$	1				
5	0	0	$\frac{1}{720}$	$\frac{1}{48}$	$-\frac{1}{3}$	1			
6	0	$-\frac{1}{90720}$	$-\frac{1}{8640}$	$\frac{13}{8640}$	$\frac{1}{24}$	$-\frac{5}{12}$	1		
7	0	0	$-\frac{1}{45360}$	$-\frac{1}{2880}$	$\frac{1}{2160}$	$\frac{5}{72}$	$-\frac{1}{2}$	1	
8	0	$\frac{1}{4838400}$	$\frac{101}{43545600}$	$-\frac{1}{53760}$	$-\frac{67}{103680}$	$-\frac{1}{432}$	$\frac{5}{48}$	$-\frac{7}{12}$	1

$$q_0(n) = 1, \quad q_1(n) = \frac{-n}{12}, \quad q_2(n) = \frac{(-1+n)n}{288}, \quad q_3(n) = \frac{26n + 15n^2 - 5n^3}{51840}.$$

The sequence of numbers in the jth diagonal of the original table is given by $\left(\frac{q_j(\ell)}{2^\ell \ell!}\right)_{\ell \geq 0}$. We can use its row n to determine the polynomial $x\sigma_n(x)$ for symbolic n. The leftmost coefficient is the beginning and hence at position 0 of diagonal n. So, it has value $\left(\frac{q_n(0)}{2^0 0!}\right)$. The coefficient of x^1 pertains to diagonal $n-1$. It is at position 1 of that diagonal so has value $\frac{q_{n-1}(1)}{2^1 1!}$. In general, the coefficient pertaining to x^j is at position j of diagonal $n-j$ and, therefore, has value $\frac{q_{n-j}(j)}{2^j j!}$. Thus, we obtain $x\sigma_n(x) = \sum_{j=0}^{n} \frac{q_{n-j}(j)}{2^j j!} x^j$, as claimed in the abstract.

As it happens, the fact $q_j(0) = 0$ for $j \geq 1$ implies $n | q_j(n)$ so that $(n-j)|q_j(n-j)$. This means that by putting $\tilde{q}_j(n) := q_j(n)/n$ and using the polynomials q_0, q_1, q_2, and q_3 computed above, we find

$$\begin{aligned}
x\sigma_n(x) &= \sum_{j=0}^{n-4} \frac{q_{n-j}(j)x^j}{2^j j!} + \frac{q_3(n-3)x^{n-3}}{2^{n-3}(n-3)!} + \frac{q_2(n-2)x^{n-2}}{2^{n-2}(n-2)!} + \frac{q_1(n-1)x^{n-1}}{2^{n-1}(n-1)!} + \frac{q_0(n)x^n}{2^n n!} \\
&= \sum_{j=0}^{n-4} \frac{\tilde{q}_{n-j}(j)x^j}{2^j (j-1)!} + \frac{\tilde{q}_3(n-3)x^{n-3}}{2^{n-3}(n-4)!} + \frac{\tilde{q}_2(n-2)x^{n-2}}{2^{n-2}(n-3)!} + \frac{\tilde{q}_1(n-1)x^{n-1}}{2^{n-1}(n-2)!} + \frac{x^n}{2^n n!} \\
&= \sum_{j=0}^{n-4} \frac{\tilde{q}_{n-j}(j)x^j}{2^j (j-1)!} + \frac{(-64 + 45n - 5n^2)x^{n-3}}{51840 \cdot (2^{n-3}(n-4)!)} + \frac{(-3+n)x^{n-2}}{288(2^{n-2}(n-3)!)} + \frac{-1x^{n-1}}{12(2^{n-1}(n-2)!)} + \frac{x^n}{2^n n!},
\end{aligned}$$

which shows that we could also write $x\sigma_n(x) = \sum_{j=0}^{n-1}(2^j(j-1)!)^{-1}\hat{q}_{n-j-1}(n)x^j + \frac{x^n}{2^n n!}$ for certain polynomials $\hat{q}_l(n)$ of degree l in n. A "really simple closed" expression $f(j,n)$ such that $x\sigma_n(x) = \sum_{j=0}^{n} f(j,n)x^{n-j}$ for all $j, n \in \mathbb{Z}_{\geq 0}$ probably does not exist because it would, for example, via the identity $B_m = -m\ m!\sigma_m(0)$, imply a simple formula for the Bernoulli numbers.

In Section 2, we collect a number of results on Stirling numbers and Stirling polynomials. In Section 3, we assume the representation $x\sigma_n(x) = \sum_{k=0}^{n}(-1)^k a_{n,k}x^{n-k}$ and prove that the sequence $\mathbb{Z}_{\geq k} \ni n \mapsto 2^{n-k}(n-k)!a_{n,k}$ is polynomial of degree k; a fact equivalent to the representation claimed for $x\sigma_n(x)$ given in the abstract. The latter should have some importance for refined asymptotic analyses of the Stirling numbers of the second kind. To obtain that result, we have to solve a first-order difference equation with polynomial coefficients.

In Section 4, we deduce the identities mentioned in the abstract. In Section 5, we provide reasons to conjecture that the second of the tables shown above is actually an example of a Riordan array and indicate other possibilities for perhaps further fruitful work.

More important than the particular polynomial identity, which we derive, might be the methods which we employ. They should be applicable in a number of similarly looking identities. But we admit that it would be desirable to first simplify our proof significantly. In this vein, we also note that by introducing the notation $x^{[k]} := x^k/k!$, the identities assume a more convenient form.

2 Stirling numbers, Stirling polynomials, and some known auxiliary facts

We collect here facts on Stirling numbers and Stirling polynomials. Our main sources are an article by Gessel and Stanley [5] and the book by Graham et al. [6, pp. 257–272]. An informative article with historically interesting remarks on Stirling numbers of the second kind is the article by Boyadzhiev [1]. Some more recent articles introducing generalisations are mentioned further.

Stirling polynomials are born from investigations into Stirling numbers. Stirling numbers, in a notation proposed by Jovan Karamata and promoted by [6], are defined for integers $n, k \geq 0$ and come in two kinds. Stirling numbers of the first kind are denoted by $\begin{bmatrix} n \\ k \end{bmatrix}$, and verbalised by "$n$ cycle k." They count the number of partitions of $[n] = \{1, 2, \ldots, n\}$ into k nonempty cycles. Stirling numbers of the second kind are denoted by $\begin{Bmatrix} n \\ k \end{Bmatrix}$ and verbalised by "n subset k." Classically, $\begin{Bmatrix} n \\ k \end{Bmatrix}$ is defined as the number of partitions of $[n] = \{1, 2, \ldots, n\}$ into k nonempty subsets; but Stirling numbers of the second kind also crop up quite differently. Assume that a and a^+ are operators satisfying $aa^+ - a^+a = 1$, then these Stirling numbers occur as coefficients when we write $(aa^+)^k$ as a linear combination of the powers $(a^+)^l a^l$, $l = 0, 1, 2, \ldots, k$, namely one has $(aa^+)^k = \sum_{l=0}^{k}\begin{Bmatrix} k \\ l \end{Bmatrix}(a^+)^l a^l$. The a and a^+ notations come from physics where they denote creation and annihilation operators in quantum mechanics, but, e.g. $a = \mathrm{d}/\mathrm{d}x$ and $a^+ = x$. i.e., multiplication with x allow interpretation as operators on polynomial space. See Kim and Kim [10,11] for this and generalisations of Stirling numbers of the both kinds. On the other hand, Stirling polynomials $\sigma_n(x)$ have a close relation to Bernoulli polynomials. The article by Choi [4] mentions generalised Bernoulli polynomials $B_n^{(\alpha)}(x)$ given through the development $(z/(e^z-1))^\alpha e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{z^n}{n!}$. The classical Bernoulli polynomials are given as follows: $B_n(x) = B_n^{(1)}(x)$. One finds then for the Stirling polynomials that $\sigma_n(x) = \frac{B_n^{(x)}(x)}{n!\,x}$.

With the supplementary conditions $\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{Bmatrix} n \\ 0 \end{Bmatrix} = \delta_{n,0}$, there hold for $n > 0$ the following recursions; for easy combinatorial explanations see [6]:

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \quad \begin{Bmatrix} n \\ k \end{Bmatrix} = k\begin{Bmatrix} n-1 \\ k \end{Bmatrix} + \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}.$$

Jekuthiel Ginsburg discovered in 1928 that there is a way to meaningfully define, for $n \geq 0$, $\begin{bmatrix} x \\ x-n \end{bmatrix}$, and $\begin{Bmatrix} x \\ x-n \end{Bmatrix}$ as polynomials in x of degree $2n$ (so that whenever x is an integer $>n$ there occur the usual Stirling numbers). This is explained in [6], where it is also observed that when $x \in \{0, 1, 2, \ldots, n\}$, then these polynomials are zero, and hence, we find that with the exception of the case $n = 0$, the expressions

$$\sigma_n(x) = \begin{bmatrix} x \\ x-n \end{bmatrix} / x(x-1)(x-2)\cdots(x-n)$$

are polynomials, called Stirling polynomials. The exception is $\sigma_0(x) = 1/x$. We have $\deg \sigma_n(x) = n - 1$.

The authors of [5] approach the topic of Stirling polynomials differently. They are interested in the sequences $\mathbb{Z}_{n\geq 1} \mapsto f_k(n) := S(n+k, n)$, where $S(n,k) = \begin{Bmatrix} n \\ k \end{Bmatrix}$, not so much for its own sake but for giving a combinatorial interpretation to the coefficients of the power series $(1-x)^{2k+1}\sum_{k\geq 0} f_k(n)x^n$. In this context, they establish that the functions f_k are polynomial of degree $2k$ with leading coefficient $(2^k k!)^{-1}$, a fact attributed to C. Jordan' s book on difference equations which is not available to us. For $k = 0$, the claim is clear since $f_0(n) = S(n, n) = 1$. The general case is done by induction on k. It is observed, with the not completely trivial proof left to the reader, that the recursion for the second kind Stirling numbers can be recast into the equation $(\Delta f_k)(n) = (n+1)f_{k-1}(n+1)$, valid for all $n \geq k \in \mathbb{Z}_{>0}$, where Δ is the forward difference operator. Using the elementary fact that a sequence $\{(\Delta f)(n)\}_{n\geq 0}$ is polynomial of degree d if and only if the sequence $\{f(n)\}_{n\geq 0}$ is polynomial of degree $d+1$ and that when the corresponding leading coefficients stand in the relation $\mathrm{lc}(\Delta f) = \deg f \cdot \mathrm{lc}(f)$, one obtains the claim.

Once one has that the map $n \mapsto f_k(n)$ coincides on the infinitely many points constituting $\mathbb{Z}_{\geq 1}$ with the values of a polynomial of degree $2k$, the authors can define $f_k(x)$ as being this polynomial. Observing that the difference equation $f_k(x+1) - f_k(x) = (x+1)f_{k-1}(x+1)$ holds for all x and supposing $f_{k-1}(0) = f_{k-1}(-1) = \cdots f_{k-1}(1-k) = 0$ and $f_k(0) = 0$ one derives successively $0 = f_k(0) = f_k(-1) = f_k(-2) = \cdots = f_k(-k)$. From this, in turn, it follows that $f_k(x) = x(x+1)\cdots(x+k) \cdot$ (a monic polynomial of degree $k-1$)$\cdot(1/2^k k!)$. In [5], it is the $f_k(x)$ that are called Stirling polynomials.

The "Stirling polynomials" of [6] and the "Stirling polynomials" of [5] are not the same but they are easily transformed to each other. By [6, p. 267], for all $k, n \in \mathbb{Z}$, $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{Bmatrix} -k \\ -n \end{Bmatrix}$. It follows, first for integer x and then, by the usual polynomial argument on the formal level, that

$$\begin{aligned} f_n(x) = \begin{Bmatrix} x+n \\ x \end{Bmatrix} &= \begin{bmatrix} -x \\ -x-n \end{bmatrix} \\ &= \sigma_n(-x)\cdot(-x)(-x-1)\cdots(-x-n) \\ &= \sigma_n(-x)(-1)^{n+1}x(x+1)\cdots(x+n). \end{aligned}$$

(We should mention that we learned of at least two other notions of Stirling polynomials, which have a loose connection to the polynomials $f_k(x)$ or $\sigma_m(x)$.)

We shall need the following recursion formula for the σ_n, mentioned in [6, Exercise 6.18].

Lemma 1. *For $n \geq 1$, one has*

$$(x+1)\sigma_n(x+1) = (x-n)\sigma_n(x) + x\sigma_{n-1}(x).$$

Proof. Substituting, for the left- and right-hand sides of this equation, respectively, the definitions of the σ_n, we obtain

$$\begin{aligned} \text{lhs} &= \begin{bmatrix} x+1 \\ x+1-n \end{bmatrix} \Big/ x(x-1)\cdots(x+1-n), \\ \text{rhs} &= \begin{bmatrix} x \\ x-n \end{bmatrix} \Big/ x(x-1)\cdots(x+1-n) + \begin{bmatrix} x \\ x+1-n \end{bmatrix} \Big/ (x-1)\cdots(x+1-n). \end{aligned}$$

Multiplying everything with $x(x-1)\cdots(x+1-n)$, we obtain that the claim is equivalent to

$$\begin{bmatrix} x+1 \\ x+1-n \end{bmatrix} = \begin{bmatrix} x \\ x-n \end{bmatrix} + x\begin{bmatrix} x \\ x+1-n \end{bmatrix}.$$

Now, this is simply an instance of the recursion formula for Stirling polynomials of the first kind. □

In Section 4, we will also use the following known facts.

Proposition 2. *Let* $a = a(x) = \sum_{j\geq 0} a_j x^j$ *be any polynomial and let* $p_n(x) := (-1)^n(-x)\sigma_n(-x)$. *Then,*
(*a*) *One has the following equivalent identities of finite sums*

$$\sum_{k\geq 0} \frac{(-1)^k}{k!(m-k)!} a(k) = (-1)^m \sum_{j\geq 0} a_j \begin{Bmatrix} j \\ m \end{Bmatrix}; \qquad \sum_{k+l=m} \frac{(-1)^l}{k!l!} a(k) = \sum_{j\geq 0} a_j \begin{Bmatrix} j \\ m \end{Bmatrix}.$$

(*b*) *For* n *and* k *nonnegative integers, there holds*

$$\begin{Bmatrix} n+k \\ k \end{Bmatrix} = \frac{(n+k)!}{k!} p_n(k).$$

Proof. (a) The sums are finite because the a_j for $j > \deg a$ are 0 and because for a negative integer s, one has $1/s! = 0$. The left formula is then essentially mentioned for polynomials $a(x)$ that are of the form x^l as [6, formula (6.19)], namely,

$$m! \begin{Bmatrix} n \\ m \end{Bmatrix} = \sum_k \binom{m}{k} k^n (-1)^{m-k}.$$

The formula given follows as any polynomial is a linear combination of monomials. The right formula follows by multiplying both sides with $(-1)^m$ and using that $(-1)^{m+k} = (-1)^{m-k}$.

(b) From the relation mentioned before Lemma 1, we see that $\begin{Bmatrix} n+k \\ k \end{Bmatrix} = \sigma_n(-k)(-1)^{n+1}k(k+1)\cdots(k+n)$. Use the definition of p_n to conclude the proof. □

Remark. In part (a) of the proposition, note that if $\deg a < m$, then in the equalities of sums, the ones at the right-hand sides, and hence at the left-hand sides, are 0. In addition, if $a(x) = x^l$, then the right-hand sides reduce to $(-1)^m \begin{Bmatrix} l \\ m \end{Bmatrix}$ and $\begin{Bmatrix} l \\ m \end{Bmatrix}$, respectively. The identities in part (a) are usually proved by applying the forward difference operator. A multivariate generalisation based on completely different reasoning can be found in [17].

The following two lemmas will be necessary only in the first part of Section 4.

Lemma 3. *If* $0 \leq \kappa \leq n$ *and* $l \geq 0$ *are integers, then*

$$\sum_{\substack{a+c=\kappa \\ b+d=n-\kappa}} \frac{(-1)^d}{a!b!c!d!} (a+b)^l = \sum_{a+c=\kappa} \frac{1}{a!c!} \sum_{\nu=0}^{l} \binom{l}{\nu} a^{l-\nu} \begin{Bmatrix} \nu \\ n-\kappa \end{Bmatrix}.$$

Proof. Using Proposition 2(a), we find

$$\begin{aligned} \text{lhs} &= \sum_{a+c=\kappa} \frac{1}{a!c!} \sum_{b+d=n-\kappa} \frac{(-1)^d}{b!d!} \sum_{\nu=0}^{l} \binom{l}{\nu} a^{l-\nu} b^{\nu} \\ &= \sum_{a+c=\kappa} \frac{1}{a!c!} \sum_{\nu=0}^{l} \binom{l}{\nu} a^{l-\nu} \sum_{b+d=n-\kappa} \frac{(-1)^d}{b!d!} b^{\nu} = \text{rhs}. \end{aligned}$$ □

From the proof, it follows that here 0^0, occurring as a special case of $a^{l-\nu}$, has to be interpreted as 1, because only with this understanding is the use of the binomial theorem correct.

In accordance with the notation used in [6], in the following lemma we use for integer $i \geq 0$ the notation $x^{\underline{i}} = x(x-1)\cdots(x-i+1)$ for falling factorials.

Lemma 4. *If n and k are nonnegative integers, then*

$$\sum_{l=0}^{n}\binom{n}{l}l^k = 2^n\sum_{i=0}^{k}\left\{{k \atop i}\right\}n^{\underline{i}}2^{-i}, \quad \textit{or, equivalently}, \quad \sum_{l+h=n}\frac{1}{l!h!}l^k = \sum_{i=0}^{k}\left\{{k \atop i}\right\}\frac{2^{n-i}}{(n-i)!}.$$

Proof. It is known that $l^k = \sum_i\left\{{k \atop i}\right\}l^{\underline{i}}$, see [6, equation (6.10)]. Hence, the left equality can be deduced as follows:

$$\text{lhs} = \sum_{i=0}^{k}\left\{{k \atop i}\right\}\sum_{l=0}^{n}\binom{n}{l}l^{\underline{i}} = \sum_{i=0}^{k}\left\{{k \atop i}\right\}\sum_{l=0}^{n}n^{\underline{i}}\binom{n-i}{l-i} = \sum_{i=0}^{k}\left\{{k \atop i}\right\}n^{\underline{i}}2^{n-i} = \text{rhs}.$$

The right equality follows by dividing by $n!$. □

3 The main result on the diagonals of the modified coefficient table of Stirling polynomials

We transform the recursion of Lemma 1 into a matrix equation for the coefficients.

Proposition 5. *Writing*

$$\sigma_n(x) = \sum_{j=0}^{n-1}a_jx^j \quad \textit{and} \quad x\sigma_{n-1}(x) = \sum_{j=0}^{n-1}b_jx^j, \quad n = 2, 3, 4 \ldots,$$

there holds the following $(n-1)\times(n-1)$ *matrix equation*:

$$\begin{pmatrix} (1+n) & \binom{2}{0} & \binom{3}{0} & \binom{4}{0} & \cdots & \binom{n-1}{0} \\ & 2+n & \binom{3}{1} & \binom{4}{1} & \cdots & \binom{n-1}{1} \\ & & 3+n & \binom{4}{2} & \cdots & \binom{n-1}{2} \\ & & & \ddots & & \vdots \\ & & & & (2n-2) & \binom{n-1}{n-3} \\ & & & & & 2n-1 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-2} \end{pmatrix} = \begin{pmatrix} b_0 - \binom{n}{0}a_{n-1} \\ b_1 - \binom{n}{1}a_{n-1} \\ b_2 - \binom{n}{2}a_{n-1} \\ \vdots \\ b_{n-2} - \binom{n}{n-2}a_{n-1} \end{pmatrix}.$$

Proof. With the understanding that $a_{-1} = a_n = 0$, we have

$$\begin{aligned} (x+1)\sigma_n(x+1) &= \sum_{j=0}^{n-1}a_j(1+x)^{j+1} = \sum_{j=0}^{n-1}a_j\sum_{l=0}^{j+1}\binom{j+1}{l}x^l \\ &= \sum_{l=0}^{n-1}\left(\sum_{j=l}^{n-1}a_j\binom{j+1}{l}\right)x^l + \sum_{j=0}^{n-1}a_jx^{j+1} \\ &= \sum_{l=0}^{n}\left(\sum_{j=l}^{n-1}a_j\binom{j+1}{l} + a_{l-1}\right)x^l; \\ (x-n)\sigma_n(x) &= \sum_{l=0}^{n}(a_{l-1} - na_l)x^l, \end{aligned}$$

and hence, since below the expression within the large parentheses is 0 if evaluated for $l = n$,

$$(x+1)\sigma_n(x+1)-(x-n)\sigma_n(x)=\sum_{l=0}^{n-1}\left(\sum_{j=l}^{n-1}a_j\binom{j+1}{l}+na_l\right)x^l.$$

By Lemma 1, we have that $\sum_{j=l}^{n-1}a_j\binom{j+1}{l}+na_l=b_l$ for $l=0,1,\dots,n-1$. It is easy to see that these equations can be encoded in the above matrix equation: for example, for $l=n-2$ we obtain $a_{n-2}\binom{n-1}{n-2}+a_{n-1}\binom{n}{n-2}+na_{n-2}=b_{n-2}$, which is equivalent to the last encoded equation. (The case $l=n-1$ need not be encoded since it expresses $2na_{n-1}=b_{n-1}$, which is a consequence of the known fact that the leading coefficients for σ_n and σ_{n-1} are, as mentioned earlier, $a_{n-1}=1/(2^n n!)$ and $\mathrm{lc}(\sigma_{n-1}(x))=\mathrm{lc}(x\cdot\sigma_{n-1}(x))=b_{n-1}=1/2^{n-1}(n-1)!$.) □

In this proposition, σ_n was fixed and a_{n-k} is the coefficient of x^{n-k} in σ_n and hence the coefficient of $x^{n+1-k}=x^{n-(k-1)}$ of $x\sigma_n(x)$. Similarly, b_{n-k} is the coefficient of x^{n-k} in $x\sigma_{n-1}(x)$. Since we have to consider the dependence on n as well in the sequel, we define

$$a_{n,k}:=(-1)^k\cdot\text{ coefficient of } x^{n-k} \text{ of } x\sigma_n(x).$$

The matrix equation of the previous proposition says that, for $k=2,3,\dots,n$,

$$(2n-k+1)a_{n-k}+\binom{n-k+2}{n-k}a_{n-k+1}+\binom{n-k+3}{n-k}a_{n-k+2}+\dots+\binom{n-1}{n-k}a_{n-2}=b_{n-k}-\binom{n}{n-k}a_{n-1}.$$

Doing the proper replacements according to $a_{n-k}\to(-1)^{k-1}a_{n,k-1}$ and $b_{n-k}\to(-1)^{k-1}a_{n-1,k-1}$, we obtain, after a rearrangement, the following equation that is valid for $k=2,\dots,n$:

$$(2n-k+1)a_{n,k-1}-a_{n-1,k-1}=(-1)^k\sum_{j=0}^{k-2}(-1)^j\binom{n-j}{n-k}a_{n,j}.$$

The last two lines in the following formula being clear and writing now k for $k-1$, this can also be written as follows:

$$a_{n,k}=\begin{cases}\dfrac{1}{2n-k}\left(a_{n-1,k}+(-1)^{k+1}\displaystyle\sum_{j=0}^{k-1}(-1)^j\binom{n-j}{n-k-1}a_{n,j}\right) & \text{for } n\ge k>0 \text{ or } n>k=0\\ 0 & \text{for } n<k \text{ or } k<0\\ 1 & \text{for } n=k=0.\end{cases}$$

The main line is valid at first for $k=1,\dots,n-1$ but as it happens it is also valid for $k=0$; in which case, it reproduces that $a_{n,0}=1/(2^n n!)$. By systematically checking the nine cases $n\varepsilon_1 k$ and $k\varepsilon_2 0$ for $\varepsilon_1,\varepsilon_2\in\{<,=,>\}$ one finds that any $(n,k)\in\mathbb{Z}^2$ satisfies exactly one of the three conditions given by the "for…"; and beginning with any n and k, the base of the recursion will satisfy the second or third condition. Thus, the recursion is well defined.

The recursion serves well if one would desire a rapid computation of the polynomials σ_n or $f_n(x)/(x+1)\cdots(x+n)$. The following Mathematica© code can be used to define $a_{n,k}$ (as a[n, k]). Then, e.g. $(-1)^3$a[5, 3] gives the coefficient of x^2 in $x\sigma_5(x)$.

```
a[n_,k_]:=(-k+2*n)^(-1)*((-1)^(k+1)*
Sum[(-1)^j*a[n,j]*Binomial[n-j,-k+n-1],{j,0,k-1}]+
  a[n-1,k])/;n>=k>0||n>k==0;
a[0,0]:=1;a[n_,k_]:=0/;n<k||k<0;
```

Alternatively, one may also use generating function approaches like the identity [6, (6.50)], which reads $((ze^z)/(e^z-1))^x=\sum_n x\sigma_n(x)z^n$ and the Series[...] command to obtain the polynomials σ_n.

Recall that one of our main goals is to show that the sequence

$$\mathbb{Z}_{\ge k}\ni n\mapsto f_{n,k}:=2^{n-k}(n-k)!a_{n,k}$$

is the sequence of values at the integers larger or equal k of a polynomial of degree k. Motivated by this, one may feel that it is simpler to work with a recursion for $f_{n,k}$ instead of $a_{n,k}$.

Multiplying the main line of the recursion above with $2^{n-k}(n-k)!$, the replacements $a_{n,j} \to f_{n,j}/(2^{n-j}(n-j)!)$ and simplification yield

$$\begin{aligned} f_{n,k} &= \frac{2^{n-k}(n-k)!}{2n-k} \cdot \frac{f_{n-1,k}}{2^{n-1-k}(n-1-k)!} + \frac{1}{2n-k} \sum_{j=0}^{k-1} (-1)^{1+k+j} \binom{n-j}{n-k-1} \cdot \frac{2^{n-k}(n-k)!}{2^{n-j}(n-j)!} f_{n,j} \\ &= \frac{2(n-k)}{2n-k} f_{n-1,k} + \frac{1}{2n-k} \sum_{j=0}^{k-1} (-1)^{1+k+j} \binom{n-j}{n-k-1} \frac{f_{n,j}}{2^{k-j}(n-k+1)\cdots(n-j)}. \end{aligned}$$

One more simplification now yields:

Corollary 6. *The numbers $f_{n,k}$ satisfy the following recursion*:

$$f_{n,k} = \begin{cases} \dfrac{(n-k)}{(2n-k)} \left(2f_{n-1,k} - \displaystyle\sum_{j=0}^{k-1} (-1/2)^{k-j} \frac{f_{n,j}}{(k+1-j)!} \right) & \textit{for } n \geq k > 0 \textit{ or } n > k = 0 \\ 0 & \textit{for } n < k \textit{ or } k < 0 \\ 1 & \textit{for } n = k = 0. \end{cases}$$

Our guiding principle for proving the theorem below was the following observation.

Observation. Assume p_{11} and p_{12} are two polynomials of degree 1 with the same leading coefficient and assume that q is a polynomial of degree k. Then, "in general" there will exist a particular solution $\mathbb{Z}_{\geq 0} \ni n \mapsto (f_n)$ for the difference equation $p_{11}(n) f_n = p_{12}(n) f_{n-1} + q(n)$, which is a polynomial of degree k.

The "proof" for this goes as follows: assume say, $p_{11}(n) = a + bn$ and $p_{12}(n) = c + bn$, then the equation can be written as $(a + bn)(f_n - f_{n-1}) + (a - c) f_{n-1} = q(n)$. Up to names of variables, this is a special case of the equation $q_n(x)\Delta^n u + q_{n-1}(x)\Delta^{n-1} u + \cdots + q_0(x) u = q(x)$ in [14, p. 377]. Here, $q_i(x)$ is supposed to be a polynomial of degree $\leq i$, $i = 0, 1, \ldots, n$; $q(x)$ is a polynomial of degree m; and Δ is the forward difference operator. In this case, the book says, we can "in general find a particular solution by assuming that $u(x) = \sum_{l=0}^{m} b_l \binom{x}{l}$ and equating coefficients." This is true, but one needs to prove that the linear forms in the b_i for the coefficients of the powers of x obtained on the left-hand side by substituting the mentioned *ansatz* are sufficiently generic to have a solvable linear system of equations. This is what we do next in our specific case.

Theorem 7. *Let k be a nonnegative integer. Then, the sequence*

$$\mathbb{Z}_{\geq k} \ni n \mapsto f_{n,k}$$

is polynomial of degree k.

Proof. The main line of the recursion for the $f_{n,k}$ can be rewritten as follows:

$$*_1: \quad (2n-k) f_{n,k} - (2n-2k) f_{n-1,k} = 2(n-k) \sum_{j=0}^{k-1} ((-2)^{1+k-j}(k+1-j)!)^{-1} f_{n,j}.$$

This is a necessary condition that the $f_{n,k}$, uniquely and well defined by the recursion, must satisfy.

We know that $a_{n,0} = (2^n n!)^{-1}$, and so by definitions, $f_{n,0} = 1$ for all n. (This can also be deduced from the recursion that reduces for the case $k = 0$ to $f_{n,0} = f_{n-1,0}$ and uses $f_{0,0} = 1$.) So, $f_{n,0}$ is a polynomial of degree 0. We now fix $k > 0$ and assume that, for $j = 0, 1, 2, \ldots, k-1$, the sequences $\mathbb{Z}_{\geq j} \ni n \mapsto f_{n,j}$ are polynomial of degree j. The right-hand side of the recursion shown is then a polynomial, and it must be of degree k since there exists only one polynomial sequence of degree $k-1$ in the sum, namely $\mathbb{Z}_{\geq k-1} \ni n \mapsto f_{n,k-1}$, and all other sequences $f_{n,j}$ occurring have lower degree. We will denote the polynomial (sequence) defining the right-hand side by $q(n) = \sum_{j=0}^{k} c_j n^j$ and have $c_k \neq 0$.

Now, we make the ansatz

$$f_{n,k} = a_0 + a_1 n + \cdots + a_k n^k,$$

and (again) with the understanding that $a_{-1} = a_{k+1} = 0$ and recalling $\binom{j}{-1} = 0$, $\binom{k}{k+1} = 0$, we find that

$$\begin{aligned}
(2n-k)f_{n,k} &= (2n-k)\sum_{i=0}^{k} a_i n^i = \sum_{i=0}^{k+1}(2a_{i-1} - ka_i)n^i;\\
(2n-2k)f_{n-1,k} &= (2n-2k)\sum_{j=0}^{k} a_j(n-1)^j\\
&= (2n-2k)\sum_{j=0}^{k} a_j\left(\sum_{i=0}^{j}\binom{j}{i}n^i(-1)^{j-i}\right)\\
&= (2n-2k)\sum_{i=0}^{k}\left(\sum_{j=i}^{k} a_j\binom{j}{i}(-1)^{j-i}\right)n^i\\
&= \sum_{i=0}^{k}\left(\sum_{j=i}^{k} 2a_j\binom{j}{i}(-1)^{j-i}\right)n^{i+1} - \sum_{i=0}^{k}\left(\sum_{j=i}^{k} 2ka_j\binom{j}{i}(-1)^{j-i}\right)n^i\\
&= \sum_{i=0}^{k+1}\left(\sum_{j=i-1}^{k} 2a_j\binom{j}{i-1}(-1)^{j-i+1} - \sum_{j=i}^{k+1} 2ka_j\binom{j}{i}(-1)^{j-i}\right)n^i\\
&= \sum_{i=0}^{k+1}\left(2a_{i-1} - \sum_{j=i}^{k+1} 2a_j(-1)^{j-i}\left(\binom{j}{i-1} + k\binom{j}{i}\right)\right)n^i.
\end{aligned}$$

Thus, the left-hand side of the recursion $*_1$ is

$$\begin{aligned}
(2n-k)f_{n,k} - (2n-2k)f_{n-1,k} &= \sum_{i=0}^{k+1}\left(\sum_{j=i}^{k+1} 2a_j(-1)^{j-i}\left(\binom{j}{i-1} + k\binom{j}{i}\right) - ka_i\right)n^i\\
&= \sum_{i=0}^{k}\left(a_i(2i+k) + 2\sum_{j=i+1}^{k}(-1)^{j-i}\left(\binom{j}{i-1} + k\binom{j}{i}\right)a_j\right)n^i,
\end{aligned}$$

and so we have to solve, for $a_0, a_1, \ldots, a_k$, the system

$$l_i(a_{0:k}) := a_i(2i+k) + 2\sum_{j=i+1}^{k}(-1)^{j-i}\left(\binom{j}{i-1} + k\binom{j}{i}\right)a_j = c_i, \quad i = 0, 1, \ldots, k.$$

Solvability is obvious because the linear form l_i actually depends only on $a_i, \ldots, a_k$, and the coefficient of a_i is $(2i + k) \neq 0$. So, in matrix form, the system would be upper triangular $k + 1 \times k + 1$ without zeros on the diagonal.

So far, we have shown that the equation $*_1$ at the beginning of the proof allows for a polynomial solution of degree k. As yet our reasoning did not take into account the hypothesis $n \geq k$ of any initial values. The general solution to the equation is obtained as the family of all sequences $(f_{n,k} + \dot{f}_n)_{n\in\mathbb{Z}}$, where $f_{n,k}$ is the polynomial sequence obtained above and $(\dot{f}_n)_{n\in\mathbb{Z}}$ is any solution to the homogeneous equation $(2n-k)\dot{f}_n - (2n-2k)\dot{f}_{n-1} = 0$. Now, at $n = k$, this equation degenerates to $k\dot{f}_k = 0$ so that $\dot{f}_k = 0$. But then we see by that $\dot{f}_n = 0$ by putting successively $n = k+1, k+2, \ldots$. Therefore, the *only* solution to the equation $*_1$ possible for $n \geq k$ is the polynomial solution found. By putting $n = k$ in that equation, we obtain $(2k-k)f_{k,k} = 0$, i.e. $f_{k,k} = 0$. It so happens that the recursion of Corollary 6 tells us $f_{k,k} = 0$ for the $k \geq 1$. Therefore, the sequence $\mathbb{Z}_{\geq k} \ni n \mapsto f_{n,k}$ coincides indeed with the polynomial sequence of degree k found for $*_1$. □

Recall that in Proposition 2, we introduced the polynomials $p_n(x) = (-1)^n(-x)\sigma_n(-x)$, and after Proposition 5, we introduced $a_{n,k} = (-1)^k\cdot$ (coefficient of x^{n-k} in $x\sigma_n(x)$).

Lemma 8. *Let m, n, μ, and ν be integers for which $0 < n \le m \le 2n-1$ and $0 \le \mu \le \nu \le m-n$. Then,*

$$\sum_{i=0}^{n}(-1)^i\frac{p_{m-n-\nu}(i)p_\mu(n-i)}{i!(n-i)!}2^{-i}a_{n-i+\nu,\nu-\mu}(n-i+\mu)! = 0.$$

Proof. The sequence $\{0,1,2,\dots,n\}\ni i \mapsto 2^{-i}a_{n-i+\nu,\nu-\mu}(n-i+\mu)!$ is well defined and nontrivial in the sense that for the i used, the subindices of a occurring are nonnegative and that the first one is larger than or equal to the second one and the factorial occurring is also nonnegative. By Theorem 7, the sequence $\mathbb{Z}_{\ge\nu-\mu}\ni n \mapsto f_{n,\nu-\mu} = 2^{n-\nu+\mu}(n-\nu+\mu)!a_{n,\nu-\mu}$ is a polynomial of degree $\nu-\mu$. If we replace in a polynomial $p = p(n)$ with coefficients in $\mathbb{R}$ the n by $n+\nu-i$ we obtain a polynomial expression $p(n+\nu-i)$ which we may view as a polynomial in i. Its leading coefficient as a polynomial in i will be a real number equal to $\pm$ its leading coefficient as a polynomial in n. In particular, its degree in i will be equal to its degree in n. Thus, in particular, the sequence $\{0,1,\dots,n+\nu\}\ni i\mapsto f_{n-i+\nu,\nu-\mu}$ and, for that matter, the sequence on $\{0,\dots,n\}$ above defined at the beginning of the proof will be polynomial of degree $\nu-\mu$. The sum of the lemma is of the form $\sum_{i=0}^{n}\frac{(-1)^i}{i!(n-i)!}q(i)$, where q is a polynomials of degree $\le(m-n-\nu)+\mu+(\nu-\mu) = m-n < n$, and therefore, 0 as follows from the remark to Proposition 2. □

Lemma 9. *If $m\in\{n,n+1,\dots,2n-1\}$ and $\nu\in\{0,1,\dots,m-n\}$, then*

$$\sum_{i=0}^{n}2^{-i}\frac{p_{m-n-\nu}(i)}{i!}\sum_{k=0}^{n-i}(-1)^k\frac{p_{n-i+\nu}(k)}{k!(n-i-k)!} = 0.$$

Proof. It is sufficient to establish the claim by substituting the polynomial $p_{n-i+\nu}(k)$ of degree $\le n-i+\nu$ in the sum by any term of this polynomial. By the definitions of $a_{n,l}$ and p_n, we have $p_n(x) = \sum_{l=0}^{n}a_{n,l}x^{n-l}$. So, we find that a typical term in $p_{n-i+\nu}(k)$ is given by $a_{n-i+\nu,l}k^{n-i+\nu-l}$, $l = 0,1,\dots,n-i+\nu$, or, introducing $\mu = \nu-l$, it is given by $a_{n-i+\nu,\nu-\mu}k^{n-i+\mu}$ with $\mu\le\nu$. In addition, note $\nu\le m-n\le n-1$. Thus, we have the computation

$$\begin{aligned}
&\sum_{i=0}^{n}2^{-i}\frac{p_{m-n-\nu}(i)}{i!}\sum_{k=0}^{n-i}(-1)^k\frac{a_{n-i+\nu,\nu-\mu}k^{n-i+\mu}}{k!(n-i-k)!}\\
&=\sum_{i=0}^{n}\frac{p_{m-n-\nu}(i)}{i!}2^{-i}a_{n-i+\nu,\nu-\mu}\sum_{k=0}^{n-i}(-1)^k\frac{k^{n-i+\mu}}{k!(n-i-k)!}\\
&=\sum_{i=0}^{n}\frac{p_{m-n-\nu}(i)}{i!}2^{-i}a_{n-i+\nu,\nu-\mu}(-1)^{n-i}\left\{\begin{matrix}n-i+\mu\\ n-i\end{matrix}\right\}\\
&=\sum_{i=0}^{n}\frac{p_{m-n-\nu}(i)}{i!}2^{-i}a_{n-i+\nu,\nu-\mu}(-1)^{n-i}\frac{(n-i+\mu)!}{(n-i)!}p_\mu(n-i)\\
&=(-1)^n\sum_{i=0}^{n}(-1)^i\frac{p_{m-n-\nu}(i)p_\mu(n-i)}{i!(n-i)!}2^{-i}a_{n-i+\nu,\nu-\mu}(n-i+\mu)! = 0,
\end{aligned}$$

where in the second and third equality we used Proposition 2(a) (with $m = n-i$ and $a(k) = k^{n-i+\mu}$) and (b) (replacing there n by μ, k by $n-i$), and in the last equality we used Lemma 8. So we have what we wanted. □

A trivial, but at the end important, corollary is as follows:

Theorem 10. *If $m\in\{n,n+1,\dots,2n-1\}$, then*

$$\sum_{i=0}^{n}2^{-i}\sum_{\nu=0}^{m-n}\frac{p_{m-n-\nu}(i)}{i!}\sum_{k=0}^{n-i}(-1)^k\frac{p_{n-i+\nu}(k)}{k!(n-i-k)!} = 0.$$

Proof. Take the sum over all $\nu \in \{0, 1, \dots, m-n\}$ of the expression above and interchange the two outer sums obtained. □

4 Origin, proof, and impact of the zero identities

Our research originated in a new approach to the odd-dimensional uniform random flight problem: assume a particle at instant 0 at the origin of an odd dimensional Euclidean space jumps exactly one unit from its current position in a random direction at each tick of the clock. (Here, the directions are defined as position vectors to uniformly distributed points on the origin-centred unit sphere.) Question: What is – as a function of r – the probability to encounter the particle after exactly n random jumps within the 0-centred ball $B = B(0, r)$ of radius r? This question was solved by García-Pelayo [7] using advanced analytical tools like Fourier analysis and the Abel transform. The article [16] shows the main result that the mentioned probabilities are piecewise polynomial by more elementary means. Based on [7], Borwein and Sinnamon [8] gave explicit formulas but these are quite complicated.

In particular, in an attempt to find an alternative formula for the case of dimension 5, the second author was led to conjecture that, putting

$$c_{t,l} = (-1)^l \sum_{\mu} (-1)^{\mu} \binom{n}{\mu, l-\mu, t-\mu, \mu+n-l-t}$$

with $t, l \in \mathbb{Z}$, the finite sum

$$\sum\nolimits_{t,l} \frac{(-1)^t c_{t,l}}{(3n-t-1)!}(x+n-2l)^{3n-t-1}$$

(certainly a polynomial in x of degree $\le 3n-1$) should be actually 0.

Here, the notation $\binom{n}{a,b,c,d}$ stands for the multinomial coefficient $\frac{n!}{a!b!c!d!}$ if $a+b+c+d=n$, and 0 otherwise.

Let us cast the formulation of this proposition into a more manageable form. First, note that the sum is actually finite. Assume the expression for $c_{t,l}$ incorporated in the second sum. Then, we can speak of an outer and an inner sum. Recall that $k \in \mathbb{Z}_{<0}$, which implies $1/k! = 0$. Assume, we choose in the outer sum $t > n$. Then $n - t < 0$ and so at least one of $l-\mu, \mu+n-l-t$ is negative. Hence, by the definition of multinomial coefficients, each of the multinomial coefficients associated with the inner sum is 0. Thus, since the roles of l and t are symmetrical we can limit the outer sum and assume it is written as $\sum_{0\le t,l\le n}\dots$. It follows that the inner sum can also be limited as $\sum_{\mu=(t+l-n)^+}^{\min\{l,t\}}\dots$. So, the sum is finite, and under these conditions the four lower indices of the multinomial coefficients are nonnegative and define a composition of n, i.e., their sum is n. Now, assume that four nonnegative integers a, b, c, and d define a composition of n. Then, let $\mu = a$, $l = a+b$, $t = a+c$; we also have $n = a+b+c+d$. Then clearly $t, l \in \{0, 1, \dots, n\}$, $0 \le \mu \le \min\{t, l\}$, and $t + l - n = 2a + b + c - n = a - d \le a = \mu$. So, using the notation $s^+ = \max\{s, 0\}$, we have $(t+l-n)^+ \le \mu \le \min\{t, l\}$. This entails that within the double sum $\sum_{0\le t,l\le n}\sum_{\mu=(t+l-n)^+}^{\min\{l,t\}}\dots$, the quadruple $(\mu, l-\mu, t-\mu, \mu+n-l-t)$ ranges precisely over the compositions (a, b, c, d) of n. What concerns the power $(-1)^{t+l+\mu} = (-1)^{3a+b+c} = (-1)^{n-d}$ occurring in the double sum, in the context of what we wish to prove, it can evidently be replaced by $(-1)^d$. Finally, we may also replace $x + n$ by x in the proposition above, and after dividing by $n!$ we see that the conjecture can be rewritten as claiming the following theorem.

Theorem 11. *Let n be an integer larger than* 1. *Then, there holds the identity*

$$\sum_{a+b+c+d=n} (-1)^d \frac{(x-2a-2b)^{3n-a-c-1}}{a!b!c!d!(3n-a-c-1)!} = 0.$$

The rest of this section is dedicated to a proof of this theorem.

Clearly, the expression above is a polynomial in x of degree at most $3n-1$. It is 0 as claimed if and only if all its coefficients are 0. The claim in the abstract is obtained simply by taking the $(s-1)$-st derivative of the identity shown. So, we focus on the identity shown here, which corresponds to the case $s=1$.

Now, for any positive integer λ, we have that

$$\text{coefficient of } x^t \text{ of } (x-2a-2b)^\lambda = \begin{cases} 0 & \text{if } \lambda < t \\ 1 & \text{if } \lambda = t \\ \dfrac{\lambda!(-2)^{\lambda-t}}{t!(\lambda-t)!}(a+b)^{\lambda-t} & \text{if } \lambda > t. \end{cases}$$

We use this for $\lambda = \lambda(a,c) = 3n-a-c-1$, which in the dynamic environment of the above sum is always $\geq 2n-1 > 0$. It can happen, though, that $a+b=0$, and at the same time, $\lambda(a,c)-t \leq 0$. In this case, we obtain, within the sum, undefined terms of form $0 \cdot 1/0$ or 0^0. The latter has to be interpreted as 1 for reasons mentioned after Lemma 3. To steer clear from any interpretation problems, we argue using the given case distinction. Having chosen any $t \in \{0,1,2,\ldots,3n-1\}$, we wish to show that the following expression is 0:

$$\begin{aligned}
&\sum_{a+b+c+d=n} (-1)^d \frac{(-1)^d}{a!b!c!d!\lambda(a,c)!} \cdot \text{coefficient of } x^t \text{ of } (x-2a-2b)^{\lambda(a,c)} \\
&= \sum_{\substack{a+b+c+d=n \\ \lambda(a,c)<t}} \ldots + \sum_{\substack{a+b+c+d=n \\ \lambda(a,c)=t}} \ldots + \sum_{\substack{a+b+c+d=n \\ \lambda(a,c)>t}} \ldots \\
&= 0 + \sum_{\substack{a+b+c+d=n \\ \lambda(a,c)=t}} \frac{(-1)^d}{a!b!c!d!\lambda(a,c)!} \cdot 1 + \sum_{\substack{a+b+c+d=n \\ \lambda(a,c)>t}} \frac{(-1)^d}{a!b!c!d!\lambda(a,c)!} \frac{\lambda(a,c)!(-2)^{\lambda(a,c)-t}}{t!(\lambda(a,c)-t)!}(a+b)^{\lambda(a,c)-t} \\
&= \frac{1}{t!}\left(\underbrace{\sum_{\substack{a+b+c+d=n \\ \lambda(a,c)=t}} \frac{(-1)^d}{a!b!c!d!}}_{S_1(t)} + \underbrace{\sum_{\substack{a+b+c+d=n \\ \lambda(a,c)>t}} \frac{(-1)^d(-2)^{\lambda(a,c)-t}}{a!b!c!d!(\lambda(a,c)-t)!}(a+b)^{\lambda(a,c)-t}}_{S_2(t)} \right).
\end{aligned}$$

Since $\lambda(a,c)=t$ iff $a+c=3n-1-t$, the restrictions in the first sum $S_1(t)$ impose $b+d=1-2n+t$ and so $2n-1 \leq t \leq 3n-1$. Otherwise, by nonnegativity of a, b, c, and d, the sum is empty and hence 0. Assuming now $2n-1 \leq t \leq 3n-1$ satisfied, via the binomial theorem applied to powers of (1 ± 1), the first sum is

$$S_1(t) = \sum_{a+c=3n-t-1} \frac{1}{a!c!} \sum_{b+d=t-(2n-1)} \frac{(-1)^d}{b!d!} = \sum_{a+c=3n-t-1} \frac{1}{a!c!}\delta_{t,2n-1} = \frac{2^n}{n!}\delta_{t,2n-1},$$

and as we have seen, this formula holds true for all $t = 0,1,2,\ldots,3n-1$.

For simplified writing of the further arguments, we introduce $t_{\text{new}} = 3n-1-t$. We have $\lambda(a,c)=t \Leftrightarrow t_{\text{new}} = a+c$ and $\lambda(a,c) > t \Leftrightarrow t_{\text{new}} > a+c$. Accordingly, we find,

$$S_1(t_{\text{new}}) = \sum_{\substack{a+b+c+d=n \\ a+c=t_{\text{new}}}} \frac{(-1)^d}{a!b!c!d!} \quad \text{and} \quad S_2(t_{\text{new}}) = \sum_{\substack{a+b+c+d=n \\ a+c<t_{\text{new}}}} \frac{(-1)^d(-2)^{t_{\text{new}}-a-c}}{a!b!c!d!(t_{\text{new}}-a-c)!}(a+b)^{t_{\text{new}}-a-c}.$$

Hence, we write t for t_{new} and have to prove $S_1(t)+S_2(t)=0$, i.e., $\frac{2^n}{n!}\delta_{t,n} + S_2(t) = 0$ for all $t = 0,1,\ldots,3n-1$.

Now,

$$\begin{aligned}
S_2(t) &= \sum_{\substack{a+b+c+d=n\\ a+c<t}} \frac{(-1)^d(-2)^{t-a-c}}{a!b!c!d!(t-a-c)!}(a+b)^{t-a-c}\\
&= \sum_{\kappa=0}^{\min\{t-1,n\}} \sum_{\substack{a+c=\kappa\\ b+d=n-\kappa}} \frac{(-1)^d(-2)^{t-\kappa}}{a!b!c!d!(t-\kappa)!}(a+b)^{t-\kappa}\\
&= \sum_{\kappa=0}^{\min\{t-1,n\}} \frac{(-2)^{t-\kappa}}{(t-\kappa)!} \sum_{\substack{a+c=\kappa\\ b+d=n-\kappa}} \frac{(-1)^d}{a!b!c!d!}(a+b)^{t-\kappa}.\\
&\text{Using Lemma 3 with } l=t-\kappa, \text{ we obtain}\\
&= \sum_{\kappa=0}^{\min\{t-1,n\}} \frac{(-2)^{t-\kappa}}{(t-\kappa)!} \sum_{a+c=\kappa} \frac{1}{a!c!} \sum_{\nu=0}^{t-\kappa} \binom{t-\kappa}{\nu} a^{t-\kappa-\nu} \left\{ {\nu \atop n-\kappa} \right\}.\\
&\text{If } \nu < n-k, \text{ then } \left\{ {\nu \atop n-\kappa} \right\} = 0. \text{ We may also cancel } (t-\kappa)!. \text{ Thus,}\\
&= \sum_{\kappa=0}^{\min\{t-1,n\}} (-2)^{t-\kappa} \sum_{a+c=\kappa} \frac{1}{a!c!} \sum_{\nu=n-\kappa}^{t-\kappa} \frac{a^{t-\kappa-\nu}}{\nu!(t-\kappa-\nu)!} \left\{ {\nu \atop n-\kappa} \right\}.
\end{aligned}$$

If $t<n$, then the inner sum is empty, so the whole sum is 0, hence $S_1(t)+S_2(t)=0$. Furthermore, we have

$$S_2(n) = \sum_{\kappa=0}^{n-1} (-2)^{n-\kappa} \sum_{a+c=\kappa} \frac{1}{a!c!} \frac{a^0}{(n-\kappa)!} = \sum_{\kappa=0}^{n-1} (-2)^{n-\kappa} \frac{2^\kappa}{\kappa!(n-\kappa)!} = 2^n \sum_{\kappa=0}^{n-1} \frac{(-1)^{n-\kappa}}{\kappa!(n-\kappa)!} = 2^n(0-\frac{1}{n!}) = -\frac{2^n}{n!}.$$

So, it follows that $S_1(n)+S_2(n)=0$, and we have demonstrated the conjecture for $t=0,1,2,\ldots,n$ and now we must demonstrate that for $t=n+1,\ldots,3n-1$, $S_2(t)=0$. Now for these t we can write the expression we obtained for $S_2(t)$ as

$$S_2(t) = \sum_{\kappa=0}^{n} (-2)^{t-\kappa} \sum_{\nu=n-\kappa}^{t-\kappa} \frac{\left\{ {\nu \atop n-\kappa} \right\}}{\nu!(t-\kappa-\nu)!} \sum_{a+c=\kappa} \frac{a^{t-\kappa-\nu}}{a!c!}.$$

The next steps show that this sum is 0 by virtue of Theorem 10. By intending to avoid a long chain of equalities with complicated expressions, we leave the guided check of some details concerning the various equivalent descriptions of sets K and K' below to the reader. There letters n and t have the fixed meaning given to them above, but the values of k, ν and i may vary (similarly as, say, the sets $\{k : 0 \le k \le 3\}$ and $\{k+2 : -2 \le k \le 1\}$ are the same, but the values of k in one set are not the same as in the other set).

First, we define the function f and the set K by

$$\begin{aligned}
f(\kappa,\nu,i) &= 2^t(-1)^{t-\kappa} \frac{\left\{ {\nu \atop n-\kappa} \right\}}{\nu!(t-\kappa-\nu)!} \frac{\left\{ {t-\kappa-\nu \atop i} \right\}}{2^i(\kappa-i)!},\\
K &= \{(\kappa,\nu,i) : 0 \le \kappa \le n, \quad n-\kappa \le \nu \le t-\kappa, \quad 0 \le i \le t-\kappa-\nu\}.
\end{aligned}$$

Next, we replace in Lemma, 4 n by κ, h by c, l by a, and k by $t-\kappa-\nu$ and note $(-2)^{t-\kappa}2^{\kappa-i} = (-1)^{t-\kappa}2^{t-i}$. Then, we find that $S_2(t) = \sum_{(\kappa,\nu,i)\in K} f(\kappa,\nu,i)$. Putting $m=t-n$, we can represent K as follows:

$$\begin{aligned}
K &= \{(n-k,\nu+k,i) : 0 \le n-k \le n, \quad k \le \nu+k \le m+k, \quad 0 \le i \le m-\nu\}\\
&\quad \text{saying } n-k \text{ ranges through } 0,1,\ldots,n \text{ is saying the same for } k; \text{ thus}\\
&= \{(n-k,\nu+k,i) : 0 \le k \le n, \quad 0 \le \nu \le m, \quad 0 \le i \le m-\nu\}\\
&\supseteq \{(n-k,\nu+k,i) : 0 \le \nu \le m, \quad 0 \le i \le m-\nu, \quad 0 \le k \le n-i\}.\\
&\quad \text{The second and third inequalities here occurring imply, e.g. } 0 \le i \le n, \text{ in the next line.}\\
&\quad \text{Similarly, show the other inequalities below; and conversely, to obtain}\\
&= \{(n-k,\nu+k,i) : 0 \le i \le n, \quad 0 \le \nu \le m-i, \quad 0 \le k \le n-i\}\\
&\quad \text{For next line, replace } \nu \text{ by } \nu-i+n\\
&= \{(n-k,\nu+k+n-i,i) : 0 \le i \le n, \quad i-n \le \nu \le m-n, \quad 0 \le k \le n-i\} =: K'
\end{aligned}$$

Note that the expression $1/(\kappa - i)!$ occurring as a multiplicative factor in $f(\kappa, v, i)$ transforms in $f(n-k, v+k+n-i, i)$ into $1/(n-k-i)!$. If here $k > n-i$, then this factor is 0. In other words, summing over all values $f(\kappa, v, i)$, where v, κ, and i are defined by the inequalities in the original representation of K is the same thing as summing over the values $f(n-k, v+k+n-i, i)$, where v, k, and i are defined by the inequalities in K'. Thus, using in the second step below that, by Proposition 2(b), $\left\{ {a \atop b} \right\}/a! = p_{a-b}(b)/b!$, we have

$$\begin{aligned} S_2(t) &= \sum\{f(n-k, v+k+n-i, i) : 0 \le i \le n, \quad i-n \le v \le m-n, \quad 0 \le k \le n-i\} \\ &= 2^t(-1)^m \sum_{i=0}^{n} 2^{-i} \sum_{v=i-n}^{m-n} \frac{p_{m-v-n}(i)}{i!} \sum_{k=0}^{n-i} (-1)^k \frac{p_{n-i+v}(k)}{k!(n-i-k)!}. \end{aligned}$$

Now, if $v < 0$, then the degree of p_{n-i+v} is smaller than $n-i$, which means that the rightmost sum is 0 by Proposition 2a and the remarks following it. Hence, $\sum_{v=i-n}^{m-n}$ can be replaced by $\sum_{v=0}^{m-n}$, and Theorem 10 implies that $S_2(t) = 0$. Theorem 11 is proved.

With this result established, we hope to publish a note soon with the more natural formula for a uniform random flight in dimension 5 than the one given in [8]. We also should mention that we have another conjecture similar (but even more complicated) than Theorem 1, which would yield probably a simplification of the formula in [8] for random flights in dimension 7 but for this conjecture to be established maybe it is worthwhile to first work towards a more transparent proof of Theorem 1.

5 Concluding remarks and open problems

After having finished the bulk of this article, we decided to comb once more through the combinatorial literature for results possibly related to ours. From this search arise some remarks and questions.

(a) There is a probable connection with Riordan arrays. Proper Riordan arrays can be characterised as those infinite lower triangular tables of complex numbers $(d_{n,k})_{n,k\ge 0}$ (with $k > n$ implying $d_{n,k} = 0$) for which there exists a sequence $A = (a_0 \ne 0, a_1, a_2, \ldots)$ such that for all $n, k \in \mathbb{Z}_{\ge 0}$ there holds $d_{n+1,k+1} = \sum_{j=0}^{\infty} a_j d_{n,k+j}$; see, e.g. He and Sprugnoli [9, Theorem 2.1] for a proof of this discovery attributed to D. G. Rogers in 1978. Numerical experiments very soon, and somewhat surprisingly, revealed that Table 2 seems to be a Riordan array with the sequence $A = (a_0 = 1, -\frac{1}{12}, -\frac{1}{144}, -\frac{1}{2160}, -\frac{1}{103680}, \frac{11}{2177280}, \frac{43}{29030400}, \frac{1}{3483648}, \ldots)$. One obtains these numbers by using the following scheme, which is a consequence of above equations and triangularity of the array:

$$\begin{array}{rcl} d_{n+1,n+1} &=& a_0 d_{n,n} \\ d_{n+1,n} &=& a_0 d_{n,n-1} + a_1 d_{n,n} \\ d_{n+1,n-1} &=& a_0 d_{n,n-2} + a_1 d_{n,n-1} + a_2 d_{n,n} \\ d_{n+1,n-2} &=& a_0 d_{n,n-3} + a_1 d_{n,n-2} + a_2 d_{n,n-1} + a_3 d_{n,n} \\ \vdots && \vdots \; \vdots \end{array}$$

Since Table 2 shows $d_{n,n} = 1$ for all n, we have $a_0 = 1$. Choosing $n = 5$, we find $-\frac{5}{12} = d_{6,5} = 1d_{5,4} + a_1 d_{5,5} = = -\frac{1}{3} + a_1 \cdot 1$ and thus $a_1 = -5/12 + 1/3 = -1/12$. Next, $a_2 = d_{6,4} - a_0 d_{5,3} - a_1 d_{5,4} = 1/24 - 1/48 - (-1/12)(-1/3) = -1/144$, etc.

(The explicit Riordan arrays published in the literature we have seen seem to be almost all integer tables and mostly serve purposes in enumerative combinatorics.) We then tried to prove that our array is indeed Riordan using the above characterisation. Although this might not be too hard, it seems not quite trivial. Perhaps one of the other characterisations given in [13] can help. Another way to approach the topic of Riordanicity of our array might be to have a characterisation of the Riordan arrays for which for each k the sequence $(d_{n+k,n})_{n\ge 0}$ is polynomial of degree k. Interestingly, as is not hard to see that if the A-sequence of a Riordan array with constant diagonal ≠0 satisfies $a_1 \neq 0$, then its kth diagonal is necessarily a polynomial of degree k. But the converse is false.

(b) Note that an array $\left(\left\{ {n \atop k} \right\}\right)_{n,k\geq 0}$ of Stirling numbers of the second kind is an infinite triangular array with 1s on the diagonal. By the cited fact from [5], its kth diagonal is a sequence that is polynomial of degree $2k$. Is there any close connection between this fact and the fact that the diagonals of our modified coefficient table are polynomials of degree k? An article by Carlitz [3] made us think so for a short while, but in the authors hands, it did not pan out. The question intends to spur an effort to find an "immediate" proof of Theorem 3.3 using some facts from the immense body of already known results on Stirling numbers or polynomials.

(c) The online encyclopaedia for integer sequences [15] tells us that the sequence of integers 1, 12, 288, 51,840, which the reader can see in the denominators of polynomials q_0, q_1, q_2, and q_3 in Section 2 actually also occurs as the sequence of denominators of an asymptotic series of the Gamma function. Indeed, see, e.g. [14, Exercise 17, p. 269]. So the full story concerning the q_i has yet to be discovered.

(d) The polynomials $x\sigma_n(x)$ occur in the series development of $((ze^z)/(e^z - 1))^x$ mentioned after the MATHEMATICA code in Section 3. In the study by Koparal et al. [12, Section 1 and Theorem 3], we learn that the numbers $\rho(n, k)$ definable by $\sum_{n\geq 0} \frac{\rho(n,k)}{n!} x^n = \left(\frac{x}{1-e^{-x}}\right)^k$ are, for $n \geq k$, equal to $n!k\sigma_n(k)$ and that these same numbers and the (signed) Stirling numbers of the first kind $s(n, k) = (-1)^{n-k} \left[{n \atop k} \right]$ are related to the Daehee numbers of order m, via the equality $\sum_{k=0}^{n} (-1)^k s(n, k)\rho(k, m) = D_n^m$. A database search easily convinces that Daehee numbers figure prominently in recent articles.

May these questions and remarks spur further research!

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