Locally Sierpinski Quotients of the Sorgenfrey Line

F. J. Craveiro de Carvalho

We will start with a quotation from [1].

> The Sorgenfrey line is an elementary example of a topological space almost always introduced and studied in the first basic topology course. Many of the interesting properties of this space can be discussed with a minimum exposure to topology so it is interesting that there are still open questions concerning the Sorgenfrey line.

In this note we will investigate which locally Sierpinski spaces can arise as quotients R_S/G, where R_S is the Sorgenfrey line and G is a group of homeomorphisms. In fact we will show that any locally Sierpinski space with countably many components, all of them finite, can be obtained as such a quotient. Naturally this result is in sharp contrast to the one obtained for R in [3]. We recall that if R/G is locally Sierpinski then it has, at most, three elements.

1. Locally Sierpinski Spaces

A topological space X is a is said to be a *locally Sierpinski space*, *l.S. space* in short, if every point $x \in X$ has an open neighbourhood U_x homeomorphic to the Sierpinski space $(\{0,1\}, \{\emptyset, \{0\}, \{0,1\}\})$.

If in an l. S. space X a unitary set $\{x\}$ is open we say that x is a *centre*. The centres and the open sets homeomorphic to the Sierpinski space form a basis for the topology of X. Therefore X is locally (path-)connected and its (path-) connected components are open.

Moreover there is a bijection between the set of centres and the set of components of X. Such a bijection associates to each centre the component which contains it.

Examples of connected l. S. spaces are often referred to in topology textbooks [5]. They are obtained as follows. Let X be a set with, at least, two points. Fix $p \in X$ and define a set to be open if it is either the empty set or contains p.

L. S. spaces can be characterized as locally (path-)connected spaces whose (path-)components are subspaces of the same type as X. However it is perhaps worth pointing out that there are topological spaces whose (path-)components are of the same type as X but are not locally (path-)connected. Take X as above and consider its product with any non-discrete, totally disconnected space Y. For Y we may choose, for instance, the rationals or the Sorgenfrey line. The (path-)connected components of $X \times Y$ are the sets $X \times \{y\}, y \in Y$ and one of them, at least, is not open. Therefore $X \times Y$ is not l. S.

2. The Sorgenfrey Line

We will use R to denote the set of real numbers with or without the usual topological structure. If R is the underlying set but the topology has the set of intervals $[a, b), a < b, a, b \in R$, as a basis then we have the *Sorgenfrey line* which will be denoted by R_S.

The following observation will be an essential ingredient in the proofs of what is to follow:

Let $f : I \subset R \to J \subset R$ be an increasing R-homeomorphism between intervals. Then $f : I \subset R_S \to J \subset R_S$ is also an R_S-homeomorphism.

Let $H(R_S)$ denote the group of homeomorphisms of R_S. Recall that the action of a group G on a set X is *highly transitive* if it is k-fold transitive , for every $k \geq 1$.

Proposition 1: $H(R_S)$ is highly transitive on R_S.

Proof: We are indebted to [4] whose reading enabled us to present a much neater proof than our previous one.

It is enough to show that, for $k \geq 1$, given distinct points $x_1, \ldots, x_{k-1} \in R_S$ and x, y distinct from $x_1, \ldots, x_{k-1}$ there is $h \in H(R_S)$ such that $h(x_i) = x_i, i = 1, \ldots, k-1$, and $h(x) = y$.

Choose an $\epsilon > 0$ such that $[x, x+\epsilon), [y, y+\epsilon)$ are disjoint and contain none of the points $x_1, \ldots, x_{k-1}$. Now define $h : R_S \to R_S$ using a homeomorphism $f : [x, x+\epsilon) \subset R_S \to [y, y+\epsilon) \subset R_S$ and its inverse and letting $f(z)$ to be z for $z \notin [x, x+\epsilon) \cup [y, y+\epsilon)$.

For f, in view of the observation we made above, we can use any increasing homeomorphism $f : [x, x+\epsilon) \subset R \to [y, y+\epsilon) \subset R$. $\bowtie$

We can deduce immediately

Corollary 1: For $k \geq 2$, there is a subgroup G of $H(R_S)$ such that R_S/G is a connected l. S. space with k points.

Proof: Let $x_1, \ldots, x_{k-1}$ be distinct points in R_S and denote by G the subgroup of $H(R_S)$ formed by the homeomorphisms which keep those points fixed. Then R_S/G is a connected l. S. space with k points. $\bowtie$

Proposition 2: Let $r_i, i = 1, \ldots, k, k \geq 2$ (respectively, $r_i, i \in N$), be integers greater than or equal to 2. Then there is a subgroup G of $H(R_S)$ such that R_S/G has exactly k components (repectively, the set of components of R_S/G is denumerable). The $r_i\ 's$ are the cardinals of the components.

Proof: Write $R_S = (-\infty, 0) \cup [0,1) \cup [1,2) \cup \ldots \cup [k-2, +\infty)$ and fix $r_1 - 1$ points in $(-\infty, 0)$, $r_2 - 2$ points in $(0,1)$, ..., $r_k - 2$ in $(k-2, +\infty)$. If the indexing set is N then the decomposition of R_S will be $R_S = (-\infty, 0) \cup [0,1) \cup \ldots \cup [n, n+1) \cup \ldots$ and the arguments below are slightly modified.

Consider the subgroup G of $H(R_S)$ formed by the homeomorphisms f such that $f((-\infty,0)) = (-\infty,0), f([0,1) = [0,1), \ldots, f([k-2,+\infty)) = [k-2,+\infty), f(0) = 0, f(1) = 1, \ldots, f(k-2) = k-2$ and keep the points mentioned above fixed. Then R_S/G is an l. S. space with the required properties.

The equivalence classes are described as follows. For $(-\infty, 0)$ they are the unitary sets formed by the points we fixed in $(-\infty, 0)$ and the remaining subset. For an interval of the type $[r, r+1)$ they are $\{r\}$, the unitary subsets formed by the points fixed in the interval and the remaining subset. Any homeomorphism $h : R_S \to R_S$ that may be needed to prove that may be constructed applying to a particular interval of the decomposition of R_S a procedure analogous to the one we used in the proof of Proposition 1 and letting h to be the identity in the complement of the interval. $\bowtie$

References

[1] D. K. Burke and J. Tatch Moore, *Subspaces of the Sorgenfrey line*, Topology Appl. 90 (19989, 57-68.

[2] F. J. Craveiro de Carvalho and A. M. d'Azevedo Breda, *Alguns passos em volta do espaço de Sierpinski*, Boletim da SPM 28 (1994), 59-63.

[3] F. J. Craveiro de Carvalho and B. Wegner, *Locally Sierpinski Spaces as Interval Quotients*, to appear.

[4] Peter M. Neumann, *Automorphisms of the Rational World*, J. London Math. Soc. 32 (1985), 439-448.

[5] S. Willard, General Topology, Addison-Wesley (1970).

Departamento de Matemática,
Universidade de Coimbra,
3000 Coimbra, PORTUGAL.
e-mail : fjcc@mat.uc.pt