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On Lax Idempotent Monads in Topology

Tese no âmbito do Programa Interuniversitário de Doutoramento em Matemática, orientada pela Professora Doutora Maria Manuel Clementino e apresentada ao Departamento de Matemática da Faculdade de Ciências e Tecnologia da Universidade de Coimbra

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Abstract

The main source of inspiration for this work is Lawvere's seminal paper [28]: *Metric spaces, generalized logic, and closed categories*. We wish to highlight two surprising examples of the deep relation between category theory and classic metric theory, from Lawvere's paper:

1. a (quasi)metric space X is a simply a category whose objects are the points of X and the distance $d(x,y) \in [0,\infty]$ is the hom-set $X(x,y)$.

2. Cauchy sequences correspond to adjoint distributors and their convergence corresponds to representability of such distributors.

Not only are these examples striking, they are also useful in guiding mathematical research. As Lawvere himself pointed out: *"this connection is more fruitful than a mere analogy, because it provides a sequence of mathematical theorems so that enriched category theory can suggest new directions of research in metric space theory and conversely"*.

The first observation is the fundamental motivation behind the study of quantale-enriched categories as a generalisation of metric spaces, while the second insight can be generalised to give us a categorical notion of Cauchy completeness in terms of weighted colimits in enriched category theory.

We recall a few fundamental notions about V-**Cat**, V-**Rel** and V-**Dist**, where V is a quantale, which will be essential for the remainder of the work.

Then we present a new proof that $V\text{-}\mathbf{Cat}^{\mathrm{op}}$ is a quasivariety, making use of certain monadicity conditions, comma categories and the category V-**ccd** which generalises the category **ccd** of constructively completely distributive lattice.

We will follow Lawvere's suggestion and generalise the formal ball monad, which is a central tool in the study of metric theory, to a quantale-enriched setting. Our work proved fruitful, as we obtained a characterisation of the (Eilenberg-Moore) algebras for the formal ball monad as the V-categories with a particular class of weighted colimits. By restricting our focus to separated V-categories we also obtained an interesting characterisation of a certain class of embeddings.

Later we present two characterisations of the submonads of the presheaf monad: one in terms of a special class of V-distributors; and another as those monads which are fully BC*, lax idempotent and satisfy certain full faithfulness conditions. By BC*, we mean a new Beck-Chevalley type condition which gives us an interaction between V-**Cat** and V-**Dist**, analogous to the interaction between **Set** and **Rel** given by the usual Beck-Chevalley condition on **Set**. After doing a brief survey about characterisations of the (Eilenberg-Moore) algebras for these submonads we present a new characterisation motivated by our results on the special case of the formal ball monad.

The presheaf monad is deeply related to Cauchy completeness, in fact, the Cauchy completion of a category can be obtained from “choosing" the correct presheaves. Alternatively we can describe the Cauchy completion of a category through its Karoubi envelope (or idempotent completion). In this setting, we show how the Karoubi envelope of V-**Rel** is equivalent to the Karoubi envelope of V-**Dist**, which parallels nicely the classical equivalence between the Karoubi envelopes of **Rel** and **Idl**. Finally, we use the equivalence between V-**Dist** and the Kleisli category of the presheaf monad on V-**Cat** to obtain a new equivalence relating the Karoubi envelope of V-**Rel** and the split idempotent completion of the category of (Eilenberg-Moore) algebras of V-**Cat**.

Resumo

A principal fonte de inspiração para este trabalho é o artigo seminal de Lawvere [28]: *Metric spaces, generalized logic, and closed categories*. Queremos destacar dois exemplos surpreendentes deste artigo de Lawvere sobre a profunda relação entre a teoria das categorias e a teoria métrica clássica:

1. um espaço (quasi)métrico X é simplesmente uma categoria cujos objetos são os pontos de X e a distância $d(x,y) \in [0,\infty]$ é o hom-conjunto $X(x,y)$.

2. As sucessões de Cauchy correspondem a distribuidores adjuntos e a convergência destas sucessões corresponde à representabilidade de tais distribuidores.

Estes exemplos não são apenas impressionantes, mas também úteis para orientar a pesquisa matemática. Como o próprio Lawvere salientou: *"this connection is more fruitful than a mere analogy, because it provides a sequence of mathematical theorems so that enriched category theory can suggest new directions of research in metric space theory and conversely"*.

A primeira observação é a motivação fundamental por trás do estudo de categorias enriquecidas num quantale como uma generalização de espaços métricos, enquanto que a segunda pode ser generalizada para nos dar uma noção categórica da completude de Cauchy em termos de colimites ponderados na teoria de categorias enriquecidas.

Relembramos algumas noções fundamentais sobre V-**Cat**, V-**Rel** e V-**Dist**, onde V é um quantale, que serão essenciais para o resto deste trabalho.

De seguida apresentamos uma nova prova que V-$\mathbf{Cat}^{\mathrm{op}}$ é uma quasivariedade, utilizando certas condições de monadicidade, categorias *comma* e a categoria V-**ccd** que generaliza a categoria **ccd** dos reticulados construtivamente completamente distributivos.

Seguimos a sugestão de Lawvere e generalizamos a mónada das bolas formais, que é uma ferramenta central no estudo da teoria métrica, para um contexto enriquecido. O nosso trabalho mostrou-se frutífero, pois obtivemos uma caracterização das álgebras (de Eilenberg-Moore) para a mónada das bolas formais como as V-categorias com uma certa classe de colimites ponderados. Ao restringir a nossa atenção a V-categorias separadas, também obtivemos uma caracterização interessante de uma certa classe de imersões.

Posteriormente apresentamos ainda duas caracterizações para as submónadas da mónada dos pré-feixes: uma em termos de uma classe especial de V-distribuidores; e outra como aquelas mónadas que são totalmente BC*, lax idempotentes e que satisfazem certas condições de fidelidade plena. Por BC*, queremos dizer uma nova condição do tipo Beck-Chevalley que nos dá uma interação entre V-**Cat** e V-**Dist**, análoga à interação entre **Set** e **Rel** dada pela usual condição de Beck-Chevalley em

Set. Depois de relembrarmos algumas caracterizações das álgebras (de Eilenberg-Moore) para estas submónadas apresentamos uma nova caracterização motivada pelos nossos resultados no caso especial da mónada das bolas formais.

A mónada dos pré-feixes está profundamente relacionada com a completude de Cauchy, de facto, a completude de Cauchy de uma categoria pode ser obtida a partir de uma “escolha correta" de pré-feixes. Neste contexto, mostraremos como o envelope de Karoubi de V-**Rel** é equivalente ao envelope de Karoubi de V-**Dist**, o que está em paralelo com a equivalência clássica entre os envelopes de Karoubi de **Rel** e **Idl**. Finalmente, usamos a equivalência entre V-**Dist** e a categoria Kleisli da mónada dos pré-feixes em V-**Cat** para obter uma nova equivalência relacionando o envelope Karoubi de V-**Rel** e a completude de idempotentes cindidos da categoria das álgebras (de Eilenberg-Moore) de V-**Cat**.

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Chapter 1

Introduction

Lax idempotent monads are defined in an **Ord**-enriched setting, where the notion of adjunction between morphisms can be defined. Our work will mostly focus on lax idempotent monads in the **Ord**-enriched category V-**Cat** of quantale-enriched categories. The main motivation for this setting is, of course, Lawvere's observation that (quasi)metric spaces can readily be generalised to quantale-enriched categories. Following Lawvere's insight we generalised some of the recent work of Goubault-Larrecq [19] on the lax idempotent formal ball monad on quasi(metric) spaces to V-**Cat**. Another very important class of examples of lax idempotent monads in V-**Cat** is the presheaf monad and its submonads, which we study and characterise.

Now we present an outline of this thesis:

In Chapter 2 we start with a few fundamental concepts in an **Ord**-enriched setting. We define (Eilbenberg-Moore) algebras and pseudo-algebras and present Escardó's characterisation of them in terms of injectivity and pseudo-injectivity with respect to T-embeddings, respectively. Lastly, we briefly highlight a few of Escardó's most notable characterisations.

In Chapter 3 we define (commutative and unital) quantales and study the most important categories for this work: V-**Cat**, V-**Rel** and V-**Dist**. In particular we prove that V-**Cat** is a symmetric monoidal-closed category, present the Yoneda lemma in this quantale-enriched setting, and explore some of the deep relations between V-**Cat** and V-**Dist**. We also define weighted colimits and show that a V-category is cocomplete precisely when it has a specific class of weighted colimits.

In Chapter 4 we present a new proof that V-$\mathbf{Cat}^{\mathrm{op}}$ is a quasivariety, making use of certain monadicity conditions, comma categories and the category V-**ccd**, which generalises the category **ccd** of constructively completely distributive lattices. This result is part of a new paper (which has a broader scope) currently in preparation [10].

In Chapter 5 we define the presheaf monad and prove our main results. We start by studying two monads which generalise the formal ball monad to a quantale-enriched setting. In particular, we show that the (Eilenberg-Moore) algebras for one of these monads are precisely the *tensored* V-categories (in the sense of Borceux and Kelly [4]). Then we shift our focus to the presheaf monad and start by showing an interesting equivalence between V-**Dist** and the Kleisli category of the presheaf monad on V-**Cat**. Next we present two characterisations of the submonads of the presheaf monad: one in terms of a special class of V-distributors; and another as those monads which are fully BC*, lax idempotent and satisfy certain full and faithfulness conditions. By BC* we mean a new Beck-Chevalley type condition which gives us an interaction between V-**Cat** and V-**Dist**, analogous to the interaction between **Set** and **Rel** given by the usual Beck-Chevalley condition on **Set**. We use our results on weighted colimits to provide a characterisation of the (Eilenberg-Moore) algebras for these submonads which was motivated by our results on the special case of the formal ball monad. These results were published recently in [6].

In Chapter 6 we show that the Karoubi envelope of V-**Rel** is equivalent to the Karoubi envelope of V-**Dist**, which parallels nicely the classical equivalence between the Karoubi envelopes of **Rel** and **Idl**. Next, we use the equivalence between V-**Dist** and the Kleisli category of the presheaf monad on V-**Cat**, proved in the previous chapter, to obtain another equivalence relating the Karoubi envelope of V-**Rel** and the split idempotent completion of the category of (Eilenberg-Moore) algebras of V-**Cat**. This result is a product of joint research with Professor Dirk Hofmann.

Chapter 2

Setting

We start by briefly recalling a few concepts which are the foundation for the rest of our work.

2.1 Monads

First we recall the notion of a *monad*: this fundamental tool gives us a way to formalise and generalise the classic notion of an *algebra* to a categorical setting through the Eilenberg-Moore and Kleisli constructions.

A **monad** $\mathbb{T} = (T, \mu, \eta)$ on a category $\mathbf{C}$ is an endofunctor $T : \mathbf{C} \to \mathbf{C}$ equipped with natural transformations $\mu : TT \to T$ and $\eta : 1 \to T$ such that the following diagrams commute

$$\begin{tikzcd} TTT \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "T\eta"] \arrow[d, "\eta T"'] \arrow[dr, equal] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"'] & T. \end{tikzcd}$$

2.1.1 Monad morphisms and submonads

Given two monads $\mathbb{T} = (T, \mu, \eta)$, $\mathbb{T}' = (T', \mu', \eta')$ on a category $\mathbf{C}$, a **monad morphism** $\sigma \colon \mathbb{T} \to \mathbb{T}'$ is a natural transformation $\sigma \colon T \to T'$ such that the following diagrams commute

$$\begin{tikzcd} 1 \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & T \arrow[d, "\sigma"] \\ & T' \end{tikzcd} \qquad \begin{tikzcd} TT \arrow[r, "\sigma_T"] \arrow[d, "\mu"'] & T'T \arrow[r, "T'\sigma"] & T'T' \arrow[d, "\mu'"] \\ T \arrow[rr, "\sigma"'] & & T'. \end{tikzcd}$$

By **submonad of** $\mathbb{T}'$ we mean a monad $\mathbb{T} = (T, \mu, \eta)$ on $\mathbf{C}$ with a monad morphism $\sigma : \mathbb{T} \to \mathbb{T}'$ such that σ_X is an extremal monomorphism for every object X of $\mathbf{C}$.

2.1.2 Eilenberg-Moore and Kleisli Categories

Given a monad $\mathbb{T} = (T, \mu, \eta)$ on a category $\mathbf{C}$ the **Eilenberg-Moore category** $\mathbf{C}^{\mathbb{T}}$ of $\mathbb{T}$ has as objects the **$\mathbb{T}$-algebras**, that is, objects X of $\mathbf{C}$ equipped with a morphism, called **$\mathbb{T}$-algebra structure**, $\alpha : TX \to X$ such that the following diagrams commute:

$$\begin{tikzcd} TTX \arrow[r, "T\alpha"] \arrow[d, "\mu_X"'] & TX \arrow[d, "\alpha"] \\ TX \arrow[r, "\alpha"'] & X \end{tikzcd} \qquad \begin{tikzcd} TX \arrow[d, "\alpha"'] & X \arrow[l, "\eta_X"'] \arrow[dl, equal] \\ X & \end{tikzcd} .$$

In $\mathbf{C}^{\mathbb{T}}$ the morphisms $f : (X, \alpha) \to (Y, \beta)$ are the **$\mathbb{T}$-homomorphisms**, given by the morphisms $f : X \to Y$ of $\mathbf{C}$ such that the following diagram commutes:

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "\alpha"'] & TY \arrow[d, "\beta"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

Given a monad $\mathbb{T} = (T, \mu, \eta)$ on a category $\mathbf{C}$ the **Kleisli category** $\mathbf{C}_{\mathbb{T}}$ of $\mathbb{T}$ has as objects the objects of $\mathbf{C}$ and $f : X \rightharpoonup Y$ is a morphism in $\mathbf{C}_{\mathbb{T}}$ precisely when $f : X \to TY$ is a morphism in $\mathbf{C}$. In $\mathbf{C}_{\mathbb{T}}$, the composition $g \circ f : X \rightharpoonup Z$ of morphisms $f : X \rightharpoonup Y$, $g : Y \rightharpoonup Z$ is given by the following morphism in $\mathbf{C}$:

$$\mu_Z \cdot Tg \cdot f : X \xrightarrow{f} TY \xrightarrow{Tg} TTZ \xrightarrow{\mu_Z} TZ.$$

The identity morphism for each object X of $\mathbf{C}_{\mathbb{T}}$ is given by the monad unit η_X.

Given an object X of $\mathbf{C}$ it follows from the monad axioms that (TX, μ_X) is a $\mathbb{T}$-algebra. Such $\mathbb{T}$-algebras are called **free** $\mathbb{T}$-algebras. By naturality of μ, any morphism $Tf : TX \to TY$ of $\mathbf{C}$ is a $\mathbb{T}$-homomorphism between the free algebras (TX, μ_X) and (TY, μ_Y) (not all $\mathbb{T}$-homomorphisms are of this form however).

Moreover, we have a full and faithful functor $\iota : \mathbf{C}_{\mathbb{T}} \to \mathbf{C}^{\mathbb{T}}$ which maps every object X of $\mathbf{C}_{\mathbb{T}}$ to the free algebra (TX, μ_X) and every morphism $f : X \rightharpoonup Y$ to the $\mathbb{T}$-homomorphism $\mu_Y \cdot Tf$, making the following diagram commute:

$$\begin{tikzcd} \mathbf{C}_{\mathbb{T}} \arrow[r, "\iota"] \arrow[d, "U_{\mathbb{T}}"'] & \mathbf{C}^{\mathbb{T}} \arrow[dl, "U^{\mathbb{T}}"] \\ \mathbf{C} & \end{tikzcd}$$

where

- $U_{\mathbb{T}} : \mathbf{C}_{\mathbb{T}} \to \mathbf{C}$ is defined by $U_{\mathbb{T}}X = TX$ and $U_{\mathbb{T}}(f : X \rightharpoonup Y) = (\mu_Y \cdot Tf : TX \to TY)$;
- $U^{\mathbb{T}} : \mathbf{C}^{\mathbb{T}} \to \mathbf{C}$ is defined by $U^{\mathbb{T}}(X, \alpha) = X$ and $U^{\mathbb{T}}(f : (X, \alpha) \to (Y, \beta)) = (f : X \to Y)$.

Therefore the Kleisli category $\mathbf{C}_{\mathbb{T}}$ is equivalent to the full subcategory of the Eilenberg-Moore category $\mathbf{C}^{\mathbb{T}}$ defined by the free $\mathbb{T}$-algebras.

2.1.3 Monadicity

Given a pair of adjoint functors $(F \dashv U) : \mathbf{D} \underset{U}{\overset{F}{\leftrightarrows}} \mathbf{C}$ with unit η and counit ε we have the following diagram:

$$
\begin{tikzcd}
 & \mathbf{D} \arrow[d, bend left, "U"] \arrow[dr, "K"] & \\
\mathbf{C}_{\mathbb{T}} \arrow[ur, hook] \arrow[r, bend left, "U_{\mathbb{T}}"] \arrow[r, phantom, "\top"] & \mathbf{C} \arrow[u, bend left, "F"] \arrow[u, phantom, "\dashv"] \arrow[l, bend left, "F_{\mathbb{T}}"] \arrow[r, bend left, "F^{\mathbb{T}}"] \arrow[r, phantom, "\perp"] & \mathbf{C}^{\mathbb{T}} \arrow[l, bend left, "U^{\mathbb{T}}"]
\end{tikzcd}
$$

where

- $\mathbb{T} = (T, \mu, \eta)$ with $T = UF$, $\mu = U\varepsilon F$, is the monad induced by the adjunction $(F \dashv U)$;
- $F_{\mathbb{T}} : \mathbf{C} \to \mathbf{C}_{\mathbb{T}}$ is defined by $F_{\mathbb{T}}X = X$ and $F_{\mathbb{T}}(f : X \to Y) = (\eta_Y \cdot f : X \rightharpoonup Y)$;
- $F^{\mathbb{T}} : \mathbf{C} \to \mathbf{C}^{\mathbb{T}}$ is defined by $F^{\mathbb{T}}X = (TX, \mu_X)$ and $F_{\mathbb{T}}(f : X \to Y) = (Tf : (TX, \mu_X) \to (TY, \mu_Y))$;
- $K : \mathbf{D} \to \mathbf{C}^{\mathbb{T}}$ is the **comparison functor**, which is defined by $KD = (UD, U\varepsilon_D)$ and $K(f : X \to Y) = (Uf : UX \to UY)$.

If the comparison functor K is an equivalence of categories we say that the adjunction $F \dashv U$ is a **monadic adjunction**. In such a case, we also say that $U : \mathbf{D} \to \mathbf{C}$ is **monadic** and that $\mathbf{D}$ is **monadic over C**.

Definition 2.1.1. A category $\mathbf{C}$ is called a **variety** if it is monadic over Set, that is, it is equivalent to $\mathbf{Set}^{\mathbb{T}}$ for some monad $\mathbb{T}$.

Definition 2.1.2. A full subcategory $\iota : \mathbf{D} \hookrightarrow \mathbf{C}$ is **reflective** if the inclusion functor ι has a left adjoint F:

$$
\begin{tikzcd}
\mathbf{D} \arrow[r, bend left, "\iota"] \arrow[r, phantom, "\top"] & \mathbf{C} \arrow[l, bend left, "F"]
\end{tikzcd}
$$

The left adjoint F is called the **reflector**, and a functor which is a reflector (or has a fully faithful right adjoint, which is the same up to equivalence) is called a **reflection**.

Definition 2.1.3. A reflective subcategory $(F \dashv \iota) : \mathbf{D} \underset{\iota}{\overset{F}{\leftrightarrows}} \mathbf{C}$ is **(regular epi)-reflective** when each component of the unit of the adjunction is a regular epimorphism.

Definition 2.1.4. A category $\mathbf{D}$ is called a **quasivariety** if and only if it is, up to equivalence, a (regular epi)-reflective subcategory of a variety.

2.2 Ord-enriched categories

The general theory of enriched categories on a monoidal category **V** can be found in [24]. Since we will only be interested in categories enriched in **Ord**, the category of (pre)orders and monotone maps, we will only present the relevant enriched category theory when $\mathbf{V} = \mathbf{Ord}$.

An **Ord-enriched category C** is a (locally small) category such that each hom-set $\mathbf{C}(X,Y)$ is equipped with an order (that is, a reflexive and transitive relation $\leq$) that is preserved by composition on either side:

$$f \leq f' \qquad \Rightarrow \qquad h \cdot f \cdot g \leq h \cdot f' \cdot g,$$

for all $f, f' : X \to Y,\ h : Y \to Z$ and $g : W \to X$.

An **Ord**-enriched category where each hom-set is equipped with an antisymmetric (pre)order, that is, a partial order, is called a **PoSet-enriched category.**

Given an **Ord**-enriched category **C**:

- its **dual** $\mathbf{C}^{\mathrm{op}}$ is defined by reversing the morphisms of **C** while preserving the order: $\mathbf{C}^{\mathrm{op}}(X,Y) = (\mathbf{C}(Y,X), \leq)$, for all objects X,Y in **C**;
- its **conjugate** $\mathbf{C}^{\mathrm{co}}$ is defined by reversing the order of each hom-set of **C** while preserving the morphisms: $\mathbf{C}^{\mathrm{co}}(X,Y) = (\mathbf{C}(X,Y), \geq)$, for all objects X,Y in **C**.

Moreover

$$\mathbf{C}^{\mathrm{op\,op}} = \mathbf{C}, \qquad \mathbf{C}^{\mathrm{co\,co}} = \mathbf{C} \qquad \text{and} \qquad \mathbf{C}^{\mathrm{co\,op}} = \mathbf{C}^{\mathrm{op\,co}}.$$

Given two **Ord**-enriched categories **C** and **D**, an **Ord**-functor is a monotone functor, that is, a functor $F : \mathbf{C} \to \mathbf{D}$ such that

$$f \leq f' \qquad \Rightarrow \qquad Ff \leq Ff',$$

for all $f, f' : X \to Y$.

An **Ord**-natural transformation is just a natural transformation between **Ord**-functors.

Let $f, f' : X \to Y$, $g : Y \to X$ be morphisms in an **Ord**-enriched category **C**. We will write $f \simeq f'$ when $f \leq f'$ and $f' \leq f$.

Moreover, if $1_X \leq g \cdot f$ and $f \cdot g \leq 1_Y$ we will say that the pair (f,g) is an **adjunction** and denote it as $f \dashv g$. In this case we say that f is a **left adjoint** of g and that g is a **right adjoint** of f.

A morphism $f : X \to Y$ in an **Ord**-enriched category **C** is called **order-monic** if,

$$f \cdot g \leq f \cdot g' \qquad \Rightarrow \qquad g \leq g'$$

for every pair of morphisms $g, g' : W \to X$.

An **Ord**-functor F is called **order-faithful** if

$$Ff \leq Ff' \qquad \Rightarrow \qquad f \leq f',$$

for all $f, f' : X \to Y$.

2.2.1 Ord-monads

Let $\mathbf{C}$ be an **Ord**-enriched category. An **Ord-monad** $\mathbb{T} = (T, \mu, \eta)$ on $\mathbf{C}$ is a monad $\mathbb{T} = (T, \mu, \eta)$ on $\mathbf{C}$ such that T is an **Ord**-endofunctor $T : \mathbf{C} \to \mathbf{C}$ and $\mu : TT \to T$ and $\eta : 1 \to T$ are **Ord**-natural transformations.

Let us consider a very important example:

Consider the category $\mathbf{Top}_0$ of T_0 topological spaces and continuous maps. Endowing each T_0 space X with the order given by

$$x \leq y \quad \text{if} \quad y \in \overline{\{x\}},$$

for any $x, y \in X$, that is, $\leq$ is the dual of the specialisation order, $\mathbf{Top}_0$ becomes an **Ord**-enriched category. Given a T_0 topological space X, we denote its lattice of open sets by ΩX. The **filter space** FX has as points the filters of ΩX and its open sets are generated by the sets $\{\varphi \in FX \mid U \in \varphi\}$, $U \in \Omega X$. It is easy to check that in FX the order is given by the reverse of the usual inclusion, that is,

$$\varphi \leq \psi \quad \text{if} \quad \psi \subseteq \varphi.$$

Given a continuous map $f : X \to Y$ we define $Ff : FX \to FY$ by $Ff(\varphi) = \{V \in \Omega Y \mid f^{-1}(V) \in \varphi\}$. Then F is an **Ord**-functor and we have the **Ord**-natural transformations $\eta_X : X \to FX$ and $\mu_X : FFX \to X$ given by

$$\eta_X(x) = \{U \in \Omega X \mid x \in U\}, \qquad \mu_X(\Phi) = \cup\{\cap\mathcal{U} \mid \mathcal{U} \in \Phi\}.$$

This defines the **filter monad** $\mathbb{F} = (F, \mu, \eta)$. Note that this monad is actually **PoSet**-enriched because we are considering T_0 spaces, where the specialisation order is antisymmetric.

2.2.2 Eilenberg-Moore pseudo-algebras

In the context of **Ord**-enriched monads we can slightly generalise the notion of Eilenberg-Moore algebra by requiring isomorphisms in place of equalities in the definition of $\mathbb{T}$-algebra structure:

Definition 2.2.1. Let $\mathbb{T} = (T, \mu, \eta)$ be an **Ord**-monad on $\mathbf{C}$. A **pseudo-$\mathbb{T}$-algebra** is an object X of $\mathbf{C}$ equipped with a morphism, called **pseudo-$\mathbb{T}$-algebra structure**, $\alpha : TX \to X$ such that

$$\alpha \cdot T\alpha \simeq \alpha \cdot \mu_X \qquad \text{and} \qquad \alpha \cdot \eta_X \simeq 1_X.$$

We note that given an **Ord**-enriched monad $\mathbb{T}$ in a **PoSet**-enriched category the concepts of pseudo-$\mathbb{T}$-algebra and (strict) $\mathbb{T}$-algebra trivially coincide.

2.2.3 Lax idempotency

Lax idempotent **Ord**-monads (also called KZ-monads) were introduced in [25].

Definition 2.2.2. An **Ord**-monad $\mathbb{T} = (T, \mu, \eta)$ on $\mathbf{C}$ is **lax idempotent** if $T\eta_X \leq \eta_{TX}$ holds for all objects $X \in \mathbf{C}$.

A notable example of a lax idempotent monad is that of the filter monad $\mathbb{F} = (F, \mu, \eta)$. Note that

$$\eta_{TX}(\varphi) = \{\mathcal{U} \in \Omega TX \mid \varphi \in \mathcal{U}\}, \qquad T\eta_X(\varphi) = \{\mathcal{U} \in \Omega TX \mid \eta_X^{-1}(\mathcal{U}) \in \varphi\}.$$

Let $\mathcal{U} \in \eta_{TX}(\varphi)$. Then $\varphi \in \mathcal{U}$. Since the sets of the form

$$\Box U := \{\varphi \in TX \mid U \in \varphi\}, \quad U \in \Omega X$$

give us a base for ΩTX, there is $U \in \Omega X$ such that $\varphi \in \Box U \subseteq \mathcal{U}$. Since $U = \eta_X^{-1}(\Box U) \in \varphi$, it follows that $\Box U \in T\eta_X(\varphi)$. Therefore $\mathcal{U} \in T\eta_X(\varphi)$ and we see that $\eta_{TX}(\varphi) \subseteq T\eta_X(\varphi)$.

Lax idempotent **Ord**-monads generalise the notion of idempotent monad in the following way: while in the idempotent case we have, for all objects $X \in \mathbf{C}$, isomorphisms

$$T\eta_X \cong \eta_{TX} \cong (\mu_X)^{-1},$$

in the lax idempotent case these are replaced by adjunctions:

$$T\eta_X \dashv \mu_X, \quad \text{or equivalently,} \quad \mu_X \dashv \eta_{TX},$$

The next proposition shows us that these adjunction conditions are equivalent to the definition of lax idempotency given above.

Proposition 2.2.3. *Let $\mathbb{T} = (T, \mu, \eta)$ be an* **Ord***-monad on* **C***. Then the following assertions are equivalent:*

(i) $\mathbb{T}$ *is lax idempotent;*

(ii) $T\eta_X \dashv \mu_X$*, for all objects $X \in$* **C***;*

(iii) $\mu_X \dashv \eta_{TX}$*, for all objects $X \in$* **C***.*

Proof. (i) $\Rightarrow$ (ii) Since $\mathbb{T}$ is lax idempotent and T is an **Ord**-functor we have that $TT\eta_X \leq T\eta_{TX}$. Therefore

$$T\eta_X \cdot \mu_X = \mu_{TX} \cdot TT\eta_X \leq \mu_{TX} \cdot T\eta_{TX} = 1_{TTX}.$$

Moreover $\mu_X \cdot T\eta_X = 1_{TX}$ since $\mathbb{T}$ is a monad.

(ii) $\Rightarrow$ (iii) Using our hypothesis, we have that $T\eta_{TX} \cdot \mu_{TX} \leq 1_{TTTX}$. Therefore

$$1_{TTX} = T\mu_X \cdot T\eta_{TX} \cdot \mu_{TX} \cdot \eta_{TTX} \leq T\mu_X \cdot \eta_{TTX} = \eta_{TX} \cdot \mu_X.$$

Moreover $\mu_X \cdot \eta_{TX} = 1_{TX}$ since $\mathbb{T}$ is a monad.

(iii) $\Rightarrow$ (i) Using our hypothesis, we have that $T\eta_X \leq \eta_{TX} \cdot \mu_X \cdot T\eta_X = \eta_{TX}$. $\Box$

Moreover, lax idempotency can also be characterised as follows:

Proposition 2.2.4. *Let $\mathbb{T} = (T, \mu, \eta)$ be an* **Ord***-monad on* **C***. Then the following assertions are equivalent:*

(i) $\mathbb{T}$ *is lax idempotent;*

(ii) *if a morphism* $\alpha : TX \to X$ *in* **C** *is such that* $\alpha \cdot \eta_X \simeq 1_X$ *then* $\alpha \dashv \eta_X$ *and* $\alpha \cdot T\alpha \simeq \alpha \cdot \mu_X$.

Proof. (i) $\Rightarrow$ (ii) Note that $\eta_X \cdot \alpha = T\alpha \cdot \eta_{TX} \geq T\alpha \cdot T\eta_X = T(\alpha \cdot \eta_X) \simeq 1_{TX}$ and therefore $\alpha \dashv \eta_X$. Moreover from $\alpha \dashv \eta_X$ we obtain that $T\alpha \dashv T\eta_X$ and then, by composing these two adjunctions, we have $\alpha \cdot T\alpha \dashv T\eta_X \cdot \eta_X$. Using now Proposition 2.2.3, we have also the adjunction $\mu_X \dashv \eta_{TX}$, which we also compose with $\alpha \dashv \eta_X$ to obtain $\alpha \cdot \mu_X \dashv \eta_{TX} \cdot \eta_X = T\eta_X \cdot \eta_X$. It follows that $\alpha \cdot T\alpha \simeq \alpha \cdot \mu_X$, since these morphisms are both left adjoints to $T\eta_X \cdot \eta_X$.

(ii) $\Rightarrow$ (i) Note that $\mu_X : TTX \to TX$ is a morphism in **C** such that $\mu_X \cdot \eta_{TX} = 1_{TX}$ and therefore $\mu_X \dashv \eta_{TX}$. □

This result gives us an immediate characterisation for the pseudo-$\mathbb{T}$-algebras of an **Ord**-enriched category using only the unit η of a lax idempotent **Ord**-monad $\mathbb{T} = (T, \mu, \eta)$.

Corollary 2.2.5. *Let* $\mathbb{T} = (T, \mu, \eta)$ *be a lax idempotent* **Ord**-*monad on* **C**. *Then the following assertions are equivalent:*

(i) X *has a pseudo-*$\mathbb{T}$*-algebra structure* $\alpha : TX \to X$;

(ii) *There exists a morphism* $\alpha : TX \to X$ *in* **C** *such that* $\alpha \cdot \eta_X \simeq 1_X$. □

Moreover all morphisms on a **Ord**-enriched category must satisfy the following condition:

Proposition 2.2.6. *Let* $\mathbb{T} = (T, \mu, \eta)$ *be a lax idempotent* **Ord**-*monad on* **C**. *If* (X, α) *and* (Y, β) *are pseudo-*$\mathbb{T}$*-algebras, then*

$$\beta \cdot Tf \leq f \cdot \alpha$$

holds for any morphism $f : X \to Y$ *in* **C**.

Proof. Since $\alpha \dashv \eta_X$, we have $1_{TX} \leq \eta_X \cdot \alpha$. Therefore

$$\beta \cdot Tf = \beta \cdot Tf \cdot 1_{TX} \leq \beta \cdot Tf \cdot \eta_X \cdot \alpha = \beta \cdot \eta_Y \cdot f \cdot \alpha \simeq 1_Y \cdot f \cdot \alpha = f \cdot \alpha,$$

by naturality of η and noting that $\beta \cdot \eta_Y \simeq 1_Y$ since β is a pseudo-$\mathbb{T}$-algebra structure. □

2.2.4 T-embeddings

In [16] Escardó presents an interesting characterisation of the $\mathbb{T}$-algebras of a lax idempotent monad $\mathbb{T} = (T, \mu, \eta)$ in terms of an injectivity condition on a special class of embeddings, in the **PoSet**-enriched setting. Given our focus on lax idempotent monads, these results are of obvious interest. We will present a very slight generalisation of Escardó's result to the **Ord**-enriched case.

We start with some important elementary concepts.

Definition 2.2.7. Let X, Y be objects in a category **C**, and let $s : X \to Y$ and $r : Y \to X$ be morphisms in **C** such that $r \cdot s = 1_X$.

$$\begin{tikzcd} X \arrow[loop above, "1_X"] \arrow[r, shift left, "s"] & Y \arrow[l, shift left, "r"] \end{tikzcd}$$

Then r is called a **retraction of** s, X is called a **retract of** Y and s is called a **section of** r.

Clearly the morphism r is a split epimorphism and the morphism s is a split monomorphism. Equivalently, one may say that a split epimorphism is a retraction and that a split monomorphism is a morphism that has a retraction.

Definition 2.2.8. An object X in a category **C** is **injective with respect to a** morphism $f : W \to Y$ if, for any morphism $g : W \to X$, there is a morphism $h : Y \to X$ such that $h \cdot f = g$.

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[dl, dotted, "h"] \\ X & \end{tikzcd}$$

In the **Ord**-enriched setting we also will find useful the following concept:

Definition 2.2.9. An object X in a **Ord**-enriched category **C** is **pseudo-injective with respect to a** morphism $f : W \to Y$ if, for any morphism $g : W \to X$, there is a morphism $h : Y \to X$ such that $h \cdot f \simeq g$.

Note that in a **PoSet**-enriched category the above notions clearly coincide.

Lemma 2.2.10. *Let X be a retract of an object Y in a category* **C**. *Then if Y is injective with respect to a morphism $f : W \to V$, so is X.*

Proof. Assume that Y is injective with respect to a morphism $f : W \to V$ and let $g : W \to X$ be a morphism in **C**. Then we have a morphism $s \cdot g : W \to Y$ and since Y is injective with respect to f, there is a morphism $h : V \to Y$ such that $h \cdot f = s \cdot g$.

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "g"] \arrow[dd, bend right=60, "s\cdot g"'] & V \arrow[dl, dotted, "r\cdot h"'] \arrow[ddl, dotted, "h"] \\ X \arrow[d, shift right, "s"'] & \\ Y \arrow[u, shift right, "r"'] & \end{tikzcd}$$

Therefore, we have a morphism $r \cdot h : V \to X$ such that

$$(r \cdot h) \cdot f = r \cdot (h \cdot f) = r \cdot (s \cdot g) = (r \cdot s) \cdot g = 1_X \cdot g = g,$$

and we conclude that X is injective with respect to f. □

Definition 2.2.11. Let $\mathbb{T} = (T, \mu, \eta)$ be an **Ord**-monad on **C**. A morphism $f \colon X \to Y$ is a T**-embedding** if Tf is a left adjoint right inverse; that is, there exists a morphism $(Tf)_\sharp$ such that $Tf \dashv (Tf)_\sharp$ and $(Tf)_\sharp \cdot Tf = 1_{TX}$.

Proposition 2.2.12. *Let $\mathbb{T} = (T, \mu, \eta)$ be a lax idempotent* **Ord**-*monad on* **C**. *Then η_X is a T-embedding for any X in* **C**.

Proof. Take $(T\eta_X)_\sharp = \mu_X$. Then $T\eta_X \dashv \mu_X$, by Proposition 2.2.3. Moreover $\mu_X \cdot T\eta_X = 1_{TX}$ since $\mathbb{T}$ is a monad. □

While simple, this proposition allows us to obtain the following characterisations:

Proposition 2.2.13. *For a lax idempotent* **Ord**-*monad $\mathbb{T} = (T, \mu, \eta)$ on* **C** *the following conditions are equivalent:*

(i) *T-embeddings are order-monic;*

(ii) *η_X is order-monic, for any X in* **C**;

(iii) *T is order-faithful;*

(iv) *for every morphism f in* **C**, *f is order-monic whenever Tf is.*

Proof. (i) $\Rightarrow$ (ii) Trivial.

(ii) $\Rightarrow$ (iii) Let $f, f' : X \to Y$ be morphisms in **C** such that $Tf \leq Tf'$. Then, by naturality of η, we have

$$\eta_Y \cdot f = Tf \cdot \eta_X \leq Tf' \cdot \eta_X = \eta_Y \cdot f'.$$

Therefore $f \leq f'$, by hypothesis.

(iii) $\Rightarrow$ (iv) Let $f : X \to Y$, $g, g' : W \to X$ be morphisms in **C**. Assume that Tf is order-monic and that $f \cdot g \leq f \cdot g'$. Then $Tf \cdot Tg = T(f \cdot g) \leq T(f \cdot g') = Tf \cdot Tg'$. Since Tf is order-monic, it follows that $Tg \leq Tg'$ and therefore, since T is order-faithful, $g \leq g'$.

(iv) $\Rightarrow$ (i) Let $f : X \to Y$ be a T-embedding and $g, g' : W \to TX$ be morphisms in **C** such that $Tf \cdot g \leq Tf \cdot g'$. Therefore

$$g = (Tf)_\sharp \cdot Tf \cdot g \leq (Tf)_\sharp \cdot Tf \cdot g' = g'.$$

So Tf is order-monic and the result follows from the hypothesis. □

Proposition 2.2.14. *Let $\mathbb{T} = (T, \mu, \eta)$ be a lax idempotent* **Ord**-*monad on* **C**. *Then, for any object X of* **C**, *the following assertions are equivalent:*

(i) *X is pseudo-injective with respect to T-embeddings;*

(ii) *There exists a morphism $\alpha : TX \to X$ in* **C** *such that $\alpha \cdot \eta_X \simeq 1_X$;*

(iii) *X has a pseudo-$\mathbb{T}$-algebra structure.*

Proof. (i) $\Rightarrow$ (ii) By Proposition 2.2.12, η_X is a T-embedding. Since X is pseudo-injective with respect to T-embeddings, there is a morphism $\alpha : TX \to X$ in **C** such that $\alpha \cdot \eta_X \simeq 1_X$.

(ii) $\Rightarrow$ (iii) Using Proposition 2.2.4 we have that $\alpha \cdot T\alpha \simeq \alpha \cdot \mu_X$. Therefore α is a pseudo-$\mathbb{T}$-algebra structure.

(iii) $\Rightarrow$ (i) Let $f : W \to Y$ be a T-embedding, $\alpha : TX \to X$ be a pseudo-$\mathbb{T}$-algebra structure on X and $g : W \to X$ be a morphism in $\mathbf{C}$. Take $h = \alpha \cdot Tg \cdot (Tf)_\sharp \cdot \eta_Y$.

Then

$$\begin{aligned} h \cdot f &= \alpha \cdot Tg \cdot (Tf)_\sharp \cdot (\eta_Y \cdot f) \\ &= \alpha \cdot Tg \cdot (Tf)_\sharp \cdot (Tf \cdot \eta_W) \\ &= \alpha \cdot Tg \cdot ((Tf)_\sharp \cdot Tf) \cdot \eta_W \\ &= \alpha \cdot (Tg \cdot \eta_W) \\ &= \alpha \cdot (\eta_X \cdot g) \\ &= (\alpha \cdot \eta_X) \cdot g \\ &\simeq 1_X \cdot g = g \end{aligned}$$

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[r, "\eta_Y"] \arrow[dl, dotted, "h"] & TY \arrow[dl, "(Tf)_\sharp"] \\ X & TW \arrow[dl, "Tg"] & \\ TX \arrow[u, "\alpha"] & & \end{tikzcd}$$

Therefore X is indeed pseudo-injective with respect to T-embeddings. $\square$

Of course, in the **PoSet**-enriched case this result is simply the characterisation found in [16].

Corollary 2.2.15. *Let $\mathbb{T} = (T, \mu, \eta)$ be a lax idempotent* **Ord***-monad on a* **PoSet***-enriched category* $\mathbf{C}$. *Then, for any object X of $\mathbf{C}$ the following assertions are equivalent:*

(i) *X is injective with respect to T-embeddings;*

(ii) *There exists a morphism $\alpha : TX \to X$ in $\mathbf{C}$ such that $\alpha \cdot \eta_X = 1_X$;*

(iii) *X has a $\mathbb{T}$-algebra structure.* $\square$

With this characterisation of $\mathbb{T}$-algebras as precisely the injective objects with respect to T-embeddings it is now easy to show the following proposition.

Corollary 2.2.16. *Let $\mathbb{T} = (T, \mu, \eta)$ be a lax idempotent* **Ord***-monad on a* **PoSet***-enriched category* $\mathbf{C}$. *Then, for any object X of $\mathbf{C}$ the following assertions are equivalent:*

(i) *$\alpha \colon TX \to X$ is a $\mathbb{T}$-algebra structure on X;*

(ii) *X is a retract of TX;*

(iii) *X is a retract of a free $\mathbb{T}$-algebra;*

(iv) *X is a retract of a $\mathbb{T}$-algebra.*

Proof. (i) $\Rightarrow$ (ii) Note that $\alpha \cdot \eta_X = 1_X$.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) Trivial.

(iv) $\Rightarrow$ (i) By Corollary 2.2.15, $\mathbb{T}$-algebras are injective with respect to T-embeddings. By Lemma 2.2.10, so is X. Therefore X is a $\mathbb{T}$-algebra, using again Corollary 2.2.15. $\square$

We finish this section highlighting a few of the most remarkable uses of Escardó's characterisation. Let $\mathbb{F}_\alpha$, for $\alpha = 0, 1, \omega, \Omega$, denote, respectively, the filter monad of all filters, proper filters, prime filters and completely prime filters. Note that these monads are lax idempotent. In [17] Escardó and Flagg proved that:

- $\mathbb{F}_0$-embeddings are precisely all the embeddings;
- $\mathbb{F}_1$-embeddings are precisely the dense embeddings;
- $\mathbb{F}_\omega$-embeddings are precisely the flat embeddings;
- $\mathbb{F}_\Omega$-embeddings are precisely the completely flat embeddings.

Making use of Escardó's [16] characterisation of the algebras of a lax idempotent monad, the result above and some previously known characterisations of $\mathbb{F}_\alpha$, for $\alpha = 0, 1, \omega, \Omega$, Escardó and Flagg concluded that, if X is an object in $\mathbf{Top}_0$, then:

- X is injective with respect to all embeddings if and only if it is a continuous lattice;
- X is injective with respect to dense embeddings if and only if it is a continuous Scott domain;
- X is injective with respect to flat embeddings if and only if it is a stably compact space;
- X is injective with respect to completely flat embeddings if and only if it is a sober space.

We also note that, following Escardó's results, Cagliari, Clementino and Mantovani [5] obtained a characterisation of fibrewise injectivity in topological T_0-spaces, that is, a characterisation of the injectivity of continuous maps (instead of spaces). To obtain this characterisation, they introduced the notion of *fibrewise* filter monads which are defined in the comma category $\mathbf{Top}_0 \downarrow Z$, for any space Z. These categories are again **PoSet**-enriched when endowed with the order inherited from the order defined in $\mathbf{Top}_0$ and these fiberwise filter monads are also lax idempotent. We will denote the $\mathbf{Top}_0 \downarrow Z$ version of the filter monads $\mathbb{F}_\alpha$ by $\hat{\mathbb{F}}_\alpha$, for $\alpha = 0, 1, \omega, \Omega$.

In the same article, they [5] go on to show that $\hat{\mathbb{F}}_\alpha$-embeddings are precisely the $\mathbb{F}_\alpha$-embeddings, for $\alpha = 0, 1, \omega, \Omega$. Applying Escardó's [16] characterisation we obtain that a continuous map $f : X \to Z$ is $\hat{\mathbb{F}}_\alpha$-injective if and only if $f : X \to Z$ has an $\hat{\mathbb{F}}_\alpha$-algebra structure, for $\alpha = 0, 1, \omega, \Omega$.

Chapter 3

Quantale-enriched categories

In this section we survey the basic theory behind the categories V-**Rel**, V-**Cat**, and V-**Dist**.

3.1 Quantales

A (commutative and unital) **quantale** $(V,\leq,\otimes,k)$ is simultaneously a complete lattice $(V,\leq)$ and a commutative monoid $(V,\otimes,k)$ such that $\otimes$ commutes with joins, that is:

$$u\otimes\bigvee_{i\in I} v_i = \bigvee_{i\in I}(u\otimes v_i)$$

for all $u,v_i\in V$.

As a category, a quantale V is a thin symmetric monoidal-closed category whose internal hom-objects $\hom(u,v)$ are given by

$$\hom(u,v) = \bigvee\{x\in V \mid x\otimes u\leq v\}$$

for all $u,v\in V$. Equivalently, $(-)\otimes u \dashv \hom(u,-)$, for all $u\in V$.

A quantale V is called **integral** if $k=\top$, where $\top$ denotes the top element of V.

Throughout this text, V will always denote such a quantale.

Examples 3.1.1.

- The trivial quantale $\mathbf{1}=\{\perp\}$;
- The **Boolean quantale** $\mathbf{2}=(\{\perp,\top\},\leq,\wedge,\top)$;
- Any set X defines a quantale $(\mathcal{P}X,\subseteq,\cap,X)$, where $\mathcal{P}X$ denotes the power set of X. Given $A,B\in\mathcal{P}X$, the set $\hom(A,B)=(X\setminus A)\cup B$.
- More generally, any frame $(X,\leq)$, that is, a complete lattice satisfying

$$u\wedge\bigvee_{i\in I} v_i = \bigvee_{i\in I}(u\wedge v_i)$$

for all $u, v_i \in X$, defines a quantale $(X, \leq, \wedge, \top)$, where $\top$ denotes the greatest element of the lattice X.

- The **Lawvere quantale** $[0,\infty]_+ = ([0,\infty], \geq, +, 0)$. The underlying set is the extended non-negative real half-line, equipped with the reverse of the natural order $\leq$. The tensor is given by the usual addition, extended by $u + \infty = \infty + u = \infty$, for all $u \in [0,\infty]$. Given that the order is reversed, joins and meets in the Lawvere quantale become respectively infima and suprema with respect to the usual order. For all $u, v \in [0,\infty]$, $\hom(u,v)$ is given by the **truncated subtraction** $v \ominus u := \max\{v - u, 0\}$, extended by $u - \infty = 0$ and $\infty - v = \infty$, when $v \neq \infty$.

- Considering the lattice isomorphism $[0,1] \to [0,\infty], u \mapsto -\ln(u)$, where $-\ln(0) = \infty$, we obtain an isomorphic presentation of the Lawvere quantale as $[0,1]_* = ([0,1], \leq, *, 1)$, where $*$ denotes the usual multiplication of real numbers, and $\leq$ denotes the usual order. In this quantale the internal hom is given by $\hom(u,v) = v \oslash u := \min\{\frac{v}{u}, 1\}$, extended by $v \oslash 0 = 1$, for any $u, v \in [0,1],\ u \neq 0$.

- The **Łukasiewicz quantale** $[0,1]_\odot = ([0,1], \leq, \odot, 1)$, where $u \odot v = \max\{u + v - 1, 0\}$. Note that, unlike the Lawvere quantale, this quantale has the usual order. For all $u, v \in [0,1]$, $\hom(u,v) = \min\{1 - u + v, 1\} = 1 - \max\{u - v, 0\}$.

- Considering the lattice isomorphism $[0,1] \to [0,1], u \mapsto 1 - u$. we obtain an isomorphic presentation of the Łukasiewicz quantale as the quantale $[0,1]_\oplus = ([0,1], \geq, \oplus, 0)$, where the tensor is given by **truncated addition** $u \oplus v = \min\{u + v, 1\}$.

- Consider the set

$$\Delta = \{\varphi : [0,\infty] \to [0,1] \mid \text{ for all } \alpha \in [0,\infty] : \ \varphi(a) = \bigvee_{\beta < \alpha} \varphi(\beta)\},$$

  of **distribution functions**. Δ is a complete lattice with the pointwise order. Let

$$\varphi \otimes \psi(\alpha) = \bigvee_{\beta + \gamma \leq \alpha} \varphi(\beta) * \psi(\gamma).$$

  It is easy to see that $\otimes : \Delta \times \Delta \to \Delta$ is associative, commutative, commutes with joins and that

$$\kappa : [0,\infty] \to [0,1], \ \alpha \mapsto \begin{cases} 0 \text{ if } \alpha = 0, \\ 1 \text{ else} \end{cases}$$

  is a neutral element for $\otimes$.

3.2 V-Rel

For any sets X, Y, a V**-relation** $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ is a map $X \times Y \to V$.

Given two V-relations $r : X \longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\; Y,\ s : Y \longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\; Z$, their composite $s \cdot r : X \longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\; Z$ is given by

$$(s \cdot r)(x,z) = \bigvee_{y \in Y} r(x,y) \otimes s(y,z).$$

V-**Rel** is the category of sets and V-relations. Given any set X its identity morphism $1_X : X \longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\; X$ is the V-relation which sends all diagonal elements (x,x) to k and all other elements to the bottom element $\perp$ of V.

The order from V induces an order on the hom-sets V-$\mathbf{Rel}(X,Y)$,

$$r \leq r' \quad \Leftrightarrow \quad r(x,y) \leq r'(x,y), \text{ for all } x \in X,\, y \in Y$$

making V-**Rel** an **Ord**-enriched category. Furthermore, because the quantale V is complete, each hom-set of V-**Rel** is also complete. Using the fact that the tensor of the quantale V commutes with arbitrary joins we can easily show that V-relational composition preserves joins on either side:

$$(\bigvee_{i \in I} s_i) \cdot r = \bigvee_{i \in I} (s_i \cdot r) \qquad \text{and} \qquad s \cdot (\bigvee_{i \in I} r_i) = \bigvee_{i \in I} (s \cdot r_i).$$

This allows us to define the following:

Definition 3.2.1. Let $r : X \longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\; Y,\ s : Y \longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\; Z$ be V-relations. Since V-relational composition preserves joins on either side, the maps

$$(\ ) \cdot r : V\text{-}\mathbf{Rel}(Y,Z) \to V\text{-}\mathbf{Rel}(X,Z) \qquad \text{and} \qquad s \cdot (\ ) : V\text{-}\mathbf{Rel}(X,Y) \to V\text{-}\mathbf{Rel}(X,Z)$$

have right adjoints, which we will denote by $[r,-]$ and $\{s,-\}$, respectively. Given a V-relation $t : X \longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\; Z$, $[r,t]$ is called the **extension of** t **along** r and $\{s,t\}$ is called the **lifting of** t **along** s.

$$\begin{tikzcd} X \arrow[r, "t", "\shortmid"{marking}] \arrow[d, "r"', "\shortmid"{marking}] \arrow[dr, phantom, "\leq"] & Z \\ Y \arrow[ur, "{[r,t]}"', "\shortmid"{marking}] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "t", "\shortmid"{marking}] \arrow[dr, "{\{s,t\}}"', "\shortmid"{marking}] \arrow[dr, phantom, "\geq", shift left=3] & Z \\ & Y \arrow[u, "s"', "\shortmid"{marking}] \end{tikzcd}$$

The next proposition gives us an explicit way to compute extensions and liftings.

Proposition 3.2.2. *Let $r : X \longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\; Y,\ s : Y \longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\; Z,\ t : X \longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\; Z$ be V-relations. Then*

$$[r,t](y,z) = \bigwedge_{x' \in X} \hom(r(x',y), t(x',z)) \quad \textit{and} \quad \{s,t\}(x,y) = \bigwedge_{z' \in Z} \hom(s(y,z'), t(x,z')),$$

for any $x \in X,\ y \in Y,\ z \in Z$.

Proof. Let

$$s'(y,z) := \bigwedge_{x' \in X} \hom(r(x',y), t(x',z)),$$

for any $y \in Y$, $z \in Z$. To verify that $[r,t] = s'$ it suffices to show that $s \leq s' \Leftrightarrow s \cdot r \leq t$ for any $s : Y \relbar\joinrel\mapsto Z$. Note that, for any $s : Y \relbar\joinrel\mapsto Z$, we have

$$\begin{aligned}
s \leq s' &\Leftrightarrow s(y,z) \leq s'(y,z), && \text{for all } y \in Y,\ z \in Z \\
&\Leftrightarrow s(y,z) \leq \hom(r(x,y), t(x,z)), && \text{for all } x \in X,\ y \in Y,\ z \in Z \\
&\Leftrightarrow r(x,y) \otimes s(y,z) \leq t(x,z), && \text{for all } x \in X,\ y \in Y,\ z \in Z \\
&\Leftrightarrow (s \cdot r)(x,z) \leq t(x,z), && \text{for all } x \in X,\ z \in Z \\
&\Leftrightarrow s \cdot r \leq t
\end{aligned}$$

Analogously we can show the other equality. □

V-**Rel** also comes with an involution $(-)^\circ$ defined by

$$r^\circ(y,x) = r(x,y),$$

for any $x \in X$, $y \in Y$.

Each map $f : X \to Y$ induces the V-relation $f_\circ : X \relbar\joinrel\mapsto Y$

$$f_\circ(x,y) = \begin{cases} k & \text{if } f(x) = y \\ \perp & \text{otherwise.} \end{cases}$$

We can use this to define the functor $(-)_\circ : \textbf{Set} \to V\text{-}\textbf{Rel}$ which maps objects identically. This functor is faithful precisely when V is not trivial. When no confusion arises, we will denote $f_\circ$ by f.

Since V-**Rel** is an **Ord**-enriched category we can also study adjointness between V-relations.

For every map $f : X \to Y$, we have that

$$f_\circ \cdot f^\circ \leq 1_Y \qquad \text{and} \qquad 1_X \leq f^\circ \cdot f_\circ,$$

that is, $f_\circ \dashv f^\circ$ in the **Ord**-enriched category V-**Rel**.

3.3 V-Cat

A V**-category** $X = (X,a)$ is a set X equipped with a reflexive and transitive V-relation, called a **structure**, $a : X \relbar\joinrel\mapsto X$, that is, a relation such that

$$1_X \leq a \qquad \text{and} \qquad a \cdot a \leq a.$$

Equivalently

$$k \leq a(x,x) \qquad \text{and} \qquad a(x,y) \otimes a(y,z) \leq a(x,z)$$

holds for any $x,y,z \in X$. We will also denote $a(x,y)$ by $X(x,y)$ when no confusion arises.

A V**-functor** $f : (X,a) \to (Y,b)$ is a map $f : X \to Y$ satisfying

$$a(x,y) \leq b(f(x), f(y)),$$

for all $x, y \in X$.

V-functoriality can also be described in terms of V-relational composition as

$$a \leq f^{\circ} \cdot b \cdot f_{\circ},$$

or, diagrammatically, as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"', "|" marking] \arrow[dr, phantom, "\leq"] & Y \arrow[d, "b", "|" marking] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

The category of V-categories and V-functors will be denoted by V-**Cat**.

Examples 3.3.1.

- For $V = \mathbf{1}$, each set has only one possible structure and any map is trivially a V-functor. So **1-Cat** is (isomorphic to) **Set**.

- A **2**-category is a set equipped with a reflexive and transitive relation, that is, a (pre)ordered set. A **2**-functor is a monotone map. As such, **2-Cat** is the category **Ord** of (pre)orders.

- Consider the Lawvere quantale $[0,\infty]_+$:

  A $[0,\infty]_+$-category is a set X equipped with a structure $d : X \times X \to [0,\infty]$, such that

  $$0 \geq d(x,x) \qquad \text{and} \qquad d(x,y) + d(y,z) \geq d(x,z)$$

  holds for any $x, y, z \in X$. Note how these conditions parallel the axioms of a (quasi)metric space. These $[0,\infty]_+$-categories are the so called **Lawvere metric spaces**.

  The $[0,\infty]_+$-functors $f : (X,d) \to (Y,d')$ are the **non-expansive maps**, that is, the maps $f : X \to Y$ satisfying, for all $x, y \in X$,

  $$d(x,y) \geq d'(f(x), f(y)).$$

  $[0,\infty]_+$**-Cat** will be denoted by **Met**.

- $[0,1]_\oplus$**-Cat** is the full subcategory **BMet** of **Met** defined by the Lawvere metric spaces with a distance function bounded by 1. Note that any $[0,1]_\oplus$-category (X,a) can be regarded as a $[0,\infty]_+$-category since $a(x,y) \oplus a(y,z) \leq a(x,y) + a(y,z)$, for any $x, y, z \in X$. Moreover any map defining a $[0,1]_\oplus$-functor clearly defines a $[0,1]_+$-functor, since the orders of each quantale agree for any $x, y \in [0,1]$. Since $[0,1]_\oplus$ is isomorphic to the Łukasiewicz quantale $[0,1]_\odot$ we can also describe **BMet** as $[0,1]_\odot$**-Cat**.

- A Δ-category is a **probabilistic Lawvere metric space** and a Δ-functor is a **probabilistic non-expansive map.** These spaces were studied deeply in [22].

The following proposition is a very simple but fundamental observation:

Proposition 3.3.2. *A quantale V is itself a V-category when equipped with its internal hom.*

Proof. Note that $u \otimes k = u \leq u$. Therefore $k \leq \bigvee\{x \in V \mid u \otimes x \leq u\} = \hom(u,u)$.

Moreover

$$u \otimes \hom(u,u') \otimes \hom(u',u'') \leq u' \otimes \hom(u',u'') \leq u''$$

and it follows that $\hom(u,u') \otimes \hom(u',u'') \leq \hom(u,u'')$, as required. □

Perhaps not surprisingly, the V-category $(V,\hom)$ is very important in the study of V-**Cat**. The next propositions give us some elementary V-functorial interactions between V and a given V-category.

Proposition 3.3.3. *For any V-category (X,a) and $x \in X$ the map $a(x,-) : X \to V, y \mapsto a(x,y)$ is a V-functor.*

Proof. Note that, for any $y,z \in X$,

$$a(y,z) \leq \hom(a(x,y),a(x,z)) \quad \Leftrightarrow \quad a(x,y) \otimes a(y,z) \leq a(x,z),$$

which is true, by the transitivity axiom of a V-category structure. □

Given a V-category (X,a) its **opposite** V-category is defined by $(X,a)^{\mathrm{op}} = (X,a^\circ)$.

Proposition 3.3.4. *For any V-category (X,a) and $x \in X$ the map $a(-,x) : X^{\mathrm{op}} \to V, y \mapsto a(y,x)$ is a V-functor.*

Proof. Note that, for any $y,z \in X$,

$$a^\circ(y,z) \leq \hom(a(y,x),a(z,x)) \quad \Leftrightarrow \quad a(z,y) \otimes a(y,x) \leq a(z,x)$$

which is true, by the transitivity axiom of a V-category structure. □

Moreover, the order from V induces an order in each V-category (X,a) as follows

$$x \leq y \quad \Leftrightarrow \quad k \leq a(x,y) \quad \Leftrightarrow \quad a(y,z) \leq a(x,z), \text{ for all } z \in X.$$

Note that this order is respected by all V-functors.

Proposition 3.3.5. *Let $f : (X,a) \to (Y,b)$ be a V-functor. Then f is monotone.*

Proof. Note that, for any $x,y \in X$,

$$x \leq y \Leftrightarrow k \leq a(x,y)$$

and therefore

$$k \leq a(x,y) \leq b(f(x),f(y)) \Rightarrow f(x) \leq f(y).$$

□

Given V-categories $(X,a),\ (Y,b)$, we order the hom-set V-**Cat**(X,Y) pointwise using the order inherited from Y :

$$f \le g \Leftrightarrow f(x) \le g(x),\ \text{for all}\ x \in X,$$

for any V-functors $f,g : X \to Y$.

Given that all V-functors are monotone, equipping each hom-set of V-**Cat** with this order makes it an **Ord**-enriched category.

Note that this order is in general not antisymmetric. A V-category Y is said to be **separated** if, for $f,g\colon X \to Y$, $f = g$ whenever $f \simeq g$; equivalently, if, for all $x,y \in Y$, $x \simeq y$ implies $x = y$. We will also be interested in studying the full subcategory of V-**Cat** defined by the separated V-categories which we denote by V-$\mathbf{Cat}_{\mathrm{sep}}$.

Remark 3.3.6. We note that the V-category $(V,\hom)$ is separated since its induced order is the same as the order of the quantale V which is a complete lattice and therefore has an antisymmetric order.

3.3.1 V-Cat is a symmetric monoidal-closed category

As a category, a quantale V is a thin symmetric monoidal-closed category. The category V-**Cat** inherits a substantial amount of this structure. As we show next V-**Cat** is itself also a symmetric monoidal-closed category.

Given two V-categories $(X,a),\ (Y,b)$, we define its **tensor product** $X \otimes Y$ as the V-category with the underlying set $X \times Y$ and the structure

$$(X \otimes Y)((x,y),(x',y')) := X(x,x') \otimes Y(y,y') = a(x,x') \otimes b(y,y'),$$

for any $(x,y),(x',y') \in X \times Y$. Note that $X \otimes Y$ is indeed a V-category since

$$k = k \otimes k \le a(x,x) \otimes b(y,y) = (X \otimes Y)((x,y),(x,y))$$

and, making repeated use of the commutativity and associativity of the tensor of V,

$$\begin{aligned}(X \otimes Y)((x,y),(x',y')) \otimes (X \otimes Y)((x',y'),(x'',y'')) &= (a(x,x') \otimes b(y,y')) \otimes (a(x',x'') \otimes b(y',y'')) \\ &= (a(x,x') \otimes a(x',x'')) \otimes (b(y,y') \otimes b(y',y'')) \\ &\le a(x,x'') \otimes b(y,y'') \\ &= (X \otimes Y)((x,y),(x'',y''))\end{aligned}$$

for any $x,x',x'' \in X$ and $y,y',y'' \in Y$.

Moreover, given any two V-functors $f : (X,a) \to (Y,b),\ g : (Z,c) \to (W,d)$, the map $f \times g : X \times Z \to Y \times W$ defines a V-functor $f \otimes g : X \otimes Z \to Y \otimes W$. Indeed, note that, by V-functoriality of f

and g,

$$\begin{aligned}(X\otimes Z)((x,z),(x',z')) &= a(x,x')\otimes c(z,z')\\ &\leq b(f(x),f(x'))\otimes d(g(z),g(z'))\\ &= (Y\otimes W)((f\otimes g)(x,z),(f\otimes g)(x',z'))\end{aligned}$$

for any $x,x'\in X,\ z,z'\in Z$.

Combining the two previous observations we can define the **tensor product functor**

$$(-)\otimes(-): V\text{-}\mathbf{Cat}\times V\text{-}\mathbf{Cat}\to V\text{-}\mathbf{Cat}.$$

The V-category $E=(\{*\},k)$, where k denotes the structure

$$k(*,*)=k,$$

is the **tensor unit** of V-**Cat**. It is simple to verify that the tensor unit V-category E together with the tensor product functor make V-**Cat** a symmetric monoidal category.

Furthermore, given two V-categories (X,a), (Y,b), their hom-set V-$\mathbf{Cat}(X,Y)$ is again a V-category when equipped with the structure

$$V\text{-}\mathbf{Cat}(f,g)=[\![f,g]\!]=\bigwedge_{x\in X} b(f(x),g(x)),$$

for all V-functors $f,g: X\to Y$. One readily verifies that $[\![-,-]\!]$ indeed defines a structure since

$$\begin{aligned}&k\leq b(f(x),f(x)),\ \text{for all } x\in X\\ \Leftrightarrow\ &k\leq \bigwedge_{x\in X} b(f(x),f(x))=[\![f,f]\!].\end{aligned}$$

Moreover

$$\begin{aligned}&b(f(x),g(x))\otimes b(g(x),h(x))\leq b(f(x),h(x)),\ \text{for all } x\in X\\ \Rightarrow\ &\bigwedge_{x\in X}\big(b(f(x),g(x))\otimes b(g(x),h(x))\big)\leq \bigwedge_{x\in X} b(f(x),h(x))=[\![f,h]\!]\end{aligned}$$

and

$$\begin{aligned}&[\![f,g]\!]\otimes[\![g,h]\!]\leq b(f(x),g(x))\otimes b(g(x'),h(x')),\ \text{for all } x,x'\in X\\ \Rightarrow\ &[\![f,g]\!]\otimes[\![g,h]\!]\leq \bigwedge_{x,x'\in X}\big(b(f(x),g(x))\otimes b(g(x'),h(x'))\big)\leq \bigwedge_{x\in X}\big(b(f(x),g(x))\otimes b(g(x),h(x))\big).\end{aligned}$$

Therefore $[\![f,g]\!]\otimes[\![g,h]\!]\leq[\![f,h]\!]$, as required.

We will also denote the V-category V-$\mathbf{Cat}(X,Y)$ by $(Y,b)^{(X,a)}$ or simply Y^X, when no confusion arises.

Finally we show that we have the adjunction $(-\otimes Y) \dashv V\text{-}\mathbf{Cat}(Y,-)$ by providing the following natural isomorphism

$$\begin{aligned} \varphi : V\text{-}\mathbf{Cat}(X\otimes Y,Z) &\to V\text{-}\mathbf{Cat}(X,V\text{-}\mathbf{Cat}(Y,Z)) \\ f &\mapsto (\varphi f : x \mapsto (\varphi f(x) : y \mapsto f(x,y))) \end{aligned}$$

for any V-categories $(X,a),(Y,b)$ and (Z,c). We will skip the proof of naturality because it is completely straightforward and merely show that φ is an isomorphism. First we show that φ is well defined. Consider a V-functor $f : X\otimes Y \to Z$. Then $\varphi f(x) : Y \to Z$ is a V-functor since:

$$b(y,y') \le a(x,x)\otimes b(y,y') = (X\otimes Y)((x,y),(x,y')) \le c(f(x,y),f(x,y'))$$

for any $x \in X$, $y,y' \in Y$. Similarly one can show that $\varphi f : X \to V$-$\mathbf{Cat}(Y,Z)$ is also a V-functor. Therefore φ is indeed well defined.

Moreover φ is also a V-functor since

$$\begin{aligned} V\text{-}\mathbf{Cat}(X\otimes Y,Z)(f,f') &= \bigwedge_{(x,y)\in X\times Y} c(f(x,y),f'(x,y)) \\ &= \bigwedge_{x\in X}\bigwedge_{y\in Y} c(f(x,y),f'(x,y)) \\ &= \bigwedge_{x\in X}\bigwedge_{y\in Y} c(\varphi f(x)(y),\varphi f'(x)(y)) \\ &= \bigwedge_{x\in X} V\text{-}\mathbf{Cat}(Y,Z)(\varphi f(x),\varphi f'(x)) \\ &= V\text{-}\mathbf{Cat}(X,V\text{-}\mathbf{Cat}(Y,Z))(\varphi f,\varphi f') \end{aligned}$$

for any V-functors $f,f' : X\otimes Y \to Z$. We also conclude that φ is injective. Lastly, we will show that φ is surjective. Let $g : X \to V$-$\mathbf{Cat}(Y,Z)$ be a V-functor. Then clearly the map $f : X\otimes Y \to Z$ given by $f(x,y) = g(x)(y)$ is such that $\varphi f = g$. We verify that f is also a V-functor:

$$\begin{aligned} (X\otimes Y)((x,y),(x',y')) &= a(x,x')\otimes b(y,y') \\ &\le V\text{-}\mathbf{Cat}(Y,Z)(g(x),g(x'))\otimes b(y,y') \\ &= \bigwedge_{y''\in Y} \big(c(g(x)(y''),g(x')(y''))\big)\otimes b(y,y') \\ &\le c(g(x)(y),g(x')(y))\otimes b(y,y') \\ &\le c(g(x)(y),g(x')(y))\otimes c(g(x')(y),g(x')(y')) \\ &\le c(g(x)(y),g(x')(y')) = c(f(x,y),f(x',y')) \end{aligned}$$

for any $x,x' \in X$, $y,y' \in Y$. We conclude that V-**Cat** is indeed a symmetric monoidal-closed category.

3.3.2 V-Cat is a topological category

Proposition 3.3.7. *Given a set X, the collection of V-categorical structures on X, that is the reflexive and transitive V-relations in V-*$\mathbf{Rel}(X,X)$*, forms a complete lattice, with the meet operation defined like in V-*$\mathbf{Rel}(X,X)$*. The bottom element it given by the* ***discrete structure*** *1_X, and the top element is given by the* ***chaotic structure*** *$\top_X$, defined by $\top_X(x,y)=\top$, for any $x,y\in X$.*

Proof. It's immediate that 1_X and $\top_X$ are the bottom and top elements, respectively. Let $a_i, i\in I$, be V-categorical structures on X. We have

$$1_X \leq \bigwedge_{i\in I} a_i$$

and

$$\bigwedge_{i\in I} a_i \cdot \bigwedge_{i\in I} a_i \leq \bigwedge_{i\in I} a_i \cdot a_i \leq \bigwedge_{i\in I} a_i.$$

Therefore the reflexive and transitive relations in V-$\mathbf{Rel}(X,X)$ form a complete lattice. □

Moreover, given a V-relation $a : X \relbar\joinrel\mapstochar\joinrel\rightarrow X$ there is a least V-categorical structure $\overline{a}$ on X such that $a\leq\overline{a}$. This V-relation can be calculated directly:

$$\overline{a} = \bigvee_{n\in\mathbb{N}} a^n,$$

where $a^0 = 1_X$ and $a^{n+1} = a^n \cdot a$, for all $n\in\mathbb{N}$.

Proposition 3.3.8. *Let $f : X \to Y$ be a* $\mathbf{Set}$ *map. For any V-categorical structure b on Y there is a largest V-categorical structure a on X that makes $f : (X,a)\to(Y,b)$ a V-functor. Moreover*

$$a = f^\circ \cdot b \cdot f_\circ$$

and, for any V-category (Z,c), a $\mathbf{Set}$ *map $h : Z\to X$ is a V-functor $(Z,c)\to(X,a)$ precisely when $f\cdot h : (Z,c)\to(Y,b)$ is a V-functor.*

Proof. Note that

$$1_X \leq f^\circ \cdot f_\circ = f^\circ \cdot 1_Y \cdot f_\circ \leq f^\circ \cdot b \cdot f_\circ$$

and

$$(f^\circ \cdot b \cdot f_\circ)\cdot(f^\circ \cdot b \cdot f_\circ) = f^\circ \cdot b \cdot (f_\circ \cdot f^\circ)\cdot b \cdot f_\circ \leq f^\circ \cdot b \cdot b \cdot f_\circ \leq f^\circ \cdot b \cdot f_\circ.$$

Therefore $f^\circ\cdot b\cdot f_\circ$ is a V-categorical structure on X. Since V-functoriality of the **Set** map $f : X\to Y$ between V-categories (X,a) and (Y,b) is characterized by the inequality $a\leq f^\circ\cdot b\cdot f_\circ$, it follows immediately that $f^\circ\cdot b\cdot f_\circ$ is indeed the largest structure on X that makes f a V-functor.

Moreover, $f\cdot h$ is a V-functor precisely when

$$c \leq (f\cdot h)^\circ \cdot b \cdot (f\cdot h)_\circ = h^\circ \cdot (f^\circ \cdot b \cdot f_\circ)\cdot h_\circ,$$

that is, when h is a V-functor. □

From the two previous propositions it is now easy to obtain the following result:

Proposition 3.3.9. *Let* (Y_i, b_i) *be* V*-categories and* $f_i : X \to Y_i$, $i \in I$ *be a family of* **Set** *maps. Then there is a largest* V*-categorical structure* a *in* X *such that* $f_i : (X,a) \to (Y_i, b_i)$, $i \in I$, *is a family of* V*-functors. This structure is called the* ***initial structure with respect to*** $(f_i)_{i\in I}$ ***at*** $(b_i)_{i\in I}$, *and the family of* V*-functors is called the* ***initial lifting with respect to*** $(f_i)_{i\in I}$ ***at*** $(b_i)_{i\in I}$. *Moreover*

$$a = \bigwedge_{i\in I} f_i^\circ \cdot b_i \cdot (f_i)_\circ$$

and, for any V*-category* (Z,c), *a* **Set** *map* $h : Z \to X$ *is a* V*-functor* $h : (Z,c) \to (X,a)$ *precisely when every* $f_i \cdot h : (Z,c) \to (Y_i, b_i)$, $i \in I$, *is a* V*-functor.*

This corollary gives us, in particular, that the forgetful functor $\mathcal{U} : V$-**Cat** $\to$ **Set**, which maps a V-category to its underlying set, is *topological over* **Set**.

Definition 3.3.10. A faithful functor $U : \mathbf{C} \to \mathbf{D}$ is said to be **topological over D** when, given objects $X \in \mathbf{D}$ and $Y_i' \in \mathbf{C}$ and morphisms $f_i : X \to UY_i'$, $i \in I$, there exists an **initial lifting**, that is:

- an object $X' \in \mathbf{C}$ such that $UX' = X$, and morphisms $f_i' : X' \to Y_i'$ in **C** such that $Uf_i' = f_i$, satisfying the following condition:
- for any object $Z' \in \mathbf{C}$, morphisms $h : UZ' \to X$ in **D** and $g_i' : Z' \to Y_i'$ in **C**, if $h \cdot f_i = Ug_i'$, then there exists a morphism $h' : Z' \to X'$ in **C** such that $Uh' = h$ and $h' \cdot f_i' = g_i'$.

In such a case, we also say that the category **C** is **topological over D**.

Clearly, from the Corollary 3.3.9, we see that $\mathcal{U} : V$-**Cat** $\to$ **Set** is indeed topological over **Set** and therefore V-**Cat** is topological over **Set**.

Therefore, we conclude immediately that V-**Cat** admits *final structures* and *final liftings*:

Corollary 3.3.11. *Let* (X_i, a_i) *be* V*-categories and* $f_i : X_i \to Y$, $i \in I$ *be a family of* **Set** *maps. Then there is a least* V*-categorical structure* b *in* Y *such that* $f_i : (X_i, a_i) \to (Y, b)$, $i \in I$, *is a family of* V*-functors. This structure is called the* ***final structure with respect to*** $(f_i)_{i\in I}$ ***at*** $(a_i)_{i\in I}$ *and the family of* V*-functors is called the* ***final lifting with respect to*** $(f_i)_{i\in I}$ ***at*** $(a_i)_{i\in I}$. *Moreover*

$$b = \bigvee_{i\in I} \overline{(f_i)_\circ \cdot a_i \cdot f_i^\circ},$$

and, for any V*-category* (Z,c), *a* **Set** *map* $g : Y \to Z$ *is a* V*-functor* $g : (Y,b) \to (Z,c)$ *precisely when every* $g \cdot f_i : (X_i, a_i) \to (Z,c)$, $i \in I$ *is a* V*-functor.*

Considering an empty family of **Set** maps in the propositions 3.3.9 and 3.3.11 we get immediately the following corollary:

Corollary 3.3.12. *The forgetful functor* $\mathcal{U} : V$-**Cat** $\to$ **Set** *has a left adjoint (which equips every set* X *with its discrete structure* 1_X*), and has a right adjoint (which equips every set* X *with its chaotic structure* $\top_X$*).*

Since V-**Cat** is topological over **Set** it inherits many useful properties from **Set**. In particular, we have that V-**Cat** is complete and cocomplete since **Set** is so.

Proposition 3.3.13. *The category V-**Cat** is complete and cocomplete. Limits in V-**Cat** are obtained as initial liftings with respect to the underlying limit cone in **Set**, and colimits in V-**Cat** are obtainted as final lifting with respect to the underlying colimit cocone in **Set**.*

Therefore, we can compute explicitly products, coproducts, equalisers, coequalisers, etc. in V-**Cat** as these are obtained like in **Set** and then equipped with the corresponding initial/final structures.

Another property that V-**Cat** inherits from **Set** of special interest to us is the (regular epi, mono) factorisation system in V-**Cat** which can be obtained from the usual (surjective, injective) factorisation system in **Set**. Given a V-functor $f : (X,a) \to (Y,b)$, its underlying map $f : X \to Y$ admits the usual (surjective, injective) factorisation in **Set**, given by

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\overline{f}"'] & & Y \\ & f(X) \arrow[ur, hook] & \end{tikzcd},$$

where $\overline{f} : X \to f(X)$ is given by f. Equipping $f(X)$ with the adequate final structure, we obtain a (regular epi, mono) factorisation for the V-functor f in V-**Cat**.

Proposition 3.3.14. *V-**Cat** admits a (regular epi, mono) orthogonal factorisation system.*

For a proof of the previous result we point the reader to (Theorem 21.16, [1]) of the classic book *Abstract and concrete categories*.

3.4 V-Dist

We will make substantial use of the category V-**Dist** of V-categories and V-distributors, which we define and study next.

Given two V-categories (X,a), (Y,b), a V**-distributor** $\varphi : X \multimap\!\!\to Y$ is a V-relation $\varphi : X \relbar\!\!\!\!\to Y$ satisfying

$$b \cdot \varphi \cdot a \leq \varphi.$$

Equivalently

$$a(x',x) \otimes \varphi(x,y) \otimes b(y,y') \leq \varphi(x',y'),$$

holds for any $x, x' \in X,\ y, y' \in Y$.

Under relational composition, V-categories and V-distributors form the category V-**Dist**, where the identity morphism for each V-category is its structure. Moreover a V-distributor $\varphi : X \multimap\!\!\to Y$ can also be characterised as a V-relation $\varphi : X \relbar\!\!\!\!\to Y$ satisfying

$$\varphi \cdot a = \varphi = b \cdot \varphi.$$

In sum, we have the following proposition:

Proposition 3.4.1. *Let* (X,a), (Y,b), (Z,c) *be* V*-categories and* $\varphi : X \,{\circ\!\!\!\longrightarrow}\, Y$, $\psi : Y \,{\circ\!\!\!\longrightarrow}\, Z$ *be* V*-distributors. Then*

(1) $\psi \cdot \varphi : X \,{\circ\!\!\!\longrightarrow}\, Z$ *is a* V*-distributor;*

(2) $a : X \,{\circ\!\!\!\longrightarrow}\, X$ *is a* V*-distributor and* $\varphi \cdot a = \varphi = b \cdot \varphi$.

Proof. (1) Note that $c \cdot (\psi \cdot \varphi) \cdot a = (c \cdot \psi \cdot 1_Y) \cdot (1_Y \cdot \varphi \cdot a) \leq (c \cdot \psi \cdot b) \cdot (b \cdot \varphi \cdot a) \leq \psi \cdot \varphi$.

(2) $a \cdot a \leq a$ gives us that $a \cdot a \cdot a \leq a$. Moreover

$$\varphi = \varphi \cdot 1_X \leq \varphi \cdot a = 1_Y \cdot \varphi \cdot a \leq b \cdot \varphi \cdot a \leq \varphi.$$

Therefore $\varphi \cdot a = \varphi$. Analogously one can show that $b \cdot \varphi = \varphi$. $\square$

Moreover, V-**Dist** inherits the **Ord**-enriched categorical structure of V-**Rel**. Like the hom-sets of V-**Rel**, the hom-sets of V-**Dist** are also complete:

Proposition 3.4.2. *Let* (X,a), (Y,b) *be* V*-categories and* $\varphi_i : X \,{\circ\!\!\!\longrightarrow}\, Y$, $i \in I$, *be* V*-distributors. Then* $\bigvee_{i \in I} \varphi_i : X \,{\circ\!\!\!\longrightarrow}\, Y$ *is a* V*-distributor.*

Proof. Note that $b \cdot (\bigvee_{i\in I} \varphi_i) \cdot a = \bigvee_{i \in I} (b \cdot \varphi_i \cdot a) \leq \bigvee_{i \in I} \varphi_i$. $\square$

3.4.1 V-functors as V-distributors and the Yoneda lemma

Proposition 3.4.3. *Every* V*-functor* $f : (X,a) \to (Y,b)$ *induces a pair of* V*-distributors*

$$f_* := b \cdot f_\circ : X \,{\circ\!\!\!\longrightarrow}\, Y \qquad \textit{and} \qquad f^* := f^\circ \cdot b : Y \,{\circ\!\!\!\longrightarrow}\, X,$$

that is, $f_*(x,y) = b(f(x),y)$ *and* $f^*(y,x) = b(y, f(x))$, *for all* $x \in X$ *and* $y \in Y$.

Proof. Note that, for all $x,x' \in X$ and $y,y' \in Y$,

$$a(x,x') \otimes f_*(x',y) = a(x,x') \otimes b(f(x'),y) \leq b(f(x),f(x')) \otimes b(f(x'),y) \leq b(f(x),y) = f_*(x,y),$$

that is $f_* \cdot a \leq f_*$. Moreover

$$f_*(x,y') \otimes b(y',y) = b(f(x),y') \otimes b(y',y) \leq b(f(x),y) = f_*(x,y),$$

that is $b \cdot f_* \leq f_*$. Therefore $b \cdot f_* \cdot a \leq f_*$, as required. Similarly, f^* is also a V-distributor. $\square$

The next proposition will establish that we can define **Ord**-functors

$$(-)_* : V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Dist}^{\mathrm{co}} \quad \text{and} \quad (-)^* : V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Dist}^{\mathrm{op}},$$

which map objects identically.

Proposition 3.4.4. *Let $f, f' : (X,a) \to (Y,b)$ and $g : (Y,b) \to (Z,c)$ be V-functors. Then*

(1) $(1_X)_* = a = (1_X)^*$;

(2) $(g \cdot f)_* = g_* \cdot f_*$ *and* $(g \cdot f)^* = f^* \cdot g^*$;

(3) $f \leq f' \quad \Leftrightarrow \quad (f')_* \leq f_* \quad \Leftrightarrow \quad f^* \leq (f')^*$.

Proof. (1) Trivial.

(2) $g_* \cdot f_* = (c \cdot g_\circ) \cdot (b \cdot f_\circ) = (c \cdot g_\circ \cdot b) \cdot f_\circ = (c \cdot g_\circ) \cdot f_\circ = c \cdot (g_\circ \cdot f_\circ) = c \cdot (g \cdot f)_\circ = (g \cdot f)_*$. Analogously one obtains that $(g \cdot f)^* = f^* \cdot g^*$.

(3) $f \leq f' \Leftrightarrow b \cdot (f')_\circ \leq b \cdot f_\circ \Leftrightarrow (f')_* \leq f_*$. On the other hand

$$(f')_* \leq f_* \quad \Rightarrow \quad k \leq b(f'(x), f'(x)) \leq b(f(x), f'(x)) \quad \Rightarrow \quad f(x) \leq f'(x),$$

for any $x \in X$. Similarly we can show that $f^* \leq (f')^*$. □

Moreover the following diagram commutes:

$$\begin{tikzcd}
V\text{-}\mathbf{Cat} \arrow[r, "(-)^{\mathrm{op}}", "\cong"'] \arrow[d, "(-)^*"'] & V\text{-}\mathbf{Cat}^{\mathrm{co}} \arrow[d, "(-)^{\mathrm{co}}_*"] \\
V\text{-}\mathbf{Dist}^{\mathrm{op}} \arrow[r, "(-)^{\mathrm{op}}", "\cong"'] & V\text{-}\mathbf{Dist}
\end{tikzcd}$$

Note that, for any V-functor $f \colon (X,a) \to (Y,b)$,

$$f_* \cdot f^* = b \cdot f_\circ \cdot f^\circ \cdot b \leq b \cdot b = b \text{ and } f^* \cdot f_* = f^\circ \cdot b \cdot b \cdot f_\circ = f^\circ \cdot b \cdot f_\circ \geq a;$$

hence every V-functor induces a pair of adjoint V-distributors, $f_* \dashv f^*$. A V-functor $f \colon X \to Y$ is said to be **fully faithful** if $f^* \cdot f_* = a$, that is, $a(x,x') = b(f(x), f(x'))$ for all $x, x' \in X$, while it is **fully dense** if $f_* \cdot f^* = b$, that is,

$$b(y,y') = \bigvee_{x \in X} b(y, f(x)) \otimes b(f(x), y'),$$

for all $y, y' \in Y$. Given separated V-categories X and Y, a fully faithful V-functor $f \colon X \to Y$ is always an injective map, but it may not be so in general.

The following proposition gives us a deep relation between V-**Cat** and V-**Dist** and will be very useful.

Theorem 3.4.5. *Given two V-categories (X,a), (Y,b) and a V-relation $\varphi : X {\longrightarrow\!\!\!\!\!\!\mapsto} Y$ the following assertions are equivalent:*

(i) $\varphi : (X,a) {-\!\!\circ\!\!\rightarrow} (Y,b)$ *is a V-distributor;*

(ii) $\varphi : (X,a)^{\mathrm{op}} \otimes (Y,b) \to (V, \hom)$ *is a V-functor.*

Proof. (i) $\Rightarrow$ (ii): Let $x,x' \in X$ and $y,y' \in Y$, then

$$\begin{aligned}\varphi(x,y)\otimes a^{\circ}(x,x')\otimes b(y,y') &= a(x',x)\otimes\varphi(x,y)\otimes b(y,y')\\ &\leq \varphi(x',y'),\end{aligned}$$

therefore $a^{\circ}(x,x')\otimes b(y,y') \leq \hom(\varphi(x,y),\varphi(x',y'))$.

(ii) $\Rightarrow$ (i): Let $x,x' \in X$ and $y,y' \in Y$,

$$\begin{aligned}a(x,x')\otimes\varphi(x',y) &= \varphi(x',y)\otimes a^{\circ}(x',x)\\ &= \varphi(x',y)\otimes a^{\circ}(x',x)\otimes k\\ &\leq \varphi(x',y)\otimes a^{\circ}(x',x)\otimes b(y,y)\\ &\leq \varphi(x,y), \quad \text{(because } a^{\circ}(x',x)\otimes b(y,y)\leq\hom(\varphi(x',y),\varphi(x,y)),\end{aligned}$$

that is $\varphi\cdot a \leq \varphi$. Similarly one can prove that $b\cdot\varphi\leq\varphi$. It follows that $b\cdot\varphi\cdot a\leq b\cdot\varphi\leq\varphi$. $\square$

The previous theorem allows us to define a V-category structure on V-$\mathbf{Dist}(X,Y)$ by using the structure of V-$\mathbf{Cat}(X^{\mathrm{op}}\otimes Y,V)$. Note that, in particular

$$V\text{-}\mathbf{Dist}(X,E)\cong V\text{-}\mathbf{Cat}(X^{\mathrm{op}}\otimes E,V)\cong V\text{-}\mathbf{Cat}(X^{\mathrm{op}},V)=V^{X^{\mathrm{op}}}.$$

The V-categorical structure a of (X,a) is a V-distributor $a\colon (X,a)\,{-\!\!\circ\!\!\rightarrow}\,(X,a)$, and therefore a V-functor $a\colon (X,a)^{\mathrm{op}}\otimes(X,a)\to(V,\hom)$, which induces, via the closed monoidal structure of V-**Cat**, the **Yoneda V-functor**

$$\mathrm{y}_X\colon (X,a)\to(V,\hom)^{(X,a)^{\mathrm{op}}}$$

Moreover, $(V,\hom)^{(X,a)^{\mathrm{op}}}$ can be equivalently described as:

$$PX := \{\varphi\colon X\,{-\!\!\circ\!\!\rightarrow}\,E \mid \varphi\ V\text{-distributor}\}.$$

Then the structure on PX is given by

$$PX(\varphi,\psi)=[\![\varphi,\psi]\!]=\bigwedge_{x\in X}\hom(\varphi(x),\psi(x)),$$

for every $\varphi,\psi\colon X\,{-\!\!\circ\!\!\rightarrow}\,E$, where by $\varphi(x)$ we mean $\varphi(x,*)$, or, equivalently, we consider the associated V-functor $\varphi\colon X\to V$. The Yoneda functor $\mathrm{y}_X\colon X\to PX$ assigns to each $x\in X$ the V-distributor $x^*\colon X\,{-\!\!\circ\!\!\rightarrow}\,E$, where we identify again $x\in X$ with the V-functor $x\colon E\to X$ assigning the (unique) element of E to x.

Theorem 3.4.6 (Yoneda lemma). *Let (X,a) be a V-category. Then for every $\varphi \in PX$ and $x \in X$, we have that*

$$[\![y_X(x), \varphi]\!] = \varphi(x).$$

Proof. Note that

$$\begin{aligned}
[\![y_X(x), \varphi]\!] &= \bigwedge_{y \in X} \hom(y_X(x)(y), \varphi(y)) \\
&= \bigwedge_{y \in X} \hom(a(y,x), \varphi(y)) \\
&\leq \hom(a(x,x), \varphi(x)) \\
&\leq \hom(k, \varphi(x)) \qquad (\text{since } k \leq a(x,x)) \\
&= \varphi(x).
\end{aligned}$$

On the other hand

$$a(y,x) \leq \hom(\varphi(x), \varphi(y)) \quad \Leftrightarrow \quad \varphi(x) \otimes a(y,x) \leq \varphi(y) \quad \Leftrightarrow \quad \varphi(x) \leq \hom(a(y,x), \varphi(y)),$$

for any $y \in X$. It follows that $\varphi(x) \leq [\![y_X(x), \varphi]\!]$. □

Corollary 3.4.7. *Let (X,a) be a V-category. Then y_X is a fully faithful V-functor.*

Proof. By the Yoneda Lemma it is immediate that $[\![y_X(x), y_X(y)]\!] = y_X(y)(x) = a(x,y)$. □

3.4.2 V-distributors and adjunctions in V-Cat

In V-**Cat** adjointness between V-functors

$$\begin{tikzcd} (Y,b) \arrow[r, bend left, "g"] \arrow[r, phantom, "\top"] & (X,a) \arrow[l, bend left, "f"] \end{tikzcd}$$

can be readily characterised in terms of V-distributors.

Proposition 3.4.8. *Let $f : (X,a) \to (Y,b)$, $g : (Y,b) \to (X,a)$ be V-functors. Then the following conditions are equivalent:*

(i) $f \dashv g$, *that is,* $1_X \leq g \cdot f$ *and* $f \cdot g \leq 1_Y$;

(ii) $f_* = g^*$;

(iii) $g^* \dashv f^*$;

(iv) $(\forall x \in X)\ (\forall y \in Y) \quad a(x, g(y)) = b(f(x), y)$.

Proof. (i) $\Rightarrow$ (ii) Let $x \in X$ and $y \in Y$. We have $b(f(x),y) \leq a(g(f(x)),g(y))$ by V-functoriality of g. Using the first inequality in our hypothesis we have $k \leq a(x,g(f(x)))$. It follows that

$$f_*(x,y) = b(f(x),y) \leq a(x,g(f(x))) \otimes a(g(f(x)),g(y)) \leq a(x,g(y)) = g^*(x,y).$$

Analogously one obtains that $g^* \leq f_*$.

(ii) $\Leftrightarrow$ (iii) Simply note that $f_* \dashv f^*$, and that left adjoints are unique in V-**Dist**.

(ii) $\Leftrightarrow$ (iv) Trivial.

(iv) $\Rightarrow$ (i) Clearly for every $x \in X,\ y \in Y$ we have

$$k \leq b(f(x),f(x)) = a(x,g(f(x))) \quad \text{and} \quad k \leq a(g(y),g(y)) = b(f(g(y)),y)$$

that is, $x \leq g(f(x))$ and $y \leq f(g(y))$. $\square$

In fact the latter condition also encodes V-functoriality of f and g:

Proposition 3.4.9. *Let (X,a), (Y,b) be V-categories and $f\colon X \to Y$, $g\colon Y \to X$ be maps satisfying the condition*

$$(\forall x \in X)\ (\forall y \in Y) \quad a(x,g(y)) = b(f(x),y).$$

Then f and g are V-functors.

Proof. Note that

$$\begin{aligned} a = 1_X \cdot a \leq (f^\circ \cdot f_\circ) \cdot a = (f^\circ \cdot 1_Y \cdot f_\circ) \cdot a \\ \leq f^\circ \cdot (b \cdot f_\circ) \cdot a = f^\circ \cdot (g^\circ \cdot a) \cdot a = f^\circ \cdot (g^\circ \cdot a) = f^\circ \cdot (b \cdot f_\circ) \end{aligned}$$

and therefore f is a V-functor. Analogously one can show that g is a V-functor. $\square$

Furthermore, we have the following result:

Proposition 3.4.10. *Let (X,a), (Y,b) be V-categories and $f: X \to Y$ be a map. Then, whenever f_* is a V-distributor (or whenever f^* is a V-distributor) f is a V-functor.*

Proof. Assume that f_* is a V-distributor. Then

$$\begin{aligned} a = 1_X \cdot a \leq (f^\circ \cdot f_\circ) \cdot a = (f^\circ \cdot 1_Y \cdot f_\circ) \cdot a \\ \leq (f^\circ \cdot b \cdot f_\circ) \cdot a = f^\circ \cdot (b \cdot f_\circ) \cdot a = f^\circ \cdot f_* \cdot a = f^\circ \cdot f_* = f^\circ \cdot (b \cdot f_\circ) \end{aligned}$$

and therefore f is a V-functor. Similarly one can show that f is a V-functor whenever f^* is a V-distributor. $\square$

3.4.3 Weighted colimits and cocompleteness

The extensions and liftings we defined in V-**Rel** behave well in V-**Dist** :

Proposition 3.4.11. *Let $\varphi : (X,a) \mathrel{-\!\!\circ\!\!\rightarrow} (Y,b)$, $\chi : (Y,b) \mathrel{-\!\!\circ\!\!\rightarrow} (Z,c)$, $\psi : (X,a) \mathrel{-\!\!\circ\!\!\rightarrow} (Z,c)$ be V-distributors. Then the extension $[\varphi, \psi]$ and the lifting $\{\chi, \psi\}$ are V-distributors.*

Proof. Since φ is a V-distributor we have that, for any $x \in X,\ y, y' \in Y,\ z \in Z$,

$$\varphi(x,y') \otimes b(y',y) \otimes \hom(\varphi(x,y), \psi(x,z)) \leq \varphi(x,y) \otimes \hom(\varphi(x,y), \psi(x,z)) \leq \psi(x,z).$$

Therefore

$$b(y',y) \otimes [\varphi, \psi](y,z) \leq b(y',y) \otimes \hom(\varphi(x,y), \psi(x,z)) \leq \hom(\varphi(x,y'), \psi(x,z)).$$

Now it follows that $b(y',y) \otimes [\varphi, \psi](y,z) \leq [\varphi, \psi](y',z)$, that is, $[\varphi, \psi] \cdot b \leq [\varphi, \psi]$. Using the fact that ψ is a V-distributor one can show similarly that $c \cdot [\varphi, \psi] \leq [\varphi, \psi]$. It follows that $[\varphi, \psi]$ is a V-distributor. The proof for liftings is analogous. $\square$

Definitions 3.4.12.

1. Given a V-functor $f : X \to Z$ and a V-distributor (here called **weight**) $\varphi : X \mathrel{-\!\!\circ\!\!\rightarrow} Y$, a **$\varphi$-weighted colimit** of f (or simply a *φ-colimit* of f), whenever it exists, is a V-functor $g : Y \to Z$ such that $g_* = [\varphi, f_*]$. One says then that g **represents** $[\varphi, f_*]$.
2. A V-category Z is called **φ-cocomplete** if it has a colimit for each weighted diagram with weight $\varphi : (X,a) \mathrel{-\!\!\circ\!\!\rightarrow} (Y,b)$; i.e. for each V-functor $f : X \to Z$, the φ-colimit of f exists.
3. Given a class Φ of V-distributors, a V-category Z is called **Φ-cocomplete** if it is φ-cocomplete for every $\varphi \in \Phi$. When $\Phi = V$-**Dist**, then Z is said to be **cocomplete**.

$$\begin{tikzcd} X \arrow[r, "f_*"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\leq"] & Z \\ Y \arrow[ur, "{g_*=[\varphi,f_*]}"'] & \end{tikzcd}$$

The next proposition gives us a way to prove that a V-category X is Φ-cocomplete by showing the existence of some special weighted colimits.

Proposition 3.4.13. *Let $\varphi : X \mathrel{-\!\!\circ\!\!\rightarrow} Y$ be a V-distributor and $f : X \to Z$ a V-functor. The following assertions are equivalent:*

(i) *the φ-colimit of f exists;*

(ii) *the $(\varphi \cdot f^*)$-colimit of 1_Z exists;*

(iii) *for each $y \in Y$, the $(y^* \cdot \varphi)$-colimit of f exists.*

Proof. (i) $\Leftrightarrow$ (ii): Direct calculation shows that

$$[\varphi, f_*] = [\varphi \cdot f^*, (1_Z)_*].$$

(i) $\Leftrightarrow$ (iii): Since $[\varphi, f_*]$ is defined pointwise, it is easily checked that, if g represents $[\varphi, f_*]$, then, for each $y \in Y$, the V-functor $E \xrightarrow{y} Y \xrightarrow{g} Z$ represents $[y^* \cdot \varphi, f_*]$.

On the other hand, if, for each $y\colon E \to Y$, $g_y\colon E \to Z$ represents $[y^* \cdot \varphi, f_*]$, then the map $g\colon Y \to Z$ defined by $g(y) = g_y(*)$ is such that $g_* = [\varphi, f_*]$. Note that g is a V-functor, by Proposition 3.4.10, as required. $\square$

Chapter 4

Quasivarieties

In this chapter we will present a new proof that V-$\mathbf{Cat}^{\mathrm{op}}$ is a quasivariety. We note that this result is already known ([21]), yet our proof is fundamentally different and therefore noteworthy enough to include in this text.

We will make use of the category V-**ccd** which generalises the category of constructively completely distributive lattices **ccd**.

It is known that a V-category (X,a) is cocomplete if and only if its corresponding Yoneda embedding $y_X : X \to PX$ admits a left adjoint $\vee_X : PX \to X$ (for details we point the reader to [35]). When this left adjoint $\vee_X : PX \to X$ itself admits a left adjoint $\Downarrow_X : X \to PX$, we say that (X,a) is a **constructively completely distributive V-category**. These are the objects of the category V-**ccd**. The morphisms in this category are the **bicontinuous** V-functors, that is, the V-functors that preserve weighted colimits and weighted limits.

Another category that we will need is the category of complete atomic Boolean algebras, **CABA**. We note that $\mathbf{CABA} \cong \mathbf{Set}^{\mathrm{op}}$ (assuming excluded middle).

The notion of comma categories was introduced by Lawvere in his Ph.D. thesis [27] and will also be essential in our proof.

Definition 4.0.1. Let $F : \mathbf{B} \to \mathbf{D}$ and $G : \mathbf{C} \to \mathbf{D}$ be functors. Then their **comma category** $F \downarrow G$ is defined as follows:

- the objects are triples (B,C,f), where $B \in \mathbf{B}$, $C \in \mathbf{C}$ and

$$f : F(B) \to G(C),$$

  is a morphism in $\mathbf{D}$.

- given two objects (B_1,C_1,f) and (B_2,C_2,g) in $F \downarrow G$,

$$f : F(B_1) \to G(C_1) \quad \text{and} \quad g : F(B_2) \to G(C_2),$$

  a morphism from (B_1,C_1,f) to (B_2,C_2,g) in $F \downarrow G$ is a pair $(u,v) : (B_1,C_1,f) \to (B_2,C_2,g)$, where $u : B_1 \to B_2$ and $v : C_1 \to C_2$ are morphisms in $\mathbf{B}$ and $\mathbf{C}$, respectively, such that the following diagram commutes

$$\begin{tikzcd} F(B_1) \arrow[r, "f"] \arrow[d, "Fu"'] & G(C_1) \arrow[d, "Gv"] \\ F(B_2) \arrow[r, "g"] & G(C_2). \end{tikzcd}$$

Composition is defined component-wise.

If $F : \mathbf{B} \to \mathbf{D}$ is the identity functor (in which case $\mathbf{B} = \mathbf{D}$), we will also denote the comma category $F{\downarrow}G$ by $\mathbf{D}{\downarrow}G$.

Definition 4.0.2. Let $F : \mathbf{B} \to \mathbf{D}$ and $G : \mathbf{C} \to \mathbf{D}$ be functors. Then $F{\downarrowtail}G$ denotes the full subcategory of $F{\downarrow}G$ defined by the monomorphisms in $\mathbf{D}$

$$f : F(B) \rightarrowtail G(C),$$

where $B \in \mathbf{B}$, $C \in \mathbf{C}$.

We will also use the following theorem which is due to Rydeheard and Burstall [33]:

Theorem 4.0.3. *Let $F : \mathbf{B} \to \mathbf{D}$ and $G : \mathbf{C} \to \mathbf{D}$ be functors with F preserving colimits. Then if $\mathbf{B}$ and $\mathbf{C}$ are cocomplete so is their comma category $F \downarrow G$.*

4.1 VTT, CTT and PTT

The last ingredient we require is an interesting result regarding monadicity. It is well known that the composition of monadic functors needs not to be monadic. Yet, certain related conditions (for instance, limit creation,) are preserved by composition. In [2] M. Barr obtains a way to compose functors satisfying certain monadicity conditions and still obtain a monadic functor.

Before introducing the relevant conditions, we will need to introduce some terminology.

Definition 4.1.1. A functor $U : \mathbf{A} \to \mathbf{B}$ is said to **create limits** if for any diagram $D : \mathbf{D} \to \mathbf{A}$ such that a limit cone $(\pi_d : B \to UD(d))_{d \in \mathbf{D}}$ exists in $\mathbf{B}$, there is a unique, up to isomorphism, cone $(\rho_d : A \to D(d))_{d \in \mathbf{D}}$ in $\mathbf{A}$, such that its image under U is a limit cone of UD in $\mathbf{B}$, and moreover, this unique cone is a limit cone.

Definition 4.1.2. A parallel pair of morphisms $f, g : X \to Y$ in a category $\mathbf{C}$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

is called a **split pair** if there is an object $Z \in \mathbf{C}$ and morphisms $h : Y \to Z$, $s : Z \to Y$ and $s' : Y \to X$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[l, bend left=60, "s'"] \arrow[r, shift left, "h"] & Z \arrow[l, shift left, "s"] \end{tikzcd}$$

such that
$$h \cdot f = h \cdot g,\;\; f \cdot s' = 1_Y,\;\; h \cdot s = 1_Z \text{ and } g \cdot s' = s \cdot h.$$

One readily verifies that, given such a split pair, we have the following coequaliser,

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{h} Z\,.$$

Definition 4.1.3. Let **A** and **B** be categories. Given a functor $U : \mathbf{A} \to \mathbf{B}$ we say that $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is a U**-split pair** if $UX \underset{Ug}{\overset{Uf}{\rightrightarrows}} UY$ is a split pair.

We can now define the promised monadicity conditions:

Definition 4.1.4. A functor $U : \mathbf{A} \to \mathbf{B}$ is said to satisfy:

- the **VTT** (Vulgar Tripleability Theorem) if it creates limits and every U-split pair is a split pair.
- the **CTT** (Crude Tripleability Theorem) if it creates limits, **A** has all coequalisers and they are preserved by U.
- the **PTT** (Precise Tripleability Theorem) if it creates limits and if the coequalisers of all U-split pairs exist and are preserved by U.

Remark 4.1.5. By Beck's monadicity theorem, a functor is monadic precisely when it satisfies the PTT and admits a left adjoint.

Theorem 4.1.6 ([2])**.** *Let $U = U_1U_2U_3$ such that U_1 satisfies the VTT, U_2 satisfies the PTT and U_3 satisfies the CTT. Then U satisfies the PTT.*

Proof. Let
$$U : A \xrightarrow{U_3} B \xrightarrow{U_2} C \xrightarrow{U_1} D$$

satisfy the conditions of the theorem. Clearly U creates limits since it is the composite of functors that do so. Consider a U-split pair $X \underset{g}{\overset{f}{\rightrightarrows}} Y$. Then $UX \underset{Ug}{\overset{Uf}{\rightrightarrows}} UY$ is a split pair, and therefore $U_2U_3X \underset{U_2U_3g}{\overset{U_2U_3f}{\rightrightarrows}} U_2U_3Y$ is a U_1-split pair. Since U_1 satisfies the VTT, $U_2U_3X \underset{U_2U_3g}{\overset{U_2U_3f}{\rightrightarrows}} U_2U_3Y$ is a split pair and it follows that $U_3X \underset{U_3g}{\overset{U_3f}{\rightrightarrows}} U_3Y$ is a U_2-split pair. Because U_2 satisfies the PTT we conclude that $U_3X \underset{U_3g}{\overset{U_3f}{\rightrightarrows}} U_3Y$ has a coequaliser which is preserved by U_2. Moreover since U_3 satisfies the CTT we know that $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ has a coequaliser $h : Y \to Z$ which is preserved by U_3. It follows that U_2U_3h is (isomorphic to) the coequaliser of the split pair $U_2U_3X \underset{U_2U_3g}{\overset{U_2U_3f}{\rightrightarrows}} U_2U_3Y$. Therefore Uh is (isomorphic to) the coequaliser of $UX \underset{Ug}{\overset{Uf}{\rightrightarrows}} UY$ since the coequaliser of split pairs is preserved by any functor. $\square$

4.2 V-**Cat**$^{\mathrm{op}}$ is a quasivariety

We start by showing that V-**ccd**$\downarrow I$ is a variety, where $I : \mathbf{Set}^{\mathrm{op}} \to V\text{-}\mathbf{ccd}$ is the functor that sends a set X to V^X. A $\mathbf{Set}^{\mathrm{op}}$ map $f : X \to Y$, that is a **Set** map $f : Y \to X$, is sent by the functor I to $(\ )\cdot f : V^X \to V^Y$. With this goal in mind, we will define three functors and show they are in the conditions of Theorem 4.1.6 :

$$V\text{-}\mathbf{ccd}{\downarrow}I \xrightarrow{U} V\text{-}\mathbf{ccd} \times \mathbf{CABA} \xrightarrow{H} (\mathbf{Set}\times\mathbf{Set})^* \xrightarrow{P} \mathbf{Set} \qquad .$$

Let $(\mathbf{Set}\times\mathbf{Set})^*$ be the full subcategory of $\mathbf{Set}\times\mathbf{Set}$ defined by the **pure** objects of $\mathbf{Set}\times\mathbf{Set}$, that is, the objects (X,Y) such that both X and Y are simultaneously not empty, or simultaneously empty. Making use of another result by Barr (Theorem 2, [2]), we immediately get the following lemma:

Lemma 4.2.1. *The functor $P : (\mathbf{Set}\times\mathbf{Set})^* \to \mathbf{Set}$ which maps each pair to its product satisfies the VTT and admits a left adjoint.*

Lemma 4.2.2. *The functor $H : V\text{-}\mathbf{ccd} \times \mathbf{CABA} \to (\mathbf{Set}\times\mathbf{Set})^*$, which is the product of the functors that send a V-category to its underlying set and an algebra of* **CABA** *to its underlying set satisfies the PTT.*

Proof. Note that the sets in the image of H are always non-empty. It is well known that the forgetful functor $\mathbf{CABA} \to \mathbf{Set}$ which maps an algebra of **CABA** to its underlying set is monadic. Moreover the forgetful functor $V\text{-}\mathbf{ccd} \to \mathbf{Set}$ is also monadic as proved by Pu and Zhang [30]. Therefore H is also monadic being the product of monadic functors. By Remark 4.1.5 it satisfies the PTT. □

Lemma 4.2.3. *The functor $I : \mathbf{Set}^{\mathrm{op}} \to V\text{-}\mathbf{ccd}$ admits a left adjoint.*

Proof. Note that we have the following commutative diagram:

$$\begin{tikzcd}
\mathbf{Set}^{\mathrm{op}} \arrow[rr, "I"] \arrow[dr, "\mathcal{P}_V"'] & & V\text{-}\mathbf{ccd} \arrow[dl, "\mathcal{U}"] \\
& \mathbf{Set} &
\end{tikzcd}$$

where

- $\mathcal{U} : V\text{-}\mathbf{ccd} \to \mathbf{Set}$ is the forgetful functor, which is monadic as proved by Pu and Zhang [30];
- $\mathcal{P}_V : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ is the V-power set functor, which is defined by $\mathcal{P}_V X = V^X$ and $\mathcal{P}_V(f : X \to Y) = ((\ )\cdot f : V^X \to V^Y)$. Sobral [34] proved that $\mathcal{P}_V$ is monadic.

Therefore, by an argument from Linton (Corollary 1 [29]), there exists a functor $L : V\text{-}\mathbf{ccd} \to \mathbf{Set}^{\mathrm{op}}$ left adjoint to I. □

Lemma 4.2.4. *The functor $U : V\text{-}\mathbf{ccd}{\downarrow}I \to V\text{-}\mathbf{ccd} \times \mathbf{CABA}$, which maps objects $f : A \to IX$ to (A,X) and morphisms $(u,v) : (f : A \to IX) \to (g : B \to IY)$ to $(u,v) : (A,X) \to (B,Y)$, admits a left and a right adjoint.*

Proof. By Lemma 4.2.3 $I : \mathbf{Set}^{\mathrm{op}} \to V$-**ccd** has a left adjoint $L : V$-**ccd** $\to \mathbf{Set}^{\mathrm{op}}$. Let $\eta : 1 \to IL$ and $\varepsilon : LI \to 1$ denote, respectively, the unit and counit of the adjunction $L \dashv I$.

Define $\tilde{L}(A,X) = (\ A \xrightarrow{\eta_A} ILA \xrightarrow{I\iota_{LA}} I(LA+X)\ )$, where $\iota_{LA} : LA \to LA+X$ is the coprojection, and define $\tilde{L}(f,g) :\ \tilde{L}(A,X) \longrightarrow \tilde{L}(B,Y)$ as

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & ILA \arrow[r, "I\iota_{LA}"] \arrow[d, "ILf"] & I(LA+X) \arrow[d, "I(Lf+g)"] \\ B \arrow[r, "\eta_B"] & ILB \arrow[r, "I\iota_{LB}"] & I(LB+Y). \end{tikzcd}$$

To show that we have the adjunction $\tilde{L} \dashv U$, we show that its unit $\tilde{\eta} : 1 \to U\tilde{L}$ is given by

$$\tilde{\eta}_{(A,X)} :\ (A,X) \xrightarrow{(1_A,\iota_X)} U\tilde{L}(A,X)\ = (A, LA+X).$$

Given any morphism $(f,g) :\ (A,X) \longrightarrow U(B \xrightarrow{\beta} IY)$, the unique morphism $\tilde{L}(A,X) \xrightarrow{\overline{(f,g)}} U(B \xrightarrow{\beta} IY)$ in V-**ccd**$\downarrow I$ making the diagram commute

$$\begin{tikzcd} (A,X) \arrow[r, "(1_A,\iota_X)"] \arrow[dr, "(f,g)"'] & U\tilde{L}(A,X) \arrow[d, "U\overline{(f,g)}"] \\ & U(B \xrightarrow{\beta} IY) \end{tikzcd}$$

is given by

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & ILA \arrow[r, "I\iota_{LA}"] & I(LA+X) \arrow[d, "I\langle \varepsilon_Y \cdot L\beta \cdot Lf, g\rangle"] \\ B \arrow[rr, "\beta"] & & IY \end{tikzcd}$$

where $\langle \varepsilon_Y \cdot L\beta \cdot Lf, g\rangle :\ LA+X \longrightarrow Y$ is given by the universal property of the coproduct:

$$\begin{tikzcd} LA \arrow[r, "Lf"] \arrow[dr, "\iota_{LA}"] & LB \arrow[r, "L\beta"] & LIY \arrow[dr, "\varepsilon_Y"] & \\ & LA+X \arrow[rr, "\langle \varepsilon_Y \cdot L\beta \cdot Lf, g\rangle"] & & Y \\ X \arrow[ur, "\iota_X"] \arrow[urrr, "g"'] & & & \end{tikzcd}$$

.

Suppose there is another such morphism (f', g') making the required diagram commute. Then it is immediate that $f' = f$ and $g' \cdot \iota_X = g$. Lastly, note that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & ILA \arrow[r, "I\iota_{LA}"] \arrow[dr, "I(\varepsilon_Y \cdot L\beta \cdot Lf)"'] & I(LA+X) \arrow[d, "I(g')"] \\ B \arrow[rr, "\beta"'] & & IY \end{tikzcd}$$

and therefore, by the universality of η, we have that $g' \cdot \iota_{LA} = \varepsilon_Y \cdot L\beta \cdot Lf$ so that $g' = \langle \varepsilon_Y \cdot L\beta \cdot Lf, g \rangle$.

Next, define $\tilde{R}(A,X) = (\ A \times IX \xrightarrow{\pi_2} IX\ )$, where $\pi_2 : A \times IX \to IX$ is the projection, and define $\tilde{R}(f,g) :\ \tilde{R}(A,X) \longrightarrow \tilde{R}(B,Y)$ as

$$\begin{tikzcd} A \times IX \arrow[r, "\pi_2"] \arrow[d, "f \times Ig"'] & IX \arrow[d, "Ig"] \\ B \times IY \arrow[r, "\rho_2"] & IY. \end{tikzcd}$$

To show that we have the adjunction $U \dashv \tilde{R}$, we show that its unit $\hat{\eta} : 1 \to \tilde{R}U$ is given, for any $\alpha : A \to IX$ in V-**ccd**$\downarrow I$, by $\hat{\eta}_\alpha = ((1_A, \alpha), 1_X)$:

$$\begin{tikzcd} A \arrow[r, "(1_A,\alpha)"] \arrow[d, "\alpha"'] & A \times IX \arrow[d, "\pi_2"] \\ IX \arrow[r, "I1_X"] & IX. \end{tikzcd}$$

Given a morphism $(f,g) : \alpha \to \tilde{R}(B,Y)$ the unique morphism $\overline{(f,g)}$ such that $\tilde{R}\overline{(f,g)}$ makes the following diagram commute

$$\begin{tikzcd} & & B \times IY \arrow[dd, "\rho_2"] \\ A \arrow[r, "(1_A,\alpha)"'] \arrow[urr, "f"] \arrow[d, "\alpha"'] & A \times IX \arrow[ur, "\rho_1 \cdot f \times Ig"'] \arrow[d, "\pi_2"] & \\ IX \arrow[r, "I1_X"] \arrow[rr, bend right, "Ig"'] & IX \arrow[r, "Ig"] & IY \end{tikzcd}$$

is $(\rho_1 \cdot f, g)$, where $\rho_1 : B \times IY \to B$ is the projection.

Suppose there is another such morphism (f', g') making the required diagram commute. Then the following diagram commutes

$$\begin{tikzcd}
 & A \arrow[rr, "\rho_1 \cdot f"] & & B \\
A \arrow[ur, "1_A"] \arrow[r, "{(1_A,\alpha)}"] \arrow[dr, "\alpha"'] & A \times IX \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[rr, "f' \times I(g')"] & & B \times IY \arrow[u, "\rho_1"'] \arrow[d, "\rho_2"] \\
 & IX \arrow[rr, "Ig"] & & IY
\end{tikzcd}$$

and therefore, by the universality of the product of morphisms, we have that $f' = \rho_1 \cdot f$ and $I(g') = Ig$.

$\square$

Lemma 4.2.5. *The functor $U : V\text{-}\mathbf{ccd}{\downarrow}I \to V\text{-}\mathbf{ccd} \times \mathbf{CABA}$ satisfies the CTT.*

Proof. By Theorem 4.0.3 it is immediate that V-$\mathbf{ccd}{\downarrow}I$ is cocomplete and therefore has coequalisers.

Note that U is a left adjoint by Lemma 4.2.4 and therefore preserves colimits. Moreover, U creates limits as we show next. Consider a diagram $D : \mathbf{D} \to V\text{-}\mathbf{ccd}{\downarrow}I$ and denote $D(d)$ by $(f_d : A_d \to IX_d)$ for each $d \in \mathbf{D}$. Take a limit cone

$$(\pi_d, \rho_d) : (A, X) \to (A_d, X_d) = U(f_d : A_d \to IX_d)$$

such that $(\pi_d : A \to A_d)$ is a limit cone in V-$\mathbf{ccd}$ and $(\rho_d : X \to X_d)$ is a limit cone in $\mathbf{CABA}$. Then $(I\rho_d : IX \to IX_d)$ is a limit cone in V-$\mathbf{ccd}$ and the morphism $f : A \to IX$ induced by the universal property of the limit

$$\begin{tikzcd}
A \arrow[d, "\pi_d"'] & IX \arrow[d, "I\rho_d"] \\
A_d \arrow[r, "f_d"'] & IX_d
\end{tikzcd}$$

is easily seen to be the desired limit of $\mathbf{D}$. Therefore U satisfies the CTT. $\square$

Proposition 4.2.6. *V-$\mathbf{ccd}{\downarrow}I$ is a variety.*

Proof. By the lemmas 4.2.1, 4.2.2 and 4.2.5 we have that P, H and U satisfy the VTT, PTT and CTT, respectively. Therefore, by Theorem 4.1.6, PHU satisfies the PTT. Since each of the functors P, H and U admit a left adjoint, we conclude, by Remark 4.1.5, that $PHU : V\text{-}\mathbf{ccd}{\downarrow}I \to \mathbf{Set}$ is monadic. $\square$

Next, we show that V-$\mathbf{ccd}{\mapsdown}I$ is a (regular epi)-reflective subcategory of V-$\mathbf{ccd}{\downarrow}I$.

Proposition 4.2.7. *V-$\mathbf{ccd}{\mapsdown}I$ is a (regular epi)-reflective subcategory of V-$\mathbf{ccd}{\downarrow}I$.*

Proof. Given an object $f : A \to IX$ in V-**ccd**$\downarrow I$, let Ff be the monomorphism obtained by the (regular epi,mono) factorisation of f in V-**ccd**

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "e_f"'] & & IX \\
& M \arrow[ur, tail, "Ff"'] &
\end{tikzcd}$$

Given a morphism $(u,v) : (f : A \to IX) \to (g : B \to IY)$ in V-**ccd**$\downarrow I$, let $F(u,v) = (d,v)$ where d is the unique morphism such that $d \cdot e_f = e_g \cdot u$ and $Fg \cdot d = Iv \cdot Ff$, which exists by the lifting property of the orthogonal factorisation system.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "e_f"] \arrow[ddd, "u"'] & & IX \arrow[ddd, "Iv"] \\
& M \arrow[ur, tail, "Ff"] \arrow[d, dotted, "d"] & \\
& N \arrow[dr, tail, "Fg"] & \\
B \arrow[rr, "g"'] \arrow[ur, "e_g"] & & IY
\end{tikzcd}$$

This defines a functor $F : V$-**ccd**$\downarrow I \to V$-**ccd**$\mathrel{\rotatebox[origin=c]{-90}{$\rightarrowtail$}} I$.

Clearly $F\iota = 1$ and we have a natural transformation $\eta : 1 \Rightarrow \iota F$ given by $\eta_f = (e_f, 1_X) : f \to Ff$, for each object $f : A \to IX$ of V-**ccd**$\downarrow I$. Therefore $F \dashv \iota$. Moreover each component $\eta_f = (e_f, 1_X)$ of the unit $\eta : 1 \Rightarrow \iota F$ is a regular epimorphism, since it is so component-wise. $\square$

Lastly, we show that V-**Cat**$^{\mathrm{op}} \cong V$-**ccd**$\mathrel{\rotatebox[origin=c]{-90}{$\rightarrowtail$}} I$.

Proposition 4.2.8. *We have the following categorical equivalence* V-**Cat**$^{\mathrm{op}} \cong V$-**ccd**$\mathrel{\rotatebox[origin=c]{-90}{$\rightarrowtail$}} I$.

Proof. For each V-category (X,a), consider the monomorphism of V-**ccd** given by

$$V^X \rightarrowtail V^{|X|}$$
$$\varphi \mapsto \varphi$$

where $|X|$ denotes the V-category equipped with its discrete structure. Clearly this defines a functor $F : V$-**Cat**$^{\mathrm{op}} \to V$-**ccd**$\mathrel{\rotatebox[origin=c]{-90}{$\rightarrowtail$}} I$. On the other hand, given a monomorphism $m : M \rightarrowtail V^X = IX$ in V-**ccd**$\mathrel{\rotatebox[origin=c]{-90}{$\rightarrowtail$}} I$, define a structure a on X as the initial structure with respect to the cone $(m(f) : X \to V)_{f \in M}$. This defines the functor $G : V$-**ccd**$\mathrel{\rotatebox[origin=c]{-90}{$\rightarrowtail$}} I \to V$-**Cat**$^{\mathrm{op}}$.

On one hand, for any (X,a), as a consequence of the Yoneda lemma, the cone $(a(x,-) : X \to V)_{x \in X}$ is initial, and therefore $GFX = X$.

On the other hand, given a monomorphism $m : M \rightarrowtail V^X$, there is $N \subseteq V^X$ such that $N \cong M$. Moreover N is closed under weighted limits and weighted colimits. Let a be the initial structure on X

with respect to N. Clearly $N \subseteq \hom((X,a),V)$. Therefore, for each $x \in X$,

$$a(x,-) = \bigwedge_{f \in M} \hom(f(x), f(-)) \in N.$$

Let now $\varphi : (X,a) \to V$ be a V-functor. Then

$$\varphi(-) = \bigvee_{y \in Y} \varphi(y) \otimes a(y,-) \in N \cong M.$$

□

We can finally state our result:

Proposition 4.2.9. *V-**Cat**$^{\mathrm{op}}$ is a quasivariety.*

Proof. By Proposition 4.2.8 and Proposition 4.2.7 we have immediately that V-**Cat**$^{\mathrm{op}}$ is indeed a quasivariety, since it is equivalent to a (regular epi)-reflective subcategory of V-**ccd**$\downarrow I$, which is a variety since $PHU : V\text{-}\mathbf{ccd}{\downarrow}I \to \mathbf{Set}$ is monadic by Proposition 4.2.6.

$$\begin{tikzcd}
V\text{-}\mathbf{Cat}^{\mathrm{op}} \arrow[r, "\cong"] & V\text{-}\mathbf{ccd}{\downarrow}I \arrow[rr, bend left, "\iota"] \arrow[rr, phantom, "\top"] & & V\text{-}\mathbf{ccd}{\downarrow}I \arrow[ll, bend left, "F"] \arrow[r, "U"] & V\text{-}\mathbf{ccd} \times \mathbf{CABA} \arrow[r, "H"] & (\mathbf{Set} \times \mathbf{Set})^{*} \arrow[r, "P"] & \mathbf{Set}
\end{tikzcd}$$

□

Chapter 5

The presheaf monad P

The **category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of (set-valued) presheaves** on $\mathbf{C}$ has as objects the presheaves $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ and as morphisms the natural transformations between the presheaf functors. In this setting the *Yoneda embedding*

$$y : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}},$$

which maps each object $x \in \mathbf{C}$ to the presheaf $\mathbf{C}(-,x) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, is especially noteworthy.
We are interested in the case where $\mathbf{Set}$ is replaced by a quantale V and $\mathbf{C}$ is a V-category. As we already have seen in the previous section, the Yoneda embedding remains a fundamental tool in our setting.

5.1 The formal ball monad B

Before studying the presheaf monad and its submonads we will consider the formal ball monad. As we will see later, this monad is actually a submonad of the presheaf monad in a suitable setting.

Formal balls were first introduced by Weihrauch and Schreiber [37]. These spaces are an important tool in the study of (quasi-)metric spaces.

Given a metric space (X,d) its **space of formal balls** is simply the collection of all pairs (x,r), where $x \in X$ and $r \in [0,\infty[$.

This space can itself be equipped with a (quasi-)metric. This construction can naturally be made into a monad on the category of (quasi-)metric spaces (cf. [19, 26] and references there).

This monad can readily be generalised to V-categories, using a V-categorical structure instead of the (quasi-)metric. We will start by considering an extended version of the formal ball monad, which we define below.

The **extended formal ball monad** $\mathbb{B}_\bullet = (B_\bullet, \mu, \eta)$ is given by the following:

- a functor $B_\bullet : V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}$ which maps each V-category (X,a) to $B_\bullet X$ with underlying set $X \times V$ and

$$B_\bullet X((x,r),(y,s)) = \hom(r, a(x,y) \otimes s)$$

and every V-functor $f\colon X \to Y$ to the V-functor $B_\bullet f\colon B_\bullet X \to B_\bullet Y$ with $B_\bullet f(x,r) = (f(x),r)$;

- natural transformations $\eta\colon 1 \to B_\bullet$ and $\mu\colon B_\bullet B_\bullet \to B_\bullet$ with $\eta_X(x) = (x,k)$ and $\mu_X((x,r),s) = (x, r\otimes s)$, for every V-category (X,a), $x \in X$, $r,s \in V$.

We will also be interested in another very similar monad obtained when we only consider balls with radius different from $\perp$. This monad is only defined when the quantale V is *disjunctive*, that is, when the implication

$$u \otimes v = \perp \Rightarrow u = \perp \text{ or } v = \perp$$

holds in V.

Given a non trivial disjunctive quantale V, the **formal ball monad** $\mathbb{B} = (B,\mu,\eta)$ is given by:

- a functor $B\colon V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}$ which maps each V-category (X,a) to BX with underlying set $X \times (V \setminus \{\perp\})$ and

$$BX((x,r),(y,s)) = \hom(r, a(x,y)\otimes s)$$

and every V-functor $f\colon X \to Y$ to the V-functor $Bf\colon BX \to BY$ with $Bf(x,r) = (f(x),r)$;

- natural transformations $\eta\colon 1 \to B$ and $\mu\colon BB \to B$ with $\eta_X(x) = (x,k)$ and $\mu_X((x,r),s) = (x, r\otimes s)$, for every V-category (X,a), $x \in X$, $r,s \in V \setminus \{\perp\}$.

We also note that, when defined, $\mathbb{B}$ is a submonad of $\mathbb{B}_\bullet$.

Our first result establishes the lax idempotency of both of these monads:

Proposition 5.1.1. *The extended formal ball monad $\mathbb{B}_\bullet$ is lax idempotent.*

Proof. For any V-category (X,a) and any $x \in X,\ r,s \in V$ we have

$$(\eta_{B_\bullet X}\cdot \mu_X)((x,r),s) = \eta_{B_\bullet X}(x, r\otimes s) = ((x, r\otimes s),k).$$

Now, note that

$$\begin{aligned} ((x,r),s) \leq ((x,r\otimes s),k) &\Leftrightarrow k \leq \hom(s, B_\bullet X((x,r),(x,r\otimes s))\otimes k) \\ &\Leftrightarrow s \leq B_\bullet X((x,r),(x,r\otimes s)) \\ &\Leftrightarrow s \leq \hom(r, a(x,x)\otimes r\otimes s) \\ &\Leftrightarrow r\otimes s \leq a(x,x)\otimes r \otimes s, \end{aligned}$$

which holds. Therefore $\eta_{B_\bullet X}\cdot \mu_X \leq 1_{B_\bullet B_\bullet X}$. Since $B_\bullet$ is a monad, we have $\mu_X \cdot \eta_{B_\bullet X} = 1_X$, and it follows that $\mu_X \dashv \eta_{B_\bullet X}$. By Proposition 2.2.3, we conclude that $B_\bullet$ is indeed lax idempotent. □

Corollary 5.1.2. *The formal ball monad $\mathbb{B}$ is lax idempotent.* □

Next, we present a new characterisation of the tensored V-categories as the Eilenberg-Moore algebras for the extended formal ball monad $\mathbb{B}_\bullet$.

5.1.1 $\mathbb{B}_\bullet$-algebras as tensored V-categories

Proposition 5.1.3. *For a V-category (X,a), the following conditions are equivalent:*

(i) *X has a $\mathbb{B}_\bullet$-algebra structure $\alpha\colon B_\bullet X \to X$;*

(ii) $(\forall x \in X)\ (\forall r \in V)\ (\exists x \oplus r \in X)\ (\forall y \in X)\ \ a(x\oplus r, y) = \hom(r, a(x,y))$*;*

(iii) *for all $(x,r) \in B_\bullet X$, all diagrams of the form*

$$\begin{tikzcd} X \arrow[r, "(1_X)_*", "\circ"{marking}] \arrow[d, "X(-{,}x)\otimes r"', "\circ"{marking}] & X \\ E \arrow[ur, "{[X(-{,}x)\otimes r{,}(1_X)_*]}"', "\circ"{marking}] \arrow[u, phantom, shift left=3, "\leq"] & \end{tikzcd}$$

have a (weighted) colimit.

Proof. (i) $\Rightarrow$ (ii): From the adjunction $\alpha \dashv \eta_X$ we get that

$$a(\alpha(x,r),y) = B_\bullet X((x,r),(y,k)) = \hom(r, a(x,y)).$$

Now, simply take $x \oplus r := \alpha(x,r)$, and the result follows.

(ii) $\Rightarrow$ (iii): Direct computation of the extension $[X(-,x)\otimes r, (1_X)_*]$ gives us that it is represented by $x \oplus r$:

$$[X(-,x)\otimes r, (1_X)_*](*,y) = \hom(r, a(x,y)).$$

(iii) $\Rightarrow$ (i) For any $(x,r) \in B_\bullet X$, we denote the V-functor that represents $[X(-,x)\otimes r, (1_X)_*]$ by $x \oplus r$. If $r = k$, we pick $x \oplus k = x$ to represent the corresponding V-distributor (any other $x' \simeq x$ would work but x is the correct choice for our goal). Then $\alpha\colon B_\bullet X \to X$ defined by $\alpha(x,r) = x \oplus r$ is clearly left adjoint to η_X and $\alpha \cdot \eta_X = 1_X$, as required. □

Borceux and Kelly introduced in [4] a notion of **tensored** V-categories as the V-categories satisfying condition (iii) in the context of general V-categories. (For the quantaloid enriched setting we point the reader to [35]).

We also immediately get the following characterisation of tensored categories making use of condition (ii) and Proposition 3.4.8:

Corollary 5.1.4. *A V-category X is tensored if, and only if, for every $x \in X$, $a(x,-) : X \to V$ admits a left adjoint, which we will denote by $x \oplus - : V \to X$,*

$$\begin{tikzcd} X \arrow[r, bend left, "{a(x,-)}"] \arrow[r, phantom, "\top"] & V \arrow[l, bend left, "x\oplus -"] \end{tikzcd}$$

*in V-**Cat**.* □

The characterisation of $\mathbb{B}_\bullet$-algebras we proved in Proposition 5.1.3 can be readily used to obtain a characterisation of $\mathbb{B}$-algebras. In fact, the algebra structures for these monads only differ with regards to $(x,\perp) \in B_\bullet X$, which, of course, does not belong to BX. Note also that $x \oplus \perp$, when it exists, is necessarily the bottom element of X, with respect to its induced order, since

$$\perp \leq a(x,y) \quad \Leftrightarrow \quad k \leq \hom(\perp, a(x,y)) = a(x \oplus \perp, y),$$

for any $x,y \in X$.

Corollary 5.1.5. *1. A V-category X has a $\mathbb{B}$-algebra structure if, and only if, $x \oplus r$ exists for every $x \in X$ and $r \in V$ with $r \neq \perp$. If it is so, then the $\mathbb{B}$-algebra structure is given by $\oplus : BX \to X, (x,r) \mapsto x \oplus r$.*

2. If X and Y are $\mathbb{B}$-algebras, a V-functor $f: X \to Y$ is a $\mathbb{B}$-homomorphism if, and only if, $f(x \oplus r) = f(x) \oplus r$, for every $x \in X$ and $r \in V$ with $r \neq \perp$. □

The characterisation of $\mathbb{B}$-algebras given in [19, Proposition 3.4] can readily be generalised to V-**Cat** as follows.

Proposition 5.1.6. *For a V-functor $\alpha : BX \to X$ the following conditions are equivalent.*

(i) *α is a $\mathbb{B}$-algebra structure.*

(ii) *For every $x \in X$, $r,s \in V \setminus \{\perp\}$, $\alpha(x,k) = x$ and $\alpha(x, r \otimes s) = \alpha(\alpha(x,r),s)$.*

(iii) *For every $x \in X$, $r \in V \setminus \{\perp\}$, $\alpha(x,k) = x$ and $X(x, \alpha(x,r)) \geq r$.*

(iv) *For every $x \in X$, $\alpha(x,k) = x$.*

Proof. By definition of $\mathbb{B}$-algebra, (i) $\Leftrightarrow$ (ii), while (i) $\Leftrightarrow$ (iv) follows from Corollary 2.2.15, since $\mathbb{B}$ is lax idempotent. (iii) $\Rightarrow$ (iv) is obvious, and so it remains to prove that, if α is a $\mathbb{B}$-algebra structure, then $X(x,\alpha(x,r)) \geq r$, for $r \neq \perp$. But

$$X(x,\alpha(x,r)) \geq r \Leftrightarrow k \leq \hom(r, X(x,\alpha(x,r))) = X(\alpha(x,r), \alpha(x,r)),$$

because $\alpha(x,-) \dashv X(x,-)$ by Corollary 5.1.4. □

Since we know that, if X has a $\mathbb{B}$-algebra structure α, then $\alpha(x,r) = x \oplus r$, we may state the conditions above as follows.

Corollary 5.1.7. *If $BX \xrightarrow{-\oplus-} X$ is a $\mathbb{B}$-algebra structure, then, for $x \in X$, $r,s \in V \setminus \{\perp\}$:*

1. *$x \oplus k = x$;*

2. *$x \oplus (r \otimes s) = (x \oplus r) \oplus s$;*

3. *$X(x, x \oplus r) \geq r$.* □

Lemma 5.1.8. *Let X and Y be V-categories equipped with $\mathbb{B}$-algebra structures $BX \xrightarrow{-\oplus-} X$ and $BY \xrightarrow{-\oplus-} Y$. Then a map $f : X \to Y$ is a V-functor if and only if*

$$f \text{ is monotone and } f(x) \oplus r \leq f(x \oplus r),$$

for all $(x,r) \in BX$.

Proof. Assume that f is a V-functor. Then f is monotone and, in particular, for any $(x,r) \in BX$ we have

$$X(x, x \oplus r) \leq Y(f(x), f(x \oplus r)).$$

Since $y \oplus - \dashv Y(y,-)$, for any $y \in Y$, it follows that

$$f(x) \oplus X(x, x \oplus r) \leq f(x \oplus r).$$

Moreover

$$f(x) \oplus r \leq f(x) \oplus X(x, x \oplus r).$$

and therefore $f(x) \oplus r \leq f(x \oplus r)$.

Conversely, assume that f is monotone and that $f(x) \oplus r \leq f(x \oplus r)$, for all $(x,r) \in BX$. Let $x, x' \in X$. Then $x \oplus X(x,x') \leq x'$ since $(x \oplus -) \dashv X(x,-)$ by Corollary 5.1.4, and then

$$\begin{aligned} f(x) \oplus X(x,x') &\leq f(x \oplus X(x,x')) && \text{(by hypothesis)} \\ &\leq f(x') && \text{(by monotonicity of } f\text{).} \end{aligned}$$

Now, using the adjunction $f(x) \oplus - \dashv Y(f(x), -))$, we conclude that

$$X(x,x') \leq Y(f(x), f(x')).$$

□

The following results are now immediate:

Corollary 5.1.9.
1. *Let $(X,\oplus),(Y,\oplus)$ be $\mathbb{B}$-algebras. Then a map $f\colon X \to Y$ is a $\mathbb{B}$-algebra morphism if and only if, for all $(x,r) \in BX$,*

$$f \text{ is monotone and } f(x \oplus r) = f(x) \oplus r.$$

2. *Let $(X,\oplus),(Y,\oplus)$ be $\mathbb{B}$-algebras. Then a V-functor $f\colon X \to Y$ is a $\mathbb{B}$-algebra morphism if and only if, for all $(x,r) \in BX$,*
$$f(x \oplus r) \leq f(x) \oplus r.$$

□

Example 5.1.10. Let V be the Lawvere quantale. If $X \subseteq [0,\infty]$, with the V-category structure inherited from hom, then

1. X is a $\mathbb{B}_{\bullet}$-algebra if, and only if, $X = [a,b]$ for some $a,b \in [0,\infty]$.

2. X is a $\mathbb{B}$-algebra if, and only if, $X =]a,b]$ or $X = [a,b]$ for some $a,b \in [0,\infty]$.

Let X be a $\mathbb{B}_{\bullet}$-algebra. From Proposition 5.1.3 one has

$$(\forall x \in X)\ (\forall r \in [0,\infty])\ (\exists x \oplus r \in X)\ (\forall y \in X)\ \ y \ominus (x \oplus r) = (y \ominus x) \ominus r = y \ominus (x + r).$$

This implies that, if $y \in X$, then $y > x \otimes r \Leftrightarrow y > x + r$. Therefore, if $x + r \in X$, then $x \oplus r = x + r$, and, moreover, X is an interval: given $x,y,z \in [0,\infty]$ with $x < y < z$ and $x,z \in X$, then, with $r = y - x \in [0,\infty]$, $x + r = y$ must belong to X:

$$z \ominus (x \oplus r) = z - (x + r) = z - y > 0 \Rightarrow z \ominus (x \oplus r) = z - (x \oplus r) = z - y \Leftrightarrow y = x \oplus r \in X.$$

In addition, X must have bottom element (that is a maximum with respect to the classical order of the real half-line): for any $x \in X$ and $b = \sup X$, $x \oplus (b - x) = \sup\{z \in X\,;\, z \leq b\} = b \in X$. For $r = \infty$ and any $x \in X$, $x \oplus \infty$ must be the top element of X, so $X = [a,b]$ for $a,b \in [0,\infty]$.

Conversely, if $X =]a,b]$, for $x \in X$ and $r \in [0,\infty[$, define $x \oplus r = x + r$ if $x + r \in X$ and $x \oplus r = b$ elsewhere. It is easy to check that condition (ii) of Proposition 5.1.3 is satisfied for $r \neq \infty$.

Analogously, if $X = [a,b]$, for $x \in X$ and $r \in [0,\infty]$, we define $x \oplus r$ as before in case $r \neq \infty$ and $x \oplus \infty = a$.

5.2 The presheaf monad P

As we already noted in 3.4.1, given a V-category X, the V-valued V-functors $X^{\mathrm{op}} \to V$ are fundamental for the study of X. In fact, we have a functor

$$\begin{aligned}
P : V\text{-}\mathbf{Cat} &\to V\text{-}\mathbf{Cat} \\
X &\mapsto PX \\
(\, X \xrightarrow{f} Y \,) &\mapsto (\,)\cdot f^* : PX \to PY \\
&\qquad (\, X \xrightarrow{\varphi}\!\!\!\!\!\!\circ\ \ E \,) \mapsto (\, Y \xrightarrow{f^*}\!\!\!\!\!\!\circ\ \ X \xrightarrow{\varphi}\!\!\!\!\!\!\circ\ \ E \,)
\end{aligned}$$

It is easy to verify that the Yoneda functors $(y_X : X \to PX)_X$, define a natural transformation $y\colon 1 \to P$. Moreover, since, for every V-functor f, the adjunction $f_* \dashv f^*$ gives us an adjunction

$$Pf = (\,)\cdot f^* \dashv (\,)\cdot f_* =: Qf,$$

Py_X has a right adjoint, which we denote by $m_X : PPX \to PX$. It is now straightforward to show that $\mathbb{P} = (P, m, y)$ is a monad on V-$\mathbf{Cat}$. This is the so called **presheaf monad**.

Alternatively, we may describe the presheaf monad $\mathbb{P} = (P, m, y)$ on V-**Cat** as the monad induced by the following adjunction:

$$\begin{tikzcd} V\text{-}\mathbf{Cat} \arrow[r, bend left, "(-)^*"] \arrow[r, phantom, "\top"] & V\text{-}\mathbf{Dist}^{\mathrm{op}} \arrow[l, bend left, "V\text{-}\mathbf{Dist}(-,E)"] \end{tikzcd}$$

- $PX = V\text{-}\mathbf{Cat}(X^{\mathrm{op}}, V) \cong V\text{-}\mathbf{Dist}(X, E)$;

- $P(f)(\varphi) = V\text{-}\mathbf{Dist}(f^*, E)(\varphi) = \varphi \cdot f^* = (\ Y \overset{f^*}{-\!\!\circ\!\!\to} X \overset{\varphi}{-\!\!\circ\!\!\to} E\ )$;

- $y_X(x) = a(-, x)$;

- $m_X(\Psi) = \Psi \cdot (y_X)_* = (\ X \overset{(y_X)_*}{-\!\!\circ\!\!\to} PX \overset{\Psi}{-\!\!\circ\!\!\to} E\ )$;

for any $x \in X$, $f : (X, a) \to (Y, b)$, $\varphi \in PX$ and $\Psi \in PPX$. Note that $P = V\text{-}\mathbf{Dist}(-, E) \circ (-)^*$ is a monotone functor:

$$f \leq g \quad \Rightarrow \quad f^* \leq g^* \quad \Rightarrow \quad \varphi \cdot f^* \leq \varphi \cdot g^*,\ \forall \varphi \in PX \quad \Rightarrow \quad Pf \leq Pg,$$

for any $f, g : X \to Y$ in V-**Cat**. Therefore the monad induced by this adjunction is an **Ord**-monad. Moreover, by construction of m_X as the right adjoint to Py_X, this monad is lax idempotent (see [20] for details).

Remark 5.2.1. Observe that

$$\begin{aligned} V\text{-}\mathbf{Dist}(X, Y) &\cong V\text{-}\mathbf{Cat}(X^{\mathrm{op}} \otimes Y, V) \\ &\cong V\text{-}\mathbf{Cat}(Y, V\text{-}\mathbf{Cat}(X^{\mathrm{op}}, V)) \\ &= V\text{-}\mathbf{Cat}(Y, PX) \end{aligned}$$

and therefore, it follows that any V-distributor $X \,-\!\!\circ\!\!\to\, Y$ can be seen as a morphism $Y \rightharpoonup X$ in the Kleisli category $V\text{-}\mathbf{Cat}_{\mathbb{P}}$ and vice versa.

In fact, we have the following proposition:

Proposition 5.2.2. *The category V-**Dist** is isomorphic to the dual of the Kleisli category of the presheaf monad $\mathbb{P}$ on V-**Cat**:*

$$V\text{-}\mathbf{Dist} \cong V\text{-}\mathbf{Cat}_{\mathbb{P}}^{\mathrm{op}}$$

Proof. Given two morphisms $\varphi : X \rightharpoonup Y$ and $\psi : Y \rightharpoonup Z$ in the Kleisli category $V\text{-}\mathbf{Cat}_{\mathbb{P}}$ we have two corresponding V-distributors $\varphi :\ Y \,-\!\!\circ\!\!\to\, X$ and $\psi :\ Z \,-\!\!\circ\!\!\to\, Y$ by Remark 5.2.1. To show the equivalence between the dual of the Kleisli category $V\text{-}\mathbf{Cat}_{\mathbb{P}}$ and V-**Dist** it suffices to show that the Kleisli composition $\psi \circ \varphi : X \rightharpoonup Y \rightharpoonup Z$ coincides with the composition $\varphi \cdot \psi :\ Z \,-\!\!\circ\!\!\to\, Y \,-\!\!\circ\!\!\to\, X$

on V-**Dist**. Note that

$$\begin{aligned}(\psi\circ\varphi)(x)(z) &= (\mathfrak{m}_Z\cdot P\psi\cdot\varphi)(x)(z)\\ &= P\psi(\varphi(x))\cdot(y_Z)_*(z)\\ &= \varphi(x)\cdot\psi^*\cdot(y_Z)_*(z)\\ &= \bigvee_{y\in Y}\varphi(x)(y)\otimes\psi^*\cdot(y_Z)_*(z,y)\\ &= \bigvee_{y\in Y}\varphi(x)(y)\otimes[\![(y_Z)_*(z),\psi(y)]\!]\\ &= \bigvee_{y\in Y}\varphi(x)(y)\otimes\psi(y)(z) && \text{(by the Yoneda Lemma (3.4.6))}\end{aligned}$$

□

5.3 Submonads of the presheaf monad

Next we will focus our attention into the study of the submonads of the presheaf monad $\mathbb{P}$.

5.3.1 In terms of admissible classes

Clementino and Hofmann [8] used certain classes of V-distributors to generalise some injectivity results of Escardó and Flagg [17] to a V-enriched context. We will use some of those classes of V-distributors to characterise the submonads of the presheaf submonad $\mathbb{P}$.

Definition 5.3.1. Given a class Φ of V-distributors, for every V-category X let

$$\Phi X = \{\varphi\colon X{\circ\!\!\!\longrightarrow}E \mid \varphi\in\Phi\}$$

have the V-category structure inherited from the one of PX. We say that Φ is **admissible** if, for every V-functor $f\colon X\to Y$ and V-distributors $\varphi\colon Z{\circ\!\!\!\longrightarrow}Y$ and $\psi\colon X{\circ\!\!\!\longrightarrow}Z$ in Φ,

(1) $f^*\in\Phi$;

(2) $\psi\cdot f^*\in\Phi$ and $f^*\cdot\varphi\in\Phi$;

(3) $\varphi\in\Phi \;\Leftrightarrow\; (\forall y\in Y)\; y^*\cdot\varphi\in\Phi$;

(4) for every V-distributor $\gamma\colon PX{\circ\!\!\!\longrightarrow}E$, if the restriction of γ to ΦX belongs to Φ, then $\gamma\cdot(y_X)_*\in\Phi$.

Remark 5.3.2. We point out that condition (2), in the presence of the remaining conditions, could have been replaced with the following condition [36]:

(2') Φ is closed under composition.

Lemma 5.3.3. *Every admissible class Φ of V-distributors induces a submonad $\Phi = (\Phi, m^\Phi, y^\Phi)$ of $\mathbb{P}$.*

Proof. For each V-category X, equip ΦX with the initial structure induced by the inclusion $\sigma_X : \Phi X \to PX$, that is, for every $\varphi, \psi \in \Phi X$, $\Phi X(\varphi, \psi) = PX(\varphi, \psi)$. For each V-functor $f : X \to Y$ and $\varphi \in \Phi X$, by condition (2), $\varphi \cdot f^* \in \Phi$, and so Pf (co)restricts to $\Phi f : \Phi X \to \Phi Y$.

Condition (1) guarantees that $y_X : X \to PX$ corestricts to $y_X^\Phi : X \to \Phi X$.

Finally, condition (4) guarantees that $m_X : PPX \to PX$ also (co)restricts to $m_X^\Phi : \Phi\Phi X \to \Phi X$: if $\gamma : \Phi X \to\!\!\!\!\circ\; E$ belongs to Φ, then $\widetilde{\gamma} := \gamma \cdot (\sigma_X)^* : PX \to\!\!\!\!\circ\; E$ belongs to Φ by (2), and then, since γ is the restriction of $\widetilde{\gamma}$ to ΦX, by (4) $m_X(\widetilde{\gamma}) = \gamma \cdot (\sigma_X)^* \cdot (y_X)_* = \gamma \cdot (\sigma_X)^* \cdot (\sigma_X)_* \cdot (y_X^\Phi)_* = \gamma \cdot (y_X^\Phi)_* \in \Phi$.

By construction, $(\sigma_X)_X$ is a natural transformation, each σ_X is an embedding, and σ satisfies the required commutation conditions for a monad morphism. $\square$

Theorem 5.3.4. *For an* **Ord**-*monad* $\mathbb{T} = (T, \mu, \eta)$ *on* V-**Cat**, *the following assertions are equivalent:*

(i) $\mathbb{T}$ *is isomorphic to* Φ, *for some admissible class of* V-*distributors* Φ.

(ii) $\mathbb{T}$ *is a submonad of* $\mathbb{P}$.

Proof. (i) $\Rightarrow$ (ii) follows from the lemma above.

(ii) $\Rightarrow$ (i): Let $\sigma : \mathbb{T} \to \mathbb{P}$ be a monad morphism, with σ_X an embedding for every V-category X, which, for simplicity, we assume to be an inclusion. First we show that

$$\Phi = \{\varphi : X \to\!\!\!\!\circ\; Y \mid \forall y \in Y \; y^* \cdot \varphi \in TX\} \tag{5.3.i}$$

is admissible. In the sequel $f : X \to Y$ is a V-functor.

(1) For each $x \in X$, $x^* \cdot f^* = f(x)^* \in TY$, and so $f^* \in \Phi$.

(2) If $\psi : X \to\!\!\!\!\circ\; Z$ is a V-distributor in Φ, and $z \in Z$, since $z^* \cdot \psi \in TX$, $Tf(z^* \cdot \psi) = z^* \cdot \psi \cdot f^* \in TY$, and therefore $\psi \cdot f^* \in \Phi$ by definition of Φ. Now, if $\varphi : Z \to\!\!\!\!\circ\; Y \in \Phi$, then, for each $x \in X$, $x^* \cdot f^* \cdot \varphi = f(x)^* \cdot \varphi \in TZ$ because $\varphi \in \Phi$, and so $f^* \cdot \varphi \in \Phi$.

(3) follows from the definition of Φ.

(4) If the restriction of $\gamma : PX \to\!\!\!\!\circ\; E$ to TX, i.e., $\gamma \cdot (\sigma_X)_*$, belongs to Φ, then $\mu_X(\gamma \cdot (\sigma_X)_*) = \gamma \cdot (\sigma_X)_* \cdot (\eta_X)_* = \gamma \cdot (y_X)_*$ belongs to TX. $\square$

5.3.2 A new type of Beck-Chevalley conditions

Beck-Chevalley conditions were originally introduced by Bénabou and Roubaud [3] in the context of monadic descent theory. Generally, Beck-Chevalley conditions describe a form of commutation of adjoints.

In this text, the Beck-Chevalley condition we will be interested parallels nicely the work done in [11], where such a condition is used to obtain an interaction between **Set** and **Rel**. We will define a similar Beck-Chevalley like condition to obtain an analogous interaction between V-**Cat** and V-**Dist** and show that the presheaf monad $\mathbb{P}$ satisfies it. We recall from [11] that a commutative square in **Set**

$$\begin{tikzcd} W \arrow[r, "l"] \arrow[d, "g"'] & Z \arrow[d, "h"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is said to be a **BC-square** if the following diagram commutes in **Rel**

$$\begin{tikzcd} W \arrow[r, "l_\circ", "\shortmid"{marking}] & Z \\ X \arrow[u, "g^\circ", "\shortmid"{marking}] \arrow[r, "f_\circ"', "\shortmid"{marking}] & Y, \arrow[u, "h^\circ"', "\shortmid"{marking}] \end{tikzcd}$$

where, given a map $t\colon A \to B$, $t_\circ\colon A \nrightarrow B$ denotes the relation defined by t and $t^\circ\colon B \nrightarrow A$ its opposite. Since $t_\circ \dashv t^\circ$ in **Rel**, this is in fact a kind of Beck-Chevalley condition. A **Set**-endofunctor T is said to satisfy BC if it preserves BC-squares, while a natural transformation $\alpha\colon T \to T'$ between two **Set**-endofunctors satisfies BC if, for each map $f\colon X \to Y$, its naturality square

$$\begin{tikzcd} TX \arrow[r, "\alpha_X"] \arrow[d, "Tf"'] & T'X \arrow[d, "T'f"] \\ TY \arrow[r, "\alpha_Y"'] & T'Y \end{tikzcd}$$

is a BC-square.

We can generalise these notions by replacing the functor $(-)_\circ : \mathbf{Set} \to \mathbf{Rel}$ with any **sinister Ord**-functor, that is, an **Ord**-functor $F : \mathbf{C} \to \mathbf{D}$ such that, for any morphism $f : X \to Y$ in $\mathbf{C}$, $Ff : FX \to FY$ admits a right adjoint $\overline{Ff} : FY \to FX$ in $\mathbf{D}$. For more on sinister functors we point the reader to [14].

Definition 5.3.5. Let $F : \mathbf{C} \to \mathbf{D}$ be a sinister **Ord**-functor. Then a commutative square in $\mathbf{C}$:

$$\begin{tikzcd} W \arrow[r, "l"] \arrow[d, "g"'] & Z \arrow[d, "h"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is said to be a **BC-square with respect to** F if the following diagram commutes in $\mathbf{D}$

$$\begin{tikzcd} FW \arrow[r, "Fl"] & FZ \\ FX \arrow[u, "\overline{Fg}"] \arrow[r, "Ff"'] & FY \arrow[u, "\overline{Fh}"'] \end{tikzcd}$$

where $Fg \dashv \overline{Fg}$ and $Fh \dashv \overline{Fh}$.

The presheaf monad $\mathbb{P}$ verifies some interesting conditions of Beck-Chevalley type, that resemble the BC conditions studied in [11]. More precisely, we will be interested exclusively in BC-squares with respect to $(-)_* : V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Dist}^{\mathrm{co}}$. We will call such commutative square **BC*-square**. Equivalently these commutative squares may be defined as follows:

Definition 5.3.6. A commutative square in V-**Cat**

$$\begin{tikzcd} (W,d) \arrow[r, "l"] \arrow[d, "g"'] & (Z,c) \arrow[d, "h"] \\ (X,a) \arrow[r, "f"'] & (Y,b) \end{tikzcd}$$

is said to be a **BC*-square** if the following diagram commutes in V-**Dist**

$$\begin{tikzcd} (W,d) \arrow[r, "l_*", "\circ" marking] & (Z,c) \\ (X,a) \arrow[u, "g^*", "\circ" marking] \arrow[r, "f_*"', "\circ" marking] & (Y,b) \arrow[u, "h^*"', "\circ" marking] \end{tikzcd}$$

(or, equivalently, $h^* \cdot f_* \leq l_* \cdot g^*$).

Remarks 5.3.7. 1. For a V-functor $f\colon (X,a) \to (Y,b)$, to be fully faithful is equivalent to

$$\begin{tikzcd} (X,a) \arrow[r, "1_X"] \arrow[d, "1_X"'] & (X,a) \arrow[d, "f"] \\ (X,a) \arrow[r, "f"'] & (Y,b) \end{tikzcd}$$

being a BC*-square (exactly in parallel with the characterisation of monomorphisms via BC-squares).

2. We point out that, contrary to the case of BC-squares, in BC*-squares the horizontal and the vertical arrows play different roles; that is, the fact that the diagram

$$\begin{tikzcd} (W,d) \arrow[r, "l"] \arrow[d, "g"'] & (Z,c) \arrow[d, "h"] \\ (X,a) \arrow[r, "f"'] & (Y,b) \end{tikzcd}$$

is a BC*-square is not equivalent to

$$\begin{tikzcd} (W,d) \arrow[r, "g"] \arrow[d, "l"'] & (X,a) \arrow[d, "f"] \\ (Z,c) \arrow[r, "h"'] & (Y,b) \end{tikzcd}$$

being a BC*-square; direct calculation shows it is equivalent to its **dual**

$$\begin{tikzcd}
(W,d^\circ) \arrow[r, "g"] \arrow[d, "l"'] & (X,a^\circ) \arrow[d, "f"] \\
(Z,c^\circ) \arrow[r, "h"'] & (Y,b^\circ)
\end{tikzcd}$$

being a BC*-square.

Definitions 5.3.8. 1. A functor $T\colon V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}$ **satisfies BC*** if it preserves BC*-squares.

2. Given two endofunctors T, T' on V-**Cat**, a natural transformation $\alpha\colon T \to T'$ **satisfies BC*** if the naturality diagram
$$\begin{tikzcd}
TX \arrow[r, "\alpha_X"] \arrow[d, "Tf"'] & T'X \arrow[d, "T'f"] \\
TY \arrow[r, "\alpha_Y"'] & T'Y
\end{tikzcd}$$
is a BC*-square for every morphism f in V-**Cat**.

3. An **Ord**-monad $\mathbb{T} = (T, \mu, \eta)$ on V-**Cat** is said to **fully satisfy BC*** if T, μ, and η satisfy BC*.

These conditions provide us with an interesting characterisation of lax idempotency for an **Ord**-monad $\mathbb{T}$ on V-**Cat**:

Proposition 5.3.9. *Let $\mathbb{T} = (T, \mu, \eta)$ be an* **Ord**-*monad on V-***Cat**.

1. *The following assertions are equivalent:*

   (i) *$\mathbb{T}$ is lax idempotent.*

   (ii) *For each V-category X, the diagram*

$$\begin{tikzcd}
TX \arrow[r, "T\eta_X"] \arrow[d, "\eta_{TX}"'] & TTX \arrow[d, "\mu_X"] \\
TTX \arrow[r, "\mu_X"'] & TX
\end{tikzcd}$$

   is a BC-square.*

2. *If $\mathbb{T}$ is lax idempotent, then μ satisfies BC*.*

Proof. (1) (i) $\Rightarrow$ (ii): The monad $\mathbb{T}$ is lax idempotent if, and only if, for every V-category X, $T\eta_X \dashv \mu_X$, or, equivalently, $\mu_X \dashv \eta_{TX}$. These two conditions are equivalent to $(T\eta_X)_* = (\mu_X)^*$ and $(\mu_X)_* = (\eta_{TX})^*$. Hence $(\mu_X)^*(\mu_X)_* = (T\eta_X)_*(\eta_{TX})^*$ as claimed.

(ii) $\Rightarrow$ (i): From $(\mu_X)^*(\mu_X)_* = (T\eta_X)_*(\eta_{TX})^*$ it follows that

$$(\mu_X)_* = (\mu_X)_*(\mu_X)^*(\mu_X)_* = (\mu_X \cdot T\eta_X)_*(\eta_{TX})^* = (\eta_{TX})^*,$$

that is, $\mu_X \dashv \eta_{TX}$.

(2) BC* for μ follows directly from lax idempotency of $\mathbb{T}$, since

$$\begin{tikzcd} TTX \arrow[r, "(\mu_X)_*", "\circ" marking] & TX \\ TTY \arrow[u, "(TTf)^*", "\circ" marking] \arrow[r, "(\mu_Y)_*"', "\circ" marking] & TY \arrow[u, "(Tf)^*"', "\circ" marking] \end{tikzcd} \quad = \quad \begin{tikzcd} TTX \arrow[r, "(\eta_{TX})^*", "\circ" marking] & TX \\ TTY \arrow[u, "(TTf)^*", "\circ" marking] \arrow[r, "(\eta_{TY})^*"', "\circ" marking] & TY \arrow[u, "(Tf)^*"', "\circ" marking] \end{tikzcd}$$

and the latter diagram commutes trivially. $\square$

Moreover, we get a similar characterisation of oplax idempotent **Ord**-monad: $\mathbb{T}$ is oplax idempotent if, and only if, the diagram

$$\begin{tikzcd} TX \arrow[r, "\eta_{TX}"] \arrow[d, "T\eta_X"'] & TTX \arrow[d, "\mu_X"] \\ TTX \arrow[r, "\mu_X"'] & TX \end{tikzcd}$$

is a BC*-square.

Theorem 5.3.10. *The presheaf monad $\mathbb{P} = (P, m, y)$ fully satisfies BC*.*

Proof. (1) P **satisfies BC***: Given a BC*-square

(5.3.ii)
$$\begin{tikzcd} (W,d) \arrow[r, "l"] \arrow[d, "g"'] & (Z,c) \arrow[d, "h"] \\ (X,a) \arrow[r, "f"'] & (Y,b) \end{tikzcd}$$

in V-**Cat**, we want to show that

$$\begin{tikzcd} PW \arrow[r, "(Pl)_*", "\circ" marking] \arrow[dr, phantom, "\geq"] & PZ \\ PX \arrow[u, "(Pg)^*", "\circ" marking] \arrow[r, "(Pf)_*"', "\circ" marking] & PY. \arrow[u, "(Ph)^*"', "\circ" marking] \end{tikzcd}$$

For each $\varphi \in PX$ and $\psi \in PZ$, we have

$$\begin{aligned}
(Ph)^*(Pf)_*(\varphi,\psi) &= (Ph)^\circ \cdot \widetilde{b} \cdot Pf(\varphi,\psi)\\
&= \widetilde{b}(Pf(\varphi), Ph(\psi))\\
&= \bigwedge_{y\in Y} \hom(\varphi \cdot f^*(y), \psi \cdot h^*(y))\\
&\leq \bigwedge_{x\in X} \hom(\varphi \cdot f^* \cdot f_*(x), \psi \cdot h^* \cdot f_*(x))\\
&\leq \bigwedge_{x\in X} \hom(\varphi(x), \psi \cdot l_* \cdot g^*(x)) && (\varphi \leq \varphi \cdot f^* \cdot f_*, \text{(5.3.ii) is BC*})\\
&= \widetilde{a}(\varphi, \psi \cdot l_* \cdot g^*)\\
&\leq \widetilde{a}(\varphi, \psi \cdot l_* \cdot g^*) \otimes \widetilde{c}(\psi \cdot l_* \cdot l^*, \psi) && (\text{because } \psi \cdot l_* \cdot l^* \leq \psi)\\
&= \widetilde{a}(\varphi, Pg(\psi \cdot l_*)) \otimes \widetilde{c}(Pl(\psi \cdot l_*), \psi)\\
&\leq \bigvee_{\gamma\in PW} \widetilde{a}(\varphi, Pg(\gamma)) \otimes \widetilde{c}(Pl(\gamma), \psi)\\
&= (Pl)_*(Pg)^*(\varphi,\psi).
\end{aligned}$$

(2) m **satisfies BC***: For each V-functor $f\colon X \to Y$, from the naturality of y it follows that the following diagram

$$\begin{tikzcd}
PPX \arrow[r, "(y_{PX})^*", "\circ" marking] & PX \\
PPY \arrow[u, "(PPf)^*", "\circ" marking] \arrow[r, "(y_{PY})^*"', "\circ" marking] & PY \arrow[u, "(Pf)^*"', "\circ" marking]
\end{tikzcd}$$

commutes. Lax idempotency of $\mathbb{P}$ means in particular that $m_X \dashv y_{PX}$, or, equivalently, $(m_X)_* = (y_{PX})^*$, and therefore the commutativity of this diagram shows BC* for m.

(3) y **satisfies BC***: Once again, for each V-functor $f\colon (X,a) \to (Y,b)$, we want to show that the diagram

$$\begin{tikzcd}
X \arrow[r, "(y_X)_*", "\circ" marking] & PX \\
Y \arrow[u, "f^*", "\circ" marking] \arrow[r, "(y_Y)_*"', "\circ" marking] & PY \arrow[u, "(Pf)^*"', "\circ" marking]
\end{tikzcd}$$

commutes. Let $y \in Y$ and $\varphi\colon X \multimap E$ belong to PX. Then

$$\begin{aligned}
((Pf)^*(y_Y)_*)(y,\varphi) &= ((Pf)^\circ \cdot \widetilde{b} \cdot y_Y)(y,\varphi) \\
&= \widetilde{b}(y_Y(y), Pf(\varphi)) \\
&= Pf(\varphi)(y) \\
&= \bigvee_{x\in X} b(y,f(x)) \otimes \varphi(x) \\
&= \bigvee_{x\in X} b(y,f(x)) \otimes \widetilde{a}(y_X(x),\varphi) \\
&= (\widetilde{a} \cdot y_X \cdot f^\circ \cdot b)(y,\varphi) = (y_X)_* \cdot f^*(y,\varphi),
\end{aligned}$$

as claimed. □

Corollary 5.3.11. *Let $\mathbb{T} = (T,\mu,\eta)$ on V-**Cat** be an **Ord**-monad on V-**Cat**, and $\sigma\colon \mathbb{T} \to \mathbb{P}$ be a monad morphism, pointwise fully faithful. Then $\mathbb{T}$ is lax idempotent.*

Proof. We know that $\mathbb{P}$ is lax idempotent, and so, for every V-category X, $(m_X)_* = (y_{PX})^*$. By definition of monad morphism, we have that $(\mu_X)_* = (\sigma_X)^*(\sigma_X)_*(\mu_X)_* = (\sigma_X)^*(m_X)_*(P\sigma_X)_*(\sigma_{TX})_*$; using the equality above, and preservation of fully faithful V-functors by $\mathbb{P}$ we have

$$\begin{aligned}
(\mu_X)_* &= (\sigma_X)^*(y_{PX})^*(P\sigma_X)_*(\sigma_{TX})_* \\
&= (\sigma_X)^*(\eta_{PX})^*(\sigma_{PX})^*(P\sigma_X)_*(\sigma_{TX})_* \\
&= (\eta_{TX})^* \cdot (\sigma_{TX})^*(P\sigma_X)^*(P\sigma_X)_*(\sigma_{TX})_* = (\eta_{TX})^*.
\end{aligned}$$

□

5.3.3 Submonads in terms of BC* conditions

Next we show how these new BC* conditions give us an alternative way to characterise the submonads of the presheaf monad $\mathbb{P}$. More precisely, we will show that the existence of a monad morphism $\mathbb{T} \to \mathbb{P}$ for a general **Ord**-monad $\mathbb{T} = (T,\mu,\eta)$ on V-**Cat** is dependent on the BC* properties we defined above.

We also point out that a necessary condition for $\mathbb{T}$ to be a submonad of $\mathbb{P}$ is that TX is separated for every V-category X, since PX is separated and separated V-categories are stable under monomorphisms.

Theorem 5.3.12. *For an **Ord**-monad $\mathbb{T} = (T,\mu,\eta)$ on V-**Cat** with TX separated for every V-category X, the following assertions are equivalent:*

(i) *$\mathbb{T}$ is a submonad of $\mathbb{P}$.*

(ii) $\mathbb{T}$ *is lax idempotent and fully satisfies BC*, and both* η_X *and* $Q\eta_X \cdot y_{TX}$ *are fully faithful, for each* V*-category* X.

(iii) $\mathbb{T}$ *is lax idempotent,* μ *and* η *satisfy BC*, and both* η_X *and* $Q\eta_X \cdot y_{TX}$ *are fully faithful, for each* V*-category* X.

(iv) $\mathbb{T}$ *is lax idempotent,* η *satisfies BC*, and both* η_X *and* $Q\eta_X \cdot y_{TX}$ *are fully faithful, for each* V*-category* X.

Proof. (i) $\Rightarrow$ (ii): By (i) there exists a monad morphism $\sigma\colon \mathbb{T} \to \mathbb{P}$ with σ_X an embedding for every V-category X. By Corollary 5.3.11, with $\mathbb{P}$, also $\mathbb{T}$ is lax idempotent. Moreover, from $\sigma_X \cdot \eta_X = y_X$, with y_X, also η_X is fully faithful. (In fact this is valid for any monad with a monad morphism into $\mathbb{P}$.)

To show that $\mathbb{T}$ satisfies BC* we use the characterisation of Theorem 5.3.4; that is, we know that there is an admissible class Φ of V-distributors so that $\mathbb{T} = \Phi$. Then BC* for T follows directly from the fact that Φf is a (co)restriction of Pf, for every V-functor f.

BC* for η follows from BC* for y and full faithfulness of σ since, for any commutative diagram in V-**Cat**

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{1}"] & \cdot \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\boxed{2}"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

with $\boxed{1\,|\,2}$ satisfying BC*, and f and g fully faithful, also $\boxed{1}$ satisfies BC*.

Thanks to Proposition 5.3.9, BC* for μ follows directly from lax idempotency of $\mathbb{T}$.

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious.

(iv) $\Rightarrow$ (i): For each V-category (X,a), we denote by $\widehat{a}$ the V-category structure on TX, and define the V-functor

$$(\,TX \xrightarrow{\sigma_X} PX\,) = (\,TX \xrightarrow{y_{TX}} PTX \xrightarrow{Q\eta_X} PX\,);$$

that is, $\sigma_X(\mathfrak{x}) = (\,X \xrightarrow{\eta_X} TX \overset{\widehat{a}}{\longrightarrow\!\!\!\!\!\!\!\!\!\!\!\!\;\;\;\;\;\;\;\;\;\;\,\circ} TX \overset{\mathfrak{x}^\circ}{\longrightarrow\!\!\!\!\!\!\!\!\!\!\!\!\;\;\;\;\;\;\;\;\;\;\,\circ} E\,) = \widehat{a}(\eta_X(\ ),\mathfrak{x})$. By hypothesis σ_X is fully faithful; moreover, it is an embedding because, by hypothesis, TX and PX are separated V-categories.

To show that $\sigma = (\sigma_X)_X\colon T \to P$ is a natural transformation, that is, for each V-functor $f\colon X \to Y$, the outer diagram

$$\begin{tikzcd} TX \arrow[r, "y_{TX}"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\boxed{1}"] & PTX \arrow[r, "Q\eta_X"] \arrow[d, "PTf"] \arrow[dr, phantom, "\boxed{2}"] & PX \arrow[d, "Pf"] \\ TY \arrow[r, "y_{TY}"'] & PTY \arrow[r, "Q\eta_Y"'] & PY \end{tikzcd}$$

commutes, we only need to observe that $\boxed{1}$ is commutative and BC* for η implies that $\boxed{2}$ is commutative.

It remains to show σ is a monad morphism : for each V-category (X,a) and $x \in X$,

$$(\sigma_X \cdot \eta_X)(x) = \widehat{a}(\eta_X(\ ),\eta_X(x)) = a(-,x) = x^* = y_X(x),$$

and so $\sigma \cdot \eta = \mathsf{y}$. To check that, for every V-category (X,a), the following diagram commutes

$$\begin{tikzcd} TTX \arrow[r, "\sigma_{TX}"] \arrow[d, "\mu"'] & PTX \arrow[r, "P\sigma_X"] & PPX \arrow[d, "m_X"] \\ TX \arrow[rr, "\sigma_X"'] & & PX, \end{tikzcd}$$

let $\mathfrak{X} \in TTX$. We have

$$m_X \cdot P\sigma_X \cdot \sigma_{TX}(\mathfrak{X}) = (\ X \xrightarrow{\mathsf{y}_X} PX \overset{\widetilde{a}}{\relbar\joinrel\mapstochar\rightarrow} PX \overset{\sigma_X^\circ}{\relbar\joinrel\mapstochar\rightarrow} TX \xrightarrow{\eta_{TX}} TTX \overset{\widehat{\widehat{a}}}{\relbar\joinrel\mapstochar\rightarrow} TTX \overset{\mathfrak{X}^\circ}{\relbar\joinrel\mapstochar\rightarrow} E\ )$$
$$= (\ X \xrightarrow{\eta_X} TX \overset{\widehat{a}}{\relbar\joinrel\mapstochar\rightarrow} TX \xrightarrow{\eta_{TX}} TTX \overset{\widehat{\widehat{a}}}{\relbar\joinrel\mapstochar\rightarrow} TTX \overset{\mathfrak{X}^\circ}{\relbar\joinrel\mapstochar\rightarrow} E\ ),$$

since $\sigma_X^\circ \cdot \widetilde{a} \cdot \mathsf{y}_X(x,\mathfrak{x}) = \widetilde{a}(\mathsf{y}_X(x),\sigma_X(\mathfrak{x})) = \sigma_X(\mathfrak{x})(x) = \widehat{a} \cdot \eta_X(x,\mathfrak{x})$, and

$$\sigma_X \cdot \mu_X(\mathfrak{X}) = (\ X \xrightarrow{\eta_X} TX \overset{\widehat{a}}{\relbar\joinrel\mapstochar\rightarrow} TX \overset{\mu_X^\circ}{\relbar\joinrel\mapstochar\rightarrow} TTX \overset{\mathfrak{X}^\circ}{\relbar\joinrel\mapstochar\rightarrow} E\ ).$$

Hence the commutativity of the diagram follows from the equality $\widehat{\widehat{a}} \cdot \eta_{TX} \cdot \widehat{a} \cdot \eta_X = \mu_X^\circ \cdot \widehat{a} \cdot \eta_X$, which we show next. Indeed,

$$\widehat{\widehat{a}} \cdot \eta_{TX} \cdot \widehat{a} \cdot \eta_X = (\eta_{TX})_*(\eta_X)_* = (\eta_{TX} \cdot \eta_X)_* = (T\eta_X \cdot \eta_X)_* = (T\eta_X)_*(\eta_X)_* = \mu_X^*(\eta_X)_* = \mu_X^\circ \cdot \widehat{a} \cdot \eta_X.$$

$\square$

The proof of the theorem allows us to conclude immediately the following corollary.

Corollary 5.3.13. *Given an* **Ord***-monad* $\mathbb{T} = (T,\mu,\eta)$ *on* V-**Cat** *such that* η *satisfies BC*, there is a monad morphism* $\mathbb{T} \to \mathbb{P}$ *if, and only if,* η *is pointwise fully faithful.* $\square$

In the particular case of the formal ball monad, using corollaries 5.3.11 and 5.3.13, we have the following result:

Proposition 5.3.14. *There is a pointwise fully faithful monad morphism* $\sigma\colon \mathbb{B}_\bullet \to \mathbb{P}$.

Proof. First of all let us check that η satisfies BC*, i.e., for any V-functor $f\colon X \to Y$,

$$\begin{tikzcd} X \arrow[r, "(\eta_X)_*", "\circ" marking] \arrow[dr, phantom, "\geq"] & B_\bullet X \\ Y \arrow[u, "f^*", "\circ" marking] \arrow[r, "(\eta_Y)_*"', "\circ" marking] & B_\bullet Y \arrow[u, "(B_\bullet f)^*"', "\circ" marking] \end{tikzcd}$$

For $y \in Y$, $(x,r) \in B_\bullet X$,

$$\begin{aligned}((B_\bullet f)^*(\eta_Y)_*)(y,(x,r)) &= B_\bullet Y((y,k),(f(x),r)) = Y(y,f(x)) \otimes r \\ &\leq \bigvee_{z\in X} Y(y,f(z)) \otimes X(z,x) \otimes r \\ &= \bigvee_{z\in X} Y(y,f(z)) \otimes B_\bullet X((z,k),(x,r)) \\ &= ((\eta_X)_* f^*)(y,(x,r)).\end{aligned}$$

Then, by Corollary 5.3.13, for each V-category X, σ_X is defined as in the proof of Theorem 5.3.12, i.e. for each $(x,r) \in B_\bullet X$, $\sigma_X(x,r) = B_\bullet X((-,k),(x,r))\colon X \to V$; more precisely, for each $y \in X$, $\sigma_X(x,r)(y) = X(y,x) \otimes r$. Moreover, σ_X is fully faithful: for each $(x,r),(y,s) \in B_\bullet X$,

$$\begin{aligned}B_\bullet X((x,r),(y,s)) &= \hom(r, X(x,y) \otimes s) \\ &\geq \hom(X(x,x) \otimes r, X(x,y) \otimes s) \\ &\geq \bigwedge_{z\in X} \hom(X(z,x) \otimes r, X(z,y) \otimes s) = PX(\sigma_X(x,r),\sigma_X(y,s)).\end{aligned}$$

$\square$

Note however that $\sigma\colon \mathbb{B}_\bullet \to \mathbb{P}$ is not pointwise monic; indeed, if $r = \bot$, then $\sigma_X(x,\bot)\colon X \,{-\!\!\circ\!\!\to}\, E$ is the V-distributor that is constantly $\bot$, for any $x \in X$. Therefore $\mathbb{B}_\bullet$ is not a submonad of $\mathbb{P}$. Yet, $\mathbb{B}$ is a submonad of $\mathbb{P}$ in the adequate setting, as we will see in Subsection 5.5.1.

5.4 $\mathbb{T}$-algebras and T-embeddings for submonads of the presheaf monad $\mathbb{P}$

Next we will study the $\mathbb{T}$-algebras and $\mathbb{T}$-embeddings for a given submonad $\mathbb{T}$ of the presheaf monad $\mathbb{P}$.

Let $\Phi(\mathbb{T})$ denote the class of admissible V-distributors that induces the monad $\mathbb{T}$ (as defined in (5.3.i)). We show that for any submonad $\mathbb{T}$ of $\mathbb{P}$, the class $\Phi(\mathbb{T})$ allows us to readily identify T-embeddings.

Proposition 5.4.1. *For a V-functor $h\colon X \to Y$, the following assertions are equivalent:*

(i) *h is a T-embedding;*

(ii) *h is fully faithful and h_* belongs to $\Phi(\mathbb{T})$.*

In particular, P-embeddings are exactly the fully faithful V-functors.

Proof. (ii) $\Rightarrow$ (i): Let h be fully faithful with $h_* \in \Phi(\mathbb{T})$. As in the case of the presheaf monad, $\Phi h : \Phi X \to \Phi Y$ has always a right adjoint whenever $h_* \in \Phi(\mathbb{T})$, $\Phi^{\dashv} h := (-)\cdot h_*\colon \Phi Y \to \Phi X$; that is, for each V-distributor $\psi : Y \,{-\!\!\circ\!\!\to}\, E$ in ΦY, $\Phi^{\dashv} h(\psi) = \psi \cdot h_*$, which is well defined because by hypothesis $h_* \in \Phi(\mathbb{T})$. If h is fully faithful, that is, if $h^* \cdot h_* = (1_X)^*$, then $(\Phi^{\dashv} h \cdot \Phi h)(\varphi) = \varphi \cdot h^* \cdot h_* = \varphi$.

(i) $\Rightarrow$ (ii): If $\Phi^{\dashv}h$ is well-defined, then $y^* \cdot h_*$ belongs to $\Phi(\mathbb{T})$ for every $y \in Y$, hence $h_* \in \Phi(\mathbb{T})$, by 5.3.1(3), and so $h_* \in \Phi(\mathbb{T})$. Moreover, if $\Phi^{\dashv}h \cdot \Phi h = 1_{\Phi X}$, then in particular $x^* \cdot h^* \cdot h_* = x^*$, for every $x \in X$, which is easily seen to be equivalent to $h^* \cdot h_* = (1_X)^*$. □

Next we present characterisations of $\mathbb{T}$-algebras using the existence of certain weighted colimits.

Theorem 5.4.2. *Given a submonad $\mathbb{T}$ of $\mathbb{P}$, for a V-category X the following assertions are equivalent:*

(i) *X is a $\mathbb{T}$-algebra.*

(ii) *X is $\Phi(\mathbb{T})$-cocomplete.*

The result above could already be found in [8].

From this result and Proposition 3.4.13 we get immediately that:

Corollary 5.4.3. *Given a submonad $\mathbb{T}$ of $\mathbb{P}$, a V-category X is a $\mathbb{T}$-algebra if, and only if, $[\varphi,(1_X)_*]$ has a colimit for every $\varphi \in TX$.*

Remark 5.4.4. Given $\varphi\colon X \mathrel{\circ\!\!\rightarrow} E$ in TX, in the diagram

$$\begin{tikzcd} X \arrow[r, "a", "\circ"{marking}] \arrow[d, "\varphi"', "\circ"{marking}] \arrow[dr, phantom, "\leq"] & X \\ E \arrow[ur, "{[\varphi,a]}"', "\circ"{marking}] & \end{tikzcd}$$

$$[\varphi,a](*,x) = \bigwedge_{x' \in X} \hom(\varphi(x',*), a(x',x)) = TX(\varphi, x^*).$$

Therefore, if $\alpha\colon TX \to X$ is a $\mathbb{T}$-algebra structure, then

$$[\varphi,a](*,x) = TX(\varphi,x^*) = X(\alpha(\varphi),x),$$

that is, $[\varphi,a] = \alpha(\varphi)_*$; this means that α assigns to each V-distributor $\varphi\colon X \mathrel{\circ\!\!\rightarrow} E$ the representative of $[\varphi,(1_X)_*]$.

Hence, we may describe the category of $\mathbb{T}$-algebras as follows.

Theorem 5.4.5.
1. *A map $\alpha\colon TX \to X$ is a $\mathbb{T}$-algebra structure if, and only if, for each V-distributor $\varphi\colon X \mathrel{\circ\!\!\rightarrow} E$ in TX, $\alpha(\varphi)_* = [\varphi,(1_X)_*]$.*
2. *If X and Y are $\mathbb{T}$-algebras, then a V-functor $f\colon X \to Y$ is a $\mathbb{T}$-homomorphism if, and only if, f preserves φ-weighted colimits for any $\varphi \in TX$, i.e., if $x \in X$ represents $[\varphi,(1_X)_*]$, then $f(x)$ represents $[\varphi \cdot f^*,(1_Y)_*]$.*

5.5 Restriction to V-$\mathbf{Cat}_{\mathrm{sep}}$

In this section we will restrict our focus to V-$\mathbf{Cat}_{\mathrm{sep}}$. This will allow us to obtain a few simple but still noteworthy results. So, in this subsection let $\mathbb{T} = (T,\mu,\eta)$ be a submonad of the presheaf monad $\mathbb{P}$ in

V-$\mathbf{Cat}_{\mathrm{sep}}$. For simplicity we will assume that the injective and fully faithful components of the monad morphism $\sigma : T \to P$ are inclusions. We get immediately from Corollary 2.2.15 that:

Proposition 5.5.1. *Let (X,a) be a V-category and $\alpha : TX \to X$ be a V-functor. The following assertions are equivalent:*

1. *(X,α) is a $\mathbb{T}$-algebra;*
2. *$\forall x \in X : \alpha(x^*) = x$.*

We would like to identify the $\mathbb{T}$-algebras directly, as we did for $\mathbb{B}_\bullet$ or $\mathbb{B}$ in Proposition 5.1.3. First of all, we point out that a $\mathbb{T}$-algebra structure $\alpha\colon TX \to X$ must satisfy, for every $\varphi \in TX$ and $x \in X$,

$$X(\alpha(\varphi),x) = TX(\varphi,x^*),$$

and so, in particular,

$$\alpha(\varphi) \leq x \;\Leftrightarrow\; \varphi \leq x^*;$$

hence α must assign to each $\varphi \in TX$ an $x_\varphi \in X$ so that

$$x_\varphi = \min\{x \in X\,;\, \varphi \leq x^*\}.$$

Moreover, for such map $\alpha\colon TX \to X$, α is a V-functor if, and only if,

$$\begin{aligned}
&(\forall \varphi,\rho \in TX)\;\; TX(\varphi,\rho) \leq X(x_\varphi,x_\rho) = TX(X(-,x_\varphi),X(-,x_\rho))\\
\Leftrightarrow\; &(\forall \varphi,\rho \in TX)\;\; TX(\varphi,\rho) \leq \bigwedge_{x\in X} \hom(X(x,x_\varphi),X(x,x_\rho))\\
\Leftrightarrow\; &(\forall x \in X)\;(\forall \varphi,\rho \in TX)\;\; X(x,x_\varphi)\otimes TX(\varphi,\rho) \leq X(x,x_\rho).
\end{aligned}$$

Therefore we have the following proposition:

Proposition 5.5.2. *A V-category X is a $\mathbb{T}$-algebra if, and only if:*

1. *for all $\varphi \in TX$ $\min\{x \in X\,;\, \varphi \leq x^*\}$ exists;*
2. *for all $\varphi,\rho \in TX$ and for all $x \in X$, $X(x,x_\varphi)\otimes TX(\varphi,\rho) \leq X(x,x_\rho)$.*

We note that condition (2) is equivalent to:

(2') for each $\rho \in TX$, the V-distributor $\rho_1 = \bigvee_{\varphi\in TX} X(-,x_\varphi)\otimes TX(\varphi,\rho)$ satisfies $x_{\rho_1} = x_\rho$,

which corresponds to condition (2) of Corollary 5.1.7.

Moreover, as for the formal ball monad, we have the following characterisation of $\mathbb{T}$-algebra morphisms.

Corollary 5.5.3. *Let $(X,\alpha),(Y,\beta)$ be $\mathbb{T}$-algebras. Then a V-functor $f : X \to Y$ is a $\mathbb{T}$-algebra morphism if and only if*

$$(\forall \varphi \in TX)\;\; \beta(\varphi\cdot f^*) \geq f(\alpha(\varphi)).$$

5.5.1 The formal ball monad $\mathbb{B}$ as a submonad of the presheaf monad $\mathbb{P}$

$B_{\bullet}X$ is not separated if X has more than one element (note that $(x,\perp) \simeq (y,\perp)$ for any $x,y \in X$) we must dispense with the formal balls with a radius of $\perp$. This will allows us to obtain some interesting results. Unfortunately X being separated does not entail BX being so. Because of this we will need to restrict our attention to the *cancellative* quantales which we define and characterise next.

Cancellative quantales

Definition 5.5.4. A quantale V is said to be **cancellative** if $\forall\, r,s \in V$, $r \neq \perp$:

$$r = s \otimes r \;\Rightarrow\; s = k.$$

We note that this new notion of cancellative quantale does not coincide with the notion of cancellable ccd quantale introduced in [9]. On the one hand cancellative quantales are quite special, since, for instance, when V is the quantale defined by a frame with more than 2 elements, V is not cancellative since, by definition, we would have $r = s \wedge r \;\Rightarrow\; s = \top$, for any $r,s \in V$, $r \neq \perp$. On the other hand, the Łukasiewicz quantale $[0,1]_{\odot}$ is cancellative but not cancellable. Moreover we point out that every *value quantale* [18, 26] is cancellative.

For integral quantales we have the following characterization:

Proposition 5.5.5. *Let V be an integral quantale. The following assertions are equivalent:*

(i) *BV is separated;*

(ii) *V is cancellative;*

(iii) *If X is separated then BX is separated.*

Proof. (i) $\Rightarrow$ (ii): Let $r,s \in V$, $r \neq \perp$ and $r = s \otimes r$. Note that

$$BV((k,r),(s,r)) = \hom(r,\hom(k,s)\otimes r) = \hom(r,s\otimes r) = \hom(r,r) = k$$

and

$$BV((s,r),(k,r)) = \hom(r,\hom(s,k)\otimes r) = \hom(r,\hom(s,k)\otimes s\otimes r) = \hom(s\otimes r, s\otimes r) = k.$$

Therefore, since BV is separated, $(s,r) = (k,r)$ and it follows that $s = k$.

(ii) $\Rightarrow$ (iii): If $(x,r) \simeq (y,s)$ in BX, then

$$BX((x,r),(y,s)) = k \Leftrightarrow r \leq X(x,y)\otimes s, \text{ and}$$

$$BX((y,s),(x,r)) = k \Leftrightarrow s \leq X(y,x)\otimes r.$$

Therefore $r \leq s$ and $s \leq r$, that is $r = s$. Moreover, since $r \leq X(x,y)\otimes r \leq r$ we have that $X(x,y) = k$. Analogously, $X(y,x) = k$ and we conclude that $x = y$.

(iii) $\Rightarrow$ (i): Since V is separated it follows immediately from (iii) that BV is separated. $\square$

We are now ready to show that $\mathbb{B}$ is a submonad of $\mathbb{P}$ in the suitable setting.

Proposition 5.5.6. *Let V be a cancellative and integral quantale. Then $\mathbb{B}$ is a submonad of $\mathbb{P}$ in V-$\mathbf{Cat}_{\mathrm{sep}}$.*

Proof. By Proposition 5.3.14, all that remains is to show that σ_X is injective on objects, for any V-category X. Let $\sigma_X(x,r) = \sigma_X(y,s)$, or, equivalently, $X(-,x)\otimes r = X(-,y)\otimes s$. Then, in particular,

$$r = X(x,x)\otimes r = X(x,y)\otimes s \leq s = X(y,y)\otimes s = X(y,x)\otimes r \leq r.$$

Therefore $r = s$ and $X(y,x) = X(x,y) = k$. We conclude that $(x,r) = (y,s)$. $\square$

B-embeddings

We end this section studying B-embeddings. In this subsection we will restrict our attention to a cancellative and integral quantale V, and $\mathbb{B}$ will be the (co)restriction of the formal ball monad to V-$\mathbf{Cat}_{\mathrm{sep}}$. Since we are working in V-$\mathbf{Cat}_{\mathrm{sep}}$, a B-embedding $h\colon X \to Y$, being fully faithful, is injective on objects. Therefore, for simplicity, we may think of it as an inclusion. With $Bh_\sharp\colon BY \to BX$ the right adjoint and left inverse of $Bh\colon BX \to BY$, we denote $Bh_\sharp(y,r)$ by (y_r, r_y).

Lemma 5.5.7. *Let $h\colon X \to Y$ be a B-embedding. Then:*

1. $(\forall y \in Y)\ (\forall x \in X)\ (\forall r \in V)\colon BY((x,r),(y,r)) = BY((x,r),(y_r,r_y))$*;*
2. $(\forall y \in Y)\colon k_y = Y(y_k, y)$*;*
3. $(\forall y \in Y)\ (\forall x \in X)\colon\quad Y(x,y) = Y(x,y_k)\otimes Y(y_k,y)$.

Proof. (1) From $Bh_\sharp \cdot Bh = 1_{BX}$ and $Bh\cdot Bh_\sharp \leq 1_{BY}$ one gets, for any $(y,r)\in BY$, $(y_r,r_y)\leq (y,r)$, i.e. $k \leq BY((y_r,r_y),(y,r))$. Therefore, for all $x\in X$, $y\in Y$, $r\in V$,

$$\begin{aligned} BY((x,r),(y,r)) &\leq BX((x,r),(y_r,r_y)) = BY((x,r),(y_r,r_y)) \\ &\leq BY((x,r),(y_r,r_y))\otimes BY((y_r,r_y),(y,r)) \leq BY((x,r),(y,r)), \end{aligned}$$

that is

$$BY((x,r),(y,r)) = BY((x,r),(y_r,r_y)).$$

(2) Let $y\in Y$. Then

$$Y(y_k,y) = BY((y_k,k),(y,k)) = BY((y_k,k),(y_k,k_y)) = k_y.$$

(3) Let $y\in Y$ and $x\in X$. Then

$$Y(x,y) = BY((x,k),(y,k)) = BY((x,k),(y_k,k_y)) = Y(x,y_k)\otimes k_y = Y(x,y_k)\otimes Y(y_k,y).$$

$\square$

Proposition 5.5.8. *Let X and Y be V-categories. A V-functor $h\colon X \to Y$ is a B-embedding if and only if h is fully faithful and*

$$(\forall y \in Y)\ (\exists! z \in X)\ (\forall x \in X) \quad Y(x,y) = Y(x,z) \otimes Y(z,y). \tag{5.5.i}$$

Proof. If h is a B-embedding, then it is fully faithful by Proposition 5.4.1 and, for each $y \in Y$, $z = y_k \in X$ fulfils the required condition. To show that such z is unique, assume that $z, z' \in X$ verify the equality of condition (5.5.i). Then

$$Y(z,y) = Y(z,z') \otimes Y(z',y) \le Y(z',y) = Y(z',z) \otimes Y(z,y) \le Y(z,y),$$

and therefore, because V is cancellative, $Y(z',z) = k$; analogously one proves that $Y(z,z') = k$, and so $z = z'$ because Y is separated.

To prove the converse, for each $y \in Y$ we denote by $\bar{y}$ the only $z \in X$ satisfying (5.5.i), and define

$$Bh_\sharp(y,r) = (\bar{y}, Y(\bar{y},y) \otimes r).$$

When $x \in X$, it is immediate that $\bar{x} = x$, and so $Bh_\sharp \cdot Bh = 1_{BX}$. Using Remark 3.4.9, to prove that $Bh_\sharp$ is a V-functor and $Bh \dashv Bh_\sharp$ it is enough to show that

$$BX((x,r), Bh_\sharp(y,s)) = BY(Bh(x,r),(y,s)),$$

for every $x \in X$, $y \in Y$, $r,s \in V$. By definition of $Bh_\sharp$ this means

$$BX((x,r),(\bar{y}, Y(\bar{y},y) \otimes s)) = BY((x,r),(y,s)),$$

that is,

$$\hom(r, Y(x,\bar{y}) \otimes Y(\bar{y},y) \otimes s) = \hom(r, Y(x,y) \otimes s),$$

which follows directly from (5.5.i). □

Corollary 5.5.9. *In* **Met**, *if $X \subseteq [0,\infty]$, then its inclusion $h\colon X \to [0,\infty]$ is a B-embedding if, and only if, X is a closed interval.*

Proof. If $X = [x_0, x_1]$, with $x_0, x_1 \in [0,\infty]$, $x_0 \le x_1$, then it is easy to check that, defining $\bar{y} = x_0$ if $y \le x_0$, $\bar{y} = y$ if $y \in X$, and $\bar{y} = x_1$ if $y \ge x_1$, for every $y \in [0,\infty]$, condition (5.5.i) is fulfilled.

We divide the proof of the converse in two cases:

(1) If X is not an interval, i.e. if there exists $x, x' \in X$, $y \in [0,\infty] \setminus X$ with $x < y < x'$, then either $\bar{y} < y$, and then

$$0 = y \ominus x' \neq (y \ominus x') + (y \ominus \bar{y}) = y - \bar{y},$$

or $\bar{y} > y$, and then

$$y - x = y \ominus x \neq (\bar{y} \ominus x) + (y \ominus \bar{y}) = \bar{y} - x.$$

(2) If $X = [x_0, x_1[$ and $y > x_1$, then there exists $x \in X$ with $\bar{y} < x < y$, and so

$$y - x = y \ominus x \neq (\bar{y} \ominus x) + (y \ominus \bar{y}) = y - \bar{y}.$$

An analogous argument works for $X =]x_0, x_1]$. $\square$

5.6 The Lawvere monad and Cauchy completeness

In [8] Clementino and Hofmann study a particularly noteworthy submonad of $\mathbb{P}$. This submonad is directly related to one of the most fundamental remarks of Lawvere in [28]: that Cauchy completeness for metric spaces is akin to cocompleteness for V-categories. Consider the submonad $\mathbb{L}$ of $\mathbb{P}$ induced by the following class of V-distributors:

$$\Phi = \{\varphi \colon X \,{-\!\!\circ\!\!\to}\, Y \,;\, \varphi \text{ is a right adjoint } V\text{-distributor}\}$$

The $\mathbb{L}$-algebras are the **Lawvere complete V-categories**. These algebras were also studied in [7] and in [23].

We will briefly explore the submonad $\mathbb{L}$, when $V = [0, \infty]_+$ is the Lawvere quantale.

Note that V-distributors $\varphi \colon X \,{-\!\!\circ\!\!\to}\, E$, $\psi \colon E \,{-\!\!\circ\!\!\to}\, X$ are adjoint

$$\begin{tikzcd} X \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\top"] & E \arrow[l, bend left, "\psi"] \end{tikzcd}$$

precisely when

$$0 \geq \inf_{x \in X} (\psi(x) + \varphi(x)),$$
$$(\forall x, x' \in X)\ \ X(x, x') \leq \varphi(x) + \psi(x').$$

In particular

$$(\forall n \in \mathbb{N})\ (\exists x_n \in X)\ \ \psi(x_n) + \varphi(x_n) \leq \frac{1}{n},$$

and

$$X(x_n, x_m) \leq \varphi(x_n) + \psi(x_m) \leq \frac{1}{n} + \frac{1}{m}.$$

So we have a **Cauchy sequence** $(x_n)_n$:

$$(\forall \varepsilon > 0)\ (\exists p \in \mathbb{N})\ (\forall n, m \in \mathbb{N})\ n \geq p \wedge m \geq p \ \Rightarrow\ X(x_n, x_m) + X(x_m, x_n) < \varepsilon.$$

Therefore, any pair of adjoint V-distributors induces an equivalence class of Cauchy sequences $(x_n)_n$, and a representative for

$$\begin{tikzcd} X \arrow[r, "(1_X)_*", "\circ" marking] \arrow[d, "\varphi"', "\circ" marking] \arrow[dr, phantom, "\leq"] & X \\ E \arrow[ur, "{[\varphi,(1_X)_*]}"', "\circ" marking] & \end{tikzcd}$$

is just a limit point for $(x_n)_n$. On the other hand, it is readily verified that every Cauchy sequence $(x_n)_n$ in X defines a pair of adjoint V-distributors

$$\varphi = \lim_n X(-,x_n) \text{ and } \psi = \lim_n X(x_n,-).$$

We note that the $\mathbb{L}$-embeddings, i.e. the fully faithful and fully dense V-functors $f\colon X \to Y$ do not coincide with the $\mathbb{L}$-dense ones, i.e. those such that f_* is a right adjoint. For instance, assuming for simplicity that V is integral, a V-functor $y\colon E \to X$ ($y \in X$) is fully dense if and only if $y \simeq x$ for all $x \in X$, while it is an $\mathbb{L}$-embedding if and only if $y \leq x$ for all $x \in X$. Indeed, $y\colon E \to X$ is $\mathbb{L}$-dense if, and only if,

1. there is a V-distributor $\varphi\colon X \,{\circ\!\!\to}\, E$, i.e.

$$(\forall x,x' \in X)\;\; X(x,x') \otimes \varphi(x') \leq \varphi(x),$$

such that

2. $k \geq \varphi \cdot y_*$, which is trivially true, and $a \leq y_* \cdot \varphi$, i.e.

$$(\forall x,x' \in X)\;\; X(x,x') \leq \varphi(x) \otimes X(y,x').$$

Therefore

$$y \text{ is } \mathbb{L}\text{-dense} \;\;\Leftrightarrow\;\; (\forall x,x' \in X)\;\; X(x,x') \leq \varphi(x) \otimes X(y,x').$$

In particular, when $x = x'$, this gives $k \leq \varphi(x) \otimes X(y,x)$, and so we can conclude that, for all $x \in X$, $y \leq x$ and $\varphi(x) = k$. The converse is also true; that is

$$y \text{ is } \mathbb{L}\text{-dense} \;\;\Leftrightarrow\;\; (\forall x \in X)\;\; y \leq x.$$

We also note that in [23] it was shown that injectivity with respect to fully dense and fully faithful V-functors characterises the $\mathbb{L}$-algebras.

Chapter 6

Idempotent completeness

Given a category **C**, the **Karoubi envelope** of **C** is the “smallest" category which contains **C** such that every idempotent is a split idempotent. Such a category is also called the idempotent completion of **C**. This concept can be generalised to a notion of Cauchy completion in an enriched category setting, but we choose to retain the classical terminology because it highlights better how our results parallel the classic theory.

Our main goal in this section is to show that the Karoubi envelope of the category V-**Rel** of V-relations is equivalent to the category $\mathrm{spl}(V\text{-}\mathbf{Cat}^{\mathbb{P}})$ of the split $\mathbb{P}$-algebras we describe in 6.2.2. To that end we prove a few elementary results and give a fully detailed proof of a theorem by Rosebrugh and Wood [32], which we require.

6.1 Idempotent splittings

Let X be a retract of Y, that is, let there be morphisms $r : Y \to X$ and $s : X \to Y$, such that $r \cdot s = 1_X$.

$$1_X :\ X \xrightarrow{s} Y \xrightarrow{r} X$$

Then the morphism $s \cdot r$

$$s \cdot r :\ Y \xrightarrow{r} X \xrightarrow{s} Y$$

is clearly idempotent:

$$(s \cdot r) \cdot (s \cdot r) = s \cdot (r \cdot s) \cdot r = s \cdot 1_X \cdot r = s \cdot r.$$

This is the fundamental example of a split idempotent. More precisely:

Definition 6.1.1. A morphism $e : Y \to Y$ in a category **C** is called a **split idempotent** if there is an object X which is a retract of Y

$$\begin{tikzcd} X \arrow[loop above, "1_X"] \arrow[r, shift left, "s"] & Y \arrow[loop above, "e"] \arrow[l, shift left, "r"] \end{tikzcd}$$

and $s \cdot r = e$. In such a case we say that e admits a **splitting** or that e **splits**.

The splitting of an idempotent e is given by the equaliser (or coequaliser) of the diagram

$$Y \underset{1_Y}{\overset{e}{\rightrightarrows}} Y$$

of two parallel morphisms $e : Y \to Y$ and $1_Y : Y \to Y$. In particular, given a category with all equalisers (or coequalisers) all idempotents split. We also note that splittings of idempotents are absolute limits and colimits, that is, they are preserved by any functor.

Definition 6.1.2. A category in which all idempotents split is called **idempotent complete**.

Definition 6.1.3. Let $\mathbf{C}$ be a category. We say that $\mathrm{kar}(\mathbf{C})$ is the **idempotent completion** of $\mathbf{C}$ if

- $\mathrm{kar}(\mathbf{C})$ is an idempotent complete category;
- there is a fully faithful functor $\iota : \mathbf{C} \to \mathrm{kar}(\mathbf{C})$
- every object in $\mathrm{kar}(\mathbf{C})$ is a retract of $\iota(X)$ for some object X in $\mathbf{C}$.

We also call $\mathrm{kar}(\mathbf{C})$ the **Karoubi envelope of C**.

The idempotent completion of a category always exists and is unique up to equivalence of categories.

Next we present a simple way to explicitly construct the idempotent completion of a given category $\mathbf{C}$.

Take as the objects of $\mathrm{kar}(\mathbf{C})$ the pairs (X,e), where X is an object of $\mathbf{C}$ and $e : X \to X$ is an idempotent morphism in $\mathbf{C}$. The morphisms $\varphi : (X,e) \to (Y,\tilde{e})$ of $\mathrm{kar}(\mathbf{C})$ are precisely the morphisms $\varphi : X \to Y$ of $\mathbf{C}$ such that

$$\tilde{e} \cdot \varphi = \varphi = \varphi \cdot e,$$

or equivalently, such that $\varphi = \tilde{e} \cdot \varphi \cdot e$. They compose as in $\mathbf{C}$ and the identity on (X,e) in $\mathrm{kar}(\mathbf{C})$ is the morphism $e : X \to X$.

There is a functor $\iota : \mathbf{C} \to \mathrm{kar}(\mathbf{C})$ which maps each object X of $\mathbf{C}$ to $(X,1_X)$ and each morphism $f : X \to Y$ of $\mathbf{C}$ to $f : (X,1_X) \to (Y,1_Y)$. Clearly the functor ι is fully faithful.

Given an idempotent $e : X \to X$ of $\mathbf{C}$ we define the following morphisms:

$$r : (X,1_X) \to (X,e) \qquad \text{and} \qquad s : (X,e) \to (X,1_X)$$

both given by e.

Clearly $r \cdot s : (X,e) \to (X,e)$ is the identity $e : (X,e) \to (X,e)$ and $s \cdot r : (X,1_X) \to (X,1_X)$ is $\iota(e) : \iota(X) \to \iota(X)$. Therefore $\iota(e)$ splits in $\mathrm{kar}(\mathbf{C})$, for every idempotent e in $\mathbf{C}$. It also follows that every object (X,e) in $\mathrm{kar}(\mathbf{C})$ is a retract of $\iota(X)$. All that is left is to show that $\mathrm{kar}(\mathbf{C})$ is idempotent complete. Let $e : (Y,\overline{e}) \to (Y,\overline{e})$ be an idempotent in $\mathrm{kar}(\mathbf{C})$. Since $(Y,\overline{e})$ is a retract of $\iota(X)$ for some X in $\mathbf{C}$, we have morphisms $r : \iota(X) \to (Y,\overline{e})$ and $s : (Y,\overline{e}) \to \iota(X)$ such that $r \cdot s = \overline{e}$.

$$\begin{tikzcd} (Y,\overline{e}) \arrow[loop above, "\overline{e}"] \arrow[r, shift left, "s"] & \iota(X) \arrow[l, shift left, "r"] \end{tikzcd}$$

Consider the morphism $f : \imath(X) \to \imath(X)$ given by

$$f : \ \imath(X) \xrightarrow{r} (Y,\overline{e}) \xrightarrow{e} (Y,\overline{e}) \xrightarrow{s} \imath(X) \ .$$

Clearly f is idempotent

$$f \cdot f = (s \cdot e \cdot r) \cdot (s \cdot e \cdot r) = s \cdot e \cdot (r \cdot s) \cdot e \cdot r = s \cdot e \cdot \overline{e} \cdot e \cdot r = s \cdot e \cdot r = f.$$

Since $\imath$ is fully faithful there is a morphism $g : X \to X$ in **C** such that $\imath(g) = f$. Moreover

$$\imath(g \cdot g) = \imath(g) \cdot \imath(g) = f \cdot f = f = \imath(g)$$

and therefore $g \cdot g = g$, that is, g is also idempotent. Since $\imath(g)$ splits for any idempotent g in **C** we may split $f = \imath(g)$ as

$$\begin{tikzcd} (Z,\hat{e}) \arrow[loop above, "\hat{e}"] \arrow[r, shift left, "t"] & \imath(X) \arrow[loop above, "f"] \arrow[l, shift left, "u"] \end{tikzcd}$$

for some object $(Z,\hat{e})$ in kar(**C**). Lastly, we show that $u \cdot s : (Y,\overline{e}) \to (Z,\hat{e})$ and $r \cdot t : (Z,\hat{e}) \to (Y,\overline{e})$ split e.

$$\begin{tikzcd} (Z,\hat{e}) \arrow[loop above, "\hat{e}"] \arrow[r, shift left, "t"] & \imath(X) \arrow[l, shift left, "u"] \arrow[r, shift left, "r"] & (Y,\overline{e}) \arrow[loop above, "e"] \arrow[l, shift left, "s"] \end{tikzcd}$$

Simply note that

$$(r \cdot t) \cdot (u \cdot s) = r \cdot (t \cdot u) \cdot s = r \cdot f \cdot s = r \cdot (s \cdot e \cdot r) \cdot s = (r \cdot s) \cdot e \cdot (r \cdot s) = \overline{e} \cdot e \cdot \overline{e} = e$$

and, since $u \cdot s \cdot e \cdot r = u \cdot f = u \cdot t \cdot u = \hat{e} \cdot u = u$, we have

$$(u \cdot s) \cdot (r \cdot t) = (u \cdot s \cdot e \cdot r \cdot s) \cdot (r \cdot t) = u \cdot s \cdot e \cdot \overline{e} \cdot r \cdot t = (u \cdot s \cdot e \cdot r) \cdot t = u \cdot t = \hat{e}$$

as required.

One of the most basic results in this setting is the equivalence between the Karoubi envelopes of **Rel** and **Idl**.

Proposition 6.1.4. kar(**Rel**) $\cong$ kar(**Idl**). ☐

Above, **Rel** denotes the category with sets as objects and relations as morphisms and **Idl** denotes the category with (pre)ordered sets as objects and (pre)ordered ideals, that is, relations $f : (X,a) \longrightarrow\!\!\!\!\!\!\mapsto (Y,b)$ such that $b \cdot f = f = f \cdot a$, as morphisms. Clearly, we may regard **Rel** as **2**-**Rel** and **Idl** as **2**-**Dist**, where **2** denotes the Boolean quantale. Therefore, the result above may be restated as

$$\text{kar}(\mathbf{2}\text{-}\mathbf{Rel}) \cong \text{kar}(\mathbf{2}\text{-}\mathbf{Dist}).$$

In fact, this result is readily generalisable to any quantale, as we show next.

Proposition 6.1.5. *The Karoubi envelopes of V-**Rel** and V-**Dist** are equivalent:*

$$\mathrm{kar}(V\text{-}\mathbf{Rel}) \cong \mathrm{kar}(V\text{-}\mathbf{Dist}).$$

Proof. We can embed V-**Rel** in V-**Dist** by equipping each set with its identity. Therefore kar(V-**Rel**) embeds in kar(V-**Dist**).

We will assume, without loss of generality, that kar(V-**Rel**) is presented like we described above.

Consider the functor $\iota : V\text{-}\mathbf{Dist} \to \mathrm{kar}(V\text{-}\mathbf{Rel})$ which maps each V-category (X,a) to the object (X,a) in kar(V-**Rel**) and each V-distributor $\varphi : (X,a) \multimap\!\!\!\!\rightarrow (Y,b)$ to the morphism $\varphi : (X,a) \to (Y,b)$ in kar(V-**Rel**). (Note that this functor is well defined since the V-relations equipping each V-category are transitive and reflexive, and therefore, idempotent. Moreover each V-distributor can be mapped to a morphism defined identically given that the necessary condition is precisely the condition that characterises V-distributors.) It is immediate that ι is a fully faithful embedding.

Given an idempotent $\varphi : (X,a) \multimap\!\!\!\!\rightarrow (X,a)$ we have the morphisms in kar(V-**Rel**)

$$r : (X,a) \to (X,\varphi) \qquad \text{and} \qquad s : (X,\varphi) \to (X,a)$$

given by φ. (Note that idempotent V-distributors are idempotent V-relations.)

Clearly $r \cdot s : (X,\varphi) \to (X,\varphi)$ is the identity $\varphi : (X,\varphi) \to (X,\varphi)$ and $s \cdot r : (X,a) \to (X,a)$ is $\iota(\varphi) : \iota(X,a) \to \iota(X,a)$.

This shows that $\iota(e)$ splits in kar(V-**Rel**) for every idempotent e in V-**Dist**. In particular we also have that every object in kar(V-**Rel**) is a retract of $\iota(X,a)$ for some (X,a) in V-**Dist**. □

6.2 Idempotent completeness and Eilenberg-Moore algebras

Next we will explore how idempotent completeness is related to the Kleisli and Eilenberg-Moore categories.

Proposition 6.2.1. *Let $\mathbb{T} = (T,\mu,\eta)$ be a monad on a idempotent complete category* **C**. *Then* $\mathbf{C}^{\mathbb{T}}$ *is also idempotent complete.*

Proof. Let $e : (Y,\alpha) \to (Y,\alpha)$ be an idempotent $\mathbb{T}$-homomorphism. Since **C** is idempotent complete we have morphisms $r : Y \to X$ and $s : X \to Y$ that split $e : Y \to Y$

$$\begin{tikzcd} X \arrow[loop above, "1_X"] \arrow[r, shift left, "s"] & Y \arrow[loop above, "e"] \arrow[l, shift left, "r"] \end{tikzcd} .$$

Next, we show that $(X, r\cdot\alpha\cdot Ts)$ is a $\mathbb{T}$-algebra. Indeed, using the naturality of η and the fact that α is a $\mathbb{T}$-algebra structure, we have

$$(r\cdot\alpha\cdot Ts)\cdot\eta_X = r\cdot\alpha\cdot(Ts\cdot\eta_X) = r\cdot\alpha\cdot(\eta_Y\cdot s) = r\cdot(\alpha\cdot\eta_Y)\cdot s = r\cdot 1_Y\cdot s = r\cdot s = 1_X,$$

and

$$\begin{aligned}
(r\cdot\alpha\cdot Ts)\cdot\mu_X &= r\cdot\alpha\cdot(Ts\cdot\mu_X)\\
&= r\cdot\alpha\cdot(\mu_Y\cdot TTs) && \text{(by naturality of } \mu)\\
&= r\cdot(\alpha\cdot\mu_Y)\cdot TTs\\
&= r\cdot(\alpha\cdot T\alpha)\cdot TTs && (\alpha \text{ is a } \mathbb{T}\text{-algebra structure})\\
&= 1_X\cdot r\cdot\alpha\cdot T\alpha\cdot TTs\\
&= r\cdot s\cdot r\cdot\alpha\cdot T\alpha\cdot TTs\\
&= r\cdot(e\cdot\alpha)\cdot T\alpha\cdot TTs\\
&= r\cdot(\alpha\cdot Te)\cdot T\alpha\cdot TTs && (e \text{ is a } \mathbb{T}\text{-homomorphism})\\
&= r\cdot\alpha\cdot T(s\cdot r)\cdot T\alpha\cdot TTs\\
&= (r\cdot\alpha\cdot Ts)\cdot T(r\cdot\alpha\cdot Ts).
\end{aligned}$$

Note that $r:(Y,\alpha)\to(X,r\cdot\alpha\cdot Ts)$ and $s:(X,r\cdot\alpha\cdot Ts)\to(Y,\alpha)$ are $\mathbb{T}$-homomorphisms since

$$(r\cdot\alpha\cdot Ts)\cdot Tr = r\cdot\alpha\cdot T(s\cdot r) = r\cdot(\alpha\cdot Te) = r\cdot(e\cdot\alpha) = r\cdot s\cdot r\cdot\alpha = 1_X\cdot r\cdot\alpha = r\cdot\alpha$$

and

$$s\cdot(r\cdot\alpha\cdot Ts) = (e\cdot\alpha)\cdot Ts = (\alpha\cdot Te)\cdot Ts = \alpha\cdot T(s\cdot r)\cdot Ts = \alpha\cdot Ts\cdot T(r\cdot s) = \alpha\cdot Ts\cdot 1_{TX} = \alpha\cdot Ts.$$

Clearly the $\mathbb{T}$-homomorphisms r and s split $e:(Y,\alpha)\to(Y,\alpha)$, as required. □

Definition 6.2.2. Let $\mathbb{T}=(T,\mu,\eta)$ be a monad on a category $\mathbf{C}$. The **category of split $\mathbb{T}$-algebras** spl($\mathbf{C}^{\mathbb{T}}$) has as objects the triples (X,α,b), where (X,α) is a $\mathbb{T}$-algebra and $b:(X,\alpha)\to(TX,\mu_X)$ is such that $\alpha\cdot b=1_X$, and as morphisms $f:(X,\alpha,b)\to(\tilde{X},\tilde{\alpha},\tilde{b})$ the $\mathbb{T}$-homomorphisms $f:(X,\alpha)\to(\tilde{X},\tilde{\alpha})$.

Equivalently, we could have defined spl($\mathbf{C}^{\mathbb{T}}$) to be the full subcategory of $\mathbf{C}^{\mathbb{T}}$ defined by the algebras (X,α) which admit a $\mathbb{T}$-homomorphism $b:(X,\alpha)\to(TX,\mu_X)$, such that $\alpha\cdot b=1_X$ in $\mathbf{C}^{\mathbb{T}}$, because given two distinct $\mathbb{T}$-homomorphisms, say b and $\tilde{b}$, such that $b\cdot\alpha=1_X=\tilde{b}\cdot\alpha$ the objects (X,α,b) and $(X,\alpha,\tilde{b})$ are isomorphic in spl($\mathbf{C}^{\mathbb{T}}$).

In the remainder of this section we will consider a monad $\mathbb{T}$ on an idempotent complete category $\mathbf{C}$ and the Kleisli category $\mathbf{C}_{\mathbb{T}}$ as the full subcategory of $\mathbf{C}^{\mathbb{T}}$ defined by the free $\mathbb{T}$-algebras (TX,μ_X).

Let $e:(TY,\mu_Y)\to(TY,\mu_Y)$ be an idempotent $\mathbb{T}$-homomorphism. Then $((TY,\mu_Y),e)$ is an object of kar($\mathbf{C}_{\mathbb{T}}$). Moreover, as in the proof of Proposition 6.2.1, we can split e in $\mathbf{C}^{\mathbb{T}}$ as

$$\begin{tikzcd} X \arrow[loop above, "1_X"] \arrow[r, shift left, "s"] & TY \arrow[loop above, "e"] \arrow[l, shift left, "r"] \end{tikzcd}$$

where the $\mathbb{T}$-algebra structure equipping X is given by $\alpha := r \cdot \mu_Y \cdot Ts$. Consider now

$$b := Tr \cdot T\eta_Y \cdot s : (X, \alpha) \to (TX, \mu_X).$$

One readily verifies that b is a $\mathbb{T}$-homomorphism and

$$\alpha \cdot b = (r \cdot \mu_Y \cdot Ts) \cdot (Tr \cdot T\eta_Y \cdot s) = r \cdot \mu_Y \cdot Te \cdot T\eta_Y \cdot s = r \cdot e \cdot \mu_Y \cdot T\eta_Y \cdot s = r \cdot e \cdot s = 1_X.$$

Therefore (X, α, b) is an object of $\mathrm{spl}(\mathbf{C}^{\mathbb{T}})$.

Now, let $f : ((TY, \mu_Y), e) \to ((T\tilde{Y}, \mu_{\tilde{Y}}), \tilde{e})$ be a morphism in $\mathrm{kar}(\mathbf{C}_{\mathbb{T}})$. Then, exactly as we did for e, we may split $\tilde{e}$ in $\mathbf{C}^{\mathbb{T}}$ as

$$\begin{tikzcd} \tilde{X} \arrow[loop above, "1_{\tilde{X}}"] \arrow[r, shift left, "\tilde{s}"] & T\tilde{Y} \arrow[loop above, "\tilde{e}"] \arrow[l, shift left, "\tilde{r}"] \end{tikzcd}$$

where the $\mathbb{T}$-algebra structure equipping $\tilde{X}$ is given by $\tilde{\alpha} := \tilde{r} \cdot \mu_{\tilde{Y}} \cdot T\tilde{s}$ and $(\tilde{X}, \tilde{\alpha}, \tilde{b} := T\tilde{r} \cdot T\eta_{\tilde{Y}} \cdot \tilde{s})$ is an object of $\mathrm{spl}(\mathbf{C}^{\mathbb{T}})$. Then $\tilde{r} \cdot f \cdot s : (X, \alpha, b) \to (\tilde{X}, \tilde{\alpha}, \tilde{b})$ is a $\mathbb{T}$-homomorphism:

$$\begin{aligned}
\tilde{\alpha} \cdot T(\tilde{r} \cdot f \cdot s) &= \tilde{r} \cdot \mu_{\tilde{Y}} \cdot T\tilde{s} \cdot T(\tilde{r} \cdot f \cdot s) \\
&= \tilde{r} \cdot \mu_{\tilde{Y}} \cdot T(\tilde{s} \cdot \tilde{r}) \cdot Tf \cdot Ts \\
&= \tilde{r} \cdot (\mu_{\tilde{Y}} \cdot T\tilde{e}) \cdot Tf \cdot Ts \\
&= \tilde{r} \cdot (\tilde{e} \cdot \mu_{\tilde{Y}}) \cdot Tf \cdot Ts && (\tilde{e} \text{ is a } \mathbb{T}\text{-homomorphism}) \\
&= \tilde{r} \cdot \tilde{e} \cdot (\mu_{\tilde{Y}} \cdot Tf) \cdot Ts \\
&= \tilde{r} \cdot \tilde{e} \cdot (f \cdot \mu_Y) \cdot Ts && (f \text{ is a } \mathbb{T}\text{-homomorphism}) \\
&= \tilde{r} \cdot (\tilde{e} \cdot f) \cdot \mu_Y \cdot Ts \\
&= \tilde{r} \cdot (f \cdot e) \cdot \mu_Y \cdot Ts && (f \text{ is a morphism in } \mathrm{kar}(\mathbf{C}_{\mathbb{T}})) \\
&= \tilde{r} \cdot f \cdot (s \cdot r) \cdot \mu_Y \cdot Ts \\
&= (\tilde{r} \cdot f \cdot s) \cdot (r \cdot \mu_Y \cdot Ts) \\
&= (\tilde{r} \cdot f \cdot s) \cdot \alpha
\end{aligned}$$

and therefore is a morphism in $\mathrm{spl}(\mathbf{C}^{\mathbb{T}})$.

We may now define a functor $\mathcal{F} : \mathrm{kar}(\mathbf{C}_{\mathbb{T}}) \to \mathrm{spl}(\mathbf{C}^{\mathbb{T}})$ by taking

$$\mathcal{F}((TY, \mu_Y), e) = (X, \alpha, b) \qquad \text{and} \qquad \mathcal{F}(f) = \tilde{r} \cdot f \cdot s$$

as constructed above, for any $((TY, \mu_Y), e)$, $((T\tilde{Y}, \mu_{\tilde{Y}}), \tilde{e})$ and $f : ((TY, \mu_Y), e) \to ((T\tilde{Y}, \mu_{\tilde{Y}}), \tilde{e})$ in $\mathrm{kar}(\mathbf{C}_{\mathbb{T}})$. This functor is indeed well defined since

$$\mathcal{F}(g) \cdot \mathcal{F}(f) = (\hat{r} \cdot g \cdot \tilde{s}) \cdot (\tilde{r} \cdot f \cdot s) = \hat{r} \cdot g \cdot (\tilde{s} \cdot \tilde{r}) \cdot f \cdot s = \hat{r} \cdot g \cdot \tilde{e} \cdot f \cdot s = \hat{r} \cdot g \cdot f \cdot s = \mathcal{F}(g \cdot f)$$

and

$$\mathcal{F}(1_{TY}) = r \cdot 1_{TY} \cdot s = r \cdot s = 1_X = 1_{\mathcal{F}(TY)}$$

for any $((TY,\mu_Y),e)$, $((T\tilde{Y},\mu_{\tilde{Y}}),\tilde{e})$, $((T\hat{Y},\mu_{\hat{Y}}),\hat{e})$ and $f : ((TY,\mu_Y),e) \to ((T\tilde{Y},\mu_{\tilde{Y}}),\tilde{e})$, $g : ((T\tilde{Y},\mu_{\tilde{Y}}),\tilde{e}) \to ((T\hat{Y},\mu_{\hat{Y}}),\hat{e})$ in $\mathrm{kar}(\mathbf{C}_{\mathbb{T}})$. We note that the choice of splitting for e in the construction we described above is not important, as all the possible choices are isomorphic.

On the other hand, let (X,α,b) be an object in $\mathrm{spl}(\mathbf{C}^{\mathbb{T}})$. Then $((TX,\mu_X),b\cdot\alpha)$ clearly is an object in $\mathrm{kar}(\mathbf{C}_{\mathbb{T}})$, since $b\cdot\alpha$ is idempotent. It is also immediate that, given a morphism $f : (X,\alpha,b) \to (\tilde{X},\tilde{\alpha},\tilde{b})$ in $\mathrm{spl}(\mathbf{C}^{\mathbb{T}})$, $\tilde{b}\cdot\tilde{\alpha}\cdot Tf\cdot b\cdot\alpha : ((TX,\mu_X),b\cdot\alpha) \to ((T\tilde{X},\mu_{\tilde{X}}),\tilde{b}\cdot\tilde{\alpha})$ is a morphism in $\mathrm{kar}(\mathbf{C}_{\mathbb{T}})$ since

$$\tilde{b}\cdot\tilde{\alpha}\cdot\tilde{b}\cdot\tilde{\alpha}\cdot Tf\cdot b\cdot\alpha\cdot b\cdot\alpha = \tilde{b}\cdot\tilde{\alpha}\cdot Tf\cdot b\cdot\alpha.$$

Therefore we may define a functor $\mathcal{G} : \mathrm{spl}(\mathbf{C}^{\mathbb{T}}) \to \mathrm{kar}(\mathbf{C}_{\mathbb{T}})$ by taking

$$\mathcal{G}(X,\alpha,b) = ((TX,\mu_X),b\cdot\alpha) \qquad \text{and} \qquad \mathcal{G}(f) = \tilde{b}\cdot\tilde{\alpha}\cdot Tf\cdot b\cdot\alpha$$

for any (X,α,b), $(\tilde{X},\tilde{\alpha},\tilde{b})$ and $f : (X,\alpha,b) \to (\tilde{X},\tilde{\alpha},\tilde{b})$ in $\mathrm{spl}(\mathbf{C}^{\mathbb{T}})$. This functor is indeed well defined since

$$\begin{aligned}
\mathcal{G}(g)\cdot\mathcal{F}(f) &= (\hat{b}\cdot\hat{\alpha}\cdot Tg\cdot\tilde{b}\cdot\tilde{\alpha})\cdot(\tilde{b}\cdot\tilde{\alpha}\cdot Tf\cdot b\cdot\alpha) \\
&= (\hat{b}\cdot g\cdot\tilde{\alpha}\cdot\tilde{b}\cdot\tilde{\alpha})\cdot(\tilde{b}\cdot\tilde{\alpha}\cdot Tf\cdot b\cdot\alpha) && (g \text{ is a } \mathbb{T}\text{-homomorphism}) \\
&= (\hat{b}\cdot g\cdot\tilde{\alpha}\cdot\tilde{b}\cdot\tilde{\alpha})\cdot(\tilde{b}\cdot f\cdot\alpha\cdot b\cdot\alpha) && (f \text{ is a } \mathbb{T}\text{-homomorphism}) \\
&= \hat{b}\cdot g\cdot f\cdot\alpha\cdot b\cdot\alpha && (\tilde{\alpha}\cdot\tilde{b} = 1_{\tilde{X}}) \\
&= \hat{b}\cdot\hat{\alpha}\cdot T(g\cdot f)\cdot b\cdot\alpha && (g\cdot f \text{ is a } \mathbb{T}\text{-homomorphism}) \\
&= \mathcal{G}(g\cdot f)
\end{aligned}$$

and

$$\mathcal{G}(1_X) = b\cdot\alpha\cdot 1_X\cdot b\cdot\alpha = b\cdot\alpha = 1_{((TX,\mu_X),b\cdot\alpha)} = 1_{\mathcal{G}(X,\alpha,b)}$$

for any objects (X,α,b), $(\tilde{X},\tilde{\alpha},\tilde{b})$, $(\hat{X},\hat{\alpha},\hat{b})$ and morphisms $f : (X,\alpha,b) \to (\tilde{X},\tilde{a},\tilde{b})$, $g : (\tilde{X},\tilde{\alpha},\tilde{b}) \to (\hat{X},\hat{\alpha},\hat{b})$ in $\mathrm{spl}(\mathbf{C}^{\mathbb{T}})$.

The next result is due to Rosebrugh and Wood [32] and is going to be critical for our result connecting the Karoubi envelope of the category of V-relations V-**Rel** and the category of the split $\mathbb{P}$-algebras $\mathrm{spl}(V\text{-}\mathbf{Cat}^{\mathbb{P}})$.

Theorem 6.2.3. *Let $\mathbb{T} = (T,\mu,\eta)$ be a monad on an idempotent complete category* **C**. *Then the functors $\mathcal{F}$ and $\mathcal{G}$ give us the following categorical equivalence*

$$\mathrm{kar}(\mathbf{C}_{\mathbb{T}}) \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathrm{spl}(\mathbf{C}^{\mathbb{T}})$$

Proof. Take an object $((TY,\mu_Y),e)$ in $\mathrm{kar}(\mathbf{C}_{\mathbb{T}})$. Then

$$\mathcal{G}\mathcal{F}((TY,\mu_Y),e) = \mathcal{G}(X,\alpha,b) = ((TX,\mu_X),b\cdot\alpha),$$

as we constructed above. Consider the natural transformation

$$\gamma\colon 1_{\mathrm{kar}(\mathbf{C}_{\mathbb{T}})} \Rightarrow \mathcal{G}\mathcal{F}$$

with components given by $\gamma_{((TY,\mu_Y),e)} = b\cdot r\colon ((TY,\mu_Y),e) \to ((TX,\mu_X),b\cdot\alpha)$. Clearly

$$(b\cdot\alpha)(b\cdot r) = b\cdot r = b\cdot 1_X\cdot r = b\cdot r\cdot s\cdot r = (b\cdot r)\cdot e$$

so $b\cdot r$ is indeed a morphism in $\mathrm{kar}(\mathbf{C}_{\mathbb{T}})$ and

$$\begin{aligned}
\mathcal{G}\mathcal{F}(f)\cdot\gamma_{((TY,\mu_Y),e)} &= \mathcal{G}(\tilde{r}\cdot f\cdot s)\cdot\gamma_{((TY,\mu_Y),e)} \\
&= (\tilde{b}\cdot\tilde{\alpha}\cdot T(\tilde{r}\cdot f\cdot s)\cdot b\cdot\alpha)\cdot\gamma_{((TY,\mu_Y),e)} \\
&= (\tilde{b}\cdot\tilde{\alpha}\cdot T(\tilde{r}\cdot f\cdot s)\cdot b\cdot\alpha)\cdot b\cdot r && (\alpha\cdot b = 1_X) \\
&= \tilde{b}\cdot\tilde{\alpha}\cdot T(\tilde{r}\cdot f\cdot s)\cdot b\cdot r \\
&= \tilde{b}\cdot\tilde{r}\cdot f\cdot s\cdot\alpha\cdot b\cdot r && (\tilde{r}\cdot f\cdot s \text{ is a } \mathbb{T}\text{-homomorphism}) \\
&= \tilde{b}\cdot\tilde{r}\cdot f\cdot s\cdot 1_X\cdot r \\
&= \tilde{b}\cdot\tilde{r}\cdot f\cdot e \\
&= \tilde{b}\cdot\tilde{r}\cdot f && (e = 1_{((TY,\mu_Y),e)}) \\
&= \gamma_{((T\tilde{Y},\mu_{\tilde{Y}}),\tilde{e})}\cdot f
\end{aligned}$$

for any morphism $f\colon ((TY,\mu_Y),e) \to ((T\tilde{Y},\mu_{\tilde{Y}}),\tilde{e})$ in $\mathrm{kar}(\mathbf{C}_{\mathbb{T}})$, verifying that γ is indeed a natural transformation. Moreover γ is actually a natural isomorphism, since each of its components $\gamma_{((TY,\mu_Y),e)} = b\cdot r\colon ((TY,\mu_Y),e) \to ((TX,\mu_X),b\cdot\alpha)$ has $s\cdot\alpha\colon ((TX,\mu_X),b\cdot\alpha) \to ((TY,\mu_Y),e)$ as its inverse:

$$(s\cdot\alpha)\cdot(b\cdot r) = s\cdot r = e = 1_{((TY,\mu_Y),e)} \qquad \text{and} \qquad (b\cdot r)\cdot(s\cdot\alpha) = b\cdot\alpha = 1_{((TX,\mu_X),b\cdot\alpha)}.$$

Now, let (X,α,b) be an object in $\mathrm{spl}(\mathbf{C}^{\mathbb{T}})$. Then $\mathcal{F}\mathcal{G}(X,\alpha,b) = \mathcal{F}((TX,\mu_X),b\cdot\alpha)$. Noting that α and b split the idempotent $b\cdot\alpha$ and that

$$\alpha\cdot\mu_X\cdot Tb = \alpha\cdot b\cdot\alpha = \alpha \qquad \text{and} \qquad T\alpha\cdot T\eta_X\cdot b = 1_{TX}\cdot b = b,$$

we may take $\mathcal{F}((TX,\mu_X),b\cdot\alpha) = (X,\alpha,b)$. Lastly, we have

$$\mathcal{F}\mathcal{G}(f) = \mathcal{F}(\tilde{b}\cdot\tilde{\alpha}\cdot Tf\cdot b\cdot\alpha) = \tilde{\alpha}\cdot(\tilde{b}\cdot\tilde{\alpha}\cdot Tf\cdot b\cdot\alpha)\cdot b = \tilde{\alpha}\cdot Tf\cdot b = f\cdot\alpha\cdot b = f,$$

for any $\mathbb{T}$-homomorphism $f\colon (X,\alpha,b) \to (\tilde{X},\tilde{\alpha},\tilde{b})$ in $\mathrm{spl}(\mathbf{C}^{\mathbb{T}})$. Therefore $\mathcal{F}\mathcal{G} = 1_{\mathrm{spl}(\mathbf{C}^{\mathbb{T}})}$. □

6.2.1 The equivalence $\mathrm{kar}(V\text{-}\mathbf{Rel}) \cong \mathrm{spl}(V\text{-}\mathbf{Cat}^{\mathbb{P}})$

Now we obtain our main result for this section, as promised:

Corollary 6.2.4. *The Karoubi envelope of the category of V-relations V-$\mathbf{Rel}$ is equivalent to the category of the split $\mathbb{P}$-algebras* $\mathrm{spl}(V\text{-}\mathbf{Cat}^{\mathbb{P}})$.

$$\mathrm{kar}(V\text{-}\mathbf{Rel}) \cong \mathrm{spl}(V\text{-}\mathbf{Cat}^{\mathbb{P}}).$$

Proof. Since V-$\mathbf{Cat}$ is complete (Remark 3.3.13) it is also idempotent complete. Therefore we may use Theorem 6.2.3 to obtain immediately that

$$\mathrm{kar}(V\text{-}\mathbf{Cat}_{\mathbb{P}}) \cong \mathrm{spl}(V\text{-}\mathbf{Cat}^{\mathbb{P}}).$$

Note that

$$\begin{aligned} \mathrm{kar}(V\text{-}\mathbf{Rel}) &\cong \mathrm{kar}(V\text{-}\mathbf{Dist}) && \text{(Prop. 6.1.5)} \\ &\cong \mathrm{kar}(V\text{-}\mathbf{Cat}_{\mathbb{P}}) && (V\text{-}\mathbf{Dist} \cong V\text{-}\mathbf{Cat}_{\mathbb{P}}, \text{ by Prop. 5.2.2)} \\ &\cong \mathrm{spl}(V\text{-}\mathbf{Cat}^{\mathbb{P}}) \end{aligned}$$

□

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