A multi-input/multi-output system representation of generalized splines in $\mathbb{R}^n$

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Abstract

Scalar generalized splines have been introduced in the late 50's by Ahlberg et al. These curves generalize the well known polynomial splines, the most used one being the cubic spline, associated to minimization of the average acceleration. Scalar generalized splines have recently been connected with optimal control. It has been proven that they can be realized as the optimal output function of a SISO classical control system. In this paper we describe how to extend the definition of a scalar generalized spline to arbitrary Euclidean space $\mathbb{R}^n$ and present its minimality properties. This generalization is motivated by the connection with optimal control of a MIMO higher order dynamical system.

1 Introduction

To define scalar generalized splines one considers a differential operator of the form $L = D^p - a_{p-1}D^{p-1} - \cdots - a_1 D - a_0 D^0$ and its adjoint $L^* = (-1)^p D^p + (-1)^p a_{p-1} D^{p-1} + \cdots + a_1 D - a_0 D^0$ with $a_i \in \mathbb{R}$. Then, for a given partition $\Delta : t_0 < t_1 < \cdots < t_m = \mathrm{T}$ of the time interval $[t_0, \mathrm{T}]$ and given real numbers α_k, η_0^i and η_m^i, $k = 1, \ldots, m-1$ and $i = 0, \ldots, p-1$, we have that a generalized spline is the unique real function s defined in $[t_0, \mathrm{T}]$ that satisfies simultaneously the following

$$s \in \mathrm{C}^{2p-2}[t_0, \mathrm{T}] \quad \text{and} \quad s \in \mathrm{C}^{2p}[t_{k-1}, t_k] \quad \text{for all} \quad k = 1, \ldots, m\,,$$

$$s \text{ satisfies} \quad L^* L s = 0 \quad \text{in each} \quad [t_{k-1}, t_k]\,, \quad k = 1, \ldots, m\,,$$

$$s \text{ fulfils the boundary conditions} \quad D^i s(t_0) = \eta_0^i\,, \quad D^i s(\mathrm{T}) = \eta_m^i\,, \quad i = 0, \ldots, p-1\,,$$

$$\text{and the interpolation conditions} \quad s(t_k) = \alpha_k\,, \quad k = 1, \ldots, m-1\,.$$

If one chooses for example $L = D^2$ then the resulting spline is the well known natural cubic spline. In the literature (a reference about the subject is Ahlberg *et al.* [1])

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the spline function described above is precisely known as generalized spline of type I. This is due to the fact that other boundary conditions can be chosen and that the resulting function is also called a generalized spline.

The relationship between scalar generalized splines and optimal control of SISO systems is now well understood. (See C. Martin *et al.* [3] and R. Rodrigues *et al.* [5]).

The above definition may be extended to $\mathbb{R}^n$ $(n > 1)$, simply replacing scalar functions by vector function and maintaining the operator L. This generalization is rather simple once the scalar case is understood, since each component of the vector spline is a scalar generalized spline. For the particular case $n = 2$, this approach has been used quite often in engineering applications, namely in path planning of mechanical systems. However, in general, such generalization doesn't have a counterpart in optimal control. And it turns out that the connection between optimal control of MIMO systems and splines requires another more challenging generalization, that we are going to introduce. We claim that this is the correct way of extending splines from $\mathbb{R}$ to $\mathbb{R}^n$, and to our knowledge this has not appeared in the literature before.

The organization of the paper is as follows. The main concepts are introduced at the beginning of section 2, followed by existence and uniqueness results. In section 2.2 generalized splines appear as solutions of a certain variational problem. The corresponding Euler-Lagrange equation is shown to have a special structure which allows a significant reduction in complexity. Connections between generalized splines and optimal control of MIMO systems appear in section 3. We also contrast our approach with other existing methods.

2 Generalized splines in $\mathbb{R}^n$

The main feature in our approach is the appearance of differential operators with matrix coefficients. This adds a significant amount of complexity, when compared with the traditional way of generalizing splines to higher dimensional spaces.

Throughout the paper we consider $\mathbb{R}^n$ equipped with the Euclidean inner product $\langle .,. \rangle$.

Let L be the following differential operator

$$L = D^p - A_{p-1}D^{p-1} - \cdots - A_1 D - A_0 D^0 \tag{2.1}$$

where $p \in \mathbb{N}$, A_i, $i = 0, \ldots, p-1$, are $n \times n$ real matrices and L^* the adjoint of L defined as

$$L^* = (-1)^p D^p + (-1)^p A_{p-1}^\top D^{p-1} + (-1)^{p-1} A_{p-2}^\top D^{p-2} + \cdots + A_1^\top D - A_0^\top D^0 \,,$$

where $A^\top$ indicates the transpose of matrix A. Also, let $\Delta : t_0 < t_1 < \cdots < t_m = \mathrm{T}$ be a partition of the time interval $[t_0, \mathrm{T}]$ and Ω the set of all functions f defined on $[t_0, \mathrm{T}]$, having values in $\mathbb{R}^n$ and such that

$$f \in \mathrm{C}^{2p-2}[t_0, \mathrm{T}] \quad \text{and} \quad f \in \mathrm{C}^{2p}[t_{k-1}, t_k] \quad \text{for all} \quad k = 1, \ldots, m \,.$$

We now present the definition of a generalized spline function in $\mathbb{R}^n$ defined in the time interval $[t_0, \mathrm{T}]$ and associated to the operator L.

Definition 2.1 (Generalized Spline).
The function $q : [t_0, \mathrm{T}] \longrightarrow \mathbb{R}^n$ is a generalized spline associated with the matrix operator L and the partition Δ of $[t_0, \mathrm{T}]$ if

$$q \in \Omega\,,$$

$$q \text{ is a solution of the differential equation } L^*L\mathrm{x} = 0 \text{ in each } [t_{k-1}, t_k] \tag{2.2}$$

and q satisfies
the boundary conditions

$$D^i q(t_0) = \eta_0^i\,, \quad D^i q(\mathrm{T}) = \eta_m^i\,, \quad i = 0, \ldots, p-1\,, \tag{2.3}$$

and the interpolation conditions

$$q(t_k) = \mathrm{q}_k\,, \quad k = 1, \ldots, m-1\,, \tag{2.4}$$

with $\eta_0^0, \mathrm{q}_1, \ldots, \mathrm{q}_{m-1}, \eta_m^0$, $m+1$ distinct points of $\mathbb{R}^n$ and $\eta_0^i, \eta_m^i \in \mathbb{R}^n$ for all $i = 1, \ldots, p-1$.

Notice that the problem of finding a generalized spline is well defined in the sense that there is the same number of unknowns and initial conditions. Indeed, Since q is locally a solution of a linear homogeneous matrix differential equation of order $2p$, there is $2np$ constants to be found in each $[t_{k-1}, t_k]$, that give a total of $2npm$ unknowns. The required boundary conditions provide $2np$ equations, the interpolation conditions generate $n(m-1)$ equations and the continuity requirements give rise to $n(2p-1)(m-1)$ equations, leading to a total of $2npm$ linear algebraic equations in the $2npm$ unknowns.

2.1 Existence and uniqueness

The first result establishes a useful relation between the operators L and L^*.

Lemma 2.2.
If x *and* y *are two vector functions defined on* $[a, b] \subset \mathbb{R}$ *and possessing derivatives up to order* p *in* $[a, b]$*, then*

$$\langle L\mathrm{x}, \mathrm{y}\rangle - \langle \mathrm{x}, L^*\mathrm{y}\rangle = \tfrac{d}{dt} B(\mathrm{x}, \mathrm{y}), \tag{2.5}$$

where $B(\mathrm{x}, \mathrm{y})$ *is defined by*

$$B(\mathrm{x}, \mathrm{y}) = \sum_{i=1}^{p} \sum_{j=0}^{i-1} (-1)^{j+1} \langle D^{i-1-j}\mathrm{x}, A_i^\top D^j \mathrm{y}\rangle.$$

Proof. Defining $A_p = -I$ we have

$$\begin{aligned}
\langle L\mathrm{x}, \mathrm{y}\rangle - \langle \mathrm{x}, L^*\mathrm{y}\rangle &= -\sum_{i=0}^{p} \langle A_i D^i \mathrm{x}, \mathrm{y}\rangle - \sum_{i=0}^{p} \langle \mathrm{x}, (-1)^{i+1} A_i^\top D^i \mathrm{y}\rangle \\
&= -\sum_{i=1}^{p} \langle A_i D^i \mathrm{x}, \mathrm{y}\rangle + (-1)^{i+1} \langle \mathrm{x}, A_i^\top D^i \mathrm{y}\rangle \\
&= -\sum_{i=1}^{p} \langle D^i \mathrm{x}, A_i^\top \mathrm{y}\rangle + (-1)^{i+1} \langle \mathrm{x}, A_i^\top D^i \mathrm{y}\rangle \,.
\end{aligned}$$

Notice that, for functions satisfying the required conditions, one has

$$\langle D^i \mathrm{x}, \mathrm{y}\rangle + (-1)^{i+1}\langle \mathrm{x}, D^i \mathrm{y}\rangle = \frac{d}{dt} \sum_{j=0}^{i-1} (-1)^j \langle D^{i-1-j}\mathrm{x}, D^j \mathrm{y}\rangle$$

for all $i = 1, \ldots, p$. Using this identity we get finally

$$\langle L\mathrm{x}, \mathrm{y}\rangle - \langle \mathrm{x}, L^*\mathrm{y}\rangle = -\sum_{i=1}^{p} \frac{d}{dt} \sum_{j=0}^{i-1} (-1)^j \langle D^{i-1-j}\mathrm{x}, A_i^\top D^j \mathrm{y}\rangle = \frac{d}{dt} B(\mathrm{x}, \mathrm{y}) \,.$$

□

Lemma 2.3.
If $g \in \Omega$, satisfies (2.2) and is such that

$$\begin{aligned}
&D^i g(t_0) = 0\,, \quad D^i g(\mathrm{T}) = 0\,, \quad i = 0, \ldots, p-1\,, \\
&g(t_k) = 0\,, \quad k = 1, \ldots, m-1\,,
\end{aligned}$$

then g is a solution of the differential equation $L\mathrm{x} = 0$ in each interval $[t_{k-1}, t_k]$.

Proof. It is clear that there exists at least one such function, namely $g(t) = 0$ for all $t \in [t_0, \mathrm{T}]$. We show that if g satisfies all the required conditions, then

$$\int_{t_0}^{\mathrm{T}} \langle Lg, Lg\rangle \, ds = 0\,,$$

which clearly implies that $Lg = 0$ in each subinterval $[t_{k-1}, t_k]$.

Using equation (2.5) with $\mathrm{x} = g$ and $\mathrm{y} = Lg$ and since g satisfies (2.2), we may write

$$\begin{aligned}
\int_{t_0}^{\mathrm{T}} \langle Lg, Lg\rangle \, ds &= \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \langle Lg, Lg\rangle \, ds \\
&= \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} \langle g, L^* Lg\rangle \, ds + B(g, Lg)\,\big]_{t_{k-1}}^{t_k} \right) \\
&= \sum_{k=1}^{m} B(g, Lg)\,\big]_{t_{k-1}}^{t_k} \,.
\end{aligned}$$

Finally, because $g \in \Omega$ and satisfies null boundary and null interpolation conditions the last expression also vanishes. Now it is also clear that

$$\int_{t_{k-1}}^{t_k} \langle Lg, Lg\rangle \, ds = 0\,, \qquad \text{for all} \quad k = 1, \ldots, m\,.$$

□

Lemma 2.4.
There exists a unique function $g \in \Omega$ that satisfies (2.2) and has null boundary and null interpolation conditions. This is the zero function.

Proof. We only have to show that $g(t) \equiv 0$, $t \in [t_0, \mathrm{T}]$, is the unique such function. We know by lemma 2.3 that g is a solution of the differential equation $L\mathrm{x} = 0$ in each subinterval $[t_{k-1}, t_k]$. Therefore, there exist linearly independent functions $v_1, v_2, \ldots, v_{np}$, such that $g(t) = c_1^k v_1(t) + \cdots + c_{np}^k v_{np}(t)$ in each $[t_{k-1}, t_k]$. In $[t_0, t_1]$ the boundary conditions $D^i g(t_0) = 0$, $i = 0, \ldots, p-1$, lead to the following algebraic system of np equations in np unknowns

$$\begin{pmatrix} v_1(t_0) & \cdots & v_{np}(t_0) \\ Dv_1(t_0) & \cdots & Dv_{np}(t_0) \\ \vdots & & \\ D^{p-1}v_1(t_0) & \cdots & D^{p-1}v_{np}(t_0) \end{pmatrix} \cdot \begin{pmatrix} c_1^1 \\ c_2^1 \\ \vdots \\ c_{np}^1 \end{pmatrix} = 0\,.$$

Since the coefficient matrix is the Wronskian of $v_1, v_2, \ldots, v_{np}$ evaluated at t_0, it is invertible and this system only has the trivial solution $c_1^1 = \cdots = c_{np}^1 = 0$, which in turn gives $g(t) \equiv 0$ in $[t_0, t_1]$. The same conclusion can be extended to all other subintervals, repeating the same arguments and using the fact that $g \in \mathrm{C}^{2p-2}[t_0, \mathrm{T}]$.
□

Theorem 2.5.
Given the matrix operator L and the partition Δ of $[t_0, \mathrm{T}]$, there exists a unique function $q \in \Omega$ that satisfies (2.2), the boundary conditions (2.3) and the interpolation conditions (2.4).

Proof. We know by (2.2) that there is $v_1, v_2, \ldots, v_{2np}$ linearly independent functions such that $q(t) = c_1^k v_1(t) + \cdots + c_{2np}^k v_{2np}(t)$ in each $[t_{k-1}, t_k]$. Forcing this function to satisfy the boundary conditions, the interpolation constraints, and the continuity conditions at each t_k, gives rise to and algebraic system of the form $Ac = b$, with $2npm$ equations and $2npm$ unknowns. The existence and uniqueness of q follows from the invertibility of the matrix A. To see that A is indeed nonsingular, just notice that for null boundary and null interpolation conditions we get $Ac = 0$, which has a unique solution by lemma 2.4. □

2.2 Minimality properties of generalized splines

The result in this lemma is required to prove minimality properties of generalized splines.

Lemma 2.6.
If q is a generalized spline and g is a function that belongs to Ω and satisfies the same boundary and interpolation conditions, then

$$\sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \langle L(g-q), Lq \rangle \, ds = 0 \, .$$

Proof. Using equation (2.5) with $\mathrm{x} = g - q$ and $\mathrm{y} = Lq$ we have

$$\begin{aligned} \int_{t_{k-1}}^{t_k} \langle L(g-q), Lq \rangle \, ds &= B(g-q, Lq)\,]_{t_{k-1}}^{t_k} + \int_{t_{k-1}}^{t_k} \langle g-q, L^* Lq \rangle \, ds \\ &= B(g-q, Lq)\,]_{t_{k-1}}^{t_k} \end{aligned}$$

since q satisfies equation (2.2). Therefore,

$$\sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \langle L(g-q), Lq \rangle \, ds = \sum_{k=1}^{m} B(g-q, Lq)\,]_{t_{k-1}}^{t_k} \, .$$

Due to the fact that q and g belong to Ω and satisfy the same boundary and interpolation conditions the right hand side of the previous equation also vanishes, which concludes the proof. □

Theorem 2.7.
The generalized spline function minimizes the functional $\int_{t_0}^{\mathrm{T}} \langle L\mathrm{x}, L\mathrm{x} \rangle \, ds$ *among all functions $g \in \Omega$ that satisfy the same boundary and interpolation conditions.*

Proof.

$$\begin{aligned} \int_{t_0}^{\mathrm{T}} \langle L(g-q), L(g-q) \rangle \, ds &= \int_{t_0}^{\mathrm{T}} \langle Lg, Lg \rangle \, ds - 2 \int_{t_0}^{\mathrm{T}} \langle Lg, Lq \rangle \, ds \\ &\quad + \int_{t_0}^{\mathrm{T}} \langle Lq, Lq \rangle \, ds \\ &= \int_{t_0}^{\mathrm{T}} \langle Lg, Lg \rangle \, ds - 2 \int_{t_0}^{\mathrm{T}} \langle L(g-q), Lq \rangle \, ds \\ &\quad - \int_{t_0}^{\mathrm{T}} \langle Lq, Lq \rangle \, ds \, . \end{aligned}$$

Using the previous lemma we get the following equation

$$\int_{t_0}^{\mathrm{T}} \langle Lg, Lg \rangle \, ds = \int_{t_0}^{\mathrm{T}} \langle Lq, Lq \rangle \, ds + \int_{t_0}^{\mathrm{T}} \langle L(g-q), L(g-q) \rangle \, ds \, ,$$

from which the minimality property of q follows immediately. □

It must be clear that among the functions in Ω that fulfill the same boundary and interpolation conditions, the one that minimizes the functional must be a solution of the differential equation $L^* L\mathrm{x} = 0$.

We next formulate an alternative definition of generalized spline that will be important to understand the connection with optimal control. Although, in contrast with the definition 2.1, no information about the expression of q appears explicitly in this new definition, it will be recovered from the Euler-Lagrange equation associated to the corresponding variational problem.

Definition 2.8.
Let $\eta_0^0, \mathrm{q}_1, \ldots, \mathrm{q}_{m-1}, \eta_m^0$, be $m+1$ distinct points of $\mathbb{R}^n$ and η_0^i, $\eta_m^i \in \mathbb{R}^n$ for all $i = 1, \ldots, p-1$. The function $q : [t_0, \mathrm{T}] \longrightarrow \mathbb{R}^n$ is a generalized spline associated with the matrix operator L and the partition Δ of $[t_0, \mathrm{T}]$ if $q \in \Omega$, satisfies
the boundary conditions

$$D^i q(t_0) = \eta_0^i\,, \quad D^i q(\mathrm{T}) = \eta_m^i\,, \quad i = 0, \ldots, p-1\,,$$

the interpolation conditions

$$q(t_k) = \mathrm{q}_k\,, \quad k = 1, \ldots, m-1\,,$$

and minimizes the functional

$$J(\mathrm{x}) = \int_{t_0}^{\mathrm{T}} \langle L\mathrm{x}, L\mathrm{x} \rangle \, ds\,. \tag{2.6}$$

We have to show that the definitions 2.1 and 2.8 are in fact equivalent. The last definition indicates that q can be found as the solution of a variational problem which consists in minimizing the functional (2.6).

Let Λ be the class of all admissible function x. That is, Λ consists of the functions $\mathrm{x} \in \Omega$ which satisfy the prescribed boundary and interpolation conditions. A necessary condition for x to be an extrema of functional (2.6) is that

$$\delta J(\mathrm{x}, h) = 0\,, \qquad \text{for all function } h \text{ such that } \mathrm{x} + h \in \Lambda\,, \tag{2.7}$$

where $\delta J(\mathrm{x}, h)$ is the Gateaux differential of the functional J, defined by

$$\delta J(\mathrm{x}, h) = \left(\frac{d}{d\alpha} J(\mathrm{x} + \alpha h) \right)_{|\alpha = \mathbf{0}}\,.$$

Hence,

$$\begin{aligned} \delta J(\mathrm{x}, h) &= \left(\frac{d}{d\alpha} \int_{t_0}^{\mathrm{T}} \langle L(\mathrm{x} + \alpha h), L(\mathrm{x} + \alpha h) \rangle \, ds \right)_{|\alpha = \mathbf{0}} \\ &= \left(\int_{t_0}^{\mathrm{T}} 2 \langle Lh, L\mathrm{x} + \alpha Lh \rangle \, ds \right)_{|\alpha = \mathbf{0}} \\ &= 2 \int_{t_0}^{\mathrm{T}} \langle Lh, L\mathrm{x} \rangle \, ds\,. \end{aligned}$$

Therefore, by (2.7), we get

$$\sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \langle Lh, L\mathrm{x} \rangle \, ds = 0\,.$$

Using identity (2.5) we can say that

$$\sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \langle h, L^* L\mathrm{x} \rangle \, ds + \sum_{k=1}^{m} B(h, L\mathrm{x})\,]_{t_{k-1}}^{t_k} = 0 \,,$$

and since $h \in \Omega$ and satisfies null boundary and null interpolation conditions, the previous equation reduces to

$$\sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \langle h, L^* L\mathrm{x} \rangle \, ds = 0 \,.$$

Since this must hold for all h such that $\mathrm{x} + h \in \Lambda$ we find finally that x must satisfy

$$L^* L\mathrm{x} = 0 \tag{2.8}$$

in each interval $[t_{k-1}, t_k]$.

Since we know, by theorem 2.5, that q is the unique function that simultaneously belongs to Ω, satisfies given boundary and interpolation conditions and is a solution of (2.8) in each subinterval, it is clear that the condition $L^* Lq = 0$ in each $[t_{k-1}, t_k]$ is necessary and sufficient for q to minimize the functional (2.6) and, consequently, the equivalence between definitions 2.1 and 2.8 is clarified.

The Euler-Lagrange equation (2.8) is a homogeneous differential equation of order $2p$ with matrix coefficients. Solving such differential equation is very hard even for low order. Tisseur and Meerbergen [8] give an account of results and main difficulties for general second order matrix differential equations, which are closely related with *the quadratic eigenvalue problem.* Also, in a recent article, Mehrmann and Watkins [4] discuss the numerical solution of eigenvalue problems for matrix polynomials where the coefficient matrices are alternating symmetric and skew-symmetric or Hamiltonian and skew-Hamiltonian. Next we will show that the coefficients of equation (2.8) have a particular alternating structure which may be used to reduce the computational effort in solving that Euler-Lagrange equation.

Proposition 2.9.
The even coefficients of the Euler-Lagrange equation $L^* Lq = 0$ are symmetric matrices while the odd coefficients are skew-symmetric matrices.

Proof. To simplify notation, replace $-A_k$ by B_k, $k = 0, \dots, p-1$ and let $I = B_p$. Then, the operators L and L^* reduce to

$$L = B_p D^p + B_{p-1} D^{p-1} + \cdots + B_1 D + B_0 D^0 \,,$$

$$L^* = (-1)^p B_p^\top D^p + (-1)^{p-1} B_{p-1}^\top D^{p-1} + \cdots - B_1^\top D + B_0^\top D^0 \,,$$

and a trivial calculation shows that

$$L^* L = C_{2p} D^{2p} + C_{2p-1} D^{2p-1} + \cdots + C_1 D + C_0 D^0,$$

where the even coefficients are given by

$$C_{2k} = (-1)^k \{ B_k^\top B_k + \sum_{\substack{i+j=2k \\ i>j}} (-1)^\delta (B_i^\top B_j + B_j^\top B_i) \} \,, \quad k = 0, \dots, p \,,$$

with $\delta = \frac{1}{2}|i-j|$ and the odd coefficients are given by

$$C_{2k-1} = (-1)^k \sum_{\substack{i+j=2k-1 \\ i>j}} (-1)^\sigma (B_i^\top B_j - B_j^\top B_i)\,, \quad k=1,\ldots,p\,,$$

with $\sigma = \frac{1}{2}(|i-j|-1)$.

It is now clear that $(C_{2k})^\top = C_{2k}$ and $(C_{2k-1})^\top = -C_{2k-1}$, as claimed. □

If $P(\lambda) = I\lambda^{2p} + C_{2p-1}\lambda^{2p-1} + C_{2p-2}\lambda^{2p-2} + \cdots + C_1\lambda + C_0$ denotes the matrix polynomial associated to the Euler-Lagrange equation (2.8), the alternating structure of its coefficients implies that $P^\top(\lambda) = P(-\lambda)$, so that, as in Mehrmann and Watkins [4], $P(\lambda)\,v = 0 \iff v^\top P(-\lambda) = 0$, i.e., v is a right eigenvector associated to the eigenvalue λ if and only if $v^\top$ is a left eigenvector associated to the eigenvalue $-\lambda$. The following Hamiltonian property of this polynomial eigenvalue problem is now an immediate consequence of the last remarks.

Proposition 2.10.
The eigenvalues of the matrix polynomial $P(\lambda)$ associated to the Euler-Lagrange equation (2.8) are symmetrically placed with respect to both real and imaginary axes.

Remark 2.11.

1. Note that if $n=1$ our approach reduces to the scalar generalized splines presented in section 1.
2. For general n and $L = D^p$, the solutions of $L^*Lq = 0$ reduce to the polynomial splines (of degree $2p-1$) in $\mathbb{R}^n$, for which all components are scalar polynomial splines of the same degree $2p-1$. This is the only particular case that has been considered in the literature before.

2.3 Explicit solutions

Finding explicit solutions of the Euler-Lagrange equation (2.8) is now the crucial point. Due to the Hamiltonian structure of the coefficient matrices of our problem, the polynomial $P(\lambda)$ factorizes as $P(\lambda) = Q^\top(-\lambda)\,Q(\lambda)$, where $Q(\lambda) = \lambda^p - A_{p-1}\lambda^{p-1} - \cdots - A_1\lambda - A_0$. So, solving the Euler-Lagrange equation of order $2p$, reduces to finding explicit solutions of matrix polynomial equations of order p.

The quadratic eigenvalue problem, which corresponds to the case $p=1$, can be easily tackled as we now show.

Theorem 2.12.
The generalized spline in $\mathbb{R}^n$ associated to the operator $L = D - AD^0$ is given in each subinterval $[t_{k-1}, t_k]$ explicitly by

$$q(t) \;=\; e^{tA}\left(\mathrm{c}_0 + \left(\int_{t_{k-1}}^{t} e^{-sA}\, e^{-sA^\top}\, ds\right)\mathrm{c}_1\right),$$

with $\mathrm{c}_0 = e^{-t_{k-1}A}\,\mathrm{q}_{k-1}$, $\mathrm{c}_1 = e^{t_{k-1}A^\top}(Dq(t_{k-1}) - A\mathrm{q}_{k-1})$, $\mathrm{q}_0 = \eta_0^0$ *and* $\mathrm{q}_m = \eta_m^0$.

Proof. In this case, the Euler-Lagrange equation $L^*Lq = 0$ reduces to

$$\ddot{q} + (A^\top - A)\,\dot{q} - (A^\top A)\,q = 0\,. \tag{2.9}$$

Notice again that the coefficient matrices $A^\top - A$ and $A^\top A$ are respectively skew-symmetric and symmetric matrices.

To solve this equation simply define $Lq = \mathrm{z}$, so that z is the solution of the first order differential equation $L^*\mathrm{z} = 0$, or equivalently $\dot{\mathrm{z}} + A^\top \mathrm{z} = 0$. So, z is given explicitly by

$$\mathrm{z}(t) = e^{-(t-t_{k-1})A^\top}\mathrm{z}(t_{k-1})\,, \qquad t \in [t_{k-1}, t_k]\,.$$

Since $\dot{q} - Aq = \mathrm{z}$, it follows immediately that

$$\begin{aligned} q(t) &= e^{(t-t_{k-1})A}\,\mathrm{q}_{k-1} + \int_{t_{k-1}}^{t} e^{(t-s)A}\,e^{(t_{k-1}-s)A^\top}\mathrm{z}(t_{k-1})\,ds \\ &= e^{(t-t_{k-1})A}\left(\mathrm{q}_{k-1} + \left(\int_{t_{k-1}}^{t} e^{(t_{k-1}-s)A}\,e^{(t_{k-1}-s)A^\top}\,ds\right)\mathrm{z}(t_{k-1})\right) \\ &= e^{tA}\left(\mathrm{c}_0 + \left(\int_{t_{k-1}}^{t} e^{-sA}\,e^{-sA^\top}\,ds\right)\mathrm{c}_1\right)\,, \end{aligned} \tag{2.10}$$

with $\mathrm{c}_0 = e^{-t_{k-1}A}\mathrm{q}_{k-1}$ and $\mathrm{c}_1 = e^{t_{k-1}A^\top}\mathrm{z}(t_{k-1}) = e^{t_{k-1}A^\top}(Dq(t_{k-1}) - A\mathrm{q}_{k-1})$ in each subinterval $[t_{k-1}, t_k]$. □

3 Connections with optimal control

Consider the following optimal control problem defined on the interval $[t_0, \mathrm{T}]$, of a linear system of higher order differential equations:

$$\min_{u\,\in\,\Omega'} \quad J(u) = \int_{t_0}^{\mathrm{T}} \langle B\,u, B\,u\rangle\,ds$$

subject to the dynamics

$$\begin{cases} L\mathrm{x} = B\,u(t) \\ y = \mathrm{x} \end{cases}, \tag{3.1}$$

the interpolation conditions

$$y(t_k) = \mathrm{y}_k\,, \quad k = 1, \dots, m-1$$

and the boundary conditions

$$D^i y(t_0) = \eta_0^i\,, \quad D^i y(\mathrm{T}) = \eta_m^i\,, \quad i = 0, \dots, p-1\,,$$

where, as usual, $\Delta : t_0 < t_1 < \cdots < t_m = \mathrm{T}$ is a partition of the time interval $[t_0, \mathrm{T}]$, $\eta_0^0, \mathrm{y}_1, \dots, \mathrm{y}_{m-1}, \eta_m^0$, are $m+1$ distinct points of $\mathbb{R}^n$, η_0^i, $\eta_m^i \in \mathbb{R}^n$ for all $i = 1, \dots, p-1$, L is the matrix differential operator (2.1) and Ω' is the set of all functions v defined on $[t_0, \mathrm{T}]$, with values in $\mathbb{R}^{n_1}$ and such that

$$v \in \mathrm{C}^{p-2}[t_0, \mathrm{T}] \quad \text{and} \quad v \in \mathrm{C}^p[t_{k-1}, t_k] \quad \text{for all} \quad k = 1, \dots, m\,.$$

Remark 3.1. The system (3.1) is fully observable ($y = \mathrm{x}$) and so the boundary conditions could be given for the state. However we prefer the statement above for the optimal control problem in order to obtain the scalar case as a particular case of this.

The natural procedure to turn (3.1) into a first order system of differential equations by means of new variables $w_i = D^{i-1}\mathrm{x}$ for $i = 1, \dots, p$ leads to the control system $\dot{\mathrm{w}} = \widetilde{A}\,\mathrm{w} + \widetilde{B}\,u(t)$, with

$$\mathrm{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_{p-1} \\ w_p \end{pmatrix}, \quad \widetilde{A} = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & I \\ A_0 & A_1 & A_2 & \cdots & A_{p-1} \end{pmatrix} \quad \text{and} \quad \widetilde{B} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ B \end{pmatrix}.$$

And since $\langle \widetilde{B}\,u, \widetilde{B}\,u \rangle = u^\top \widetilde{B}^\top \widetilde{B} u = u^\top B^\top B u = \langle Bu, Bu \rangle$, the above optimal control problem may be equivalently formulated as:

$$\min_{u \in \Omega'} \int_{t_0}^{\mathrm{T}} \langle \widetilde{B}\,u, \widetilde{B}\,u \rangle \, ds$$

subject to the dynamics

$$\begin{cases} \dot{\mathrm{w}} = \widetilde{A}\,\mathrm{w} + \widetilde{B}\,u(t) \\ \zeta = w_1 \end{cases}, \tag{3.2}$$

the interpolation conditions

$$w_1(t_k) = \mathrm{y}_k\,, \quad k = 1, \dots, m-1$$

and the boundary conditions

$$\mathrm{w}(t_0) = \begin{pmatrix} \eta_0^0 \\ \vdots \\ \eta_0^{p-1} \end{pmatrix}, \qquad \mathrm{w}(\mathrm{T}) = \begin{pmatrix} \eta_m^0 \\ \vdots \\ \eta_m^{p-1} \end{pmatrix}.$$

Assuming that the system (3.2) is completely state controllable, one can prove that the optimal control problem has a unique solution. The proof can be found in appendix A. The connection between generalized splines in $\mathbb{R}^n$ and optimal control of MIMO systems of the form (3.1) is also clear from the observation that if $L\mathrm{x} = Bu$ both problems (the optimal control problem and the variational problem in definition 2.8) have the same energy function. The following is a consequence of these remarks.

Theorem 3.2. *If the pair $(\widetilde{A}, \widetilde{B})$ is controllable, then the optimal control problem formulated at the beginning of this section has a unique solution, and the optimal output function, i.e., the optimal state trajectory, is the unique generalized spline in $\mathbb{R}^n$ satisfying the required boundary and interpolation conditions.*

Remark 3.3.

1. For a SISO system $\dot{\mathrm{x}} = A\mathrm{x} + bu(t)$, $y = x_1$ (x_1 being the first componente of the state) one obtains the generalized scalar splines (defined at the introduction) as optimal outputs.
2. When $L\mathrm{x} = D^p\,\mathrm{x}$, one gets the polynomial splines in $\mathbb{R}^n$.

4 Conclusion

We have presented a new approach to produce splines in $\mathbb{R}^n$, motivated by the relationship between SISO optimal control and scalar generalized splines.

This paper also clarifies the connection between scalar generalized splines and the optimal control problem (for single-input single-output time invariant linear system) studied in Martin *et al.* [3] and Rodrigues *et al.* [5] when the matrices A and B are not in controllability canonical form.

Finally, one can define L-splines in $\mathbb{R}^n$ (for references about scalar L-splines see Schultz & Varga [7]) and establish the connection with control theory as it was done in Rodrigues & Leite [6].

A Solution of the optimal control problem (3.2)

We interpret the optimal control problem as a minimum norm problem in the Hilbert space $L_2[t_0, \mathrm{T}]$ of vector functions with inner product

$$\langle v, u\rangle = \int_{t_0}^{\mathrm{T}} v^\top u \, ds$$

and use the next result that can be found in [2].

Theorem A.1.
Let X be a Hilbert space with inner product $\langle .,.\rangle$ and $\{y_1, y_2, \ldots, y_q\}$ a set of linearly independent vectors in X. There is a unique minimum norm vector v_0 among all vectors $v \in X$ satisfying

$$\langle v, y_1\rangle = \alpha_1,\ \langle v, y_2\rangle = \alpha_2,\ \ldots,\ \langle v, y_q\rangle = \alpha_q\,,$$

with real constants $\alpha_1, \ldots, \alpha_q$. Vector v_0 is given by

$$v_0 = \sum_{k=1}^{q} \gamma_k\, y_k$$

where the coefficients γ_k satisfy the equations

$$\begin{gathered}
\langle y_1, y_1\rangle\gamma_1 + \langle y_2, y_1\rangle\gamma_2 + \cdots + \langle y_q, y_1\rangle\gamma_q = \alpha_1 \\
\langle y_1, y_2\rangle\gamma_1 + \langle y_2, y_2\rangle\gamma_2 + \cdots + \langle y_q, y_2\rangle\gamma_q = \alpha_2 \\
\vdots \\
\langle y_1, y_q\rangle\gamma_1 + \langle y_2, y_q\rangle\gamma_2 + \cdots + \langle y_q, y_q\rangle\gamma_q = \alpha_q\,.
\end{gathered}$$

The next lemma will also be necessary.

Lemma A.2.
Functions

$$g_i^k(t) = \begin{cases} e_i^\top e^{(t_k - t)\,\widetilde{A}}\widetilde{B}\,, & t \in [t_0, t_k] \\ 0\,, & t > t_k \end{cases}\ ,$$

$i = 1, \cdots, np$ *and* $k = 1, \cdots, m$, *where* e_i *identifies the* ith *vector of the canonical basis of* $\mathbb{R}^{np}$, *are linearly independent in each interval* $[t_0, \mathrm{T}]$.

Proof. We first show that functions $g_i^m(t) = e_i^\top e^{(\mathrm{T}-t)\widetilde{A}}\widetilde{B}$, $i = 1\ldots, np$ and $t \in [t_0, \mathrm{T}]$, are linearly independent on the interval $[t_{m-1}, \mathrm{T}]$ (and therefore linearly independent on $[t_0, \mathrm{T}]$).

Define $G(t) = \theta_1\, g_1^m(t) + \cdots + \theta_{np}\, g_{np}^m(t)$.

We have

$$\begin{aligned} G(t) &= \theta_1\, e_1{}^\top e^{(\mathrm{T}-t)\widetilde{A}}\widetilde{B} + \cdots + \theta_{np}\, e_{np}{}^\top e^{(\mathrm{T}-t)\widetilde{A}}\widetilde{B} \\ &= \theta^\top e^{(\mathrm{T}-t)\widetilde{A}}\widetilde{B} \end{aligned}$$

with $\theta = \sum \theta_i e_i$.

If $G(t) = 0$ for all $t \in [t_{m-1}, \mathrm{T}]$ then $G^{(r)}(\mathrm{T}) = (-1)^r \theta^\top \widetilde{A}^r \widetilde{B} = 0$ for all $r \in \mathbb{N}$.

We get immediately that $\theta^\top[\widetilde{B}\;\widetilde{A}\widetilde{B}\;\ldots\widetilde{A}^{np-1}\widetilde{B}] = 0$ but since the control system (3.2) is completely state controllable we must have $\theta = 0$ and the conclusion follows.

A similar reasoning leads to the conclusion that for each $k = 1, \ldots, m-1$ functions $g_1^k, \ldots, g_{np}^k$ are linearly independent on the time interval $[t_{k-1}, t_k]$.

Now we show that functions $g_1^{m-1}, \ldots, g_{np}^{m-1}, g_1^m, \ldots, g_{np}^m$ are linearly independent on interval $[t_{m-2}, \mathrm{T}]$.

Define equation

$$\mu_1 g_1^{m-1}(t) + \cdots \mu_{np} g_{np}^{m-1}(t) + \nu_1 g_1^m(t) + \cdots + \nu_{np} g_{np}^m(t) = 0, \quad t \in [t_{m-2}, \mathrm{T}]\,.$$

If $t > t_{m-1}$ we have for every $i = 1, \ldots, np$ that $g_i^{m-1}(t) = 0$ and the previous equation reduces to

$$\nu_1 g_1^m(t) + \cdots + \nu_{np} g_{np}^m(t) = 0, \quad t \in [t_{m-1}, \mathrm{T}]$$

which clearly gives $\nu_1 = \cdots = \nu_{np} = 0$.

Hence,

$$\mu_1 g_1^{m-1}(t) + \cdots + \mu_{np} g_{np}^{m-1}(t) = 0, \quad t \in [t_{m-2}, \mathrm{T}]\,,$$

but since functions $g_1^{m-1} \ldots, g_{np}^{m-1}$ are linearly independent on $[t_{m-2}, t_{m-1}]$ we get $\mu_1 = \ldots = \mu_{np} = 0$.

Applying the same procedure repeatedly leads to the conclusion that functions

$$g_1^1, \ldots, g_{np}^1, \ldots, g_1^m, \ldots, g_{np}^m$$

are linearly independent on $[t_0, \mathrm{T}]$. □

Remark A.3. We will be interested only in functions g_i^k, $i = 1, \ldots, n$, $k = 1, \ldots, m-1$ and g_i^m, $i = 1, \ldots, np$, which are also linearly independent in $[t_0, \mathrm{T}]$.

The solution of the differential equation $\dot{\mathrm{w}} = \widetilde{A}\,\mathrm{w} + \widetilde{B}\,u(t)$ that satisfies the initial condition $\mathrm{w}(t_0) = (\eta_0^0, \ldots, \eta_0^{p-1})^\top$ is given by

$$\mathrm{w}(t) = e^{(t-t_0)\widetilde{A}}\,\mathrm{w}(t_0) + \int_{t_0}^{t} e^{(t-s)\widetilde{A}}\,\widetilde{B}\,u(s)\,ds\,.$$

Therefore, each interpolation condition $w_1(t_k) = \mathrm{y}_k$, $k = 1, \ldots, m-1$, is equivalent to n equations of the form

$$e_i^\top \left(\int_{t_0}^{t_k} e^{(t_k-s)\widetilde{A}}\,\widetilde{B}\,u(s)\,ds \right) = e_j^\top\,\mathrm{y}_k - e_i^\top\,e^{(t_k-t_0)\widetilde{A}}\,\mathrm{w}(t_0)$$

for $i = j = 1, \ldots, n$, where e_i identifies the ith vector of the canonical basis of $\mathbb{R}^{np}$ and e_j identifies the jth vector of the canonical basis of $\mathbb{R}^n$. For the boundary condition $\mathrm{w}(\mathrm{T}) = (\eta_m^0, \ldots, \eta_m^{p-1})^\top$ we have

$$e_i^\top \left(\int_{t_0}^{t_m} e^{(t_m-s)\widetilde{A}}\,\widetilde{B}\,u(s)\,ds \right) = e_i^\top \left(\mathrm{w}(\mathrm{T}) - e^{(\mathrm{T}-t_0)\widetilde{A}}\,\mathrm{w}(t_0) \right)\,,$$

$i = 1, \ldots, np$.

Hence, the interpolation conditions give rise to $(m-1) \times n$ equations of the form

$$\langle (g_i^k(t))^\top, u(t) \rangle = \alpha_i^k$$

where

$$\alpha_i^k = e_j^\top\,\mathrm{y}_k - e_i^\top\,e^{(t_k-t_0)\widetilde{A}}\,\mathrm{w}(t_0)$$

for each $i = j = 1, \ldots, n$ and $k = 1, \ldots, m-1$. We also have another np equations

$$\langle (g_i^m(t))^\top, u(t) \rangle = \alpha_i^m$$

where

$$\alpha_i^m = e_i^\top \left(\mathrm{w}(\mathrm{T}) - e^{(\mathrm{T}-t_0)\widetilde{A}}\,\mathrm{w}(t_0) \right)\,,$$

$i = 1, \ldots, np$.

Finally, using the previous remark and applying theorem A.1, the following result follows immediately.

Theorem A.4.
Problem (3.2) has a unique optimal solution of the form

$$\widehat{u}(t) = \sum_{k=1}^{m-1} \sum_{i=1}^{n} \gamma_i^k\,(g_i^k(t))^\top + \sum_{i=1}^{np} \gamma_i^m\,(g_i^m(t))^\top \tag{A.1}$$

where $\gamma_i^k \in \mathbb{R}$ *for all* i *and* k.

The next result further characterizes the optimal control given above.

Corollary A.5.
The optimal control given by equation (A.1) has the following expression in each interval $[t_{j-1}, t_j]$.

$$\widehat{u}(t) = \widetilde{B}^{\top} e^{(t_{j-1}-t)\,\widetilde{A}^{\top}}\, S^{-1}\,(e^{(t_{j-1}-t_j)\,\widetilde{A}}\,\mathrm{w}_j - \mathrm{w}_{j-1}) = \widetilde{B}^{\top} e^{-t\,\widetilde{A}^{\top}}\rho \tag{A.2}$$

where $\mathrm{w}_j = \mathrm{w}(t_j)$,

$$S = S(t_{j-1}, t_j) = \int_{t_{j-1}}^{t_j} e^{(t_{j-1}-s)\widetilde{A}}\widetilde{B}\,\widetilde{B}^{\top} e^{(t_{j-1}-s)\widetilde{A}^{\top}}\,ds$$

and ρ *is a constant matrix of dimension* $np \times 1$.

Proof. We know that there exist points w_j and w_{j-1} that belong to $\mathbb{R}^{np}$ and are such that

$$\mathrm{w}_j = e^{(t_j-t_{j-1})\widetilde{A}}\,\mathrm{w}_{j-1} + \int_{t_{j-1}}^{t_j} e^{(t_j-s)\widetilde{A}}\,\widetilde{B}\,u(s)\,ds\,. \tag{A.3}$$

Also, in each interval $[t_{j-1}, t_j]$ the optimal control can be rewritten as

$$\widehat{u}(t) = \sum_{k=j}^{m-1}\left(\widetilde{B}^{\top} e^{(t_k-t)\widetilde{A}^{\top}}\left(\sum_{i=1}^{n}\gamma_i^k\,e_i\right)\right) + \widetilde{B}^{\top} e^{(\mathrm{T}-t)\widetilde{A}^{\top}}\left(\sum_{i=1}^{np}\gamma_i^m\,e_i\right). \tag{A.4}$$

Replacing this expression in equation (A.3) we have

$$\begin{aligned}
\mathrm{w}_j &= e^{(t_j-t_{j-1})\widetilde{A}}\,\mathrm{w}_{j-1} + \sum_{k=j}^{m-1}\left(\int_{t_{j-1}}^{t_j} e^{(t_j-s)\widetilde{A}}\,\widetilde{B}\,\widetilde{B}^{\top} e^{(t_k-s)\widetilde{A}^{\top}}\,ds\left(\sum_{i=1}^{n}\gamma_i^k\,e_i\right)\right) \\
&\quad + \left(\int_{t_{j-1}}^{t_j} e^{(t_j-s)\widetilde{A}}\,\widetilde{B}\,\widetilde{B}^{\top} e^{(\mathrm{T}-s)\widetilde{A}^{\top}}\,ds\right)\left(\sum_{i=1}^{np}\gamma_i^m\,e_i\right) \\
&= e^{(t_j-t_{j-1})\widetilde{A}}\,\mathrm{w}_{j-1} + \sum_{k=j}^{m-1}\left(e^{(t_j-t_{j-1})\widetilde{A}}\,S\;e^{(t_k-t_{j-1})\widetilde{A}^{\top}}\left(\sum_{i=1}^{n}\gamma_i^k\,e_i\right)\right) \\
&\quad + e^{(t_j-t_{j-1})\widetilde{A}}\,S\;e^{(\mathrm{T}-t_{j-1})\widetilde{A}^{\top}}\left(\sum_{i=1}^{np}\gamma_i^m\,e_i\right).
\end{aligned}$$

Since the control system (3.2) is completely state controllable matrix S is invertible and it follows that

$$\begin{aligned}
S^{-1}\,(e^{(t_{j-1}-t_j)\,\widetilde{A}}\,\mathrm{w}_j - \mathrm{w}_{j-1}) &= \sum_{k=j}^{m-1}\left(e^{(t_k-t_{j-1})\widetilde{A}^{\top}}\left(\sum_{i=1}^{n}\gamma_i^k\,e_i\right)\right) \\
&\quad + e^{(\mathrm{T}-t_{j-1})\widetilde{A}^{\top}}\left(\sum_{i=1}^{np}\gamma_i^m\,e_i\right).
\end{aligned}$$

It is now sufficient to multiply both members of the previous equation on the left hand side by $\widetilde{B}^{\top} e^{(t_{j-1}-t)\widetilde{A}^{\top}}$ and use equation (A.4). □

A.1 Structure of the components of the optimal trajectory

When $p = 1$ and $L = D - AD^0$ in problem (3.1) some information can also be given about the structure of the optimal components $y^i(t)$, $i = 1\ldots,n$, of the optimal output function $y(t) = (y^1(t)\ y^2(t)\cdots y^n(t))^\top$. We can say that there is at most $2n$ linearly independent function such that each component are linear combination of some of those $2n$ functions. It must be clear that there is at least one case in which at least one component y^i is a linear combination of all $2n$ functions. This extreme situation reveals when the control system has a single input and a single output. This idea is explained in the following.

Let $p(\lambda) = \lambda^n - a_{n-1}\lambda^{n-1} - \cdots - a_1\lambda - a_0$ be the characteristic polynomial of the coefficient matrix A,

$$F \equiv D^n - a_{n-1}D^{n-1} - \cdots - a_1 D - a_0 D^0$$

the associated linear operator and F^* it's adjoint. The following result characterizes the structure of each component.

Proposition A.6.
In each interval $[t_{k-1}, t_k]$, the components of the optimal output function y belong to the kernel of the operator F^*F.

Proof. Using the dynamic equation $\dot{\mathrm{x}} = A\,\mathrm{x} + Bu(t)$ we deduce that

$$D^k\mathrm{x} = A^k\mathrm{x} + \sum_{j=0}^{k-1} A^j B D^{k-1-j} u$$

for all $k \in \mathbb{N}$. Hence,

$$Fy = (A^{n-1} - a_{n-1}A^{n-2} - \cdots - a_1 id)\, B\widehat{u} + \cdots + (A - a_{n-1}id)\, BD^{n-2}\widehat{u} + BD^{n-1}\widehat{u}$$

and

$$F^*Fy = (A^{n-1} - a_{n-1}A^{n-2} - \cdots - a_1 id)\, F^*B\widehat{u} + \cdots + F^*BD^{n-1}\widehat{u}$$

in each interval $[t_{k-1}, t_k]$. Now, the function $B\widehat{u}$, where $\widehat{u}$ is given by equation (A.2), is a solution of $F^*v = 0$ in each interval $[t_{k-1}, t_k]$. To see this, just apply the operator F^* to the function $B\widehat{u}$ and use Cayley-Hamilton's theorem. Therefore, $F^*Fy = 0$, or equivalently, $F^*Fy^i = 0$, $i = 1\ldots,n$, in each interval $[t_{k-1}, t_k]$. □

It is clear that there is $2n$ linearly independent function such that each component y^i is a linear combination of some of those $2n$ functions. Notice also that in order to find those $2n$ linearly independent functions one only needs to find the spectrum of matrix A.

It is important to observe that $F^*Fy^i = 0$ indicates what kind of function is y^i but doesn't help if one wants to really determine y^i. In general there will be less conditions than necessary.

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