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CURRENT ISSUES ON SINGULAR AND DEGENERATE EVOLUTION EQUATIONS

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Contents

1. Introduction

Let Ω be an open set in $\mathbf{R}^N$ and consider the quasilinear, parabolic, partial differential equation of the second order

$$(1.1)_p \qquad \begin{cases} u \in L^\infty_{\rm loc}(0,T;W^{1,p}_{\rm loc}(\Omega)) \ , \quad p>1; \\ u_t - \operatorname{div} \mathbf{A}(x,t,u,\nabla u) = B(x,t,u,\nabla u) \quad \text{weakly in } \Omega_T \ . \end{cases}$$

Here $T>0$ is given, $\Omega_T = \Omega \times (0,T)$ and ∇ denotes the gradient with respect to the space variables $x=(x_1,\ldots,x_N)$. The functions $\mathbf{A}=(A_1,\ldots,A_N)$ and B are real valued, measurable with respect to their arguments, and satisfying the structure conditions

$$(1.2)_p \qquad \begin{cases} C_0|\nabla u|^{p-2}|\nabla u|^2 - C \leq \ \mathbf{A}(x,t,u,\nabla u)\cdot \nabla u \ ; \\ |\mathbf{A}(x,t,u,\nabla u)| + |B(x,t,u,\nabla u)| \leq C(1+|\nabla u|^{p-1}) \ , \end{cases}$$

where C_0 and C are given positive constants. The quantity $C_0|\nabla u|^{p-2}$ is the modulus of ellipticity of the equation. If $p>2$ it vanishes whenever $|\nabla u|=0$ and the equation is said to be *degenerate* at those $(x,t)\in\Omega_T$ where this occurs. If $1<p<2$ the modulus of ellipticity becomes infinity whenever $|\nabla u|=0$ and the equation is said to be *singular* at those $(x,t)\in\Omega_T$ where $|\nabla u|=0$.

Along with $(1.1)_p$–$(1.2)_p$, consider also the quasilinear equation,

$$(1.1)_m \qquad \begin{cases} u \in L^\infty_{\rm loc}(0,T;W^{1,2}_{\rm loc}(\Omega)) \ ; \\ u_t - \operatorname{div} \mathbf{A}(x,t,u,\nabla u) = B(x,t,u,\nabla u) \quad \text{weakly in } \Omega_T \ , \end{cases}$$

with structure conditions,

$$(1.2)_m \qquad \begin{cases} C_0|u|^{m-1}|\nabla u|^2 - C \leq \ \mathbf{A}(x,t,u,\nabla u)\cdot \nabla u \ ; \\ |\mathbf{A}(x,t,u,\nabla u)| + |B(x,t,u,\nabla u)| \leq C|u|^{m-1}(1+|\nabla u|) \ , \end{cases}$$

where m is a given positive number. The prototype example of $(1.1)_m$–$(1.2)_m$ is

$$(1.3)_m \qquad u_t - \Delta |u|^{m-1}u = 0 \quad \text{for some } m>0 \ ,$$

and a the prototype example of $(1.1)_p$–$(1.2)_p$ is,

$$(1.3)_p \qquad u_t - \operatorname{div} |\nabla u|^{p-2}\nabla u = 0 \quad \text{for some } p>1 \ .$$

The first is called the *porous medium* equation. If $m>1$ the modulus of ellipticity vanishes for $u=0$ and the equation is degenerate at those points of Ω_T where the solution u vanishes. If $0 < m < 1$ the modulus of ellipticity is infinity whenever $u = 0$ and the equation is *singular* at those points of Ω_T where $u = 0$. If $m > 1$ the equation is referred to as the *slow diffusion* case of the porous medium equation. The case $0 < m < 1$ is the *fast diffusion*.

The equation in $(1.3)_p$ is the p–Laplacian equation and its modulus of ellipticity is $|\nabla u|^{p-2}$. If $p > 2$ such a modulus vanishes whenever $|\nabla u| = 0$ and at such points the equation is degenerate. If $1 < p < 2$ the modulus of ellipticity becomes infinity whenever $|\nabla u| = 0$ and the equation is singular at these points.

If $m = 1$ in $(1.3)_m$ or $p = 2$ in $(1.3)_p$ one recovers the classical heat equation for which information are essentially encoded in the fundamental solution

$$\Gamma(x, y; t, \tau) = \frac{1}{(t-\tau)^{N/2}} \exp\left\{-\frac{|x-y|^2}{4(t-\tau)}\right\} \quad x, y \in \mathbf{R}^N, \quad t > \tau\ . \tag{1.4}$$

The porous medium equation in $(1.3)_m$ admits an explicit similarity solution that "resembles" the fundamental solution of the heat equation. Such a solution is called the *Barenblatt* similarity solution and it is given by ([17])

$$(1.4)_m \qquad \Gamma_m(x, y; t, \tau) = \frac{1}{(t-\tau)^{N/\lambda}} \left\{1 - \gamma_m \left[\frac{|x-y|}{(t-\tau)^{1/\lambda}}\right]^2\right\}_+^{\frac{1}{m-1}} \qquad t > \tau$$

where for a real number α, $\{\alpha\}_+ = \max\{\alpha; 0\}$, and

$$(1.4)'_m \qquad \lambda = N(m-1)+2\ , \qquad \gamma_m = \frac{1}{\lambda}\,\frac{m-1}{2}\ .$$

An examination of this solution reveals that it is well defined for all positive values of m for which $\lambda > 0$. One also verifies that $\Gamma_m \to \Gamma$ as $m \to 1$. In this sense Γ_m is the *fundamental solution* of the porous medium equation.

Also the p–Laplacian equation $(1.3)_p$ admits explicit similarity solutions,

$$(1.4)_p \qquad \Gamma_p(x, y; t, \tau) = \frac{1}{(t-\tau)^{N/\kappa}} \left\{1 - \gamma_p \left[\frac{|x-y|}{(t-\tau)^{1/\kappa}}\right]^{\frac{p}{p-1}}\right\}_+^{\frac{p-1}{p-2}} \qquad t > \tau$$

where

$$(1.4)'_p \qquad \kappa = N(p-2)+p\ , \qquad \gamma_p = \left(\frac{1}{\kappa}\right)^{\frac{1}{p-1}}\,\frac{p-2}{p}\ .$$

This is well defined for all $p > 1$ such that $\kappa > 0$ and $\Gamma_p \to \Gamma$ as $p \to 2$. In this sense Γ_p is the *fundamental solution* of the p–Laplacian equation.

• Issues of compact support and regularity

Assume first $m > 1$ and $p > 2$. The first difference between these *fundamental solutions* and the fundamental solution of the heat equation is in their support with respect to the space variables. For fixed $t > \tau$ and $y \in \mathbf{R}^N$, the functions $x \mapsto \Gamma_m(x), \Gamma_p(x)$ are compactly supported in $\mathbf{R}^N$, whereas $x \mapsto \Gamma(x)$ is positive in the whole $\mathbf{R}^N$. The *moving boundaries* separating the regions where Γ_m and Γ_p are positive from the regions where they vanish are,

$$\left.\begin{array}{rcll} (t-\tau)^{2/\lambda} & = & \gamma_m |x-y|^2 & \text{for } \Gamma_m \\ (t-\tau)^{p/\kappa} & = & \gamma_p^{p-1}|x-y|^p & \text{for } \Gamma_p \end{array}\right. \qquad \tau \in \mathbf{R},\ y \in \mathbf{R}^N \text{ fixed} .$$

The second difference is in their degree of regularity. For fixed $t > \tau$ and $y \in \mathbf{R}^N$ the function $x \mapsto \Gamma_m(x)$ is Hölder continuous with Hölder exponent $1/(m-1)$. The function $x \mapsto \Gamma_p(x)$ is differentiable and its partial derivatives are Hölder continuous with Hölder exponent $1/(p-2)$. Such a modest degree of regularity is in contrast to the fundamental solution of the heat equation which is analytic in the space variables.

Let now $0 < m < 1$ by keeping $\lambda > 0$. Then Γ_m is positive and locally analytic in the whole $\mathbf{R}^N \times \{t > \tau\}$. Thus Γ_m seems to share the same properties as Γ.

If $1 < p < 2$ by keeping $\kappa > 0$ then Γ_p is positive in the whole $\mathbf{R}^N \times \{t > \tau\}$ but still it maintains a limited degree of regularity.

• Issues of Harnack inequalities

Non–negative, local solutions of the heat equation in Ω_T satisfy the Harnack inequality. This is a celebrated result of Hadamard [89] and Pini [144] and it takes the following form. Fix $(x_0, t_0) \in \Omega_T$ and for $\rho > 0$ consider the ball $B_\rho(x_0)$ centered at x_0 and with radius ρ and the cylindrical domain

$$Q_\rho(x_0, t_0) = B_\rho(x_0) \times (t_0 - \rho^2 , t_0 + \rho^2) . \tag{1.5}$$

These are the *parabolic cylinders* associated with the heat equation and also to $(1.3)_m$ since these equations remain unchanged under a similarity transformation of the space–time variables that keeps constant the ratio $|x|^2/t$.

There exists a constant C depending only upon N and independent of (x_0, t_0) and ρ, such that

$$C\, u(x_0, t_0) \ge \sup_{B_\rho(x_0)} u(x_0, t_0 - \rho^2) \qquad \text{provided} \quad Q_{2\rho}(x_0, t_0) \subset \Omega_T \,. \tag{1.6}$$

The proof is based on local representations by means of the heat potentials (1.4). In particular, for fixed $\tau \in \mathbf{R}$ and $y \in \mathbf{R}^N$, the fundamental solution $(x, t) \mapsto \Gamma(x, t)$ satisfies such a Harnack estimate. It is then natural to ask whether the *fundamental solution* $(1.4)_m$ would satisfy a Harnack estimate. Take for example the Γ_m for $\tau = 0$ and y the origin of $\mathbf{R}^N$. Assume first that $m > 1$ and fix (x_0, t_0) on the moving boundary, so that $\Gamma_m(x_0, t_0) = 0$. If ρ and $|x_0|$ are sufficiently large, the ball $B_\rho(x_0)$ intersects the support of Γ_m at $t = t_0 - \rho^2$. Therefore for such choices the left hand hand side of (1.6) is zero and the right hand side would be positive.

If $0 < m < 1$ and $\lambda > 0$, take $x_0 = 0$ and $t_0 > 4\rho^2$. Then a direct calculation shows that (1.6) cannot be verified for a constant C independent of ρ and t_0.

Consider now $(1.3)_p$ and the corresponding Γ_p. The similarity rescaling that keeps $(1.3)_p$ invariant is $|x|^p/t = \text{const}$. Therefore the natural *parabolic cylinders* associated with $(1.3)_p$ are of the type

$$Q_\rho(x_0, t_0) = B_\rho(x_0) \times (t_0 - \rho^p\,,\ t_0 + \rho^p)\,. \tag*{$(1.5)_p$}$$

Similar arguments show that Γ_p does not satisfy the Harnack inequality for all $p > 1$ such that $\kappa > 0$, even in its own natural parabolic geometry $(1.5)_p$.

• Local behavior of solutions

These issues suggest a unifying theory of the local behavior of weak solutions of degenerate or singular parabolic equations. A cornerstone of such a unifying theory would be that weak solutions of $(1.1)_{m,p}$–$(1.2)_{m,p}$ are Hölder continuous.

Another key component would be an understanding of the Harnack estimate in the degenerate or singular setting of $(1.1)_{m,p}$–$(1.2)_{m,p}$. Whether for example there is a form of such an estimate that replaces (1.6) and that would reduce to it when either $m \to 1$ or $p \to 2$. The general structure in $(1.1)_{m,p}$–$(1.2)_{m,p}$ is not an artificial requirement. To illustrate this point we return briefly on the issue of regularity of solutions of $(1.3)_p$. We have observed that the space–gradient of the *fundamental solution* Γ_p for $p > 2$ is locally Hölder continuous in $\mathbf{R}^N \times (\tau, \infty)$. It is then natural to conjecture that the same would be true for local solutions u of $(1.3)_p$. For such solutions, it turns out that $v = |\nabla u|^2$ formally satisfies ([55],[165])

$$v_t - \left(a_{ij} v^{\frac{p-2}{2}} v_{x_i}\right)_{x_j} \le 0 \qquad \text{where} \qquad a_{ij} = \delta_{ij} + (p-2)\frac{u_{x_i} u_{x_j}}{|\nabla u|^2} \; .$$

This is a quasilinear version of $(1.3)_m$ with $m = p/2$. Thus an investigation of the local regularity of solutions of $(1.3)_p$ requires an understanding of degenerate or singular equations with the general quasilinear structure $(1.1)_m$-$(1.2)_m$.

1.1. Historical background on regularity and Harnack estimates. Considerable progress was made in the early 1950s and mid 1960s in the theory of elliptic equations, due to the discoveries of DeGiorgi [47] and Moser [132], [133]. Consider local weak solutions of

$$\left(a_{ij} u_{x_i}\right)_{x_j} = 0 \quad \text{weakly in } \Omega\,; \qquad u \in W^{1,2}_{\rm loc}(\Omega) \tag{1.7}$$

where the coefficients $x \mapsto a_{ij}(x)$, $i, j = 1, 2, \ldots, N$ are only bounded and measurable and satisfying the ellipticity condition

$$a_{ji}\xi_i\xi_j \ge C_0|\xi|^2 \quad \text{a.e. in } \Omega\,, \quad \forall \xi \in \mathbf{R}^N, \quad \text{for some } C_0 > 0 \; . \tag{1.8}$$

DeGiorgi established that local solutions are Hölder continuous and Moser proved that non-negative solutions satisfy the Harnack inequality. Such inequality can be used, in turn, to prove the Hölder continuity of solutions. Both authors worked with *linear* p.d.e.'s. However the linearity has no bearing in the proofs. This permits an extension of these results to elliptic quasilinear equations of the type

$$\operatorname{div} \mathbf{A}(x, u, \nabla u) + B(x, u, \nabla u) = 0 \quad \text{weakly in } \Omega\,; \quad u \in W^{1,p}_{\rm loc}(\Omega)\,; \quad p > 1 \tag{1.9}$$

with structure conditions

$$\begin{cases} C_0|\nabla u|^p - C \le \mathbf{A}(x, u, \nabla u) \cdot \nabla u \; ; \\ |\mathbf{A}(x, u, \nabla u)| + |B(x, u, \nabla u)| \le C(1 + |\nabla u|^{p-1}) \; , \end{cases} \tag{1.10}$$

for a given $C_0 > 0$ and a given non–negative constant C. By using the methods of DeGiorgi, Ladyzhenskaja and Ural'tzeva [120] established that weak solutions of (1.9)–(1.10) are Hölder continuous, whereas Serrin [157] and Trudinger [163], following the methods of Moser, proved that non-negative solutions satisfy a Harnack principle. The generalisation is twofold i.e., the principal part $\mathbf{A}(x, u, \nabla u)$ is permitted to have a non-linear dependence with respect to ∇u, and a *non-linear growth* with respect to $|\nabla u|$. The latter is of particular interest since the equation in (1.9) might be either degenerate or singular.

A striking result of Moser [134] is that the Harnack estimate (1.6) continues to hold for non-negative, local, weak solutions of

$$\begin{cases} u \in L^\infty\left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; W^{1,2}(\Omega)\right) \ ; \\ u_t - \left(a_{ij}(x,t) u_{x_i}\right)_{x_j} = 0 \qquad \text{in } \Omega_T \ , \end{cases} \tag{1.11}$$

where $a_{ij} \in L^\infty(\Omega_T)$ satisfy the analog of the ellipticity condition (1.8). As before, it can be used to prove that weak solutions are locally Hölder continuous in Ω_T. Since the linearity of (1.11) is immaterial to the proof, one might expect, as in the elliptic case, an extension of these results to quasilinear equations of the type

$$u_t - \operatorname{div} \mathbf{A}(x, t, u, \nabla u) = B(x, t, u, \nabla u) \qquad \text{in } \Omega_T \tag{1.12}$$

where the structure condition is as in (1.10). Surprisingly however, Moser's proof could be extended only for the case $p = 2$, i.e., for equations whose principal part has a *linear growth* with respect to $|\nabla u|$. This appears in the work of Aronson and Serrin [16] and Trudinger [164]. The methods of De-Giorgi also could not be extended. Ladyzenskaja et al. [121] proved that solutions of (1.12) are Hölder continuous, provided the principal part has exactly a *linear growth* with respect to $|\nabla u|$. Analogous results were established by Kruzkov [110], [111], [112] and by Nash [136] by entirely different methods. Thus it appears that unlike the elliptic case, the degeneracy or singularity of the principal part plays a peculiar role, and for example, for the p–Laplacian equation in $(1.3)_p$ one could not establish whether non-negative weak solutions satisfy the Harnack estimate or whether a solution is locally Hölder continuous.

In the mid-1980, some progress was made in the theory of degenerate p.d.e.'s of the type of (1.12), for $p > 2$. It was shown that the solutions are locally Hölder continuous (see [51]). Surprisingly, the same techniques can be suitably modified to establish the *local* Hölder continuity of *any local* solution of quasilinear porous medium-type equations. These modified methods in turn, are crucial in proving that weak solutions of the p–Laplacian equation $(1.3)_p$ are of class $C^{1,\alpha}_{\text{loc}}(\Omega_T)$.

Therefore understanding the local structure of the solutions of (1.12) has implications to the theory of equations with degeneracies quite different than (1.12).

In the early 1990s the theory was completed ([37]) by establishing that solutions of $(1.1)_{m,p}$–$(1.2)_{m,p}$ are Hölder continuous for all $p > 1$ and all $m > 0$. For a complete account see [55].

1.2. A new approach to regularity. These results follow, one way or another, from a single unifying idea which we call *intrinsic rescaling.* The diffusion processes in $(1.3)_{m,p}$ evolve in a time scale determined instant by instant by the solution itself, so that, loosely speaking, they can be regarded as the heat equation in their own intrinsic time-configuration. A precise description of this fact as well as its effectiveness is linked to its technical implementations which we will present in §2.

The indicated regularity results assume the solutions to be locally or globally bounded. A theory of boundedness of weak solutions of $(1.1)_{m,p}$–$(1.2)_{m,p}$ is quite different from the linear theory and it is presented in §3. For example weak solutions of $(1.1)_p$–$(1.2)_p$ are locally bounded only if $\kappa = N(p-2)+p > 0$ and weak solutions of $(1.1)_m$–$(1.2)_m$ are locally bounded only if $\lambda = N(m-1)+2 > 0$. It is shown by counterexamples that these conditions are sharp.

The same notion of intrinsic rescaling is at the basis of a new notion of Harnack inequality for non–negative solutions of $(1.3)_{m,p}$ established in the late 1980s and early 1990s ([54], [66]). Consider non–negative weak solution of $(1.3)_m$ for $m > 1$. The Harnack inequality (1.6) continues to hold for such solutions provided the time is rescaled by the quantity $u^{m-1}(x_0,t_0)$. Similar statements hold for $(1.3)_p$ in their intrinsic parabolic geometry $(1.5)_p$. In §4 we present these *intrinsic* versions of the Harnack inequality and trace their connection to the Hölder continuity of solutions.

A major open problem is to establish the Harnack estimate for non–negative solutions of $(1.1)_{m,p}$ with the full quasilinear structure $(1.2)_{m,p}$. The proofs in ([54], [66]) use in an essential way the structure of $(1.3)_{m,p}$ as well as their corresponding *fundamental solutions* $\Gamma_{m,p}$. The leap forward of Moser's Harnack inequality was in bypassing the classical approaches based on heat potentials, by introducing new harmonic analysis methods and techniques. It is our belief that a proof of the intrinsic Harnack estimate for non–negative solutions of $(1.1)_{m,p}$–$(1.2)_{m,p}$ that would bypass the *potentials* $\Gamma_{m,p}$, would have the same impact.

The values of $p > 1$ for which non–negative solutions of $(1.3)_p$ satisfy Harnack's inequality are those for which $\kappa = N(p-2)+p > 0$. Likewise the values of $m > 0$ for which non–negative solutions of $(1.3)_m$ satisfy Harnack's inequality are those for which $\lambda = N(m-1)+2 > 0$. These limitations are sharp for a Harnack estimate to hold (§4).

1.3. Limiting cases and miscellanea remarks. The cases $\kappa, \lambda \leq 0$ are not well understood and form the object of current investigations. The case $1 < p \leq 2N/(N+1)$ seems to suggest questions similar to those of the limiting Sobolev exponent for elliptic equations (see Brézis [30]) and questions in differential geometry. As $p \searrow 1$, $(1.3)_p$ tends *formally* to a p.d.e. of the type of motion by mean curvature. Investigations in this directions are due to Evans and Spruck ([79]). As $m \to 0$ the porous medium equation $(1.3)_m$ for $u \geq 0$ tends to the singular equation

$$u_t - \Delta \ln u = 0 \quad \text{weakly in } \Omega_T \; . \tag{1.13}$$

When $N = 2$ the Cauchy problem for this equation is related to the Ricci flow associated to a complete metric in $\mathbf{R}^2$ ([90], [182], [59], [60]). A characterization of the initial data for which (1.13) is solvable has been identified and the theory seems fairly complete ([59], [46], [78]). The case $N \geq 3$ however is still not understood and while solvability has been established for a rather large class of data ([59]), a precise characterization of such a class still eludes the investigators.

Degenerate and singular elliptic and parabolic equations are one of the branches of modern analysis both in view of the physical significance of the equations at hand ([8], [10], [11], [91], [92], [113], [114], [119], [126], [127], [161], [183]) and the novel analytical techniques that they generate ([55]).

The class of such equations is large, ranging from flows by mean curvature to Monge–Ampére equations to infinity-Laplacian. These are *implicitly* degenerate or singular equations in that the solution itself determines, implicitly, the regions of degeneracy. Explicitly degenerate equations would be those for which the degeneracy or singularity is *a priori* prescribed in the coefficients. For example if the modulus of ellipticity C_0 in (1.8) were a non–negative function of x vanishing at some specified value x_*, such a point would be a point of explicit degeneracy. There is a vast literature on all these aspects of degenerate equations. We have chosen to present a subsection of the theory that has a unifying set of techniques, issues, physical relevance, and future directions.

1.4. Singular equations of the Stefan–type. In this framework fall singular parabolic evolution equations where the singularity occurs on the time–part of the operator. These take the form

$$\begin{cases} u \in C_{\rm loc}(0,T;W^{1,2}_{\rm loc}(\Omega_T)) \ ; \\ \beta(u)_t - \operatorname{div} \mathbf{A}(x,t,u,\nabla u) \ni B(x,t,u,\nabla u) \quad \text{in } \mathcal{D}'(\Omega_T) \ , \end{cases} \tag{1.14}$$

where $\mathbf{A}$ and B have the same structure conditions as $(1.2)_m$ for $m = 1$ and $\beta(\cdot)$ is a coercive, maximal monotone graph in $\mathbf{R} \times \mathbf{R}$. The prototype example is

$$\beta(u)_t - \Delta u \ni 0 \quad \text{in } \mathcal{D}'(\Omega_T) \tag{1.15}$$

for a $\beta(\cdot)$ given by

$$\beta(s) = \begin{cases} s & \text{for} \quad s < 0 \, ; \\ [0,1] & \text{for} \quad s = 0 \, ; \\ 1 + s & \text{for} \quad s > 0 \, . \end{cases} \tag{1.16}$$

Graphs $\beta(\cdot)$ such as this one, i.e., exhibiting a single jump at the origin, arise from a weak formulation of the classical Stefan problem modelling a solid/liquid the phase transition such as water–ice. In the latter case a natural question would be to ask whether the transition of phase occurs with a continuous temperature across water/ice interface. This issue, raised initially by Oleinik in the 1950s and reported in the book [LSU] is at the origin of the modern and current theory of local regularity and local behaviour of solutions of degenerate and/or singular evolution equations. The coercivity of $\beta(\cdot)$ for a solution to be continuous is essential, as pointed out by examples and counterexamples in [61].

It was established in [31], [48], [149], [150], [184] that for $\beta(\cdot)$ exhibiting a single jump, the solutions of (1.14) are continuous with a given quantitative modulus of continuity (not Hölder). This raises naturally the question of a graph $\beta(\cdot)$ exhibiting multiple jumps and or singularities of other nature (§5). For these rather general graphs, in the mid 1990s it was established in [72] that solutions of (1.14) are continuous provided $N = 2$. For dimension $N \geq 3$ the same conclusion holds provided the principal part of the differential equation is *exactly* the Laplacian, as in the first of (1.15). Several recent investigations have extended and improved these results for specific graphs ([86], [87]). It is still an open question however, whether solutions of (1.14) with its full quasilinear structure and for a general coercive maximal monotone graph $\beta(\cdot)$ and for $N \geq 3$, are continuous in their domain of definition.

1.5. Outline of these notes. The issues touched on here will be expanded in the next sections. We will provide precise statements and self–sufficient structure of proofs.

In section 2 we deal with the question of the regularity of the weak solutions of singular and degenerate quasilinear parabolic equations, proving their Hölder character. We start with the precise definition of weak solution and the derivation of the building blocks of the theory: the local energy and logarithmic estimates. In §2.2 we briefly present the classical approach of De Giorgi to uniformly elliptic equations. We introduce De Giorgi's class and show that functions in De Giorgi's class are Hölder continuous. The two main sections §2.3 and §2.4 deal, respectively, with the degenerate and the singular case. There we present in full detail the idea of intrinsic scaling and, at least in the degenerate case, prove all the results leading to the Hölder continuity. We have decided to present the theory for the model case of the p-Laplace equation to bring to light what is really essential in the method, leaving aside technical refinements needed to deal with more general equations. We close the section with remarks on the possible generalisations, namely to porous medium type equations.

Section 3 addresses the boundedness of weak solutions. The theory discriminates between the degenerate and the singular case. If $p > 2$, a local bound for the solution is implicit in the notion of weak solution. If $1 < p < 2$, local or global solutions need not be bounded in general.

In section 4, we first give a review about classical results concerning Harnack inequalities. Then we consider the degenerate case and we point out the differences with respect to the nondegenerate one. We sketch a proof of the Harnack inequality both in the degenerate and singular case. We show that for positive solutions of the singular p-Laplace equation an "elliptic" Harnack inequality holds. We also analyze the phenomenon of the extinction of the solution in finite time. Through a suitable use of the Raleigh quotient, we are also able to give sharp estimates on the extinction time and to describe the asymptotic profile of the extinction. In the whole section we point out the major open questions about Harnack inequalities for singular and degenerate parabolic equations.

In section 5 we give physical motivations concerning Stefan-like equations and show, through the Kruzkov- Sukorjanski transformation, the deep links between degenerate equations and Stefan-like equations. Then, we describe the approaches made by Aronson, Caffarelli, DiBenedetto, Sachs and Ziemer in the 1980's. Thanks to their contributions the case of only one singularity was completely solved. Lastly we analyze the new pioneering approach of [72] where, through a lemma of measure theory, the case of multiple singularities was totally solved in the case $N = 2$. Moreover we show that this approach also works in the case $N \geq 3$ but only under strong assumptions. In this section we also point out the major open questions.

We have chosen not to present existence theorems for boundary value problems associated with these equations. Theorems of this kind are mostly based on Galerkin approximations and appear in the literature in a variety of forms. We refer, for example, to [121] or [123]. Given the a priori estimates presented here these can be obtained alternatively by a limiting process in a family of approximating problems and an application of Minty's Lemma.

These notes can be ideally divided in three parts:

- **1.:** Hölder continuity and boundedness of solutions (§2-3)
- **2.:** Harnack type estimates (§4)
- **3.:** Stefan-like problems (§5)

These parts are technically linked but they are conceptually independent, in the sense that they deal with issues that have developed in independent directions. We have attempted to present them in such a way that they can be approached independently.

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2. Regularity of weak solutions

We address the question of the regularity of weak solutions of singular and degenerate parabolic equations by proving that they are Hölder continuous.

We will concentrate on quasilinear parabolic equations, with principal part in divergence form, of the type

$$u_t - \operatorname{div} |\nabla u|^{p-2}\nabla u = 0 \ , \qquad p > 1 \ . \tag{2.1}$$

If $p > 2$, the equation is degenerate in the space part, due to the vanishing of its modulus of ellipticity $|\nabla u|^{p-2}$ at points where $|\nabla u| = 0$. The singular case corresponds to $1 < p < 2$: the modulus of ellipticity becomes infinity at points where $|\nabla u| = 0$.

The results in this section extend to a variety of equations and, in particular, to equations with general principal parts satisfying appropriate structure assumptions and with lower order terms. We have chosen to present the results and the proofs for the particular model case (2.1) to bring to light what we feel are the essential features of the theory. Remarks on generalisations, which in some way or another correspond to more or less sophisticated technical improvements, are made at the end of the section.

Results on the continuity of solutions at a point consist basically in constructing a sequence of nested and shrinking cylinders with vertex at that point, such that the essential oscillation of the function in those cylinders converges to zero when the cylinders shrink to zero. At the basis of the proof is an iteration technique, that is a refinement of the technique by De-Giorgi and Moser (cf. [47], [132] and [121]), based on energy (also known as Cacciopolli) and logarithmic estimates for the solution, that we briefly review in §2.2. In the degenerate or singular cases these estimates are not homogeneous in the sense that they involve integral norms corresponding to different powers, namely the powers 2 and p. The key idea is then to look at the equation in its own geometry, i.e., in a geometry dictated by its intrinsic structure. This amounts to rescale the standard parabolic cylinders by a factor that depends on the oscillation of the solution. This procedure, which can be called accommodation of the degeneracy, allows one to recover the homogeneity in the energy estimates written over these rescaled cylinders. We can say heuristically that the equation behaves in its own geometry like the heat equation. In the sequel, we first treat the degenerate case in §2.3 and then the more involved singular case in section §2.4. We conclude the section with some remarks on generalisations, namely to porous medium type equations.

2.1. Weak solutions and local estimates. A local weak sub(super)-solution of (2.1) is a measurable function

$$u \in C_{\text{loc}}\left(0, T; L^2_{\text{loc}}(\Omega)\right) \cap L^p_{\text{loc}}\left(0, T; W^{1,p}_{\text{loc}}(\Omega)\right)$$

such that, for every compact $K \subset \Omega$ and for every subinterval $[t_1, t_2]$ of $(0, T]$,

$$\int_K u\varphi \,\mathrm{d}x\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\!\!\int_K \left\{u\varphi_t + |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\right\} \mathrm{d}x\,\mathrm{d}t \le (\ge)\; 0 \;, \tag{2.2}$$

for all $\varphi \in W^{1,2}_{\mathrm{loc}}\big(0,T; L^2(K)\big) \cap L^p_{\mathrm{loc}}\big(0,T; W^{1,p}_0(K)\big)$, $\varphi \ge 0$. A function that is both a local subsolution and a local supersolution of (2.1) is a local solution of (2.1).

It would be technically convenient to have at hand a formulation of weak solution involving the time derivative u_t. Unfortunately, solutions of (2.1), whenever they exist, possess a modest degree of time-regularity and in general u_t has a meaning only in the sense of distributions. To overcome this limitation we introduce the Steklov average of a function $v \in L^1(\Omega_T)$, defined for $0 < h < T$ by

$$v_h = \begin{cases} \frac{1}{h}\int_t^{t+h} v(\cdot,\tau)\,\mathrm{d}\tau & \text{if} \quad t \in (0, T-h] \\[2ex] \qquad 0 & \text{if} \quad t \in (T-h, T] \end{cases} \;,$$

and observe that the notion (2.2) of solution is equivalent to:

for every compact $K \subset \Omega$ and for all $0 < t < T - h$,

$$\int_{K\times\{t\}} \left\{(u_h)_t\,\varphi + \left(|\nabla u|^{p-2}\nabla u\right)_h \cdot \nabla\varphi\right\} \mathrm{d}x \le (\ge)\; 0 \;, \tag{2.3}$$

for all $\varphi \in W^{1,p}_0(K) \cap L^\infty_{\mathrm{loc}}(\Omega)$, $\varphi \ge 0$.

We will show that locally bounded solutions of (2.1) are locally Hölder continuous within their domain of definition. No specific boundary or initial values need to be prescribed for u. Although the arguments below are of local nature, to simplify the presentation we assume that u is a.e. defined and bounded in Ω_T and set

$$M \equiv \|u\|_{L^\infty(\Omega_T)} \;.$$

See section 3 for results on the boundedness of weak solutions.

2.1.1. *Local energy and logarithmic estimates.* Given a point $x_0 \in \mathbf{R}^N$, denote by $K_\rho(x_0)$ the N-dimensional cube with centre at x_0 and wedge 2ρ:

$$K_\rho(x_0) := \left\{x \in \mathbf{R}^N : \max_{1\le i\le N} |x_i - x_{0i}| < \rho\right\} \;;$$

given a point $(x_0, t_0) \in \mathbf{R}^{N+1}$, the cylinder of radius ρ and height $\tau > 0$ is

$$(x_0, t_0) + Q(\tau, \rho) := K_\rho(x_0) \times (t_0 - \tau, t_0) \ .$$

Consider a cylinder $(x_0, t_0) + Q(\tau, \rho) \subset \Omega_T$ and let $0 \le \zeta \le 1$ be a piecewise smooth cutoff function in $(x_0, t_0) + Q(\tau, \rho)$ such that

$$|\nabla \zeta| < \infty \qquad \text{and} \qquad \zeta(x, t) = 0 \ , \quad x \notin K_\rho(x_0) \ . \tag{2.4}$$

We start with the energy estimates. Without loss of generality, we will state them for cylinders with "vertex" at the origin $(0, 0)$, the changes being obvious for cylinders with "vertex" at a generic (x_0, t_0).

Proposition 2.1. *Let u be a local weak solution of (2.1). There exists a constant $C \equiv C(p) > 0$ such that for every cylinder $Q(\tau, \rho) \subset \Omega_T$,*

$$\begin{aligned}
\sup_{-\tau < t < 0} \int_{K_\rho \times \{t\}} (u-k)_\pm^2 \, \zeta^p \, dx + \int_{-\tau}^0 \int_{K_\rho} |\nabla (u-k)_\pm \zeta|^p \, dx \, dt& \\
\le \int_{K_\rho \times \{-\tau\}} (u-k)_\pm^2 \, \zeta^p \, dx + C \int_{-\tau}^0 \int_{K_\rho} (u-k)_\pm^p \, |\nabla \zeta|^p \, dx \, dt& \\
+ p \int_{-\tau}^0 \int_{K_\rho} (u-k)_\pm^2 \, \zeta^{p-1} \, \partial_t \zeta \, dx \, dt \ .&
\end{aligned} \tag{2.5}$$

Proof. Use $\varphi = \pm (u_h - k)_\pm \zeta^p$ as a testing function in (2.3) and perform standard energy estimates (cf. [55, pages 24–27]).

■

Given constants a, b, c, with $0 < c < a$, define the nonnegative function

$$\begin{aligned}
\psi^\pm_{\{a,b,c\}}(s) \ &\equiv \ \left(\ln \left\{ \frac{a}{(a+c) - (s-b)_\pm} \right\} \right)_+ \\
&= \begin{cases} \ln \left\{ \frac{a}{(a+c) \pm (b-s)} \right\} & \text{if } \ b \pm c \ \lesseqgtr \ s \ \lesseqgtr \ b \pm (a+c) \\[2ex] \qquad 0 & \text{if } \ s \ \lesseqqgtr \ b \pm c \end{cases}
\end{aligned}$$

whose first derivative is

$$\left(\psi^\pm_{\{a,b,c\}} \right)'(s) = \begin{cases} \frac{1}{(b-s) \pm (a+c)} & \text{if } \ b \pm c \ \lesseqgtr \ s \ \lesseqgtr \ b \pm (a+c) \\[2ex] \qquad 0 & \text{if } \ s \ \lesseqgtr \ b \pm c \end{cases} \qquad \gtreqqless \ 0 \ ,$$

and second derivative, off $s = b \pm c$, is $\left(\psi^\pm_{\{a,b,c\}} \right)'' = \left\{ \left(\psi^\pm_{\{a,b,c\}} \right)' \right\}^2 \ge 0$.

Now, given a bounded function u in a cylinder $(x_0,t_0)+Q(\tau,\rho)$ and a number k, define the constant

$$H^{\pm}_{u,k} \equiv \operatorname*{ess\,sup}_{(x_0,t_0)+Q(\tau,\rho)} \left|(u-k)_{\pm}\right| \ .$$

The following function was introduced in [48] and since then has been used as a recurrent tool in the proof of results concerning the local behaviour of solutions of degenerate PDE's:

$$\Psi^{\pm}\left(H^{\pm}_{u,k},(u-k)_{\pm},c\right) \equiv \psi^{\pm}_{\left\{H^{\pm}_{u,k},k,c\right\}}(u) \ , \quad 0<c<H^{\pm}_{u,k} \ .$$

From now on, when referring to this function we will write it as $\psi^{\pm}(u)$, omitting the subscripts whose meaning will be clear from the context.

Let $x \mapsto \zeta(x)$ be a time-independent cutoff function in $K_\rho(x_0)$ satisfying (2.4). The logarithmic estimates in cylinders $Q(\tau,\rho)$ with "vertex" at $(0,0)$, are

Proposition 2.2. *Let u be a local weak solution of (2.1), $k\in\mathbf{R}$ and $0<c<H^{\pm}_{u,k}$. There exists a constant $C>0$ such that for every cylinder $Q(\tau,\rho)\subset\Omega_T$,*

$$\sup_{-\tau<t<0}\int_{K_\rho\times\{t\}}\left[\psi^{\pm}(u)\right]^2\zeta^p\,\mathrm{d}x \le \int_{K_\rho\times\{-\tau\}}\left[\psi^{\pm}(u)\right]^2\zeta^p\,\mathrm{d}x$$
$$+\,C\int_{-\tau}^{0}\int_{K_\rho}\psi^{\pm}(u)\,\left|(\psi^{\pm})'(u)\right|^{2-p}|\nabla\zeta|^p\,\mathrm{d}x\,\mathrm{d}t \ . \qquad (2.6)$$

Proof. Take $\varphi = 2\,\psi^{\pm}(u_h)\,\left[(\psi^{\pm})'(u_h)\right]\,\zeta^p$ as a testing function in (2.3) and integrate in time over $(-\tau,t)$ for $t\in(-\tau,0)$. Since $\partial_t\zeta\equiv 0$,

$$\int_{-\tau}^{t}\int_{K_\rho}\partial_t u_h\,\left\{2\,\psi^{\pm}(u_h)\,\left[(\psi^{\pm})'(u_h)\right]\,\zeta^p\right\}\mathrm{d}x\,\mathrm{d}t = \int_{-\tau}^{t}\int_{K_\rho}\partial_t\left\{\left[\psi^{\pm}(u_h)\right]^2\right\}\zeta^p\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{K_\rho\times\{t\}}\left[\psi^{\pm}(u_h)\right]^2\zeta^p\,\mathrm{d}x - \int_{K_\rho\times\{-\tau\}}\left[\psi^{\pm}(u_h)\right]^2\zeta^p\,\mathrm{d}x \ .$$

From this, letting $h\to 0$,

$$\int_{-\tau}^{t}\int_{K_\rho}\partial_t u_h\,\left\{2\,\psi^{\pm}(u_h)\,\left[(\psi^{\pm})'(u_h)\right]\,\zeta^p\right\}\mathrm{d}x\,\mathrm{d}t \ \longrightarrow\ \int_{K_\rho\times\{t\}}\left[\psi^{\pm}(u)\right]^2\zeta^p\,\mathrm{d}x$$
$$-\ \int_{K_\rho\times\{-\tau\}}\left[\psi^{\pm}(u)\right]^2\zeta^p\,\mathrm{d}x \ .$$

As for the remaining term, we first let $h\to 0$, to obtain

$$\int_{-\tau}^{t}\int_{K_\rho}|\nabla u|^{p-2}\nabla u\cdot\nabla\left\{2\,\psi^{\pm}(u)\,\left[(\psi^{\pm})'(u)\right]\,\zeta^p\right\}\mathrm{d}x\,\mathrm{d}t$$

$$
\begin{aligned}
&= \int_{-\tau}^{t}\int_{K_\rho} |\nabla u|^p \Big\{2\,\big(1+\psi^{\pm}(u)\big)\big[(\psi^{\pm})'(u)\big]^2\,\zeta^p\Big\}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + p\int_{-\tau}^{t}\int_{K_\rho} |\nabla u|^{p-2}\nabla u\cdot\nabla\zeta\,\Big\{2\,\psi^{\pm}(u)\,\big[(\psi^{\pm})'(u)\big]\,\zeta^{p-1}\Big\}\,\mathrm{d}x\,\mathrm{d}t \\
&\geq \int_{-\tau}^{t}\int_{K_\rho} |\nabla u|^p \Big\{2\,\big(1+\psi^{\pm}(u)-\psi^{\pm}(u)\big)\,\big[(\psi^{\pm})'(u)\big]^2\,\zeta^p\Big\}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - 2(p-1)^{p-1}\int_{-\tau}^{t}\int_{K_\rho} \psi^{\pm}(u)\,\big|(\psi^{\pm})'(u)\big|^{2-p}|\nabla\zeta|^p\,\mathrm{d}x\,\mathrm{d}t \\
&\geq -\,C\int_{-\tau}^{t}\int_{K_\rho} \psi^{\pm}(u)\,\big|(\psi^{\pm})'(u)\big|^{2-p}|\nabla\zeta|^p\,\mathrm{d}x\,\mathrm{d}t\ .
\end{aligned}
$$

Since $t \in (-\tau, 0)$ is arbitrary, we can combine both estimates to obtain (2.6).

■

2.1.2. *Some technical tools.* We gather a few technical facts that, although marginal to the theory, are essential in establishing its main results.

Given a continuous function $v : \Omega \to \mathbf{R}$ and two real numbers $k < l$, we define

$$
\begin{aligned}
[v > l] \ &\equiv\ \{x\in\Omega\ :\ v(x) > l\}\ , \\
[v < k] \ &\equiv\ \{x\in\Omega\ :\ v(x) < k\}\ , \\
[k < v < l] \ &\equiv\ \{x\in\Omega\ :\ k < v(x) < l\}\ .
\end{aligned}
\tag{2.7}
$$

Lemma 2.3 (DeGiorgi, [47]). *Let $v \in W^{1,1}(B_\rho(x_0)) \cap C(B_\rho(x_0))$, with $\rho > 0$ and $x_0 \in \mathbf{R}^N$ and $k < l \in \mathbf{R}$. There exists a constant C, depending only on N and p (so independent of ρ, x_0, v, k and l), such that*

$$
(l-k)\ |[v>l]| \leq C\,\frac{\rho^{N+1}}{|[v<k]|}\int_{[k<v<l]} |\nabla v|\,\mathrm{d}x\ .
$$

Remark 2.4. The conclusion of the lemma remains valid, provided Ω is convex, for functions $v \in W^{1,1}(\Omega) \cap C(\Omega)$. We will use it in the case Ω is a cube. In fact, the continuity is not essential for the result to hold. For a function merely in $v \in W^{1,1}(\Omega)$, we define the sets (2.7) through any continuous representative in the equivalence class. It can be shown that the conclusion of the lemma is independent of that choice.

The following lemma concerns the geometric convergence of sequences.

Lemma 2.5. *Let $\{X_n\}$, $n = 0, 1, 2, \ldots$, be a sequence of positive real numbers satisfying the recurrence relation*

$$X_{n+1} \leq C\, b^n\, X_n^{1+\alpha}$$

where $C, b > 1$ and $\alpha > 0$ are given. If

$$X_0 \leq C^{-1/\alpha}\, b^{-1/\alpha^2}$$

then $X_n \to 0$ as $n \to \infty$.

Let $V_0^p(\Omega_T)$ denote the space

$$V_0^p(\Omega_T) = L^\infty\left(0,T;L^p(\Omega)\right) \cap L^p\left(0,T;W_0^{1,p}(\Omega)\right)$$

endowed with the norm

$$\|u\|^p_{V^p(\Omega_T)} = \operatorname*{ess\,sup}_{0\leq t\leq T} \|u(\cdot,t)\|^p_{p,\Omega} + \|\nabla u\|^p_{p,\Omega_T}\ ,$$

for which the following embedding theorem holds (cf. [55, page 9]):

Theorem 2.6. *Let $p > 1$. There exists a constant γ, depending only upon N and p, such that for every $v \in V_0^p(\Omega_T)$,*

$$\|v\|^p_{p,\Omega_T} \leq \gamma\, |\, |v| > 0\, |^{\frac{p}{N+p}} \|v\|^p_{V^p(\Omega_T)}\ .$$

With C or C_j we denote constants that depend only on N and p and that might be different in different contexts.

2.2. The classical approach of De Giorgi. Results concerning the Hölder continuity of weak solutions u consist essentially in showing that for every point $(x_0,t_0) \in \Omega_T$ we can find a sequence of nested and shrinking cylinders $(x_0,t_0) + Q(\tau_n,\rho_n)$ such that the essential oscillation of u in these cylinders approaches zero as $n \to \infty$ in a way that can be quantified.

The approach to regularity introduced by De Giorgi is based on the following embedding theorem (see [47] for the elliptic case and [121] for the parabolic case):

Proposition 2.7. *Assume that $u \in L^2_{\mathrm{loc}}(0,T;W^{1,2}_{\mathrm{loc}}(\Omega)) \cap W^{1,2}_{\mathrm{loc}}(0,T;L^2_{\mathrm{loc}}(\Omega))$ is locally bounded and satisfies the Caccioppoli inequalities (2.5) with $p = 2$. Then u is locally Hölder continuous, with the modulus of continuity depending only upon the data.*

A solution of a non-degenerate parabolic equation with the full quasilinear structure satisfies these inequalities. One uses the structure of the equation to prove the Caccioppoli estimates for a solution u and this is the only role of the equation. Once such inequalities are derived, the Hölder continuity of u is solely a consequence of (2.5) for $p = 2$.

Alternative approaches are in Kruzkov ([110], [111], [112]) and, through the use of the Harnack inequality, in Moser ([132], [134]), Trudinger ([164]), Aronson-Serrin ([16]) and, for equations in non-divergence form, in Krylov-Safonov ([117]).

Set $Q_r = Q(r^2, r)$, fix a point $(x_0, t_0) \in \Omega_T$, and let ρ_0 be the largest radius so that $(x_0, t_0) + Q_{\rho_0}$ is contained in Ω_T. For a constant $\delta \in (0, 1)$, consider the sequence of decreasing radii,

$$\rho_n = \delta^n \rho_0 \ , \qquad n = 0, 1, 2, \dots \tag{2.8}$$

and the family of nested shrinking cylinders, with the same "vertex"

$$(x_0, t_0) + Q_{\rho_n} \ , \qquad n = 0, 1, 2, \dots \tag{2.9}$$

Set

$$\mu_n^- := \operatorname*{ess\,inf}_{(x_0,t_0)+Q_{\rho n}} u \ \ ; \qquad \mu_n^+ := \operatorname*{ess\,sup}_{(x_0,t_0)+Q_{\rho n}} u \ \ ; \qquad \omega_n := \operatorname*{ess\,osc}_{(x_0,t_0)+Q_{\rho n}} u = \mu_n^+ - \mu_n^- \ .$$

Proposition 2.8. *Let u satisfy the Caccioppoli inequalities (2.5) for $p = 2$. Then there exist constants $C > 1$ and $\delta, \eta \in (0, \frac{1}{2})$, that can be determined a priori only in terms of the data, such that for every $(x_0, t_0) \in \Omega_T$ and every $n \in \mathbf{N}$, at least one of the following two inequalities holds*

$$\operatorname*{\mathit{ess\ sup}}_{(x_0,t_0)+Q_{\rho_{n+1}}} u \ \le \mu_n^+ - \eta\omega_n, \tag{2.10}$$

$$\operatorname*{\mathit{ess\ inf}}_{(x_0,t_0)+Q_{\rho_{n+1}}} u \ \ge \mu_n^- + \eta\omega_n. \tag{2.11}$$

A nontrivial proof can be found in [121]. This proposition can be interpreted as a weak maximum principle. For example (2.10) asserts that the supremum of u over the cylinder $Q_{\rho_{n+1}}$ is strictly less than the supremum of u over the larger coaxial cylinder Q_{ρ_n}. In other words, the supremum of u over Q_{ρ_n} can only be achieved in the parabolic shell $Q_{\rho_n} \setminus Q_{\rho_{n+1}}$ that can be considered as a sort of parabolic boundary of Q_{ρ_n}.

A consequence of such a weak maximum principle is:

Proposition 2.9. *Let u be as above. Then there exist constants $C > 1$ and $\delta, \eta \in (0, \frac{1}{2})$, that can be determined a priori only in terms of the data, such that for every $(x_0, t_0) \in \Omega_T$ and every $n \in \mathbf{N}$,*

$$\omega_{n+1} \leq (1 - \eta)\omega_n \ . \tag{2.12}$$

This in turn implies that u is locally Hölder continuous in Ω_T.

Proof. Fix $(x_0, t_0) \in \Omega_T$. Assume that (2.10) holds. By subtracting μ_{n+1}^- from the left hand side and μ_n^- from the right hand side,

$$\omega_{n+1} = \mu_{n+1}^+ - \mu_{n+1}^- \leq \mu_n^+ - \mu_n^- - \eta\omega_n = (1 - \eta)\omega_n \ .$$

If (2.11) holds one can argue in a similar way. By iteration,

$$\omega_n \leq (1 - \eta)^n \omega_0 \ , \qquad \forall n \in \mathbf{N} \ . \tag{2.13}$$

The numbers η and δ are related by $(1 - \eta) = \delta^\alpha$ and $\alpha = \frac{\ln(1-\eta)}{\ln \delta} \in (0, 1)$. Therefore,

$$\omega_n \leq \omega_0 \left(\frac{\rho_n}{\rho_0}\right)^\alpha \ , \qquad \forall n \in \mathbf{N} \ . \tag{2.14}$$

Since $(x_0, t_0) \in \Omega_T$ is arbitrary, we conclude that u is locally Hölder continuous in Ω_T with exponent α. ∎

Remark 2.10. The cylinder $(x_0, t_0) + Q_{\rho_0}$ must be contained in Ω_T. Thus from (2.14) it follows that the Hölder continuity can be claimed only within compact subsets of Ω_T and that the Hölder constant $\omega_0 \rho_0^{-\alpha}$ deteriorates as (x_0, t_0) approaches the parabolic boundary of Ω_T.

2.3. The degenerate case $p > 2$. We go back to equation (2.1) and focus on the degenerate case $p > 2$. The energy and logarithmic estimates of §2.1 are not homogeneous in the space and time parts due to the presence of the power $p \neq 2$. To go about this difficulty we will consider the equation in a geometry dictated by its own structure, which is designed, roughly speaking, to restore the homogeneity of the various parts of the Caccioppoli inequalities (2.5). This means that, instead of the usual cylinders, we have to work in cylinders whose dimensions take the degeneracy of the equation into account, in a process that we call *intrinsic rescaling*. Let's make this idea precise.

2.3.1. *The geometric setting and the alternative.* Consider $R > 0$ such that $Q(R^2, R) \subset \Omega_T$, define

$$\mu_+ := \operatorname*{ess\,sup}_{Q(R^2,R)} u \quad ; \qquad \mu_- := \operatorname*{ess\,inf}_{Q(R^2,R)} u \quad ; \qquad \omega := \operatorname*{ess\,osc}_{Q(R^2,R)} u = \mu_+ - \mu_-$$

and construct the cylinder

$$Q(a_0 R^p, R) \equiv K_R(0) \times (-a_0 R^p, 0) \qquad \text{with} \qquad a_0 = \left(\frac{\omega}{2^\lambda}\right)^{2-p} , \tag{2.15}$$

where $\lambda > 1$ is to be fixed later depending only on the data (see (2.55)). Note that for $p = 2$, i.e., in the non-degenerate case, these are the standard parabolic cylinders reflecting the natural homogeneity between the space and time variables.

We will assume, without loss of generality, that

$$R < \frac{\omega}{2^\lambda} \tag{2.16}$$

because if this doesn't hold there is nothing to prove since the oscillation is comparable to the radius.

Now, (2.16) implies the inclusion

$$Q(a_0 R^p, R) \subset Q(R^2, R)$$

and the relation

$$\operatorname*{ess\,osc}_{Q(a_0 R^p,R)} u \leq \omega \tag{2.17}$$

which will be the starting point of an iteration process that leads to our main results. Note that we had to consider the cylinder $Q(R^2, R)$ and assume (2.16), so that (2.17) would hold for the rescaled cylinder $Q(a_0 R^p, R)$. This is in general not true for a given cylinder since its dimensions would have to be intrinsically defined in terms of the essential oscillation of the function within it - the stretching procedure is commonly referred to as an *accommodation of the degeneracy.*

We now consider subcylinders of $Q(a_0 R^p, R)$ of the form

$$(0, t^*) + Q(dR^p, R) \ , \qquad \text{with} \quad d = \left(\frac{\omega}{2}\right)^{2-p} \tag{2.18}$$

that are contained in $Q(a_0 R^p, R)$ if

$$\left(2^{p-2} - 2^{\lambda(p-2)}\right) \frac{R^p}{\omega^{p-2}} < t^* < 0 \ . \tag{2.19}$$

The proof now follows from the analysis of two complementary cases. We briefly describe them in the following way: in the first case we assume that there is a cylinder of the type $(0,t^*)+Q(dR^p,R)$ where u is essentially away from its infimum. We show that going down to a smaller cylinder the oscillation decreases by a small factor that we can exhibit. If that cylinder can not be found then u is essentially away from its supremum in all cylinders of that type and we can compound this information to reach the same conclusion as in the previous case. We state this in a precise way.

For a constant $\nu_0 \in (0,1)$, that will be determined depending only on the data, we will assume that either

The first alternative:

There is a cylinder of the type $(0,t^*)+Q(dR^p,R)$ for which

$$\frac{\left|\left\{(x,t)\in(0,t^*)+Q(dR^p,R)\ :\ u(x,t)<\mu_-+\frac{\omega}{2}\right\}\right|}{|Q(dR^p,R)|}\le \nu_0 \tag{2.20}$$

or

The second alternative:

For every cylinder of the type $(0,t^*)+Q(dR^p,R)$

$$\frac{\left|\left\{(x,t)\in(0,t^*)+Q(dR^p,R)\ :\ u(x,t)>\mu_+-\frac{\omega}{2}\right\}\right|}{|Q(dR^p,R)|}< 1-\nu_0 \tag{2.21}$$

2.3.2. *Analysis of the first alternative.*

Lemma 2.11. *Assume (2.20) holds for some t^* as in (2.19) and that (2.16) is in force. There exists a constant $\nu_0\in(0,1)$, depending only on the data, such that*

$$u(x,t)>\mu^-+\frac{\omega}{4}\qquad a.e.\qquad (x,t)\in(0,t^*)+Q\left(d\left(\frac{R}{2}\right)^p,\frac{R}{2}\right)\ .$$

Proof. Take the cylinder for which (2.20) holds and assume, by translation, that $t^*=0$. Let

$$R_n=\frac{R}{2}+\frac{R}{2^{n+1}}\ ,\quad n=0,1,\ldots,$$

and construct the family of nested and shrinking cylinders $Q(dR_n^p, R_n)$. Consider piecewise smooth cutoff functions $0 < \zeta_n \le 1$, defined in these cylinders, and satisfying the following set of assumptions

$$\zeta_n = 1 \quad \text{in} \quad Q\left(dR_{n+1}^p, R_{n+1}\right) \qquad \zeta_n = 0 \quad \text{on} \quad \partial_p Q\left(dR_n^p, R_n\right)$$

$$|\nabla \zeta_n| \le \tfrac{2^{n+1}}{R} \qquad\qquad 0 \le \partial_t \zeta_n \le \tfrac{2^{p(n+1)}}{dR^p} \quad .$$

Write the energy inequality (2.5) for the functions $(u-k_n)_-$, with

$$k_n = \mu_- + \frac{\omega}{4} + \frac{\omega}{2^{n+2}}\ , \quad n = 0, 1, \ldots,$$

in the cylinders $Q\left(dR_n^p, R_n\right)$ and with $\zeta = \zeta_n$. They read

$$\begin{aligned}
&\sup_{-dR_n^p<t<0} \int_{K_{R_n}\times\{t\}} (u-k_n)_-^2 \zeta_n^p \,\mathrm{d}x + \int_{-dR_n^p}^0 \int_{K_{R_n}} |\nabla (u-k_n)_- \zeta_n|^p \,\mathrm{d}x\,\mathrm{d}t \\
&\le C \int_{-dR_n^p}^0 \int_{K_{R_n}} (u-k_n)_-^p |\nabla \zeta_n|^p \,\mathrm{d}x\,\mathrm{d}t + p \int_{-dR_n^p}^0 \int_{K_{R_n}} (u-k_n)_-^2 \zeta_n^{p-1} \partial_t \zeta_n \,\mathrm{d}x\,\mathrm{d}t \\
&\le C\, \frac{2^{p(n+1)}}{R^p} \left\{ \int_{-dR_n^p}^0 \int_{K_{R_n}} (u-k_n)_-^p \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{d} \int_{-dR_n^p}^0 \int_{K_{R_n}} (u-k_n)_-^2 \,\mathrm{d}x\,\mathrm{d}t \right\} .
\end{aligned} \tag{2.22}$$

Next, observing that

$$(u-k_n)_- = (\mu_- - u) + \frac{\omega}{4} + \frac{\omega}{2^{n+2}} \le \frac{\omega}{2}$$

and

$$(u-k_n)_-^2 = (u-k_n)_-^{2-p}\,(u-k_n)_-^p \ge \left(\frac{\omega}{2}\right)^{2-p}\ (u-k_n)_-^p\ ,$$

we obtain from (2.22)

$$\begin{aligned}
&\left(\frac{\omega}{2}\right)^{2-p} \sup_{-dR_n^p<t<0} \int_{K_{R_n}\times\{t\}} (u-k_n)_-^p \zeta_n^p \,\mathrm{d}x + \int_{-dR_n^p}^0 \int_{K_{R_n}} |\nabla (u-k_n)_- \zeta_n|^p \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le C\, \frac{2^{p(n+1)}}{R^p}\ \left\{ \left(\frac{\omega}{2}\right)^p + \frac{1}{d}\left(\frac{\omega}{2}\right)^2 \right\} \int_{-dR_n^p}^0 \int_{K_{R_n}} \chi_{\{(u-k_n)_->0\}} \,\mathrm{d}x\,\mathrm{d}t\ .
\end{aligned} \tag{2.23}$$

Recall that $d = \left(\frac{\omega}{2}\right)^{2-p}$ and divide (2.23) by d to get

$$\begin{aligned}
&\sup_{-dR_n^p<t<0} \int_{K_{R_n}\times\{t\}} (u-k_n)_-^p \zeta_n^p \,\mathrm{d}x + \frac{1}{d} \int_{-dR_n^p}^0 \int_{K_{R_n}} |\nabla (u-k_n)_- \zeta_n|^p \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad \le C\, \frac{2^{p(n+1)}}{R^p}\ \left(\frac{\omega}{2}\right)^p\ \frac{1}{d} \int_{-dR_n^p}^0 \int_{K_{R_n}} \chi_{\{(u-k_n)_->0\}} \,\mathrm{d}x\,\mathrm{d}t\ .
\end{aligned} \tag{2.24}$$

Now we perform a change of the time variable in (2.24), putting $\overline{t} = t/d$ and defining

$$\overline{u}(\cdot, \overline{t}) = u(\cdot, t) \qquad \text{and} \qquad \overline{\zeta_n}(\cdot, \overline{t}) = \zeta_n(\cdot, t) \ ,$$

and obtain the simplified inequality

$$\left\|(\overline{u} - k_n)_- \ \overline{\zeta_n}\right\|^p_{V_0^p(Q(R_n^p, R_n))} \leq C \ \frac{2^{pn}}{R^p} \ \left(\frac{\omega}{2}\right)^p \int_{-R_n^p}^0 \int_{K_{R_n}} \chi_{\{(\overline{u}-k_n)_- > 0\}} \, \mathrm{d}x \, \mathrm{d}t \ . \tag{2.25}$$

Define, for each n,

$$A_n = \int_{-R_n^p}^0 \int_{K_{R_n}} \chi_{\{(\overline{u}-k_n)_- > 0\}} \ \mathrm{d}x \ \mathrm{d}\overline{t}$$

and observe that

$$\begin{aligned} \frac{1}{2^{p(n+2)}} \ \left(\frac{\omega}{2}\right)^p \ A_{n+1} &\leq |k_n - k_{n+1}|^p \ A_{n+1} \\ &\leq \|(\overline{u} - k_n)_-\|^p_{p, Q(R_{n+1}^p, R_{n+1})} \\ \leq \left\|(\overline{u} - k_n)_- \overline{\zeta_n}\right\|^p_{p, Q(R_n^p, R_n)} &\leq C \ \left\|(\overline{u} - k_n)_- \ \overline{\zeta_n}\right\|^p_{V_0^p(Q(R_n^p, R_n))} \ A_n^{\frac{p}{N+p}} \\ &\leq C \ \frac{2^{pn}}{R^p} \ \left(\frac{\omega}{2}\right)^p \ A_n^{1 + \frac{p}{N+p}} \ . \end{aligned} \tag{2.26}$$

The first three of these inequalities follow from the definition of A_n and $k_{n+1} < k_n$; the fourth inequality is a consequence of Theorem 2.6 and the last one follows from (2.25). Next, define the numbers

$$X_n = \frac{A_n}{|Q(R_n^p, R_n)|} \ ,$$

divide (2.26) by $|Q(R_{n+1}^p, R_{n+1})|$ and obtain the recursive relation

$$X_{n+1} \leq C \, 4^{pn} \ X_n^{1 + \frac{p}{N+p}} \ ,$$

for a constant C depending only upon N and p. By Lemma 2.5 on fast geometric convergence, if

$$X_0 \leq C^{-\frac{N+p}{p}} 4^{-\frac{(N+p)^2}{p}} \equiv \nu_0 \tag{2.27}$$

then

$$X_n \longrightarrow 0 \ . \tag{2.28}$$

Therefore,

$$\left|\left\{(x,t) \in Q\left(d(\tfrac{R}{2})^p, \tfrac{R}{2}\right) \ : \ u(x,t) \leq \mu_- + \frac{\omega}{4}\right\}\right| = 0 \ .$$

■

Our next aim is to show that the conclusion of lemma 2.11 holds in a full cylinder $Q(\tau,\rho)$. The idea is to use the fact that at the time level

$$-\hat{t} := t^* - d\left(\tfrac{R}{2}\right)^p \tag{2.29}$$

the function $u(x,-\hat{t}\,)$ is strictly above the level $\mu^- + \frac{\omega}{4}$ in the cube $K_{\frac{R}{2}}$, and look at this time level as an initial condition to make the conclusion hold up to $t=0$. As an intermediate step we need the following lemma.

Lemma 2.12. *Assume (2.20) holds for some t^* as in (2.19) and that (2.16) is in force. Given $\nu_* \in (0,1)$, there exists $s_* \in \mathbf{N}$, depending only on the data, such that*

$$\left|\left\{x \in K_{\frac{R}{4}} \ : \ u(x,t) < \mu_- + \frac{\omega}{2^{s_*}}\right\}\right| \le \nu_* \left|K_{\frac{R}{4}}\right| \ , \quad \forall t \in (-\hat{t},0) \ .$$

Proof. We use the logarithmic estimate (2.6) applied to the function $(u-k)_-$ in the cylinder $Q(\hat{t},\frac{R}{2})$, with the choices

$$k = \mu_- + \frac{\omega}{4} \qquad \text{and} \qquad c = \frac{\omega}{2^{n+2}}$$

where $n \in \mathbf{N}$ will be chosen later. We have

$$k - u \le H^-_{u,k} = \operatorname*{ess\,sup}_{Q(\hat{t},\frac{R}{2})} \left|\left(u - \mu_- - \frac{\omega}{4}\right)_-\right| \le \frac{\omega}{4} \ . \tag{2.30}$$

If $H^-_{u,k} \le \dfrac{\omega}{8}$ the result is trivial for the choice $s^* = 3$. Assuming $H^-_{u,k} > \dfrac{\omega}{8}$ the logarithmic function is defined in the whole $Q(\hat{t},\frac{R}{2})$ and it is given by

$$\Psi^- = \psi^-_{\left\{H^-_{u,k},k,\frac{\omega}{2^{n+2}}\right\}}(u) = \begin{cases} \ln\left\{\dfrac{H^-_{u,k}}{H^-_{u,k} + u - k + \frac{\omega}{2^{n+2}}}\right\} & \text{if } \ u < k - \frac{\omega}{2^{n+2}} \\[2ex] 0 & \text{if } \ u \ge k - \frac{\omega}{2^{n+2}} \end{cases} .$$

From (2.30), we estimate

$$\Psi^- \le n \ln 2 \qquad \text{since} \qquad \frac{H^-_{u,k}}{H^-_{u,k} + u - k + \frac{\omega}{2^{n+2}}} \le \frac{\frac{\omega}{4}}{\frac{\omega}{2^{n+2}}} = 2^n \tag{2.31}$$

and

$$\left|\left(\psi^-\right)'(u)\right|^{2-p} = \left(H^-_{u,k} + u - k + c\right)^{p-2} \le \left(\frac{\omega}{2}\right)^{p-2} \ . \tag{2.32}$$

Now observe that as a consequence of Lemma 2.11, we have $u(x,-\hat{t}) > k$ in the cube $K_{\frac{R}{2}}$, which implies that

$$\Psi^-(x,-\hat{t}) = 0\ , \quad x \in K_{\frac{R}{2}}\ .$$

Choosing a piecewise smooth cutoff function $0 < \zeta(x) \leq 1$, defined in $K_{\frac{R}{2}}$ and such that

$$\zeta = 1 \ \text{ in } \ K_{\frac{R}{4}} \qquad \text{and} \qquad |\nabla\zeta| \leq \tfrac{8}{R}\ ,$$

inequality (2.6) reads

$$\sup_{-\hat{t}<t<0} \int_{K_{\frac{R}{2}}\times\{t\}} \left[\psi^-(u)\right]^2 \zeta^p\,\mathrm{d}x \leq C\int_{-\hat{t}}^0\int_{K_{\frac{R}{2}}} \psi^-(u)\ \left|(\psi^-)'(u)\right|^{2-p} |\nabla\zeta|^p\ \mathrm{d}x\,\mathrm{d}t\ . \tag{2.33}$$

The right hand side is estimated above, using (2.31) and (2.32), by

$$C\,n(\ln 2)\ \left(\frac{\omega}{2}\right)^{p-2}\left(\frac{8}{R}\right)^p\ \hat{t}\ \left|K_{\frac{R}{2}}\right| \leq C\,n\,2^{\lambda(p-2)}\ \left|K_{\frac{R}{4}}\right|\ ,$$

since, by (2.29)

$$\hat{t} \leq a_0 R^p = \left(\frac{\omega}{2^\lambda}\right)^{2-p} R^p\ .$$

We estimate below the left hand side of (2.33) by integrating over the smaller set

$$S(t) = \left\{x \in K_{\frac{R}{4}}\ :\ u(x,t) < \mu_- + \frac{\omega}{2^{n+2}}\right\} \subset K_{\frac{R}{2}}$$

and observing that in S, $\zeta = 1$ and

$$\frac{H^-_{u,k}}{H^-_{u,k} + u - k + \frac{\omega}{2^{n+2}}} \geq \frac{\frac{\omega}{8}}{\left(H^-_{u,k} - \frac{\omega}{4}\right) + \frac{\omega}{2^{n+2}}} \geq \frac{\frac{\omega}{2^3}}{\frac{\omega}{2^{n+2}}} = 2^{n-1}\ ,$$

since $\left(H^-_{u,k} - \frac{\omega}{4}\right) \leq 0$. Therefore,

$$\left[\psi^-(u)\right]^2 \geq \left[\ln\left(2^{n-1}\right)\right]^2 = (n-1)^2(\ln 2)^2 \quad \text{on} \quad S(t)\ .$$

Combining these estimates in (2.33) we get

$$\left|\left\{x \in K_{\frac{R}{4}}\ :\ u(x,t) < \mu_- + \frac{\omega}{2^{n+2}}\right\}\right| \leq C\,\frac{n}{(n-1)^2}\,2^{\lambda(p-2)}\ \left|K_{\frac{R}{4}}\right|\ ,$$

for all $t \in (-\hat{t}, 0)$ and to prove the lemma we choose

$$s_* = n+2 \qquad \text{with} \qquad n > 1 + \frac{2C}{\nu_*}\,2^{\lambda(p-2)}\ . \tag{2.34}$$

■

We now state the main result of this section.

Proposition 2.13. *Assume (2.20) holds for some t^* as in (2.19) and that (2.16) is in force. There exist constants $\nu_0 \in (0,1)$, $s_1 \in \mathbf{N}$, depending only on the data, such that,*

$$u(x,t) > \mu^- + \frac{\omega}{2^{s_1+1}}\ , \quad a.e. \quad (x,t) \in Q\left(\hat{t}, \frac{R}{8}\right)\ .$$

Proof. Consider the cylinder for which (2.20) holds, let

$$R_n = \frac{R}{8} + \frac{R}{2^{n+3}}\ , \quad n = 0, 1, \dots$$

and construct the family of nested and shrinking cylinders $Q(\hat{t}, R_n)$, where $\hat{t}$ is given by (2.29). Take piecewise smooth cutoff functions $0 < \zeta_n(x) \le 1$, independent of t, defined in K_{R_n} and satisfying

$$\zeta_n = 1 \ \text{ in } \ K_{R_{n+1}} \qquad |\nabla \zeta_n| \le \tfrac{2^{n+4}}{R}\ .$$

Write the local energy inequalities (2.5) for the functions $(u-k_n)_-$ in the cylinders $Q\left(\hat{t}, R_n\right)$, with

$$k_n = \mu_- + \frac{\omega}{2^{s_1+1}} + \frac{\omega}{2^{s_1+1+n}}\ , \quad n = 0, 1, \dots,$$

and $\zeta = \zeta_n$. Observing that, due to Lemma 2.11, we have $u(x, -\hat{t}) > \mu_- + \frac{\omega}{4} \ge k_n$ in the cube $K_{\frac{R}{2}} \supset K_{R_n}$, which implies that

$$(u-k_n)_-(x, -\hat{t}) = 0\ , \quad x \in K_{R_n}\ , \quad n = 0, 1, \dots\ ,$$

they read

$$\sup_{-\hat{t}<t<0} \int_{K_{R_n}\times\{t\}} (u-k_n)_-^2 \zeta_n^p \,\mathrm{d}x + \int_{-\hat{t}}^0 \int_{K_{R_n}} |\nabla (u-k_n)_- \zeta_n|^p \,\mathrm{d}x\,\mathrm{d}t$$

$$\le C \int_{-\hat{t}}^0 \int_{K_{R_n}} (u-k_n)_-^p |\nabla \zeta_n|^p \,\mathrm{d}x\,\mathrm{d}t \le C \frac{2^{p(n+4)}}{R^p} \int_{-\hat{t}}^0 \int_{K_{R_n}} (u-k_n)_-^p \,\mathrm{d}x\,\mathrm{d}t\ . \quad (2.35)$$

From (2.29) we estimate $\hat{t} \le a_0 R^p = \left(\frac{\omega}{2^\lambda}\right)^{2-p} R^p$ where a_0 is defined in (2.15). From this,

$$\begin{aligned} (u-k_n)_-^2 \ge \left(\frac{\omega}{2^{s_1}}\right)^{2-p} (u-k_n)_-^p \ &\ge \ \left(\frac{2^{s_1}}{2^\lambda}\right)^{p-2} \frac{\hat{t}}{R^p} (u-k_n)_-^p \\ &\ge \ \frac{\hat{t}}{R^p} (u-k_n)_-^p\ , \end{aligned}$$

provided $s_1 > \lambda$. Dividing now (2.35), by $\dfrac{\widehat{t}}{(\frac{R}{2})^p}$ gives

$$\sup_{-\widehat{t}<t<0} \int_{K_{R_n}\times\{t\}} (u-k_n)_-^p \zeta_n^p \,\mathrm{d}x + \frac{R^p}{\widehat{t}} \int_{-\widehat{t}}^0 \int_{K_{R_n}} |\nabla(u-k_n)_-\zeta_n|^p \,\mathrm{d}x\,\mathrm{d}t$$
$$\leq C\,\frac{2^{pn}}{\widehat{t}} \int_{-\widehat{t}}^0 \int_{K_{R_n}} (u-k_n)_-^p \,\mathrm{d}x\,\mathrm{d}t\ .$$

The change of the time variable $\overline{t} = t\frac{(\frac{R}{2})^p}{\widehat{t}}$, along with the new function

$$\overline{u}(\cdot,\overline{t}) = u(\cdot,t)\ ,$$

leads to the simplified inequality

$$\|(\overline{u}-k_n)_-\ \zeta_n\|^p_{V_0^p\left(Q\left((\frac{R}{2})^p,R_n\right)\right)} \leq C\,\frac{2^{pn}}{(\frac{R}{2})^p}\left(\frac{\omega}{2^{s_1}}\right)^p \int_{-(\frac{R}{2})^p}^0 \int_{K_{R_n}} \chi_{\{(\overline{u}-k_n)_->0\}}\,\mathrm{d}x\,\mathrm{d}\overline{t}\ .$$

Define, for each n,

$$A_n = \int_{-(\frac{R}{2})^p}^0 \int_{K_{R_n}} \chi_{\{(\overline{u}-k_n)_->0\}}\ \mathrm{d}x\ \mathrm{d}\overline{t}\ .$$

By a reasoning similar to the one leading to (2.26):

$$\frac{1}{2^{p(n+2)}}\ \left(\frac{\omega}{2^{s_1}}\right)^p\ A_{n+1} \leq |k_n - k_{n+1}|^p\ \ A_{n+1}$$
$$\leq \|(\overline{u}-k_n)_-\|^p_{p,Q\left((\frac{R}{2})^p,R_{n+1}\right)}$$
$$\leq \|(\overline{u}-k_n)_-\zeta_n\|^p_{p,Q\left((\frac{R}{2})^p,R_n\right)} \leq C\ \ \|(\overline{u}-k_n)_-\ \zeta_n\|^p_{V_0^p\left(Q\left((\frac{R}{2})^p,R_n\right)\right)}\ \ A_n^{\frac{p}{N+p}}$$
$$\leq C\,\frac{2^{pn}}{(\frac{R}{2})^p}\left(\frac{\omega}{2^{s_1}}\right)^p\ A_n^{1+\frac{p}{N+p}}\ .$$

Next, define the numbers

$$X_n = \frac{A_n}{\left|Q\left((\frac{R}{2})^p, R_n\right)\right|}\ ,$$

divide the previous inequality by $\left|Q\left((\frac{R}{2})^p, R_{n+1}\right)\right|$ to obtain the recursive relations

$$X_{n+1} \leq C\,4^{pn}\ X_n^{1+\frac{p}{N+p}}\ .$$

By Lemma 2.5 on fast geometric convergence, if

$$X_0 \leq C^{-\frac{N+p}{p}} 4^{-\frac{(N+p)^2}{p}}\ \ \equiv \nu_* \in (0,1) \tag{2.36}$$

then

$$X_n \longrightarrow 0 \ . \tag{2.37}$$

To verify (2.36), apply Lemma 2.12 with such a ν_* and conclude that there exists $s_* \equiv s_1$, depending only on the data, such that

$$\left|\left\{x \in K_{\frac{R}{4}} \ : \ u(x,t) < \mu_- + \frac{\omega}{2^{s_1}}\right\}\right| \leq \nu_* \ \left|K_{\frac{R}{4}}\right| \ , \quad \forall t \in (-\hat{t}, 0) \ .$$

Since (2.37) implies that $A_n \to 0$, we conclude that

$$\begin{aligned} &\left|\left\{(x,\bar{t}) \in Q\left(\left(\tfrac{R}{2}\right)^p, \frac{R}{8}\right) \ : \ \overline{u}(x,\bar{t}) \leq \mu_- + \frac{\omega}{2^{s_1+1}}\right\}\right| \\ &= \left|\left\{(x,t) \in Q\left(\hat{t}, \frac{R}{8}\right) \ : \ u(x,t) \leq \mu_- + \frac{\omega}{2^{s_1+1}}\right\}\right| \ = 0 \ . \end{aligned}$$

■

Corollary 2.14. *Assume (2.20) holds for some t^* as in (2.19) and that (2.16) is in force. There exist constants $\nu_0, \sigma_0 \in (0,1)$, depending only on the data, such that*

$$\underset{Q\left(d\left(\frac{R}{8}\right)^p, \frac{R}{8}\right)}{\mathit{ess\ osc}} \ u \ \leq \ \sigma_0 \ \omega \ . \tag{2.38}$$

Proof. By Proposition 2.13 there exists $s_1 \in \mathbf{N}$ such that

$$\underset{Q(\hat{t}, \frac{R}{8})}{\operatorname{ess\,inf}} \ u \geq \mu^- + \frac{\omega}{2^{s_1+1}} \ .$$

From this,

$$\underset{Q(\hat{t}, \frac{R}{8})}{\operatorname{ess\,osc}} \ u \ \leq \left(1 - \frac{1}{2^{s_1+1}}\right) \omega \ .$$

Since $d\left(\frac{R}{8}\right)^p \leq \hat{t} = -t^* + d\left(\frac{R}{2}\right)^p$, $t^* < 0$, we have

$$Q\left(d\left(\frac{R}{8}\right)^p, \frac{R}{8}\right) \subset Q\left(\hat{t}, \frac{R}{8}\right) \ ,$$

and the corollary follows with $\sigma_0 = \left(1 - \frac{1}{2^{s_1+1}}\right)$.

■

2.3.3. *Analysis of the second alternative.* Assume now that the second alternative (2.21) holds true. We will show that a conclusion similar to (2.38) can be reached. Recall that the constant ν_0 has already been quantitatively determined by (2.27), and it is fixed. We continue to assume that (2.16) is in force.

Lemma 2.15. *Assume (2.21) holds and let (2.16) be in force. Fix a cylinder* $(0,t^*)+Q(dR^p,R)\subset Q(a_0R^p,R)$ *for which (2.21) holds. There exists a time level*

$$t^\circ\in\left[t^*-dR^p,t^*-\frac{\nu_0}{2}dR^p\right]$$

such that

$$\left|\left\{x\in K_R\ :\ u(x,t^\circ)>\mu_+-\frac{\omega}{2}\right\}\right|\le\left(\frac{1-\nu_0}{1-\nu_0/2}\right)\,|K_R|\ .$$

Proof. Suppose not. Then, for all $t\in\left[t^*-dR^p,t^*-\frac{\nu_0}{2}dR^p\right]$,

$$\left|\left\{(x,t)\in(0,t^*)+Q(dR^p,R)\ :\ u(x,t)>\mu_+-\frac{\omega}{2}\right\}\right|$$

$$\ge\int_{t^*-dR^p}^{t^*-\frac{\nu_0}{2}dR^p}\left|\left\{x\in K_R\ :\ u(x,\tau)>\mu_+-\frac{\omega}{2}\right\}\right|\,\mathrm{d}\tau>(1-\nu_0)\,\left|Q(dR^p,R)\right|\ ,$$

which contradicts (2.21).

■

The next lemma asserts that the set where $u(\cdot,t)$ is close to its supremum is small, not only at the specific time level t°, but for all time levels near the top of the cylinder $(0,t^*)+Q(dR^p,R)$.

Lemma 2.16. *Assume (2.21) holds and let (2.16) be in force. There exists* $1<s_2\in\mathbf{N}$*, depending only on the data, such that*

$$\left|\left\{x\in K_R\ :\ u(x,t)>\mu_+-\frac{\omega}{2^{s_2}}\right\}\right|\le\left(1-\left(\frac{\nu_0}{2}\right)^2\right)\,|K_R|\ ,$$

for all $t\in\left[t^*-\frac{\nu_0}{2}dR^p,t^*\right]$.

Proof. The proof consists in using the logarithmic inequalities (2.6) applied to the function $(u-k)_+$ in the cylinder $K_R\times(t^\circ,t^*)$, with the choices

$$k=\mu_+-\frac{\omega}{2}\qquad\text{and}\qquad c=\frac{\omega}{2^{n+1}}$$

where $n \in \mathbf{N}$ will be chosen later. We have

$$u - k \leq H_{u,k}^{+} = \operatorname*{ess\ sup}_{K_R \times (t^\circ, t^*)} \left| \left(u - \mu_+ + \frac{\omega}{2} \right)_+ \right| \leq \frac{\omega}{2} \ . \tag{2.39}$$

If $H_{u,k}^{+} \leq \frac{\omega}{8}$ the result is trivial for the choice of $s_2 = 3$. Assuming $H_{u,k}^{+} > \frac{\omega}{8}$ the logarithmic function Ψ^+ is defined in the whole $K_R \times (t^\circ, t^*)$, and it is given by

$$\Psi^+ = \psi^+_{\left\{H_{u,k}^{+}, k, \frac{\omega}{2^{n+1}}\right\}}(u) = \begin{cases} \ln \left\{ \dfrac{H_{u,k}^{+}}{H_{u,k}^{+} - u + k + \frac{\omega}{2^{n+1}}} \right\} & \text{if} \quad u > k + \frac{\omega}{2^{n+1}} \\ \\ 0 & \text{if} \quad u \leq k + \frac{\omega}{2^{n+1}} \end{cases}$$

From (2.39), estimate

$$\Psi^+ \leq n \ln 2 \qquad \text{since} \qquad \frac{H_{u,k}^{+}}{H_{u,k}^{+} - u + k + \frac{\omega}{2^{n+1}}} \leq \frac{\frac{\omega}{2}}{\frac{\omega}{2^{n+1}}} = 2^n \ , \tag{2.40}$$

and,

$$\left| \left(\psi^+\right)'(u) \right|^{2-p} = \left(H_{u,k}^{+} - u + k + c \right)^{p-2} \leq \left(\frac{\omega}{2} \right)^{p-2} \ . \tag{2.41}$$

Choosing a piecewise smooth cutoff function $0 < \zeta(x) \leq 1$, defined in K_R and such that, for some $\sigma \in (0,1)$,

$$\zeta = 1 \quad \text{in} \quad K_{(1-\sigma)R} \qquad \text{and} \qquad |\nabla \zeta| \leq (\sigma R)^{-1} \ ,$$

inequality (2.6) reads

$$\sup_{t^\circ < t < t^*} \int_{K_R \times \{t\}} \left[\psi^+(u) \right]^2 \zeta^p \, \mathrm{d}x \leq \int_{K_R \times \{t^\circ\}} \left[\psi^+(u) \right]^2 \zeta^p \, \mathrm{d}x$$
$$+ C \int_{t^\circ}^{t^*} \int_{K_R} \psi^+(u) \ \left| (\psi^+)'(u) \right|^{2-p} |\nabla \zeta|^p \, \mathrm{d}x \, \mathrm{d}t \ . \tag{2.42}$$

The first integral on the right hand side can be bounded above using Lemma **2.15** and taking into account that Ψ^+ vanishes on the set

$$\{ x \in K_R \ : \ u(x, \cdot) < k \} \ .$$

This gives

$$\int_{K_R \times \{t^\circ\}} \left[\psi^+(u) \right]^2 \, \mathrm{d}x \leq n^2 (\ln 2)^2 \ \left(\frac{1 - \nu_0}{1 - \nu_0/2} \right) \ |K_R| \ .$$

To bound the second integral we use (2.40) and (2.41):

$$\begin{aligned}
C\int_{t^\circ}^{t^*}\int_{K_R}\psi^+(u)\,\left|(\psi^+)'(u)\right|^{2-p}|\nabla\zeta|^p\,\mathrm{d}x\,\mathrm{d}t
&\le C\,n(\ln 2)\left(\frac{\omega}{2}\right)^{p-2}(\sigma R)^{-p}\,(t^*-t^\circ)\ |K_R| \\
&\le C\,n\left(\frac{\omega}{2}\right)^{p-2}\left(\frac{1}{\sigma R}\right)^{p} dR^p\ |K_R| \\
&\le C\,n\,\frac{1}{\sigma^p}\ |K_R|\ ,
\end{aligned}$$

since $t^*-t^\circ\le dR^p$. The left hand side is estimated below by integrating over the smaller set

$$S=\left\{x\in K_{(1-\sigma)R}\ :\ u(x,t)>\mu_+-\frac{\omega}{2^{n+2}}\right\}\subset K_R$$

and observing that in S, $\zeta=1$ and

$$\frac{H^+_{u,k}}{H^+_{u,k}-u+k+\frac{\omega}{2^{n+1}}}\ge\frac{\frac{\omega}{8}}{\left(H^+_{u,k}-\frac{\omega}{2}\right)+\frac{\omega}{2^{n+2}}}\ge\frac{\frac{\omega}{2}}{\frac{\omega}{2^n}}=2^{n-1}\ ,$$

since $\left(H^+_{u,k}-\frac{\omega}{2}\right)\le 0$. Therefore

$$\left[\psi^+(u)\right]^2\ge\left[\ln\left(2^{n-1}\right)\right]^2=(n-1)^2(\ln 2)^2\ .$$

From this

$$\sup_{t^\circ<t<t^*}\int_{K_R\times\{t\}}\left[\psi^+(u)\right]^2\,\zeta^p\,\mathrm{d}x\ge(n-1)^2(\ln 2)^2\,|S|\ .$$

Combining these three estimates, we arrive at

$$\begin{aligned}
|S| &\le \left(\frac{n}{n-1}\right)^2\left(\frac{1-\nu_0}{1-\nu_0/2}\right)\ |K_R|+C\,\frac{n}{(n-1)^2}\,\frac{1}{\sigma^p}\ |K_R| \\
&\le \left(\frac{n}{n-1}\right)^2\left(\frac{1-\nu_0}{1-\nu_0/2}\right)\ |K_R|+\frac{C}{n\sigma^p}\ |K_R|\ .
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\left|\left\{x\in K_R\ :\ u(x,t)>\mu_+-\frac{\omega}{2^{n+1}}\right\}\right| \\
&\le\left|\left\{x\in K_{(1-\sigma)R}\ :\ u(x,t)>\mu_+-\frac{\omega}{2^{n+1}}\right\}\right|+\left|K_R\setminus K_{(1-\sigma)R}\right| \\
&\quad\le|S|+N\sigma\ |K_R|
\end{aligned}$$

thus

$$\left|\left\{x \in K_R \ : \ u(x,t) > \mu_+ - \frac{\omega}{2^{n+1}}\right\}\right|$$
$$\leq \left\{\left(\frac{n}{n-1}\right)^2 \left(\frac{1-\nu_0}{1-\nu_0/2}\right) + \frac{C}{n\sigma^p} + N\sigma\right\} |K_R| \ ,$$

for all $t \in (t^\circ, t^*)$. Choose σ so small that $N\sigma \leq \frac{3}{8}\nu_0^2$ and n so large that

$$\left(\frac{n}{n-1}\right)^2 \leq \left(1-\frac{\nu_0}{2}\right) (1+\nu_0) \equiv \beta \qquad \text{and} \qquad \frac{C}{n\sigma^p} \leq \frac{3}{8}\nu_0^2 \ . \tag{2.43}$$

Note that $\beta > 1$. With this choice of n, the lemma follows with $s_2 = n+1$.

■

The same type of conclusion holds in an upper portion of the full cylinder $Q(a_0R^p, R)$, say for all $t \in \left(-\frac{a_0}{2}R^p, 0\right)$. Indeed, (2.21) holds for *all* cylinders of the type $(0,t^*) + Q(dR^p, R)$ so that the conclusion of the previous lemma holds true for all time levels

$$t \geq -(a_0 - d)R^p - \frac{\nu_0}{2}dR^p \ .$$

So if we choose

$$2^{(\lambda-1)(p-2)} \geq 2 \tag{2.44}$$

we get $\dfrac{a_0}{d} \geq 2 - \nu_0$, which is equivalent to

$$-(a_0 - d)R^p - \frac{\nu_0}{2}dR^p \leq -\frac{a_0}{2}R^p \ .$$

Corollary 2.17. *Assume (2.21) holds and let (2.16) be in force. For all* $t \in \left(-\frac{a_0}{2}R^p, 0\right)$,

$$\left|\left\{x \in K_R \ : \ u(x,t) > \mu_+ - \frac{\omega}{2^{s_2}}\right\}\right| \leq \left(1 - \left(\frac{\nu_0}{2}\right)^2\right) |K_R| \ .$$

The main result of this section states that in fact u is strictly below its supremum μ^+ in a smaller cylinder with the same vertex and axis as $Q\left(\frac{a_0}{2}R^p, R\right)$.

Proposition 2.18. *Assume (2.21) holds and let (2.16) be in force. The choice of λ can be made so that*

$$u(x,t) \leq \mu_+ - \frac{\omega}{2^{\lambda+1}} \qquad a.e. \qquad (x,t) \in Q\left(\frac{a_0}{2}\left(\frac{R}{2}\right)^p, \frac{R}{2}\right) \ . \tag{2.45}$$

Proof. Define

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}}\ , \quad n = 0, 1, \dots ,$$

and construct the family of nested shrinking cylinders $Q\left(\frac{a_0}{2}R_n^p, R_n\right)$. Consider piecewise smooth cutoff functions $0 < \zeta_n \leq 1$, defined in these cylinders and satisfying the following set of assumptions:

$$\zeta_n = 1 \ \text{ in } \ Q\left(\tfrac{a_0}{2}R_{n+1}^p, R_{n+1}\right) \qquad \zeta_n = 0 \ \text{ in } \ \partial_p Q\left(\tfrac{a_0}{2}R_n^p, R_n\right)$$

$$|\nabla \zeta_n| \leq \tfrac{2^{n+1}}{R} \qquad\qquad 0 \leq \partial_t \zeta_n \leq \frac{2^{p(n+1)}}{\frac{a_0}{2}R^p} \qquad .$$

The energy inequality (2.5) for the functions $(u-k_n)_+$, with

$$k_n = \mu_+ - \frac{\omega}{2^{\lambda+1}} - \frac{\omega}{2^{\lambda+1+n}}\ , \quad n = 0, 1, \dots ,$$

in the cylinders $Q\left(\frac{a_0}{2}R_n^p, R_n\right)$ and with $\zeta = \zeta_n$, read

$$\sup_{-\frac{a_0}{2}R_n^p < t < 0} \int_{K_{R_n}\times\{t\}} (u-k_n)_+^2 \zeta_n^p \,\mathrm{d}x + \int_{-\frac{a_0}{2}R_n^p}^0 \int_{K_{R_n}} |\nabla(u-k_n)_+\zeta_n|^p \,\mathrm{d}x\,\mathrm{d}t$$

$$\leq C \int_{-\frac{a_0}{2}R_n^p}^0 \int_{K_{R_n}} (u-k_n)_+^p |\nabla\zeta_n|^p \,\mathrm{d}x\,\mathrm{d}t + C \int_{-\frac{a_0}{2}R_n^p}^0 \int_{K_{R_n}} (u-k_n)_+^2 \zeta_n^{p-1}\partial_t\zeta_n \,\mathrm{d}x\,\mathrm{d}t$$

$$\leq C\, \frac{2^{p(n+1)}}{R^p} \left\{ \int_{-\frac{a_0}{2}R_n^p}^0 \int_{K_{R_n}} (u-k_n)_+^p \,\mathrm{d}x\,\mathrm{d}t + \frac{2}{a_0} \int_{-\frac{a_0}{2}R_n^p}^0 \int_{K_{R_n}} (u-k_n)_+^2 \,\mathrm{d}x\,\mathrm{d}t \right\} . \tag{2.46}$$

Observe that

$$(u-k_n)_+^2 = (u-k_n)_+^{2-p}\,(u-k_n)_+^p \geq \left(\frac{\omega}{2^\lambda}\right)^{2-p}\,(u-k_n)_+^p\ .$$

Therefore, from (2.46),

$$\left(\frac{\omega}{2^\lambda}\right)^{2-p} \sup_{-\frac{a_0}{2}R_n^p < t < 0} \int_{K_{R_n}\times\{t\}} (u-k_n)_+^p \zeta_n^p \,\mathrm{d}x + \int_{-\frac{a_0}{2}R_n^p}^0 \int_{K_{R_n}} |\nabla(u-k_n)_+\zeta_n|^p \,\mathrm{d}x\,\mathrm{d}t$$

$$\leq C\, \frac{2^{p(n+1)}}{R^p}\ \left\{ \left(\frac{\omega}{2^\lambda}\right)^p + \frac{2}{a_0}\left(\frac{\omega}{2^\lambda}\right)^2 \right\} \int_{-\frac{a_0}{2}R_n^p}^0 \int_{K_{R_n}} \chi_{\{(u-k_n)_+>0\}} \,\mathrm{d}x\,\mathrm{d}t\ . \tag{2.47}$$

Now recall that $a_0 = \left(\frac{\omega}{2^\lambda}\right)^{2-p}$, and divide (2.47) by a_0 to get

$$\sup_{-\frac{a_0}{2}R_n^p < t < 0} \int_{K_{R_n}\times\{t\}} (u-k_n)_+^p \zeta_n^p \,\mathrm{d}x + \frac{1}{a_0} \int_{-\frac{a_0}{2}R_n^p}^0 \int_{K_{R_n}} |\nabla(u-k_n)_+\zeta_n|^p \,\mathrm{d}x\,\mathrm{d}t$$

$$\leq \frac{C2^{pn}}{R^p} \left(\frac{\omega}{2^\lambda}\right)^p \frac{1}{a_0} \int_{-\frac{a_0}{2}R_n^p}^0 \int_{K_{R_n}} \chi_{\{(u-k_n)_+>0\}} \, \mathrm{d}x \, \mathrm{d}t \ . \tag{2.48}$$

Next, perform a change in the time variable in (2.48), putting

$$\overline{t} = \frac{t}{a_0/2}$$

and defining

$$\overline{u}(\cdot, \overline{t}) = u(\cdot, t) \qquad \text{and} \qquad \overline{\zeta_n}(\cdot, \overline{t}) = \zeta_n(\cdot, t) \ ,$$

and obtain the simplified inequality

$$\left\| (\overline{u} - k_n)_+ \, \overline{\zeta_n} \right\|^p_{X_0^p(Q(R_n^p, R_n))} \leq \frac{C2^{pn}}{R^p} \left(\frac{\omega}{2^\lambda}\right)^p \int_{-R_n^p}^0 \int_{K_{R_n}} \chi_{\{(\overline{u}-k_n)_+>0\}} \, \mathrm{d}x \, \mathrm{d}\overline{t} \ . \tag{2.49}$$

Define, for each n,

$$A_n = \int_{-R_n^p}^0 \int_{K_{R_n}} \chi_{\{(\overline{u}-k_n)_+>0\}} \, \mathrm{d}x \ \mathrm{d}\overline{t}$$

and estimate

$$\begin{aligned} \frac{1}{2^{p(n+2)}} \left(\frac{\omega}{2^\lambda}\right)^p A_{n+1} &\leq |k_{n+1} - k_n|^p \ A_{n+1} \\ &\leq \left\| (\overline{u} - k_n)_+ \right\|^p_{p, Q(R_{n+1}^p, R_{n+1})} \\ \leq \left\| (\overline{u} - k_n)_+ \overline{\zeta_n} \right\|^p_{p, Q(R_n^p, R_n)} &\leq C \ \left\| (\overline{u} - k_n)_+ \, \overline{\zeta_n} \right\|^p_{X_0^p(Q(R_n^p, R_n))} \ A_n^{\frac{p}{N+p}} \\ &\leq \frac{C2^{pn}}{R^p} \left(\frac{\omega}{2^\lambda}\right)^p \ A_n^{1+\frac{p}{N+p}} \ . \end{aligned} \tag{2.50}$$

The first three inequalities are obvious; and the last is a consequence of the imbedding theorem 2.6. Define the numbers

$$X_n = \frac{A_n}{|Q(R_n^p, R_n)|} \ ,$$

divide (2.50) by $|Q(R_{n+1}^p, R_{n+1})|$, and obtain the recursive relation

$$X_{n+1} \leq C \, 4^{pn} \ X_n^{1+\frac{p}{N+p}} \ .$$

By Lemma 2.5 on fast geometric convergence, if

$$X_0 \leq C^{-\frac{N+p}{p}} 4^{-\frac{(N+p)^2}{p}} \equiv \nu_* \tag{2.51}$$

then

$$X_n \longrightarrow 0 \ . \tag{2.52}$$

Thus if (2.51) holds,

$$\left|\left\{(x,t)\in Q\left(\frac{a_0}{2}\left(\frac{R}{2}\right)^p,\frac{R}{2}\right)\ :\ u(x,t)>\mu_+-\frac{\omega}{2^{\lambda+1}}\right\}\right|=0$$

and the result follows.

We are left to prove (2.51). To simplify the symbolism introduce the sets

$$B_\sigma(t)=\left\{x\in K_R\ :\ u(x,t)>\mu^+-\frac{\omega}{2^\sigma}\right\}$$

and

$$B_\sigma=\left\{(x,t)\in Q\left(\frac{a_0}{2}R^p,R\right)\ :\ u(x,t)>\mu_+-\frac{\omega}{2^\sigma}\right\}\ .$$

With this notation (2.51) reads

$$|B_\lambda|\le\nu_*\left|Q\left(\frac{a_0}{2}R^p,R\right)\right|\ .$$

We'll use the information contained in Corollary 2.17 to show that this holds, i.e., that the subset of the cylinder $Q\left(\frac{a_0}{2}R^p,R\right)$ where u is close to its supremum μ^+ can be made arbitrarily small. Consider the local energy inequalities (2.5) for the functions $(u-k)_+$ in the cylinders $Q\left(a_0R^p,2R\right)$, with

$$k=\mu_+-\frac{\omega}{2^s}\ ,$$

where s will be chosen later satisfying $s_2\le s\le\lambda$ (recall that s_2 was chosen on Lemma 2.16). Take a piecewise smooth cutoff function $0<\zeta\le 1$, defined in $Q\left(a_0R^p,2R\right)$, and such that

$$\zeta=1\ \text{ in }\ Q\left(\tfrac{a_0}{2}R^p,R\right)\qquad \zeta=0\ \text{ in }\ \partial_pQ\left(a_0R^p,2R\right)$$

$$|\nabla\zeta|\le\tfrac{1}{R}\qquad\qquad 0\le\partial_t\zeta\le\tfrac{2}{a_0R^p}\qquad .$$

Neglecting the first term on the left hand side of the estimates, we obtain for the indicated choices,

$$\int_{-\frac{a_0}{2}R^p}^0\int_{K_R}|\nabla(u-k)_+|^p\ \mathrm{d}x\,\mathrm{d}t$$

$$\le\frac{C}{R^p}\int_{-a_0R^p}^0\int_{K_{2R}}(u-k)_+^p\,\mathrm{d}x\,\mathrm{d}t+\frac{C}{a_0R^p}\int_{-a_0R^p}^0\int_{K_{2R}}(u-k)_+^2\,\mathrm{d}x\,\mathrm{d}t\ . \tag{2.53}$$

We estimate the two terms on the right hand side of this inequality as follows:

$$\frac{C}{R^p}\int_{-a_0R^p}^0\int_{K_{2R}}(u-k)_+^p\,\mathrm{d}x\,\mathrm{d}t\le\frac{C}{R^p}\left(\frac{\omega}{2^s}\right)^p\left|Q\left(\frac{a_0}{2}R^p,R\right)\right|$$

and, recalling the definition of a_0,

$$\frac{C}{a_0 R^p}\int_{-a_0R^p}^{0}\int_{K_{2R}}(u-k)_+^2\,\mathrm{d}x\,\mathrm{d}t \le \frac{C}{R^p}\left(\frac{\omega}{2^\lambda}\right)^{p-2}\left(\frac{\omega}{2^s}\right)^2\left|Q\left(\frac{a_0}{2}R^p,R\right)\right|$$
$$\le \frac{C}{R^p}\left(\frac{\omega}{2^s}\right)^p\left|Q\left(\frac{a_0}{2}R^p,R\right)\right| \ ,$$

since $s \le \lambda$. Putting this in (2.53) gives,

$$\iint_{B_s}|\nabla u|^p\,\mathrm{d}x\,\mathrm{d}t \le \frac{C}{R^p}\left(\frac{\omega}{2^s}\right)^p\left|Q\left(\frac{a_0}{2}R^p,R\right)\right| \ . \tag{2.54}$$

We next apply Lemma **2.3** to the function $u(\cdot,t)$, for all $-\frac{a_0}{2}R^p \le t \le 0$, and with

$$k=\mu_+-\frac{\omega}{2^s}\ , \qquad l=\mu_+-\frac{\omega}{2^{s+1}}\ , \qquad l-k=\frac{\omega}{2^{s+1}}\ .$$

Observing that, owing to Corollary 2.17,

$$\left|\left\{x\in K_R\ :\ u(x,t)<\mu_+-\frac{\omega}{2^s}\right\}\right| \equiv |K_R|-|B_s(t)| \ge \left(\frac{\nu_0}{2}\right)^2\ |K_R|\ ,$$

we obtain

$$\frac{\omega}{2^{s+1}}\,|B_{s+1}(t)| \le \frac{4C\ R^{N+1}}{\nu_0^2\ |K_R|}\int_{B_s(t)\setminus B_{s+1}(t)}|\nabla u|\,\mathrm{d}x\,\mathrm{d}t\ ,$$

for $t\in\left(-\frac{a_0}{2}R^p,0\right)$. Integrating over this interval, we conclude that

$$\begin{aligned}\frac{\omega}{2^{s+1}}\,|B_{s+1}| &\le \frac{CR}{\nu_0^2}\iint_{B_s\setminus B_{s+1}}|\nabla u|\,\mathrm{d}x\,\mathrm{d}t\\ &\le \frac{CR}{\nu_0^2}\left(\iint_{B_s}|\nabla u|^p\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac1p}\ |B_s\setminus B_{s+1}|^{\frac{p-1}{p}}\\ &\le \frac{C}{\nu_0^2}\left(\frac{\omega}{2^s}\right)\left|Q\left(\frac{a_0}{2}R^p,R\right)\right|^{\frac1p}\ |B_s\setminus B_{s+1}|^{\frac{p-1}{p}}\ ,\end{aligned}$$

using also (2.54). Taking the $\frac{p}{p-1}$ power and dividing through by $\left(\frac{\omega}{2^{s+1}}\right)^{\frac{p}{p-1}}$, we obtain

$$|B_{s+1}|^{\frac{p}{p-1}} \le C\,(\nu_0)^{-\frac{2p}{p-1}}\ \left|Q\left(\frac{a_0}{2}R^p,R\right)\right|^{\frac{1}{p-1}}\ |B_s\setminus B_{s+1}|\ .$$

Since these inequalities are valid for $s_2\le s\le\lambda$, we add them for

$$s=s_2,s_2+1,s_2+2,\dots,\lambda-1\ ,$$

and since the sum on the right hand side can be bounded above by the quantity $\left|Q\left(\frac{a_0}{2}R^p, R\right)\right|$, we obtain

$$(\lambda - s_2)\ |B_\lambda|^{\frac{p}{p-1}} \le C\,(\nu_0)^{-\frac{2p}{p-1}}\ \left|Q\left(\frac{a_0}{2}R^p, R\right)\right|^{\frac{p}{p-1}}\ ,$$

that is,

$$|B_\lambda| \le \frac{C}{(\lambda - s_2)^{\frac{p-1}{p}}}\,(\nu_0)^{-2}\ \left|Q\left(\frac{a_0}{2}R^p, R\right)\right|\ .$$

We obtain (2.51) if λ is chosen so large that

$$\frac{C}{\nu_0^2\,(\lambda - s_2)^{\frac{p-1}{p}}} \le \nu_*\ .$$

We finally make the choice

$$\lambda = \max\left\{s_2 + \left(\frac{C}{\nu_0^2\nu_*}\right)^{\frac{p}{p-1}}\ ,\ 1 + \frac{1}{p-2}\right\} \tag{2.55}$$

(recall that s_2 is given through (2.43), ν_0 is given by (2.27), and ν_* is given by (2.51)) thus concluding the proof of the proposition. ■

Remark 2.19. Observe that the choice of λ was made so that (2.44) holds, and a larger λ is admissible. Choosing λ determines the length of the cylinder $Q(a_0R^p, R)$, since $a_0 = \left(\frac{\omega}{2^\lambda}\right)^{2-p}$. The proposition has a double scope: we determine a level $\mu_+ - \frac{\omega}{2^{\lambda+1}}$ and a cylinder (fixing λ and consequently a_0) such that the conclusion holds in that particular cylinder.

Corollary 2.20. *Assume (2.21) holds and let (2.16) be in force. There exist constants* $\nu_0, \sigma_1 \in (0,1)$, *depending only on the data, such that if (2.21) holds then*

$$\operatorname*{ess\ osc}_{Q\left(\frac{a_0}{2}\left(\frac{R}{2}\right)^p, \frac{R}{2}\right)} u\ \le\ \sigma_1\ \omega\ .$$

Proof. It is similar to the proof of Corollary 2.14 for the choice

$$\sigma_1 = \left(1 - \frac{1}{2^{\lambda+1}}\right)\ .$$

■

2.3.4. *The Hölder continuity.* We finally prove the Hölder continuity of weak solutions. An immediate consequence of Corollaries 2.14 and 2.20 is

Proposition 2.21. *There exists a constant $\sigma \in (0,1)$, that depends only on the data, such that*

$$\operatorname*{\mathit{ess\ osc}}_{Q\left(d(\frac{R}{8})^p, \frac{R}{8}\right)} u \ \leq\ \sigma\, \omega\ .$$

Proof. Assume (2.16) is in force. Then by Corollaries 2.14 and 2.20

$$\operatorname*{ess\ osc}_{Q\left(d(\frac{R}{8})^p, \frac{R}{8}\right)} u \ \leq\ \sigma\, \omega \qquad \text{where} \qquad \sigma = \max\{\sigma_0, \sigma_1\}\ , \tag{2.56}$$

since

$$d\left(\frac{R}{8}\right)^p \leq \frac{a_0}{2}\left(\frac{R}{2}\right)^p\ .$$

■

We define now an iteration process that will lead to the Hölder continuity of u.

Proposition 2.22. *There exists a positive constant C, depending only on the data, such that defining the sequences*

$$R_n = C^{-n} R \qquad\qquad \mathit{and} \qquad\qquad \omega_n = \sigma^n \omega\ ,$$

for $n = 0, 1, 2, \ldots$, where $\sigma \in (0,1)$ is given by (2.56), and constructing the family of cylinders

$$Q_n = Q(a_n R_n^p, R_n)\ , \qquad \mathit{with} \qquad a_n = \left(\frac{\omega_n}{2^\lambda}\right)^{2-p}\ ,$$

where $\lambda > 1$ is given by (2.55), we have

$$Q_{n+1} \subset Q_n \qquad\quad \mathit{and} \qquad\quad \operatorname*{\mathit{ess\ osc}}_{Q_n} u \leq \omega_n\ , \tag{2.57}$$

for all $n = 0, 1, 2, \ldots$

Proof. Recall the definition of $a_0 = \left(\frac{\omega}{2^\lambda}\right)^{2-p}$ and the construction of the initial cylinder so that the starting relation

$$\operatorname*{ess\ osc}_{Q_0} u \leq \omega \tag{2.58}$$

holds. We find

$$\begin{aligned}
d\left(\frac{R}{8}\right)^p &= \left(\frac{\omega}{2}\right)^{2-p}\frac{R^p}{8^p}\\
&= \left(\frac{\omega}{2}\right)^{2-p}\left(\frac{2^\lambda}{\omega_1}\right)^{2-p}\left(\frac{\omega_1}{2^\lambda}\right)^{2-p}\frac{R^p}{8^p}\\
&= \left(\frac{\omega}{\omega_1}\right)^{2-p}\left(\frac{2^\lambda}{2}\right)^{2-p}\left(\frac{\omega_1}{2^\lambda}\right)^{2-p}\frac{R^p}{8^p}\\
&= \sigma^{p-2}\,2^{(\lambda-1)(2-p)-3p}\;a_1\,R^p\\
&= a_1 R_1^p\ ,
\end{aligned}$$

where $R_1 = C^{-1}R$, provided C is chosen from

$$C = \sigma^{\frac{2-p}{p}}\,2^{\frac{(\lambda-1)(p-2)}{p}+3} > 8\ .$$

From Proposition 2.21, we conclude

$$\operatorname*{ess\,osc}_{Q_1} u \le \operatorname*{ess\,osc}_{Q\left(d(\frac{R}{8})^p,\frac{R}{8}\right)} u \ \le\ \sigma\,\omega = \omega_1$$

which puts us back to the setting of (2.58). The entire process can now be repeated inductively starting from Q_1.

■

Remark 2.23. The proof of Proposition 2.22 shows that it would have been sufficient to work with a number ω and a cylinder $Q(a_0R^p, R)$ linked by (2.17). This relation is in general not verifiable *a priori* for a given cylinder, since its dimensions would have to be intrinsically defined in terms of the essential oscillation of u within it.

The role of having introduced the cylinder $Q(R^2, R)$ and having assumed (2.16) is that (2.17) holds true for the *constructed* box $Q(a_0R^p, R)$. It is part of the proof of proposition 2.22 to show that, at each step, the cylinders Q_n and the essential oscillation of u within them satisfy the intrinsic geometry dictated by (2.17).

Lemma 2.24. *There exist constants $\gamma > 1$ and $\alpha \in (0,1)$, that can be determined a priori in terms of the data, such that for all the cylinders*

$$Q(a_0\rho^p, \rho)\ , \qquad \textit{with} \quad 0 < \rho \le R\ ,$$

$$\operatorname*{\textit{ess osc}}_{Q(a_0\rho^p,\rho)} u \le \gamma\,\omega\left(\frac{\rho}{R}\right)^\alpha\ .$$

Proof. Let $0 < \rho \le R$ be fixed. There exists a non-negative integer n such that

$$C^{-(n+1)} R \le \rho \le C^{-n} R$$

so, putting $\alpha = -\dfrac{\ln \sigma}{\ln C}$, we deduce

$$C^{-(n+1)} \le \frac{\rho}{R} \Leftrightarrow \sigma^{\frac{n+1}{\alpha}} \le \frac{\rho}{R} \Leftrightarrow \sigma^{n+1} \le \left(\frac{\rho}{R}\right)^{\alpha} \ .$$

Thus

$$\omega_n = \sigma^n \omega \le \gamma\, \omega \left(\frac{\rho}{R}\right)^{\alpha} \ , \qquad \text{with} \quad \gamma = \sigma^{-1} \ .$$

To conclude the proof, observe that the cylinder $Q(a_0\rho^p, \rho)$ is contained in the cylinder $Q_n \equiv Q(a_n R_n^p, R_n)$ since $\omega_n \le \omega$ and $\rho \le C^{-n} R \equiv R_n$.

■

Let $\Gamma = \partial_p \Omega_T$ be the parabolic boundary of Ω_T. Introduce the degenerate intrinsic parabolic p-distance from a compact set $K \subset \Omega_T$ to Γ, by

$$p - \mathrm{dist}(K; \Gamma) \equiv \inf_{\substack{(x,t)\in K \\ (y,s)\in\Gamma}} \left(|x-y| \wedge M^{\frac{p-2}{p}} |t-s|^{\frac{1}{p}} \right) \ .$$

Theorem 2.25. *Let u be a bounded local weak solution of (2.1) in Ω_T and $M = \|u\|_{\infty,\Omega_T}$. Then u is locally Hölder continuous in Ω_T, i.e., there exist constants $\gamma > 1$ and $\alpha \in (0,1)$, depending only upon the data, such that, for every compact subset K of Ω_T,*

$$|u(x_1,t_1) - u(x_2,t_2)| \le \gamma M \left(\frac{|x_1 - x_2| + M^{\frac{p-2}{p}} |t_1 - t_2|^{\frac{1}{p}}}{p - \mathrm{dist}(K;\Gamma)} \right)^{\alpha}$$

for every pair of points $(x_i, t_i) \in K$, $i = 1, 2$.

Proof. Fix $(x_i, t_i) \in K$, $i = 1, 2$, such that $t_2 > t_1$ and construct the cylinder $(x_2, t_2) + Q\left(M^{2-p} R^p, R\right) \subset \Omega_T$. This is realized if we choose

$$R \le \inf_{\substack{x\in K \\ y\in\partial\Omega}} |x-y| \ , \qquad \text{and} \qquad M^{\frac{2-p}{p}} R \le \inf_{t\in K} t^{\frac{1}{p}} \ .$$

Thus in particular we may choose

$$2R = p - \mathrm{dist}\,(K;\Gamma) \ .$$

To prove the Hölder continuity in the t–variable assume first that $(t_2 - t_1) < M^{2-p}R^p$. Then there exists $\rho \in (0, R)$ such that $(t_2 - t_1) = M^{2-p}\rho^p$, i.e., $\rho = M^{\frac{p-2}{p}}|t_2 - t_1|^{\frac{1}{p}}$. The oscillation inequality of Lemma 2.24, applied in the cylinder $(x_2, t_2) + Q(a_0\rho^p, \rho)$ implies

$$|u(x_2, t_2) - u(x_2, t_1)| \le \gamma M \left(\frac{M^{\frac{p-2}{p}}|t_2 - t_1|^{\frac{1}{p}}}{p - \mathrm{dist}(K; \Gamma)} \right)^{\alpha} .$$

If $(t_2 - t_1) \ge M^{2-p}R^p$ we have

$$|u(x_2, t_2) - u(x_2, t_1)| \le 2M \le 4M \left(\frac{M^{\frac{p-2}{p}}|t_2 - t_1|^{\frac{1}{p}}}{p - \mathrm{dist}(K; \Gamma)} \right) .$$

The Hölder continuity in the space variables is proved analogously.

■

Remark 2.26. The theory extends to full quasilinear equations and includes statements of regularity up to the parabolic boundary of Ω_T (see [55]).

2.4. The singular case $1 < p < 2$. We now turn to the singular case $1 < p < 2$, still for the model equation

$$u_t - \mathrm{div}\, |\nabla u|^{p-2}\nabla u = 0 \ , \quad 1 < p < 2 \ . \tag{2.59}$$

The analysis for this case is somehow more involved, but several of the previous techniques apply. As before the Hölder continuity of u will be solely a consequence of the Caccioppoli inequalities (2.5) and the logarithmic inequalities (2.6). Throughout this section the function u is merely assumed to satisfy such inequalities. To avoid repetition of arguments, we will outline the approach to regularity emphasizing the main differences with respect to the degenerate situation.

A key point is again the choice of the appropriate intrinsic geometry in which to carry the iteration argument. Fix a point $(x_0, t_0) \in \Omega_T$ and, as before, assume $(x_0, t_0) = (0, 0)$. Consider a cylinder

$$B_{R^{\frac{p}{2}}}(0) \times (-R^p, 0) \equiv Q(R^p, R^{\frac{p}{2}}) \subset \Omega_T$$

where $R > 0$ is taken such that the inclusion holds. Now, let

$$\mu_- := \operatorname*{ess\,inf}_{Q(R^p, R^{\frac{p}{2}})} u \ ; \quad \mu_+ := \operatorname*{ess\,sup}_{Q(R^p, R^{\frac{p}{2}})} u \ ; \quad \omega := \operatorname*{ess\,osc}_{Q(R^p, R^{\frac{p}{2}})} u = \mu_+ - \mu_-$$

and construct the cylinder

$$Q(R^p, c_0R) \qquad \text{with} \qquad c_0 = \left(\frac{\omega}{2^\lambda}\right)^{\frac{p-2}{p}} \tag{2.60}$$

where λ is to be determined only in terms of the data (we use the same letter as in the case $p > 2$ but the λ's are different in the two cases).

To start the iteration, we assume

$$\left(\frac{\omega}{2^\lambda}\right)^{\frac{p-2}{p}} < R^{\frac{p-2}{2}} \tag{2.61}$$

(otherwise, $\omega \leq 2^\lambda R^{\frac{p}{2}}$ and there is nothing to prove) so that

$$Q(R^p, c_0R) \subset Q(R^p, R^{\frac{p}{2}})$$

and

$$\operatorname*{ess\,osc}_{Q(R^p,c_0R)} u \leq \omega. \tag{2.62}$$

Cylinders of the type (2.60) have the space variables stretched by a factor $\left(\frac{\omega}{2^\lambda}\right)^{\frac{p-2}{p}}$ which is intrinsically determined by the solution. Note that if $p = 2$ these are just the standard parabolic cylinders. The geometry chosen is not the only possible. We could have introduced, for example, a scaling with different parameters in the space and the time variables. Another possibility would be to work with a scaling formally identical to the one used in the degenerate case. Our option here was dictated only by a matter of simplicity.

The main result leading to the Hölder continuity of solutions is

Proposition 2.27. *There exist constants $\eta \in (0,1)$ and $C, \lambda > 1$, that can be determined only in terms of the data, satisfying the following. Construct the sequences*

$$\left.\begin{array}{rcl} R_n & = & C^{-n}R \\ \omega_n & = & \eta^n \omega \end{array}\right. \qquad n = 0, 1, 2, \ldots$$

and the cylinders

$$Q_n \equiv Q(R_n^p, c_nR_n) \quad \text{with} \quad c_n = \left(\frac{\omega_n}{2^\lambda}\right)^{\frac{p-2}{p}}, \quad n = 0, 1, 2, \ldots$$

Then, for all $n = 0, 1, 2, \ldots$,

$$Q_{n+1} \subset Q_n \qquad \text{and} \qquad \operatorname*{ess\,osc}_{Q_n} u \leq \omega_n .$$

This proposition implies an analogue to Lemma 2.24 from which the Hölder continuity follows as in the proof of Theorem 2.25.

The proof of the proposition is, as in the degenerate case, based on the analysis of an alternative. To begin, consider inside of $Q(R^p, c_0 R)$ subcylinders of smaller size

$$(\overline{x}, 0) + Q(R^p, d_0 R) \ , \qquad d_0 = \left(\frac{\omega}{2}\right)^{\frac{p-2}{p}} \ . \tag{2.63}$$

These cylinders are contained in $Q(R^p, c_0 R)$ if $\overline{x}$ ranges over the cube $K_{\mathcal{R}(\omega)}$ where

$$\mathcal{R}(\omega) \equiv \left\{ \left(2^{\lambda-1}\right)^{\frac{2-p}{p}} - 1 \right\} \left(\frac{\omega}{2}\right)^{\frac{p-2}{p}} R = L_0 d_0 R \ , \tag{2.64}$$

for $L_0 = \left(2^{\lambda-1}\right)^{\frac{2-p}{p}} - 1$.

We can regard these cylinders as boxes moving inside $Q(R^p, c_0 R)$ as the coordinates $\overline{x}$ of their centres range over the cube $K_{\mathcal{R}(\omega)}$. We may arrange L_0 to be an integer and consider the cube $K_{c_0 R}$ as the union, up to a set of measure zero, of L_0^N disjoint cubes each of them congruent to $K_{d_0 R}$. Analogously, $Q(R^p, c_0 R)$ is the disjoint union, up to a set of measure zero, of L_0^N open boxes each congruent to $Q(R^p, d_0 R)$. Then we can view $(\overline{x}, 0) + Q(R^p, d_0 R)$ as the blocks of a partition of $Q(R^p, c_0 R)$ (see [55, fig. 3.1, page 82]).

Let $\nu_0 \in (0,1)$; then either

The first alternative:

There exists a cylinder of the type $(\overline{x}, 0) + Q(R^p, d_0 R)$ for which

$$\frac{\left|\left\{(x,t) \in (\overline{x}, 0) + Q(R^p, d_0 R) \ : \ u(x,t) < \mu_- + \frac{\omega}{2}\right\}\right|}{|Q(R^p, d_0 R)|} \leq \nu_0 \tag{2.65}$$

or

The second alternative:

For every cylinder of the type $(\overline{x}, 0) + Q(R^p, d_0 R)$

$$\frac{\left|\left\{(x,t) \in (\overline{x}, 0) + Q(R^p, d_0 R) \ : \ u(x,t) < \mu_- + \frac{\omega}{2}\right\}\right|}{|Q(R^p, d_0 R)|} > \nu_0 \tag{2.66}$$

In both cases, we will conclude that the essential oscillation of u within a smaller cylinder, centered at the origin, decreases in a way that can be quantitatively measured.

2.4.1. *Rescaled iterations.* The study of both alternatives makes crucial use of a rescaled iteration technique which applies to any subcylinder of Ω_T. Let $m > 0$ be given by

$$m = m_1 + m_2 \,, \qquad \text{where} \quad m_1 \geq 1 \,, \text{ and } m_2 \geq 0$$

and consider the cube

$$K_{d_1 R} \equiv \left\{ x \in \mathbf{R}^N : \max_{1 \leq i \leq N} |x_i| < d_1 R \right\} \,, \qquad d_1 = \left(\frac{\omega}{2^{m_1}} \right)^{\frac{p-2}{p}}$$

and the box

$$\mathcal{Q}_R(m_1, m_2) \equiv K_{d_1 R} \times \left(-2^{m_2(p-2)} R^p, 0 \right) \,. \tag{2.67}$$

Fix $(\overline{x}, \overline{t}) \in \Omega_T$, and let $R > 0$ be so small that

$$(\overline{x}, \overline{t}) + \mathcal{Q}_R(m_1, m_2) \subset \Omega_T \,.$$

Remark 2.28. Note that, for $(\overline{x}, \overline{t}) = (0,0)$, $m_1 = \lambda$ and $m_2 = 0$, the cylinder $(\overline{x}, \overline{t}) + \mathcal{Q}_R(m_1, m_2)$ coincides with the cylinder $Q(R^p, c_0 R)$. Analogously, if $m_2 = 0$, $m_1 = 1$ and $\overline{t} = 0$ then, for a suitable choice of $\overline{x}$, the cylinder $(\overline{x}, \overline{t}) + \mathcal{Q}_R(m_1, m_2)$ coincides with one of the boxes $(\overline{x}, 0) + Q(R^p, d_0 R)$ making up the partition of $Q(R^p, c_0 R)$.

Lemma 2.29. *There exists a number ν_0 that can be determined a priori only in terms of the data and independent of ω, R and m_1, m_2 such that:*

- *If u is a super-solution of (2.59) in $(\overline{x}, \overline{t}) + \mathcal{Q}_R(m_1, m_2)$ satisfying*

$$\operatorname*{ess\,osc}_{(\overline{x}, \overline{t}) + \mathcal{Q}_R(m_1, m_2)} u \leq \omega$$

and

$$\left| \left\{ (x,t) \in (\overline{x}, \overline{t}) + \mathcal{Q}_R(m_1, m_2) : u(x,t) < \mu_- + \frac{\omega}{2^m} \right\} \right| \leq \nu_0 \, |\mathcal{Q}_R(m_1, m_2)|$$

then

$$u(x,t) \geq \mu_- + \frac{\omega}{2^{m+1}} \,, \qquad \forall \, (x,t) \in (\overline{x}, \overline{t}) + \mathcal{Q}_{\frac{R}{2}}(m_1, m_2) \,.$$

Analogously,

- *If u is a sub-solution of (2.59) in $(\overline{x},\overline{t})+\mathcal{Q}_R(m_1,m_2)$ satisfying*

$$\operatorname*{ess\,osc}_{(\overline{x},\overline{t})+\mathcal{Q}_R(m_1,m_2)} u \le \omega$$

and

$$\left|\left\{(x,t)\in(\overline{x},\overline{t})+\mathcal{Q}_R(m_1,m_2) : u(x,t)>\mu_+-\frac{\omega}{2^m}\right\}\right| \le \nu_0\,|\mathcal{Q}_R(m_1,m_2)|$$

then

$$u(x,t)\le \mu_+-\frac{\omega}{2^{m+1}}\,, \qquad \forall\,(x,t)\in(\overline{x},\overline{t})+\mathcal{Q}_{\frac{R}{2}}(m_1,m_2)\ .$$

Proof. We only prove the statement concerning super-solutions (for sub-solutions the proof is similar). For simplicity, assume $(\overline{x},\overline{t})=(0,0)$ and construct the decreasing sequences of numbers

$$R_n=\frac{R}{2}+\frac{R}{2^{n+1}}\ ; \qquad k_n=\mu_-+\frac{\omega}{2^{m+1}}+\frac{\omega}{2^{m+1+n}}\ ; \qquad n=0,1,2,\dots\ ,$$

and the families of nested cubes and cylinders

$$K_n\equiv K_{d_1R_n}\,, \qquad d_1=\left(\frac{\omega}{2^{m_1}}\right)^{\frac{p-2}{p}}\ ,$$

$$\mathcal{Q}_n\equiv\mathcal{Q}_{R_n}(m_1,m_2)=K_n\times\left(-2^{(p-2)m_2}R_n^p,0\right)\ .$$

Consider the energy estimate (2.5), written for the functions $(u-k_n)_-$ over the boxes $\mathcal{Q}_n$, taking as cutoff functions ξ_n

$$\begin{cases} 0<\xi_n\le 1 \quad \text{in} \quad Q_n \qquad \text{and} \quad \xi_n\equiv 1 \quad \text{in} \quad Q_{n+1} \\ \xi_n\equiv 0 \quad \text{on} \quad \partial_p Q_n \\ |\nabla\xi_n|\le \dfrac{2^{n+2}}{R}\left(\dfrac{\omega}{2^{m_1}}\right)^{\frac{2-p}{p}}\ , \quad 0\le\xi_{n,t}\le 2^{(2-p)m_2}\,\dfrac{2^{p(n+2)}}{R^p}\ . \end{cases}$$

In this context, the energy inequalities read

$$\sup_{-2^{(p-2)m_2}R_n^p<t<0}\int_{K_n}(u-k_n)_-^2\xi_n^p\,\mathrm{d}x+\iint_{\mathcal{Q}_n}|\nabla(u-k_n)_-\xi_n|^p\,\mathrm{d}x\,\mathrm{d}t$$

$$\le C\frac{2^{np}}{R^p}\left(\frac{\omega}{2^{m_1}}\right)^{2-p}\iint_{\mathcal{Q}_n}(u-k_n)_-^p\,\mathrm{d}x\,\mathrm{d}t+C\frac{2^{np}}{R^p}2^{(2-p)m_2}\iint_{\mathcal{Q}_n}(u-k_n)_-^2\,\mathrm{d}x\,\mathrm{d}t\ .$$

Since

$$(u-k_n)_-\le\sup_{\mathcal{Q}_n}(u-k_n)_-\le\frac{\omega}{2^{m+1}}+\frac{\omega}{2^{m+1+n}}\le\frac{\omega}{2^m}$$

and

$$\left(\frac{\omega}{2^{m_1}}\right)^{2-p}\left(\frac{\omega}{2^m}\right)^p=\left(\frac{\omega}{2^m}\right)^2 2^{(2-p)m_2}$$

the two terms on the right hand side of the inequality are estimated above by

$$C\frac{2^{np}}{R^p}2^{(2-p)m_2}\left(\frac{\omega}{2^m}\right)^2\iint_{\mathcal{Q}_n}\chi_{\{(u-k_n)_->0\}}\,\mathrm{d}x\,\mathrm{d}t\ .$$

To estimate below the two integrals on the left hand side, we introduce the level

$$\overline{k_n}\equiv\frac{k_n+k_{n+1}}{2}\in(k_{n+1},k_n)\ .$$

Then, for all $t\in\left(-2^{(p-2)m_2}R_n^p,0\right)$,

$$\begin{aligned}\int_{K_n}(u-k_n)_-^2\,\xi_n^p\,\mathrm{d}x\ &\geq\ \int_{K_n}(k_n-\overline{k_n})^{2-p}(u-\overline{k_n})_-^p\,\xi_n^p\,\mathrm{d}x\\ &=\ \left(\frac{\omega}{2^m}\right)^{2-p}2^{(n+3)(p-2)}\int_{K_n}(u-\overline{k_n})_-^p\,\xi_n^p\,\mathrm{d}x\ ,\end{aligned}$$

and, since $k_n>\overline{k_n}$,

$$\iint_{\mathcal{Q}_n}|\nabla(u-k_n)_-\xi_n|^p\,\mathrm{d}x\,\mathrm{d}t\geq\int\int_{\mathcal{Q}_n}|\nabla(u-\overline{k_n})_-\xi_n|^p\,\mathrm{d}x\,\mathrm{d}t\ .$$

Combining these estimates and dividing through by

$$\left(\frac{\omega}{2^m}\right)^{2-p}2^{(n+3)(p-2)}$$

we obtain

$$\begin{gathered}\sup_{-2^{(p-2)m_2}R_n^p<t<0}\int_{K_n}(u-\overline{k_n})_-^p\xi_n^p\,\mathrm{d}x+\left(\frac{\omega}{2^m}\right)^{p-2}2^{(n+3)(2-p)}\iint_{\mathcal{Q}_n}|\nabla(u-\overline{k_n})_-\xi_n|^p\,\mathrm{d}x\,\mathrm{d}t\\ \leq C\,\frac{2^{2n}}{R^p}\,2^{(2-p)m_2}\left(\frac{\omega}{2^m}\right)^p\iint_{\mathcal{Q}_n}\chi_{\{(u-k_n)_->0\}}\,\mathrm{d}x\,\mathrm{d}t\end{gathered}$$

Next consider the change of variables

$$y=d_1^{-1}x\,,\qquad z=2^{(2-p)m_2}t$$

which maps the cylinder $\mathcal{Q}_n$ into the cylinder $Q_n\equiv K_{R_n}\times(-R_n^p,0)$, and define new functions

$$v(y,z)=u\left(d_1y,2^{(p-2)m_2}z\right)\ ,\qquad \widehat{\xi_n}(y,z)=\xi_n\left(d_1y,2^{(p-2)m_2}z\right)$$

and the sets

$$A_n(z) \equiv \{y \in K_{R_n} : v(y,z) < k_n\} \ , \qquad \text{with} \quad |A_n| \equiv \int_{-R_n^p}^0 |A_n(z)| \, \mathrm{d}z \ .$$

Since $1 < p < 2$ and $\omega < 1$, the coefficient

$$\left(\frac{\omega}{2^m}\right)^{p-2} 2^{(n+3)(2-p)+m_2(p-2)} = \omega^{p-2} 2^{(2-p)(m_1+n+3)} > 1,$$

and we obtain

$$\left\| (v - \overline{k_n})_- \widehat{\xi_n} \right\|_{V^p(Q_n)}^p \leq C \, \frac{2^{2n}}{R^p} \left(\frac{\omega}{2^m}\right)^p |A_n| \ .$$

Noting that $k_{n+1} < \overline{k_n}$, by the imbedding theorem 2.6,

$$\begin{aligned}
2^{-(n+3)p} \left(\frac{\omega}{2^m}\right)^p |A_{n+1}| &= \left(\overline{k_n} - k_{n+1}\right)^p |A_{n+1}| \\
&= \iint_{Q_{n+1}} \left(\overline{k_n} - k_{n+1}\right)^p \chi_{\{v < k_{n+1}\}} \, \mathrm{d}y \, \mathrm{d}z \\
&\leq \iint_{Q_{n+1}} \left(v - \overline{k_n}\right)_-^p \, \mathrm{d}y \, \mathrm{d}z \\
&\leq \left\| \left(v - \overline{k_n}\right)_- \widehat{\xi_n} \right\|_{p,Q_n}^p \\
&\leq C \, |A_n|^{\frac{p}{N+p}} \left\| \left(v - \overline{k_n}\right)_- \widehat{\xi_n} \right\|_{V^p(Q_n)}^p \\
&\leq C \, \frac{2^{2n}}{R^p} \left(\frac{\omega}{2^m}\right)^p |A_n|^{1+\frac{p}{N+p}} \ .
\end{aligned}$$

Thus, setting

$$Y_n = \frac{|A_n|}{|Q_n|}$$

we obtain the recursive relation

$$Y_{n+1} \leq C \, 4^{np} \, Y_n^{1+\frac{p}{N+p}} \ .$$

The result now follows from Lemma 2.5. In fact if

$$Y_0 = \frac{|A_0|}{|Q_0|} = \frac{|v < k_0|}{|Q_0|} = \frac{|u < \mu^- + \frac{\omega}{2^m}|}{|\mathcal{Q}_R(m_1, m_2)|} \leq C^{\frac{-1}{\alpha}} (4^p)^{\frac{-1}{\alpha^2}} \equiv \nu_0 \ , \tag{2.68}$$

where $\alpha = \frac{p}{N+p}$, the lemma guarantees that $Y_n \to 0$ when $n \to \infty$. But this is nothing but the conclusion of the Lemma and (2.68) is precisely the hypothesis.

■

Remark 2.30. The proof shows that ν_0 depends upon p, but it is stable as $p \nearrow 2$ in the sense that $\nu_0(p) \to \nu_0(2)$ when $p \nearrow 2$.

Remark 2.31. The conclusion of Lemma 2.29 continues to hold for cylinders of the type

$$Q_R(m,\beta) \equiv K_r \times (-\beta R^p, 0) \ , \qquad r = \left(\frac{\omega}{2^m}\right)^{\frac{p-2}{p}} R\,, \quad \beta > 0 \ ,$$

provided β is independent of ω and R. In such a case we take $m_1 = m$ and ν_0 will also depend upon β.

2.4.2. *The first alternative.* Assume that there exists a cylinder of the type $(\overline{x}, 0) + Q(R^p, d_0 R)$, making up the partition of $Q(R^p, c_0 R)$, for which (2.65) holds. Applying Lemma 2.29 with $m_1 = 1$ and $m_2 = 0$ we conclude that

$$u(x,t) \ge \mu_- + \frac{\omega}{4}\,, \quad \forall\,(x,t) \in (\overline{x}, 0) + Q\left(\left(\frac{R}{2}\right)^p, d_0\frac{R}{2}\right) \ . \tag{2.69}$$

We view the box $(\overline{x}, 0) + Q((\frac{R}{2})^p, d_0\frac{R}{2})$ as a block inside $Q(R^p, c_0 R)$. The location of $\overline{x}$ in the cube $K_{\mathcal{R}(\omega)}$, where $\mathcal{R}(\omega)$ is defined by (2.64), is only known qualitatively. However, the *positivity* of u as stated in (2.69) *spreads* over the full cube $K_{c_o R}$, for all times

$$-\left(\frac{R}{8}\right)^p \le t \le 0 \ .$$

More precisely, we will prove

Proposition 2.32. *Assume that (2.69) holds for some* $\overline{x} \in K_{\mathcal{R}(\omega)}$*. There exists a positive number* s_1 *that can be determined a priori only in terms of the data and the number* λ *in the definition (2.60) of* $Q(R^p, c_0 R)$*, such that*

$$u(x,t) \ge \mu_- + \frac{\omega}{2^{s_1+1}}\,, \quad \forall\,(x,t) \in Q\left(\left(\frac{R}{8}\right)^p, c_0 R\right) \ . \tag{2.70}$$

As a consequence we may rephrase the first alternative in the following way

Corollary 2.33. *Assume (2.65) holds for some cylinder of the type* $(\overline{x}, 0) + Q(R^p, d_0 R)$ *making up the partition of* $Q(R^p, c_0 R)$*. There exists a positive* *number s_1 that can be determined a priori only in terms of the data and the number λ in the definition (2.60) of $Q(R^p, c_0R)$, such that*

$$\operatorname*{ess\,osc}_{Q(\rho^p,c_0\rho)} u \le \eta_1\omega\,, \quad \forall\,\rho \in \left(0, \frac{R}{8}\right) \tag{2.71}$$

where $\eta_1 \equiv 1 - \dfrac{1}{2^{s_1+1}}$.

To prove Proposition 2.32 we regard $\overline{x}$ as the centre of a larger cube $\overline{x} + K_{8c_0R}$ which we may assume to be contained in $K_{R^{\frac{p}{2}}}$. Otherwise we would have

$$16\,c_0\,R \ge R^{\frac{p}{2}} \;\Rightarrow\; \omega \le 16^{\frac{p}{2-p}}\,2^\lambda\,R^{\frac{p}{2}}\ .$$

We work within the box

$$(\overline{x}, 0) + Q\left(\left(\frac{R}{2}\right)^p, 8c_0R\right)$$

and show that the conclusion of Proposition 2.32 holds within the cylinder

$$(\overline{x}, 0) + Q\left(\left(\frac{R}{8}\right)^p, 2c_0R\right)\ .$$

This contains $Q\left((\frac{R}{8})^p, c_0R\right)$, regardless of the location of $\overline{x}$ in the cube $K_{\mathcal{R}(\omega)}$.

The proof begins by introducing of the change of variables

$$x \mapsto \frac{x-\overline{x}}{2c_0R}\ , \qquad t \mapsto \frac{4^pt}{(\frac{R}{2})^p}\ ,$$

which maps $(\overline{x}, 0) + Q\left((\frac{R}{2})^p, 8c_0R\right)$ into $Q_4 \equiv K_4 \times (-4^p, 0)$, and the new unknown function

$$v = (u - \mu^-)\frac{2}{\omega}\ . \tag{2.72}$$

Denoting again with x and t the new variables, the function v satisfies the PDE

$$v_t - c\,\mathrm{div}\,|\nabla v|^{p-2}\nabla v = 0 \quad \text{in} \quad \mathcal{D}'(Q_4)\ , \tag{2.73}$$

where

$$c = \frac{1}{2^{4p}}\left(\frac{2}{2^\lambda}\right)^{2-p} = 2^{(\lambda-1)(p-2)-4p}\ .$$

The information (2.69) now translates into

$$v(x,t) \ge \frac{1}{2} \qquad \text{a.e.} \quad (x,t) \in Q(h_0) \equiv \{x : |x| < h_0\} \times (-4^p, 0) \tag{2.74}$$

where

$$h_0 = \frac{d_0}{4c_0} = \frac{1}{4}\left(\frac{2}{2^\lambda}\right)^{\frac{2-p}{p}} = 2^{\frac{(\lambda-1)(p-2)}{p}-2} < 1. \tag{2.75}$$

We regard $Q(h_0)$ as a thin cylinder sitting at the centre of Q_4. We prove that the relative largeness of v in $Q(h_0)$ spreads sidewise over $Q_2 \equiv K_2 \times (-2^p, 0)$, thus obtaining the desired result. Indeed, we want to show that

$$u(x,t) \geq \mu_- + \frac{\omega}{2^{1+s_1}}\,, \quad (x,t) \in (\overline{x}, 0) + Q\left(\left(\frac{R}{8}\right)^p, 2c_0 R\right)$$

which, according to the change of variables and the new function, is the same as

$$v(x,t) \geq \frac{1}{2^{s_1}}\,, \quad (x,t) \in Q_1 \equiv K_1 \times (-1, 0) \subset Q_2\ .$$

Proposition 2.32 will then be a consequence of the following

Lemma 2.34. *For every $\nu \in (0,1)$ there exists a positive number $\delta^* \in (0,1)$, that can be determined a priori only in terms of ν, N, p and the data, such that*

$$|\{x \in K_2 : v(x,t) \leq \delta^*\}| \leq \nu\, |K_2|\ , \tag{2.76}$$

for all time levels $t \in [-2^p, 0]$.

Remark 2.35. The key feature of the lemma is that the set where v is small can be made arbitrarily small for *every* time level in $[-2^p, 0]$.

The proof of this lemma is rather technical, involving the manipulation of appropriate integral inequalities, and will be omitted; the interested reader is referred to the book [55, Chap. IV – §6-9] for a detailed proof.

Proof of Proposition 2.32 assuming Lemma 2.34. Let ν_0 be the number claimed by Lemma 2.29, take $\nu = \nu_0$ in Lemma 2.34 and determine the corresponding $\delta^* = \delta^*(\nu_0)$. Let m_2 be defined by

$$2^{-m_2} = \delta^*(\nu_0)$$

and apply lemma 2.29 with $\mu_- = 0$, $\omega = 1$, $m_1 = 0$, $R = 2$, over the boxes

$$(0, \overline{t}) + K_2 \times \left(-2^{m_2(p-2)}2^p, 0\right) \equiv (0, \overline{t}) + \mathcal{Q}_2(0, m_2)$$

as long as they are contained in Q_2, i.e., for $\overline{t}$ satisfying

$$2^{m_2(p-2)}2^p - 2^p \leq \overline{t} \leq 0\ .$$

Since (2.76) holds true for *all* time levels in $t \in [-2^p, 0]$, each such box satisfies

$$\left|\left\{(x,t) \in (0,\bar{t}) + \mathcal{Q}_2(0,m_2) : v(x,t) \le 2^{-m_2}\right\}\right| \le \nu_0 \left|\mathcal{Q}_2(0,m_2)\right| .$$

Therefore, by Lemma 2.29,

$$v(x,t) \ge 2^{-(m_2+1)} , \qquad \forall\, (x,t) \in (0,\bar{t}) + \mathcal{Q}_1(0,m_2) ,$$

for all $\bar{t} \in \left(2^{m_2(p-2)}2^p - 2^p, 0\right)$. Since

$$\left(2^{m_2(p-2)}2^p - 2^p - 2^{m_2(p-2)}, 0\right) \supset (-1,0)$$

we conclude that

$$v(x,t) \ge 2^{-(m_2+1)} , \qquad \forall\, (x,t) \in Q_1 .$$

Returning to the original coordinates and redefining the various constants accordingly, we arrive at

$$u(x,t) \ge \mu_- + \frac{\omega}{2^{m_2+2}} , \qquad \forall\, (x,t) \in (\bar{x},0) + Q\left(\left(\frac{R}{8}\right)^p, 2c_0R\right)$$

and Proposition 2.32 follows with

$$s_1 = m_2 + 1 , \qquad m_2 = -\log_2\left(\delta^*(\nu_0)\right) , \qquad \nu_0 = \left(C4^{N+p}\right)^{-\frac{N+p}{p}} .$$

■

Remark 2.36. We comment further on the expansion of positivity of Proposition 2.32. A crucial point in the proof of Lemma 2.34 (which was omitted) is the use of the information contained in (2.74)–(2.75) to apply a Poincaré inequality. But for this it is not truly necessary to know that the set $\{v \ge \frac{1}{2}\}$ is concentrated in a cylinder centered at the origin; it suffices to have the following information:

$$\exists\, \alpha_0, k_0 > 0 \ : \ \left|\{x \in K_2 : v > k_0\}\right| \ge \alpha_0 , \qquad \forall\, t \in (-4^p, 0) .$$

2.4.3. *The second alternative.* We will omit most of the proofs in this section; for the details see [55].

Assume that (2.66) holds for all cylinders $(\bar{x},0) + Q(R^p, d_0R)$, making up the partition of $Q(R^p, c_0R)$. Since

$$\mu_+ - \frac{\omega}{2} = \mu_- + \frac{\omega}{2} ,$$

we can rephrase (2.66) as

$$\frac{\left|\left\{(x,t) \in (\overline{x},0) + Q(R^p, d_0 R) \ : \ u(x,t) > \mu_+ - \frac{\omega}{2}\right\}\right|}{|Q(R^p, d_0 R)|} \le 1 - \nu_0 \tag{2.77}$$

for *all* boxes $(\overline{x},0) + Q(R^p, d_0 R)$ making up the partition of $Q(R^p, c_0 R)$.

Let n be a positive number to be chosen and arrange that $2^{n\frac{2-p}{p}}$ is an integer. Then we combine $2^{\frac{n(2-p)}{p}N}$ of these cylinders to form boxes congruent to

$$Q(R^p, d_* R) \equiv K_{d_* R} \times (-R^p, 0), \qquad d_* = \left(\frac{\omega}{2^{n+1}}\right)^{\frac{p-2}{p}} = d_0 \left(2^n\right)^{\frac{2-p}{p}} \ . \tag{2.78}$$

We next consider cylinders of the type $(\tilde{x},0) + Q(R^p, d_* R)$. These are contained in $Q(R^p, c_0 R)$ if the abscissa $\tilde{x}$ of their vertices ranges over the cube $K_{\mathcal{R}_1(\omega)}$, where

$$\begin{aligned}
\mathcal{R}_1(\omega) &= \left\{2^{\lambda(2-p)} - 2^{(n+1)\frac{2-p}{p}}\right\} \omega^{\frac{p-2}{p}} R \\
&= \left\{\left(2^{\lambda-(n+1)}\right)^{\frac{2-p}{p}} - 1\right\} \left(\frac{\omega}{2^{n+1}}\right)^{\frac{p-2}{p}} R \\
&= L_1 \, d_* \, R \ , \qquad \text{where} \qquad L_1 \equiv \left(2^{\lambda-(n+1)}\right)^{\frac{2-p}{p}} - 1 \ .
\end{aligned}$$

We will take $\lambda > n+1$ and arrange that L_1 is an integer. Then we regard $Q(R^p, c_0 R)$ as the union, up to a set of measure zero, of L_1^N pairwise disjoint boxes each congruent to $Q(R^p, d_* R)$. Since each box $(\tilde{x},0)+Q(R^p, d_* R)$ is the pairwise disjoint union of boxes $(\overline{x},0) + Q(R^p, d_0 R)$, each of them satisfying (2.77), we can rephrase again (2.66), this time as

$$\frac{\left|\left\{(x,t) \in (\tilde{x},0) + Q(R^p, d_* R) \ : \ u(x,t) > \mu_+ - \frac{\omega}{2}\right\}\right|}{|Q(R^p, d_* R)|} \le 1 - \nu_0 \tag{2.79}$$

for all cylinders $(\tilde{x},0) + Q(R^p, d_* R)$ making up the partition of $Q(R^p, c_0 R)$.

Remark 2.37. The need of introducing a larger cylinder than the one involved in (2.66) is justified by the appearance of the factor $2^{n(2-p)}$ in the logarithmic estimates (2.6) employed in the proofs. The use of a geometry in which the space dimensions are stretched by this factor accommodates the singularity and restores the homogeneity in (2.5)–(2.6).

Lemma 2.38. *Let $(\tilde{x}, 0) + Q(R^p, d_* R)$ be any box contained in $Q(R^p, c_0 R)$ and satisfying (2.79). There exists a time level*

$$t^* \in \left(-R^p, -\frac{\nu_0}{2} R^p\right) ,$$

such that, for all $s \geq 2$,

$$\left| x \in \tilde{x} + K_{d_* R} \,:\, u(x, t^*) > \mu_+ - \frac{\omega}{2^s} \right| \leq \left(\frac{1 - \nu_0}{1 - \nu_0/2} \right) |K_{d_* R}| \ . \tag{2.80}$$

Proof. If not,

$$\left| x \in \tilde{x} + K_{d_* R} \,:\, u(x, t^*) > \mu_+ - \frac{\omega}{2^s} \right| > \left(\frac{1 - \nu_0}{1 - \nu_0/2} \right) |K_{d_* R}|$$

for all $t \in \left(-R^p, -\frac{\nu_0}{2} R^p\right)$. Then

$$\begin{aligned} &\left| (x,t) \in (\tilde{x}, 0) + Q(R^p, d_* R) \,:\, u(x,t) > \mu_+ - \frac{\omega}{2} \right| \\ &\quad = \int_{-R^p}^{0} \left| x \in \tilde{x} + K_{d_* R} \,:\, u(x,t) > \mu_+ - \frac{\omega}{2} \right| \, \mathrm{d}t \\ &\quad \geq \int_{-R^p}^{-\frac{\nu_0}{2} R^p} \left| x \in \tilde{x} + K_{d_* R} \,:\, u(x,t) > \mu_+ - \frac{\omega}{2^s} \right| \, \mathrm{d}t \\ &\quad > \int_{-R^p}^{-\frac{\nu_0}{2} R^p} \left(\frac{1 - \nu_0}{1 - \nu_0/2} \right) |K_{d_* R}| \, \mathrm{d}t \\ &\quad = (1 - \nu_0) \, |Q(R^p, d_* R)| \end{aligned}$$

which contradicts (2.79). ∎

The next lemma asserts that a property similar to (2.80) still holds for all time levels from t^* up to 0. The proof of the lemma, that we omit, will also determine the number n.

Lemma 2.39. *There exists a positive integer n such that for all $t^* < t < 0$,*

$$\left| x \in \tilde{x} + K_{d_* R} \,:\, u(x,t) > \mu_+ - \frac{\omega}{2^{n+1}} \right| \leq \left(1 - \left(\frac{\nu_0}{2} \right)^2 \right) \, |K_{d_* R}| \ . \tag{2.81}$$

The information of Lemma 2.39 will be exploited to show that in a small cylinder about $(0,0)$, the solution u is strictly bounded above by

$$\mu_+ - \frac{\omega}{2^m}\ , \qquad \text{for some} \quad m > n+1\ .$$

The process also determines the number λ that defines the size of $Q(R^p, c_0 R)$. To make this quantitative consider the box

$$Q(\beta R^p, c_0 R)\ , \qquad \beta = \frac{\nu_0}{2}\ , \qquad c_0 = \left(\frac{\omega}{2^\lambda}\right)^{\frac{p-2}{p}}\ .$$

We view $Q(\beta R^p, c_0 R)$ as being partitioned into sub-boxes $(\tilde{x}, 0) + Q(\beta R^p, d_* R)$ where $\tilde{x}$ takes finitely many points within the cube $K_{\mathcal{R}_1(\omega)}$. For each of these cylinders Lemma 2.39 holds.

Lemma 2.40. *For every $\nu \in (0,1)$ there exists a number m depending only on the data and independent of ω and R such that, for all cylinders $(\tilde{x}, 0) + Q(\beta R^p, d_* R)$ making up the partition of $Q(\beta R^p, c_0 R)$,*

$$\left| (x,t) \in (\tilde{x}, 0) + Q(\beta R^p, d_* R)\ :\ u(x,t) > \mu_+ - \frac{\omega}{2^m} \right| \le \nu \left| Q(\beta R^p, d_* R) \right|\ . \tag{2.82}$$

Remark 2.41. The proof shows that m must be chosen so that

$$m \ge n + 1 + \frac{C}{\nu^{\frac{p}{p-1}}}\ .$$

This estimate deteriorates as $p \searrow 1$, *i.e*, $m \nearrow \infty$ as $p \searrow 1$. Nevertheless the choice of m is stable as $p \nearrow 2$.

To proceed we return to the box $Q(\beta R^p, c_0 R)$ and recall that it is the finite union, up to a set of measure zero, of pairwise disjoint boxes $(\tilde{x}, 0) + Q(\beta R^p, d_* R)$. Therefore lemma 2.40 implies

Corollary 2.42. *For every $\nu \in (0,1)$ there exists a number m depending only upon the data and independent of ω and R such that*

$$\left| (x,t) \in Q(\beta R^p, c_0 R)\ :\ u(x,t) > \mu_+ - \frac{\omega}{2^m} \right| \le \nu \left| Q(\beta R^p, c_0 R) \right|\ . \tag{2.83}$$

We finally determine the size of the cylinder $Q(\beta R^p, c_0 R)$ and consequently the number λ. First, in Corollary 2.42, take $\nu = \nu_0$ and determine m accordingly. Then let m_2 be given by

$$\beta = \frac{\nu_0}{2} = 2^{m_2(p-2)} \tag{2.84}$$

and assume that $m \geq m_2$ (if necessary take m even larger). Determine λ from

$$\lambda = m_1 \qquad \text{and} \qquad m = m_1 + m_2 \ . \tag{2.85}$$

With these choices, the cylinder $Q(\beta R^p, c_0 R)$ coincides with the cylinder $\mathcal{Q}_R(m_1, m_2)$ introduced in (2.67). By Corollary 2.42, we have

$$\left|(x,t) \in \mathcal{Q}_R(m_1,m_2) \ : \ u(x,t) > \mu_+ - \frac{\omega}{2^m}\right| \leq \nu_0 \, |\mathcal{Q}_R(m_1,m_2)|$$

which implies, using Lemma 2.29,

$$u(x,t) \leq \mu_+ - \frac{\omega}{2^{m+1}} \ , \qquad \forall \, (x,t) \in \mathcal{Q}_{\frac{R}{2}}(m_1,m_2) \ .$$

We summarize:

Proposition 2.43. *Assume that (2.66) holds true for all cylinders* $(\overline{x}, 0) + Q(R^p, d_0 R)$ *making up the partition of* $Q(R^p, c_0 R)$. *Then, for all* $0 < \rho < \frac{R}{2}$

$$\mathop{\mathit{ess\ osc}}_{Q(\beta\rho^p, c_0\rho)} u \leq \eta_0 \, \omega \ , \qquad \mathit{where} \quad \eta_0 \equiv 1 - \frac{1}{2^{m+1}}. \tag{2.86}$$

2.4.4. *Proof of the main proposition.* The proof of Proposition 2.27 follows by combining the two alternatives. The conclusion of the first alternative is that

$$\mathop{\text{ess osc}}_{Q(\rho^p, c_0\rho)} u \leq \eta_1 \, \omega \ , \qquad \forall \, \rho \in \left(0, \frac{R}{8}\right) \ , \tag{2.87}$$

where $\eta_1 = 1 - \frac{1}{2^{m_2+2}}$. The conclusion of the second alternative is that

$$\mathop{\text{ess osc}}_{Q(\beta\rho^p, c_0\rho)} u \leq \eta_0 \, \omega \ , \qquad \forall \, \rho \in \left(0, \frac{R}{2}\right) \ , \tag{2.88}$$

where $\eta_0 = 1 - \frac{1}{2^{m+1}}$.

Set

$$\eta = \max\{\eta_0, \eta_1\} \qquad \text{and} \qquad C = \frac{\beta^{\frac{1}{p}}}{4} = \left(\frac{\nu_0}{2^{2p+1}}\right)^{\frac{1}{p}} \ .$$

Observe that, assuming $\nu_0 \le \frac{1}{2}$,

$$C \le \frac{1}{2^{2+\frac{2}{p}}} < \frac{1}{8} < \frac{1}{2} \qquad \text{and} \qquad C < \frac{\beta^{\frac{1}{p}}}{2} \ .$$

Define

$$R_1 = C\,R \ ; \qquad \omega_1 = \eta\,\omega$$

and the cylinder

$$Q_1 = Q(R_1^p, c_1 R_1) \ , \qquad c_1 = \left(\frac{\omega_1}{2^\lambda}\right)^{\frac{p-2}{p}} \ .$$

Since $\eta < 1$,

$$c_1 R_1 = \left(\frac{\omega_1}{2^\lambda}\right)^{\frac{p-2}{p}} R_1 = \left(\frac{\omega}{2^\lambda}\right)^{\frac{p-2}{p}} \eta^{\frac{p-2}{p}} R_1 \le c_0\, R_1 \ .$$

Therefore, combining both alternatives,

$$\operatorname*{ess\,osc}_{Q_1} \ u \le \omega_1 \ .$$

The process can now be repeated inductively starting from such relation. This yields the Hölder continuity of u as in the degenerate case $p > 2$.

2.5. The porous medium equation and other generalisations. As indicated earlier, the Hölder continuity of u is solely a consequence of the Caccioppoli inequalities (2.5) and the logarithmic inequalities (2.6). For this reason the techniques just presented are rather flexible and adjust to a variety of singular and degenerate parabolic partial differential equations. The first generalisation we want to mention is to equations with the full p-Laplacian type quasilinear structure

$$u_t - \operatorname{div} \mathbf{a}(x,t,u,\nabla u) = b(x,t,u,\nabla u) \qquad \text{in} \quad \mathcal{D}'(\Omega_T) \ , \tag{2.89}$$

where $\mathbf{a} : \Omega_T \times \mathbf{R}^{N+1} \to \mathbf{R}^N$ and $b : \Omega_T \times \mathbf{R}^{N+1} \to \mathbf{R}$ are measurable and satisfy the structure assumptions

($\mathbf{A}_1$): $\quad \mathbf{a}(x,t,u,\nabla u) \cdot \nabla u \ge C_0 |\nabla u|^p - \varphi_0(x,t)$;
($\mathbf{A}_2$): $\quad |\mathbf{a}(x,t,u,\nabla u)| \ge C_1 |\nabla u|^{p-1} + \varphi_1(x,t)$;
($\mathbf{A}_3$): $\quad |b(x,t,u,\nabla u)| \ge C_2 |\nabla u|^p + \varphi_2(x,t)$,

for $p>1$ and a.e. $(x,t)\in\Omega_T$. The C_i, $i=0,1,2$, are given positive constants and the φ_i, $i=0,1,2$, are given non-negative functions, defined in Ω_T and subject to the integrability conditions

$$\varphi_0\ ,\quad \varphi_1^{\frac{p}{p-1}}\ ,\quad \varphi_2\ \in\ L^{q,r}(\Omega_T)$$

with $q,r\geq 1$ satisfying

$$\frac{1}{r}+\frac{N}{pq}\in(0,1)\qquad\qquad (1<p\leq N)\ .$$

See [55] for the details.

Another family of equations to which the theory applies are degenerate or singular equations of porous medium type that can be cast in the form (2.89), for the structure assumptions

(B$_1$): $\quad \mathbf{a}(x,t,u,\nabla u)\cdot\nabla u\geq C_0|u|^{m-1}|\nabla u|^2-\varphi_0(x,t),\qquad m>0;$
(B$_2$): $\quad |\mathbf{a}(x,t,u,\nabla u)|\geq C_1|u|^{m-1}|\nabla u|+\varphi_1(x,t);$
(B$_3$): $\quad |b(x,t,u,\nabla u)|\geq C_2\left|\nabla|u|^m\right|^2+\varphi_2(x,t),$

and the functions φ_i, $i=0,1,2$, satisfy the same conditions as before with $p=2$. We require

$$u\in L^\infty_{\rm loc}\big(0,T;L^2_{\rm loc}(\Omega)\big)\qquad\text{and}\qquad |u|^m\in L^2_{\rm loc}\big(0,T;W^{1,2}_{\rm loc}(\Omega)\big)\ .$$

There is a wide literature concerning this problem. We refer the reader to the Proceedings [12], [29], [82], [94], as well as the references therein.

Further generalisations can be obtained by replacing s^{m-1}, $s>0$ with a function that blows up like a power when $s\searrow 0$ and is regular otherwise. To be specific, consider doubly degenerate equations of the form (2.89) with structure assumptions

(C$_1$): $\quad \mathbf{a}(x,t,u,\nabla u)\cdot\nabla u\geq C_0\Phi(|u|)|\nabla u|^p-\varphi_0(x,t);$
(C$_2$): $\quad |\mathbf{a}(x,t,u,\nabla u)|\geq C_1\Phi(|u|)|\nabla u|^{p-1}+\Phi^{\frac{1}{p}}(u)\varphi_1(x,t);$
(C$_3$): $\quad |b(x,t,u,\nabla u)|\geq C_2\Phi(|u|)|\nabla u|^p+\varphi_2(x,t).$

Here φ_i, $i=0,1,2$, satisfy the same conditions as before and the function $\Phi(\cdot)$ is degenerate near the origin in the sense that

$$\exists\,\sigma>0\ :\ \gamma_1 s^{\beta_1}\leq\Phi(s)\leq\gamma_2 s^{\beta_2}\ ,\qquad \forall\,s\in(0,\sigma)\ ,$$

for given constants $0<\gamma_1\leq\gamma_2$ and $0\leq\beta_2\leq\beta_1$. For $s>\sigma$, *i.e.*, away from zero it is assumed that Φ is bounded above and below by given positive constants. We require that

$$u \in C_{\mathrm{loc}}\left(0,T;L^2_{\mathrm{loc}}(\Omega)\right) \qquad \text{and} \qquad \Phi^{\frac{1}{p-1}}(u)|\nabla u| \in L^p_{\mathrm{loc}}(\Omega_T)$$

and, denoting with $F(\cdot)$ the primitive of $\Phi^{\frac{1}{p-1}}(\cdot)$, that

$$F(u) \in L^p_{\mathrm{loc}}\left(0,T;W^{1,p}_{\mathrm{loc}}(\Omega)\right) \ ,$$

which allows for an interpretation of the equation in the weak sense. One recognizes that if $\Phi(s) \equiv 1$ the equation is of p-Laplacian type and if $\Phi(s) = s^{m-1}$ and $p = 2$ the equation is of porous medium type. The Hölder continuity of solutions was obtained independently in [147] and [96].

3. Boundedness of weak solutions

The regularity theorems of the previous section apply to <u>bounded</u> weak solutions of (2.1). The theory of local boundedness discriminates between the degenerate case $p > 2$ and the singular case $1 < p < 2$. If $p > 2$, a local bound for the solution is implicit in the notion of weak solution. If $1 < p < 2$, local or global solutions need not be bounded in general. Another substantial difference between the two cases surfaces when studying the Dirichlet problem

$$\begin{cases} u \in C\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;W^{1,p}(\Omega)\right) \equiv V^{2,p}\left(\Omega_T\right) \\ u_t - \operatorname{div}|\nabla u|^{p-2}\nabla u = 0 \quad \text{in} \quad \Omega_T \\ u(\cdot,t)\mid_{\partial\Omega} = g(\cdot,t) \quad \text{traces of functions in } \ V^{2,p}\left(\Omega_T\right) \\ u(\cdot,0) = u_0 \quad \text{in the sense of } L^2(\Omega) \ , \end{cases} \tag{3.1}$$

or the Cauchy problem

$$\begin{cases} u \in C\left(0,T;L^1_{loc}(\mathbf{R}^N)\right) \cap L^2\left(0,T;W^{1,p}_{loc}(\mathbf{R}^N)\right) \\ u_t - \operatorname{div}|\nabla u|^{p-2}\nabla u = 0 \quad \text{in} \quad \Sigma_T \equiv \mathbf{R}^N \times (0,T) \\ u(\cdot,0) = u_o \quad \text{in the sense of } L^1_{loc}(\mathbf{R}^N) \ . \end{cases} \tag{3.2}$$

The following weak maximum principle is common to both cases.

Theorem 3.1. *Let $p > 1$ and let u be a weak solution of (3.1) and assume that $g \in L^\infty(\partial\Omega \times (0,T))$ and $u_0 \in L^\infty(\Omega)$. Then*

$$\operatorname*{ess\,sup}_{\Omega_T} |u| \leq \max\left\{\operatorname*{ess\,sup}_{\Omega} |u_0| \,;\, \operatorname*{ess\,sup}_{\partial\Omega\times(0,T)} |g|\right\} .$$

Theorem 3.2. *Let $p > 1$ and let u be a weak solution of (3.2). Then if $u_0 \in L^\infty(\mathbf{R}^N)$,*

$$\operatorname*{ess\,sup}_{\Sigma_T} |u| \leq \operatorname*{ess\,sup}_{\mathbf{R}^N} |u_0| .$$

In the next sections we let u be a non–negative weak subsolution of (2.1) and will state several upper bounds for it. The assumption that u is non–negative is not essential and is used here only to deduce that u is locally or globally bounded. If u is a subsolution, not necessarily bounded below, our results supply *a priori* bounds *above* for u. Analogous statements hold for non–positive local supersolutions and in particular for solutions.

3.1. The degenerate case $p > 2$.

Theorem 3.3. *Let $p > 2$. Every non–negative, local weak subsolution u of (2.1) in Ω_T is locally bounded in Ω_T. Moreover for all $\varepsilon \in (0,2]$, there exists a constant γ depending only upon N, p, Λ and ε, such that $\forall\,(x_0,t_0)+Q(\tau,\rho) \subset \Omega_T$ and $\forall\,\sigma \in (0,1)$,*

$$\sup_{(x_0,t_0)+Q(\sigma\tau,\sigma\rho)} u \leq \frac{\gamma\,(\tau/\rho^p)^{1/\varepsilon}}{(1-\sigma)^{\frac{N+p}{\varepsilon}}} \left(\iint_{(x_0,t_0)+Q(\tau,\rho)} u^{p-2+\varepsilon} \mathrm{d}x\mathrm{d}t\right)^{1/\varepsilon} \wedge \left(\frac{\rho^p}{\tau}\right)^{\frac{1}{p-2}} .$$

Remark 3.4. *In the linear case $p = 2$ and $\tau = \rho^2$ such an estimate holds for any positive number ε (see* [134]*). In our case ε is restricted in the range* $(0,2]$.

It is of interest to have sup–estimates that involve “low” integral norms of the solution. The next theorem is a result in this direction.

Theorem 3.5. *Let $p > 2$ and let u be a non–negative, local subsolution of (2.1) in Ω_T. There exists a constant $\gamma = \gamma(data)$, such that $\forall\,(x_0,t_0) +$*

$Q(\tau,\rho)\subset\Omega_T$ and $\forall\,\sigma\in(0,1)$,

$$\sup_{(x_0,t_0)+Q(\sigma\tau,\sigma\rho)} u \le \frac{\gamma\sqrt{\tau/\rho^p}}{(1-\sigma)^{\frac{N(p+1)+p}{2}}}\left(\sup_{t_0-\tau<t<t_0}\int_{B_\rho(x_0)} u(x,t)\,\mathrm{d}x\right)^{p/2}\wedge\left(\frac{\rho^p}{\tau}\right)^{\frac{1}{p-2}} .$$

3.1.1. *Global estimates for solutions of the Dirichlet problem.* Consider a non–negative weak subsolution of the Dirichlet problem (3.1) and let $p>2$. If the boundary data are bounded then the weak maximum principle of Theorem 3.1 holds true. If however u_0^+ is not bounded, it is of interest to investigate how the supremum of u behaves when $t\to 0$.

Theorem 3.6. *Let u be a non–negative weak subsolution of the Dirichlet problem (3.1). There exists a constant $\gamma=\gamma(data)$, such that $\forall t\in(0,T)$,*

$$\sup_\Omega u(\cdot,t)\le\sup_{S_T} g+\frac{\gamma}{t^{N/\lambda}}\left(\int_0^t\!\!\int_\Omega u\,\mathrm{d}x\mathrm{d}s\right)^{p/\lambda} ,\qquad \lambda=N(p-2)+p\ .$$

Results of this kind could be used to construct solutions of the Dirichlet problem with initial data in $L^1(\Omega)$ or even finite measures. Indeed the regularity results of the previous section supply the necessary compactness to pass to the limit in a sequence of approximating problems.

3.1.2. *Estimates in Σ_T.* Consider a non–negative weak subsolution u of the Cauchy problem (3.2) in the whole strip Σ_T. By this we mean that u is a *local* weak subsolution of the p.d.e. in (3.2) in Ω_T for every *bounded* domain $\Omega\subset\mathbf{R}^N$. To derive global sup–estimates, we must impose some control on the behaviour of u as $|x|\to\infty$. We assume that the quantity

$$|\!|\!|u|\!|\!|_{\{r,t\}}\equiv\sup_{0<s<t}\sup_{\rho\ge r}\int_{B_\rho}\frac{u(x,s)}{\rho^{\lambda/(p-2)}}\,\mathrm{d}x\ ,\qquad \lambda=N(p-2)+p\ , \tag{3.3}$$

is finite for some $r>0$ and for all $t\in(0,T)$. This assumption is not restrictive. It is shown in [64] that it is necessary and sufficient for a non–negative solution of (3.2) to exist in Σ_T.

The subsolution u at hand, is not necessarily bounded. However it is locally bounded and as $|x|\to\infty$ grows no faster that $|x|^{\frac{p}{p-2}}$. This is the content of the next Theorem.

Theorem 3.7. *Let u be a non–negative subsolution of (3.2) in Σ_T, and assume (3.3) holds. There exist constants γ_* and γ, depending only upon N, p and Λ, such that*

$$\|u(\cdot,t)\|_{\infty,B_\rho} \le \gamma \frac{\rho^{p/(p-2)}}{t^{N/\lambda}} \|u\|_{\{r,t\}}^{p/\lambda} , \qquad \lambda = N(p-2)+p ,$$

for all $0 < t < \gamma_* \|u\|_{\{r,t\}}^{2-p}$ and all $\rho \ge r$.

Information of this kind are of interest in investigating the behaviour of the solutions for t near zero and in studying the structure of the non–negative solutions in Σ_T. In this estimate, the functional dependence as $t \searrow 0$ is sharp as it can be verified from the explicit Barenblatt solution (recall $(1.4)_p$ in the introduction). The functional dependence as $|x| \to \infty$ is also optimal as it follows from the explicit solution

$$\mathcal{D}(x,t) = \left\{ A \left(\frac{T}{T-t}\right)^{\frac{N(p-2)}{\lambda(p-1)}} + \left(\frac{p-2}{p}\right) \lambda^{-\frac{1}{p-1}} \left(\frac{|x|^p}{T-t}\right)^{\frac{1}{p-1}} \right\}^{\frac{p-1}{p-2}} , \tag{3.4}$$

where A and T are two positive parameters.

3.2. The singular case $1 < p < 2$. We will give below an example of a solution with $p = \frac{2N}{N+1}$, that is unbounded. Thus in the singular range $1 < p < 2$, the boundedness of a weak solutions is not a purely local fact and, if at all true, it must be deduced from some global information. One of them is the weak maximum principle of Theorems 3.1 and 3.2. Another is a sufficiently high order of integrability. A sharp sufficient condition can be given in terms of the numbers

$$\lambda_r = N(p-2) + rp , \qquad r \ge 1 .$$

We assume that u satisfies

$$u \in L^r_{loc}(\Omega_T) , \qquad \text{for some} \quad r \ge 1 \quad \text{such that} \quad \lambda_r > 0 . \tag{3.5}$$

The global information needed here is

$$\left\{ \begin{array}{l} u \text{ can be constructed as the weak limit in } L^r_{loc}(\Omega_T) \text{ of a} \\ \text{sequence of non–negative } \textit{bounded} \text{ subsolutions of (2.1) .} \end{array} \right. \tag{3.6}$$

The notion of weak subsolution requires u to be in the class $u \in V^{2,p}_{\rm loc}(\Omega_T)$. This space is embedded into $L^q_{\rm loc}(\Omega_T)$, where $q = p\frac{N+2}{N}$. Therefore if p is so close to one that $\lambda_q \le 0$, the order of integrability in (3.5) is not implicit in the notion of subsolution and must be imposed.

Theorem 3.8. *Let u be a non–negative local weak subsolution of (2.1) in Ω_T and assume that (3.5) and (3.6) hold. There exists a constant $\gamma = \gamma(data, r)$, such that $\forall\,(x_0, t_0) + Q(\tau, \rho) \subset \Omega_T$ and $\forall\,\sigma \in (0,1)$,*

$$\sup_{(x_0,t_0)+Q(\sigma\tau,\sigma\rho)} u \le \frac{\gamma\,(\rho^p/\tau)^{N/\lambda_r}}{(1-\sigma)^{\frac{p}{\lambda_r}(N+p)}} \left(\iint_{(x_0,t_0)+Q(\tau,\rho)} u^r \mathrm{d}x\mathrm{d}t \right)^{p/\lambda_r} \wedge \left(\frac{\tau}{\rho^p}\right)^{\frac{1}{2-p}} . \tag{3.7}$$

3.2.1. *Estimates near $t = 0$.* Fix $t \in (0, T)$ and let us rewrite (3.7) for the pair of boxes

$$B_{\sigma\rho} \times (\sigma t, t)\ , \quad B_\rho \times (0, t)\ .$$

Corollary 3.9. *Let u be a non–negative local weak subsolution of (2.1) in Ω_T and let (3.5)–(3.6) hold. There exists a constant $\gamma = \gamma(data, r)$ such that for all $0 < t \le T$ and for all $\sigma \in (0,1)$,*

$$\sup_{B_{\sigma\rho}} u(\cdot, t) \le \frac{\gamma t^{-N/\lambda_r}}{(1-\sigma)^{\frac{N+p}{\lambda_r}}} \left(\int_0^t\!\!\int_{B_\sigma} u^r \mathrm{d}x\mathrm{d}s \right)^{p/\lambda_r} \wedge \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}} . \tag{3.8}$$

Remark 3.10. *Assume that (3.5) holds with $r = 1$, i.e.,*

$$p > \frac{2N}{N+1}\ .$$

Then the behaviour of the supremum of u as $t \searrow 0$ is formally the same as that of solutions of the Dirichlet problem (3.1) for degenerate equations as in Theorem 3.6.

3.2.2. *Global estimates: Dirichlet data.* A peculiar phenomenon of these equations is that, unlike their degenerate counterparts, local and global estimates take essentially the same form. This appears for example by comparing (3.8) with the next global estimate.

Theorem 3.11. *Let u be a non–negative weak subsolution of the Dirichlet problem (3.1) and let (3.5)–(3.6) hold. There exists a constant $\gamma = \gamma(data, r)$ such that $\forall\, t \in (0, T)$,*

$$\sup_{\Omega} u(\cdot, t) \le \sup_{S_T} g + \frac{\gamma}{t^{N/\lambda_r}} \left(\int_0^t\!\!\int_\Omega u^r \mathrm{d}x\mathrm{d}s \right)^{p/\lambda_r} .$$

3.2.3. *A counterexample.* Let $a \in (0,1)$ be a positive constant and let B_a denote the ball of radius a in $\mathbf{R}^N$ centered at the origin. Consider the functions

$$z = \frac{\left(a^2 - |x|^2\right)_+^2}{|x|^N \left|\ln |x|^2\right|^\beta} \qquad \text{and} \qquad v = (1-ht)_+ \, z \ ,$$

where $\beta, h > 1$ are parameters to be chosen. One verifies that

$$z \in L^1(B_a) \ , \quad \text{and} \quad z \notin L^{1+\varepsilon}(B_a) \ , \quad \forall \varepsilon > 0 \ .$$

Consider also the Cauchy problem

$$\begin{cases} u_t - \operatorname{div} |\nabla u|^{p-2} \nabla u = 0 & \text{in} \quad \Sigma_1 \\ u(\cdot, 0) = z \ . \end{cases} \tag{3.9}$$

Lemma 3.12. *Assume that $N(p-2)+p=0$. The constants $a \in (0,1)$ and $\beta, h > 1$ can be determined a priori so that v is a non–negative, weak subsolution of (3.9) in Σ_1.*

Next we return to (3.9) and observe that by the comparison principle $u \geq v$ and therefore u is not bounded.

4. Intrinsic Harnack Estimates

In this section we present some results about Harnack inequalities. More precisely we consider nonnegative weak solutions of the type:

$$\begin{cases} u \in C_{\mathrm{loc}}\left(0,T; L^2_{\mathrm{loc}}(\Omega)\right) \cap L^p_{\mathrm{loc}}\left(0,T; W^{1,p}_{\mathrm{loc}}(\Omega)\right), \ p > 1 \\ \\ u_t - \operatorname{div} |\nabla u|^{p-2} \nabla u = 0 \ , \qquad \text{in } \Omega_T \end{cases} \tag{4.1}$$

The first parabolic version of the Harnack inequality is due to Hadamard ([89]) and Pini ([144]). Their result is the following:

Let u be a non-negative solution of the heat equation in Ω_T. Let $(x_0, t_0) \in \Omega_T$ and assume that the cylinder $(x_0,t_0)+Q_{2\rho} \subset \Omega_T$ where $Q_\rho \equiv B_\rho \times (-\rho^2, 0)$. Then there exists a constant γ, depending only upon N, such that

$$u(x_0, t_0) \geq \gamma \sup_{B_\rho(x_0)} u(x, t_0 - \rho^2) \ . \tag{4.2}$$

The proof is based on local representations by means of heat potentials. A breakthrough in the theory is due to Moser, who in his celebrated paper [134] proved that (4.2) continues to hold for nonnegative weak solution of the type

$$\begin{cases} u \in C_{\mathrm{loc}}\left(0,T;L^2_{\mathrm{loc}}(\Omega)\right) \cap L^2_{\mathrm{loc}}\left(0,T;W^{1,2}_{\mathrm{loc}}(\Omega)\right) \\ u_t - \sum_{i,j=1}^{N} D_i(a_{ij}(x,t)D_j u) = 0 \ , \qquad \text{in } \Omega_T \end{cases} \tag{4.3}$$

where $a_{ij} \in L^\infty(\Omega_T)$ and satisfy the ellipticity condition

$$\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq \nu|\xi|^2 \ , \qquad \forall \xi \in \mathbf{R}^N \tag{4.4}$$

with ν a positive constant. The result of Moser can be extended (see [16], [164] and [163]) to nonnegative weak solutions of the quasilinear parabolic equation

$$u_t - \mathrm{div}\mathbf{a}(x,t,u,\nabla u) + b(x,t,u,\nabla u) = 0 \ , \qquad \text{in } \Omega_T \ , \tag{4.5}$$

where the diffusion field $\mathbf{a}$ and the forcing term b are real valued and measurable over $\Omega_T \times \mathbf{R} \times \mathbf{R}^N$ and satisfy the structure conditions considered in Section 2.4 for $m = 1$ or for $p = 2$.

The proof of Moser's result is based on suitable integral estimates for powers and logarithm of the solution u; the general structure follows the same one Moser used in his earlier work on Harnack's inequality in the elliptic case ([133]) and it is basically articulated in three steps.

First Step: Estimates on positive powers of u - Let u be a nonnegative solution; then for all $\varepsilon > 0$ there exists a positive constant γ, depending only upon N and ε, such that for every cylinder $Q_\rho(x_0,t_0) \subset \Omega_T$ and for every $\sigma \in (0,1)$

$$\sup_{Q_{\sigma\rho}(x_0,t_0)} u \leq \frac{\gamma}{(1-\sigma)^{\frac{N+2}{2\varepsilon}}} \left(\frac{1}{|Q_\rho(x_0,t_0)|} \int\int_{Q_\rho(x_0,t_0)} u^\varepsilon \,\mathrm{d}x\mathrm{d}\tau \right)^{\frac{1}{\varepsilon}} \ . \tag{4.6}$$

Let us remark that this estimate holds also for nonnegative *subsolutions*.

Second Step: Estimates on negative powers of u - Let u be a positive solution; then for all $\varepsilon > 0$ there exists a positive constant γ, depending only upon N and ε, such that for every cylinder $Q_\rho(x_0,t_0) \subset \Omega_T$ and for every $\sigma \in (0,1)$

$$\sup_{Q_{\sigma\rho}(x_0,t_0)} \frac{1}{u} \leq \frac{\gamma}{(1-\sigma)^{\frac{N+2}{2\varepsilon}}} \left(\frac{1}{|Q_\rho(x_0,t_0)|} \int\int_{Q_\rho(x_0,t_0)} \frac{1}{u^\varepsilon} \,\mathrm{d}x\mathrm{d}\tau \right)^{\frac{1}{\varepsilon}} \ . \tag{4.7}$$

Quite analogously to what happened in the first step, this estimate holds also for positive *supersolutions*.

If we consider the mean values

$$M(p, D) = \left(\frac{1}{|D|} \int\int_D u^p \, \mathrm{d}x\mathrm{d}\tau \right)^{\frac{1}{p}},$$

it is obvious that (4.6) and (4.7) can be rewritten as

$$M(+\infty, Q_{\sigma\rho}) \leq \gamma_1 \, M(\epsilon, Q_\rho), \qquad M(-\epsilon, Q_\rho) \leq \gamma_2 \, M(-\infty, Q_{\sigma\rho}).$$

The main point is then to establish a so - called "crossover inequality", namely

$$M(p, D_-) \leq \gamma_3 \, M(-p, D_+)$$

for sufficiently small $p > 0$ and appropriate domains D_-,D_+. Indeed this is the result of the

Third Step: Crossover Lemma - Let u be a positive solution, $D_+ = \{|x| < 1, \ \frac{1}{2} < t < 1\}$ and $D_+ = \{|x| < 1, \ -1 < t < -\frac{1}{2}\}$. Then there exist constants $\delta > 0$ and $C > 0$, depending only on N such that

$$\left(\int\int_{D_-} u^\delta \mathrm{d}x\mathrm{d}\tau \right) \left(\int\int_{D_+} u^{-\delta} \mathrm{d}x\mathrm{d}\tau \right) \leq C. \tag{4.8}$$

We stated (4.8) in a normalized form just for the sake of simplicity.

It is worth saying that the third step is the most difficult part of Moser's proof. Inequality (4.8) is a straightforward consequence of an adaptation to the parabolic case of the well - known lemma of F. John and L. Nirenberg, which concerns the exponential decay of the distribution function of a function with bounded mean oscillation. Going from the elliptic to the parabolic situation the difficulty lies in the special role played by the time variable. In fact as clearly stated in (4.2), Harnack inequality for a nonnegative solution of a parabolic equation is an *inf-bound* on the value of such a solution at a given time in terms of its value at a previous time and this necessary time lag has to be reflected in a proper parabolic John-Nirenberg Lemma. This is precisely what Moser did in his Main Lemma in [134]. Indeed Moser's proof is hard to follow and the need for a possible simplification was immediately felt.

Moser himself published a new proof of Harnack inequality in 1971 (see [135]), with the expressed purpose to avoid the use of his parabolic John-Nirenberg Lemma; through estimates on the logarithm of the solution and via a measure lemma based on a result of Bombieri ([27], [28]), he showed that it is possible to estimate in a quantitative way the supremum of u. Repeating the same argument for u^{-1} one gets the quantitative estimates for the infimum of u. By combining these results the Harnack estimates are proved.

A few years later Fabes and Garofalo ([80]) came back to Moser's Main Lemma and gave a simplified proof, using Calderon's proof of the original John-Nirenberg lemma, see also ([81]).

In Moser's approach the main feature that makes the method work is the homogeneity of the time and space terms of the equation; in fact Trudinger ([164] - Section 5) shows that things run in the same way in the proof of Harnack inequality for doubly nonlinear equations of the type

$$(u^{p-1})_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \ , \qquad \text{in } \Omega_T \tag{4.9}$$

which is p-homogeneous, exactly as (4.3) is 2-homogeneous.

On the other hand, coming back to equation (4.1), quite surprisingly Moser's method does not work when $p \neq 2$ and this is not simply a matter of technique. As already discussed in the Introduction, as (4.1) is invariant by the scaling $x \to hx$ and $t \to h^p t$, one would guess that Harnack estimates would hold in the cylinder $[(x_0,t_0) + B_\rho \times (-\rho^p, 0)]$, but this is not the case. Let us consider the explicit solution of (4.1) introduced by Barenblatt in [17]:

$$\mathcal{B}(x,t) = t^{-\frac{N}{\lambda}} \left\{ 1 - \gamma_p \left(\frac{|x|}{t^{1/\lambda}} \right)^{\frac{p}{p-1}} \right\}_+^{\frac{p-1}{p-2}} , \quad t > 0,\ p > 2 \ , \tag{4.10}$$

where $\gamma_p = \lambda^{\frac{1}{1-p}}(p-2)/p$ and $\lambda = N(p-2)+p$. Let (x_0,t_0) be a point of the *free boundary* $\{t = |x|^\lambda\}$. If t_0 is large enough, the ball $B_\rho(x_0)$ taken at the time level $t_0 - \rho^p$ intersects the support of $x \to \mathcal{B}(x, t_0 - \rho^p)$ in an open set. Hence

$$\sup_{B_\rho(x_0)} \mathcal{B}(x, t_0 - \rho^p) > 0 \quad \text{and} \quad \mathcal{B}(x_0,t_0) = 0$$

which contradicts (4.2) and we conclude that things must be more complicated.

However a comparison between (4.4) and (4.1) suggests that one may heuristically regard (4.1) as if it were written in a time scale intrinsic to the solution itself and, loosely speaking, of the order of $t[u(x,t)]^{2-p}$. Indeed if one looks at the Barenblatt solutions once more, one realizes that for such specific functions a Harnack estimate holds with an intrinsic time scale exactly of the order $u(x_0,t_0)^{2-p}$.

A general result of this kind for (4.1) is proved in [54], [40] and [66]:

Theorem 4.1. *Let u be a nonnegative weak solution of (4.1) and let $p > \frac{2N}{N+1}$. Fix $(x_0,t_0) \in \Omega_T$ and assume that $u(x_0,t_0) > 0$. There exists constants $\gamma > 1$ and $C > 0$, depending only upon N and p, such that*

$$u(x_0,t_0) \leq \gamma \inf_{B_\rho(x_0)} u(\cdot, t_0 + \theta) \ , \tag{4.11}$$

where

$$\theta \equiv \frac{C\rho^p}{[u(x_0,t_0)]^{p-2}} \tag{4.12}$$

provided that the cylinder $(x_0,t_0) + B_{4\rho} \times (-4\theta, 4\theta)$ is contained in Ω_T.

In the next sections we will give a sketch of the proof both for the degenerate and singular cases. For the moment let us make some general remarks and point out some open questions.

Remark 4.2. There is a big difference between the degenerate case ($p > 2$) and the singular case ($p < 2$) and this is due to the different behaviour of the modulus of ellipticity $|Du|^{p-2}$. In the degenerate situation the modulus vanishes when Du is zero; hence the evolution phenomenon dominates over the diffusion process and this holds more and more as p grows to infinity. We have a direct consequence in (4.11), as the constant C is larger than one when $p > 2$; moreover $C \to \infty$ as $p \to \infty$. Roughly speaking Harnack inequality states that the original positivity of u at (x_0,t_0) is spread over the full ball $B_\rho(x_0)$ and is preserved for a large time. On the other hand, when $p \to 2^+$, $\gamma(N,p)$, $C(N,p) \to \gamma(N,2)$, $C(N,2)$ so that, at least formally, we recover the classical Harnack inequality for nonnegative solutions of the heat equation. On the contrary, in the singular case the modulus blows up when Du vanishes, so that now the previous situation is reversed, the diffusion dominates over the evolution phenomenon and this is felt more and more as $p \to \frac{2N}{N+1}$. Once more we can see this clearly expressed in (4.11), since $C \in (0,1)$ and $C \to 0^+$ as $p \to \frac{2N}{N+1}$: under a geometrical point of view, we can say that the original positivity of u at (x_0,t_0) spreads over the full ball $B_\rho(x_0)$ but is now preserved only for a relatively small time. Exactly as in the degenerate case, when $p \to 2^-$ we have that $\gamma(N,p)$, $C(N,p) \to \gamma(N,2)$, $C(N,2)$, so that once again we recover the classical results in the limit situation. Finally one may naturally ask about the lower bound for p: why $p > \frac{2N}{N+1}$ and not just $p > 1$? Indeed $p > \frac{2N}{N+1}$ is optimal for Harnack inequality to hold, as we will see in the next sections, discussing the phenomenon of extinction in finite time.

Remark 4.3. Let apart the intrinsic height of the cylinder, inequality (4.11) is obviously equivalent to (4.2). Let us just remark that in the case of parabolic equations in non-divergence form, Krylov and Safonov gave to Harnack inequality exactly the same formulation as in (4.11).

In Theorem 4.1 the level θ is defined in terms of $u(x_0, t_0)$ by (4.12). Notwithstanding the previous discussion about the intrinsic scaling, it is natural to ask if an estimate holds where the geometry can be a priori prescribed independent of the solution. In [54] a positive answer is given when $p > 2$ by the following

Theorem 4.4. *Let u be a nonnegative weak solution of (4.1) and let $p > 2$. There exists a constant $B = B(N, p) > 1$ such that*

$$\forall (x_0, t_0) \in \Omega_T,\ \forall \rho,\ \theta > 0 \text{ s.t. } (x_0, t_0) + B_{4\rho} \times (-4\theta, 4\theta) \subset \Omega_T$$

we have

$$u(x_0, t_0) \le B \left\{ \left(\frac{\rho^p}{\theta} \right)^{\frac{1}{p-2}} + \left(\frac{\theta}{\rho^p} \right)^{\frac{N}{p}} \left[\inf_{B_\rho(x_0)} u(\cdot, t) \right]^{\frac{\lambda}{p}} \right\} , \tag{4.13}$$

where $\lambda = N(p-2) + p$.

At a first glance Theorem 4.1 and Theorem 4.4 could look markedly different, as in the second one the positivity of $u(x_0, t_0)$ is not required and $\theta > 0$ can be assumed arbitrarily. Indeed (4.13) holds trivially when $u(x_0, t_0) = 0$ and both statements are equivalent when $u(x_0, t_0) > 0$ in the sense that (4.11) $\Rightarrow$ (4.13) in any case and (4.13) $\Rightarrow$ (4.11) with a constant $\gamma(N, p)$ which may not be stable as $p \to 2^+$.

Assuming $u(x_0, t_0) > 0$, let us prove the second implication. Under the hypothesis that (4.13) is valid for all $\theta > 0$ s. t. $(x_0, t_0) + B_{4\rho} \times (-4\theta, 4\theta) \subset \Omega_T$, if we choose

$$\theta = \frac{(2B)^{p-2} \rho^p}{[u(x_0, t_0)]^{p-2}}$$

we immediately conclude that

$$u(x_0, t_0) \le \gamma \inf_{B_\rho(x_0)} u(\cdot, t_0 + \theta) \quad \text{with} \quad \gamma = 2B^{N(p-2)+\lambda}.$$

We postpone the proof of the opposite implication to the next section.

A consequence of Theorem 4.4 is

Corollary 4.5. *Let u be a nonnegative weak solution of (4.1) and let $p > 2$. There exists a constant $B = B(N,p) > 1$ such that*

$$\forall (x_0,t_0) \in \Omega_T,\ \forall \rho,\ \theta > 0 \text{ s.t. } (x_0,t_0) + B_{4\rho} \times (-4\theta, 4\theta) \subset \Omega_T$$

we have

$$\frac{1}{\rho^N} \int_{B_\rho(x_0)} u(x,t_0)dt \le \gamma \left\{ \left(\frac{\rho^p}{\theta}\right)^{\frac{1}{p-2}} + \left(\frac{\theta}{\rho^p}\right)^{\frac{N}{p}} [u(x_0,t)]^{\frac{\lambda}{p}} \right\} . \tag{4.14}$$

Harnack inequalities like the ones stated in Theorems 4.1 and 4.4 hold for nonnegative solutions of the porous medium equation

$$\begin{cases} u \in C_{\mathrm{loc}}\left(0,T; L^2_{\mathrm{loc}}(\Omega)\right),\ u^m \in L^2_{\mathrm{loc}}\left(0,T; W^{1,2}_{\mathrm{loc}}(\Omega)\right), \\ \\ u_t - \triangle u^m = 0 , \qquad \text{in } \Omega_T,\ m > 1. \end{cases} \tag{4.15}$$

In particular Theorem 4.1 becomes

Theorem 4.6. *Let u be a nonnegative weak solution of (4.15) and let $m > \frac{(N-2)_+}{N+2}$. Fix $(x_0,t_0) \in \Omega_T$ and assume that $u(x_0,t_0) > 0$. There exists constants $\gamma > 1$ and $C > 0$, depending only upon N and m, such that*

$$u(x_0,t_0) \le \gamma \inf_{B_\rho(x_0)} u(\cdot, t_0 + \theta) , \tag{4.16}$$

where

$$\theta \equiv \frac{C\rho^2}{[u(x_0,t_0)]^{m-1}} \tag{4.17}$$

provided that the cylinder $(x_0,t_0) + B_{4\rho} \times (-4\theta, 4\theta)$ is contained in Ω_T.

The different behaviour of the constant C dependent upon $p > 2$ or $\frac{2N}{N+1} < p < 2$ discussed in Remark 4.2 comes up in this context too, where the degenerate case is given by $m > 1$ and the singular case by $\frac{(N-2)_+}{N+2} < m < 1$.

Remark 4.7. Similar estimates have been proved also for the solutions of doubly nonlinear parabolic equations of the type

$$u_t = \mathrm{div}(|\nabla u|^{p-2}|u|^{m-1}\nabla u)$$

(see [176]). Equations of this type are classified as doubly nonlinear ([123]) or with implicit nonlinearity ([103]). This class of equations have their own mathematical interest (the porous medium equation and the p-Laplacian equations belong to this larger class) and physical interest (see the review paper [103]). Also for this larger class, local Hölder continuity results hold (see [96], [97], [147] and [177]).

Remark 4.8. The theory of Harnack estimates is fragmented and incomplete. The estimates for $p \neq 2$ hold only for homogeneous pdes and this strongly depends on the method we will present in the following, which basically relies on the construction of special solutions and subsolutions. The shortcoming of such a technique is evident even in the framework of homogeneous equations since a Harnack-type estimate is not known to hold for nonnegative weak solutions of (see [123])

$$u_t = \sum_{i=1}^{N} D_i(|D_i u|^{p-2} D_i u) \ .$$

The open question is if it is possible to extend the Harnack estimates to the case of parabolic equations with the full quasilinear structure, as it happens when $p = 2$. Results of this kind would probably require a new method independent of local representations and local subsolutions. Whenever developed, such a technique would parallel the discovery of the Moser estimates [134], based on real and harmonic analysis tools, versus the estimates by Hadamard [89] and Pini [144], based on local representations.

4.1. Harnack estimates: the degenerate case. First of all, let us briefly comment upon the assumption that the cylinder $(x_0, t_0) + B_{4\rho} \times (-4\theta, 4\theta)$ is contained in Ω_T. Under a geometrical point of view, this means that t_0 should be of the order of θ and this is essential. In fact if we consider the Barenblatt solution given in (4.10) with $x_0 = 0$ and t_0 arbitrarily close to the origin, it is evident that it cannot satisfy (4.13). One might think that this is due to the pointwise nature of (4.12) and (4.13), but this is not the case and the reason actually lies in the local character of the solutions we are considering. Indeed quite surprisingly also the averaged form of the Harnack inequality (4.14) does not hold without the assumption that the cylinder $(x_0, t_0) + B_{4\rho} \times (-4\theta, 4\theta)$ is contained in Ω_T. To see this, let u be the unique weak solution of the boundary value problem

$$\begin{cases} u_t - (|u_x|^{p-2}u_x)_x = 0 & \text{in } Q \equiv (0,1)\times(0,\infty), \\ u(0,t) = u(1,t) = 0 & \text{for all } t \geq 0 \\ u(x,0) = u_0(x) \in C_0^\infty(0,1) \\ u_0(x) \in [0,1],\ \forall x \in (0,1) \quad \text{and } u_0(x) = 1 \text{ for } x \in (\frac{1}{4}, \frac{3}{4}). \end{cases} \tag{4.18}$$

Thanks to the results of [20], we can say that

$$u_t \geq -\frac{1}{p-2}\frac{u}{t} \qquad \text{in } \mathcal{D}'(Q).$$

Since $0 \leq u \leq 1$, by the comparison principle we have

$$-(|u_x|^{p-2}u_x)_x \leq \frac{1}{(p-2)t}, \qquad t > 0.$$

At any fixed level t, the function $x \to u(x,t)$ is majorised by

$$v(x,t) = \frac{\gamma x^\delta}{t^{\frac{1}{p-1}}}, \quad \delta \in \left(\frac{p-1}{p}, 1\right), \quad (\gamma\delta)^{p-1}(1-\delta)(p-1) \geq \frac{1}{p-2},$$

as

$$-(|v_x|^{p-2}v_x)_x \geq \frac{1}{(p-2)t} \quad \text{and} \quad v(0,t) = 0, \quad v(1,t) > 0.$$

Therefore for every $\delta \in \left(\frac{p-1}{p}, 1\right)$ there exists a constant $C = C(\delta)$, such that

$$u(\frac{1}{2}, t) \leq \frac{C(\delta)}{t^{\frac{1}{p-1}}}.$$

Now if (4.14) held for $t_0 = 0$, $x_0 = \frac{1}{2}$ and $\rho = \frac{1}{4}$, for $t > 1$ we would have

$$1 \leq \text{const}\left(t^{-\frac{1}{p-2}} + t^{-\frac{1}{p}}\right) \to 0 \quad \text{as} \quad t \to +\infty.$$

In the sequel we will see that the limitation on t_0 in (4.14) can be dropped when (4.1) is considered in the whole $\mathbf{R}^N$.

We can now finally come to the proof of Theorem 4.1 when $p > 2$. The technical tools used in the proof are only two: the Hölder continuity of solutions as proved in Section 2 and the comparison principle. This point of view is somehow reversed with respect to Moser's approach where the Hölder continuity is implied by the Harnack estimate. Even though not so explicitly stated, a method similar to ours is already present in the work of Krylov and Safonov [117].

We can basically recognize four steps.

First Step: Renormalization of the solution - Let $(x_0, t_0) \in \Omega_T$ and $\rho > 0$ be fixed, assume $u(x_0, t_0) > 0$ and consider the box

$$Q_{4\rho} = \{|x - x_0| < 4\rho\} \times \{t_0 - \frac{4C\rho^p}{[u(x_0, t_0)]^{p-2}}, t_0 + \frac{4C\rho^p}{[u(x_0, t_0)]^{p-2}}\}$$

where C is a positive constant to be determined later. We render the equation *dimensionless* by the change of variables

$$\xi = \frac{x - x_0}{\rho}, \quad \tau = \frac{(t - t_0)[u(x_0, t_0)]^{p-2}}{\rho^p}, \quad v = \frac{u}{u(x_0, t_0)}.$$

This maps $Q_{4\rho}$ into $Q = Q^+ \cup Q^-$ where $Q^+ \equiv B(4) \times [0, 4C)$, $Q^- \equiv B(4) \times (-4C, 0]$. We denote again the new variables with x and t and observe that the rescaled function v is a bounded nonnegative solution of the equation

$$v_t - \mathrm{div}|\nabla v|^{p-2}\nabla v = 0 \qquad \text{in } Q$$

with $v(0, 0) = 1$. To prove the Harnack inequality it is enough to find constants $0 < \gamma_0 < 1$ and $C > 1$, depending only upon N and p, such that for each $x \in B(1)$ we have

$$v(x, C) \geq \gamma_0 \ . \tag{4.19}$$

As a matter of fact if $u(x_0, t_0) = 0$, no rescaling is possible and we are led to consider Theorem 4.4, which is trivially satisfied in this case as already remarked.

Second Step: Determination of the largest value of v in Q^- - Construct the family of nested boxes $Q_\tau \equiv B_\tau \times (-\tau^p, 0]$. Define the numbers $M_\tau = \sup_{Q_\tau} v$ and $N_\tau = (1 - \tau)^{-\beta}$ where $\beta > 1$ will be chosen later. Let $0 \leq \tau_0 < 1$ be the largest root of the equation $M_\tau = N_\tau$. Such a root is well defined, since $M_0 = N_0$ and as $\tau \to 1^-$ M_τ remain bounded and N_τ blow up. By construction $\sup_{Q_\tau} v \leq N_\tau$ for all $\tau > \tau_0$ Moreover, from the continuity of v in Q, there exists at least a point $(x_1, t_1) \in N_{\tau_0}$ where $v(x_1, t_1) = (1 - \tau_0)^{-\beta}$.

Third Step: Lower bound on v at the same time - level t_1 - Relying on the Hölder continuity of v ([51]), we can determine a small ball of radius r_0 about (x_1, t_1) where $v \geq \frac{(1-\tau_0)^{-\beta}}{2}$. Roughly speaking, we have found a small ball $B_{r_0}(x_1)$ at time t_1, close to $(0, 0)$, where the largeness of $v(\cdot, t_1)$ is *qualitatively* determined. The proof is concluded once we choose the constant $\beta > 1$ and $C > 1$ in such a way that we come up with a quantitative lower bound on v over the full ball B_1 at a later time C: otherwise stated, we have to spread the positivity of v and this is the crucial step in the proof of the Harnack inequality.

Fourth Step: Expansion of the positivity set - The spread of positivity is achieved by means of a proper comparison function. For $t \geq t_1$ consider the function

$$\mathcal{B}_{k,\rho}(x,t;x_1,t_1) \equiv \frac{k\rho^n}{S^{N/\lambda}(t)} \left[1 - \left(\frac{|x-x_1|}{S^{1/\lambda}(t)}\right)^{\frac{p}{p-1}}\right]_+^{\frac{p-1}{p-2}}$$

where as usual $\lambda = N(p-2)+p$, $S(t) = B(N,p)k^{p-2}\rho^{N(p-2)}(t-t_1)+\rho^\lambda$, $b(N,p) = \lambda(\frac{p}{p-2})^{p-1}$ and choose $k = \frac{(1-\tau_0)^{-\beta}}{2}$ and $\rho = r_0$. By direct calculation one verifies that $\mathcal{B}_{k,\rho}(x,t;x_1,t_1)$ is a weak solution of (4.1) in $\mathbf{R}^N \times \{t_1, t\}$. This comparison function was introduced in [17], [143] and [54].

By a proper choice of β and C, the support of $\mathcal{B}_{k,\rho}(\cdot, C; x_1, t_1)$ contains B_2 and by the comparison principle

$$\inf_{B_1} v(x,C) \geq \inf_{B_1} \mathcal{B}_{k,\rho}(x,C;x_1,t_1) \geq \gamma_0$$

for a suitable value of γ_0 and we are finished. ■

Remark 4.9. The constant γ_0 tends to 0 as $p \to 2^+$. Therefore in order to have the constants under control as p approaches the non - degenerate case, a comparison function other than $\mathcal{B}_{k,\rho}$ is used for p close to 2.

Once we have proved Theorem 4.1, we can show how it implies Theorem 4.4 and therefore conclude about the equivalence between the two different Harnack estimates. Let $(x_0,t_0) \in \Omega_T$, $\rho > 0$ and $\theta > 0$ be fixed in such a way that the cylinder $(x_0,t_0) + B_{4\rho} \times (-4\theta, 4\theta)$ is contained in Ω_T. Without loss of generality we can assume that $(x_0,t_0) \equiv (0,0)$ and set $u_* = u(0,0)$. With C and γ as determined in Theorem 4.1, we can assume that

$$t^* \equiv \frac{C\rho^p}{u_*^{p-2}} \leq \frac{\theta}{2}$$

otherwise there is nothing to prove. Relying on this and using the comparison principle with the function $\mathcal{B}_{k,\rho}(x,t;0,t^*)$ and $k=\gamma^{-1}u_*$, we can prove that

$$u(x,\theta) \geq \gamma_1\, u_*^{p/\lambda} \left(\frac{\rho^p}{\theta}\right)^{N/\lambda} \quad \text{with} \quad \gamma_1 \equiv \gamma_1(N,p).$$

We have finished once we set $B=\max\{\gamma_1^{-\lambda/p};(2C)^{\frac{1}{p-2}}\}$. ■

The assumption that the cylinder $Q_{4\rho}(\theta)$ be contained in the domain of definition of the solution is essential for the Harnack estimates of Theorems 4.1 and 4.4 to hold. When the solution is defined in $\mathbf{R}^N$ any restriction on t_0 can be avoided because we do not need to impose any restriction on ρ to have that the cylinder $Q_{4\rho}(\theta)$ belongs to the domain if definition. More precisely, if we consider nonnegative weak solutions of the type

$$\begin{cases} u \in C_{\text{loc}}\left(0,T;L^2_{\text{loc}}(\mathbf{R}^N)\right) \cap L^p_{\text{loc}}\left(0,T;W^{1,p}_{\text{loc}}(\mathbf{R}^N)\right)\ , \qquad p>2 \\ \\ u_t - \operatorname{div}|\nabla u|^{p-2}\nabla u = 0\ , \qquad \text{in } \Sigma_T\ , \end{cases} \tag{4.20}$$

where $\Sigma_T \equiv \mathbf{R}^N \times (0,T]$, we have

Theorem 4.10. *Let u be a nonnegative weak solution of (4.20). Let $(x_0,t_0)\in \Sigma_T$, $\rho>0$ and $t>t_0$. Then*

$$\frac{1}{\rho^N}\int_{B_\rho(x_0)} u(x,t_0)dt \leq \gamma \left\{ \left(\frac{\rho^p}{t-t_0}\right)^{\frac{1}{p-2}} + \left(\frac{t-t_0}{\rho^p}\right)^{\frac{N}{p}} \left[\inf_{B_\rho(x_0)} u(\cdot,t)\right]^{\frac{\lambda}{p}} \right\}\ , \tag{4.21}$$

where $\gamma>1$ is depending only upon N, p and $\lambda = N(p-2)+p$.

Remark 4.11. Even if we are still dealing with local estimates as with the previous Harnack inequalities, it is the switch from Ω_T to Σ_T that gives us a useful piece of *global* information and allows us to get arbitrarily close to 0 with t_0. Estimate (4.21) contains information on the initial data of (4.20). Let $x_0\in\mathbf{R}^N$, $r>0$ and $\varepsilon>0$. Apply (4.21) with $(t-t_0)=T-\varepsilon$, divide by $\rho^{\frac{p}{p-2}}$ and take the supremum of both sides for $\rho>r$ and $\tau\in(0,T-\varepsilon)$. In this way one obtains that

$$\sup_{0<\tau\leq T-\varepsilon}\ \sup_{\rho>r} \int_{B_\rho(x_0)} \frac{u(x,\tau)}{\rho^{\frac{\lambda}{p}}}\,\mathrm{d}x \leq \frac{\gamma}{\varepsilon^{\frac{1}{p-2}}}\left[1+\left(\frac{T}{r^p}\right)^{\frac{1}{p-2}} u(x_0,T-\varepsilon)\right]^{\frac{\lambda}{p}}\ . \tag{4.22}$$

The previous estimate implies that the nonnegative solutions of (4.21) are locally bounded and, as $|x|\to+\infty$ they cannot grow faster than $|x|^{\frac{p}{p-2}}$.

4.2. Harnack estimates: the singular case. The proof of the singular case is quite similar to the degenerate case, except for the last part, relative to the spread of positivity. For the sake of completeness we recall all the steps.

First Step: Renormalization of the solution - Let $(x_0, t_0) \in \Omega_T$ and $\rho > 0$ be fixed, assume $u(x_0, t_0) > 0$ and consider the box

$$Q_{4\rho} = \{|x - x_0| < 4\rho\} \times \{t_0 - [u(x_0, t_0)]^{2-p}(4\rho)^p, t_0 + [u(x_0, t_0)]^{2-p}(4\rho)^p\}.$$

We render the equation *dimensionless* by the change of variables

$$\xi = \frac{x - x_0}{\rho}, \quad \tau = \frac{(t - t_0)[u(x_0, t_0)]^{p-2}}{\rho^p}, \quad v = \frac{u}{u(x_0, t_0)}.$$

This maps $Q_{4\rho}$ into $Q = Q^+ \cup Q^-$ where $Q^+ \equiv B(4) \times [0, 4^p)$, $Q^- \equiv B(4) \times (-4^p, 0]$. We denote again the new variables with x and t and observe that the rescaled function v is a bounded nonnegative solution of the equation

$$v_t - \operatorname{div}|\nabla v|^{p-2}\nabla v = 0 \qquad \text{in } Q$$

with $v(0, 0) = 1$. To prove the Harnack inequality it is enough to find constants $0 < \gamma_0 < 1$ and $0 < C < 1$, depending only upon N and p, such that for each $x \in B(1)$ we have

$$v(x, C) \geq \gamma_0 \ . \tag{4.23}$$

As a matter of fact if $u(x_0, t_0) = 0$, no rescaling is possible and we are led to consider the Elliptic - type Harnack inequality, we will discuss in the next Section.

Second Step: Determination of the largest value of v in Q^- - Construct the family of nested boxes $Q_\tau \equiv B_\tau \times (-\delta\tau, 0]$. Define the numbers $M_\tau = \sup_{Q_\tau} v$ and $N_\tau = (1 - \tau)^{-\frac{p}{2-p}}$ where $0 < \delta < 1$ will be chosen later and has the effect of making flat the boxes Q_τ. If we compare the situation with the analogous one for $p > 2$, we notice that the cylinders Q_τ are rather thin in the t-variable and the exponent for N_τ is fixed and depends only on the singularity of the equation. Let $0 \leq \tau_0 < 1$ be the largest root of the equation $M_\tau = N_\tau$. Such a root is well defined, since $M_0 = N_0$ and as $\tau \to 1^-$ M_τ remain bounded and N_τ blow up. By construction

$$M_{\tau_0} = (1 - \tau_0)^{-\frac{p}{2-p}}, \qquad M_{\frac{1-\tau_0}{2}} \leq 2^{\frac{p}{2-p}}(1 - \tau_0)^{-\frac{p}{2-p}}.$$

Moreover, from the continuity of v in Q, there exists at least a point $(x_1, t_1) \in Q_{\tau_0}$ where $v(x_1, t_1) = (1 - \tau_0)^{-\frac{p}{2-p}}$ and

$$\sup_{|x-\bar{x}|<\frac{1-\tau_0}{2}} v(x, t_1) \leq 2^{\frac{p}{2-p}}(1 - \tau_0)^{-\frac{p}{2-p}}.$$

Third Step: Lower bound on v at the same time - level t_1 - Relying on the Hölder continuity of v, we can determine a small ball of radius $r_0 = \epsilon(1 - \tau_0)$ about (x_1, t_1) where $v \geq \frac{(1-\tau_0)^{\frac{p}{p-2}}}{2}$. ϵ is a small constant that tends to 0 as $p \to 2^-$.

Fourth Step: Time - expansion of positivity - All the previous arguments are independent of the quantity δ and now it is fixed. By means of a proper comparison function, the positivity of v is spread over a small time interval *without* modifying the space size of the box we are working in. More precisely it is proved that there exist small positive numbers c_0 and δ that can be determined a priori only in terms of N and p such that

$$v(x, t) \geq c_0(1 - \tau_0)^{\frac{p}{p-2}}, \quad \forall |x - \bar{x}| < \epsilon(1 - \tau_0), \quad \forall\, \delta \leq t \leq 2\delta.$$

This step (and the next one, too) is the main difference with respect to the degenerate case when $p > 2$. Roughly speaking, in that case the positivity is spread over time <u>and</u> space in one stroke. Here we need to proceed one step at a time.

Fifth Step: Sidewise expansion of positivity - Using a new comparison function the positivity of v is spread over the full ball $\{|x| < 1\}$ at the time level $t = 2\delta$. This is done by showing that there exists a constant $\gamma_0 = \gamma_0(N, p)$ such that

$$v(x, 2\delta) \geq \gamma_0, \qquad \forall |x - \bar{x}| < 2$$

and with this we are finished. ■

Remark 4.12. The comparison functions used in the proof were introduced in [5] and [66]. We also point out that the comparison principle is a consequence of L^1-techniques if $u_t \in L^1_{\text{loc}}(\Omega_T)$. If u_t does not belong to $L^1_{\text{loc}}(\Omega_T)$ the comparison principle can be proved adapting a technique introduced by Kalashnikov, Oleinik, Yui-Lin and Chzhou [104] (see also Appendix 9 of [66]).

Remark 4.13. Exactly as in the degenerate case when $p > 2$, the constant γ_0 tends to 0 as $p \to 2^-$. Therefore in order to have the constants under control as p approaches the non-singular case, a different comparison function is used for p close to 2.

Remark 4.14. In the singular case too, it is possible to state an integral Harnack inequality that holds for all $1 < p < 2$: Let u be a nonnegative weak solution of (4.1). Then there exists a constant γ, depending only upon N and p, such that for all $(x_0, t_0) \in \Omega_T$, for all $\rho > 0$ such that $B_{4\rho}(x_0) \subset \Omega$ and for all $t > t_0$,

$$\sup_{t_0 < \tau \leq t} \int_{B_\rho(x_0)} u(x,\tau)\mathrm{d}x \leq \gamma \inf_{t_0 < \tau \leq t} \int_{B_{2\rho}(x_0)} u(x,\tau)\mathrm{d}x + \gamma \left(\frac{t - t_0}{\rho^\lambda} \right)^{\frac{1}{2-p}} \tag{4.24}$$

with $\lambda = N(p-2) + p$. Note that λ might be of either sign. Moreover, a very important difference with respect to the degenerate case is that in the singular case the L^1 norm of $u(\cdot, t)$ over a ball bounds the L^1 norm of $u(\cdot, \tau)$ over a smaller ball for any previous or later time. We stress out that in the nonsingular case it is only possible a control for later times and NOT for previous times. Accordingly the constant γ deteriorates when $p \to 2^-$.

4.3. Elliptic-type Harnack estimates and extinction time. In this subsection we focus our attention on what is peculiar of the case $p < 2$. As discussed before, in the singular case, at the points where $|\nabla u| = 0$, the modulus of ellipticity becomes infinite. Hence, roughly speaking, the elliptic nature of the diffusion dominates the time-evolution of the process itself and this implies that the positivity of u at some point (x_0, t_0) spreads at the same time level. We have already seen a hint of this property in the fifth step of the proof of Theorem 4.1 but such a feature can be made quantitatively precise and assumes the form of an elliptic - type Harnack inequality, where the infimum of u over the ball $B_\rho(x_0)$ is bound by the supremum over the same ball at the <u>same</u> time level:

Theorem 4.15. *Let u be a nonnegative weak solution of (4.1) and let $\frac{2N}{N+1} < p < 2$. Let $(x_0, t_0) \in \Omega_T$, $\rho > 0$ and $t_0 > 0$. Let $\theta = C[u(x_0, t_0)]^{2-p}\rho^p$ where the constant C is defined in (4.12). Assume that the cylinder $(x_0, t_0) + B_{4\rho} \times (t_0 - 4\theta, t_0 + 4\theta)$ is contained in Ω_T. Then*

$$\gamma^{-1} \sup_{B_\rho(x_0)} u(\cdot, t) \leq u(x_0, t_0) \leq \gamma \inf_{B_\rho(x_0)} u(\cdot, t)\ , \tag{4.25}$$

where $\gamma > 1$ *depends only upon* N *and* p *and* $\gamma \to \infty$ *when either* $p \to (\frac{2N}{N+1})^+$ *or* $p \to 2^-$.

This result is proved in [67]. See also [176] for the extension of such a result to more general operators.

Remark 4.16. Estimate (4.25) fails in the case of nonnegative solutions of the heat equation and also for nonnegative solutions of the p-Laplacian when $p > 2$. To verify this in the case of the heat equation, consider the fundamental solution in 1-space dimension

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{x^2}{4t}} \ .$$

If Theorem 4.15 were to hold, we would have for some $\rho > 0$ that $\Gamma(n,1) \le \Gamma(n+\rho,1)$. Letting $n \to \infty$ we obtain a contradiction. That is the reason why the constants in (4.25) deteriorate when p goes to 2.

The elliptic-like Harnack inequality holds also for the nonnegative solutions of the porous medium equation. In such a case we get sharp estimates on the solution

Theorem 4.17. *Let* u *be a nonnegative weak solution of*

$$u_t - \Delta(u^m) = 0 \ , \qquad \text{in } \Omega_T \ , \tag{4.26}$$

and let $\frac{(N-2)_+}{N+2} < m < 1$. *Let* $\theta = C[u(x_0,t_0)]^{1-m}\rho^2$ *where the constant* C *is defined in (4.12). Assume that the cylinder* $(x_0,t_0) + B_{4\rho} \times (t_0 - 4\theta, t_0 + 4\theta)$ *is contained in* Ω_T. *Then for each multiindex* α

$$|D^\alpha u(x_0,t_0)| \le \frac{C^{|\alpha|+1}|\alpha|!}{\rho^\alpha} u(x_0,t_0) \tag{4.27}$$

and for every nonnegative integer k

$$\left| \frac{\partial^k}{\partial t^k} u(x_0,t_0) \right| \le \frac{C^{2k+1}(k!)^2}{\rho^{2k}} u(x_0,t_0)^{1-(1-m)k} \tag{4.28}$$

where $C > 1$ *depends only upon* N *and* m.

Proof. We render the equation *dimensionless* by the change of variables

$$\xi = \frac{x - x_0}{\rho}, \quad \tau = \frac{(t-t_0)[u(x_0,t_0)]^{m-1}}{\rho^2}, \quad v = \frac{u}{u(x_0,t_0)}.$$

This maps $Q_{4\rho}$ into $Q = Q^+ \cup Q^-$ where $Q^+ \equiv B(4) \times [0, 4C)$, $Q^- \equiv B(4) \times (-4C, 0]$. We denote again the new variables with x and t and observe that the rescaled function v is a bounded nonnegative solution of the equation

$$v_t - \Delta(v^m) = 0 \qquad \text{in } Q \tag{4.29}$$

with $v(0,0) = 1$. By the integral Harnack inequality (4.24) and the elliptic-like Harnack inequality (4.25) we have that there exist positive constants r_0 and $\gamma > 1$, depending only upon N and m, such that $\gamma^{-1} \leq v(x,t) \leq \gamma$ for each $(x,t) \in Q(r_0^2, r_0)$. Using this new piece of information, we have that in such a cylinder we can apply classical results due to Friedman [83] and Kinderleher-Nirenberg [106] to the equation (4.29) to obtain the analiticity of the solution. More precisely we get that for each multiindex α

$$|D^\alpha v(0,0)| \leq \frac{C^{|\alpha|+1}}{|\alpha|!}$$

and for every nonnegative integer k

$$\left|\frac{\partial^k}{\partial t^k} v(0,0)\right| \leq \frac{C^{2k+1}}{(k!)^2}$$

where $C > 1$ depends only upon N and m.

By the reverse change of variables we deduce (4.27) and (4.28). ■

Remark 4.18. Estimate (4.27) not only implies the analiticity of the solution in the space variables but it also says that when the solution vanishes at a point then all the derivatives vanish at the same point. Therefore, for the analiticity, the solution vanishes in all the domain Ω whenever vanishes at a point of Ω. Estimate (4.27) fails in the case of nonnegative solutions of the heat equation. Indeed, consider the fundamental solution in 1-space dimension

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{x^2}{4t}} \ .$$

If (4.27) were to hold, we would have for some $C > 0$

$$|\frac{d}{dx}\Gamma(n,1)| = \frac{1}{2} n\Gamma(n,1) \leq C\Gamma(n,1).$$

Letting $n \to \infty$ we obtain a contradiction.

Remark 4.19. The previous estimates say that a bounded nonnegative solution of the singular porous medium equation is analytic in the space variables and at least Lipschitz continuous in the time variable. We stress that these estimates are optimal. Indeed let z be the nonnegative and non trivial solution of the problem $z_{xx} = \frac{1}{1-m} z^{\frac{1}{m}}$ in the interval $(0,1)$, with boundary conditions $z(0) = z(1) = 0$. Then $u(x,t) = z^{\frac{1}{m}} (T-t)_+^{\frac{1}{1-m}}$ solves the equation $u_t = \Delta(u^m)$ and satisfies the above estimates sharply.

Remark 4.20. These estimates hold also in the case of a class of quasilinear parabolic equations. More precisely for nonnegative, local weak solutions of

$$u_t = \Delta(u^m) + f(x,t,u,\nabla u),$$

with $\frac{(N-2)_+}{N+2} < m < 1$, and f locally analytic and such that

$$0 \le f(x,t,u,\nabla u) \le F u^m$$

for some positive constant F (see [67]).

Another peculiarity of the case $p < 2$ is that the solution can become extinct in a finite time. The extinction profile is defined as the set $\partial[u > 0] \cap [\Omega \times (0,\infty)]$. By the elliptic-like Harnack principle the extinction profile is a portion of the hyperplane $\Omega \times \{t = T^*\}$.

Let us first consider the case of a bounded domain.

Theorem 4.21. *Let Ω be a bounded domain of $\mathbf{R}^N$. Let u be the unique nonnegative weak solution of*

$$\begin{cases} u \in C\left(\mathbf{R}^+; L^2(\Omega)\right) \cap L^p\left(\mathbf{R}^+; W_0^{1,p}(\Omega)\right) , \qquad 1 < p < 2 \\[2ex] u_t - \operatorname{div}|\nabla u|^{p-2}\nabla u = 0 , \qquad \text{in } \Omega \times \mathbf{R}^+ \\[2ex] u(\cdot,0) = u_0(x) \in L^\infty(\Omega) \qquad \text{and} \qquad u_0 \ge 0 . \end{cases} \tag{4.30}$$

Then there is a finite time T^*, depending only upon N, p and u_0, such that $u(\cdot,t) \equiv 0$ for all $t \ge T^*$.

Moreover, if $\max(1, \frac{2N}{N+2}) < p < 2$ then

$$0 < T^* \le \gamma^* \|u_0\|_{2,\Omega}^{2-p} |\Omega|^{\frac{N(p-2)+2p}{2N}} \tag{4.31}$$

with γ^* depending only upon N and p.

If $1 < p \le \frac{2N}{N+1}$ and $N \ge 2$ then

$$0 < T^* \le \gamma^* \|u_0\|_{s,\Omega}^{2-p} \tag{4.32}$$

with γ^* depending only upon N, p and $s = \frac{N(2-p)}{p}$.

Remark 4.22. Note that there is an overlap in the range of p in the previous estimates. We stress that for $1 < p < \frac{2N}{N+1}$, the upper estimate of T^* does not depend upon the measure of Ω.

Proof. Consider first the case $\max(1, \frac{2N}{N+2}) < p < 2$. Take u as a test function in the weak formulation of the equation (4.30) to get

$$\frac{d}{dt}\|u(t)\|_{2,\Omega}^2 + 2\|\nabla u(t)\|_{p,\Omega}^p = 0.$$

By Hölder's inequality and Sobolev embedding theorem

$$\|u(t)\|_{2,\Omega} \leq |\Omega|^{\frac{N(p-2)+2p}{2Np}} \|u(t)\|_{\frac{Np}{N-p},\Omega} \leq \gamma |\Omega|^{\frac{N(p-2)+2p}{2Np}} \|\nabla u(t)\|_{p,\Omega}.$$

In a straightforward way one may deduce that $\|u(t)\|_{2,\Omega}$ satisfies the following differential inequality

$$\frac{d}{dt}\|u(t)\|_{2,\Omega} + \gamma_1 \|u(t)\|_{2,\Omega}^{p-1} \leq 0 \ ,$$

where $\gamma_1 = \gamma^{-p}|\Omega|^{-\frac{N(p-2)+2p}{2N}}$. Solving the ordinary differential equation one obtains

$$\|u(t)\|_{2,\Omega} \leq \|u_0\|_{2,\Omega} \left\{ 1 - \frac{(2-p)\gamma_1 t}{\|u_0\|_{2,\Omega}^{2-p}} \right\}_+^{\frac{1}{2-p}} \tag{4.33}$$

and

$$0 < T^* \leq \frac{1}{2-p}\gamma^p |\Omega|^{\frac{N(p-2)+2p}{2N}} \|u_0\|_{2,\Omega}^{2-p} \ .$$

Consider now the case $1 < p \leq \frac{2N}{N+1}$ and $N \geq 2$. Let $s = \frac{N(2-p)}{p}$ and take u^{s-1} as a test function in the weak formulation of the equation (4.30) to get

$$\frac{1}{s}\frac{d}{dt}\|u(t)\|_{s,\Omega}^s + \gamma_2 \|\nabla u^{\frac{s+(p-2)}{p}}(t)\|_{p,\Omega}^p = 0 \ ,$$

where $\gamma_2 = (s-1)(\frac{p}{s+(p-2)})^p$. By Sobolev embedding theorem

$$\|u(t)\|_{s,\Omega}^s \leq \gamma \|\nabla u^{\frac{s+(p-2)}{p}}(t)\|_{p,\Omega}^{\frac{Np}{N-p}} \ .$$

In a straightforward way one may deduce that $\|u(t)\|_{s,\Omega}$ satisfies the following differential inequality

$$\frac{d}{dt}\|u(t)\|_{s,\Omega} + \gamma_3 \|u(t)\|_{s,\Omega}^{p-1} \leq 0 \ ,$$

where $\gamma_3 = \gamma^{-p}\gamma_2$. Solving the ordinary differential equation one obtains

$$\|u(t)\|_{s,\Omega} \le \|u_0\|_{s,\Omega} \left\{ 1 - \frac{(2-p)\gamma_3 t}{\|u_0\|_{s,\Omega}^{2-p}} \right\}_+^{\frac{1}{2-p}} \tag{4.34}$$

and

$$0 < T^* \le \frac{1}{(2-p)\gamma_2} \gamma^p \| \|u_0\|_{s,\Omega}^{2-p} .$$

■

Remark 4.23. The estimate (4.33) is *stable* as $p \to (\frac{2N}{N+1})^+$. As $p \to 2^-$ the boundary value problem (4.30) tends to the corresponding boundary value problem for the heat equation, for which the extinction in finite time does not occur. Accordingly, if $p \to 2^-$, the estimate (4.33) becomes

$$\|u(t)\|_{2,\Omega} \le \|u_0\|_{2,\Omega} e^{-\frac{t|\Omega|^{\frac{2}{N}}}{\gamma^2}}$$

where γ is the best constant of the Sobolev embedding of $W^{1,2}$ in $L^{\frac{2N}{N-2}}$.

The estimate (4.34) deteriorates as $p \to (\frac{2N}{N+1})^-$ and is *stable* as $p \to 1^+$. However, as the regularity results of the previous sections deteriorate as $p \to 1^+$, we cannot infer the convergence of (4.30) to a boundary value problem in some reasonable topology.

Remark 4.24. Theorem 4.21 holds for solutions of variable sign. The only modification occurs in the case $1 < p < \frac{2N}{N+1}$, $N \ge 2$. For this it is enough to work with the testing functions $|u|^{s-2}u$.

The Harnack principle gives also an estimate on the way the solution approaches the extinction. Let $M = \|u\|_{\infty,\Omega_\infty}$, where $\Omega_\infty \equiv \Omega \times (0,\infty)$.

Theorem 4.25. *Let Ω be a bounded domain of $\mathbf{R}^N$. Let u be the unique nonnegative weak solution of (4.30) and let $\frac{2N}{N+1} < p < 2$. Then there exists a constant γ, depending only upon N and p, such that for all $(x,t) \in \Omega \times (\frac{T^*}{2}, T^*)$*

$$u(x,t) \le \gamma \max \left\{ M^{2-p}; \frac{T^*}{[\mathrm{dist}\{x,\partial\Omega\}]^p} \right\}^{\frac{1}{2-p}} \left(\frac{T^*-t}{T^*} \right)^{\frac{1}{2-p}} .$$

Proof. Fix $x \in \Omega$ and $\frac{T^*}{2} \le t \le T^*$. Assume that $u(x,t) > 0$ and set

$$4\rho \equiv \min \left\{ \mathrm{dist}\,\{x,\partial\Omega\}\,; \left(\frac{T^*}{2M^{2-p}} \right)^{\frac{1}{p}} \right\} . \tag{4.35}$$

Consider the cylinder

$$Q_{4\rho}(x,t) = B_{4\rho}(x) \times \{t - [u(x,t)]^{2-p}(4\rho)^p, t + [u(x,t)]^{2-p}(4\rho)^p\} \ .$$

By the choice (4.35 the cylinder is contained in Ω_∞. Apply Harnack inequality (4.11) over the ball $B_\rho(x)$ and the cylinder $Q_{4\rho}(x,t)$. We must have

$$T^* - t \geq C[u(x,t)]^{2-p}\rho^p$$

otherwise, by Harnack estimate, $u(x,t) = 0$ against the assumption.

■

Consider now the case of the extinction in $\mathbf{R}^N$.

Theorem 4.26. *Let u be the unique nonnegative weak solution of*

$$\begin{cases} u \in C\left(\mathbf{R}^+; L^2(\mathbf{R}^N)\right) \cap L^p\left(\mathbf{R}^+; W^{1,p}(\mathbf{R}^N)\right) \ , \qquad 1 < p < 2 \\ u_t - \operatorname{div}|\nabla u|^{p-2}\nabla u = 0 \ , \qquad \text{in } \mathbf{R}^N \times \mathbf{R}^+ \\ u(\cdot,0) = u_0(x) \geq 0 \ . \end{cases} \tag{4.36}$$

Assume that u_0 is continuous in a ball $B(2R)$ and vanishes outside $B(R)$. Assume $1 < p \leq \frac{2N}{N+1}$ and $N \geq 2$, and let $s = \frac{N(2-p)}{p}$. Then there is a positive number T^*, depending only upon N, p and u_0, such that $u(x,t) = 0$ for each $t \geq T^*$. Moreover, $0 < T^* \leq \gamma^* \|u_0\|_{s,\Omega}^{2-p}$, with γ^* depending only upon N and p.

Proof. The solution of (4.36) can be constructed as the uniform limit of a sequence u_n of solutions in bounded domains where n is a natural number and $n \geq R$. More precisely,

$$\begin{cases} u_n \in C\left(\mathbf{R}^+; L^2(B(n))\right) \cap L^p\left(\mathbf{R}^+; W_0^{1,p}(B(n))\right) \\ (u_n)_t - \operatorname{div}|\nabla u_n|^{p-2}\nabla u_n = 0 \ , \qquad \text{in } B(n) \times \mathbf{R}^+ \\ u_n(\cdot,0) = u_0(x) \ . \end{cases} \tag{4.37}$$

By Theorem 4.21, the extinction time T_n^* is independent of $B(n)$. Moreover, as $u_n \leq u_{n+1}$, we have that u_n converges to the solution u and T_n^* converges to T^*.

■

Remark 4.27. Theorem 4.26 holds for solutions of variable sign and for data in $L^s(\mathbf{R}^N)$ with $s = N\frac{2-p}{p}$. The only modification in the proof occurs in making precise in what sense the solutions of the approximated problems converge to the solution of (4.36).

Remark 4.28. Theorem 4.26 implies that the range $\frac{2N}{N+1} < p < 2$ is optimal for a Harnack estimate to hold. Let $\Sigma_\infty \equiv \mathbf{R}^N \times \mathbf{R}_+$. Fix $(x_0, t_0) \in \mathbf{R}^N \times (0, T^*)$, where t_0 is so close to T^* as to satisfy

$$T^* - t_0 \le \frac{C}{4^p} t_0 \ , \tag{4.38}$$

where C is the constant appearing in (4.12). Now let $\rho > 0$ be so large that

$$C[u(x_0, t_0)]^{2-p} \rho^p = T^* - t_0 \ . \tag{4.39}$$

By the choice (4.38) the cylinder

$$Q_{4\rho}(t, x) = B_{4\rho}(x) \times \{t_0 - [u(x_0, t_0)]^{2-p}(4\rho)^p, t_0 + [u(x_0, t_0)]^{2-p}(4\rho)^p\}$$

is contained in Σ_∞. If the Harnack inequality (4.11) were to hold for $1 < p < \frac{2N}{N+1}$, $N \ge 2$, for some constants C and γ independent of ρ, it would give $0 < u(x_0, t_0) \le \gamma \inf_{x \in B_\rho(x_0)} u(x, T^*) = 0$. We stress that the choice (4.39) is possible in the whole Σ_∞.

The same argument implies that no extinction in finite time can occur for solutions of (4.36) if $\frac{2N}{N+1} < p < 2$. In such a range the Harnack estimate (4.11) holds and if a finite extinction time T^* were to exists, the choices (4.38) and (4.39) would give $u(x, T^*) > 0$.

4.4. Raleigh quotient and extinction profile. The Harnack estimates play a fundamental role in analyzing the asymptotic behaviour of solutions of singular equations. The physical motivation of such an analysis comes from the modelling of plasma assuming the Okuda-Dawson diffusion model (see [74] and [139], see also [22] and [23]). In [118] and [67] this analysis was carried out through Sobolev embedding Theorem, Harnack estimates and Raleigh quotient (see also [155], where general singular operators and general boundary conditions are considered). In this subsection this application of Harnack inequalities is described.

Let Ω be a bounded set of $\mathbf{R}^N$. Consider the Cauchy-Dirichlet problem:

$$\begin{cases} u \in C\left(\mathbf{R}^+; L^2(\Omega)\right) \cap L^p\left(\mathbf{R}^+; W^{1,p}(\Omega)\right) , & \frac{2N}{N+1} < p < 2 \\ u_t - \operatorname{div}|\nabla u|^{p-2}\nabla u = 0 , \quad \text{in } \Omega \times \mathbf{R}^+ & \\ u(\cdot,0) = u_0(x) \in L^2(\Omega) \quad \text{and} \quad u_0 \geq 0 & \\ u(x,t) = 0 \quad \forall\, x \in \partial\Omega \quad \text{and} \quad \forall\, t > 0 . & \end{cases} \tag{4.40}$$

The main result of this section is the following:

Theorem 4.29. *Let u be the unique nonnegative weak solution of (4.40) and let $T^* > 0$ be the extinction time. Let $u_*(x,t) = u(x,t)(T^* - t)^{\frac{1}{p-2}}$. Then there is a sequence $t_n \to T^*$ such that $u_*(x,t) \to v(x)$, where v is a nontrivial solution of the equation*

$$\operatorname{div}|\nabla v|^{p-2}\nabla v = \frac{1}{2-p}v , \qquad \text{in } \Omega \tag{4.41}$$

satisfying homogeneous Dirichlet boundary conditions.

Remark 4.30. The proof of Theorem 4.29 is heavily based upon the Sobolev embedding of $W^{1,p}$ into L^2. For this reason we assume Ω bounded and $\frac{2N}{N+1} < p < 2$. If $p \leq \frac{2N}{N+1}$ the result is false. Indeed if the Theorem were to hold for such a range of p, it would give the existence of a nontrivial solution of (4.41) in contradiction with known results of the elliptic theory (see [145]). We recall that the existence of a nonzero solution of (4.41) when $p \leq \frac{2N}{N+1}$ is not true in general, but depends on topological properties of the set Ω (see, for instance, [30]).

Before coming to the actual proof of Theorem 4.29, we consider some auxiliary results. First of all let us introduce the Raleigh quotient

$$\mathcal{E}[u](t) = \left(\frac{\|\nabla u\|_{p,\Omega}}{\|u\|_{2,\Omega}}\right)^p .$$

Proposition 4.31. *The quantity $\mathcal{E}[u](t)$ is not increasing in time.*

Proof. Choose u as a test function in the weak form of (4.40) to get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{2,\Omega}^2 + \|\nabla u(t)\|_{p,\Omega}^p = 0 . \tag{4.42}$$

Setting $\Delta_p(u)(t) = \mathrm{div}|\nabla u(t)|^{p-2}\nabla u(t)$, we obtain

$$\int_\Omega |\nabla u(t)|^p \mathrm{d}x = -\int_\Omega u(t)\Delta_p(u)(t)\mathrm{d}x \le \|u(t)\|_{2,\Omega}\left(\int_\Omega |\Delta_p(u)(t)|^2 \mathrm{d}x\right)^{\frac{1}{2}} . \tag{4.43}$$

On the other hand,

$$\frac{d}{dt}\int_\Omega |\nabla u(t)|^p \mathrm{d}x = p\int_\Omega |\nabla u(t)|^{p-2}\nabla u(t)\nabla u_t(t)\mathrm{d}x = -p\int_\Omega u_t(t)\Delta_p(u)(t)\mathrm{d}x . \tag{4.44}$$

Hence

$$\frac{d}{dt}\int_\Omega |\nabla u(t)|^p \mathrm{d}x = -p\int_\Omega |\Delta_p(u)(t)|^2 \mathrm{d}x , \tag{4.45}$$

which gives, together with (4.43),

$$\frac{d}{dt}\|\nabla u(t)\|^p_{p,\Omega} \le -p\frac{\|\nabla u(t)\|^{2p}_{p,\Omega}}{\|u(t)\|^2_{2,\Omega}} . \tag{4.46}$$

Directly from the equation one gets

$$\frac{1}{p}\frac{\frac{d}{dt}\|\nabla u(t)\|^p_{p,\Omega}}{\|\nabla u(t)\|^p_{p,\Omega}} \le \frac{1}{2}\frac{\frac{d}{dt}\|u(t)\|^2_{2,\Omega}}{\|u(t)\|^2_{2,\Omega}}$$

and this implies that $\mathcal{E}[u](t)$ is not increasing in time.

■

Let $B_{p,\Omega}$ be the best Sobolev constant of the embedding of $W^{1,p}(\Omega)$ in $L^2(\Omega)$.

Proposition 4.32. *The inequalities*

$$B_{p,\Omega} \le \mathcal{E}[u](t) \le \mathcal{E}[u](0) ,$$

hold.

Proof. It is a straightforward consequence of Proposition 4.31 and the definition of the Raleigh quotient.

■

The following Proposition gives a sharp estimate on the decay of the solution at the extinction time.

Proposition 4.33. *Let u be the unique nonnegative weak solution of (4.40) and let $T^* > 0$ be the extinction time. Then*

$$[(2-p)B_{p,\Omega}(T^*-t)]^{\frac{1}{2-p}} \le \|u(t)\|_{2,\Omega} \le [(2-p)\mathcal{E}[u](0)(T^*-t)]^{\frac{1}{2-p}}$$

and

$$\begin{aligned} B_{p,\Omega}\left[(2-p)B_{p,\Omega}(T^*-t)\right]^{\frac{1}{2-p}} &\leq \|\nabla u(t)\|_{p,\Omega} \\ &\leq \mathcal{E}[u](0)\left[(2-p)\mathcal{E}[u](0)(T^*-t)\right]^{\frac{1}{2-p}} \end{aligned}$$

Proof. It is sufficient to note that $\|u(t)\|_{2,\Omega}$ solves the O.D.E.

$$\frac{d}{dt}\|u(t)\|_{2,\Omega}^{2-p} = -(2-p)\mathcal{E}[u](t) \ .$$

Hence

$$0 = \|u(T^*)\|_{2,\Omega}^{2-p} = \|u(t)\|_{2,\Omega}^{2-p} - (2-p)\int_t^{T^*} \mathcal{E}[u](s)\mathrm{d}s \ .$$

Now the statement follows from Proposition 4.32 and the definition of the Raleigh quotient.

■

Using the previous results we can now conclude with

Proof of Theorem 4.29. Consider the change of variables: $t = T^* - T^* e^{-\tau}$. Let

$$w(\cdot,\tau) = \frac{u(\cdot, T^* - T^* e^{-\tau})}{(T^* e^{-\tau})^{\frac{1}{2-p}}}.$$

The function w is a nonnegative bounded weak solution of

$$\begin{cases} w_t = \mathrm{div}|\nabla w|^{p-2}\nabla w - \frac{1}{2-p}w \ , & \text{in } \Omega\times\mathbf{R}^+ \\ w(\cdot,0) = u_0(x) T^{*\frac{1}{p-2}} \end{cases} \tag{4.47}$$

Consider the functional

$$F(h) = \int_\Omega \left(\frac{1}{p}|\nabla h|^p - \frac{1}{2(2-p)}|h|^2\right)dx$$

and the function

$$g(\tau) = F(w(\tau)).$$

The function $g(\tau)$ is a nonincreasing function. Indeed, by using (4.44) and (4.47), we have

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_\Omega |Dw|^p dx &= -\int_\Omega w_t^2 dx + \frac{1}{2-p}\int_\Omega w_t w dx \\ &= \int_\Omega w_t^2 dx + \frac{1}{2(2-p)}\frac{d}{dt}\int_\Omega w^2 dx \ . \end{aligned}$$

Moreover, by Proposition 4.31 $\gamma(\tau)$ is bounded from below. Therefore there exists a sequence $\tau_n n \to \infty$ such that

$$\lim_{n\to\infty} \frac{d}{dt} F(w(\tau_n)) = 0.$$

From the previous calculations, this implies that $w_t(\tau_n) \to 0$ in $L^p(\Omega)$. Hence we get that there is a sequence $t_n \to T^*$ such that

$$\frac{u(\cdot, t_n)}{(T^* - t_n)} \to v$$

where v is the solution of (4.41). On the other hand, by Proposition 4.33, there are two positive constants c_1 and c_2, such that, for each t_n

$$0 < c_1 \leq \|\frac{\nabla u(\cdot, t_n)}{(T^* - t_n)}\|_{p,\Omega} \leq c_2.$$

Therefore

$$\frac{u(\cdot, t_n)}{(T^* - t_n)} \to v$$

in $W^{1,p}(\Omega)$. Applying the regularity results of the previous sections, it is easy to show that this convergence holds also in $C^\alpha(\bar{\Omega})$. Applying the results of the previous subsection, we have that v satisfies (4.25). ■

Remark 4.34. If the asymptotic profile of the singular porous medium equation is considered, arguing as before one can prove that the limiting solution v satisfies the sharpest estimates (4.27) and (4.28).

The approach through the Raleigh quotient can be applied also in the case of degenerate parabolic equations using similar arguments (see [125]).

Theorem 4.35. *Let $p > 2$ and let u be the unique nonnegative weak solution of (4.40). Let $u_*(x,t) = u(x,t)t^{\frac{1}{2-p}}$. Then there is a sequence $t_n \to \infty$ such that $u_*(x,t) \to w(x)$, where w is a nontrivial solution of the equation (4.41), satisfying Dirichlet boundary conditions.*

Remark 4.36. In the literature there are several papers devoted to the study of the asymptotic behaviour of the solutions of the porous medium equation and the p-Laplacian equation. Among them we quote [15], [14], [24], [25] and [174]. In these papers the approach is different from what we followed here. Indeed they first study the elliptic equation (4.41), then using some comparison principles they analyze the asymptotic behaviour of the evolution equation. With this approach things are somehow reversed: the basic properties of the evolution equation allow for the study of the asymptotic behaviour and the elliptic result follows as a consequence.

Remark 4.37. The proof of Theorems 4.29 and 4.35 is based on Sobolev embedding and weak convergence arguments. For this reason one can apply this approach also in the case of initial data with variable sign and in the case of Neumann or mixed boundary conditions ([155]).

5. Stefan-like problems

In this section we present some results about the local and global behaviour of weak solution of singular parabolic equations which model physical phenomena like transitions of phase and/or the flow of immiscible fluids in a porous medium. More precisely let us consider parabolic inclusions of the type

$$\frac{\partial}{\partial t}\beta(u) - \operatorname{div}\mathbf{A}(x,t,u,\nabla u) + B(x,t,u,\nabla u) \ni 0 \ , \qquad \text{in } \Omega_T \ , \tag{5.1}$$

where β is a maximal monotone graph in $\mathbf{R}\times\mathbf{R}$. We assume the coerciveness of the graph $\beta(\cdot)$, i.e., there exists a positive constant γ_0 such that

$$\beta(s_1) - \beta(s_2) \geq \gamma_o(s_1 - s_2) \ , \qquad \forall s_i \in \mathbf{R} \ . \tag{5.2}$$

We also assume that

$$\forall\ M > 0 \ , \qquad \sup_{-M\leq s\leq M} |\beta(s)| \equiv \gamma_1 < \infty \ . \tag{5.3}$$

We stress that we do not assume any further assumptions on the behaviour of $\beta(\cdot)$. In particular, in any finite interval, the graph might exhibit infinitely many jumps or become vertical infinitely many times with any possible growth (exponentially fast or faster). Examples of such $\beta(\cdot)$ are

$$\beta(s) = \begin{cases} s & \text{if } \ s < 0 \\ [0,1] & \text{if } \ s = 0 \\ 1+s & \text{if } \ 0 < s < 1 \end{cases} \quad ; \tag{5.4}$$

$$\beta(s) = \begin{cases} s & \text{if } \ s < 0 \\ [0,1] & \text{if } \ s = 0 \\ 1+s & \text{if } \ 0 < s < 1 \\ [2,3] & \text{if } \ s = 1 \\ 2+s & \text{if } \ s > 1 \end{cases} \quad ; \tag{5.5}$$

$$\beta(s) = |s|^{\frac{1}{m}} \operatorname{sign} s \ , \quad m > 1 \ ; \tag{5.6}$$

or

$$\beta(s) = 1 + s^{\alpha_1} - (1-s)^{\alpha_2} \ , \tag{5.7}$$

where $s \in [0,1]$ and $\alpha_i \in [0,1]$.

Equation (5.4) describes the enthalpy function in the weak formulation of a Stefan-like problem modelling a transition of phase, while (5.5) could be a good prototype to model the behaviour of the enthalphy in a double transition of phase. There is a wide literature concerning the classical Stefan problem. For a summary of the main results we refer the reader to the Monographs of Meirmanov [129] and Visintin [178], the review papers by Danilyuk [45], DiBenedetto [57] and by Visintin, Fasano, Magenes and Verdi [179], the Proceedings [29], [82], [94], as well as the references therein. Here, we consider only the aspects related to the local continuity of a weak solution. Equation (5.6) is the classical porous medium equation, describing the flow of a single fluid in a porous matrix, that was already mentioned in §2.5. Equation (5.7) is a first approximation of the flows of two immiscible fluids in a porous matrix. Once more this is a widely studied problem. For instance, Van Duijn and Zhang [173] considered a 1-dimensional model in hydrology (see also [93] for an investigation from a numerical point of view). For multi-dimensional multiphase models we refer the reader to the Monographs [18], [19], [34], [44], [156] and to the references therein.

The diffusion field $\mathbf{A}$ and the forcing term B in (5.1) are real valued and measurable over $\Omega_T \times \mathbf{R} \times \mathbf{R}^N$ and satisfy the structure conditions:

$$|\mathbf{A}(x,t,v,\bar{p})| \geq \mu_0(|v|)|\bar{p}|^2 - \phi_0(x,t) \ ; \tag{5.8}$$

$$|\mathbf{A}(x,t,v,\bar{p})| \leq \mu_1(|v|)|\bar{p}| - \phi_1(x,t) \ ; \tag{5.9}$$

$$|B(x,t,v,\bar{p})| \leq \mu_2(|v|)|\bar{p}|^2 - \phi_2(x,t) \ , \tag{5.10}$$

where $\mu_0 : \mathbf{R}_+ \to \mathbf{R}_+$ is a continuous and decreasing function, $\mu_1, \mu_2(\cdot) : \mathbf{R}_+ \to \mathbf{R}_+$ are continuous and increasing functions and ϕ_i $(i = 0,1,2)$ are non-negative and satisfy

$$\|\phi_0, \phi_2\|_{\hat{q},\hat{r},\Omega_T} \ , \qquad \|\phi_1\|_{2\hat{q},2\hat{r},\Omega_T} \leq \mu_3 \ . \tag{5.11}$$

Here μ_3 is a given constant and $\hat{q}$, $\hat{r}$ are positive numbers linked by

$$\frac{1}{\hat{r}} + \frac{N}{2\hat{q}} = 1 - \kappa_1 \ , \qquad 0 < \kappa_1 < 1 \tag{5.12}$$

with

$$\hat{q} \in \left[\frac{N}{2(1-\kappa_1)}, \infty\right] , \qquad \hat{r} \in \left[\frac{1}{1-\kappa_1}, \infty\right] , \qquad \text{if } N \geq 2 \tag{5.13}$$

$$\hat{q} \in (1, \infty) , \qquad \hat{r} \in \left[\frac{1}{1-\kappa_1}, \frac{1}{1-2\kappa_1}\right] , \qquad 0 < \kappa_1 < \frac{1}{2} , \qquad \text{if } N = 1 . \tag{5.14}$$

The inclusion (5.1) is in the sense of the graphs and in the weak sense. More precisely a function

$$u \in L^2_{\rm loc}\left(0, T; W^{1,2}_{\rm loc}(\Omega)\right) \tag{5.15}$$

is a weak solution of (5.1) if there exists a measurable selection $w \subseteq \beta(u)$, such that

$$t \to w(\cdot, t) \qquad \text{is weakly continuous in } L^2_{\rm loc}(\Omega)$$

and

$$\int_\Omega w(x,\tau)\, \phi(x,\tau)\, {\rm d}x \,\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\!\!\int_\Omega \left[-w(x,\tau)\frac{\partial}{\partial \tau}\phi(x,\tau)\right] {\rm d}x\, {\rm d}\tau \tag{5.16}$$

$$+ \int_{t_1}^{t_2}\!\!\int_\Omega \left\{\mathbf{A}(x,\tau,u,\nabla u)\cdot \nabla\phi + B(x,\tau,u,\nabla u)\,\phi\,\right\} \,{\rm d}x\, {\rm d}\tau = 0 ,$$

for all $\phi \in W^{1,2}_{\rm loc}\left(0,T; L^2_{\rm loc}(\Omega)\right) \cap L^2_{\rm loc}\left(0,T; W^{1,2}_0(\Omega)\right)$ and all intervals $[t_1, t_2] \subset (0, T]$.

In the sequel, when possible and for the sake of simplicity, we will work with the simplest example of (5.1), i.e.,

$$\frac{\partial}{\partial t}\beta(u) - \Delta u \ni 0 , \quad \text{in } \Omega_T . \tag{5.17}$$

We will obviously point out the situations in which the claimed results hold only for this simplest inclusion.

5.1. The continuity of weak solutions. It is quite natural to investigate if locally bounded weak solutions of (5.2) are continuous and if their modulus of continuity can be estimated quantitatively. Let us state this more precisely. For the sake of simplicity, assume that u is a solution of (5.2) and that it is bounded in Ω_T. Set

$$\|u\|_{\infty,\Omega_T} = M . \tag{5.18}$$

We recall that this assumption is not restrictive if we define Ω_T as the domain of definition of u. In a similar way, we assume the following integrability assumption

$$\|\phi_0 + \phi_1^2 + \phi_2\|_{\bar{q},\bar{r},\Omega_T} = \Phi \ . \tag{5.19}$$

We set the numbers

$$N \ , \quad \gamma_i \ , \quad M \ , \quad \Phi \ , \quad \mu_i \qquad (i = 0, 1, 2)$$

as the data. Consequently, we say that a constant $C = C(\text{data})$ or a continuous function $\omega(\cdot) = \omega_{\text{data}}(\cdot)$ if they can be determined *a priori* only in term of the above parameters. Let $\Theta \subset\subset \Omega_T$ be an arbitrary subset. In the sequel we investigate the problem of the continuity of u in Θ with a modulus of continuity $\omega_{\text{data}}(\cdot)$ depending only upon the data and the distance from Θ and the parabolic boundary of Ω_T.

Remark 5.1. If $\beta(\cdot)$ is the identity, we are in DeGiorgi's setting so any locally bounded weak solution is Hölder continuous in Ω_T. The assumptions (5.11) - (5.14) are optimal for this result to hold (see for instance the Monograph by Ladyzhenskaja, Solonnikov, and Ural'tzeva [121], Chapters 1,2 and 5). In this section we want to study how the singularity of β affects the regularity of u.

Remark 5.2. The assumption that u is bounded is essential. Even in the most favourable case when $\beta(\cdot)$ is the identity, weak solutions need not to be bounded (even in the elliptic case; see a counterexample by Stampacchia [159]). This is due to the critical growth of the forcing term $B(x, t, u, \nabla u)$ with respect to ∇u. We recall that in the nonsingular case, if we assume that

$$B(x, t, u, \nabla u) \leq \mu_2 |\nabla u|^q + \phi_2(x, t) \ ,$$

where $0 \leq q < \frac{N+4}{N+2}$, instead of (5.10), then any weak solution is locally bounded. This statement follows by means of a simple adaptation of the method of [121] (see also [47] and [132]). Even when the solution is not necessarily locally bounded, if one has some *a priori* qualitative knowledge of the boundedness of the solution such a qualitative information can, in many cases, be turned into a quantitative one (see, for instance, [141], [26], [175] and the references therein).

In the sequel we assume in addition that the solutions of (5.1) can be constructed as the limit in the topology of (5.15) of a sequence of smooth local solutions of (5.1) with smooth $\beta(\cdot)$. This assumption is made in order to justify some of the calculations and to deal with equations instead of inclusions. We stress that the modulus of continuity of u must be independent of any approximating procedure and must depend only upon the data. Such a result gives us some compactness that, in many cases, is fundamental to obtain existence of a solution. Actually if one approximates the equation with a sequence of regular ones, through the estimate of the modulus of continuity, one obtains that the approximating solutions are uniformly continuous. Then, via the Ascoli-Arzelá Theorem one gets the convergence in the uniform norm of a subsequence of approximating functions to a continuous function v. Thus, in many cases, by applying the method exploited by Kinderlehrer and Stampacchia [107] and based on Minty's lemma [130], it is possible to prove that v is a weak solution of the original equation.

Lastly we recall that if

$$B(x,t,u,\nabla u) \leq \mu_2 |\nabla u|^q + \phi_2(x,t) \ ,$$

where $0 \leq q < \frac{N+4}{N+2}$, the questions of existence and uniqueness are well understood. We refer the readers to the Monographs [84], [121], [123], to the Proceedings [29], [82], [94] and to the references therein.

5.2. A bridge between singular and degenerate equations. The understanding of the physical model requires the analysis of the equation (5.1) in its full generality. For instance, the flow of two immiscible fluids is described by a system of two parabolic equations, written in terms of the saturations and pressures of the two fluids (see, for instance, Chap. 9 of [18], Chap. 6 of [19], Chap. 6 of [44], Chap. 10 of [156] and the article by Leverett [122]). The transformation by Kruzkov-Sukorjanski [115] reduces the above system of two equations to a system of a degenerate-elliptic equation in terms of the mean pressure and a parabolic equation of the type

$$v_t - \operatorname{div} \mathbf{a}(x,t,v,\nabla v) + b(x,t,v,\nabla v) = 0 \ , \quad \text{in } \Omega_T \tag{5.20}$$

in terms of the saturation v of only one of the two fluids. In equation (5.20), the forcing term $b(x,t,v,\nabla v)$ depends essentially on the mean pressure.

The diffusion field $\mathbf{a}$ and the forcing term b in (5.20) are real valued and measurable over $\Omega_T \times \mathbf{R} \times \mathbf{R}^N$ and satisfy the structure conditions:

$$|\mathbf{a}(x,t,v,\bar{v})| \geq \phi(|v|)|\bar{v}|^2 - \phi_0(x,t) \ ; \tag{5.21}$$

$$|\mathbf{a}(x,t,v,\bar{v})| \leq \phi(|v|)|\bar{v}| - \phi_1(x,t) \ ; \tag{5.22}$$

$$|b(x,t,v,\bar{v})| \leq \phi(|v|)|\bar{v}|^2 - \phi_2(x,t), \tag{5.23}$$

where $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ is a continuous function and ϕ_i $(i = 0,1,2)$ are non-negative and satisfy assumptions (5.11)- (5.12).

As the function $(x,t) \to v(x,t)$ represents the local relative saturation of one of the two fluids (see, for instance, [5], [18], [19], [44], [115], [156]), it is natural to assume it is bounded, for example $v \in [0,1]$. The equation is degenerate because $\phi(\cdot)$ is allowed to vanish. More precisely we assume that

$$\phi(v) > 0 \ , \ \forall\, v \in [0,1] \ ; \qquad \phi(0) = \phi(1) = 0 \ . \tag{5.24}$$

Obviously v satisfies (5.20) in the weak sense defined previously and we require that

$$\nabla(\phi(v)) \in L^2_{\text{loc}}(\Omega_T) \ . \tag{5.25}$$

The problem of proving the local continuity of the saturations was studied in [5], [53] and [170], for example. The continuity of the saturations comes from the continuity of the solution of (5.20) arguing as in [5] and [53]. Proving the continuity of the solution is difficult, non only for the double degeneracy of $\phi(\cdot)$, but also because of the lack of a precise quantitative and qualitative information on its modulus of continuity. Actually the function ϕ is related to the permeability of both fluids and the permeability vanishes as one fluid is totally replaced by the other one (that is, when $v = 0$ or $v = 1$); this is the physical origin of the degeneracy of $\phi(\cdot)$. The information on the rate of vanishing is limited because it is derived only from hydrostatic experiments (see [18], [19], [44], [156]), dimensional analysis (see [122]) and heuristic arguments. As a matter of fact, such a limited information on the nature of the degeneracy is typical of models of flows of a mixture of fluids in a porous medium. Hence $\phi(\cdot)$ could degenerate at $v = 0$ and $v = 1$ at different rates (exponentially fast or faster). By the phenomenon of connate water it might be even completely flat in a small right neighborhood of 0 or in a small left neighborhood of 1 (see Chap. 9 of [18], Chap. 2 of [44], Chaps. 3 and 10 of [156]). So the problem of the continuity of weak solutions of (5.20) consists in showing that v is continuous *whatever* the nature of the degeneracy of $\phi(\cdot)$ is.

Let $u \in [0,1]$ be a solution of (5.1) with $\beta(\cdot) \in C^1(0,1)$ and singular in 0 and 1. For example assume

$$\lim_{u\to 0^+} \beta'(u) = \lim_{u \to 1^-} \beta'(u) = +\infty \ .$$

Then, by setting $v = \beta(u)$ and $\phi(\cdot) = \beta^{-1\prime}(\cdot)$, the singular equation (5.1) in terms of u is recasted as the degenerate equation (5.20) in terms of v. Moreover, all the assumptions on $\mathbf{a}$ and b are satisfied. Viceversa, putting

$$u = \int_0^v \phi(s)\,\mathrm{d}s\ , \qquad \beta(u) = v\ ,$$

one gets the singular equation from the degenerate one. In this case, however, while the assumptions on $\mathbf{A}$ are verified, the assumption on the free term B can not be verified in the case of a superlinear growth with respect to ∇v. We refer the reader to [5] for a detailed technical analysis of this case.

5.3. Parabolic equations with one-point singularity. Let us consider the case where $\beta(\cdot)$ is singular at only one point (the prototype cases are given by the examples (5.4) and (5.6)). The local continuity of the solutions was proved in the papers [31], [48], [50], [52], [149], [150] and [184]. However the situations of (5.4) and (5.6) are very different; actually, for $\beta(\cdot)$ of the type (5.6) it is possible to repeat De Giorgi's argument (i.e. to find suitable Caccioppoli and logarithmic estimates such that an embedding in the space of Hölder continuous functions holds) while that procedure is impossible in the case of (5.4). In this case, the Caccioppoli and logarithmic estimates are more complicated than in the nonsingular case:

there exists a constant γ, only depending upon the data, such that for each

$$(y,s) + Q(\sigma\theta\rho^2, \sigma\rho) \subset (y,s) + Q(\theta\rho^2,\rho)\ , \qquad \sigma \in (0,1)$$

$$\begin{aligned} &\sup_{s-\theta\rho^2\le t\le s} \int_{y+K_{\sigma\rho}} (u-k)^2_\pm(x,t) + \int\int_{(y,s)+Q(\sigma\theta\rho^2,\sigma\rho)} |\nabla(u-k)_\pm|^2 \\ &\le \frac{\gamma}{(1-\sigma)^2\rho^2}\int\int_{(y,s)+Q(\theta\rho^2,\rho)} (u-k)^2_\pm \\ &+\frac{\gamma}{(1-\sigma)\theta\rho^2}\int\int_{(y,s)+Q(\theta\rho^2,\rho)} (u-k)_\pm\ ; \end{aligned} \tag{5.26}$$

$$\sup_{s-\theta\rho^2\le t\le s}\int_{y+K_{\sigma\rho}}(u-k)_\pm^2(x,t)+\int\int_{(y,s)+Q(\sigma\theta\rho^2,\sigma\rho)}|\nabla(u-k)_\pm|^2$$

$$\le\frac{\gamma}{(1-\sigma)^2\rho^2}\int\int_{(y,s)+Q(\theta\rho^2,\rho)}(u-k)_\pm^2 \tag{5.27}$$

$$+\gamma\int_{y+K_\rho}(u-k)_\pm(x,s-\theta\rho^2)\ ;$$

$$\sup_{s-\theta\rho^2\le t\le s}\int_{y+K_{\sigma\rho}}\Psi^2(\mathcal{H}_k^\pm,(u-k)_\pm,c)(x,t)$$

$$\le\frac{\gamma}{(1-\sigma^2)\rho^2}\int\int_{(y,s)+Q(\theta\rho^2,\rho)}\Psi(\mathcal{H}_k^\pm,(u-k)_\pm,c) \tag{5.28}$$

$$+\frac{\gamma}{c}\int_{y+K_\rho}\Psi^2(\mathcal{H}_k^\pm,(u-k)_\pm,c)(x,s-\theta\rho^2)\ .$$

Note that, for the sake of simplicity, we have written the above estimates only in the case of the prototype equation (5.17).

If $N=2$, the existence of the modulus of continuity can be derived from the above estimates (see [72]). If $N\ge 3$ there are bounded discontinuous functions satisfying the previous estimates (see [158]). So in order to prove the local continuity of the solutions one has to use the structure of the parabolic equation. In [31], [48], [50], [52], [149], [150] and [184] the proof of continuity has this common point: assume that $\beta(\cdot)$ is singular only at a point, say, for example, at 0. Fix a cylinder $(y,s)+Q_\rho$. There are two possibilities: (i) either the singularity occupies a small portion of such a cylinder (and in such a case it plays a negligible role); or (ii) the singularity occupies a large portion. In such a situation, outside the singular set, the evolution equation is uniformly parabolic so the solution remains close to 0 because it cannot grow too fast due to the classical regularity properties of non singular parabolic equations. This gives us a control on the oscillation of the solution and allows us to obtain some recursive inequalities that will imply the local continuity. As it is absolutely evident, the whole argument is based on the fact that $\beta(\cdot)$ has only a singularity. The proofs of these recursive inequalities differ in [31], [48], [50], [149], [150] and [184]. In [31], the authors use the local representation in terms of heat potentials (for this reason their approach works only in the case of the prototype equation (5.17)). In [48] and [50], De Giorgi's iterations are used following the setting of [121]. The shrinking technique introduced by Krylov and Safonov ([117]) is applied in [149] and [150]. As this method is genuinely based on the nondivergence structure of the operator, also in this case the approach works only for the prototype equation (5.17). Lastly in [184] the Harnack-type techniques of Moser are applied, following the setting of [16], [132], [134] and [164].

The singularity of $\beta(\cdot)$ changes the iterative procedure of the nonsingular case. Actually the singularity affects the sequence of the radii of the nested cylinders and the reduction of the oscillation. More precisely, let

$$\eta\ ,\delta : (0,2M] \to (0,1)\ ; \qquad \eta(0),\delta(0)=0 \tag{5.29}$$

and define

$$\begin{aligned} \omega_0 &= \max\left\{2M; C\rho_0^{\lambda}\right\}\ ; \\ \rho_{n+1} &= \delta(\omega_n)\rho_n\ ; \\ \omega_{n+1} &= \max\left\{(1-\eta(\omega_n))\omega_n; C\rho_n^{\lambda}\right\}\ , \end{aligned} \tag{5.30}$$

where $C>1$ and $\lambda \in (0,1)$ are two given constants. Define also the corresponding family of shrinking nested cylinders $(x_0,t_0)+Q_{\rho_n}$.

Proposition 5.3. *Let u be a weak solution of (5.1) with $\beta(\cdot)$ a graph of Stefan type (5.4). Then there exist constants $C>1$ and $\lambda \in (0,1)$, and two continuous functions $\delta(\cdot),\eta(\cdot)$ as in (5.29), that can be determined a priori only in terms of data, such that for every $(x_0,t_0)\in\Omega_T$ and every $n\in\mathbf{N}$,*

$$\operatorname*{\mathit{ess\ osc}}_{(x_0,t_0)+Q_{\rho_n}} u \le \omega_n\ , \tag{5.31}$$

where λ is a number determined only in terms of the integrability conditions (5.8)-(5.13) and is independent of δ and η. As a consequence u is locally continuous in Ω_T.

Proof. For a detailed proof we refer the reader to [48]. Here we assume that (5.31) holds so that to prove the proposition it is sufficient to show that $\{\omega_n\}\to 0$ when $n\to\infty$. From the definition, the sequences $\{\omega_n\}$ and $\{\rho_n\}$ are non-increasing, so their limits exist when $n\to\infty$. It is clear that $\lim\limits_{n\to\infty}\rho_n=0$ because the function $\delta(\cdot)\in(0,1)$.

Now, assume that $\lim\limits_{n\to\infty}\omega_n=\omega_\infty>0$. Then

$$\omega_{n+1}=\max\left\{(1-\eta(\omega_\infty))\omega_n; C\delta(\omega_\infty^{\lambda n}\rho^{\lambda})\right\}$$

and hence we get that $\lim_{n\to\infty}\omega_n=0$, which contradicts the assumption.

■

Remark 5.4. The constants C and λ appearing in (5.30) are due only to the functions ϕ_i appearing in the structure conditions (5.8)-(5.10), and they are zero for the prototype equation (5.17).

Remark 5.5. With respect to the case of a uniformly parabolic equation, here the modulus of continuity is not explicit but it can be derived quantitatively from (5.30). More precisely, in [48], it is proved that $\delta(s), \eta(s)$ have the form $K^{-\frac{h}{s}}$ with K and h large constants, but it is not obtained an explicit modulus of continuity for u in terms of K and h.

5.4. Parabolic equations with multiple singularities. Graphs $\beta(\cdot)$ that are singular at multiple points, besides their intrinsic mathematical interest, arise naturally in phenomena of multiple transitions of phase. An example is a water-ice-vapour triple point and another one is the Buckley-Leverett model of two immiscible fluids in a porous medium (see the previous sections for more details about these models).

The first attempt to prove continuity results in such a setting was made in [5], where some restrictions were made on the singularities. Actually, the authors considered the case of the Buckley-Leverett model, i.e., the case of only two singularities. They assumed that $\beta(\cdot)$ could be singular at any rate in one point, while in the second point the singularity allowed was only of logarithmic type. This result was improved in [53], by allowing the second singularity to have a power-like behaviour, and in [170] where Hölder continuity was proved assuming that both degeneracies are power-like. The alternative argument is quite similar to the one of only one singularity. Let Q_ρ be the cylinder where we want to reduce the oscillation of the solution. If the non restricted singularity occupies a small portion of such a cylinder it means that it plays a negligible role and the continuity results for porous medium equation hold. If the singularity occupies a large portion, outside the singular set, the solution cannot grow too fast due to the regularity properties of porous medium equations. This gives a control on the oscillation of the solution and allows one to obtain some recursive inequalities that imply the local continuity. It is clear, however, that any parabolic approach cannot face the case of two unrestricted singularities. In the pioneering paper [72], the approach is based on the energy estimates and on some measure-theoretical results. The results obtained in [72] are optimal in the case $N = 2$ and hold for the prototype equation (5.17).

Theorem 5.6. *Let $N = 2$. Let u be a locally bounded weak solution of (5.1), where $\beta(\cdot)$ is any maximal monotone graph satisfying conditions (5.2) and (5.3), and let the structure conditions (5.8)–(5.14) are satisfied. Then u is continuous in Ω_T. Moreover, for every compact subset $K \subset \Omega_T$, there exists a continuous, nonnegative, increasing function*

$$s \ \mapsto \ \omega_{data,K}(s) \ ; \qquad \omega_{data,K}(0) = 0$$

that can be determined a priori only in terms of the data and the distance from K to the parabolic boundary of Ω_T, such that

$$|u(x_1, t_1) - u(x_2, t_2)| \leq \omega_{data,K}(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}}) \ ,$$

for every pair of points $(x_i, t_i) \in K$, $i = 1, 2$.

In the case $N \geq 3$, it is necessary to assume some conditions either on the maximal graph $\beta(\cdot)$ or on the structure of the elliptic operator. In [72] the following result is proved.

Theorem 5.7. *Let $N \geq 3$. Let u be a locally bounded weak solution of the prototype equation (5.17), where $\beta(\cdot)$ is any maximal monotone graph satisfying conditions (5.2) and (5.3). Then u is continuous in Ω_T. Moreover, for every compact subset $K \subset \Omega_T$, there exists a continuous, nonnegative, increasing function*

$$s \ \mapsto \ \omega_{data,K}(s) \ ; \qquad \omega_{data,K}(0) = 0$$

that can be determined a priori only in terms of the data and the distance from K to the parabolic boundary of Ω_T, such that

$$|u(x_1, t_1) - u(x_2, t_2)| \leq \omega_{data,K}(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}}) \ ,$$

for every pair of points $(x_i, t_i) \in K$, $i = 1, 2$.

5.4.1. *The statement of the alternative.* For the moment we consider the general equation (5.1) in any number of dimensions. We assume that $\beta(\cdot)$ is any maximal monotone graph satisfying conditions (5.2) and (5.3) and that the structure conditions (5.8)–(5.14) hold. Only later we will point the differences between $N = 2$ and $N \geq 3$. Without loss of generality, we assume that the generic point (x_0, t_0) is equal to $(0, 0)$.

Consider the following coaxial cylinders with vertex in $(0,\tilde{t})$ and congruent to $Q_{4\delta\rho}$

$$(0,\tilde{t}) + Q_{4\delta\rho} = K_{4\delta\rho} \times (\tilde{t} - (4\delta\rho)^2, \tilde{t}) \ ,$$

where

$$\tilde{t} \in (-(1-16\delta^2)\rho^2, 0) \tag{5.32}$$

and $\delta \in (0, \frac{1}{4})$ is a positive number to be chosen.

By moving the time, one seeks to find a cylinder where one can apply the techniques developed in the previous sections. More precisely, we look for $\tilde{t}$ such that the subset of $(0,\tilde{t}) + Q_{4\delta\rho}$ where u is close to μ^+ or to μ^- is small. Denote by ω the oscillation of u in Q_ρ. Then, one of these two possible alternatives takes place:

there exists a $\tilde{t} \in (-(1-16\delta^2)\rho^2, 0)$ and a positive δ such that either

$$\text{Meas}\left\{(x,t) \in (0,\tilde{t}) + Q_{4\delta\rho} \ : \ u(x,t) \geq \mu^+ - \frac{1}{6}\omega\right\} \leq \nu |Q_{4\delta\rho}| \tag{5.33}$$

or

$$\text{Meas}\left\{(x,t) \in (0,\tilde{t}) + Q_{4\delta\rho} \ : \ u(x,t) \leq \mu^- + \frac{1}{6}\omega\right\} \leq \nu |Q_{4\delta\rho}| \ . \tag{5.34}$$

Otherwise,

$$\text{both (5.33) and (5.34) are violated } \forall \tilde{t} \in (-(1-16\delta^2)\rho^2, 0) \ , \tag{5.35}$$

where $\nu \in (0,1)$ is a number that will be determined a priori only in terms of the data.

Assume for the moment that one of (5.33) and (5.34) holds. By using an iterative argument based on the energy estimates, we can show that it is possible to reduce the oscillation of u in the smaller cylinder $(0,\tilde{t}) + Q_{2\delta\rho}$ if one chooses ν very small and depending upon ω (see Proposition $2.5^{\pm}$ of [72]):

Proposition 5.8. *There exists a number $\nu \in (0,1)$, that can be determined a priori only in terms of the data and ω, such that, if (5.33) holds for some $\tilde{t} \in (-(1-16\delta^2)\rho^2, 0)$, then either $\omega \leq C\rho^\lambda$ or*

$$u(x,t) \leq \mu^+ - \frac{1}{16}\omega \ , \qquad (x,t) \in (0,\tilde{t}) + Q_{2\delta\rho} \ . \tag{5.36}$$

Analogously, if (5.34) holds for some $\tilde{t} \in (-(1-16\delta^2)\rho^2, 0)$*, then either* $\omega \le C\rho^\lambda$ *or*

$$u(x,t) \ge \mu^- + \frac{1}{16}\omega\ , \qquad (x,t) \in (0,\tilde{t}) + Q_{2\delta\rho}\ . \tag{5.37}$$

We recall to the reader that C is a constant depending upon the structure conditions of equation (5.1).

By using the logarithmic estimates (see Proposition $3.2^{\pm}$ of [72]) it is possible to bring the information of the reduction of the oscillation of the solution up to the level $t = 0$. More precisely:

Proposition 5.9. *Assume that there is* $\tilde{t} \in (-(1-16\delta^2)\rho^2, -4\delta^2\rho^2)$ *such that*

$$u(x,\tilde{t}) \le \mu^+ - \frac{1}{16}\omega\ , \qquad \forall x \in K_{2\delta\rho}\ . \tag{5.38}$$

Then there are constants $\eta, \lambda \in (0,1)$ *and* $C > 1$*, depending upon the data and* δ*, but independent of* ω *and* ρ*, such that either* $\omega \le C\rho^\lambda$ *or*

$$u(x,t) \le \mu^+ - \eta\omega\ , \qquad (x,t) \in (0,0) + Q_{\delta\rho}\ . \tag{5.39}$$

Analogously, if there is $\tilde{t} \in (-(1-16\delta^2)\rho^2, -4\delta^2\rho^2)$ *such that*

$$u(x,\tilde{t}) \le \mu^- + \frac{1}{16}\omega\ , \qquad \forall x \in K_{2\delta\rho} \tag{5.40}$$

then either $\omega \le C\rho^\lambda$ *or*

$$u(x,t) \ge \mu^- + \eta\omega\ , \qquad (x,t) \in (0,0) + Q_{\delta\rho}\ . \tag{5.41}$$

Summarizing, if the alternatives (5.33) and (5.34) are satisfied, then respectively (5.38) and (5.40) hold. Therefore the oscillation of u is reduced in the smaller cylinder $(0,0) + Q_{\delta\rho}$. If we are able to prove that even when (5.35) holds we have a reduction of the oscillation of u in a smaller cylinder, by repeating the arguments of the previous sections, one can deduce the local continuity of the solution. Therefore, in the sequel, we assume that for each small δ and for each $\tilde{t} \in (-(1-16\delta^2)\rho^2, 0)$

$$\text{Meas}\left\{(x,t) \in (0,\tilde{t}) + Q_{4\delta\rho}\ :\ u(x,t) \ge \mu^+ - \frac{1}{8}\omega\right\} \ge \nu|Q_{4\delta\rho}|$$

and

$$\text{Meas}\left\{(x,t) \in (0,\tilde{t}) + Q_{4\delta\rho}\ :\ u(x,t) \ge \mu^- + \frac{1}{8}\omega\right\} \ge \nu|Q_{4\delta\rho}|\ .$$

By using a lemma of measure theory, the above inequalities will imply regions where the energy is concentrated. From this fact, we are able to find a contradiction in assuming (5.35) and hence to prove the regularity of the solutions. In the next section we state and prove the lemma of measure theory. We feel that it is of intrinsic interest and that it can be applied in different fields.

5.4.2. *A lemma of measure theory.*

Lemma 5.10. *Let v be a function in $W^{1,p}(K_\rho)$, $p > 1$, satisfying*

$$\int_{K_\rho} |\nabla v|^p \, \mathrm{d}x \le \gamma^p \rho^{N-p} \ , \tag{5.42}$$

for a given positive constant γ, and let

$$\text{Meas}\,\{x \in K_\rho \ : \ v(x) < 1\} > \alpha |K_\rho| \ , \tag{5.43}$$

for a given $\alpha \in (0,1)$. Then, for every $\eta \in (0,1)$ and $\lambda > 1$, there exists $x^* \in K_\rho$ and a number $\delta \in (0,1)$, that can be determined a priori only in terms of $N, p, \gamma, \alpha, \lambda, \eta$, such that within the cube $K_{\delta\rho}(x^*)$ centered at x^* with wedge $2\delta\rho$, there holds

$$\text{Meas}\,\{x \in K_{\delta\rho}(x^*) \ : \ v(x) < \lambda\} > (1-\eta)|K_{\delta\rho}(x^*)| \ . \tag{5.44}$$

If v were continuous, this lemma would be an easy consequence of the permanence of positivity. For the sake of simplicity, we prove this lemma assuming $N = 2$ (the case $N \ge 3$ can be proved using an inductive procedure) and $v \in C^1(K_\rho)$. Obviously we establish (5.44) with δ independent of the modulus of continuity of v. The assumption $v \in C^1(K_\rho)$ is made only to justify some calculations and can be removed via a limiting procedure. For more details on this proof, we refer the reader to Proposition A.1 of [72]. Before proving the lemma we stress that δ goes to 0 when either η goes to 0 or λ goes to 1.

Proof. Let (x, y) be the coordinates in $\mathbf{R}^2$. Denote by $Y(x)$ the cross section of the set $[v < 1] \cap K_\rho$ with lines parallel to the y-axis, i.e.,

$$Y(x) = \{y \in (-\rho, \rho) \ : \ v(x,y) < 1\} \ .$$

Hence

$$|[v < 1] \cap K_\rho| = \int_{-\rho}^{\rho} |Y(x)| \, \mathrm{d}x \ .$$

As $|[v<1]\cap K_\rho|\geq 2\alpha\rho$, there exists some $x_0\in(-\rho,\rho)$ such that

$$|Y(x_0)|\geq 2\alpha\rho\ . \tag{5.45}$$

For each $y\in Y(x_0)$, consider the segment

$$I_{\delta_0\rho}(y)=[x_0-\delta_0\rho,x_0+\delta_0\rho]\times\{y\}\ ,$$

where δ_0 will be chosen later. Denote with $Y_{\delta_0}(x_0)$ the set of $Y(x_0)$ such that, in the corresponding intervals $I_{\delta_0\rho}(y)$, the function $v(\cdot,y)$ is less than $\frac{1}{2}(1+\lambda)$, i.e.,

$$Y_{\delta_0}(x_0)=\left\{y\in Y(x_0)\ :\ v(x,y)<\frac{1}{2}(1+\lambda)\ ,\quad \forall x\in I_{\delta_0\rho}(y)\right\}\ .$$

Now, we want to prove that that, for each $\eta_0<1$ there exists a small δ_0 such that

$$|Y_{\delta_0}(x_0)|\geq(1-\eta_0)|Y(x_0)|\ .$$

Let $Y^C_{\delta_0}(x_0)$ be the complement of $Y_{\delta_0}(x_0)$. Fix $y\in Y^C_{\delta_0}(x_0)$ and some $x\in I_{\delta_0\rho}(y)$ such that $v(x,y)\geq\frac{1}{2}(1+\lambda)$. Then

$$\frac{1}{2}(\lambda-1)\leq v(x,y)-v(x_0,y)=\int_{x_0}^{x}v_x(s,y)\,\mathrm{d}s\ .$$

By integrating over $Y^C_{\delta_0}(x_0)$ and majorising the obtained result via Hölder's inequality, we get

$$\frac{1}{2}(\lambda-1)|Y^C_{\delta_0}(x_0)|\leq\int_{Y(x_0)}\int_{-\delta_0\rho}^{\delta_0\rho}|\nabla v|\,\mathrm{d}x\,\mathrm{d}y\leq\left(|Y_{\delta_0}(x_0)|2\delta_0\rho\right)^{1-\frac{1}{p}}\|\nabla v\|_{p,K_\rho}\ .$$

Therefore, using (5.42) and (5.45) one gets:

$$|Y^C_{\delta_0}(x_0)|<\frac{4\gamma\delta_0^{1-\frac{1}{p}}}{(\lambda-1)(4\alpha)^{\frac{1}{p}}}|Y(x_0)|\ .$$

So we have that

$$|Y_{\delta_0}(x_0)|\geq(1-\eta_0)|Y(x_0)|\ ,$$

choosing δ_0 such that $\frac{4\gamma\delta_0^{1-\frac{1}{p}}}{(\lambda-1)(4\alpha)^{\frac{1}{p}}}\leq\eta_0$.

Next, fix $y^*\in Y_{\delta_0}(x_0)$. We recall that $v(x,y^*)\leq\frac{(1+\lambda)}{2}$ for each $x\in I_{\delta_0\rho}(y^*)$. Consider the vertical segment

$$J_{\delta\rho}(x)=\{x\}\times[y^*-\delta\rho,y^*+\delta\rho]\ ,$$

where δ will be chosen later. Denote with $H_\delta(y^*)$ the set of $I_{\delta_0\rho}(y^*))$ such that, in the corresponding intervals $J_{\delta\rho}(x)$, the function $v(x,\cdot)$ is less than λ, i.e.,

$$H_\delta(y^*) = \{x \in I_{\delta_0\rho}(y^*) \ : \ v(x,y) < \lambda \ , \quad \forall y \in J_{\delta\rho}(x)\} \ .$$

Now, we want to prove that, for each $\eta < 1$, there exists a small δ such that

$$|H_\delta(y^*)| \geq (1-\eta)|I_{\delta_0\rho}(y^*)| \ .$$

Let $H_\delta^C(y^*)$ be the complement of $H_\delta(y^*)$. Fix $x \in H_\delta^C(y^*)$ and some $y \in J_{\delta\rho}(x)$ such that $v(x,y) \geq \lambda$. Then

$$\frac{1}{2}(\lambda-1) \leq v(x,y) - v(x,y^*) = \int_{y^*}^{y} v_y(x,s)\,\mathrm{d}s \ .$$

By integrating over $H_\delta^C(y^*)$ and majorising the obtained result via Hölder's inequality, we get

$$\frac{1}{2}(\lambda-1)|H_\delta^C(y^*)| \leq \int_{I_{\delta_0\rho}(y^*)} \int_{-\delta\rho}^{\delta\rho} |\nabla v|\,\mathrm{d}y\,\mathrm{d}x \leq (4\delta_0\delta\rho^2)^{1-\frac{1}{p}} \|\nabla v\|_{p,K_\rho} \ .$$

Therefore, using (5.42), one obtains

$$|H_\delta^C(y^*)| < \frac{8\gamma\delta^{1-\frac{1}{p}}}{(\lambda-1)(4\delta_0)^{\frac{1}{p}}} |I_{\delta_0\rho}(y^*)| \ .$$

So we have that

$$|H_\delta(y^*)| \geq (1-\eta)|I_{\delta_0\rho}(y^*)|$$

choosing δ such that $\delta_0 \leq \frac{8\gamma\delta^{1-\frac{1}{p}}}{(\lambda-1)(4\delta_0)^{\frac{1}{p}}} \leq \eta$.

Without loss of generality (assuming δ smaller) we may assume that $\frac{\delta_0}{\delta}$ is a positive integer. Consider the interval $I_{\delta_0\rho}(y^*)$ and a partition of it with $\frac{\delta_0}{\delta}$ intervals with length $2\delta\rho$. Let (x_i, y^*) be the centres of such intervals. Let x^* be one of such indices such that

$$\text{Meas}\,\{(x^*-\delta\rho, x^*+\delta\rho) \cap H_\delta(y^*)\} > (1-\eta)(2\delta\rho) \ .$$

Then, from the definition of $H_\delta(y^*)$, the cube $K_{\delta\rho}(x^*, y^*)$ satisfies the assumption of the lemma.

■

5.4.3. *The geometric approach.* In this section, we will apply the previous lemma to get some geometric information about the localization of the energy of the solution. This is, in essence, the novelty of the approach in [72]. We recall that we assumed that for each small δ and for each $\tilde{t} \in (-(1-16\delta^2)\rho^2, 0)$

$$\text{Meas}\left\{(x,t) \in (0,\tilde{t}) + Q_{4\delta\rho} \ : \ u(x,t) \geq \mu^+ - \frac{1}{8}\omega\right\} \geq \nu |Q_{4\delta\rho}|$$

and

$$\text{Meas}\left\{(x,t) \in (0,\tilde{t}) + Q_{4\delta\rho} \ : \ u(x,t) \geq \mu^- + \frac{1}{8}\omega\right\} \geq \nu |Q_{4\delta\rho}| \ .$$

Roughly speaking (for more details see sections 5-8 of [72]) it means that the region where the function u is close to μ^+ (respectively to μ^-) is relatively large. By applying the lemma of the previous section (together with energy and logarithmic estimates) one can get that the set where u is close to μ^+ (respectively, to μ^-) must have a concentration region, even though it might be scattered in the whole cylinder. Moreover, it is possible to prove that these two concentration regions are localized at the same time levels. More precisely it is possible to prove:

Proposition 5.11. *Let u be a locally bounded weak solution of (5.1) where $\beta(\cdot)$ is any maximal monotone graph satisfying conditions (5.2) and (5.3)and suppose that the structure conditions (5.8)–(5.14) are satisfied. Assume that the alternative (5.35) holds. Then there exists a time $\tilde{t} \in (-(1-16\delta^2)\rho^2, 0)$, and two points $x_1, x_2 \in K_{4\delta\rho}$ such that $[x_i + K_{\delta^2\rho}]$, $i = 1, 2$, have their cross sections mutually separated by a distance of at least $\delta^2\rho$, and*

$$u(x_1, \tilde{t}) \geq \mu^+ - \frac{1}{8}\omega \ , \qquad \forall (x,t) \in (x_1, \tilde{t}) + Q_{\delta^2\rho} \tag{5.46}$$

and

$$u(x_2, \tilde{t}) \leq \mu^- + \frac{1}{8}\omega \ , \qquad \forall (x,t) \in (x_2, \tilde{t}) + Q_{\delta^2\rho} \ . \tag{5.47}$$

Sketch of the proof. Suppose that (5.34) is violated. Then for some time levels $t \in (\tilde{t} - 16\delta^2\rho^2)$ we have

$$\text{Meas}\left\{x \in K_{\delta^2\rho^2} \ : \ u(x,t) \leq \mu^- + \frac{1}{16}\omega\right\} \geq \nu |K_{\delta^2\rho^2}| \ .$$

By setting

$$v(x,t) = \frac{16}{\omega}(u(x,t) - \mu^-)$$

we have

$$\text{Meas}\,\{x \in K_{\delta^2\rho^2} \;:\; v(x,t) \leq 1\} \geq \nu |K_{\delta^2\rho^2}| \;. \tag{5.48}$$

Changing the constant ν into a smaller positive constant α, it is possible to find a time τ such that the following inequalities hold:

$$\text{Meas}\,\{x \in K_{\delta^2\rho^2} \;:\; v(x,t) \leq 1\} \geq \alpha |K_{\delta^2\rho^2}| \;;$$

$$\int_{K_{\delta^2\rho^2}} |\nabla v(x,\tau)|^2 \,\mathrm{d}x \leq \gamma_{\text{data}}(\omega)(\rho\delta)^{2N-4} \;,$$

where γ_{data} is independent of τ. Therefore by the lemma of measure theory, there exists $\eta\delta_0 \in (0,1)$ and a cube $[x^* + K_{\delta^2\rho^2\delta_0}] \subset K_{\delta^2\rho^2}$, such that

$$\text{Meas}\,\{x \in [x^* + K_{\delta^2\rho^2\delta_0}] \;:\; v(x,\tau) \leq 2\} \geq (1-\eta)|K_{\delta^2\rho^2\delta_0}| \;.$$

It means that $u(x,\tau) \leq \mu^- + \frac{1}{16}\omega$ except, at most, at a set of measure less than $\eta|K_{\delta^2\rho^2\delta_0}|$. The information at the time τ is almost complete (with the exception of an arbitrarily small set). Removing this set of small measure through the energy and logarithmic estimates, it is possible to get (5.47).

■

Summarizing, if the alternative (5.35) holds, then (5.46) and (5.47) are verified. Therefore

$$\frac{1}{4}\omega \leq u(y_1,t) - u(y_2,t) \quad \forall y_i \in [x_i + K_{\delta^2\rho}] \;, \quad i = 1,2 \tag{5.49}$$

for all time levels

$$t \in (\tilde{t} - \delta^4\rho^2, \tilde{t}) \;. \tag{5.50}$$

In such temporal range, integrate (5.49) over a path, piecewise parallel to the coordinates axes and joining $y_1 \in [x_1 + K_{\delta^2\rho}]$ and $y_2 \in [x_2 + K_{\delta^2\rho}]$. Integrate the resulting segment-integrals over the remaining $N-1$ variables, and then over the time in the range (5.50). Therefore

$$\gamma(\omega)(\delta\rho)^N \leq \int_{\tilde{t}-\delta^4\rho^2}^{\tilde{t}} \int_{K_{\delta\rho}\setminus K_{\delta^2\rho^2}} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}\tau \;. \tag{5.51}$$

This inequality has been derived for all $\tilde{t}$ for which (5.33) and (5.34) are both violated. Noting that, in the temporal range, the number of cylinders of the type $(0,\tilde{t})+Q_{4\delta\rho}$ is of order δ^{-2}, we add (5.51) over the corresponding boxes to get

$$\gamma(\omega)\delta^{N-2}\rho^N \le \int_{-\rho^2}^0 \int_{K_{\delta\rho}\setminus K_{\delta^2\rho^2}} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}\tau \ . \tag{5.52}$$

We may repeat the argument replacing δ^2 with δ. Iterating this procedure, we have that for each $n \in \mathbf{N}$

$$\gamma(\omega)\delta^{n(N-2)}\rho^N \le \int_{-\rho^2}^0 \int_{K_{\delta^n\rho}\setminus K_{\delta^{n+1}\rho^2}} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}\tau \ . \tag{5.53}$$

For more details about the estimates of this section, we refer the reader to Sections 9-12 of [72].

5.4.4. *The case $N=2$.* Adding (5.53) for $n=1,\dots,n_0$ one obtains

$$\gamma(\omega)n_0\rho^N \le \int\int_{Q_\rho} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}\tau \ . \tag{5.54}$$

On the other hand, via a standard energy estimate, we have

$$\int\int_{Q_\rho} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}\tau \le C\rho^N \ , \tag{5.55}$$

where C is a constant only depending on the data. Therefore, combining (5.54) and (5.55), one gets

$$\gamma(\omega)n_0 \le C \ .$$

This is a contradiction if n_0 is sufficiently large depending on the data and ω. It follows that at least one of the alternatives (5.33) or (5.34) holds in the range (5.50) and for some radius $\rho_0 \in [\rho, \delta^{n_0}\rho]$. But we have already shown that if one of the alternatives (5.33) or (5.34) holds, then the oscillation of u is reduced in the cylinder $Q_{\frac{\rho_0}{2}}$. Hence, the local continuity of the solution is proved in the case $N=2$.

Remark 5.12. The same argument works in the case in which one has information that essentially reduces the space dimension N to 1 or 2.

Remark 5.13. In the case $N=2$ the previous argument works for a more general maximal monotone graph $\beta=\beta_{AC}+\beta_s$ of bounded variation, with β_{AC} an absolutely continuous and strictly increasing function, and $\beta_s \ge 0$ a nondecreasing function where, roughly speaking, the jumps occur (for more details, see [77]). This method works also for more general (other than second order) operators (see [154], [152] and [153]).

5.4.5. *The geometric approach continued.* In order to face the case $N \geq 3$, the geometric approach needs to be improved. We stress, that all the results of this section hold under the most general assumptions.

To prove the continuity of u at a point $(x,t) \in \Omega_T$ we assume that such a point coincides with the origin up to a translation. The novelty of this improved approach is that we work with a longer cylinder $Q(\theta\rho^2, \rho)$, where θ is a positive integer number to be chosen. The idea is to find a "long" cylinder where u is away from μ^- (and μ^+) and this property should allow a space extension of positivity of the solution. It is clear that the structure itself of these singular parabolic equations forces us to deal with "long" cylinders. Actually, if one tries to repeat the argument of intrinsically rescaled cylinders (used in the case of the porous medium equation) for graphs of the Stefan type, one realizes that $\beta'(\cdot)$ is the Dirac mass at the origin. As a consequence the time should be intrinsically rescaled into another one that would remain constant on the transition set $u = 0$. In other words, one has to work with cylinders whose length depends upon the singularity of $\beta(\cdot)$. We recall that a similar approach was used for the first time in [52] in the context of the boundary regularity in the case of a single singularity.

Consider the cylinders

$$(0, t_i) + Q(\rho^2, \rho) \ , \qquad t_i = -i\rho^2 \ , \quad i = 0, 1, \ldots, \theta - 1 \tag{5.56}$$

that make a partition of the cylinder $Q(\theta\rho^2, \rho)$. Analogously to what was made in the previous sections we can state an alternative:

there exists $i = 0, 1, \ldots, \theta - 1$ such that either

$$\text{Meas}\left\{(x,t) \in (0, t_i) + Q(\rho^2, \rho) \ : \ u(x,t) \geq \mu^+ - \frac{1}{6}\omega\right\} \leq \nu |Q(\rho^2, \rho)| \tag{5.57}$$

or

$$\text{Meas}\left\{(x,t) \in (0, t_i) + Q(\rho^2, \rho) \ : \ u(x,t) \leq \mu^- + \frac{1}{6}\omega\right\} \leq \nu |Q(\rho^2, \rho)| \ . \tag{5.58}$$

Otherwise

$$\text{both (5.57) and (5.58) are violated } \forall i = 0, 1, \ldots, \theta - 1 \ . \tag{5.59}$$

If (5.57) and (5.58) hold then, as shown in the previous sections, one can deduce the reduction of the oscillation of the solution. So we assume that (5.59) holds. Analogously this assumption implies that the solution has regions of concentration in any of the cylinders $(0, t_i) + Q(\rho^2, \rho)$.

Let us state this result in a different way. Let m be an integer and define $\delta = (4m)^{-1}$. Consider a partition of the original cube K_ρ into m^N pairwise disjoint subcubes of wedge $8\delta\rho$ and centered at points $x_l \in K_\rho$. That is

$$[x_l + K_{4\delta\rho}] \subset K_\rho \ , \quad \forall\, l = 1, \dots, m^N$$

$$[x_l + K_{4\delta\rho}] \cap [x_j + K_{4\delta\rho}] = \emptyset \ , \quad \text{if } j \neq l$$

$$K_\rho = \bigcup_{l=1}^{m^N} [x_l + K_{4\delta\rho}] \ .$$

Partition any cylinder $(0, t_i) + Q(\rho^2, \rho)$ into $16m^{N+2}$ subcylinders

$$(x_l, t_i + j\delta^2\rho^2) + Q_{4\delta\rho} \ , \quad \forall l = 1, \dots, m^N \ \forall j = 1, \dots, 16m^2$$

$$(x_l, t_i + j\delta^2\rho^2) + Q_{4\delta\rho} \cap (x_r, t_i + s\delta^2\rho^2) + Q_{4\delta\rho} = \emptyset \ , \quad \text{if } r \neq l, \text{or } j \neq s$$

$$Q_\rho = \bigcup_{l=1}^{m^N} \bigcup_{j=1}^{16m^2} (x_l, t_i + j\delta^2\rho^2) + Q_{4\delta\rho} \ .$$

If (5.59) holds then, for m large enough it is possible to show (see Proposition 23.1 of [72]) that for each $i = 0, 1, \dots, \theta - 1$ there are $1 \le l, r \le m^N$ and $1 \le j, s \le 16m^2$ such that

$$u(x,t) \ge \mu^+ - \frac{1}{8}\omega \ , \qquad \forall (x,t) \in (x_l, t_i + j\delta^2\rho^2) + Q_{4\delta\rho} \tag{5.60}$$

and

$$u(x,t) \le \mu^- + \frac{1}{8}\omega \ , \qquad \forall (x,t) \in (x_r, t_i + s\delta^2\rho^2) + Q_{4\delta\rho} \ . \tag{5.61}$$

Using the logarithmic estimates one can bring this kind of information up to the level zero (see sections 22–29 of [72]). Let m_0 be an integer and define $\delta_0 = (4m_0)^{-1}$. Consider the thin cylinder

$$(0,0) + Q(\rho^2, \delta_0\rho) \subset (0,0) + Q(\rho^2, \rho)$$

and consider a partition

$$(0, t_i) + Q(\rho^2, \delta_0\rho) \ , \qquad t_i = -i\delta_0^2\rho^2 \ , \quad i = 0, 1, \dots, 16m_0^2 - 1 \ .$$

Proposition 5.14. *Let u be a locally bounded weak solution of (5.1), with $\beta(\cdot)$ any maximal monotone graph satisfying conditions (5.2) and (5.3), and assume that the structure conditions (5.8)–(5.14) are satisfied. Then either the oscillation of u is reduced (in a way that can be quantitatively determined) in $Q(\delta_0^2\rho^2, \delta_0\rho)$ or there exist $\varepsilon > 0$, depending only upon the data, ω and δ_0, and $x_l, x_j \in K_{\delta_0\rho}$ such that*

$$u(x,t) \geq \mu^- + \varepsilon\omega \ , \qquad \forall (x,t) \in (x_l, 0) + Q((1-\delta_0^2)\rho^2, \delta_0\rho) \tag{5.62}$$

and

$$u(x,t) \leq \mu^+ - \varepsilon\omega \ , \qquad \forall (x,t) \in (x_j, 0) + Q((1-\delta_0^2)\rho^2, \delta_0\rho) \ . \tag{5.63}$$

Sketch of the proof. If in some of the subcylinders

$$(0, t_i) + Q(\rho^2, \delta_0\rho) \ , \qquad t_i = -i\delta_0^2\rho^2 \ , \quad i = 0, 1, \ldots, 16m_0^2 - 1$$

(5.57) or (5.58) holds, then, as shown in the previous sections, one can deduce the reduction of the oscillation of the solution in $Q(\delta_0\rho, \delta_0^2\rho^2)$. So we assume that (5.59) holds. Under such an assumption, (5.60) and (5.61) hold. Therefore, there are $1 \leq l, r \leq m^N$ and $1 \leq j, s \leq 16m^2$ such that

$$u(x,t) \geq \mu^+ - \frac{1}{8}\omega \ , \qquad \forall (x,t) \in (x_l, -(1 - j\delta^2\delta_0^2)\rho^2) + Q_{\delta_0\delta\rho}$$

and

$$u(x,t) \leq \mu^- + \frac{1}{8}\omega \ , \qquad \forall (x,t) \in (x_r, -(1 - s\delta^2\delta_0^2)\rho^2) + Q_{\delta_0\delta\rho} \ .$$

Applying the logarithmic estimates from this box up to level zero, one gets the statement.

■

If one is only interested in having, at any time level, a set where the solution is away from μ^+ and μ^- (thus losing the geometrical information that this set has the shape of a very long and thin cylinder) one can prove that the constant ε does not depend on δ_0. More precisely

Proposition 5.15. *Let u be a locally bounded weak solution of (5.1), with $\beta(\cdot)$ any maximal monotone graph satisfying conditions (5.2) and (5.3), and assume that the structure conditions (5.8)–(5.14) are satisfied. Then either the oscillation of u is reduced (in a way that can be quantitatively determined) in $Q(\delta_0^2\rho^2, \delta_0\rho)$ or there exist $\varepsilon, \eta > 0$, depending only upon the data and ω, and $x_l, x_j \in K_{\delta_0\rho}$ such that, for each time $\tau \in [0, (1-\delta_0^2)\rho^2]$,*

$$\operatorname{Meas}\left\{x \in K_{\delta_0\rho} \ : \ u(x,\tau) \geq \mu^- + \varepsilon\omega\right\} \geq \eta |K_{\delta_0\rho}| \tag{5.64}$$

and

$$\text{Meas}\big\{x \in K_{\delta_0\rho} \ : \ u(x,\tau) \le \mu^+ - \varepsilon\omega\big\} \ge \eta|K_{\delta_0\rho}| \ . \tag{5.65}$$

Sketch of the proof. Reasoning as in the previous proposition, we may assume that (5.60) and (5.61) hold. Therefore, for each $i = 0, \dots, 16m_0^2 - 1$, there are $1 \le l, r \le m^N$ and $1 \le j, s \le 16m^2$ such that

$$u(x,t) \ge \mu^+ - \frac{1}{8}\omega \ , \qquad \forall (x,t) \in (x_l, -(i - j\delta^2)\delta_0^2\rho^2) + Q_{\delta_0\delta\rho}$$

and

$$u(x,t) \le \mu^- + \frac{1}{8}\omega \ , \qquad \forall (x,t) \in (x_r, -(i - s\delta^2)\delta_0^2\rho^2) + Q_{\delta_0\delta\rho} \ .$$

Applying the logarithmic estimates from this box up to $t = -(i-2)\delta_0^2\rho^2$ one gets the statement.

■

Remark 5.16. The real open question is to prove under general assumptions that (5.62), (5.63), (5.64) and (5.65) imply a space extension of positivity. What seems to be missing is some sort of weak form of the Harnack inequality for solutions of singular parabolic equations.

Remark 5.17. The estimates (5.64) and (5.65) give us the same piece of information in the interior of Ω_T that a suitable Dirichlet condition gives on the boundary in [52]. This allows to extend the techniques introduced in [52] to the case of multiple singularities (see [87]).

If one accepts to work with cylinders whose base is very small with respect to the length, the previous results can be improved. The next Proposition is proved in [72] (see Proposition 24.1, page 291) and it is based on a tricky combinatorial argument.

Proposition 5.18. *Let u be a locally bounded weak solution of (5.1), with $\beta(\cdot)$ any maximal monotone graph satisfying conditions (5.2) and (5.3), and assume that the structure conditions (5.8)–(5.14) are satisfied. Then there exists $\varepsilon \in (0,1)$, depending only upon the data and ω, such that for each $\theta > 0$ there exists a $\delta_0 > 0$ such that either the oscillation of u is reduced (in a way that can be quantitatively determined) in $Q(\delta_0^2\rho^2, \delta_0\rho)$ or there exists $(x_i, t_i) \in Q_{(\rho^2,\delta_0\rho)}$, with $i = 1, 2$ such that the cylinders $(x_i, t_i) + Q_{(\theta\delta_0^2\rho^2,\delta_0\rho)}$ are contained in the cylinder $Q_{(\rho^2,\delta_0\rho)}$ and*

$$u(x,t) \ge \mu^- + \varepsilon\omega \ , \qquad \forall (x,t) \in (x_1, t_1) + Q_{(\theta\delta_0^2\rho^2,\delta_0\rho)} \tag{5.66}$$

and

$$u(x,t) \le \mu^+ - \varepsilon\omega\ , \qquad \forall (x,t) \in (x_2,t_2) + Q_{(\theta\delta_0^2\rho^2,\delta_0\rho)}\ . \tag{5.67}$$

Summarizing, the previous proposition says that if one wants that the concentration of the solution still has the shape of a long cylinder and does not want to pay the price of having ε depending upon the length of the cylinders, then one looses the information about the precise localization of such a cylinder.

The next proposition is based on energy estimates, a De Giorgi lemma (see Lemma 2.2 of [55]) and estimates (5.64) and (5.65). For a detailed proof see Lemma 4.10 of [87].

Proposition 5.19. *Let u be a locally bounded weak solution of (5.1), with $\beta(\cdot)$ any maximal monotone graph satisfying conditions (5.2) and (5.3), and assume that the structure conditions (5.8)–(5.14) are satisfied. Then there exist two continuous strictly increasing functions $\delta(\cdot),\eta(\cdot)$, $\delta(0)=\eta(0)=0$, that can be determined a priori only in terms of the data and ω, such that either the oscillation of u is reduced (in a way that can be quantitatively determined) in $Q(\delta^2(s)\rho^2,\delta(s)\rho)$ or*

$$\text{Meas}\left\{(x,t) \in Q_{(\rho^2,\delta(s)\rho)}\ :\ u(x,t) \le \mu^- + s\omega\right\} \ge \eta(s)|Q_{(\rho^2,\delta(s)\rho)}| \tag{5.68}$$

and

$$\text{Meas}\left\{(x,t) \in Q_{(\rho^2,\delta(s)\rho)}\ :\ u(x,t) \ge \mu^+ - s\omega\right\} \ge \eta(s)|Q_{(\rho^2,\delta(s)\rho)}|\ . \tag{5.69}$$

Summarizing, one is able to estimate the sets where the solution is close to μ^+ and μ^- paying the price of considering "thin" cylinders.

5.4.6. *The case $N \ge 3$.* As already stressed, we are not able to prove the regularity in the general case. We follow the approach of [72] (see also [86] and [160]) and we prove continuity results using a suitable comparison function.

In [72], the local continuity for weak solutions of the prototype equation (5.17) is proved. The key point in the proof of the continuity theorem are some estimates on a proper function v, which is then compared with the solution u of the original singular parabolic equation. In these estimates the radial symmetry of the problem is heavily used, but a careful examination of the whole procedure shows that this assumption can be done away with, provided the maximum principle and a Harnack inequality for the corresponding elliptic operator hold for all time levels (see [86]). Let us just remark that the basic reason to use the comparison function is to mimic a parabolic Harnack inequality, whose validity is not known in this context. For a detailed proof of the results of this section we refer the reader to [72], [86] and [160]).

Consider the singular parabolic equation

$$\beta(u)_t = \mathcal{L}u \tag{5.70}$$

where $\mathcal{L}$ is an elliptic operator with principal part in divergence form. We assume

$$\mathcal{L}u = \sum_{ij} D_i(a_{ij}(x,t)D_j u + a_i(x,t)u) + b_i(x,t)D_i u + e(x,t)u \ , \tag{5.71}$$

where $a_{ij}(x,t),\ a_i(x,t),\ b_i(x,t),\ e(x,t)$ are continuous functions with respect to the time variable and are measurable functions with respect to the spatial variables, and satisfy

$$\frac{1}{\mu_1}|\xi|^2 \le \sum_{ij} a_{ij}(x,t)\xi_i\xi_j \le \mu_1|\xi|^2 \tag{5.72}$$

$$\left\| \sum a_i^2, \sum b_i^2, e \right\|_{q,r,\Omega_T} \le \mu_2 \ , \tag{5.73}$$

with q and r such that

$$\frac{1}{r} + \frac{N}{2q} = 1 - \kappa_1$$

and $q \in \left[\dfrac{N}{2(1-\kappa_1)}, \infty\right]$, $\quad r \in \left[\dfrac{1}{1-\kappa_1}, \infty\right]$, $\quad 0 < \kappa_1 < 1, N \ge 2.$
Moreover, we suppose that

$$\forall\ 0 < t < s < T \ , \quad \int_t^s \int_\Omega \left(ev - \sum_i a_i D_i v \right) \mathrm{d}x\,\mathrm{d}t \le 0 \ , \tag{5.74}$$

for all $v \in C_0^1(\Omega \times (t,s))$, $v \ge 0$. Assumptions (5.72) and (5.73) mean that the structure conditions (5.8)–(5.14) are satisfied, while (5.74) is assumed in order to have the maximum principle in any parabolic cylinder contained in Ω_T. Before giving a sketch of the proof, we stress that the only properties of $\mathcal{L}(x,t,u,Du)$ on which we will rely are the following:

(1) $\mathcal{L}$ satisfies the maximum principle;
(2) the coefficients of $\mathcal{L}$ are continuous in t;
(3) in the case of time–dependent coefficients, the elliptic operator $\mathcal{L}$ satisfies a <u>uniform</u> Harnack inequality t by t.

The most important of the three assumptions is the last one and this shows once more how the Harnack inequality is crucial when proving regularity results for solution of partial differential equations.

For the sake of simplicity we consider only the case of time independent coefficients (the reader finds a detailed proof of the general case in [86] and [160]). Without loss of generality, assume that (5.66) holds, i.e.,

$$u(x,t) \geq \mu^- + \varepsilon\omega \ , \qquad \forall (x,t) \in (x_1,t_1) + Q_{(\theta\delta_0^2\rho^2,\delta_0\rho)}$$

(if not, it means that the oscillation of the solution is automatically reduced in a smaller cylinder). As in [72], apply the change of variables

$$x \longrightarrow \frac{4(x-x_1)}{|x_1|} \ ; \qquad t \longrightarrow \frac{t-t_1}{\delta_0^2\rho^2}$$

and introduce the function

$$\tilde{u} \equiv \frac{u-\mu^-}{\varepsilon\omega} \ .$$

We have that $\tilde{u}$ solves the differential equation

$$(\tilde{\beta}(\tilde{u}))_t = \tilde{\mathcal{L}}(\tilde{u}) \ ,$$

where $\tilde{\beta}$ and $\tilde{\mathcal{L}}$ satisfy the same structural conditions of the operator (5.71) (for more details see [72]). Moreover, $\tilde{u}(x,t) \geq 1$ in $K_{\varepsilon_0} \times (0,\theta)$. Without loss of generality, we may assume that $\varepsilon_0 \leq \frac{1}{2}$. Define $\mathcal{A}_{(d_1,d_2)} \times (t_1,t_2)$ as the annulus $\{d_1 < |x| < d_2\} \times (t_1,t_2)$. Finally, we are interested in the behaviour of $\tilde{u}$ in an annulus contained in $\mathcal{A}_{(\varepsilon_0,4d)} \times (0,\theta)$ where we may assume $d \geq 4$ (see [72]). For this reason we introduce a proper comparison function. Namely, let v solve the following boundary value problem

$$\begin{cases} (\tilde{\beta}(v))_t &= \tilde{\mathcal{L}}v \quad \text{in } \mathcal{A}_{(\varepsilon_0,4d)} \times (0,\theta) \\ v(x,t) &= 0 \quad \text{on } |x| = 4d \\ v(x,t) &= 1 \quad \text{on } |x| = \varepsilon_0 \\ v(x,0) &= 0 \end{cases} \tag{5.75}$$

By the maximum principle, $\tilde{u} \geq v$. Hence if we are able to prove that there is a level $t_0 \in (0,\theta)$ such that there is a $\sigma_0 > 0$, depending only upon the data, such that

$$v(x,t_0) \geq \sigma_0 \ , \qquad \forall 1 \leq |x| \leq 2d \ , \tag{5.76}$$

we have that a similar bound works for $\tilde{u}$. By returning to the original coordinates, we conclude that there exists a time level t_1 such that

$$u(x,t_1) > \mu_- + \sigma_0\varepsilon\omega \ , \qquad \forall x \in K_{\delta^*\rho} \ ,$$

with δ^* a positive constant that can be determined *a priori* only in terms of the data. As in [72] (sections 24 and 25), by using the logarithmic estimates, one can bring this piece of information up to level zero and in this way reduce the oscillation of u in the small cylinder $Q(\delta^*\rho)$.

In order to prove (5.76), let ζ be the solution of the elliptic problem

$$\begin{cases} \mathcal{L}\zeta &= 0 \quad \text{in} \quad \mathcal{A}_{\varepsilon_0,4d} \\ \zeta(x) &= 0 \quad \text{on} \quad |x| = 4d \\ \zeta(x) &= 1 \quad \text{on} \quad |x| = \varepsilon_0 \end{cases} \tag{5.77}$$

By well-known classical results, there is a unique solution ζ satisfying (5.77) (see [88], Theorems 8.1 and 8.3). Moreover, ζ is Hölder continuous (see [121]). Finally, the hypotheses on the coefficients ensure that w belongs to an elliptic De Giorgi's class, which in turn guarantees that w satisfies a Harnack inequality (under this point of view, see for example [124], Chapter 3). Therefore we may assume that there is σ_0 such that

$$\zeta(x) \geq 2\sigma_0 \ , \qquad \forall 1 \leq |x| \leq 2d \ . \tag{5.78}$$

The aim now is to transfer this information to v. With this purpose, put $z = v - \zeta$ and set $\Gamma(x,\cdot) = \tilde{\beta}(\cdot - \zeta(x))$. It is easy to see that z satisfies

$$\begin{cases} (\Gamma(z))_t &= \mathcal{L}z \qquad \text{in} \quad \mathcal{A}_{\varepsilon_0,4d} \times (0,k) \\ z(x,t) &= 0 \qquad \text{on} \quad |x| = 4d \\ z(x,t) &= 0 \qquad \text{on} \quad |x| = \varepsilon_0 \\ z(x,0) &= -\zeta(x) \quad \text{on} \quad x \in \mathcal{A}_{\varepsilon_0,4d} \end{cases} \tag{5.79}$$

By the energy estimates, one gets that

$$\int_0^k \int_{\mathcal{A}_{\varepsilon_0,4d}} |\nabla z|^2 \, \mathrm{d}x \, \mathrm{d}t \leq \ C \ . \tag{5.80}$$

There must exist a time level t^* such that

$$\int_{\mathcal{A}_{\varepsilon_0,4d}} |\nabla z|^2 \, \mathrm{d}x \leq \delta_0 \ , \tag{5.81}$$

with τ_0 a proper small quantity. In fact, if it were not so, integrating on $(0,k)$ we obtain

$$\int_0^k \int_{\mathcal{A}_{\varepsilon_0,4d}} |\nabla z|^2 \, \mathrm{d}x \geq k\delta_0$$

and it suffices to choose k large enough to get a contradiction.

Now we claim that, if we choose δ_0 small enough, a consequence of (5.81) is that

$$\forall\, x \in \mathcal{A}_{\varepsilon_0,2d} \ , \qquad (\zeta - v)(x,t^*) \leq \sigma_0 \ . \tag{5.82}$$

In fact, if it were not true, reproducing the same argument of [72, §6–8], and using the measure lemma, we conclude that there exist a $y^* \in \mathcal{A}_{\varepsilon_0,4d}$ and a small cube $K_\rho(y^*) \subset \mathcal{A}_{\varepsilon_0,4d}$ such that

$$\forall\, x \in K_\rho(y^*)\ , \qquad (\zeta - v)(x,t^*) > \frac{\sigma_0}{2}\ .$$

Connecting, through a path, the boundary $K_\rho(y^*)$ with the boundary of $\mathcal{A}_{\varepsilon_0,4d}$, i.e., with a portion of the boundary where $|x| = 4d$, and working as in section 9 of [72] we get a lower bound for $\int_{\mathcal{A}_{\varepsilon_0,4d}} |\nabla z(x,t^*)|^2\, dx$, thus obtaining a contradiction by choosing δ_0 small enough. Therefore (5.82) holds.

The proof of the local continuity of a bounded solution of (5.70) follows from the remark that estimate (5.76) is a direct consequence of (5.78) and (5.82).

Remark 5.20. The boundary regularity for singular equations like the ones we are dealing with here still presents several open questions. In the case of β with a single jump, interior regularity is again a matter of energy and logarithmic inequalities and this allows for a complete solution of boundary regularity for variational data, as in [48].

In the case of a general β it remains an open problem to devise a technique of proof only based on energy and logarithmic estimates. Therefore, for the moment, it is only possible to consider homogeneous Dirichlet boundary conditions under mild assumptions on $\partial\Omega$, without taking into account initial conditions. Assume, for instance, the compactness and the Lipschitz continuity of the boundary. By the compactness of the the boundary of Ω, we can cover $\partial\Omega$ with a finite number of neighborhoods centered at points of $\partial\Omega$. The Lipschitz continuity of the boundary allows us to find a map from every neighborhood into a half ball of $\mathbf{R}^N$. The transformed equation via this map has coefficients which are still measurable with respect to x, properly summable with respect to the pair (x,t), and continuous with respect to t. Now we reflect the operator through the entire ball and notice that this reflection does not affect the $(\beta(u))_t$ term, since it is done only with respect to the x variable. We have therefore reduced the study to a problem in the interior and we can apply the previous results to prove the boundary regularity.

5.4.7. *The case $N \geq 3$ continued.* In this section, we prove the regularity for particular cases of equation (5.1). Here we permit the maximal generality allowed from conditions (5.8)–(5.14), but we consider a very special β of the type

$$\beta(s) = \begin{cases} -s-\nu_1 & \text{if } \ s<0 \\ [-\nu_1, 0] & \text{if } \ s=0 \\ s & \text{if } \ 0<s<1 \\ [1, 1+\nu_2] & \text{if } \ s=1 \\ s+\nu_2 & \text{if } \ s>1 \end{cases} .$$

This special β allows us to write explicitly the logarithmic and energy estimates. The use of the estimates, together with the geometrical constructions of the previous sections, will allow to prove the local continuity of the solutions. This approach is developed in a very detailed way in [87]. A similar approach can be found in [52] in a different context. We feel that the right approach to the local continuity is this one, that is, it must be based only on logarithmic and energy estimates and geometric constructions.

We only give a sketch of the ideas on which the proof is based. Without loss of generality, we may assume that $\mu^+ = 1+\varepsilon_0$ and $\mu^- = -\varepsilon_0$, with ε_0 very "small". The idea is to reduce the oscillation of u such that either $u > 0$ or $u < 1$. In this way, the problem has been reduced to the case of only a singularity. Without loss of generality, we can focus our attention to find a subcylinder where $u > 0$. Apply proposition 5.19. Choose $s_0 > \varepsilon_0$ such that either the oscillation of u is so reduced in $Q(\delta^2(s_0)\rho^2, \delta(s_0)\rho)$ that $u > 0$ in such a cylinder or

$$\text{Meas}\left\{(x,t) \in Q_{(\rho^2, \delta(s_0)\rho)} \ : \ u(x,t) \le \mu^- + s\omega\right\} \ge \eta(s_0)|Q_{(\rho^2,\delta(s_0)\rho)}| \ . \quad (5.83)$$

Note that $\eta(s_0)$ is very small (in a quantitatively way that will be clear in the sequel) if ε_0 is chosen really small.

Partition the cylinder $Q_{(\rho^2,\delta(s_0)\rho)}$ in subcylinders $Q_{(\delta^2(s_0)\rho^2,\delta(s_0)\rho)}$. Apply the iteration method based on energy and logarithmic estimates, choosing as test function $(u-\frac{1}{4})_-$. There are two possibilities:

either in one of the subcylinders the set where u is less than or equal to zero is negligible; then the iteration method works as in the nonsingular case. So one can find a small cylinder where $u \ge \frac{1}{8}$. Then, via the logarithmic estimates, one can bring this positivity up to the level zero if ε_0 is chosen small enough;

or, in each of the subcylinders, the set where u is less or equal to zero is not negligible. Then by (5.83) one can deduce that there exists a subcylinder where the measure of the set where u is less than $\frac{1}{8}$ is a function of η_0. So if we choose η_0 small enough, the alternative (5.34) holds. So one can find a small cylinder where $u \geq \frac{1}{16}$. Then, via the logarithmic estimates, one can bring this positivity up to the level zero if ε_0 is chosen small enough.

Remark 5.21. The fact that $\beta(\cdot)$ has only two jumps is not essential. Any finite number of jumps would be acceptable.

Remark 5.22. This approach seems to work not only with jumps but also with singularities that grow in a very fast way. It could be of some interest to start an investigation on such a direction.

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