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SHAPE ANALYSIS OF AN ADAPTIVE ELASTIC ROD MODEL

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ABSTRACT: We analyse the shape semiderivative of the solution to an asymptotic nonlinear adaptive elastic rod model, derived in Figueiredo and Trabucho [8], with respect to small perturbations of the cross section. Taking advantage of the special structure of this model, which is defined by generalized Bernoulli-Navier elastic equilibrium equations coupled with an ordinary differential equation with respect to time, using the properties of the forms and data defining the model, and the regularity of its solution, we compute and completely identify, in an appropriate functional space involving time, the weak shape semiderivative.

KEYWORDS: adaptive elasticity, functional spaces, shape derivative, rod.
AMS SUBJECT CLASSIFICATION (2000): 74B20, 74K10, 74L15, 90C31.

1. Introduction

In this paper we consider a sensitivity analysis problem in shape optimization: the calculus of a derivative of the solution to an asymptotic nonlinear adaptive elastic rod model, with respect to shape variations of the cross section of the rod. More precisely, for each small parameter $s \in [0, \delta]$, we define a perturbed adaptive elastic rod $\overline{\Omega}_s = \overline{\omega}_s \times [0, L]$. The scalar $L > 0$ is its length and $\omega_s = \omega + s\,\theta(\omega)$ is a perturbation of a fixed cross section ω, in the direction of the vector field $\theta = (\theta_1, \theta_2)$, that realizes the shape variation. To each rod $\overline{\Omega}_s$, we associate the corresponding unique solution (u^s, d^s) of the asymptotic adaptive elastic rod model, derived in Figueiredo and Trabucho [8]. The purpose of this paper is to compute the limit $(\frac{u^s-u}{s}, \frac{d^s-d}{s})$, when $s \to 0^+$, where (u, d) is the solution of the rod model for the case $s = 0$. This limit is the semiderivative of the shape function $J : s \in [0, \delta] \to J(\Omega_s) = (u^s, d^s)$, at $s = 0$ in the direction of the vector field θ (in the sense of Delfour and Zolésio [6], p.289), or equivalently, the material derivative of the map J at $s = 0$ (in the sense of Haslinger and Mäkinen [9], p.111).

The difficulties that arise in the computation of the limit $(\frac{u^s-u}{s}, \frac{d^s-d}{s})$, when $s \to 0^+$, are caused by the complicated form of the asymptotic adaptive elastic rod model derived in Figueiredo and Trabucho [8]. In fact, this is a simplified adaptive elastic model (proposed for the mathematical modeling of the physiological process of bone remodeling) that couples the generalized Bernoulli-Navier elastic equilibrium equations with an ordinary differential equation with respect to time, which is the remodeling rate equation, that expresses the process of absorption and deposition of bone material due to an external stimulus (cf. Cowin and Hegedus [3, 4], Cowin and Nachlinger [5], and Monnier and Trabucho [10], for a description of the theory of adaptive elasticity and results of uniqueness and existence of three-dimensional solutions). For each $s \in [0, \delta]$, the unique solution of this asymptotic adaptive elastic rod model, is the pair (u^s, d^s), where u^s is the displacement vector field of the rod $\overline{\Omega}_s$, and d^s is a scalar field that represents the change in volume fraction of the elastic material in the rod $\overline{\Omega}_s$. Moreover, u^s is the solution of the generalized Bernoulli-Navier equilibrium equations and d^s is the solution of the remodeling rate equation. In particular, u^s and d^s are coupled in the model because the material coefficients depend on d^s and the remodeling rate equation depends on u^s.

In spite of this complicated structure of the model we succeeded in the calculus of the limit $(\frac{u^s-u}{s}, \frac{d^s-d}{s})$, when $s \to 0^+$. There are two main results in this paper. The first principal result consists in proving that, for each time t, the sequence $(\frac{u^s-u}{s}, \frac{d^s-d}{s})(.,t)$ converges weakly to $(\bar{u}, \bar{d})(.,t)$, when $s \to 0^+$, in an appropriate functional space of Sobolev type (we note that the pair (u^s, d^s) is the solution of the perturbed rod model, reformulated in the unperturbed fixed domain $\overline{\Omega}$, which is the domain of (u, d)). The second main result is the identification of the weak shape semiderivative $(\bar{u}, \bar{d})$, as the unique solution of a nonlinear problem which couples a variational equation (whose solution is $\bar{u}$) and depends on (u, d) and $\bar{d}$, and an ordinary differential equation with respect to time (whose solution is $\bar{d}$), and that depends on (u, d) and $\bar{u}$.

The reasonings used to achieve these two main results are next summarized. We show that the sequences (u^s, d^s) and $(\frac{u^s-u}{s}, \frac{d^s-d}{s})$ are bounded in appropriate functional spaces, involving time; we use the continuity, the ellipticity, the regularity properties and the special structure of the asymptotic adaptive elastic rod model. To identify the weak shape semiderivative we also apply the weak or/and strong convergence of the sequences $\{u^s\}$ and $\{d^s\}$, when $s \to 0^+$, and again, the special structure of the asymptotic adaptive elastic rod model. In particular, due to the form of the remodeling rate equation, we are able to use the integral Gronwall's inequality, which is the key to obtain the estimates for the sequences $\{d^s\}$ and $\{\frac{d^s-d}{s}\}$ and to identify the ordinary differential equation with respect to time, whose solution is $\bar{d}$.

Finally let us briefly explain the contents of the paper. After this introduction, in section 2, we describe the problem P_s, which is the asymptotic nonlinear adaptive elastic model for the perturbed rod $\overline{\Omega}_s$, we also prove a regularity property of its solution and finally we define the shape problem, that we want to solve. In section 3 we reformulate the problem P_s in the unperturbed domain $\overline{\Omega}$, which is necessary, because, in order to compute the limit of the quotient sequence $(\frac{u^s-u}{s}, \frac{d^s-d}{s})$, the vector fields u^s, u, d^s and d must be defined in the same fixed domain, independent of s. In section 4 we prove that all the sequences $\{u^s\}$, $\{d^s\}$, $\{\frac{u^s-u}{s}\}$ and $\{\frac{d^s-d}{s}\}$ are bounded in appropriate functional spaces involving time, we determine, for each time t, the weak limit of the quotient sequence $(\frac{u^s-u}{s}, \frac{d^s-d}{s})(.,t)$, when $s \to 0^+$, and we identify the weak shape semiderivative (this identification is summarized in theorems 4.14, 4.15 and corollary 4.3). Finally we present some conclusions and future work.

2. Description of the Problem

In this section we first introduce the notations used in this paper. We consider a family of rods $\overline{\Omega}_s = \overline{\omega}_s \times [0, L]$, with length L and cross section ω_s, parameterized by $s \in [0, \delta]$, which is a small parameter. We assume that $\omega_s = \omega + s\,\theta(\omega)$ is a perturbation of $\omega = \omega_0$ in the direction of the vector field θ, so $\overline{\Omega}_s$ is a perturbation of the rod $\overline{\Omega} = \overline{\omega} \times [0, L] = \overline{\Omega}_0$. Next, for each s, we describe the adaptive elastic rod model, derived in Figueiredo and Trabucho [8]. This model (denoted by P_s, for each s) is highly nonlinear. In fact, the unknown is a pair (u_s, d_s) whose components are coupled; u_s is a vector field, that represents the displacement of the rod $\overline{\Omega}_s$, and solves the generalized Bernoulli-Navier equilibrium equations, and d_s is a scalar vector field, that measures the change in volume fraction of the elastic material, and that is the solution of the remodeling rate equation, that depends on u_s. Aferwards we recall the result of existence and uniqueness of the solution of the rod model P_s (proved in Figueiredo and Trabucho [8]). Moreover, we also prove a regularity result for the displacement vector field u_s. Finally, we describe the shape problem under consideration in this paper. Defining the map $J(s) = (u_s, d_s)$, for $s \in [0, \delta]$, the shape problem consists in the calculus of the semiderivative $dJ(\Omega; \theta)$ at $s = 0$ in the direction of the vector field θ, in the sense of Delfour and Zolésio [6], p.289.

2.1. Notations. Let $\delta > 0$ be a small parameter and for each $s \in [0, \delta]$ we consider the perturbation I_s of the identity operator I in $\mathbb{R}^2$, defined by $I_s(x_1, x_2) = (I + s\theta)(x_1, x_2) = (x_{s1}, x_{s2})$, for all $(x_1, x_2) \in \mathbb{R}^2$, where $\theta = (\theta_1, \theta_2) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a vector field regular enough (at least $\theta \in [W^{2,\infty}(\mathbb{R}^2)]^2$).

Let ω be an open, bounded and connected subset of $\mathbb{R}^2$, with a boundary $\partial\omega$ regular enough. For each $s \in [0, \delta]$ we define $\omega_s = I_s(\omega)$, which is the perturbation of ω in the direction of the vector field θ. We also denote by $\overline{\Omega}_s$ the set occupied by a cylindrical adaptive elastic rod, in its reference configuration, with length $L > 0$ and cross section ω_s, that is

$$\overline{\Omega}_s = \overline{\omega}_s \times [0, L] = I_s(\overline{\omega}) \times [0, L] \subset \mathbb{R}^3. \tag{1}$$

Moreover we denote by $x_s = (x_{s1}, x_{s2}, x_3)$ a generic element of $\overline{\Omega}_s$ and define the sets

$$\Gamma_s = \partial\omega_s \times]0, L[, \qquad \Gamma_{s0} = \overline{\omega}_s \times \{0\}, \qquad \Gamma_{sL} = \overline{\omega}_s \times \{L\}, \tag{2}$$

where $\partial\omega_s$ is the boundary of ω_s. These three sets represent, respectively, the lateral boundary of the rod $\overline{\Omega}_s$ and its two extremities. The unit outer normal vector along the boundary $\partial\Omega_s$ of Ω_s is denoted by $n_s = (n_{s1}, n_{s2}, n_{s3})$. We assume that, for each $s \in [0, \delta]$ the coordinate system (O, x_{s1}, x_{s2}, x_3) is a principal system of inertia associated with the rod Ω_s. Consequently, axis Ox_3 passes through the centroid of each section $\omega_s \times \{x_3\}$ and we have

$$\int_{\omega_s} x_{s1}\, d\omega_s = \int_{\omega_s} x_{s2}\, d\omega_s = \int_{\omega_s} x_{s1} x_{s2}\, d\omega_s = 0. \tag{3}$$

We observe that the choice of the vector field θ, that realizes the shape variation of the cross section ω, must be admissible with this last condition.

The set $C^m(\overline{\Omega}_s)$ stands for the space of real functions m times continuously differentiable in $\overline{\Omega}_s$. The spaces $W^{m,q}(\Omega_s)$ and $W^{0,q}(\Omega_s) = L^q(\Omega_s)$ are the usual Sobolev spaces, where q is a real number satisfying $1 \leq q \leq \infty$ and m is a positive integer. The norms in these Sobolev spaces are denoted by $\|.\|_{W^{m,q}(\Omega_s)}$. The set

$$\mathcal{R}_s = \{v_s \in \mathbb{R}^3 : \ \ v_s = a + b \wedge x_s, \ \ a, b \in \mathbb{R}^3\} \tag{4}$$

where $\wedge$ is the exterior product in $\mathbb{R}^3$, is the set of infinitesimal rigid displacements. We denote by $[W^{m,q}(\Omega_s)]^3 \setminus \mathcal{R}_s$ the quotient space induced by the set $\mathcal{R}_s$ in the Sobolev space $[W^{m,q}(\Omega_s)]^3$.

Throughout the paper, the latin indices i, j, k, l... belong to the set $\{1, 2, 3\}$, the greek indices α, β, μ... vary in the set $\{1, 2\}$ and the summation convention with respect to repeated indices is employed, that is, for example $a_i b_i = \sum_{i=1}^{3} a_i b_i$.

Let $T > 0$ be a real parameter and denote by t the time variable in the interval $[0,T]$. If V is a topological vectorial space, the set $C^m([0,T];V)$ is the space of functions $g : t \in [0,T] \to g(t) \in V$, such that g is m times continuously differentiable with respect to t. If V is a Banach space we denote $\|.\|_{C^m([0,T];V)}$ the usual norm in $C^m([0,T];V)$. Moreover, given a function $g_s(x_s,t)$ defined in $\overline{\Omega}_s \times [0,T]$ we denote by $\dot{g}_s$ its partial derivative with respect to time, by $\partial_{s\alpha} g_s$ and $\partial_3 g_s$ its partial derivatives with respect to $x_{s\alpha}$ and x_3, respectively, that is, $\dot{g}_s = \frac{\partial g_s}{\partial t}$, $\partial_{s\alpha} g_s = \frac{\partial g_s}{\partial x_{s\alpha}}$ and $\partial_3 g_s = \frac{\partial g_s}{\partial x_3}$.

2.2. The adaptive elastic rod model. The theory of adaptive elasticity (cf. Cowin and Hegedus [3, 4]) is proposed for the mathematical modeling of the physiological process of bone remodeling (and it is a generalization of linearized elasticity). The basic system of equations governing the adaptive elasticity model couples the equilibrium and constitutive equations with an ordinary differential equation with respect to time, which is the remodeling rate equation. This latter equation mathematically expresses the process of absorption and deposition of bone material due to an external stimulus. Figueiredo and Trabucho [8], applied the asymptotic expansion method (cf. Trabucho and Viaño [11], for an explanation of the mathematical modeling of rods with the asymptotic expansion method) to the model derived in Cowin and Hegedus [3, 4], with the modifications proposed in Monnier and Trabucho [10], for a remodeling rate equation depending nonlinearly or linearly on the strain field, and for a thin rod whose cross section is a function of a small parameter. They obtained a simplified adaptive elastic rod model (that we denote also by asymptotic adaptive elastic rod model), that is a system of nonlinear coupled equations, with generalized Bernoulli-Navier equilibrium equations and a simplified remodeling rate equation. This system is defined as follows, for any perturbed rod $\overline{\Omega}_s$ (with $s \in [0,\delta]$) and in the case where the original three-dimensional remodeling rate equation depends linearly on the strain field:

$$\left[\begin{array}{l} u_s = (u_{s1}, u_{s2}, u_{s3}) : \overline{\Omega}_s \times [0,T] \to \mathbb{R}^3, \quad d_s : \overline{\Omega}_s \times [0,T] \to \mathbb{R}, \\ u_{s\alpha} : [0,L] \times [0,T] \to \mathbb{R}, \\ u_{s3} = \underline{u}_{s3} - x_{s\alpha}\partial_3 u_{s\alpha} \quad \text{and} \quad \underline{u}_{s3} : [0,L] \times [0,T] \to \mathbb{R}, \\ (u_s, d_s) \quad \text{satisfies:} \end{array}\right. \tag{5}$$

the equilibrium equations in $(0, L) \times (0, T)$,

$$\left[\begin{array}{l} \left[\begin{array}{l} -\partial_3\big(l_s(d_s)\partial_3\underline{u}_{s3} - e_{s\alpha}(d_s)\partial_{33}u_{s\alpha}\big) = \\ \quad \int_{\omega_s}\gamma_s(\xi_{s0} + P_\eta(d_s))\, f_{s3}\, d\omega_s + \int_{\partial\omega_s} g_{s3}\, d\partial\omega_s, \end{array}\right. \\ \\ \left[\begin{array}{l} \partial_{33}\big(- e_{s\beta}(d_s)\partial_3\underline{u}_{s3} + h_{s\alpha\beta}(d_s)\partial_{33}u_{s\alpha}\big) = \\ \quad \int_{\omega_s}\gamma_s(\xi_{s0} + P_\eta(d_s))\, f_{s\beta}\, d\omega_s + \int_{\partial\omega_s} g_{s\beta}\, d\partial\omega_s \\ \qquad + \int_{\omega_s} x_{s\beta}\, \partial_3[\gamma_s(\xi_{s0} + P_\eta(d_s))\, f_{s3}]\, d\omega_s + \int_{\partial\omega_s} x_{s\beta}\, \partial_3 g_{s3}\, d\partial\omega_s, \end{array}\right. \end{array}\right. \tag{6}$$

the boundary conditions for $\{\overline{x}_3\} \times (0, T)$, with $\overline{x}_3 = 0, L$,

$$\left[\begin{array}{l} \big(l_s(d_s)\partial_3\underline{u}_{s3} - e_{s\alpha}(d_s)\partial_{33}u_{s\alpha}\big)(\overline{x}_3) = \int_{\omega_s} h_{s3}(\overline{x}_3)\, d\omega_s, \\ \big(e_{s\beta}(d_s)\partial_3\underline{u}_{s3} - h_{s\alpha\beta}(d_s)\partial_{33}u_{s\alpha}\big)(\overline{x}_3) = \int_{\omega_s} x_{s\beta}\, h_{s3}(\overline{x}_3)\, d\omega_s, \\ \left[\begin{array}{l} \partial_3\big(e_{s\beta}(d_s)\partial_3\underline{u}_{s3} - h_{s\alpha\beta}(d_s)\partial_{33}u_{s\alpha}\big)(\overline{x}_3) = \\ \quad \int_{\omega_s} h_{s\beta}(\overline{x}_3)\, d\omega_s - \int_{\partial\omega_s} x_{s\beta} g_{s3}(\overline{x}_3)\, d\partial\omega_s \\ \qquad - \int_{\omega_s} x_{s\beta}\, \gamma_s(\xi_{s0} + P_\eta(d_s)) f_{s3}(\overline{x}_3)\, d\omega_s, \end{array}\right. \end{array}\right. \tag{7}$$

the remodeling rate equation,

$$\left[\begin{array}{lll} \dot{d}_s = c(d_s)e_{33}(u_s) + a(d_s), & \text{in} & \Omega_s \times (0, T), \\ c(d_s) = A_{\alpha\beta}(d_s)\frac{b_{\alpha\beta33}(d_s)}{b_{3333}(d_s)} + A_{33}(d_s), & & \\ d_s(x_s, 0) = \overline{d}_s(x_s), & \text{in} & \overline{\Omega}_s. \end{array}\right. \tag{8}$$

The unknowns of the model (5)-(8) are the displacement vector field $u_s(x_s, t)$, corresponding to the displacement of the point x_s of the rod $\overline{\Omega}_s$ at time t and the measure of change in volume fraction of the elastic material (from the reference volume fraction ξ_{s0}) $d^\varepsilon(x_s, t)$ at (x_s, t). In particular $e_{33}(u_s) = \partial_3 u_{s3} = \partial_3\underline{u}_{s3} - x_{s\alpha}\partial_{33}u_{s\alpha}$ is an element of the linear strain tensor $\big(e_{ij}(u_s)\big)$, and it is a function of u_s.

On the other hand, the data of the model (5)-(8) are the following: the open set $\Omega_s \times (0, T)$, the density $\gamma_s = \gamma$ of the full elastic material, which is supposed to be a constant independent of s, the reference volume fraction of the elastic material ξ_{s0}, that belongs to $C^1(\overline{\Omega}_s)$ for each s, the body load $f_s = (f_{si})$, such that $f_{si} \in C^1([0, T])$ and depends only on t, the normal tractions on the boundary $g_s = (g_{si})$ and $h_s = (h_{si})$, the initial value of the change in volume fraction $\overline{d}_s(x_s)$, which belongs to $C^0(\overline{\Omega}_s)$, the truncation operator $\mathcal{P}_\eta(.)$, and the coefficients $l_s(d_s)$, $e_{s\alpha}(d_s)$, $h_{s\alpha\beta}(d_s)$, $c(d_s)$, $a(d_s)$, $A_{\alpha\beta}(d_s)$, $A_{33}(d_s)$, $b_{\alpha\beta 33}(d_s)$ and $b_{3333}(d_s)$, which are all material coefficients depending upon the change in volume fraction d_s.

On these data we also suppose further conditions, which we will describe next. We assume that, for each $s \in [0, \delta]$, $0 < \xi_{s0}^{min} \leq \xi_{s0}(x_s) \leq \xi_{s0}^{max} < 1$ and that the normal tractions verify

$$g_{si} \in C^1([0,T]; W^{1-1/p,p}(\Gamma_s)), \qquad h_{si} \in C^1([0,T]; W^{1-1/p,p}(\Gamma_{s0} \cup \Gamma_{sL})), \quad (9)$$

with $p > 3$. In addition we assume that the resultant of the system of applied forces is null for rigid displacements, that is, for any $v_s = (v_{si})$, and for all $t \in [0, T]$

$$\begin{cases} \int_{\Omega_s} \gamma(\xi_{s0} + \mathcal{P}_\eta(d_s)) f_{si} v_{si} dx_s + \int_{\Gamma_s} g_{si} v_{si} d\Gamma_s \\ \qquad + \int_{\Gamma_{s0} \cup \Gamma_{sL}} h_{si} v_{si} d(\Gamma_{s0} \cup \Gamma_{sL}) = 0. \end{cases} \quad (10)$$

The truncation operator $\mathcal{P}_\eta$, of class C^1, is defined by

$$\mathcal{P}_\eta(d_s)(x_s) = \begin{cases} -\xi_{s0}(x_s) + \frac{\eta}{2}, \text{ if } d_s(x_s) \leq -\xi_{s0}(x_s) + \frac{\eta}{2} \\ d_s(x_s), \text{ if } \eta - \xi_{s0}(x_s) \leq d_s(x_s) \leq 1 - \xi_{s0}(x_s) - \eta \\ 1 - \xi_{s0}(x_s), \text{ if } d_s(x_s) \geq 1 - \xi_{s0}(x_s) \end{cases} \quad (11)$$

and satisfies

$$0 < \frac{\eta}{2} \leq (\xi_{s0} + \mathcal{P}_\eta(d_s))(x_s) \leq 1, \quad \forall x_s \in \overline{\Omega}_s. \quad (12)$$

The coefficients $b_{\alpha\beta 33}(d_s)$ and $b_{3333}(d_s)$ are continuously differentiable with respect to d_s and are elements of the matrix $\big(b_{ijkl}(d_s)\big)$ which is the inverse matrix of the three-dimensional elastic coefficients $\big(c_{ijkl}(d_s)\big)$ of the rod $\overline{\Omega}_s$ (cf. Figueiredo and Trabucho [8]). Moreover $b_{\alpha\beta 33}(d_s)$ and $b_{3333}(d_s)$ belong to the space $C^1([0,T]; C^1(\mathbb{R}^3))$ when $d_s \in C^1([0,T]; C^0(\overline{\Omega}_s))$ (cf. Monnier and Trabucho [10] and also Figueiredo and Trabucho [8]). The coefficients $A_{\alpha\beta}(d_s)$, $A_{33}(d_s)$ and $a(d_s)$ are remodeling rate coefficients and are supposed continuously differentiable with respect to d_s. In addition we also assume that there exist strictly positive constants C_1, C_2, C_3, C_4, C_5 and C_6 independent of s and t, such that for any $(x_s,t) \in \overline{\Omega} \times [0,T]$

$$0 < C_1 \leq \frac{1}{b_{3333}(d_s)} \leq C_2, \qquad \forall s \in [0,\delta], \tag{13}$$

$$|c(d_s)| \leq C_3, \qquad |a(d_s)| \leq C_4,, \qquad \forall s \in [0,\delta], \tag{14}$$

$$|c'(d_s)| \leq C_5, \qquad |a'(d_s)| \leq C_6,, \qquad \forall s \in [0,\delta], \tag{15}$$

where $c'(.)$ and $a'(.)$ are the derivatives of the scalar functions $c(.)$ and $a(.)$.

The coefficients $l_s(d_s)$, $e_{s\alpha}(d_s)$ and $h_{s\alpha\beta}(d_s)$, that depend on the elasticity coefficient $b_{3333}(d_s)$ (cf. Figueiredo and Trabucho [8]), are functions of x_3 and t, and are defined by

$$\begin{aligned} &l_s(x_3,t) = \int_{\omega_s} \tfrac{1}{b_{3333}(d_s)} d\omega_s, \\ &e_{s\alpha}(x_3,t) = \int_{\omega_s} \tfrac{x_{s\alpha}}{b_{3333}(d_s)} d\omega_s, \qquad h_{s\alpha\beta}(x_3,t) = \int_{\omega_s} \tfrac{x_{s\alpha}x_{s\beta}}{b_{3333}(d_s)} d\omega_s. \end{aligned} \tag{16}$$

Doing the variational formulation of the equilibrium equations (that is obtained multiplying the first equilibrium equation (6) by $\underline{v}_{s3} \in W^{1,2}(0,L)$ and the second and third equilibrium equations by $x_{s1}\, v_{s1}$ and $x_{s2}\, v_{s2}$, respectively, with $v_{s\beta} \in W^{2,2}(0,L)$, for $\beta = 1,2$, and subsequently integrating in $(0,L)$ and using the boundary condititons (7)), the asymptotic adaptive elastic rod model (5)-(8) is equivalent to the following nonlinear (variational and differential) system (P_s) (cf. Figueiredo and Trabucho [8])

$$P_s \left[\begin{array}{l} \text{Find } u_s : \overline{\Omega}_s \times [0,T] \to \mathbb{R}^3, \quad d_s : \overline{\Omega}_s \times [0,T] \to \mathbb{R} \text{ such that :} \\ u_s(.,t) \in V(\Omega_s) \setminus \mathcal{R}_s, \\ a_s(u_s,v_s) = L_s(v_s), \qquad \forall v_s \in V(\Omega_s) \setminus \mathcal{R}_s, \\ \dot{d}_s = c(d_s) e_{33}(u_s) + a(d_s), \quad \text{in} \quad \Omega_s \times (0,T), \\ d_s(x_s,0) = \overline{d}_s(x_s), \quad \text{in} \quad \overline{\Omega}_s. \end{array} \right. \tag{17}$$

The space $V(\Omega_s) = \Big\{ v_s \in [W^{1,2}(\Omega_s)]^3 : \quad e_{\alpha\beta}(v_s) = e_{3\beta}(v_s) = 0 \Big\}$ is identified with

$$\begin{aligned} &\Big\{ v_s = (v_{s1}, v_{s2}, v_{s3}) \in [W^{2,2}(0,L)]^2 \times W^{1,2}(\Omega_s) : \quad v_{s\alpha}(x_s) = v_{s\alpha}(x_3), \\ &\qquad v_{s3}(x_s) = \underline{v}_{s3}(x_3) - x_{s\alpha} \partial_3 v_{s\alpha}(x_3), \quad \underline{v}_{s3} \in W^{1,2}(0,L) \Big\}, \end{aligned} \tag{18}$$

and the quotient space $V(\Omega_s) \setminus \mathcal{R}_s$ is the set

$$\Big\{ v_s = z_s + a + b \wedge x_s : \quad z_s \in V(\Omega_s), \quad a \in \mathbb{R}^3, \quad b = (b_1, b_2, 0) \in \mathbb{R}^3 \Big\}. \quad (19)$$

The bilinear form $a_s(.,.)$, depending on the unknown d_s, is defined in the space $V(\Omega_s) \setminus \mathcal{R}_s$ by

$$a_s(z_s, v_s) = \int_{\Omega_s} \frac{1}{b_{3333}(d_s)} e_{33}(z_s) e_{33}(v_s) d\Omega_s, \quad \forall z_s,\ v_s \in V(\Omega_s) \setminus \mathcal{R}_s \qquad (20)$$

and $L_s(.)$ is a linear form also defined in $V(\Omega_s) \setminus \mathcal{R}_s$ by

$$\left\{ \begin{aligned} L_s(v_s) = &\textstyle\int_{\Omega_s} \gamma(\xi_{s0} + P_\eta(d_s))\, f_{si}\, v_{si}\, d\Omega_s + \int_{\Gamma_s} g_{si}\, v_{si}\, d\Gamma_s \\ &+ \textstyle\int_{\Gamma_{s0} \cup \Gamma_{sL}} h_{si}\, v_{si}\, d(\Gamma_{s0} \cup \Gamma_{sL}). \end{aligned} \right. \qquad (21)$$

Remark 2.1 (Justification of the definition of $V(\Omega_s) \setminus \mathcal{R}_s$)**.** *We must have $b_3 = 0$ in the definition (19), because otherwise the quotient space $V(\Omega_s) \setminus \mathcal{R}_s$ would not be contained in $V(\Omega_s)$. In fact developing $v_s = z_s + a + b \wedge x_s$ we have for the first component, $v_{s1} = z_{s1} + a_1 + b_2 x_3 - b_3 x_{s2}$, for the second component $v_{s2} = z_{s2} + a_2 - b_1 x_3 + b_3 x_{s1}$ and finally for the third component $v_{s3} = \underline{z}_{s3} - x_{s\alpha}\partial_3 z_{s\alpha} + a_3 + b_1 x_{s2} - b_2 x_{s1}$. Therefore if $b_3 \neq 0$, then $v_s \notin V(\Omega_s)$, and if $b_3 = 0$, we have*

$$\begin{aligned} v_{s1} &= z_{s1} + a_1 + b_2 x_3 \quad (v_{s1} \textit{ depends only on } x_3) \\ v_{s2} &= z_{s2} + a_2 - b_1 x_3 \quad (v_{s2} \textit{ depends only on } x_3) \\ v_{s3} &= \underline{z}_{s3} - x_{s\alpha}\partial_3 z_{s\alpha} + a_3 + b_1 x_{s2} - b_2 x_{s1} \\ &= \underbrace{\underline{z}_{s3} + a_3}_{\underline{v}_{s3}} - x_{s\alpha}\partial_3 v_{s\alpha} = \underline{v}_{s3} - x_{s\alpha}\partial_3 v_{s\alpha} \end{aligned} \qquad (22)$$

because $(b_1, b_2, 0) \wedge x_s = \big(b_2 x_3, -b_1 x_3, b_1 x_{s2} - b_2 x_{s1}\big)$. □

By the following Korn's type inequality in the quotient space $V(\Omega_s) \setminus \mathcal{R}_s$ (cf. Ciarlet [1] or Valent [12]) we have

$$\exists c > 0 : \quad \|v_s\|^2_{[W^{1,2}(\Omega_s)]^3} \leq c \|e_{33}(v_s)\|^2_{L^2(\Omega_s)}, \qquad (23)$$

where $v_s = (v_{s1}, v_{s2}, v_{s3})$, $v_{s\alpha} \in W^{2,2}(0, L)$ and $v_{s3} = \underline{v}_{s3} - x_{s\alpha}\partial_3 v_{s\alpha}$, with $\underline{v}_{s3} \in W^{1,2}(0, L)$

$$\begin{aligned} &e_{33}(v_s) = \partial_3 v_{s3} = \partial_3 \underline{v}_{s3} - x_{s\alpha}\partial_{33} v_{s\alpha}, \quad \text{and due to (3)}, \\ &\|e_{33}(v_s)\|^2_{L^2(\Omega_s)} = \|\partial_3 \underline{v}_{s3}\|^2_{L^2(0,L)} + \big(\textstyle\int_{\omega_s} x^2_{s\alpha} d\omega_s\big) \|\partial_{33} v_{s\alpha}\|^2_{L^2(0,L)}. \end{aligned} \qquad (24)$$

Hence, we conclude that $\|e_{33}(.)\|_{L^2(\Omega_s)}$ is a norm in the space $V(\Omega_s) \setminus \mathcal{R}_s$ equivalent to the usual norm induced in the quotient space by $\|.\|_{[W^{1,2}(\Omega_s)]^3}$. Moreover $V(\Omega_s) \setminus \mathcal{R}_s$ is a Hilbert space with the norm $\|e_{33}(.)\|_{L^2(\Omega_s)}$ and the bilinear form $a_s(.,.)$ is elliptic in $V(\Omega_s) \setminus \mathcal{R}_s$. In fact, there exists a constant $C > 0$, such that, for all $v_s \in V(\Omega_s) \setminus \mathcal{R}_s$

$$\left\{ \begin{array}{l} a_s(v_s, v_s) = \int_{\Omega_s} \frac{1}{b_{3333}(d_s)}\, e_{33}(v_s)\, e_{33}(v_s)\, d\Omega_s \geq \\ C_1 \|e_{33}(v_s)\|^2_{L^2(\Omega_s)} = C_1 \|v_s\|^2_{V(\Omega_s)\setminus\mathcal{R}_s} \geq C\|v_s\|^2_{[W^{1,2}(\Omega_s)]^3}, \end{array} \right. \tag{25}$$

where C_1 is the constant defined in condition (13).

For each $s \in [0, \delta]$, the existence and uniqueness of solution of the asymptotic adaptive elastic rod model P_s (cf. (17)) is proved in Figueiredo and Trabucho [8], and relies on the Schauder's fixed point theorem. The next theorem recalls this statement of existence and uniqueness.

Theorem 2.1 (Existence and uniqueness of solution of P_s)**.** *Let $s \in [0, \delta]$ be fixed. Assuming that, for each fixed $\hat{d}_s$, the unique solution $\hat{u}_s$ of the equilibrium problem*

$$\left\{ \begin{array}{l} \textit{Find}\ u_s(.,t) \in V(\Omega_s) \setminus \mathcal{R}_s, \quad \textit{such that}: \\ a_s(u_s, v_s) = L_s(v_s), \quad \forall v_s \in V(\Omega_s) \setminus \mathcal{R}_s, \end{array} \right. \tag{26}$$

is regular enough, then there exists a unique pair (u_s, d_s), solution of problem P_s (cf. (17)), verifying

$$u_s \in C^1([0,T]; V(\Omega_s) \setminus \mathcal{R}_s) \qquad \textit{and} \qquad d_s \in C^1([0,T]; C^0(\overline{\Omega}_s)). \quad \square \tag{27}$$

The next theorem is a regularity result, that will be important in section 4, and, to prove it, we introduce the following notations

$$\begin{array}{l} z_{s3} = l_s(d_s)\partial_3 \underline{u}_{s3} - e_{s\alpha}(d_s)\partial_{33} u_{s\alpha}, \\ z_{s\beta} = -e_{s\beta}(d_s)\partial_3 \underline{u}_{s3} + h_{s\alpha\beta}(d_s)\partial_{33} u_{s\alpha}, \\ F_{s3} = \int_{\omega_s} \gamma(\xi_{s0} + P_\eta(d_s))\, f_{s3}\, d\omega_s + \int_{\partial\omega_s} g_{s3}\, d\partial\omega_s, \\ F_{s\beta} = \left\{ \begin{array}{l} \int_{\omega_s} \gamma(\xi_{s0} + P_\eta(d_s))\, f_{s\beta}\, d\omega_s + \int_{\partial\omega_s} g_{s\beta}\, d\partial\omega_s \\ + \int_{\omega_s} x_{s\beta}\, \partial_3[\gamma(\xi_{s0} + P_\eta(d_s))\, f_{s3}]\, d\omega_s + \int_{\partial\omega_s} x_{s\beta}\, \partial_3 g_{s3}\, d\partial\omega_s, \end{array} \right. \end{array} \tag{28}$$

and the matrix M_s

$$M_s = \begin{bmatrix} l_s & -e_{s1} & -e_{s2} \\ -e_{s1} & h_{s11} & h_{s12} \\ -e_{s2} & h_{s21} & h_{s22} \end{bmatrix}. \tag{29}$$

We remark that

$$\begin{aligned} & z_{s3} \in C^1([0,T];L^2(0,L)) \quad \text{and} \quad z_{s\beta} \in C^1([0,T];L^2(0,L)), \quad \text{for } \beta = 1,2, \\ & F_{s3} \in C^1([0,T];L^2(0,L)) \quad \text{and} \quad F_{s\beta} \in C^1([0,T];L^2(0,L)), \quad \text{for } \beta = 1,2, \\ & M_s \in C^1\big([0,T];[C^1(\mathbb{R}^3)]^9\big). \end{aligned} \tag{30}$$

Theorem 2.2 (Regularity of u_s)**.** *Let (u_s, d_s) be the unique solution of problem P_s (cf. (17)). We assume that the determinant of matrix M_s is not zero,* $\det M_s \neq 0$*. Then, for each $t \in [0,T]$*

$$u_{s\beta}(.,t) \in W^{3,2}(0,L) \quad \textit{and} \quad \underline{u}_{s3}(.,t) \in W^{2,2}(0,L). \tag{31}$$

In consequence, and because $u_s = (u_{s1}, u_{s2}, \underline{u}_{s3} - x_{s\beta}\partial_3 u_{s\beta})$, we have $u_s(.,t) \in [W^{2,2}(\Omega_s)]^3$.
(Note – for example, if $\frac{1}{b_{3333}(d^s)} = c$, where c is a constant, then

$$\det M_s = [\int_{\omega_s} 1 d\omega_s \int_{\omega_s} x_{s1}^2 d\omega_s \int_{\omega_s} x_{s1}^2 d\omega_s \,]\, c^{-3} > 0) \tag{32}$$

Proof: We first remark that the equilibrium equations (6) can be written in the form

$$\begin{aligned} & -\partial_3 z_{s3} = F_{s3}, \\ & \partial_{33} z_{s\beta} = F_{s\beta}, \quad \beta = 1,2, \end{aligned} \tag{33}$$

and for each t, $F_{s3}(.,t)$ and $F_{s\beta}(.,t)$ belong to the space $L^2(0,L)$.

Since $z_{s3}(.,t)$ belongs to $L^2(0,L)$, and because of the first equilibrium equation in (33), also $\partial_3 z_{s3}(.,t)$ belongs to $L^2(0,L)$, so we conclude that $z_{s3}(.,t) \in W^{1,2}(0,L)$, for each t.

For each t, $z_{s\beta}(.,t) \in L^2(0,L)$ and so $\partial_3 z_{s\beta}(.,t) \in [W^{1,2}(0,L)]'$, where $[W^{1,2}(0,L)]'$ is the dual of $W^{1,2}(0,L)$. But from the second equilibrium equation in (33) we have that $\partial_{33} z_{s\beta}(.,t) \in L^2(0,L)$ and also $\partial_{33} z_{s\beta}(.,t) \in [W^{1,2}(0,L)]'$ because $L^2(0,L) \subset [W^{1,2}(0,L)]'$. So, because of a lemma of J.L.Lions (cf. Ciarlet [2], p. 39) $\partial_3 z_{s\beta}(.,t) \in L^2(0,L)$. Hence the elements $z_{s\beta}(.,t)$, $\partial_3 z_{s\beta}(.,t)$ and $\partial_{33} z_{s\beta}(.,t)$ belong to the space $L^2(0,L)$, which means that $z_{s\beta}(.,t) \in W^{2,2}(0,L)$.

Therefore, assembling these properties we obtain, for each t

$$\begin{aligned} z_{s3}(.,t) &= p_{s3}(.,t) \in W^{1,2}(0,L), \\ z_{s1}(.,t) &= p_{s1}(.,t) \in W^{2,2}(0,L), \\ z_{s2}(.,t) &= p_{s2}(.,t) \in W^{2,2}(0,L), \end{aligned} \tag{34}$$

where p_{si} is a primitive of F_{si} (in the sense of distributions in $W^{1,2}(0,L)$), for $i = 1,2,3$. Replacing z_{si} by its definition (cf. (28)) the system (34) is equivalent to the following system of three equations

$$\begin{bmatrix} l_s & -e_{s1} & -e_{s2} \\ -e_{s1} & h_{s11} & h_{s12} \\ -e_{s2} & h_{s21} & h_{s22} \end{bmatrix} \begin{bmatrix} \partial_3 \underline{u}_{s3} \\ \partial_{33} u_{s1} \\ \partial_{33} u_{s2} \end{bmatrix} = \begin{bmatrix} p_{s3} \\ p_{s1} \\ p_{s2} \end{bmatrix} \tag{35}$$

With the assumption $\det M_s \neq 0$, we may solve the system (35). We obtain, for $\partial_3 \underline{u}_{s3}(.,t)$

$$\partial_3 \underline{u}_{s3}(.,t) = \frac{\det \begin{bmatrix} p_{s3} & -e_{s1} & -e_{s2} \\ p_{s1} & h_{s11} & h_{s12} \\ p_{s2} & h_{s21} & h_{s22} \end{bmatrix}}{\det M_s} \in W^{1,2}(0,L) \tag{36}$$

for $\partial_{33} u_{s1}(.,t)$

$$\partial_{33} u_{s1}(.,t) = \frac{\det \begin{bmatrix} l_s & p_{s3} & -e_{s2} \\ -e_{s1} & p_{s1} & h_{s12} \\ -e_{s2} & p_{s2} & h_{s22} \end{bmatrix}}{\det M_s} \in W^{1,2}(0,L) \tag{37}$$

and for $\partial_{33} u_{s2}(.,t)$

$$\partial_{33} u_{s2}(.,t) = \frac{\det \begin{bmatrix} l_s & -e_{s1} & p_{s3} \\ -e_{s1} & h_{s11} & p_{s1} \\ -e_{s2} & h_{s21} & p_{s2} \end{bmatrix}}{\det M_s} \in W^{1,2}(0,L). \tag{38}$$

We remark that the regularity indicated in (36)-(38) depends also on the regularity of the elements of M_s that belong to the space $C^1([0,T]; C^1(\mathbb{R}^3))$.

So we conclude that the component $u_{s\beta}(.,t) \in W^{3,2}(0,L)$, for $\beta = 1,2$ and $\underline{u}_{s3}(.,t) \in W^{2,2}(0,L)$, which implies that $u_s(.,t) \in [W^{2,2}(\Omega_s)]^3$. $\square$

2.3. The shape probem. We consider now the shape map J defined by

$$\begin{aligned} J:\ [0,\delta] &\longrightarrow C^1([0,T];V(\Omega_s)\setminus\mathcal{R}_s)\times C^1([0,T];C^0(\overline{\Omega}_s)) \\ s &\longrightarrow J(\Omega_s)=(u_s,d_s) \end{aligned} \tag{39}$$

where (u_s,d_s) is the unique solution of the nonlinear adaptive rod model P_s (cf. (17)), defined in the perturbed rod Ω_s. For each s, the functions $u_s(x_s,t)$ and $d_s(x_s,t)$ are the displacement and the measure of change in volume fraction of the elastic material, at the point x_s of the rod $\overline{\Omega}_s$, and at time t. Moreover, as remarked before, the unknowns u_s and d_s are coupled in the model P_s.

We recall that $I_s(\omega)$ is a perturbation of the cross section ω of the rod $\overline{\Omega}=\overline{\omega}\times[0,L]$ (cf. (1)), so $\overline{\Omega}_s$ is a perturbation of the rod $\overline{\Omega}$, that is

$$\begin{aligned} &\overline{\Omega}_s=I_s(\overline{\omega})\times[0,L]=(I+s\,\theta)(\overline{\omega})\times[0,L], \\ &\text{and} \\ &\overline{\Omega}=\overline{\omega}\times[0,L]=I_0(\overline{\omega})\times[0,L]=\overline{\Omega}_0. \end{aligned} \tag{40}$$

The aim of this paper is to compute the shape semiderivative $dJ(\Omega;\theta)$ at $s=0$, in the direction of the vector field θ. This semiderivative is defined by (cf. Delfour and Zolésio [6], p.289)

$$dJ(\Omega;\theta)=\lim_{s\to0^+}\frac{J(\Omega_s)-J(\Omega)}{s}. \tag{41}$$

Accordingly to the definition of J we have

$$dJ(\Omega;\theta)=\lim_{s\to0^+}\frac{(u_s,d_s)-(u,d)}{s}=\Big(\lim_{s\to0^+}\frac{u_s-u}{s},\ \lim_{s\to0^+}\frac{d_s-d}{s}\Big) \tag{42}$$

where (u,d) is the solution of problem (17) but for the unperturbed rod $\overline{\Omega}_0=\overline{\Omega}=\overline{\omega}\times[0,L]$.

We also remark that the semiderivative $dJ(\Omega;\theta)$ is equivalent to the definition of the material derivative of the map J at $s=0$, in the sense of Haslinger and Mäkinen [9], p.111.

In section 4, we prove that it is possible to compute and to identify, in a weak sense and in an appropriate product space, this shape semiderivative. The final identification result is described in theorems 4.14-4.15 and corollary 4.3.

3. Equivalent Formulation of the Adaptive Rod Model

In order to be able to calculate the shape semiderivative indicated in (42) we must reformulate, for each s, the problem P_s (cf. (17)) in the domain $\overline{\Omega} \times [0,T]$, independent of s. Therefore we first formulate, in Ω, all the forms involved in the definition of problem P_s. Afterwards, we describe the resulting rod model, that is denoted by P^s, equivalent to P_s, formulated in $\overline{\Omega} \times [0,T]$.

3.1. Reformulations of the forms defining $\mathbf{P_s}$. We define the homeomorphism

$$Q(x_1,x_2,x_3) = \big(x_1 + s\theta_1(x_1,x_2), x_2 + s\theta_2(x_1,x_2), x_3\big), \tag{43}$$

that verifies

$$\Omega_s = Q_s(\Omega), \qquad \det \nabla Q_s = 1 + s\, div\, \theta + s^2 \det \nabla\theta, \tag{44}$$

where the matrices ∇Q_s and $\nabla\theta$ are the gradients of Q_s and θ, respectively, and $div\,\theta = \partial_\alpha \theta_\alpha$ is the divergence of θ.

To each function v_s defined in Ω_s we associate the corresponding function v^s (with upper index s) defined on Ω by

$$v^s = v_s \circ Q_s. \tag{45}$$

Hence and because of remark 2.1 for any $v_s \in V(\Omega_s) \setminus \mathcal{R}_s$, then

$$\begin{aligned} v_1^s &= z_1^s + a_1 + b_2 x_3 \\ v_{s2}^s &= z_2^s + a_2 - b_1 x_3 \\ v_3^s &= \underline{z}_3^s - x_\alpha \partial_3 z_\alpha^s + a_3 + b_1 x_2 - b_2 x_1 \\ &= \underbrace{\underline{z}_3^s + a_3}_{\underline{v}_3^s} - x_\alpha \partial_3 v_\alpha^s = \underline{v}_3^s - x_\alpha \partial_3 v_\alpha^s \end{aligned} \tag{46}$$

and consequently $v^s \in V(\Omega) \setminus \mathcal{R}$ (whose definition is (19), for the case $s=0$). Moreover,

$$\begin{aligned} e_{33}(v_s) &= \partial_3 \underline{v}_{s3} - x_{s\alpha} \partial_{33} v_{s\alpha} = \partial_3 \underline{v}_{s3} - (x_\alpha + s\,\theta_\alpha)\partial_{33} v_\alpha^s \\ &= \partial_3 \underline{v}_3^s - x_\alpha \partial_{33} v_\alpha^s - s\,\theta_\alpha \partial_{33} v_\alpha^s \\ &= e_{33}(v^s) - s\,\theta_\alpha \partial_{33} v_\alpha^s. \end{aligned} \tag{47}$$

Using (44)-(47) and the change of variables formula for domain and boundary integrals (cf. Delfour and Zolésio [6], p.351-353) we get for $a_s(u_s, v_s)$

$$\int_\Omega \frac{1}{b_{3333}(d^s)}\big(e_{33}(u^s) - s\,\theta_\alpha \partial_{33} u^s_\alpha\big)\big(e_{33}(v^s) - s\,\theta_\alpha \partial_{33} v^s_\alpha\big)\ \det \nabla Q_s d\Omega, \tag{48}$$

and for $L_s(v_s)$

$$\begin{cases} \int_\Omega \gamma(\xi_0^s + P_\eta(d^s))\, f_i^s\, v_i^s\, (\det \nabla Q_s)\, d\Omega \\ \quad + \int_\Gamma g_i^s\, v_i^s\, |(Cof\ \nabla Q_s)^T n|_{\mathbb{R}^3}\, d\Gamma \\ \qquad + \int_{\Gamma_0 \cup \Gamma_L} h_i^s\, v_i^s\, |(Cof\ \nabla Q_s)^T n|_{\mathbb{R}^3}\, d(\Gamma_0 \cup \Gamma_{sL}). \end{cases} \tag{49}$$

In (49) $|.|_{\mathbb{R}^3}$ is the euclidean norm in $\mathbb{R}^3$, $(Cof\ \nabla Q_s)^T$ is the transpose of the cofactor matrix of ∇Q_s, that is $(Cof\ \nabla Q_s)^T = (\det \nabla Q_s)(\nabla Q_s)^{-T}$, whose definition is

$$\begin{pmatrix} 1 + s\partial_2\theta_2 & -s\partial_1\theta_2 & 0 \\ -s\partial_2\theta_1 & 1 + s\partial_1\theta_1 & 0 \\ 0 & 0 & (1 + s\partial_2\theta_2)(1 + s\partial_1\theta_1) - s^2\partial_1\theta_2\partial_2\theta_1 \end{pmatrix} \tag{50}$$

and $n = (n_1, n_2, n_3)$ is the unit outer normal vector along the boundary $\partial\Omega$ of Ω. We remark that

$$\begin{aligned} n(x) &= (n_1(x), n_2(x), 0), && \text{for} \quad x \in \Gamma = \partial\omega \times]0, L[, \\ n(x) &= (0, 0, -1), && \text{for} \quad x \in \Gamma_0 = \overline{\omega} \times \{0\}, \\ n(x) &= (0, 0, 1), && \text{for} \quad x \in \Gamma_L = \overline{\omega} \times \{L\}. \end{aligned} \tag{51}$$

Developing (48)-(49) we obtain the following decomposition for $a_s(u_s, v_s)$ and $L_s(v_s)$

$$a_0^s(u^s, v^s) + s a_1^s(u^s, v^s) + s^2 a_2^s(u^s, v^s) + s^3 a_3^s(u^s, v^s) + s^4 a_4^s(u^s, v^s),$$

$$\begin{cases} F_0^s(v^s) + G_0^s(v^s) + H_0^s(v^s) + s\,\big(F_1^s(v^s) + G_1^s(v^s) + H_1^s(v^s)\big) + \\ s^2\,\big(F_2^s(v^s) + G_2^s(v^s) + H_2^s(v^s)\big) + s^3\,\big(F_3^s(v^s) + G_3^s(v^s) + H_3^s(v^s)\big). \end{cases} \tag{52}$$

The bilinear forms $a_i^s(.,.)$ for $i = 0, 1, 2, 3, 4$, depend on θ and d^s, and are defined by the formulas

$$
\begin{aligned}
&a_0^s(u,v) = \int_\Omega \frac{1}{b_{3333}(d^s)} e_{33}(u) e_{33}(v) d\Omega,\\
&a_1^s(u,v) = \left\{ \begin{array}{l} \int_\Omega \frac{1}{b_{3333}(d^s)} \Big[-\theta_\alpha \big(e_{33}(u)\partial_{33} v_\alpha + e_{33}(v)\partial_{33}u_\alpha\big) \\ \qquad\qquad +e_{33}(u)e_{33}(v) div\, \theta \Big] d\Omega, \end{array}\right.\\
&a_2^s(u,v) = \left\{ \begin{array}{l} \int_\Omega \frac{1}{b_{3333}(d^s)} \Big[e_{33}(u)e_{33}(v) \det \nabla\theta + \theta_\alpha\, \theta_\beta\, \partial_{33}u_\alpha\, \partial_{33}v_\beta \\ \qquad -(div\,\theta)\, \theta_\alpha \big(e_{33}(u)\partial_{33} v_\alpha + e_{33}(v)\partial_{33}u_\alpha\big) \Big] d\Omega, \end{array}\right.\\
&a_3^s(u,v) = \left\{ \begin{array}{l} \int_\Omega \frac{1}{b_{3333}(d^s)} \Big[\theta_\alpha\theta_\beta \partial_{33}u_\alpha \partial_{33}v_\beta\, div\,\theta \\ \qquad -(\det\nabla\theta)\, \theta_\alpha \big(e_{33}(u)\partial_{33} v_\alpha + e_{33}(v)\partial_{33}u_\alpha\big) \Big] d\Omega, \end{array}\right.\\
&a_4^s(u,v) = \int_\Omega \frac{1}{b_{3333}(d^s)} \Big[\theta_\alpha\, \theta_\beta\, \partial_{33}u_\alpha\, \partial_{33}v_\beta\, \det\nabla\theta \Big] d\Omega,
\end{aligned}
\tag{53}
$$

for any pair (u, v) in the space $V(\Omega) \setminus \mathcal{R}$. The linear forms F_0^s, F_1^s, F_2^s and F_3^s, depend on θ and d^s, and are also defined in the same quotient space $V(\Omega) \setminus \mathcal{R}$ by the following expressions

$$
\begin{aligned}
&F_0^s(v) = \int_\Omega \gamma(\xi_0^s + P_\eta(d^s))\, (f_\alpha^s\, v_\alpha + f_3^s\, \underline{v}_3)\, d\Omega,\\
&F_1^s(v) = \int_\Omega \gamma(\xi_0^s + P_\eta(d^s))\, \Big[(f_\alpha^s\, v_\alpha + f_3^s\, \underline{v}_3)\, div\,\theta - f_3^s\, \theta_\alpha\, \partial_3 v_\alpha \Big]\, d\Omega,\\
&F_2^s(v) = \int_\Omega \gamma(\xi_0^s + P_\eta(d^s))\, \Big[(f_\alpha^s\, v_\alpha + f_3^s\, \underline{v}_3)\, \det\nabla\theta \\
&\qquad\qquad\qquad\qquad -f_3^s\, \theta_\alpha\, \partial_3 v_\alpha\, div\,\theta \Big]\, d\Omega,\\
&F_3^s(v) = -\int_\Omega \gamma(\xi_0^s + P_\eta(d^s))\, f_3^s\, \theta_\alpha\, \partial_3 v_\alpha\, \det\nabla\theta\, d\Omega.
\end{aligned}
\tag{54}
$$

The linear forms G_0^s, G_1^s, G_2^s, G_3^s and H_0^s, H_1^s, H_2^s, H_3^s result from the change of variable in the boundary integrals (defined in Γ_s and in $\Gamma_{s0} \cup \Gamma_{sL}$, respectively). Their expressions, that depend on θ and n, but are independent of

d^s, are the following for any v in the space $V(\Omega)\setminus\mathcal{R}$:

$$\begin{aligned}
G_0^s(v) &= \int_\Gamma \left(g_\alpha^s\, v_\alpha + g_3^s\, \underline{v}_3\right) d\Gamma,\\
G_1^s(v) &= \int_\Gamma \left[\left(g_\alpha^s\, v_\alpha + g_3^s\, \underline{v}_3\right) G_1(\theta,n) - g_3^s\,\theta_\alpha\,\partial_3 v_\alpha\right] d\Gamma,\\
G_2^s(v) &= \int_\Gamma \left[\left(g_\alpha^s\, v_\alpha + g_3^s\, \underline{v}_3\right) G_2(\theta,n) - g_3^s\,\theta_\alpha\,\partial_3 v_\alpha\, G_1(\theta,n)\right] d\Gamma,\\
G_3^s(v) &= -\int_\Gamma g_3^s\,\theta_\alpha\,\partial_3 v_\alpha\, G_3(\theta,n)\, d\Gamma,
\end{aligned} \tag{55}$$

where $G_1(\theta,n)$, $G_2(\theta,n)$ and $G_3(\theta,n)$ are bounded scalar functions of θ and n, the unit outer normal vector, and

$$\begin{aligned}
H_0^s(v) &= \int_{\Gamma_0\cup\Gamma_L} \left(h_\alpha^s\, v_\alpha + h_3^s\, \underline{v}_3\right) d\Gamma_0\cup\Gamma_L,\\
H_1^s(v) &= \int_{\Gamma_0\cup\Gamma_L} \left[\left(h_\alpha^s\, v_\alpha + h_3^s\, \underline{v}_3\right) H_1(\theta) - h_3^s\,\theta_\alpha\,\partial_3 v_\alpha\right] d\Gamma_0\cup\Gamma_L,\\
H_2^s(v) &= \int_{\Gamma_0\cup\Gamma_L} \left[\left(h_\alpha^s\, v_\alpha + h_3^s\, \underline{v}_3\right) H_2(\theta) - h_3^s\,\theta_\alpha\,\partial_3 v_\alpha\, H_1(\theta)\right] d\Gamma_0\cup\Gamma_L,\\
H_3^s(v) &= -\int_{\Gamma_0\cup\Gamma_L} h_3^s\,\theta_\alpha\,\partial_3 v_\alpha\, H_2(\theta)\, d\Gamma_0\cup\Gamma_L,
\end{aligned} \tag{56}$$

where $H_1(\theta)$ and $H_2(\theta)$ are bounded scalar functions of θ.

3.2. The problem $\mathbf{P_s}$ formulated in $\overline{\boldsymbol{\Omega}}\times[\mathbf{0},\mathbf{T}]$. As a direct consequence of (52) we deduce the next theorem which formulates the problem P_s in Ω; the new equivalent problem is denoted by P^s, with upper index s.

Theorem 3.1 (Problem P^s). *We assume that for each $s\in[0,\delta]$, $d^s(x,0)=\overline{d}(x)$ in $\overline{\Omega}$, and $\overline{d}$ is independent of s. We denote by (u,d) the unique solution of problem P^s, for $s=0$, that is, (u,d) is the solution of the following system P^0 (cf. (17), for the case $s=0$), formulated in $\overline{\Omega}\times[0,T]$*

$$\left[\begin{array}{l}
\textit{Find } u:\overline{\Omega}\times[0,T]\to\mathbb{R}^3, \quad \textit{and} \quad d:\overline{\Omega}\times[0,T]\to\mathbb{R} \quad \textit{such that}:\\
u(.,t)\in V(\Omega)\setminus\mathcal{R},\\
a_0(u,v)=L_0(v)\equiv F_0(v)+G_0(v)+H_0(v), \qquad \forall v\in V(\Omega)\setminus\mathcal{R},\\
\dot{d}=c(d)e_{33}(u)+a(d), \quad \textit{in} \quad \Omega\times(0,T),\\
d(x,0)=\overline{d}(x), \quad \textit{in} \quad \overline{\Omega},
\end{array}\right. \tag{57}$$

where $a_0(.,.)$, $F_0(.)$, $G_0(.)$ and $H_0(.)$ are independent of s and are defined by

$$\begin{aligned}
&a_0(z,v) = \int_\Omega \frac{1}{b_{3333}(d)}\, e_{33}(z)\, e_{33}(v)\, d\Omega, \quad (a_0(.,.) \text{ depends on } d)\\
&F_0(v) = \int_\Omega \gamma(\xi_0 + P_\eta(d))\,(f_\alpha\, v_\alpha + f_3\, \underline{v}_3)\, d\Omega, \quad (F_0(.) \text{ depends on } d)\\
&G_0(v) = \int_\Gamma \left(g_\alpha\, v_\alpha + g_3\, \underline{v}_3\right) d\Gamma,\\
&H_0(v) = \int_{\Gamma_0\cup\Gamma_L} \left(h_\alpha\; v_\alpha + h_3\, \underline{v}_3\right) d\Gamma_0 \cup \Gamma_L, \quad \forall z,\, v \in V(\Omega)\setminus\mathcal{R}.
\end{aligned} \tag{58}$$

Then, the problem P_s (cf. (17)), for $s \neq 0$, is equivalent to the following problem P^s posed in the domain $\overline{\Omega}\times[0,T]$ independent of s

$$\left[\begin{array}{l}
\text{Find } u^s : \overline{\Omega}\times[0,T] \to \mathbb{R}^3, \quad \text{and} \quad d^s : \overline{\Omega}\times[0,T] \to \mathbb{R} \quad \text{such that}:\\
u^s(.,t) \in V(\Omega)\setminus\mathcal{R},\\
\left[\begin{array}{l}
a_0^s(u^s,v) + s\, a_1^s(u^s,v) + s^2\, a_2^s(u^s,v) + s^3\, a_3^s(u^s,v)\\
+s^4\, a_4^s(u^s,v) = F_0^s(v) + sF_1^s(v) + s^2\, F_2^s(v) + s^3\, F_3^s(v)\\
\quad +G_0^s(v) + s\, G_1^s(v) + s^2\, G_2^s(v) + s^3\, G_3^s(v)\\
\qquad +H_0^s(v) + s\, H_1^s(v) + s^2\, H_2^s(v) + s^3\, H_3^s(v),\ \forall v \in V(\Omega)\setminus\mathcal{R},
\end{array}\right.\\
\dot{d}^s = c(d^s)e_{33}(u^s) + a(d^s) - s\, c(d^s)\theta_\alpha \partial_{33} u^s_\alpha, \quad \text{in} \quad \Omega\times(0,T),\\
d^s(x,0) = \overline{d}(x), \quad \text{in} \quad \overline{\Omega},
\end{array}\right. \tag{59}$$

where the sets $V(\Omega)$ and $V(\Omega)\setminus\mathcal{R}$ are defined in (18) and (19), for the case $s = 0$.

4. Calculus and Identification of the Shape Semiderivative

In this section we first prove that the sequence $\{(u^s, d^s)\}$ is bounded in the space $C^0([0,T]; V(\Omega)\setminus\mathcal{R}) \times C^0([0,T]; L^2(\Omega))$, $\{u^s\}$ is bounded in the space $C^0([0,T]; W^{2,2}(\Omega))$, and $\{e_{33}(u^s)\}$ is bounded in the space $C^0([0,T]; C^0(\overline{\Omega}))$, independently of s. Then we show, that the quotient sequence $\{(\frac{u^s-u}{s}, \frac{d^s-d}{s})\}$, where (u,d) is the solution of problem P^0, is also bounded, independently of s, in the space $C^0([0,T]; V(\Omega)\setminus\mathcal{R}) \times C^0([0,T]; L^2(\Omega))$. These results guarantee the existence, for each t, of a pair $(\bar{u}, \bar{d})(.,t)$, which is the weak limit of a subsequence of $\{(\frac{u_s-u}{s}, \frac{d_s-d}{s})(.,t)\}$ in the reflexive Banach space $V(\Omega)\setminus\mathcal{R}\times L^2(\Omega)$, when $s \to 0^+$; consequently, $(\bar{u}, \bar{d})(.,t)$ is the weak shape semiderivative of the functional J, defined in (42), at $s=0$ in the direction of the vector field θ, for each t. We also prove that the sequence $\{e_{33}(u^s)\}$ converges strongly to 0 in $C^0([0,T];C^0(\overline{\Omega}))$, and $\{d^s\}$ converges strongly to d in $C^0([0,T];C^0(\overline{\Omega}))$, when $s\to 0^+$. Using all these estimates and convergence results, we identify the weak shape semiderivative $(\bar{u},\bar{d})$ as the unique solution of a nonlinear problem: the component $\bar{u}$ is the solution of a variational equation, depending on (u,d) and $\bar{d}$, where $\bar{d}$ is the solution of an ordinary differential equation with respect to time, depending on (u,d) and $\bar{u}$. Morover we show that $\bar{u}\in C^1([0,T];V(\Omega)\setminus\mathcal{R})$ and $\bar{d}\in C^1([0,T];C^0(\overline{\Omega}))$.

4.1. Preliminary estimates.

Theorem 4.1 (First estimates for the sequences u^s and d^s)**.** *We suppose that the conditions (13)-(14) are verified and $\overline{d}(x)\in L^2(\Omega)$, then*

$$\exists c_1>0:\qquad \|u^s\|_{C^0([0,T];V(\Omega)\setminus\mathcal{R})}\le c_1,\quad \forall s\in[0,\delta], \tag{60}$$

$$\exists c_2>0:\qquad \|d^s\|_{C^0([0,T];L^2(\Omega))}\le c_2,\quad \forall s\in[0,\delta], \tag{61}$$

where c_1 and c_2 are constants independent of s.

Proof: The pair (u^s,d^s) is the solution of problem P^s (cf. (59)), thus for each time t,

$$a_0^s(u^s,u^s)=L^s(u^s)-\sum_{i=1}^{4}s^i\,a_i^s(u^s,u^s). \tag{62}$$

By the ellipticity of $a_0^s(.,.)$ in $V(\Omega)\setminus\mathcal{R}$ (cf. (25)) we have for each t

$$a_0^s(u^s,u^s)\ge c\,\|u^s(.,t)\|^2_{V(\Omega)\setminus\mathcal{R}}, \tag{63}$$

where C is a constant independent of s and t. But applying (24)

$$\|e_{33}(u^s(.,t))\|^2_{L^2(\Omega)}=\|\partial_3\underline{u}_3^s(.,t)\|^2_{L^2(0,L)}+\Big(\int_\omega x_\alpha^2 d\omega\Big)\|\partial_{33}u_\alpha^s(.,t)\|^2_{L^2(0,L)}, \tag{64}$$

hence

$$\begin{aligned}&\|\partial_3\underline{u}_3^s(.,t)\|_{L^2(0,L)}\le\|e_{33}(u^s(.,t))\|_{L^2(\Omega)}=\|u^s(.,t)\|_{V(\Omega)\setminus\mathcal{R}}\\&\|\partial_{33}u_\alpha^s(.,t)\|_{L^2(0,L)}\le c\|e_{33}(u^s(.,t))\|_{L^2(\Omega)}=c\|u^s(.,t)\|_{V(\Omega)\setminus\mathcal{R}},\end{aligned} \tag{65}$$

where c is a constant independent of s and t. Thus using (64)-(65) we easily check that $a_i^s(.,.)$, for $i=1,2,3,4$, are continuous bilinear forms, that verify for each t

$$a_i^s(u^s,u^s)\le c_i\,\|u^s(.,t)\|^2_{V(\Omega)\setminus\mathcal{R}} \tag{66}$$

with c_i strictly positive constants, independent of s and t, but depending on θ. Analogously, using (64)-(65) we verify again, that, $L^s(.)$ is a linear form, and, for each t

$$L^s(u^s) \le c_L \, \|u^s(.,t)\|_{V(\Omega)\setminus\mathcal{R}} \tag{67}$$

where c_L is another strictly positive constant independent of s and t. So applying (63), (66) and (67) to (62), we conclude that for each time t

$$\Big(c - \sum_{i=1}^{4} s^i \, c_i\Big) \, \|u^s(.,t)\|^2_{V(\Omega)\setminus\mathcal{R}} \le c_L \, \|u^s(.,t)\|_{V(\Omega)\setminus\mathcal{R}} \tag{68}$$

and we obtain the estimate (60), since s is a very small parameter.

Taking the integral in time in the remodeling rate equation of problem (59), we get

$$d^s(x,t) = \int_0^t \Big[c(d^s)e_{33}(u^s) + a(d^s) - s\, c(d^s)\theta_\alpha \partial_{33} u^s_\alpha\Big] \, dr + \overline{d}(x). \tag{69}$$

But as the material and remodeling coefficients $c(d^s)$ and $a(d^s)$ appearing in (69) are bounded (cf. (14)) we deduce that

$$\|d^s(.,t)\|_{L^2(\Omega)} \le \int_0^T \Big[c_1 \, \|u^s(.,t)\|_{V(\Omega)\setminus\mathcal{R}} + c_2\Big] \, dr + \|\overline{d}(x)\|_{L^2(\Omega)}, \tag{70}$$

with c_1 and c_2 two strictly positive constants independent of s. Therefore and because of (60) we have the inequality (61). □

Theorem 4.2 (Second estimate for the sequence u^s)**.** *We assume that condition (13) is verified, the hypotheses of theorem 2.2 are satisfied, and* $\frac{1}{b_{3333}(d^s)} = b + \mathcal{O}(s)$*, where* $b : \mathbb{R} \to \mathbb{R}$ *is a scalar function, where* b *is independent of* s*,* $0 < |b| \le c$ *with* $c > 0$ *a constant, and* $\mathcal{O}(s)$ *is a term of order* s *(cf. Monnier and Trabucho* [10]*, formulas (6) and (2), for a justification of this latter condition on the material coefficient* $b_{3333}(d^s)$*). Then*

$$\exists c_1 > 0 : \qquad \|u^s\|_{C^0([0,T];W^{2,2}(\Omega))} \le c_1, \tag{71}$$

$$\exists c_2 > 0 : \qquad \|e_{33}(u^s)\|_{C^0([0,T];C^0(\overline{\Omega}))} \le c_2, \tag{72}$$

where c_1 *and* c_2 *are constants independent of* $s \in [0, \delta]$.

Proof: Using the relations (36)-(38) in the proof of the regularity theorem (2.2), we deduce that for each t

$$\|u^s(.,t)\|_{W^{2,2}(\Omega)} \le C(M^s; p^s_i) \tag{73}$$

where $M^s = M_s \circ Q_s$, $p_i^s = p_{si} \circ Q_s$ and $C(M^s; p_i^s)$ is a strictly positive constant depending on the $W^{1,2}(0,L)$ norms of the elements of M^s and p_i^s. As p_i^s, for $i = 1,2,3$, are data of the problem, related to the forces (cf. (34)), and due to the definition of M^s and the additional hypothesis for $\frac{1}{b_{3333}(d^s)}$, we easily deduce that

$$\exists c > 0 : \qquad C(M^s; p_i^s) \leq c, \quad \forall s \in [0, \delta], \tag{74}$$

where c is independent of s and t, and therefore we have (71).

Also from the regularity result we have, for each t,

$$e_{33}(u^s)(.,t) = \partial_3 \underline{u}_3^s(.,t) - x_\alpha \partial_{33} u_\alpha^s(.,t) \in W^{1,2}(\Omega) \cap C^0(\overline{\Omega}) \tag{75}$$

because $\partial_3 \underline{u}_3^s(.,t)$ and $\partial_{33} u_\alpha^s(.,t)$ belong to $W^{1,2}(0,L)$ which is compactly embedded in the space $C^0([0,L])$. Hence we get

$$\|e_{33}(u^s)(.,t)\|_{C^0(\overline{\Omega})} \leq c_1 \|e_{33}(u^s)(.,t)\|_{W^{1,2}(\Omega)} \leq c_2 \|u^s(.,t)\|_{W^{2,2}(\Omega)} \leq c_3 \tag{76}$$

where c_1, c_2 and c_3 are constants independent of s and t, and consequently we have (72). $\square$

Theorem 4.3 (Estimate for the sequence $\frac{u_s - u}{s}$)**.** *Let (u^s, d^s) and (u, d) be the solutions of problems P^s (cf. (59)) and P^0 (cf. (57)), respectively. We assume that conditions (13)-(15) are verified, and, for each s, $\xi_0^s = \xi_0$, $f_i^s = f_i$, $g_i^s = g_i$, $h_i^s = h_i$, where ξ_0, f_i, g_i and h_i are independent of s. Then,*

$$\|\frac{u^s - u}{s}\|_{C^0([0,T]; V(\Omega)\setminus\mathcal{R})} \leq c_1 \|\frac{d^s - d}{s}\|_{C^0([0,T]; L^2(\Omega))} + c_2, \tag{77}$$

where c_1 and c_2 are strictly positive constants independent of s and t.

Proof: In this proof we sometimes write u^s instead of $u^s(.,t)$ in order to simplify the notations. For each t we have

$$\begin{aligned} a^s(u^s, v) = L^s(v), \qquad \forall v \in V(\Omega) \setminus \mathcal{R} \\ a_0(u, v) = L_0(v), \qquad \forall v \in V(\Omega) \setminus \mathcal{R} \end{aligned} \tag{78}$$

which implies that

$$\frac{1}{s}[a^s(u^s, v) - a_0(u, v)] = \frac{L^s(v) - L_0(v)}{s}. \tag{79}$$

Developing this last equation for the choice $v = \frac{u^s-u}{s}$ and observing that, for each t,

$$\left[\begin{array}{l} \frac{1}{s}\Big[\int_\Omega \frac{1}{b_{3333}(d^s)} e_{33}(u^s)e_{33}(\frac{u^s-u}{s})d\Omega - \int_\Omega \frac{1}{b_{3333}(d)} e_{33}(u)e_{33}(\frac{u^s-u}{s})d\Omega\Big] = \\ \int_\Omega \frac{1}{b_{3333}(d^s)} e_{33}(\frac{u^s-u}{s})e_{33}(\frac{u^s-u}{s})d\Omega + \\ \qquad \int_\Omega \frac{b_{3333}(d)-b_{3333}(d^s)}{s}\big(b_{3333}(d)\, b_{3333}(d^s)\big)^{-1} e_{33}(u)e_{33}(\frac{u^s-u}{s})\, d\Omega \end{array}\right. \tag{80}$$

we obtain that (79) is equivalent to the equation

$$\left[\begin{array}{l} a_0^s(\frac{u^s-u}{s}, \frac{u^s-u}{s}) = -s^3\, a_4^s(u^s, \frac{u^s-u}{s}) \\ -\int_\Omega \frac{b_{3333}(d)-b_{3333}(d^s)}{s}\big(b_{3333}(d^s)\, b_{3333}(d)\big)^{-1} e_{33}(u)e_{33}(\frac{u^s-u}{s})\, d\Omega + \\ \frac{1}{s}\Big[F_0^s(\frac{u^s-u}{s}) - F_0(\frac{u^s-u}{s})\Big] + \\ \frac{1}{s}\Big[G_0^s(\frac{u^s-u}{s}) - G_0(\frac{u^s-u}{s}) + H_0^s(\frac{u^s-u}{s}) - H_0(\frac{u^s-u}{s})\Big] + \\ -a_1^s(u^s, \frac{u^s-u}{s}) + F_1^s(\frac{u^s-u}{s}) + G_1^s(\frac{u^s-u}{s}) + H_1^s(\frac{u^s-u}{s}) + \\ s\,\Big[-a_2^s(u^s, \frac{u^s-u}{s}) + F_2^s(\frac{u^s-u}{s}) + G_2^s(\frac{u^s-u}{s}) + H_2^s(\frac{u^s-u}{s})\Big] + \\ s^2\,\Big[-a_3^s(u^s, \frac{u^s-u}{s}) + F_3^s(\frac{u^s-u}{s}) + G_3^s(\frac{u^s-u}{s}) + H_3^s(\frac{u^s-u}{s})\Big]. \end{array}\right. \tag{81}$$

Using this last equation, the ellipticity of $a_0^s(.,.)$ and the properties of continuity of all the other remaining terms in (81) we obtain the estimate (77). We explain next these calculations in detail, analyzing (81), for each t.

Because of condition (13), we have for each t

$$|a_0^s(\frac{u^s-u}{s}, \frac{u^s-u}{s})| \geq c\, \|\frac{u^s-u}{s}(.,t)\|^2_{V(\Omega)\setminus\mathcal{R}} \tag{82}$$

where c is a strictly positive constant independent of s and t. Using the definitions of $a_i^s(.,.)$ and the estimate (60) we obviously obtain

$$\left\{\begin{array}{l} |a_i^s(u^s, \frac{u^s-u}{s})| \leq \overline{c}_{a_i}\, \|u^s(.,t)\|_{V(\Omega)\setminus\mathcal{R}}\, \|\frac{u^s-u}{s}(.,t)\|_{V(\Omega)\setminus\mathcal{R}} \leq \\ c_{a_i}\, \|\frac{u^s-u}{s}(.,t)\|_{V(\Omega)\setminus\mathcal{R}} \end{array}\right. \tag{83}$$

where $\overline{c}_{a_i}$ and c_{a_i}, for $i = 1, 2, 3, 4$, are strictly positive constants independent of s and t. Considering now the definitions of the forces F_i^s, G_i^s, and H_i^s we easily check that, for each t

$$F_i^s(\frac{u^s-u}{s}) + G_i^s(\frac{u^s-u}{s}) + H_i^s(\frac{u^s-u}{s}) \leq c_i \, \| \frac{u^s-u}{s}(.,t)\|_{V(\Omega)\backslash\mathcal{R}} \tag{84}$$

where c_i, for $i = 1, 2, 3$, are other strictly positive constants independent of s and t. In addition we have also, for each t

$$\begin{aligned} \tfrac{1}{s}\Big[G_0^s(\tfrac{u^s-u}{s}) - G_0(\tfrac{u^s-u}{s})\Big] &= 0 \\ \tfrac{1}{s}\Big[H_0^s(\tfrac{u^s-u}{s}) - H_0(\tfrac{u^s-u}{s})\Big] &= 0. \end{aligned} \tag{85}$$

Using the mean value theorem for the operator P_η, we deduce

$$\left[\begin{array}{l} \Big|\tfrac{1}{s}\Big[F_0^s(\tfrac{u^s-u}{s}) - F_0(\tfrac{u^s-u}{s})\Big]\Big| \leq \\ \quad \int_\Omega \gamma \, |\tfrac{P_\eta(d^s)-P_\eta(d)}{s}| \, |f_\alpha \, \tfrac{u^s_\alpha-u_\alpha}{s} + f_3 \, \tfrac{u^s_3-u_3}{s}| \, d\Omega = \\ \quad\quad \int_\Omega \gamma \, |\tfrac{P_\eta(d^s)-P_\eta(d)}{d^s-d}| \, |\tfrac{d^s-d}{s}| \, |f_\alpha \, \tfrac{u^s_\alpha-u_\alpha}{s} + f_3 \, \tfrac{u^s_3-u_3}{s}| \, d\Omega \leq \\ \quad\quad\quad c_0 \, \|\tfrac{d^s-d}{s}(.,t)\|_{L^2(\Omega))} \, \|\tfrac{u^s-u}{s}(.,t)\|_{V(\Omega)\backslash\mathcal{R})} \end{array}\right. \tag{86}$$

where c_0 is a strictly positive constant independent of s and t. Finally, using the mean value theorem for the material coefficient $b_{3333}(.)$, the estimate (72), for $s = 0$, and condition (13) we get

$$\left[\begin{array}{l} \Big| \int_\Omega \tfrac{b_{3333}(d)-b_{3333}(d^s)}{s} \big(b_{3333}(d^s)\, b_{3333}(d)\big)^{-1} e_{33}(u) e_{33}(\tfrac{u^s-u}{s}) \, d\Omega \Big| = \\ \Big| \int_\Omega \tfrac{b_{3333}(d)-b_{3333}(d^s)}{d^s-d} \tfrac{d^s-d}{s} \big(b_{3333}(d^s)\, b_{3333}(d)\big)^{-1} e_{33}(u) e_{33}(\tfrac{u^s-u}{s}) \, d\Omega \Big| \leq \\ \quad\quad c_b \, \|\tfrac{d^s-d}{s}(.,t)\|_{L^2(\Omega))} \, \|\tfrac{u^s-u}{s}(.,t)\|_{V(\Omega)\backslash\mathcal{R}} \end{array}\right. \tag{87}$$

where c_b is a strictly positive constant independent of s and t. Therefore using (81) and the estimates (82)-(87), we have, for each t

$$\left\{\begin{array}{l} c \, \|\tfrac{u^s-u}{s}(.,t)\|^2_{V(\Omega)\backslash\mathcal{R}} \leq \\ c_1 \, \|\tfrac{d^s-d}{s}(.,t)\|_{L^2(\Omega))} \, \|\tfrac{u^s-u}{s}(.,t)\|_{V(\Omega)\backslash\mathcal{R})} + c_2 \, \|\tfrac{u^s-u}{s}(.,t)\|_{V(\Omega)\backslash\mathcal{R})} \end{array}\right. \tag{88}$$

where c, c_1 and c_2 are strictly positive constants independent of s and t. The proof is finished, dividing (88) by $c \, \|\frac{u^s-u}{s}(.,t)\|_{V(\Omega)\backslash\mathcal{R}}$. ☐

Theorem 4.4 (Estimate for the sequence $\frac{d_s-d}{s}$)**.** *Let (u^s, d^s) and (u, d) be the solutions of problems (59) and (57), respectively. We assume that the* *hypotheses of the previous theorem 4.3 are satisfied. Then,*

$$\|\frac{d^s - d}{s}\|_{C^0([0,T];L^2(\Omega))} \leq c \tag{89}$$

where c is a strictly positive constant independent of s and t.

Proof: We write the remodeling rate equations of problems P^s (cf. (59)) and P^0 (cf. (57))

$$\begin{cases} \dot{d}^s = c(d^s)e_{33}(u^s) + a(d^s) - s\, c(d^s)\theta_\alpha \partial_{33} u^s_\alpha, \quad \text{in} \quad \overline{\Omega} \times (0,T), \\ d^s(x,0) = \overline{d}(x), \quad \text{in} \quad \overline{\Omega}, \end{cases} \tag{90}$$

$$\begin{cases} \dot{d} = c(d)e_{33}(u) + a(d), \quad \text{in} \quad \Omega \times (0,T), \\ d(x,0) = \overline{d}(x), \quad \text{in} \quad \overline{\Omega}. \end{cases} \tag{91}$$

Subtracting (91) from (90) we obtain

$$\begin{cases} \dot{d}^s - \dot{d} = c(d^s)e_{33}(u^s - u) \\ \qquad +[c(d^s) - c(d)]e_{33}(u) + a(d^s) - a(d) - s\, c(d^s)\theta_\alpha \partial_{33} u^s_\alpha \\ (d^s - d)(x,0) = 0. \end{cases} \tag{92}$$

Integrating in time between zero and t

$$\begin{cases} (d^s - d)(x,t) = \int_0^t \Big[c(d^s)e_{33}(u^s - u) + [c(d^s) - c(d)]e_{33}(u) \\ \qquad +a(d^s) - a(d) - s\, c(d^s)\theta_\alpha \partial_{33} u^s_\alpha \Big]\, dr. \end{cases} \tag{93}$$

Taking the $L^2(\Omega)$ norm in (93), using the conditions (13)-(15) and the mean value theorem for the terms $c(d^s) - c(d)$ and $a(d^s) - a(d)$, we obtain, for each t, the estimate

$$\begin{cases} \|(d^s - d)(.,t)\|_{L^2(\Omega)} \leq \int_0^t \Big[c_1 \|e_{33}(u^s - u)(.,r)\|_{L^2(\Omega)} \\ \qquad +c_2 \|(d^s - d)(.,r)\|_{L^2(\Omega)} \|e_{33}(u)(.,r)\|_{C^0(\overline{\Omega})} \\ +c_3 \|(d^s - d)(.,r)\|_{L^2(\Omega)} + s\, c_4 \|\partial_{33} u^s_\alpha(.,r)\|_{L^2(\Omega)} \Big]\, dr. \end{cases} \tag{94}$$

where c_1, c_2, c_3 and c_4 are strictly positive constants independent of s and t. But

$$\begin{aligned} &\|e_{33}(u^s-u)(.,t)\|_{L^2(\Omega)} = \|(u^s-u)(.,t)\|_{V(\Omega)\setminus\mathcal{R}} \\ &\|e_{33}(u)(.,t)\|_{C^0(\overline{\Omega})} \le c_2 \\ &\|\partial_{33}u^s_\alpha(.,t)\|_{L^2(\Omega)} \le c_0\|e_{33}u^s(.,r)\|_{L^2(\Omega)} = c_0\|u^s(.,t)\|_{V(\Omega)\setminus\mathcal{R}} \le c \end{aligned} \tag{95}$$

where c_2 is the constant defined in (72), for the case $s=0$, c_0 is defined in (65) and c is a constant depending on the constant defined in (60); these three constants are independent of s and t. So we have from (94)-(95) and theorem 4.3

$$\begin{cases} \|\frac{d^s-d}{s}(.,t)\|_{L^2(\Omega)} \le \\ \int_0^t \left[c_1\,\|\frac{u^s-u}{s}(.,r)\|_{V(\Omega)\setminus\mathcal{R}} + c_5\,\|\frac{d^s-d}{s}(.,r)\|_{L^2(\Omega)} + c_6\right] dr \le \\ c_7 + \int_0^t c_8\,\|\frac{d^s-d}{s}(.,r)\|_{L^2(\Omega)}\,dr, \end{cases} \tag{96}$$

with c_5, c_6, c_7 and c_8 other strictly positive constants independent of s and t. Then using the integral Gronwall's inequality (cf. Evans [7], p. 625)

$$\|\frac{d^s-d}{s}(.,t)\|_{L^2(\Omega)} \le c_7\,(1+tc_8e^{c_8t}), \quad \forall t\in[0,T], \tag{97}$$

which implies (89). $\square$

Corollary 4.1. *With the hypotheses of theorem 4.3*

$$\exists c>0: \qquad \|\frac{u^s-u}{s}\|_{C^0([0,T];V(\Omega)\setminus\mathcal{R})} \le c, \tag{98}$$

where c is independent of s and t. $\square$

So we have reached the conclusion that the solutions (u^s,d^s) and (u,d) of problems P^s and P^0, respectively, verify for all $s\in[0,\delta]$

$$\|\frac{u^s-u}{s}\|_{C^0([0,T];V(\Omega)\setminus\mathcal{R})} \le c_1 \quad \text{and} \quad \|\frac{d^s-d}{s}\|_{C^0([0,T];L^2(\Omega))} \le c_2 \tag{99}$$

where c_1 and c_2 are strictly positive constants independent of s.

Hence, for each t, each one of the bounded quotient sequences $\{\frac{u^s-u}{s}\}(.,t)$ and $\{\frac{d^s-d}{s}\}(.,t)$ has a subsequence, that converges weakly, when $s\to 0^+$, in $V(\Omega)\setminus\mathcal{R}$ and $L^2(\Omega)$, respectively. This statement and a consequence of it is summarized in the next theorem.

Theorem 4.5 (Weak limits of the quotient sequences). *Let (u^s, d^s) and (u, d) be the solutions of problems P^s and P^0 and we assume that the hypotheses of theorem (4.3) are verified. Then, for each t, there exists a subsequence of $\{(u^s, d^s)(., t)\}$, also denoted by $\{(u^s, d^s)(., t)\}$, and elements $\bar{u}(., t) \in V(\Omega)\backslash\mathcal{R}$ and $\bar{d}(., t) \in L^2(\Omega)$, such that, when the parameter $s \to 0^+$*

$$\frac{u^s - u}{s}(., t) \rightharpoonup \bar{u}(., t) \quad \textit{weakly in} \quad V(\Omega) \setminus \mathcal{R}, \tag{100}$$

$$e_{33}(\frac{u^s - u}{s})(., t) \rightharpoonup e_{33}(\bar{u})(., t) \quad \textit{weakly in} \quad L^2(\Omega), \tag{101}$$

$$\frac{d^s - d}{s}(., t) \rightharpoonup \bar{d}(., t) \quad \textit{weakly in} \quad L^2(\Omega), \tag{102}$$

and therefore, when $s \to 0^+$

$$(u^s - u)(., t) \longrightarrow 0 \quad \textit{strongly in} \quad V(\Omega) \setminus \mathcal{R}, \tag{103}$$

$$e_{33}(u^s - u)(., t) \longrightarrow 0 \quad \textit{strongly in} \quad L^2(\Omega), \tag{104}$$

$$(d^s - d)(., t) \longrightarrow 0 \quad \textit{strongly in} \quad L^2(\Omega). \quad \square \tag{105}$$

We finish this section with a convergence result for the sequences $\{u^s\}$ and $e_{33}(u^s)$, and a corollary concerning the convergence of the sequence $\{d^s\}$, that will be useful in subsection (4.2).

Theorem 4.6 (Strong limit of $e_{33}(u^s)$). *Let (u^s, d^s) and (u, d) be the solutions of problems P^s and P^0 and we assume that the hypotheses of theorems 2.2, 4.2 and 4.3 are verified. Then, there exists a subsequence of $\{u^s\}$, also denoted by $\{u^s\}$, that verifies the following convergence, when the parameter $s \to 0^+$,*

$$e_{33}(u^s - u) \longrightarrow 0 \quad \textit{strongly in} \quad C^0([0, T]; C^0(\overline{\Omega})). \tag{106}$$

Proof: Recalling the definition of $u^s - u$ and its regularity (cf. theorem 2.2), we have

$$u^s - u = \Big(u_1^s - u_1, u_2^s - u_2, \underline{u}_3^s - \underline{u}_3 - x_\alpha \partial_3(u_\alpha^s - u_\alpha)\Big), \tag{107}$$

where for each t, $(u_\alpha^s - u_\alpha)(., t) \in W^{3,2}(0, L)$, for $\alpha = 1, 2$ and $(\underline{u}_3^s - \underline{u}_3)(., t) \in W^{2,2}(0, L)$. The calculus of $e_{33}(u^s - u)$ gives

$$e_{33}(u^s - u) = \partial_3(u_3^s - u_3) = \partial_3(\underline{u}_3^s - \underline{u}_3) - x_\alpha \partial_{33}(u_\alpha^s - u_\alpha), \tag{108}$$

where $[\partial_3(\underline{u}_3^s-\underline{u}_3)](.,t)\in W^{1,2}(0,L)$ and $[\partial_{33}(u_\alpha^s-u_\alpha)](.,t)\in W^{1,2}(0,L)$, for $\alpha=1,2$. But, because of the strong convergence (104),

$$\|e_{33}(u^s-u)(.,t)\|_{L^2(\Omega)} \longrightarrow 0 \quad \text{when} \quad s\to 0^+. \tag{109}$$

Applying the definition of $\|e_{33}(u^s-u)(.,t)\|_{L^2(\Omega)}$ (cf. (24)), that is

$$\left\{\begin{aligned} \|e_{33}(u^s-u)(.,t)\|^2_{L^2(\Omega)} &= \|\partial_3(\underline{u}_3^s-\underline{u}_3)(.,t)\|^2_{L^2(0,L)} \\ &+\big(\textstyle\int_\omega x_\alpha^2 d\omega\big)\|\partial_{33}(u_\alpha^s-u_\alpha)(.,t)\|^2_{L^2(0,L)}, \end{aligned}\right. \tag{110}$$

we conclude that, for each t

$$\partial_3(\underline{u}_3^s-\underline{u}_3)(.,t) \longrightarrow 0 \quad \text{strongly in} \quad L^2(0,L), \tag{111}$$

$$\partial_{33}(u_\alpha^s-u_\alpha)(.,t) \longrightarrow 0 \quad \text{strongly in} \quad L^2(0,L). \tag{112}$$

On the other hand we get directly from (71)

$$\left\{\begin{aligned} &\|e_{33}(u^s-u)\|_{C^0([0,T];W^{1,2}(\Omega))} \le \|u^s-u\|_{C^0([0,T];W^{2,2}(\Omega))} \le \\ &\qquad \|u^s\|_{C^0([0,T];W^{2,2}(\Omega))} + \|u\|_{C^0([0,T];W^{2,2}(\Omega))} \le c. \end{aligned}\right. \tag{113}$$

where c is a constant independent of s. Therefore the sequences $\partial_3(\underline{u}_3^s-\underline{u}_3)$ and $\partial_{33}(u_\alpha^s-u_\alpha)$ are bounded in $C^0([0,T];W^{1,2}(0,L))$, and consequently, because of (111)-(112), we obtain the following weak convergences, when $s\to 0^+$

$$\partial_3(\underline{u}_3^s-\underline{u}_3)(.,t) \rightharpoonup 0 \quad \text{weakly in} \quad W^{1,2}(0,L), \tag{114}$$

$$\partial_{33}(u_\alpha^s-u_\alpha)(.,t) \rightharpoonup 0 \quad \text{weakly in} \quad W^{1,2}(0,L). \tag{115}$$

But as the space $W^{1,2}(0,L)$ is compactly embedded in $C^0([0,L])$, we have, when $s\to 0^+$

$$\partial_3(\underline{u}_3^s-\underline{u}_3)(.,t) \longrightarrow 0 \quad \text{strongly in} \quad C^0([0,L]), \tag{116}$$

$$\partial_{33}(u_\alpha^s-u_\alpha)(.,t) \longrightarrow 0 \quad \text{strongly in} \quad C^0([0,L]), \tag{117}$$

which implies the strong convergence of $e_{33}(u^s-u)$ to 0 in $C^0([0,T];C^0(\overline{\Omega}))$. □

Corollary 4.2 (Strong limit of d^s). *Let (u^s,d^s) and (u,d) be the solutions of problems P^s and P^0, and we assume the same hypotheses of theorem 4.6. Then, there exists a constant $c>0$ independent of s, such that when $s\to 0^+$*

$$d^s-d \longrightarrow 0 \quad \textit{strongly in} \quad C^0([0,T];C^0(\overline{\Omega})). \tag{118}$$

Proof: Using exactly the same arguments in the beginning of theorem 4.4 (cf. formulas (93), (95) and (96)), we have

$$\left\{\begin{array}{l} |(d^s-d)(x,t)| \leq \\ \int_0^t \Big[c_1\,|e_{33}(u^s-u)(x,r)| + c_2\,|(d^s-d)(x,r)|\,|e_{33}(u)(x,r)| \\ +c_3\,|(d^s-d)(x,r)| + s\,c_4\,|\partial_{33}u^s_\alpha(x,r)|\Big]\,dr \leq \\ \int_0^t \Big[c_1\,|e_{33}(u^s-u)(x,r)| + c_5\,|(d^s-d)(x,r)| + s\,c_4\Big]\,dr \end{array}\right. \tag{119}$$

where the constants c_i, for $i = 1, 2, 3, 4, 5$, are strictly positive constants independent of s and t and consequently

$$\left\{\begin{array}{l} |(d^s-d)(x,t)| \leq \\ \underbrace{T\Big[c_1\,\|e_{33}(u^s-u)\|_{C^0([0,T];C^0(\overline{\Omega}))} + s\,c_4\Big]}_{\varphi^s} + \int_0^t c_5\,|(d^s-d)(x,r)|\,dr. \end{array}\right. \tag{120}$$

Because of the strong convergence (106), the scalar $\varphi^s \to 0$, when $s \to 0^+$. Then using the integral Gronwall's inequality (cf. Evans [7], p. 625)

$$|(d^s-d)(x,t)| \leq \varphi^s\,(1 + tc_5e^{c_5t}), \quad \forall t \in [0,T], \tag{121}$$

which implies (118). $\square$

4.2. Shape semiderivatives. The objective of this section is to identify, for each t, the weak limits $\bar{u}(.,t)$ and $\bar{d}(.,t)$ of the sequences $\{\frac{u^s-u}{s}(.,t)\}$ and $\{\frac{d^s-d}{s}(.,t)\}$, defined in (100) and (102). To this end we proceed in the following way: we consider again the problems P^s and P^0, we subtract and divide by s the equilibrium variational equations and the remodeling rate equations, and then we take the limit, when the parameter $s \to 0^+$. We conclude that, for each t, the pair $(\bar{u}(.,t), \bar{d}(.,t))$ is the solution of another nonlinear problem. Finally we identify $(\bar{u}, \bar{d})$ as the unique solution of this latter problem in the space $C^1([0,T]; V(\Omega)\setminus\mathcal{R}) \times C^1([0,T]; C^0(\overline{\Omega}))$.

4.2.1. *Shape semiderivative* $\bar{u}(.,t)$. For each $t \in [0,T]$, the two equilibrium variational equations, of problems P^s and P^0, are

$$\begin{array}{ll} a^s(u^s,v) = L^s(v), & \forall v \in V(\Omega)\setminus\mathcal{R}, \\ a_0(u,v) = L_0(v), & \forall v \in V(\Omega)\setminus\mathcal{R}, \end{array} \tag{122}$$

and subtracting and dividing by s we obtain (cf. (79)-(81))

$$\left[\begin{array}{l}\int_\Omega \frac{1}{b_{3333}(d^s)} e_{33}(\frac{u^s-u}{s}) e_{33}(v) d\Omega + \\ \int_\Omega \frac{b_{3333}(d)-b_{3333}(d^s)}{s} \big(b_{3333}(d^s)\, b_{3333}(d)\big)^{-1} e_{33}(u) e_{33}(v)\, d\Omega + \\ a_1^s(u^s,v) = \\ \frac{1}{s}\Big[F_0^s(v) - F_0(v) + G_0^s(v) - G_0(v) + H_0^s(v) - H_0(v)\Big] + \\ F_1^s(v) + G_1^s(v) + H_1^s(v) + \\ s\Big[-a_2^s(u^s,v) + F_2^s(v) + G_2^s(v) + H_2^s(v)\Big] + \\ s^2\Big[-a_3^s(v) + F_3^s(v) + G_3^s(v) + H_3^s(v)\Big] - s^3\, a_4^s(u^s,v).\end{array}\right. \tag{123}$$

We compute now, for each t, the limit of each term of (123). Accordingly to the definitions of the forms involved in the last terms of this equation, it is clear that, for each t, the sum

$$\left\{\begin{array}{l} s\Big[-a_2^s(u^s,v) + F_2^s(v) + G_2^s(v) + H_2^s(v)\Big] + \\ s^2\Big[-a_3^s(v) + F_3^s(v) + G_3^s(v) + H_3^s(v)\Big] - s^3\, a_4^s(u^s,v)\end{array}\right. \tag{124}$$

converges to 0 when $s \to 0^+$, because we have the product of bounded terms by s or by a positive power of s. The calculus of the other limits requires a more careful analysis.

Theorem 4.7. *We assume the hypotheses of theorems 2.2, 4.2 and 4.3. Then for each t, when $s \to 0^+$,*

$$\int_\Omega \frac{1}{b_{3333}(d^s)} e_{33}(\frac{u^s-u}{s}) e_{33}(v) d\Omega \;\longrightarrow\; \int_\Omega \frac{1}{b_{3333}(d)} e_{33}(\bar{u}) e_{33}(v) d\Omega. \tag{125}$$

Proof: We have

$$\left\{\begin{array}{l} |\int_\Omega e_{33}(\frac{u^s-u}{s})\, \frac{1}{b_{3333}(d^s)}\, e_{33}(v) d\Omega - \int_\Omega \frac{1}{b_{3333}(d)} e_{33}(\bar{u}) e_{33}(v) d\Omega| \le \\ |\int_\Omega \frac{1}{b_{3333}(d^s)} e_{33}(\frac{u^s-u}{s} - \bar{u}) e_{33}(v) d\Omega| + \\ \qquad |\int_\Omega \left(\frac{1}{b_{3333}(d^s)} - \frac{1}{b_{3333}(d)}\right) e_{33}(\bar{u})\, e_{33}(v) d\Omega|.\end{array}\right. \tag{126}$$

The sequence $e_{33}(\frac{u^s-u}{s})(.,t)$ weakly converges to $e_{33}(\bar{u})(.,t)$ in $L^2(\Omega)$ (cf. (101)), therefore, for each t

$$|\int_\Omega \frac{1}{b_{3333}(d^s)}\, e_{33}(\frac{u^s-u}{s}-\bar{u})\, e_{33}(v)\, d\Omega| \;\longrightarrow\; 0, \quad \text{when} \quad s\to 0^+. \tag{127}$$

Because of condition (13), the mean value theorem for the function $\frac{1}{b_{3333}(.)}$ and the strong convergence of d^s to d in $C^0([0,T];C^0(\overline{\Omega}))$ (cf. (118)) we have, for each t, when $s\to 0^+$

$$\frac{1}{b_{3333}(d^s)}\, e_{33}(v) \;\longrightarrow\; \frac{1}{b_{3333}(d)} e_{33}(v), \quad \text{strongly in} \quad L^2(\Omega). \tag{128}$$

In fact, for each t, when $s\to 0^+$

$$\begin{cases} \int_\Omega |\frac{1}{b_{3333}(d^s)}-\frac{1}{b_{3333}(d)}|^2\, |e_{33}(v)|^2\, d\Omega \le \\ \quad \sup_\Omega |\frac{1}{b_{3333}(d^s)}-\frac{1}{b_{3333}(d)}|^2 \int_\Omega |e_{33}(v)|^2\, d\Omega \le \\ \qquad c\,\|d^s-d\|_{C^0([0,T];C^0(\overline{\Omega}))} \int_\Omega |e_{33}(v)|^2\, d\Omega. \end{cases} \tag{129}$$

Using again the condition (13), the mean value theorem for the function $\frac{1}{b_{3333}(.)}$ and the strong convergence of d^s to d in $C^0([0,T];C^0(\overline{\Omega}))$ we have for the last term of (126), and for each t, when $s\to 0^+$

$$\begin{cases} \int_\Omega \left|\frac{1}{b_{3333}(d^s)}-\frac{1}{b_{3333}(d)}\right|\, |e_{33}(\bar{u})\, e_{33}(v)|\, d\Omega \le \\ \quad \sup_\Omega |\frac{1}{b_{3333}(d^s)}-\frac{1}{b_{3333}(d)}|\, \|e_{33}(\bar{u})\|_{L^2(\Omega)}\, \|e_{33}(v)\|_{L^2(\Omega)} \;\longrightarrow\; 0, \end{cases} \tag{130}$$

which terminates the proof. □

Theorem 4.8. *We assume the hypotheses of theorems 2.2, 4.2 and 4.3. Then for each t, when $s\to 0^+$*

$$\begin{array}{c} \int_\Omega \frac{b_{3333}(d)-b_{3333}(d^s)}{s}\, \big(b_{3333}(d^s)\, b_{3333}(d)\big)^{-1} e_{33}(u)\, e_{33}(v)\, d\Omega \\ \downarrow \\ -\int_\Omega b'_{3333}(d)\, b_{3333}(d)^{-2}\, \bar{d}\, e_{33}(u)\, e_{33}(v)\, d\Omega \end{array} \tag{131}$$

where $b'_{3333}(.)$ is the first derivative of the scalar function $b_{3333}(.)$.

Proof: We remark that

$$\begin{cases} \int_\Omega \underbrace{\frac{b_{3333}(d)-b_{3333}(d^s)}{d^s-d}\big(b_{3333}(d^s)\,b_{3333}(d)\big)^{-1}}_{\Psi^s}\,\frac{d^s-d}{s}\,e_{33}(u)\,e_{33}(v)\,d\Omega \\ -\int_\Omega \underbrace{b'_{3333}(d)\,b_{3333}(d)^{-2}}_{\Psi}\,\bar d\,e_{33}(u)\,e_{33}(v)\,d\Omega = \\ \underbrace{\int_\Omega (\Psi^s-\Psi)\,\frac{d^s-d}{s}\,e_{33}(u)\,e_{33}(v)\,d\Omega}_{Term\,1} + \\ \qquad \underbrace{\int_\Omega \Psi\,\Big(\frac{d^s-d}{s}-\bar d\Big)\,e_{33}(u)\,e_{33}(v)\,d\Omega}_{Term\,2} \end{cases} \tag{132}$$

For each t, the sequence $(\Psi^s-\Psi)(.,t)$ converges strongly to 0 in $C^0(\overline{\Omega})$, when $s\to 0^+$, because of condition (13), the mean value theorem for the function $b_{3333}(.)$, the condition (72) for the case $s=0$, and the strong convergence of d_s to d in $C^0([0,T];C^0(\overline{\Omega}))$ (cf. corollary 4.2, formula (118)). Thus, for each t, the term 1 in equation (132) verifies, when $s\to 0^+$

$$\begin{cases} \Big|\int_\Omega (\Psi^s-\Psi)\,\frac{d^s-d}{s}\,e_{33}(u)\,e_{33}(v)\,d\Omega\Big| \leq \\ \|(\Psi^s-\Psi)\,e_{33}(u)\|_{C^0(\overline{\Omega})}\int_\Omega \big|\frac{d^s-d}{s}\,e_{33}(v)\big|\,d\Omega \leq \\ \|(\Psi^s-\Psi)\,e_{33}(u)\|_{C^0(\overline{\Omega})}\|\frac{d^s-d}{s}\|_{C^0([0,T];L^2(\Omega))}\,\|e_{33}(v)\|_{L^2(\Omega)} \;\longrightarrow 0. \end{cases} \tag{133}$$

The term 2 also converges to 0, when $s\to 0^+$, because $\Psi e_{33}(u)e_{33}(v)$ belongs to $L^2(\Omega)$, for each t, and $\frac{d^s-d}{s}(.,t)$ weakly converges to $\bar d(.,t)$ in $L^2(\Omega)$, for each t (cf. theorem 4.5, formula (102)). $\square$

Theorem 4.9. *We assume the hypotheses of theorems 2.2, 4.2 and 4.3. Then for each t, when $s\to 0^+$*

$$\begin{array}{c} \int_\Omega \frac{1}{b_{3333}(d^s)}\Big[e_{33}(u^s)e_{33}(v)div\,\theta-\theta_\alpha\big(e_{33}(u^s)\partial_{33}v_\alpha+e_{33}(v)\partial_{33}u^s_\alpha\big)\Big]d\Omega \\ \downarrow \\ \int_\Omega \frac{1}{b_{3333}(d)}\Big[e_{33}(u)e_{33}(v)div\,\theta-\theta_\alpha\big(e_{33}(u)\partial_{33}v_\alpha+e_{33}(v)\partial_{33}u_\alpha\big)\Big]d\Omega. \end{array} \tag{134}$$

Proof: It is enough to prove that, for each t, when $s \to 0^+$

$$\begin{aligned}
&\int_\Omega \frac{1}{b_{3333}(d^s)}\theta_\alpha e_{33}(u^s)\,\partial_{33}v_\alpha\,d\Omega \;\longrightarrow\; \int_\Omega \frac{1}{b_{3333}(d)}\theta_\alpha e_{33}(u)\,\partial_{33}v_\alpha\,d\Omega\\
&\int_\Omega \frac{1}{b_{3333}(d^s)}\theta_\alpha e_{33}(v)\,\partial_{33}u^s_\alpha\,d\Omega \;\longrightarrow\; \int_\Omega \frac{1}{b_{3333}(d)}\theta_\alpha e_{33}(v)\partial_{33}u_\alpha\,d\Omega\\
&\int_\Omega \frac{1}{b_{3333}(d^s)}\,e_{33}(u^s)e_{33}(v)div\,\theta\,d\Omega \;\longrightarrow\; \int_\Omega \frac{1}{b_{3333}(d)}\,e_{33}(u)e_{33}(v)div\,\theta\,d\Omega,
\end{aligned} \tag{135}$$

We only give a sketch of the proof of the first convergence indicated in (135), because the arguments for proving the remaining two are similar.

We have the following estimate, for each t

$$\begin{cases}
\left|\int_\Omega \frac{1}{b_{3333}(d^s)}\theta_\alpha e_{33}(u^s)\,\partial_{33}v_\alpha\,d\Omega - \int_\Omega \frac{1}{b_{3333}(d)}\,\theta_\alpha e_{33}(u)\,\partial_{33}v_\alpha\,d\Omega\right| \leq\\
\int_\Omega \left|\frac{1}{b_{3333}(d^s)}\,e_{33}(u^s) - \frac{1}{b_{3333}(d)}\,e_{33}(u)\right| |\theta_\alpha\,\partial_{33}v_\alpha|\,d\Omega \leq\\
\int_\Omega \Big[\big|\frac{1}{b_{3333}(d^s)}\,\big(e_{33}(u^s) - e_{33}(u)\big)\big|\\
\qquad + \big|\big(\frac{1}{b_{3333}(d^s)} - \frac{1}{b_{3333}(d)}\big)\,e_{33}(u)\big|\Big]\,|\theta_\alpha\,\partial_{33}v_\alpha|\,d\Omega.
\end{cases} \tag{136}$$

But, for each t, $\|\big(e_{33}(u^s) - e_{33}(u)\big)(.,t)\|_{L^2(\Omega)}$ converges strongly to 0, when $s \to 0^+$ (cf. theorem 4.5, formula (103)), so

$$\begin{cases}
\|\frac{1}{b_{3333}(d^s)}\,\big(e_{33}(u^s) - e_{33}(u)\big)\|_{L^2(\Omega)}\|\theta_\alpha\,\partial_{33}v_\alpha\|_{L^2(\Omega)} \leq\\
c_1\,\|e_{33}(u^s) - e_{33}(u)\|_{L^2(\Omega)}\,\|\theta_\alpha\,\partial_{33}v_\alpha\|_{L^2(\Omega)} \;\longrightarrow\; 0.
\end{cases} \tag{137}$$

Also, for each t, $\|(d^s - d)(.,t)\|_{L^2(\Omega)}$ converges strongly to 0, when $s \to 0^+$ (cf. theorem 4.5, formula (105)), hence, when $s \to 0^+$

$$\begin{cases}
\|\big(\frac{1}{b_{3333}(d^s)} - \frac{1}{b_{3333}(d)}\big)\,e_{33}(u)\|_{L^2(\Omega)}\,\|\theta_\alpha\,\partial_{33}v_\alpha\|_{L^2(\Omega)} \leq\\
c_2\|d^s - d\|_{L^2(\Omega)}\,\|e_{33}(u)\|_{C^0([0,T];C^0(\overline{\Omega}))}\,\|\theta_\alpha\,\partial_{33}v_\alpha\|_{L^2(\Omega)} \;\longrightarrow 0. \quad \square
\end{cases} \tag{138}$$

Theorem 4.10. *We assume the hypotheses of theorems 2.2, 4.2 and 4.3. Then, for each t, when $s \to 0^+$*

$$\begin{gathered}
\frac{1}{s}\Big[F_0^s(v) - F_0(v) + G_0^s(v) - G_0(v) + H_0^s(v) - H_0(v)\Big]\\
\downarrow\\
\int_\Omega \gamma\,\bar{d}\,P'_\eta(d)\,(f_\alpha\,v_\alpha + f_3\,\underline{v}_3)\,d\Omega,
\end{gathered} \tag{139}$$

where $P'_\eta(.)$ is the first derivative of the scalar function $P_\eta(.)$.

Proof: As a consequence of the hypotheses we have that $G_0^s(v) = G_0(v)$ and $H_0^s(v) = H_0(v)$. We analyse now the difference

$$\left\{\begin{array}{l} \frac{1}{s}\Big[F_0^s(v) - F_0(v)\Big] - \int_\Omega \gamma\, \bar{d}\, P'_\eta(d)\,(f_\alpha\, v_\alpha + f_3\, \underline{v}_3)\, d\Omega = \\ \int_\Omega \gamma \Big(\frac{P_\eta(d^s)-P_\eta(d)}{s} - \bar{d} P'_\eta(d)\Big)\,(f_\alpha\, v_\alpha + f_3\, \underline{v}_3)\, d\Omega = \\ \int_\Omega \gamma \,\Big[\, \frac{P_\eta(d^s)-P_\eta(d)}{d^s-d}\, \big(\frac{d^s-d}{s} - \bar{d}\big) \\ \qquad + \big(\frac{P_\eta(d^s)-P_\eta(d)}{d^s-d} - P'_\eta(d)\big)\, \bar{d}\,\Big] (f_\alpha\, v_\alpha + f_3\, \underline{v}_3)\, d\Omega. \end{array}\right. \tag{140}$$

But, for each t when $s \to 0^+$, the term

$$\int_\Omega \gamma\, \frac{P_\eta(d^s) - P_\eta(d)}{d^s - d}\, \big(\frac{d^s - d}{s} - \bar{d}\big)(f_\alpha\, v_\alpha + f_3\, \underline{v}_3)\, d\Omega \;\longrightarrow\; 0, \tag{141}$$

because $\gamma\, \frac{P_\eta(d^s)-P_\eta(d)}{d^s-d}\,(f_\alpha\, v_\alpha + f_3\, \underline{v}_3)$ converges strongly to $\gamma\, P'_\eta(d)\,(f_\alpha\, v_\alpha + f_3\, \underline{v}_3)$ in $L^2(\Omega)$ and $\frac{d^s-d}{s}$ converges weakly to $\bar{d}$ in $L^2(\Omega)$, for each t (cf. theorem 4.5, formula (102)); and the remaining last term in (140) verifies, when $s \to 0^+$

$$\int_\Omega \gamma\, \big(\frac{P_\eta(d^s) - P_\eta(d)}{d^s - d} - P'_\eta(d)\big)\, \bar{d}\,(f_\alpha\, v_\alpha + f_3\, \underline{v}_3)\, d\Omega \;\longrightarrow 0, \tag{142}$$

since for each t, $\bar{d}\,(f_\alpha\, v_\alpha + f_3\, \underline{u}_3)$ belongs to $L^1(\Omega)$ and $\gamma\, \big(\frac{P_\eta(d^s)-P_\eta(d)}{d^s-d} - P'_\eta(d)\big)$ converges strongly to 0 in $C^0([0,T]; C^0(\overline{\Omega}))$. From (141)-(142) the conclusion follows. □

Theorem 4.11. *We assume the hypotheses of theorems 2.2, 4.2 and 4.3. Then for each t, when $s \to 0^+$*

$$F_1^s(v) + G_1^s(v) + H_1^s(v)$$
$$\downarrow$$
$$\left\{\begin{array}{l} \int_\Omega \gamma(\xi_0 + P_\eta(d))\, \Big[(f_\alpha\, v_\alpha + f_3\, \underline{v}_3)\, div\, \theta - f_3\, \theta_\alpha\, \partial_3 v_\alpha\Big]\, d\Omega + \\ \int_\Gamma \Big[(g_\alpha\, v_\alpha + g_3\, \underline{v}_3)\, G_1(\theta, n) - g_3\, \theta_\alpha\, \partial_3 v_\alpha\Big]\, d\Gamma + \\ \int_{\Gamma_0 \cup \Gamma_L} \Big[(h_\alpha\, v_\alpha + h_3\, \underline{v}_3)\, H_1(\theta) - h_3\, \theta_\alpha\, \partial_3 v_\alpha\Big]\, d\Gamma_0 \cup \Gamma_L. \end{array}\right. \tag{143}$$

Proof: To prove this theorem we apply the definitions of the forms F_1^s, G_1^s, H_1^s and the strong convergence of $P_\eta(d^s)$ to $P_\eta(d)$ in the space $C^0([0,T]; C^0(\overline{\Omega}))$. □

Assembling the information of theorems 4.7- 4.11 and the equation (123) we concluded the following.

Theorem 4.12 (identification of $\bar{u}(.,t)$)**.** *The weak limit* $\bar{u}(.,t)$ *of the sequence* $\{\frac{u^s-u}{s}\}(.,t)$ *verifies the following variational equation for each* $t \in [0,T]$

$$B(\bar{u}, v) = S(v), \qquad \forall v \in V(\Omega) \setminus \mathcal{R}, \tag{144}$$

where $B(.,.)$ *is a bilinear form defined by*

$$B(z,v) = \int_\Omega \frac{1}{b_{3333}(d)}\, e_{33}(z)\, e_{33}(v)\, d\Omega, \qquad \forall z,\ v \in V(\Omega) \setminus \mathcal{R}, \tag{145}$$

and $S(.)$ *is a linear form defined for all* $v \in V(\Omega) \setminus \mathcal{R}$ *by*

$$\left\{\begin{array}{l} S(v) = \int_\Omega b'_{3333}(d)\, b_{3333}(d)^{-2}\, \bar{d}\, e_{33}(u)\, e_{33}(v)\, d\Omega \\ - \int_\Omega \frac{1}{b_{3333}(d)} \Big[- \theta_\alpha \big(e_{33}(u) \partial_{33} v_\alpha + e_{33}(v) \partial_{33} u_\alpha\big) \\ \qquad\qquad\qquad + e_{33}(u) e_{33}(v) div\, \theta \Big] d\Omega \\ + \int_\Omega \gamma\, \bar{d}\, P'_\eta(d)\, (f_\alpha\, v_\alpha + f_3\, \underline{v}_3)\, d\Omega \\ + \int_\Omega \gamma (\xi_0 + P_\eta(d)) \Big[(f_\alpha\, v_\alpha + f_3\, \underline{v}_3)\, div\, \theta - f_3\, \theta_\alpha\, \partial_3 v_\alpha \Big]\, d\Omega \\ + \int_\Gamma \Big[(g_\alpha\, v_\alpha + g_3\, \underline{v}_3)\, G_1(\theta, n) - g_3\, \theta_\alpha\, \partial_3 v_\alpha \Big]\, d\Gamma \\ + \int_{\Gamma_0 \cup \Gamma_L} \Big[(h_\alpha\, v_\alpha + h_3\, \underline{v}_3)\, H_1(\theta) - h_3\, \theta_\alpha\, \partial_3 v_\alpha \Big]\, d\Gamma_0 \cup \Gamma_L. \end{array}\right. \tag{146}$$

4.2.2. *Shape semiderivative* $\bar{d}(.,t)$. By subtracting and dividing by s the remodeling rate equations of problems P^s and P^0 and integrating in time between zero and t, we obtain

$$\left\{\begin{array}{l} \frac{(d^s-d)}{s}(.,t) = \int_0^t \Big[c(d^s)\, e_{33}(\frac{u^s-u}{s}) \\ \qquad + \frac{c(d^s)-c(d)}{s}\, e_{33}(u) + \frac{a(d^s)-a(d)}{s} - c(d^s) \theta_\alpha \partial_{33} u^s_\alpha \Big]\, dr. \end{array}\right. \tag{147}$$

Working with this equation we are able to prove the following theorem.

Theorem 4.13 (identification of $\bar{d}(.,t)$)**.** *We assume that the hypotheses of theorems 2.2, 4.2 and 4.3 are verified. For each* t*, the weak limit* $\bar{d}(.,t)$ *of the sequence* $\{\frac{d^s-d}{s}(.,t)\}$ *is the solution of the following ordinary differential* *equation with respect to time*

$$\left\{\begin{array}{l} \bar{d}^s = c(d)e_{33}(\bar{u}) + \bar{d}\,[c'(d)\,e_{33}(u) + a'(d)] \\ \qquad\qquad\qquad -c(d)\,\theta_\alpha \partial_{33} u_\alpha, \quad in \quad \Omega \times (0,T), \\ \bar{d}(x,0) = 0, \quad in \quad \overline{\Omega}, \end{array}\right. \tag{148}$$

where (u,d) *is the solution of problem* P^0 *and* $\bar{u}(.,t)$ *is the weak limit of the sequence* $\{\frac{u^s-u}{s}(.,t)\}$ *already identified in theorem 4.12.*

Proof: Using the equation (147) and considering any $v \in L^2(\Omega)$ we have, for each t,

$$\left\{\begin{array}{l} \int_\Omega \frac{d^s-d}{s}\, v\, d\Omega = \int_0^t \Big(\int_\Omega \Big[c(d^s)\, e_{33}(\frac{u^s-u}{s}) \\ \qquad + \frac{c(d^s)-c(d)}{s}\, e_{33}(u) + \frac{a(d^s)-a(d)}{s} - c(d^s)\theta_\alpha \partial_{33} u^s_\alpha \Big]\, v\, d\Omega \Big)\, dr. \end{array}\right. \tag{149}$$

On the other hand, and for each t, we have the following convergences, when $s \to 0^+$

$$\left\{\begin{array}{l} c(d^s) \;\longrightarrow\; c(d) \quad \text{strongly in} \quad C^0(\overline{\Omega}), \\ e_{33}(\frac{u^s-u}{s}) \;\rightharpoonup\; e_{33}(\bar{u}) \quad \text{weakly in} \quad L^2(\Omega), \\ \frac{c(d^s)-c(d)}{s}\, e_{33}(u) \;\rightharpoonup\; \bar{d}\, c'(d)\, e_{33}(u) \quad \text{weakly in} \quad L^2(\Omega), \\ \frac{a(d^s)-a(d)}{s} \;\rightharpoonup\; \bar{d}\, a'(d) \quad \text{weakly in} \quad L^2(\Omega), \\ \partial_{33} u^s_\alpha \;\longrightarrow\; \partial_{33} u_\alpha \quad \text{strongly in} \quad L^2(\Omega). \end{array}\right. \tag{150}$$

Hence we conclude that, for each t,

$$\left\{\begin{array}{l} \lim_{s\to 0^+} \int_\Omega \frac{d^s-d}{s}\, v\, d\Omega = \int_\Omega \Big(\int_0^t \Big[c(d) e_{33}(\bar{u}) \\ \qquad + \bar{d}\, c'(d)\, e_{33}(u) + \bar{d}\, a'(d) - c(d)\, \theta_\alpha \partial_{33} u_\alpha \Big]\, dr \Big)\, v\, d\Omega. \end{array}\right. \tag{151}$$

But, for each t, and by (102), $\frac{d^s-d}{s}(.,t)$ converges weakly to $\bar{d}(.,t)$ in $L^2(\Omega)$, when $s \to 0^+$. Therefore $\bar{d}(.,t)$ must verify (148), since the weak limit is unique. □

4.2.3. *Final identification result.* Collecting the results of theorems 4.12 and 4.13 we have the following theorem, that identifies, for each t, the (weak) shape semiderivatives $\bar{u}(.,t)$ and $\bar{d}(.,t)$.

Theorem 4.14. *We assume that the hypotheses of theorems 2.2, 4.2 and 4.3 are verified. For each $t \in [0,T]$, the weak shape semiderivative $(\bar{u}, \bar{d})(.,t)$ is an element of the space $V(\Omega) \setminus \mathcal{R} \times L^2(\Omega)$ and is the solution of the following nonlinear problem ($\bar{P}$)*

$$\bar{P} \left[\begin{array}{l} \textit{Find } \bar{u} : \overline{\Omega} \times [0,T] \to \mathbb{R}^3, \quad \bar{d} : \overline{\Omega} \times [0,T] \to \mathbb{R} \quad \textit{such that}: \\ \bar{u}(.,t) \in V(\Omega) \setminus \mathcal{R}, \\ B(\bar{u}, v) = S(v), \qquad \forall v \in V(\Omega) \setminus \mathcal{R}, \\ \dot{\bar{d}}^s = c(d) e_{33}(\bar{u}) + \bar{d}\,[c'(d)\, e_{33}(u) + a'(d)] - c(d)\, \theta_\alpha \partial_{33} u_\alpha, \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{in} \quad \Omega \times (0,T), \\ \bar{d}(x,0) = 0, \quad \textit{in} \quad \overline{\Omega}. \end{array} \right. \tag{152}$$

The bilinear form $B(.,.)$ is defined by

$$B(z,v) = \int_\Omega \frac{1}{b_{3333}(d)}\, e_{33}(z)\, e_{33}(v)\, d\Omega, \qquad \forall z,\ v \in V(\Omega) \setminus \mathcal{R}, \tag{153}$$

and $S(.)$ is a linear form defined for all $v \in V(\Omega) \setminus \mathcal{R}$, by

$$\left\{ \begin{array}{l} S(v) = \int_\Omega b'_{3333}(d)\, b_{3333}(d)^{-2}\, \bar{d}\, e_{33}(u)\, e_{33}(v)\, d\Omega \\ - \int_\Omega \frac{1}{b_{3333}(d)} \Big[- \theta_\alpha \big(e_{33}(u) \partial_{33} v_\alpha + e_{33}(v) \partial_{33} u_\alpha \big) \\ \qquad\qquad\qquad\qquad + e_{33}(u) e_{33}(v) div\, \theta \Big] d\Omega \\ + \int_\Omega \gamma\, \bar{d}\, P'_\eta(d)\, (f_\alpha\, v_\alpha + f_3\, \underline{v}_3)\, d\Omega \\ + \int_\Omega \gamma (\xi_0 + P_\eta(d)) \Big[(f_\alpha\, v_\alpha + f_3\, \underline{v}_3)\, div\, \theta - f_3\, \theta_\alpha\, \partial_3 v_\alpha \Big] d\Omega \\ + \int_\Gamma \Big[(g_\alpha\, v_\alpha + g_3\, \underline{v}_3)\, G_1(\theta, n) - g_3\, \theta_\alpha\, \partial_3 v_\alpha \Big] d\Gamma \\ + \int_{\Gamma_0 \cup \Gamma_L} \Big[(h_\alpha\, v_\alpha + h_3\, \underline{v}_3)\, H_1(\theta) - h_3\, \theta_\alpha\, \partial_3 v_\alpha \Big] d\Gamma_0 \cup \Gamma_L, \end{array} \right. \tag{154}$$

We remark $S(.)$ depends on (u,d), which is the solution of problem P^0 and also on $\bar{d}$, which is the weak shape semiderivative of d^s, at $s = 0$ and in the direction of the vector field θ. The bilinear form $B(.,.)$ depends on d, that is the measure of change in volume fraction of the elastic material of problem P^0. □

Moreover we have the following existence and uniqueness theorem for problem $\bar{P}$.

Theorem 4.15. *There exists a unique solution* $(\bar{u}, \bar{d})$ *of problem* $\bar{P}$*, defined in (152), which verifies*

$$\bar{u} \in C^1([0,T]; V(\Omega) \setminus \mathcal{R}) \qquad \bar{d} \in C^1([0,T]; C^0(\overline{\Omega})). \tag{155}$$

Proof: The arguments are analogous to those of the proof of existence and uniqueness of solution of problem P_s (cf. Figueiredo and Trabucho [8]), and rely on the Schauder's fixed point theorem. □

As a result of the previous theorem 4.15 we have the following final identification result.

Corollary 4.3. *We assume that the hypotheses of theorems 2.2, 4.2 and 4.3 are verified. For each* $t \in [0,T]$*, the entire sequence* $\{(\frac{u^s-u}{s}, \frac{d^s-d}{s})(.,t)\}$ *weakly converges to* $(\bar{u}, \bar{d})(.,t)$ *in the space* $V(\Omega) \setminus \mathcal{R} \times L^2(\Omega)$*, when* $s \to 0^+$*. Thus, the shape map* J *defined in (39) by*

$$\begin{aligned} J: \ [0,\delta] &\longrightarrow C^1([0,T]; V(\Omega) \setminus \mathcal{R}) \times C^1([0,T]; C^0(\overline{\Omega})) \\ s &\longrightarrow J(\Omega_s) = (u^s, d^s) \end{aligned} \tag{156}$$

has a weak shape semiderivative $dJ(\Omega;\theta)$*, at* $s = 0$*, in the direction of the vector field* θ *(cf. (41)), that is perfectly defined, for each* t*, by*

$$dJ(\Omega;\theta)(.,t) = (\bar{u}, \bar{d})(.,t), \tag{157}$$

where $(\bar{u}, \bar{d}) \in C^1([0,T]; V(\Omega) \setminus \mathcal{R}) \times C^1([0,T]; C^0(\overline{\Omega}))$ *is the unique solution of problem* $\bar{P}$*.* □

5. Conclusion and future work

In this paper we considered the family $\overline{\Omega}_s$, of perturbed thin rods, for $s \in [0,\delta]$, and the corresponding family of solutions (u^s, d^s) of the nonlinear asymptotic adaptive elastic model, derived in Figueiredo and Trabucho [8]. We proved that, for each t, the sequence $(\frac{u^s-u}{s}, \frac{d^s-d}{s})(.,t)$ converges weakly to $(\bar{u}, \bar{d})(.,t)$ in the space $V(\Omega) \setminus \mathcal{R} \times L^2(\Omega)$, when $s \to 0^+$. Consequently, for each t, $(\bar{u}, \bar{d})(.,t)$ is the weak shape semiderivative of the function $J(\Omega_s) = (u^s, d^s)$, at $s = 0$ in the direction of the vector field θ. Moreover, we prove that the pair $(\bar{u}, \bar{d})$ is the unique solution of another nonlinear problem that couples a variational equation, depending on (u,d) and $\bar{d}$, and an ordinary differential equation with respect to time, depending on (u,d) and $\bar{u}$. We intend to apply this methodology to analyse the weak shape semiderivative of the solution to the nonlinear adaptive elastic asymptotic model (5)-(8), but for the case where the remodeling rate equation (8) depends nonlinearly on $e_{33}(u_s)$ (cf. Figueiredo and Trabucho [8]). We think that this nonlinear term may originate some difficulties in the proof that the sequence $\{\frac{d^s-d}{s}\}$ is bounded, independently of s, and subsequently in the identification of the shape semiderivative.

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