Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 04–29

THE ALGEBRAIC PROBLEM OF THE EXACT POLE ASSIGNMENT BY STATIC OUTPUT FEEDBACK IN LINEAR SYSTEMS

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ABSTRACT: Let A, B and C be matrices of sizes $n \times n$, $n \times m$ and $p \times n$ respectively, with entries from an arbitrary field $\mathbb{K}$. Let $\mathbb{L}$ be a proper finite normal and separable extension field of $\mathbb{K}$.

In this paper we present a new method based on results of Moore [25] and Fletcher [11] giving sufficient conditions (in the case where the Galois group $\mathcal{G}_{\mathbb{L}/\mathbb{K}}$of $\mathbb{L}$ over $\mathbb{K}$ is cyclic) in order to guarantee an affirmative answer to the pole assignment by output feedback problem (PAOFP):

If (A, B) is controllable and (C, A) is observable, does there exist a matrix $K \in \mathbb{K}^{m\times p}$ such that

$$\det(A - BKC - \lambda I) = \prod_{i=1}^{n}(\lambda_i - \lambda) \tag{1}$$

for every n distinct elements $\lambda_1, \dots, \lambda_n$ of $\mathbb{L}$ that are the roots of some polynomial with coefficients from $\mathbb{K}$?

KEYWORDS: linear systems, static output control feedback, pole assignment, arbitrary field.
AMS SUBJECT CLASSIFICATION (1991): 11G25, 11G35, 12F10, 93B52, 93B55.

1. Introduction

In this work we shall consider a linear time-invariant multivariable system

$$x(k+1) = Ax(k) + Bu(k) \tag{2}$$
$$y(k) = Cx(k) \tag{3}$$

in which A, B and C are matrices with entries from a field $\mathbb{K}$ and appropriate sizes and $x(\cdot)$, $u(\cdot)$ and $y(\cdot)$ are functions defined on the nonnegative integers and with values in $\mathbb{K}^n$, $\mathbb{K}^m$ and $\mathbb{K}^p$ respectively. Throughout this report $x(k)$, $u(k)$ and $y(k)$ will be denoted by x, u and y respectively.

The PAOFP is studied when $\operatorname{rank} B = m$ and $\operatorname{rank} C = p$. As the cases where $m = n$ or $p = n$ can be solved using the state feedback problem we will suppose $m < n$ and $p < n$.

Received September 28, 2004.
The author was partially supported by ISR and by a grant from INVOTAN.

This question has received much attention since, studying the generic PAOFP, Kimura obtained a result and an inequality that we will refer in the next chapter.

In the generic case (defined using Zariski's topology) and for $\mathbb{K} = \mathbb{R}$ and $\mathbb{L} = \mathbb{C}$, Hermann and Martin [17] obtained the necessary and sufficient condition $mp \geq n$ for the PAOFP to have an affirmative answer although allowing K to have complex entries. That condition is only necessary in the case of a real K (Willems and Hesserlink [34], Eremenko and Gabrielov [9] and [10]). The condition is quite obvious using the pole placement map.

Wang [32] proved (using the notion of Grassmannian) that $mp > n$ is sufficient for the generic PAOFP.

By a compensator we mean a preliminary linear transformation after which the output is sent to the input.

When the compensator is another linear system we have the dynamic PAOFP. For this problem we refer the works of Brasch and Pearson [3], Rosenthal and Wang [28], Roca and Zaballa [26] and Cabral, Silva and Zaballa [6]. In the two references involving Zaballa output injection and state feedback are both used.

In a survey of Syrmos, Abdallah, Dorato and Grigoriadis [30] the reader can obtain information on the exact and on the generic PAOFP.

In the second section we describe the exact static pole assignment by output feedback problem (PAOFP) and give some (known) results about it. This problem appears when the compensator is just a constant matrix.

In section three we study the exact static PAOFP based on papers of Moore [25] and Fletcher [11]. We also generalize results obtained by Fletcher and Magni [12], [13] and [22] when $\mathbb{K} = \mathbb{R}$ and $\mathbb{L} = \mathbb{C}$. We will find sufficient conditions under which the PAOFP has an affirmative answer, when the Galois group of a proper finite normal and separable extension $\mathbb{L}$ of $\mathbb{K}$ is cyclic. This case is quite important because for example it is well known that the Galois group of a finite field is cyclic.

The pole assignment by state feedback problem (PASFP) is the particular case of the PAOFP when C is the identity matrix I_n. It has been studied by several authors. We refer Brunovsky [5], Brockett [4], Rosenbrock [27], Wohnam [36] and Zaballa [38] who studied the question even when (A, B) is not controllable.

The proofs of some results on the PASFP that are used in our work, can be found for example in Barnett [1] or Gonçalves [14] and [15].

The cases of the PAOFP that can not be solved with the PASFP will be treated in an algorithmic way.

2. Pole assignment by output feedback

Let A, B and C be matrices of sizes $n \times n$, $n \times m$ and $p \times n$ respectively with entries from a field $\mathbb{K}$. Let $\mathbb{L}$ be a proper finite normal and separable extension field of $\mathbb{K}$ and let $\mathcal{D}$ denote the set of $n-tuples$ of distinct elements of $\mathbb{L}$ which are the roots of some polynomial with coefficients in $\mathbb{K}$.

The set of such polynomials will be denoted by $\mathbb{K}[\lambda]$.

Definition 2.1. *Let $\mathcal{R}(B)$ denote the range of B. Then*

(1) *The pair (A, B) is* controllable *if there is no non-trivial A^T-invariant subspace of $ker B^T$.*
(2) *(C, A) is* observable *if there is no non-trivial A-invariant subspace of $ker C$.*

Considering $x(\cdot)$, $u(\cdot)$ and $y(\cdot)$ as functions defined on the nonnegative integers and with values in $\mathbb{L}^n$, $\mathbb{L}^m$ and $\mathbb{L}^p$ respectively we can easily prove the following necessary and sufficient conditions for controllability and observability.

Theorem 2.2. *Let $\mathcal{R}(B)$ denote the range of B and let*

$$<A, \mathcal{R}(B)> \; := \; \sum_{i=0}^{n-1} A^i \, \mathcal{R}(B).$$

Then (A, B) is controllable if and only if $< A, \mathcal{R}(B) > = \mathbb{L}^n$ and (C, A) is observable if and only if (A^T, C^T) is controllable.

In the previous case, controllability can be described by other conditions.

Theorem 2.3. *The following statements are equivalent:*

a) *(A, B) is controllable.*
b) *$rank\,[B|AB| \ldots |A^{n-1}B] = n$, with $[B|AB| \ldots |A^{n-1}B]$ being an $n \times nm$ matrix whose first m columns are the columns of B, the next m columns are the columns of AB, and so on.*
c) *There is no proper A-invariant subspace containing $\mathcal{R}(B)$.*
d) *$rank\,[A - \alpha I|B] = n$, for every α in the spectrum $\sigma(A)$ of A.*
e) *Let α be an element of $\mathbb{L}$. Then the only vector x satisfying $x^T A = \alpha x^T$ and $x^T B = 0$ is the null vector.*

The following four assumptions will apply throughout this section.

Assumption 2.4. *The sum of the numbers of inputs and outputs exceeds the number of states, that is*

$$m + p \;>\; n. \tag{4}$$

Assumption 2.5. *The triple* (A, B, C) *is* complete *in the sense that* (A, B) *is controllable and* (C, A) *is observable.*

Assumption 2.6. $x(\cdot)$, $u(\cdot)$ *and* $y(\cdot)$ *are functions defined on the nonnegative integers and with values in* $\mathbb{L}^n$, $\mathbb{L}^m$ *and* $\mathbb{L}^p$ *respectively.*

Assumption 2.7. $rank\, B = m$ *and* $rank\, C = p$.

Remark 2.8. *The fourth assumption is a technical one and as noted by several authors as Fletcher and Magni* [13] *or Mondié, Zagalak and Kučera* [24] *it can be done without any loss of generality.*

When we want to know if for the system given by (2) and (3) there exists an output feedback control law equation

$$u \;=\; -Ky \tag{5}$$

such that K has entries from $\mathbb{K}$ and the spectrum of the plant matrix $A - BKC$ of the closed loop system governed by (2) and (5) can be arbitrarily assigned in $\mathcal{D}$, we have the pole assignment by output feedback problem (PAOFP).

PAOFP *Does there exist an* $m \times p$ *matrix* K *with entries from* $\mathbb{K}$ *such that*

$$\det(A - BKC - \lambda I) = \prod_{i=1}^{n} (\lambda_i - \lambda) \tag{6}$$

for every n-tuple $(\lambda_1, \ldots, \lambda_n) \in \mathcal{D}$ *?*

Kimura [18] obtained for $\mathbb{K} = \mathbb{R}$ and $\mathbb{L} = \mathbb{C}$ the following result.

Theorem 2.9. *For every* $n - tuple\, (\lambda_1, \ldots, \lambda_n)$ *in* $\mathcal{D}$ *and for* $i = 1, \ldots, n$, *there exist elements* γ_i *in an arbitrary neighborhood of* λ_i *such that the spectrum of the matrix* $A - BKC$ *is* $\{\gamma_1, \ldots, \gamma_n\}$ *for some real matrix* K.

In the same paper Kimura introduced inequality (4) and gave an example to prove that Theorem 2.9 may fail if the system does not satisfy that inequality. Then, if the system does not satisfy (4), the answer to the PAOFP is negative.

The next theorem was proved by Fletcher and Magni [13].

Theorem 2.10. *If $\{\lambda_1, \dots, \lambda_n\}$ is a set of distinct complex numbers then there exists a (possibly complex) matrix K such that $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $A - BKC$.*

Remark 2.11. *The algorithmic proof of Theorem* 2.10 *does not depend on the characteristic properties of $\mathbb{C}$.*

An example in Kimura [18] shows that, even if $\{\lambda_1, \dots, \lambda_n\}$ is closed under conjugation, the matrix K of Theorem 2.10 can have some non real entries. Then

Theorem 2.12. *The PAOFP has negative answer in the absence of further conditions.*

In this work we give sufficient conditions in order that the PAOFP has an affirmative answer.

The last result of this section was proved by Fletcher and Magni [12], [13], [22]. It gives a set of necessary and sufficient conditions for the solvability of the exact PAOFP when $\mathbb{K} = \mathbb{R}$ and $\mathbb{L} = \mathbb{C}$. In order to describe those conditions we need to define the following subspaces.

Definition 2.13. *Let α be an element of $\mathbb{L}$. Then*

$$\begin{aligned}
\mathcal{S}(\alpha) &:= \{s \in \mathbb{L}^n : (A - \alpha I)s \in \mathcal{R}(B)\} \\
\mathcal{T}(\alpha) &:= \{t \in \mathbb{L}^n : t = (A - \alpha I)y,\ for\,some\ y \in ker C\} \\
\mathcal{B}_0 &:= \bigcap_{\alpha \in \mathbb{L}} \mathcal{S}(\alpha) \\
\mathcal{C}_0 &:= \sum_{\alpha \in \mathbb{L}} \mathcal{T}(\alpha) \\
m_0 &:= \dim(\mathcal{B}_0) \\
n - p_0 &:= \dim(\mathcal{C}_0).
\end{aligned}$$

Theorem 2.14. *The PAOFP has an affirmative answer for $\mathbb{K} = \mathbb{R}$ and $\mathbb{L} = \mathbb{C}$ if and only if the following three conditions do not hold simultaneously:*

a) *$n = m + p - 1$, with m and p even.*
b) *The set $\{\lambda_1, \ldots, \lambda_n\}$ has one and only one real element denoted by α.*
c) *Either $\mathcal{C}_0 = \mathcal{S}(\alpha)$ or $\mathcal{B}_0 = \mathcal{T}(\alpha)$.*

3. Exact pole assignment by output feedback over an arbitrary field allowing the poles to be in an extension with a cyclic Galois group

Having in mind Remark 2.11 we may assume that for an arbitrary field $\mathbb{K}$ the answer to the PAOFP is affirmative provided that K is allowed to have entries from $\mathbb{L}$. It means that if $(\lambda_1, \ldots, \lambda_n)$ is an $n - tuple$ of distinct elements of $\mathcal{D}$ there exists a matrix K (possibly with entries from $\mathbb{L}$) such that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A - BKC$.

Our aim in this chapter is to give sufficient conditions for the matrix K to exist and to have all its entries from $\mathbb{K}$.

3.1. Fields and Galois Theory. We need some basic results on fields and on Galois theory. They are proved in basic text books like Birkhoff and MacLane [2] and Lang [19].

Remark 3.1. *Any extension $\mathbb{N}$ of a field $\mathbb{F}$ may be considered as a vector space over F. If the vector space has a finite dimension the field $\mathbb{N}$ is math-called a finite extension of $\mathbb{F}$ and its dimension is known as the degree of the extension. This dimension will be denoted by $[\mathbb{N} : \mathbb{F}]$.*

Definition 3.2. *A finite extension $\mathbb{N}$ of a field $\mathbb{F}$ is said to be* normal *if every polynomial $p(x)$ irreducible over $\mathbb{F}$ which has one root over $\mathbb{N}$ has all its roots in $\mathbb{N}$.*

Theorem 3.3. *Every finite, normal, and separable extension of F is the root field of a separable polynomial.*

In our case $\mathbb{L}$ is then the root field of some separable polynomial.

Definition 3.4. *The* Galois group $\mathcal{G}_{\mathbb{L}/\mathbb{K}}$ *is the set of automorphisms u of $\mathbb{L}$ such that $u \in \mathcal{G}_{\mathbb{L}/\mathbb{K}} \Longrightarrow u(\alpha) = \alpha$, $\forall \alpha \in \mathbb{K}$.*

Remark 3.5. *By Artin's Theorem, $\mathbb{K} = \{\alpha \in \mathbb{L} : u(\alpha) = \alpha, \forall u \in \mathcal{G}_{\mathbb{L}/\mathbb{K}}\}$.*

Theorem 3.6. *If $g_1, \ldots, g_q$ are the elements of $\mathcal{G}_{\mathbb{L}/\mathbb{K}}$ then there exists an element $\beta \in \mathbb{L}$ such that $g_1(\beta), \ldots, g_q(\beta)$ form a basis of $\mathbb{L}$ over $\mathbb{K}$.*

Corollary 3.7. $[\mathbb{L}:\mathbb{K}]$ *is the order of the Galois group* $\mathcal{G}_{\mathbb{L}/\mathbb{K}}$.

Definition 3.8. *If* $\forall\, i \in \{1,\ldots,n\}\, \forall u \in \mathcal{G}_{\mathbb{L}/\mathbb{K}}\, \exists\, j \in \{1,\ldots,n\} \,:\, u(\beta_i) = \beta_j$ *then the set* $\{\beta_1,\ldots,\beta_n\}$ *is* closed under the action of $\mathcal{G}_{\mathbb{L}/\mathbb{K}}$.

Remark 3.9. *The set* $\{\lambda_1,\ldots,\lambda_n\}$ *is closed under the action of* $\mathcal{G}_{\mathbb{L}/\mathbb{K}}$ *since* $\lambda_1,\ldots,\lambda_n$ *are the roots of a polynomial in* $\mathbb{K}[\lambda]$.

3.2. The subspaces $\mathcal{S}(\alpha)$, $\mathcal{T}(\alpha)$, $\mathcal{B}_0$ and $\mathcal{C}_0$. In Definition 2.13 we introduced the subspaces $\mathcal{S}(\alpha)$, $\mathcal{T}(\alpha)$, $\mathcal{B}_0$ and $\mathcal{C}_0$ for $\alpha \in \mathbb{L}$.

Remark 3.10. *Moore* [25] *proved that* $\mathcal{S}(\lambda_i)$ *are the subspaces where we can choose the right eigenvectors of* $A - BKC$ *corresponding to the eigenvalues* λ_i. *By duality,* $[\mathcal{T}(\lambda_i)]^\perp$ *is the subspace where the left eigenvectors can be chosen.*

The following lemma has an easy proof so we will omit it.

Lemma 3.11. *If (A,B) is controllable then* $\dim \mathcal{S}(\alpha) = m$ *for all* $\alpha \in \mathbb{L}$.
If (C,A) *is observable then* $\dim \mathcal{T}(\alpha) = n - p$ *for all* $\alpha \in \mathbb{L}$.

Lemma 3.12. *Let* (α,μ) *be any pair of elements of* $\mathbb{L}$. *Then*

a) $\mathcal{B}_0 = \mathcal{S}(\alpha) \cap \mathcal{S}(\mu)$.
b) $\mathcal{C}_0 = \mathcal{T}(\alpha) + \mathcal{T}(\mu)$.
c) $\mathcal{B}_0 = \{b \in \mathcal{R}(B) : Ab \in \mathcal{R}(B)\} = \mathcal{R}(B) \cap \mathcal{S}(\alpha)$.
d) $\mathcal{C}_0 = ker C + A ker C = ker C + \mathcal{T}(\alpha)$.
e) $\dim \mathcal{B}_0 \geq 2m - n$ *and* $\dim \mathcal{C}_0 \leq 2(n-p)$ *so either* $\mathcal{B}_0 \neq \{0\}$ *or* $\mathcal{C}_0 \neq \mathbb{L}^n$.

The proof of the previous lemma in the case $\mathbb{K} = \mathbb{R}$ and $\mathbb{L} = \mathbb{C}$ can be found in [11]. It is very easy to verify that it is field independent.
In e) we need controllability and observability.

Lemma 3.13. *If for some* $i \in \{1,\ldots,n\}$ *we have* $\mathcal{S}(\lambda_i) \subseteq \mathcal{R}(B)$ *then* $\mathcal{R}(B) = \mathbb{L}^n$.

Proof: If there exists i such that $\mathcal{S}(\lambda_i) \subseteq \mathcal{R}(B)$ then, by Lemma 3.12 c), we can conclude that $\mathcal{B}_0 = \mathcal{R}(B)$.
As $\mathcal{B}_0$ is A-invariant and (A,B) is controllable, then, by c) of Theorem 2.3, $\mathcal{R}(B) = \mathbb{L}^n$. ∎

In the case of Lemma 3.13, since (A,B) is supposed to be controllable, then it is well known from the pole assignment by state feedback theory (see for example [1] and [14]) that there is a matrix F with entries from $\mathbb{K}$ and such that $\{\lambda_1, \ldots, \lambda_n\}$ is the set of eigenvalues of $A - BF$.

Since the spectrum of $A - BKC$ is the same as that of $A^T - C^T K^T B^T$ we may assume that $p \geq m$.

If for some i we have $\mathcal{S}(\lambda_i) \subseteq \mathcal{R}(B)$, it follows from the previous lemma that $p = m = n$.

Then the PAOFP has affirmative answer with $K = FC^{-1}$.

Because of this discussion we make the following assumption that will apply throughout the remaining chapter.

Assumption 3.14. *There is no integer i $(i = 1, \ldots, n)$ such that $\mathcal{S}(\lambda_i)$ is a subset of $\mathcal{R}(B)$.*

3.3. The Warren and Eckberg structure for $\mathcal{S}(\alpha)$. In this section we briefly describe the structure of $\mathcal{S}(\alpha)$ based on Warren and Eckberg [33].

Theorem 3.15. *In the controllable system described by the pair (A, B) there exist indices ν_i, i=1, ...,m such that*

a) $0 < \nu_1 \leq \ldots \leq \nu_m$
b) $\nu_1 + \ldots + \nu_m = n$

Theorem 3.16. *There is a basis $\{s_{10}, \ldots, s_{1,\nu_1-1}, \ldots, s_{m0}, \ldots, s_{m,\nu_m-1}\}$ of $\mathbb{L}^n$ with all the vectors in $\mathbb{K}^n$ such that*

a) *$s_{1,\nu_1-1}, s_{2,\nu_2-1}, \ldots, s_{m,\nu_m-1}$ is a basis of $\mathcal{R}(B)$;*
b) *if $\alpha \in \mathbb{L}$ and if, for any $i \in \{1, \ldots, m\}$, $s_i(\alpha)$ is defined as follows*

$$s_i(\alpha) := s_{i0} + \alpha s_{i1} + \ldots + \alpha^{\nu_i - 1} s_{i,\nu_i-1}$$

then $\{s_1(\alpha), \ldots, s_m(\alpha)\}$ is a basis of $\mathcal{S}(\alpha)$.

Theorem 3.17. *Let $\alpha \neq 0$ be an element of $\mathbb{L}$.*
If $\mathcal{M} := \{i : s_i(\alpha) = s_{i0}\}$ then $\mathcal{A} := \{s_i(\alpha) : i \in \mathcal{M}\}$ is a basis of $\mathcal{B}_0$

Corollary 3.18.

$$m_0 = \sum_{i \in \mathcal{M}} \nu_i = \# \mathcal{M}.$$

Fletcher [11] proved the following useful result.

Lemma 3.19. *Let $\mathcal{V}$ be a proper subspace of $\mathbb{L}^n$, of dimension k say, and let I be an integer such that $\nu_1 + \ldots + \nu_I > k$.*
Then there are fewer than ν_I values of α for which $\mathcal{S}(\alpha) \subseteq \mathcal{V}$.

The previous lemma has an obvious consequence.

Corollary 3.20. *Let $\mathcal{V}$ be a proper subspace of $\mathbb{L}^n$. Then there are at most $n-m$ values of α for which $\mathcal{S}(\alpha) \subseteq \mathcal{V}$.*

3.4. Sufficient conditions for an affirmative answer to the PAOFP. We will study the PAOFP in the case $\mathcal{G}_{\mathbb{L}/\mathbb{K}}$ is a cyclic group with generator g and order $[\mathbb{L} : \mathbb{K}]$.

We suppose that Assumptions 2.4, 2.5, 2.6 and 2.7 apply throughout this chapter.

<u>First case:</u> The PAOFP when $K \in \mathbb{L}^{m\times p}$

Having in mind Remark 2.11 we can conclude that

Theorem 3.21. *The PAOFP has an affirmative answer if K is allowed to have elements from $\mathbb{L}$.*

The key step of Fletcher and Magni's proof (see [13]) is the following lemma

Lemma 3.22. *Let $s_i \in \mathcal{S}(\lambda_i)$ ($i = 1, \ldots, r$) be vectors such that*

$$\begin{aligned} s_1, \ldots, s_r \quad &\text{are} \quad \text{linearly independent} && (7)\\ \dim(\mathcal{R}(B) + \operatorname{span}\{s_1, \ldots, s_r\}) \quad &= \quad \min\{n, m+r\} && (8)\\ \{\operatorname{span}\{s_1, \ldots, s_r\}\} \cap \mathcal{C}_0 \quad &= \quad \{0\}. && (9) \end{aligned}$$

Let Q be an $n \times (n-r)$ matrix satisfying $Q^TQ = I_{n-r}$ and $Q^Ts_i = 0$ $(i = 1, \ldots, r)$.

Let P be a $(p-r) \times p$ matrix such that with $\operatorname{rank} P = p - r$ and $PCs_i = 0$ $(i = 1, \ldots, r)$.

Then there is a matrix $K_r \in \mathbb{L}^{m\times p}$ such that

a) $(A - BK_rC)s_i = \lambda_i s_i \quad (i = 1, \ldots, r)$.

b) $(Q^T(A - BK_rC)Q, Q^TB, PCQ)$ *is a complete triple.*

c) $Q^T(A - BK_rC)Q \in \mathbb{L}^{(n-r)\times(n-r)}$, *rank* $Q^TB = \min\{n-r, m\}$ *and rank* $PCQ = p - r$.

Remark 3.23. *From now on P and Q will denote matrices as defined in the previous lemma.*

From the proof of Fletcher and Magni we point out some facts.

Remark 3.24. *Using, if necessary, A^T, C^T, B^T, then from c) of Lemma 3.12 and Assumption 2.4, we conclude that $\mathcal{C}_0 \neq \mathbb{L}^n$.*

Having in mind Assumption 3.14 *there exists* $s_1 \in \mathcal{S}(\lambda_1) - (\mathcal{C}_0 \cup \mathcal{R}(B))$.
Therefore, at least for $r = 1$, (7), (8) *and* (9) *can be satisfied.*

Remark 3.25. *The algorithm remains valid if* (9) *is replaced by the following two conditions:*

$$\begin{aligned} \operatorname{span}\{s_1, \ldots, s_r\} \cap ker C &= \{0\} && (10) \\ \operatorname{span}\{s_1, \ldots, s_r\} \cap \mathcal{T}(\beta) &= \{0\}. && (11) \end{aligned}$$

In (11) β *is considered in an algebraic closure of* $\mathbb{L}$.

Remark 3.26. *We note now the way conditions* (7), (8), (9), (10) *and* (11) *act in the algorithm.*

- $rank\, Q^T B = \min\{n - r, m\}$ *is proved using* (7) *and* (8).
- *From* (7) *and* (10) *we can prove that* $(A - BK_rC)s_i = \lambda_i s_i$ $(i = 1, \ldots, r)$, $rank\, PCQ = p - r$ *and* $(Q^T(A - BK_rC)Q, Q^T)$ *is controllable.*
- *The proof of the existence of* K_r *satisfying a) is based on*

$$\begin{aligned} s_i \in \mathcal{S}(\lambda i) \quad &\Rightarrow \quad (A - \lambda_i I)s_i = Bw_i \\ (10) \quad &\Rightarrow \quad Cs_i \neq 0, \ i = 1, \ldots, r \\ \exists K_r : \quad & K_r C s_i = w_i, \ i = 1, \ldots, r. \end{aligned}$$

- *The observability of* $(PCQ, Q^T(A - BK_rC)Q)$ *is proved using* (7) *and* (11).
- *Obviously using* (7) *and* (9) *we can conclude that* $(A - BK_rC)s_i = \lambda_i s_i$ $(i = 1, \ldots, r)$, *and* $(Q^T(A - BK_rC)Q, Q^T B, PCQ)$ *is a complete triple.*

Remark 3.27. *If, by induction, we suppose that there exists an* $m \times (p - r)$ *matrix* L *such that* $\lambda_{r+1}, \ldots, \lambda_n$ *are the (left) eigenvalues of* $Q^T(A - BK_rCQ) - Q^T BLPCQ$ *then* $\sigma(A - B(K_r + LP)C) = \{\lambda_1, \ldots, \lambda_n\}$.
This is the last step to complete the proof of Theorem 3.21.

<u>**Second case:**</u> **The PAOFP when** $K \in \mathbb{K}^{m \times p}$

In order to guarantee that the algorithm leads to a matrix $K \in \mathbb{K}^{m \times p}$ we need the following result.

Theorem 3.28. *If the vectors* $s_i \in \mathcal{S}(\lambda_i)$ $(i = 1, \ldots, n)$ *obtained from the successive application of Lemma* 3.22 *can be chosen to satisfy*

$$\lambda_j = g^t(\lambda_i) \ \Rightarrow \ s_j = g^t(s_i) \tag{12}$$

then $K \in \mathbb{K}^{m \times p}$.

Proof: From (7) we prove that for all $i \in \{1, \ldots, n\}$

$$\begin{aligned} g(A - BKC)g(s_i) &= g(\lambda_i)g(s_i) \\ (A - BKC)g(s_i) &= g(\lambda_i)g(s_i). \end{aligned}$$

Then $[(A - BKC) - g^{-1}(A - BKC)]s_i = 0$ and, as $\{s_1, \ldots, s_n\}$ is a basis of $\mathbb{L}^n$, we can conclude that $A - BKC = A - Bg(K)C$.

Then, by Assumption 2.7, $K = g(K)$ and from Remark 3.5, K has all its entries from $\mathbb{K}$. ■

Let $\lambda_{j_1}, \ldots, \lambda_{j_r}$ be any orbit. We suppose, without loss of generality, that $j_i = i$ $(i = 1, \ldots, r)$ and $\lambda_{i+1} = g^i(\lambda_1)$ $(i = 0, \ldots, r - 1)$.

Remark 3.29. *The reduced order system is controllable, observable and satisfies an inequality corresponding to* (4)*.*

As in Remark 3.27 *we can conclude from Fletcher and Magni's proof of Theorem* 3.21 *(see* [13]*) that if, in the reduced system, L is the feedback matrix for one orbit* $\lambda_{r+1}, \ldots, \lambda_{r_1}$ *then* $\{\lambda_1, \ldots, \lambda_{r_1}\} \subseteq \sigma(A - B(K_r + LP)C)$*.*

Then in each step of the algorithm we need to make sure that the vectors can be chosen in such a way that the condition of the previous theorem is satisfied.

The detailed proof of that possibility can be seen in Magni [23]*. We will refer again to this question in Remark* 3.54*.*

3.4.1. *The existence of vectors satisfying (7) and (8).* Suppose that the vectors $s_i \in \mathcal{S}(\lambda_i)$ $(i = 1, \ldots, r)$ are linearly independent. Then

Lemma 3.30.

$$\dim(\mathcal{R}(B) + \operatorname{span}\{s_1, \ldots, s_r\}) = m + r \;\Rightarrow\; \textit{rank}\, Q^T B = m. \tag{13}$$

Proof: As $[B|s_1|\ldots|s_r]$ has rank $m + r$ then $\mathcal{R}(B) \cap (\operatorname{span}\{s_1, \ldots, s_r\} = \{0\}$.

It is easy to prove that $\ker Q^T = \operatorname{span}\{s_1, \ldots, s_r\}$. Then $\operatorname{rank} Q^T B = m$. ■

Remark 3.31. *In the previous lemma it is obvious that* $m \leq n - r$*.*

Next assumption will apply from now on.

Assumption 3.32. $\#\mathbb{K} \geq [\mathbb{L} : \mathbb{K}]$.

We will need the following result that will be used to prove some results in this chapter and it justifies why we made the previous assumption.

Lemma 3.33. *If $\#\mathbb{K} \geq [\mathbb{L} : \mathbb{K}]$ and if $t+1$ doesn't exceed the order of $\mathcal{G}_{\mathbb{L}/\mathbb{K}}$ then for every polynomial $P(X_1, \ldots, X_{t+1})$ satisfying:*

T.1 *the degree in each variable is at most 1,*

T.2 *the coefficient of the monomial $X_1X_2\ldots X_t$ is non null,*

T.3 *the degree of the polynomial is t,*

T.4 *the constant term is null,*

there is $\alpha \in \mathbb{L}$ such that $P(\alpha, g(\alpha), \ldots, g^t(\alpha)) \neq 0$.

Proof: Let us suppose that $\forall\alpha \in \mathbb{L}, P(\alpha, \ldots, , g^t(\alpha)) = 0$ with P a polynomial satisfying T.1, T.2, T.3 and T.4.

Then, for all $\alpha \in \mathbb{K}$, $P(\alpha, \ldots, g^t(\alpha)) = P_t(\alpha)$ with P_t a t-degree polynomial.

As $\mathbb{K}$ has at least $t+1$ elements we arrive to the following contradiction: the t-degree polynomial P_t has at least $t+1$ different roots. ■

The following **important lemma** generalizes a result proved by Magni [21] but our proof is quite different from the one he produced.

Lemma 3.34. *Let $\mathcal{V}$ and $\mathcal{W}$ be subspaces of $\mathbb{L}^n$ satisfying*

$$g(\mathcal{V}) \;=\; \mathcal{V}. \tag{14}$$

$$\dim(\mathcal{V} + \sum_{i=0}^{t-1} g^i(\mathcal{W})) \;\geq\; \dim\mathcal{V} + t, \quad \forall t \leq \bar{t}. \tag{15}$$

Then there exists a vector $w \in \mathcal{W}$ such that $w, g(w), \ldots, g^{\bar{t}-1}(w)$ are linearly independent and $\mathcal{V} \cap \operatorname{span}\{w, g(w), \ldots, g^{\bar{t}-1}(w)\} = \{0\}$.

Proof: From Speiser's Galois Descent Theorem (see [35]), we can prove that there exists a basis $\{e_1, \ldots, e_k\}$ of $\mathcal{W} + g(\mathcal{W}) + \ldots + g^{[\mathbb{L}:\mathbb{K}]-1}(\mathcal{W})$ such that $e_i \in \mathbb{K}^n$ for all i in $\{1, \ldots, k\}$.

Let $\{x_1, \ldots, x_l\}$ be a basis of $\mathcal{V}$. From (14) and Speiser's Galois Descent Theorem, we can suppose that $x_i \in \mathbb{K}^n$ for all i in $\{1, \ldots, l\}$.

Renumbering if necessary it is easy to prove that there is an integer k_1 such that $\{x_1, \ldots, x_l, e_1, \ldots, e_{k_1}\}$ is a basis of $\mathcal{V} + \mathcal{W} + g(\mathcal{W}) + \ldots + g^{[\mathbb{L}:\mathbb{K}]-1}(\mathcal{W})$.

Then $1 \leq k_1 \leq k$ and there exists $x \in \mathcal{W}$ such that $\mathcal{V} \cap \operatorname{span}\{x\} = \{0\}$.

Let $\bar{t_1}$ be the least integer such that $g^{\bar{t_1}}(x) \in \mathcal{V} + \operatorname{span}\{x, g(x), \ldots, g^{\bar{t_1}-1}(x)\}$.

If $\bar{t_1} \geq \bar{t}$ the proof comes to an end. Let us then suppose that $1 \leq \bar{t_1} < \bar{t}$.

$g^{\bar{t_1}}(\mathcal{W})$ is not a subset of $\mathcal{V} + \operatorname{span}\{x, g(x), \ldots, g^{\bar{t_1}-1}(x)\}$ because otherwise

$$g^i(\mathcal{W}) \subseteq \mathcal{V} + \operatorname{span}\{x, g(x), \ldots, g^{\bar{t_1}-1}(x)\} \;\; (0 \leq i \leq \bar{t_1})$$

$$\text{and} \;\; \dim(\mathcal{V} + \mathcal{W} + g(\mathcal{W}) + \ldots + g^{\bar{t_1}}(\mathcal{W})) < \dim \mathcal{V} + (\bar{t_1} + 1),$$

contradicting (15).

Let y be a vector of $\mathcal{W}$ such that $g^{\bar{t_1}}(y) \notin \mathcal{V} + \operatorname{span}\{x, g(x), \ldots, g^{\bar{t_1}-1}(x)\}$.

The vectors $x_1, \ldots, x_l, \theta x + y, \ldots, g^{\bar{t_1}}(\theta x + y)$ can be represented in the basis $\{x_1, \ldots, x_l, e_1, \ldots, e_{k_1}\}$ as the columns of the matrix

$$Q(\theta) := \begin{bmatrix} I_{l\times l} & (A(\theta))_{l\times(\bar{t_1}+1)} \\ 0_{k_1\times l} & (B(\theta))_{k_1\times(\bar{t_1}+1)} \end{bmatrix}$$

$$\text{with} \;\; (A(\theta))_{l\times(\bar{t_1}+1)} := \begin{bmatrix} \theta\alpha_{11} + \beta_{11} & \ldots & g^{\bar{t_1}}(\theta)\alpha_{1,\bar{t_1}+1} + \beta_{1,\bar{t_1}+1} \\ \vdots & & \vdots \\ \theta\alpha_{l1} + \beta_{l1} & \ldots & g^{\bar{t_1}}(\theta)\alpha_{l,\bar{t_1}+1} + \beta_{l,\bar{t_1}+1} \end{bmatrix}$$

$$\text{and}(B(\theta))_{k_1\times(\bar{t_1}+1)} := \begin{bmatrix} \theta\gamma_{11} + \delta_{11} & \ldots & g^{\bar{t_1}}(\theta\gamma_{11} + \delta_{11}) \\ \vdots & & \vdots \\ \theta\gamma_{k_1 1} + \delta_{k_1 1} & \ldots & g^{\bar{t_1}}(\theta\gamma_{k_1 1} + \delta_{k_1 1}) \end{bmatrix}.$$

As the vectors $x_1, \ldots, x_l, x, \ldots, g^{\bar{t_1}-1}(x), g^{\bar{t_1}}(y)$ are linearly independent we can suppose without loss of generality that the following $(\bar{t_1}+1) \times (\bar{t_1}+1)$ minor is non null

$$\begin{vmatrix} \gamma_{11} & \ldots & g^{\bar{t_1}-1}(\gamma_{11}) & g^{\bar{t_1}}(\delta_{11}) \\ \vdots & & \vdots & \vdots \\ \gamma_{\bar{t_1}+1,1} & \ldots & g^{\bar{t_1}-1}(\gamma_{\bar{t_1}+1,1}) & g^{\bar{t_1}}(\delta_{\bar{t_1}+1,1}) \end{vmatrix} \neq 0. \tag{16}$$

If $x_1, \ldots, x_l, y, g(y), \ldots, g^{\bar{t_1}}(y)$ are linearly independent, then the the proof comes trivially to an end.

Let us assume that $x_1, \ldots, x_l, y, g(y), \ldots, g^{\bar{t_1}}(y)$ are linearly dependent.

Then from (3.10) we can conclude that

$$\begin{vmatrix} \theta\gamma_{11} + \delta_{11} & \ldots & g^{\bar{t_1}}(\theta\gamma_{11} + \delta_{11}) \\ \vdots & & \vdots \\ \theta\gamma_{\bar{t_1}+1,1} + \delta_{\bar{t_1}+1,1} & \ldots & g^{\bar{t_1}}(\theta\gamma_{\bar{t_1}+1,1} + \delta_{\bar{t_1}+1,1}) \end{vmatrix} = P(\theta, g(\theta), \ldots, g^{\bar{t_1}}(\theta))$$

with $P(X_1, \ldots, X_{\bar{t_1}+1})$ a polynomial in $\bar{t_1}+1$ variables satisfying conditions T.1, T.2, T.3 and T.4 of Lemma 3.33.

Then there exists $\theta \in \mathbb{L}$ such that $x_1, \ldots, x_l, \theta x + y, \ldots, g^{\bar{t_1}}(\theta x + y)$ are linearly independent vectors.

If $w := \theta x + y$ then $w \in \mathcal{W}$ and $w, g(w), \ldots, g^{\bar{t_1}}(w)$ are linearly independent vectors satisfying $\mathcal{V} \cap \operatorname{span}\{w, g(w), \ldots, g^{\bar{t_1}}(w)\} = \{0\}$. ∎

The following result is an immediate consequence of lemmas 3.12, 3.30 and 3.34.

Corollary 3.35. *If*

$$\dim(\mathcal{R}(B) + \sum_{i=1}^{t} \mathcal{S}(\lambda_i)) \ \geq \ m + t, \quad \forall t \leq r \tag{17}$$

then there exists $s \in \mathcal{S}(\lambda_1)$ such that $s_i := g^{i-1}(s)$ $(i = 1, \ldots, r)$ are linearly independent vectors satisfying $(\mathcal{R}(B) + \operatorname{span}\{s_1, \ldots, s_r\}) = m + r$.

Furthermore, the rank of $Q^T B$ is m.

The next lemma is very important in the pole assignment by state feedback problem (see [14]) and it generalizes to an arbitrary field a result obtained by Fletcher and Magni [13]. It will be necessary in this chapter namely in the proof of Lemma 3.37.

Lemma 3.36. *Let $\lambda_0, \ldots, \lambda_k$ be distinct elements of $\mathbb{L}$ satisfying*

$$\mathcal{S}(\lambda_0) \ \subseteq \ \mathcal{S}(\lambda_1) + \ldots + \mathcal{S}(\lambda_k).$$

Then $\mathcal{S}(\lambda_1) + \ldots + \mathcal{S}(\lambda_k) = \mathbb{L}^n$.

Proof: If $\mathcal{S}(\lambda_1) \subseteq \mathcal{R}(B)$ then from Lemma 3.13 we prove that $\mathcal{B}_0 = \mathcal{R}(B) = \mathbb{L}^n$ and obviously $\mathcal{S}(\lambda_1) + \ldots + \mathcal{S}(\lambda_k) = \mathbb{L}^n$.

If $\mathcal{S}(\lambda_1) \not\subseteq \mathcal{R}(B)$ let x_1 be an element of $\mathcal{S}(\lambda_1)$ not belonging to $\mathcal{R}(B)$. In this case it is easy to prove that $\dim(\mathcal{S}(\lambda_1) + \ldots + \mathcal{S}(\lambda_k)) \geq 2$ and we can suppose that there exists $x_2 \in \mathcal{S}(\lambda_1) + \ldots + \mathcal{S}(\lambda_k)$ such that x_1 and x_2 are linearly independent vectors.

Let $\{x_1, \ldots, x_n\}$ be a basis of $\mathbb{L}^n$. If $\{{x_1}^*, \ldots, {x_n}^*\}$ denotes the canonimathcal isomorphic image in $\mathbb{L}^n$ of the dual basis, we have $(x_1)^{*T} x_1 = 1$.

Then $(A - \lambda_1 I)x_1 = Bv_1$ and for $K = v_1 {x_1}^*$ we have $(A - BK)x_1 = \lambda_1 x_1$.

If $Q^* := [{x_2}^*| \ldots |{x_n}^*]$ it is trivial to prove that

$(Q^*)^T P = I_{n-1}$, $(Q^*)^T x_1 = 0$, ${x_1}^{*T} P = 0$, and $[P^*|{x_1}^*]^T [P|x_1] = I_n$.

As (A, B) is controllable then $(A - BK, B)$ and $((Q^*)^T (A - BK)P, (Q^*)^T B)$ are also controllable.

Let $\bar{A}$ and $\bar{B}$ denote $(Q^*)^T (A - BK)P$ and $(Q^*)^T B$ respectively.

If for every element α of $\mathbb{L}$ we define $\bar{\mathcal{S}}(\alpha) := \{x \in \mathbb{L}^n : [(\bar{A} - \alpha I)x] \in \mathcal{R}(\bar{B})\}$ then

$\bar{\mathcal{S}}(\alpha) = (Q^*)^T (\mathcal{S}(\alpha))$, $\forall \alpha \neq \lambda_1$ and $\bar{\mathcal{S}}(\lambda_1) \supseteq (Q^*)^T (\mathcal{S}(\lambda_1))$,

with "$\supseteq$" replaced by "$=$" if and only if $n = m$.

All we need now is to prove that $\dim\{\mathcal{S}(\lambda_1)+\ldots+\mathcal{S}(\lambda_k)\}=n$.

From the different equalities introduced for $\mathcal{B}_0$ and having in mind that x_2 is an element of $\mathcal{S}(\lambda_1)+\ldots+\mathcal{S}(\lambda_k)$ it is easy, arguing by induction on the number of columns of P, to conclude that $\dim\{(Q^*)^T(\mathcal{S}(\lambda_1)+\ldots+\mathcal{S}(\lambda_k))\}=n-1$.

Let $\{x_1,z_2,\ldots,z_{s+1}\}$ be a basis of $\mathcal{S}(\lambda_1)+\ldots+\mathcal{S}(\lambda_k)$. Then $z_{i+1}=Py_i+\alpha_i x_1$, with $y_i \in \mathbb{L}^{n-1}$ and $\alpha_i\in\mathbb{L}$ $(i=1,\ldots,s)$.

$\{(Q^*)^T Py_1,\ldots,(Q^*)^T Py_s\}$ contains a basis of $(Q^*)^T(\mathcal{S}(\lambda_1)+\ldots+\mathcal{S}(\lambda_k))$.

Then $s\geq n-1$ and obviously $n=s+1$.

Therefore $\dim\{\mathcal{S}(\lambda_1)+\ldots+\mathcal{S}(\lambda_k)\}=n$ and $\mathcal{S}(\lambda_1)+\ldots+\mathcal{S}(\lambda_k)=\mathbb{L}^n$ as required. ■

Lemma 3.37. *Let $\mu_0,\mu_1,\ldots,\mu_s$ be $s+1$ distinct elements of $\mathbb{L}$ satisfying*

$$\mathcal{S}(\mu_0)\subseteq\mathcal{R}(B)+\sum_{i=1}^{s}\mathcal{S}(\mu_i).$$

Then

$$\mathcal{R}(B)+\sum_{i=1}^{s}\mathcal{S}(\mu_i)=\mathbb{L}^n.$$

Proof: If for some integer i we have $\mathcal{S}(\mu_i)\subseteq\mathcal{R}(B)$ then, arguing as in the proof of Lemma 3.36, this lemma can be easily proved.

Let us suppose that $\mathcal{S}(\mu_i)\nsubseteq\mathcal{R}(B)$.

Then there exists a vector x_1 in $\mathcal{S}(\mu_i)$ that does not belong to $\mathcal{R}(B)$.

If

$$\dim\left(\sum_{i=1}^{s}\mathcal{S}(\mu_i)\right)\ \geq\ m+2$$

then, if the the columns of the matrix $M:=[B|x_1|\ldots|x_{n-m}]$ form a basis of $\mathbb{L}^n$ and $x_2\in\mathcal{S}(\mu_1)+\ldots+\mathcal{S}(\mu_s)$, we can define the $n\times(n-1)$ matrix $Q:=[B|x_2|\ldots|x_{n-m}]$ and we can prove the lemma arguing as in the proof of Lemma 3.36.

All we have to do now is to prove the lemma, supposing that

$$\dim\left(\mathcal{R}(B)+\sum_{i=1}^{s}\mathcal{S}(\mu_i)\right)\ =\ m+1.$$

In this case we can suppose $s=1$.

Let $\{y_1, \ldots, y_m, b\}$ be a basis of $\mathcal{R}(B) + \mathcal{S}(\mu_1)$. We can suppose that $y_i \in \mathcal{S}(\mu_1)$ $(i = 1, \ldots, m)$, $y_i \in \mathcal{B}_0$ $(i = 1, \ldots, m-1)$ and $b \in \mathcal{R}(B)$.

Again from Lemma 3.36 it is easy to prove that span$\{\mathcal{S}(\lambda), \lambda \in \mathbb{L}\} = \mathbb{L}^n$. Then there exists $\mu \in \mathbb{L}$ such that $Ab \in \mathcal{S}(\mu)$.

As $A(\mathcal{R}(B)+\mathcal{S}(\mu)) \subseteq \mathcal{R}(B)+\mathcal{S}(\mu)$ and (A, B) is controllable, then $\mathcal{R}(B)+\mathcal{S}(\mu) = \mathbb{L}^n$ and $n = m + 1$. The proof is now complete. ■

Remark 3.38. *In the previous lemma, Assumption* 3.14 *does not need to be assumed.*

Lemma 3.39. *If* (17) *does not hold for some integer* t $(1 \leq t \leq r)$ *let* $t_0 := \min\{t : 1 \leq t \leq r \text{ and (17) does not hold}\}$. *Then* $n = m + t_0 - 1$.

Proof: From Lemma 3.37, we have

$$\left(\mathcal{S}(\lambda_{t_0}) \subseteq \mathcal{R}(B) + \sum_{i=1}^{t_0-1} \mathcal{S}(\lambda_i)\right) \Rightarrow \left(\mathcal{R}(B) + \sum_{i=1}^{t_0-1} \mathcal{S}(\lambda_i) = \mathbb{L}^n\right).$$

The following inequalities are obvious

$$\begin{aligned} m + t_0 - 1 &\leq \dim\left(\mathcal{R}(B) + \sum_{i=1}^{t_0-1} \mathcal{S}(\lambda_i)\right) \\ &\leq \dim\left(\mathcal{R}(B) + \sum_{i=1}^{t_0} \mathcal{S}(\lambda_i)\right) \\ &< m + t_0. \end{aligned}$$

Then $n = m + t_0 - 1$. ■

Now we begin the study of the case suggested by Lemma 3.39.

We introduce our **main assumption** that will lead to the sufficient conditions to the PAOFP to have an affirmative answer.

Assumption 3.40. $m + p > n + [\mathbb{L} : \mathbb{K}] - 1$.

Remark 3.41. *For all that will be proved in this section* 3.4.1, *the condition* $m + p \geq n + [\mathbb{L} : \mathbb{K}] - 1$ *would be sufficient.*

Remark 3.42. *In the case* $\mathbb{L} = \mathbb{C}$ *and* $\mathbb{K} = \mathbb{R}$ *Fletcher and Magni* [12], [13], [22] *proved (see Theorem* 2.14*) that* $m + p > n + 1$ *is a sufficient condition to the PAOFP to have an affirmative answer.*

Going on with the study of the case suggested by Lemma 3.39, we have two subcases.

subcase 1: $m < r$

In this case $[\mathbb{L}:\mathbb{K}] + p > n + [\mathbb{L}:\mathbb{K}] - 1$ then $p = n$ and C is invertible.

By the pole assignment by state feedback results (see [14]) there exists F such that $\sigma(A - BF) = \{\lambda_1, \ldots, \lambda_n\}$ and (with $K = FC^{-1}$) we have $\sigma(A - BKC) = \{\lambda_1, \ldots, \lambda_n\}$.

subcase 2: $m \geq [\mathbb{L}:\mathbb{K}]$

In this case we have the following result.

Lemma 3.43. *If* (17) *does not hold for some integer* $t \leq r$ *then there is* $s \in \mathcal{S}(\lambda_1)$ *such that*

a) $s, g(s), \ldots, g^{r-1}(s)$ *are linearly independent;*
b) $\dim[\operatorname{span}\{s, g(s), \ldots, g^{r-1}(s)\} + \mathcal{R}(B)] = n.$

Proof: By Theorem 3.17 we know that if $\mathcal{B}_0 = \{0\}$ then $\nu_i > 1,\ \ i = 1.\ldots, m.$ Let $\dim \mathcal{B}_0 = a \geq 1$. From Assumption 3.14 we conclude that $a < m$.

Let $\nu_i = 1,\ \ i = 1\ldots, \dim \mathcal{B}_0$. Then

(1) $\mathcal{S}(\lambda_1) = \operatorname{span}\{s_{10}, \ldots, s_{a0}, s_{a+1}(\lambda_1), \ldots, s_m(\lambda_1)$
(2) $\nu_i > 1, \forall i \in \{a+1, \ldots, m\}$
(3) $\{s_{10},\ \ldots, s_{a0},\ s_{a+1,0} + \lambda_1 s_{a+1,1} + \ \ldots\ + {\lambda_1}^{\nu_{a+1}-2} s_{a+1,\nu_{a+1}-2},\ \ldots\ ,\ s_{m0} + \lambda_1 s_{m1} + \ \ldots\ + {\lambda_1}^{\nu_m - 2} s_{m,\nu_m - 2},\ s_{a+1,\nu_{a+1}-1},\ \ldots\ , s_{m,\nu_m-1}\}$ is a basis of the subspace $\mathcal{S}(\lambda_1) + \mathcal{R}(B)$ because
   (a) The vectors are linearly independent.
   (b) All the vectors belong to $\mathcal{S}(\lambda_1) + \mathcal{R}(B)$ (note that $s_{10}, \ldots, s_{a0}$ belong to $\mathcal{S}(\lambda_1) \cap \mathcal{R}(B)$ and $s_{a+1,\nu_{a+1}-1}, \ldots, s_{m,\nu_m-1}$ are vectors from $\mathcal{R}(B) - \mathcal{S}(\lambda_1)$).
   (c) The number of vectors is

$$a + (m-a) + (m-a) = \dim\{\mathcal{S}(\lambda_1) + \mathcal{R}(B)\}.$$

Let $w(\lambda_1) := s_{10} + \lambda_1 s_{20} + \ldots + {\lambda_1}^{a-1} s_{a0} + {\lambda_1}^{a} s_{a+1,\nu_{a+1}-1} + \ldots + {\lambda_1}^{m-1} s_{m,\nu_m-1} + w_{a+1}(\lambda_1) + w_{a+2}(\lambda_1) + \ldots + w_m(\lambda_1)$, with $w_{a+1}(\lambda_1) := {\lambda_1}^{m}(s_{a+1,0} + \lambda_1 s_{a+1,1} + \ldots + {\lambda_1}^{\nu_{a+1}-2} s_{a+1,\nu_{a+1}-2})$ and (for $a+2 \leq j \leq m$) $w_j(\lambda_1) := {\lambda_1}^{m + \sum_{i=a+1}^{j-1} \nu_i - (j-a-1)}(s_{j0} + \lambda^1 s_{j1} + \ldots + {\lambda_1}^{\nu_j - 2} s_{j,\nu_j-2})$.

As

$$\sum_{i=a+1}^{m} \nu_i > m - a - 1 + 1$$

then for $\beta := \deg w(\lambda)$ it is obvious that

$$\beta = m + \sum_{i=a+1}^{m} \nu_i - (m-a-1) - 2 > m + (m-a-1+1) - m + a - 1 = m-1.$$

Then we have $\beta \geq m \geq r$.

For $i = 1, \ldots, r$ it is easy to prove that

$$w(\lambda_i) = g^{i-1}(w(\lambda_1)) \in \mathcal{S}(\lambda_i) + \mathcal{R}(B).$$

As $\beta \geq r$ and

$$[w(\lambda_1)| \ldots |w(\lambda_r)] \;=\; MN$$

with M the matrix

$$[s_{10}| \ldots |s_{a0}|s_{a+1,\nu_{a+1}-1}| \ldots |s_{m,\nu_m-1}|s_{a+1,0}| \ldots |s_{a+1,\nu_{a+1}-2}| \ldots |s_{m0}| \ldots |s_{m,\nu_m-2}]$$

and N the Van der Monde matrix

$$N = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & g(\lambda_1) & \ldots & g^{r-1}(\lambda_1) \\ \vdots & \vdots & \vdots & \vdots \\ {\lambda_1}^{\beta} & (g(\lambda_1))^{\beta} & \ldots & (g^{r-1}(\lambda_1))^{\beta} \end{bmatrix}$$

then $w(\lambda_1), \ldots, w(\lambda_r)$ are linearly independent.

Let us now prove that $\dim[\mathrm{span}\{w(\lambda_1), \ldots, w(\lambda_r)\} + \mathcal{R}(B)] = n$.

We have already noted that $s_{10}, \ldots, s_{a0}, s_{a+1,\nu_{a+1}-1}, \ldots, s_{m,\nu_m-1}$ are vectors of the subspace $\mathcal{R}(B)$.

With all the remaining μ elements of the Warren and Eckberg basis for $\mathcal{L}^n$ we compose each of the following vectors $z(\lambda_i) := {\lambda_i}^{-m}(w(\lambda_1) + \ldots + w(\lambda_r)$ $(i = 1, \ldots, r)$.

Every vector $z(\lambda_i)$ is an element of $\mathrm{span}\{w(\lambda_1), \ldots, w(\lambda_r)\} + \mathcal{R}(B)$.

It is easy to prove that

$$\begin{aligned} \mu &= \sum_{i=a+1}^{m} \nu_i - (m-a+1) + 1 = \sum_{i=1}^{m} \nu_i - \sum_{i=1}^{a} \nu_i - (m-a) \\ &= n - a - (m-a) = m + t_0 - 1 - m = t_0 - 1 < r. \end{aligned}$$

Then $[z(\lambda_1)| \ldots |z(\lambda_\mu)] = M_1 N_1$ with

$$M_1 = [s_{a+1,0}| \ldots |s_{a+1,\nu_{a+1}-2}| \ldots |s_{m0}| \ldots |s_{m,\nu_m-2}]$$

and

$$N_1 = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & g(\lambda_1) & \dots & g^{\mu-1}(\lambda_1) \\ \vdots & \vdots & \vdots & \vdots \\ {\lambda_1}^{\mu-1} & (g(\lambda_1))^{\mu-1} & \dots & (g^{\mu-1}(\lambda_1))^{\mu-1} \end{bmatrix}.$$

As N_1 is a Van der Monde invertible matrix then

$$s_{a+1,0}, \dots, s_{a+1,\nu_{a+1}-2}, \dots, s_{m0}, \dots, s_{m,\nu_m-2}$$

are vectors of the subspace $\text{span}\{w(\lambda_1), \dots, w(\lambda_r)\} + \mathcal{R}(B)$.

If we set $s := w(\lambda_1)$ the proof comes to an end. ■

Corollary 3.44. *If* (17) *does not hold for some integer $t \leq r$ if $m \geq [\mathbb{L} : \mathbb{K}]$ and if $s \in \mathcal{S}(\lambda_1)$ is as defined in the previous lemma, then for $s_i := g^{i-1}(s)$ ($i = 1, \dots, r$), the rank of the matrix $Q^T B$ is $n - r$.*

Proof: The proof is easily deduced from the following obvious equalities
$n - r = \text{rank}\, Q^T = \text{rank}\, (Q^T[B|s|g(s)| \dots |g^{r-1}(s)]) = \text{rank}\, (Q^T B)$. ■

Remark 3.45. *In the Corollary 3.44 it is obvious that $n - r \leq m$.*

The results obtained so far in this section prove the following important theorem.

Theorem 3.46. *There exists a vector $s \in \mathcal{S}(\lambda_1)$ such that*

a) *$s, g(s), \dots, g^{r-1}(s)$ are linearly independent.*
b) *For an $n \times (n-r)$ matrix Q satisfying $Q^T Q = I_{n-r}$ and $Q^T(g_i(s)) = 0$ ($i = 0, \dots, r-1$), the rank of $Q^T B$ is* $\min(m, n - r)$.

Remark 3.47. *All what is said about the first subcase not only applies to the case suggested by Lemma* 3.39*. Then we make the following assumption that will apply throughout all the remaining chapter.*

Assumption 3.48. $m \geq [\mathbb{L} : \mathbb{K}]$.

3.4.2. *The existence of vectors satisfying (7) and (9).* The following result has an easy proof that we will omit.

Lemma 3.49. $g(ker C) = ker C$ *and* $g(\mathcal{C}_0) = \mathcal{C}_0$.

In the proof of the next result, by using, if necessary, (A^T, C^T, B^T), we will suppose, without any loss of generality, that $m - m_0 \geq p - p_0$.

Theorem 3.50. *If $m+p>n+[\mathbb{L}:\mathbb{K}]-1$ then*

$$\dim\left(\sum_{i=1}^{t}\mathcal{S}(\lambda_i)+\mathcal{C}_0\right)\geq t+\dim\,\mathcal{C}_0,\quad \forall t\leq r.$$

Proof: From Remark 3.24 there exists $s\in(\mathcal{S}(\lambda_1)-\mathcal{C}_0)$.

Let us suppose that for $k<r$ we have

$$\dim\left(\sum_{i=1}^{k}\mathcal{S}(\lambda_i)+\mathcal{C}_0\right)\geq k+\dim\,\mathcal{C}_0=k+n-p_0.$$

Then

$$\dim\left(\sum_{i=1}^{k+1}\mathcal{S}(\lambda_i)+\mathcal{C}_0\right)\geq (k+1)+\dim\,\mathcal{C}_0=k+1+n-p_0.$$

The worst case happens if

$$\dim\left(\sum_{i=1}^{k}\mathcal{S}(\lambda_i)+\mathcal{C}_0\right)=k+n-p_0\ \text{ and }\ \mathcal{S}(\lambda_{k+1})\subseteq\sum_{i=1}^{k}\mathcal{S}(\lambda_i)+\mathcal{C}_0.$$

In this case we have the following relations:

$$\begin{aligned}
\mathcal{S}(\lambda_{k+1})+\mathcal{S}(\lambda_k) &\subseteq \sum_{i=1}^{k}\mathcal{S}(\lambda_i)+\mathcal{C}_0\\
\dim\left[\mathcal{S}(\lambda_{k+1})+\mathcal{S}(\lambda_k)\right] &= m+m-m_0\leq k+n-p_0\\
m-m_0 &\leq k+n-p_0-m+p-p_0\\
(m-m_0)-(p-p_0) &\leq k+n-m-p<[\mathbb{L}:\mathbb{K}]+n-m-p.
\end{aligned}$$

The proof is now complete because we arrived to the contradiction $(m-m_0)-(p-p_0)\leq[\mathbb{L}:\mathbb{K}]-1+n-m-p<0$. ∎

Corollary 3.51. *If $m+p>n+[\mathbb{L}:\mathbb{K}]-1$ then there exists $s\in\mathcal{S}(\lambda_1)$ such that $s,g(s),\ldots,g^{r-1}(s)$ verify* (7) *and* (9)*.*

Proof: The corollary can be easily proved using lemmas 3.34 and 3.49 and Theorem 3.50. ∎

Theorem 3.52. *If $m+p\geq n+[\mathbb{L}:\mathbb{K}]-1$ then*

$$\dim\left(\sum_{i=1}^{t}\mathcal{S}(\lambda_i)+\mathit{ker}C\right)\geq t+\dim\left(\,\mathit{ker}C\,\right)=t+n-p,\quad\forall t\leq r.$$

Proof: Let us begin our inductive proof by showing that

$$\dim\,(\,\mathcal{S}(\lambda_1) + \ker C\,) \geq 1 + \dim\,(\,\ker C\,) = 1 + n - p.$$

If $\dim\,(\,\mathcal{S}(\lambda_1) + \ker C\,) = \dim\,(\,\ker C\,)$ then $\mathcal{S}(\lambda_1) \subseteq \ker C$ and $m \leq n - p$. This is a contradiction with Assumption 2.4.

Then $\dim\,(\,\mathcal{S}(\lambda_1) + \ker C\,) \geq 1 + \dim\,(\,\ker C\,) = 1 + n - p$.

Let

$$\dim\left(\sum_{i=1}^{l} \mathcal{S}(\lambda_i) + \ker\ C\right) \geq l + \dim\,(\,\ker C\,) = l + n - p\ l < r \leq [\mathbb{L} : \mathbb{K}]\,.$$

$$\text{If}\ \ \dim\left(\sum_{i=1}^{l} \mathcal{S}(\lambda_i) + \ker C\right) > l + \dim\,(\,\ker C\,)$$

or

$$\mathcal{S}(\lambda_{l+1}) \nsubseteq \left(\sum_{i=1}^{l} \mathcal{S}(\lambda_i) + \ker C\right)$$

$$\text{then}\ \ \dim\left(\sum_{i=1}^{l+1} \mathcal{S}(\lambda_i) + \ker C\right) \geq l + 1 + \dim\,(\,\ker C\,) = l + 1 + n - p.$$

This allows us to suppose

$$\dim\left(\sum_{i=1}^{l} \mathcal{S}(\lambda_i) + \ker C\right) = l + \dim\,(\,\ker C\,)$$

and

$$\mathcal{S}(\lambda_{l+1}) \subseteq \left(\sum_{i=1}^{l} \mathcal{S}(\lambda_i) + \ker C\right).$$

Therefore we have

$$\begin{aligned}
\mathcal{S}(\lambda_{l+1}) + \mathcal{S}(\lambda_l) \ &\subseteq\ \left(\sum_{i=1}^{l} \mathcal{S}(\lambda_i) + \ker C\right) \\
m + m - m_0 \ &\leq\ l + n - p \\
m - m_0 \leq l + n - p - m \ &<\ r + n - p - m \leq [\mathbb{L} : \mathbb{K}] + n - p - m \\
m - m_0 \ &\leq\ n + [\mathbb{L} : \mathbb{K}] - 1 - p - m \leq 0.
\end{aligned}$$

From the last relation we conclude that $m = m_0$ and $\mathcal{B}_0 = \mathcal{S}(\alpha),\ \forall \alpha \in \mathbb{L}$.

The proof is now over because using Lemma 3.12 we have $\mathcal{S}(\lambda_i) = \mathcal{B}_0 \subseteq \mathcal{R}(B)$, $i = 1, \ldots, n$ and this is a contradiction with Assumption 3.14. ∎

From lemmas 3.34 and 3.49 and Theorem 3.52 we have

Corollary 3.53. *If $m+p \geq n+[\mathbb{L}:\mathbb{K}]-1$ then there exists $s \in \mathcal{S}(\lambda_1)$ such that $s, g(s), \ldots, g^{r-1}(s)$ verify* (7) *and* (10).

Remark 3.54. *Now we point out some notes on conditions* (7), (8), (9), (10) *and* (11) *with $s_i = g^{i-1}(s)$ for some $s \in \mathcal{S}(\lambda_1)$.*

- *Conditions* (7) *and* (8) *hold if $m+p \geq n+[\mathbb{L}:\mathbb{K}]-1$.*
- *Conditions* (7) *and* (10) *hold if $m+p \geq n+[\mathbb{L}:\mathbb{K}]-1$.*
- *Conditions* (7) *and* (9) *hold if $m+p > n+[\mathbb{L}:\mathbb{K}]-1$.*
- *If the PAOFP has not an affirmative answer in the case where $m+p = n+[\mathbb{L}:\mathbb{K}]-1$ the problem will be condition* (11) *and, by Remark* 3.26*, the observability of $(PCQ, Q^T(A-BK_rC)Q)$ is not guaranteed.*
- *As noted in Remark* 3.24*, we may suppose that $\mathcal{C}_0 \neq \mathbb{L}^n$.*

  If for all $\lambda_i \in \{\lambda_{r+1}, \ldots, \lambda_n\}$ we have $\mathcal{S}(\lambda_i) \subseteq \mathcal{C}_0$, then, from Corollary 3.20, we conclude that $n-r \leq n-m$. But this, according to Assumption 3.48*, would lead to $r = m = [\mathbb{L}:\mathbb{K}]$.*

  Then, by Assumption 3.40*, $p = n$ and C is invertible.*

  Therefore the state feedback matrix $F \in \mathbb{K}^{m\times n}$ satisfying $\sigma(A-BF) = \{\lambda_1, \ldots, \lambda_n\}$ (see Gonçalves [14] *and* [15]*) could be used to construct $K = FC^{-1} \in \mathbb{K}^{m\times p}$ such that $\sigma(A-BKC) = \{\lambda_1, \ldots, \lambda_n\}$. This would solve the PAOFP.*

3.4.3. *The existence of vectors satisfying (7), (8) and (9).* To prove that there exists a vector $s \in \mathcal{S}(\lambda_1)$ verifying (7), (8) and (9) we need some results from algebraic geometry.

We will use an approach based on Goodwin and Fletcher [16] and on the chapter 16 of van der Waerden [31]. The set of polynomials in i variables with coefficients in $\mathbb{K}$ will be denoted by $\mathbb{K}[x_1, \ldots, x_i]$.

Definition 3.55. *A* variety *V in $\mathbb{K}^n$ is the set of common zeros of a finite set of polynomials $\Phi_1, \ldots, \Phi_k$ (in $\mathbb{K}[x_1, \ldots, x_n]$), i.e., the set of vectors $w \in \mathbb{K}^n$ such that $w = (w_1, w_2, \ldots, w_n)$ and $\Phi_i(w_1, \ldots, w_n) = 0$, for $i = 1, \ldots, k$.*

Definition 3.56. *A property P_0 on $\mathbb{K}^n$ is* generic *if it fails on and only on a proper variety in $\mathbb{K}^n$.*

Definition 3.57. *A* $\mathbb{K}$-variety *V in $\mathbb{L}^n$ is the set of vectors $w \in \mathbb{L}^n$ such that $(w, g(w), \ldots, g^{[\mathbb{L}:\mathbb{K}]-1}(w))$ is in the set of common zeros of a finite set* *of polynomials* $\Xi_1, \ldots, \Xi_k$ *(in* $\mathbb{K}[x_1, \ldots, x_{n[\mathbb{L}:\mathbb{K}]}]$*), i.e., the set of vectors* $w = (w_1, w_2, \ldots, w_n)$ *such that for* $i = 1, \ldots, k$*,*

$$\Phi_i(w_1, \ldots, w_n, g(w_1), \ldots, g(w_n), \ldots, g^{[\mathbb{L}:\mathbb{K}]-1}(w_1), \ldots g^{[\mathbb{L}:\mathbb{K}]-1}(w_n))^T = 0.$$

If the variety is proper, the lowest degree of all non identimathcally null polynomials will be denoted by $d(V)$*.*

Definition 3.58. *A property* P_0 *on* $\mathbb{L}^n$ *is* generic *if it fails on and only on a proper variety in* $\mathbb{L}^n$*.*

The next result is due to Marques de Sá.

Lemma 3.59. *Let* $L_1, \ldots, L_n$ *be infinite subsets of* $\mathbb{K}$*.*

If the polynomial $f(x_1, \ldots, x_n)$ *vanishes for every element of* $L_1 \times \ldots \times L_n$*, then* f *is the null polynomial.*

Proof: We can write f as $f = \phi_0 + \phi_1 x_1 + \ldots + \phi_t x_1^t$, where

$$\phi_i \in \mathbb{K}[x_2, \ldots, x_n], \ i = 0, \ldots, t.$$

For $c_j \in L_j$ $(j = 2, \ldots, n)$, the polynomial $f(x_1, c_2, \ldots, c_n)$ vanishes for an infinity of values of x_1. Then this polynomial, in one variable, is null, that is, $\phi_i(c_2, \ldots, c_n)$ $(i = 0, \ldots, t)$ are null. This proves the lemma in the case $n = 1$.

For $n \geq 2$ and $i = 0, \ldots, t$, then $\phi_i(x_2, \ldots, x_n)$ vanishes for every element of $L_1 \times \ldots \times L_n$. By induction on n, ϕ_i $(i = 0, \ldots, t)$ are null polynomials. Then f is the null polynomial. ∎

The following result is an obvious consequence of the previous lemma.

Corollary 3.60. *If* $\mathbb{K}$ *is infinite and the polynomial* $f(x_1, \ldots, x_i)$ *is null for every element of* $\mathbb{K}^n$*, then* f *is the null polynomial.*

Lemma 3.61. *If* $\mathbb{K}$ *is infinite and* P_1 *and* P_2 *are generic properties on* $\mathbb{L}^n$*, then there exists* $w \in \mathbb{L}^n$ *such that* P_1 *and* P_2 *hold at* w*.*

Proof: Let $\Xi_{11}, \ldots, \Xi_{k_i 1}$ be the defining polynomials of the varieties where P_i fails $(i \in \{1, 2\})$.

By Theorem 3.6 there exists $\beta \in \mathbb{L}$ such that $\{\beta, g(\beta), \ldots, g^{[\mathbb{L}:\mathbb{K}]-1}(\beta)\}$ is a basis of $\mathbb{L}$ over $\mathbb{K}$.

For each $i \in \{1, 2\}$ define a property P_i' on ${\mathbb{K}^n}^{[\mathbb{L}:\mathbb{K}]}$ such that P_i' holds at

$$(x_0, \ldots, x_{[\mathbb{L}:\mathbb{K}]-1}) \in {\mathbb{K}^n}^{[\mathbb{L}:\mathbb{K}]}$$

if and only if P_i holds at

$$\beta x_0 + g(\beta)x_1 + \ldots + g^{[\mathbb{L}:\mathbb{K}]-1}(\beta)\, x_{[\mathbb{L}:\mathbb{K}]-1} \in \mathbb{L}^n.$$

We will show that P_1' and P_2' are generic.

For $i \in \{1,2\}$, suppose that P_i' fails at $(x_0, \ldots, x_{[\mathbb{L}:\mathbb{K}]-1}) \in {\mathbb{K}^n}^{[\mathbb{L}:\mathbb{K}]}$, i.e., if and only if P_i fails at $w := \beta x_0 + g(\beta)x_1 + \ldots + g^{[\mathbb{L}:\mathbb{K}]-1}(\beta)\, x_{[\mathbb{L}:\mathbb{K}]-1}$ which happens if and only if $\Xi_{ji}(w, g(w), \ldots, g^{[\mathbb{L}:\mathbb{K}]-1}(w) = 0$, for $j = 1, \ldots, k_i$.

The previous condition is equivalent to $\Psi_{ji}(w, g(w), \ldots, g^{[\mathbb{L}:\mathbb{K}]-1}(w)) = 0$, with $\Psi_{ji}(X) \in \mathbb{K}[X]$ and $j = 1, \ldots, k_i'$.

This proves that each P_i' fails on and only on a variety in ${\mathbb{K}^n}^{[\mathbb{L}:\mathbb{K}]}$.

All these varieties are proper because otherwise the corresponding P_i would not be generic.

This proves that P_i' are generic.

Suppose now that for all $w := \beta x_0 + g(\beta)x_1 + \ldots + g^{[\mathbb{L}:\mathbb{K}]-1}(\beta)\, x_{[\mathbb{L}:\mathbb{K}]-1}$ at least one of the P_i fails.

This means that for all $(x_0, \ldots, x_{[\mathbb{L}:\mathbb{K}]-1}) \in {\mathbb{K}^n}^{[\mathbb{L}:\mathbb{K}]}$ at least one of the P_i' fails. Then every product $\Psi_{j1}\Psi_{k2}$ vanishes for all points in ${\mathbb{K}^n}^{[\mathbb{L}:\mathbb{K}]}$.

Using Lemma 3.59, we can conclude that $\Psi_{j1}\Psi_{k2}$ is the null polynomial for every $j = 1, \ldots, k_i'$ and $k = 1, \ldots, k_2'$.

If Ψ_{j1} is the null polynomial for every $j = 1, \ldots, k_1'$, then Φ_{l1} are the null polynomial for every $l = 1, \ldots, k_1$. But this contradicts the fact that P_1 is generic.

If there exists $j \in \{1, \ldots, k_1\}$ such that Ψ_{j1} is not the null polynomial, then every polynomial Ψ_{k2} $(k = 1, \ldots, k_2')$ is the null polynomial. This contradicts the fact that P_2 is generic.

Therefore there exists $w \in \mathbb{L}^n$ such that P_1 and P_2 hold at w. ■

If $\mathbb{K}$ is finite then the union of two proper varieties in $\mathbb{L}^n$ can be the whole space $\mathbb{L}^n$. This is a problem in the last part of the proof of the previous lemma.

Instead of using results about bounds for the number of points on varieties over finite fields which involve the dimension and the degree of irreducible varieties (see for example Lang and Weil [20] or Cafure and Matera [8]), we solve the question with the following two results.

The following lemma is refered by Cafure and Matera [7] and has been proved by Schmidt [29].

Lemma 3.62. *Let $f(x_1, \ldots, x_l) \in \mathbb{K}[x_1, \ldots, x_l]$ be a non identimathcally null polynomial with degree d and let $N(f)$ be the number of elements of the set $\{(\alpha_1, \ldots, \alpha_l) \,|\, f(\alpha_1, \ldots, \alpha_l) = 0)\}$.*

Then $N(f) \leq dq^{l-1}$ with q the number of elements of $\mathbb{K}$.

Lemma 3.63. *Let P_1 and P_2 be generic properties on $\mathbb{L}^n$ failing respectively on the varieties V_1 and V_2.*

If $\mathbb{K}$ is finite and has more than $(d(V_1)+d(V_2))$ elements then there exists $w \in \mathbb{L}^n$ such that P_1 and P_2 hold at w.

Proof: As in the proof of Lemma 3.61 we can show that P_i' are generic.

The other part of the proof has to be done in a different way.

Let $\Psi_{j_1 1}$ be a minimal degree non identimathcally null polynomial defining the variety V_1.

Let $\Psi_{j_2 2}$ be defined in a similar way.

Let us suppose that $\Psi_{j_1 1}\Psi_{j_2 2}$ vanishes for all elements of $\mathbb{K}^{n[\mathbb{L}:\mathbb{K}]}$. By Lemma 3.62, $N(\Psi_{j_1 1}\Psi_{j_2 2}) \leq (d(V_1) + d(V_2))q^{n[\mathbb{L}:\mathbb{K}]-1}$. Then we would have $q^{n[\mathbb{L}:\mathbb{K}]} = N(\Psi_{j_1 1}\Psi_{j_2 2}) \leq (d(V_1) + d(V_2))q^{n[\mathbb{L}:\mathbb{K}]-1}$, arriving to $q \leq (d(V_1) + d(V_2))$ which is a contradiction

Therefore $\Psi_{j_1 1}\Psi_{j_2 2}$ does not vanish for all elements of $\mathbb{K}^{n[\mathbb{L}:\mathbb{K}]}$.

In this way we conclude that $V_1 \cup V_2 \neq \mathbb{K}^{n[\mathbb{L}:\mathbb{K}]}$ and the proof is complete. ■

Lemma 3.64. *Let P_1 and P_2 be generic properties on $\mathbb{L}^n$ and $\mathcal{S}$ be a subspace such that for each $i \in \{1, 2\}$ there exists a vector $s_i \in \mathcal{S}$ such that P_i holds at s_i. Then there exists a vector $s \in \mathcal{S}$ such that P_1 and P_2 hold at s.*

Proof: Let $\dim \mathcal{S} = k$.

Then there is a bijective map M from $\mathbb{L}^k$ to $\mathcal{S}$. The proof is easy if for each $i \in \{1, 2\}$ we define a property $\tilde{P}_i$ such that $\tilde{P}_i$ holds at $t \in \mathbb{L}^k$ if and only if P_i holds at $M(t) \in \mathcal{S}$. ■

Using the notations of Lemma 3.34, we can define a property P on $\mathbb{L}^n$ such that P holds at $w \in \mathbb{L}^n$ if and only if $w, g(w), \ldots, g^{\bar{t}-1}(w)$ are linearly independent and $\mathcal{V} \cap \operatorname{span}\{w, g(w), \ldots, g^{\bar{t}-1}(w)\} = \{0\}$.

From the proof of Lemma 3.34 it is easy to deduce that the P is generic.

As corollaries 3.35 and 3.51 are proved based on Lemma 3.34 the following two results have obvious proofs.

Lemma 3.65. *The property P_3 on $\mathbb{L}^n$ defined as that property which holds at $s \in \mathbb{L}^n$ if and only if* (7) *an* (8) *hold for s, is generic.*

Lemma 3.66. *The property P_4 on $\mathbb{L}^n$ defined as that property which holds at $s \in \mathbb{L}^n$ if and only if* (7) *and* (9) *hold for s, is generic.*

From lemmas 3.64, 3.65, 3.66 and Assumption 3.32 we can conclude the following result.

Lemma 3.67. *If*

- $\mathbb{K}$ *is finite,* $p < n$, $m < n$, $m + p > n + [\mathbb{L} : \mathbb{K}] - 1$ *and* $\#\mathbb{K} > \max\{d(V_3) + d(V_4), [\mathbb{L} : \mathbb{K}]\}$ *or*
- $\mathbb{K}$ *is not finite,* $p < n$, $m < n$ *and* $m + p > n + [\mathbb{L} : \mathbb{K}] - 1$

then there exists a vector $s \in \mathcal{S}(\lambda_1)$ satisfying (7), (8) *and* (9).

3.5. Main results. From all the previous discussion mainly lemmas 3.22 and 3.67 and Theorem 3.28, we conclude the following results.

case 1: if $p < n$, $m < n$ and $\mathbb{K}$ is finite

Theorem 3.68. *Let P_3 and P_4 be the properties defined in lemmas* 3.19 *and* 3.20 *failing respectively on the varieties V_3 and V_4.*

If $m + p > n + [\mathbb{L} : \mathbb{K}] - 1$ and $\mathbb{K}$ is finite and has more than $\max\{d(V_3) + d(V_4), [\mathbb{L} : \mathbb{K}]\}$ *elements, then the PAOFP has an affirmative answer.*

Remark 3.69. *Obviously $d(V_i) \leq o(\alpha) \leq [\mathbb{L} : \mathbb{K}]$ with $i \in \{3, 4\}$ and $o(\alpha)$ the length of the greatest orbit of the elements $\lambda_1, \ldots, \lambda_n$. This allows other bounds instead of* $\max\{d(V_3) + d(V_4), [\mathbb{L} : \mathbb{K}]\}$.

case 2: if $p < n$, $m < n$ and $\mathbb{K}$ is not finite

Theorem 3.70. *If $m+p > n+[\mathbb{L} : \mathbb{K}]-1$ then the PAOFP has an affirmative answer.*

case 3: if $p = n$ or $m = n$

If $p = n$ or $m = n$ then the pole assignment by state feedback can be used to solve the PAOFP and (see Gonçalves [14] or [15]) and we need no restriction on the number of elements of $\mathbb{K}$.

Acknowledgments

The author wants to thank Doctor Leslie R. Fletcher for suggesting this problem and for the helpful discussions during the preparation of this report.

The author wants as well to thank Professor Marques de Sá for the helpful discussions that have been held, for his advices and for his quite important suggestions.

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