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PROOF OF TWO CONJECTURES ON ASKEY-WILSON POLYNOMIALS

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Abstract. We give positive answer to two conjectures posed by M. E. H Ismail in his monograph [*Classical and quantum orthogonal polynomials in one variable*, Cambridge University Press, Cambridge, 2005]. These results generalize the classical problems of Sonine and Hahn.

1. Introduction and main result

Characterize sequences of polynomials that have certain additional properties has produced some of the most useful theorems on orthogonal polynomials. Some of these results have brought together well-known families of orthogonal polynomials. For instance, the classical orthogonal polynomials of Jacobi, Laguerre and Hermite (in a more general sense also Bessel) have several properties common to all of them; particularly beautiful are those due to Al-Salam and Chihara, Bochner, Sonine and Hahn, Maroni, and McCarthy, among others (see [4] and references therein for a recent survey on the subject). But also the so-called characterization theorems have given rise to new families. For example, Al-Salam-Chihara polynomials were discovered via a characterization theorem [2] which gives positive answer to a question raised by Karlin and Szegő in their monumental paper on determinants whose elements are orthogonal polynomials [7]. The weight function for this family was found by Askey and Ismail [3], who appropriately gave these polynomials their current name. The book chapter [1] and the books [15] and [8] are devoted to characterization theorem. The most interesting problems and conjectures on the subject were collected by Ismail in his monograph on Orthogonal Polynomials and Special Functions published in 2005 and revised in 2009, see [13, Section 24.7]. This note deals with two of these conjectures (see Conjecture 1.1 and Conjecture 1.2) that extends, for Askey-Wilson operator, the classical problem of Sonine [16] and Hahn [5,6] (see also, for instance, [10], [11], [9] and [12]): *The only orthogonal polynomials whose derivatives are also orthogonal are Jacobi, Laguerre, and Hermite (in a more general sense also Bessel) polynomials and special cases of them.*

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The Askey-Wilson divided difference operator is defined by

$$(\mathcal{D}_q f)(x) = \frac{\breve{f}\big(q^{1/2}z\big) - \breve{f}\big(q^{-1/2}z\big)}{\breve{e}\big(q^{1/2}z\big) - \breve{e}\big(q^{-1/2}z\big)}, \quad z = e^{i\theta}, \tag{1}$$

where $\breve{f}(z) = f\big((z+1/z)/2\big) = f(\cos\theta)$ for each polynomial f and $e(x) = x$. Here $0 < q < 1$ and θ is not necessarily a real number (see [13, p. 300]). Hereafter, we denote $x(s) = (q^s + q^{-s})/2$ with $0 < q < 1$. Taking $e^{i\theta} = q^s$ in (1), $\mathcal{D}_q$ reads

$$\mathcal{D}_q f(x(s)) = \frac{f\big(x(s+\frac{1}{2})\big) - f\big(x(s-\frac{1}{2})\big)}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})}.$$

Set $\mathcal{D}_q^0 f = f$ and $\mathcal{D}_q^1 = \mathcal{D}_q$, and define $\mathcal{D}_q^k = \mathcal{D}_q(\mathcal{D}_q^{k-1})$ for each $k = 1, 2, \ldots$.

Conjecture 1.1. [13, Conjecture 24.7.10] *If $(p_n)_{n\geq 0}$ and $(\mathcal{D}_q p_n)_{n\geq 0}$, or the latter with a limiting case of $\mathcal{D}_q$, are two sequences of orthogonal polynomials, then $(p_n)_{n\geq 0}$ are multiples of the Askey-Wilson polynomials, or special or limiting cases of them.*

Conjecture 1.2. [13, Conjecture 24.7.11] *If $(p_n)_{n\geq 0}$ and $(\mathcal{D}_q^k p_{n+k})_{n\geq 0}$, or the latter with a limiting case of $\mathcal{D}_q$, are two sequences of orthogonal polynomials for some k, $k = 1, 2, \ldots$, then $(p_n)_{n\geq 0}$ are multiples of the Askey-Wilson polynomials, or special or limiting cases of them.*

Define the average operator $\mathcal{A}_q$ by

$$\mathcal{A}_q f(x(s)) = \frac{f\big(x(s+\frac{1}{2})\big) + f\big(x(s-\frac{1}{2})\big)}{2}$$

for every polynomial f. In 2003, Ismail proved the following result (see [13, Theorem 20.1.3]):

Theorem 1.1. *A second order operator equation of the form*

$$f(x)\mathcal{D}_q^2\, y + g(x)\mathcal{A}_q\mathcal{D}_q\, y + h(x)\, y = \lambda_n\, y \tag{2}$$

has a polynomial solution $y_n(x)$ of exact degree n for each $n = 0, 1, \ldots$, if and only if $y_n(x)$ is a multiple of the Askey-Wilson polynomials, or special or limiting cases of them. In all these cases f, g, h, and λ_n reduce to

$$f(x) = -q^{-1/2}(2(1+\sigma_4)x^2 - (\sigma_1+\sigma_3)x - 1 + \sigma_2 - \sigma_4),$$

$$g(x) = \frac{2}{1-q}(2(\sigma_4 - 1)x + \sigma_1 - \sigma_3), \quad h(x) = 0,$$

$$\lambda_n = \frac{4q(1-q^{-n})(1-\sigma_4 q^{n-1})}{(1-q)^2},$$

or a special or limiting case of it, σ_j being the jth elementary symmetric function of the Askey-Wilson parameters.

Virtually the above conjectures are summed up in one if we are able to prove Conjecture 1.2. To do this, we prove that the sequences of polynomials appearing in Conjecture 1.2 satisfy, for each k, a second order operator equation of the form (2). The important point to note here is that this argument would not lead to a satisfactory conclusion if we were not looking for the whole space of "Askey-Wilson polynomials, or special or limiting cases of them".

Theorem 1.2. *If $(p_n)_{n\geq 0}$ and $(\mathcal{D}_q^k p_{n+k})_{n\geq 0}$, or the latter with a limiting case of $\mathcal{D}_q$, are two sequences of orthogonal polynomials for some k, $k=1,2,\dots$, then, for each k, $(\mathcal{D}_q^{k-1} p_{n+k-1})_{n\geq 0}$ are multiples of the Askey-Wilson polynomials, or special or limiting cases of them.*

Fix k, $k=1,2,\dots$. It is easily seen that $(p_n)_{n\geq 0}$ is a sequence of orthogonal polynomials satisfying

$$\pi(x)\mathcal{D}_q^k p_n(x) = \sum_{j=-m}^{m} c_{n,j}\, p_{n+j}(x), \quad c_{n,-m}\neq 0, \tag{3}$$

for a polynomial π of degree at most $2k$ which does not depend on n, if and only if $(p_n)_{n\geq 0}$ and $(\mathcal{D}_q^k p_{n+k})_{n\geq 0}$ are sequences of orthogonal polynomials. Now, we can apply Theorem 1.2 to conclude that for each k, $k=1,2,\dots$, $(\mathcal{D}_q^{k-1} p_{n+k-1})_{n\geq 0}$ are multiples of the Askey-Wilson polynomials, or special or limiting cases of them. In particular, taking $k=2$ and $m=2$ in (3), we have the main result proved in [14]: $(p_n)_{n\geq 0}$ *are multiples of the Askey-Wilson polynomials, or special or limiting cases of them.* Neither in this work nor in [14] was possible to exclude the "limiting cases" in the last statement. If so, we would have positive answer to a particular case of another conjecture posed by Ismail (see [13, Conjecture 24.7.9]). However this has not been possible and this problem is still open in its full generality.

2. Proof of Theorem 1.2

The following properties are well known:

$$\mathcal{D}_q(fg) = (\mathcal{D}_q f)(\mathcal{A}_q g) + (\mathcal{A}_q f)(\mathcal{D}_q g),$$

$$\mathcal{A}_q(fg) = \mathrm{U}_2(\mathcal{D}_q f)(\mathcal{D}_q g) + (\mathcal{A}_q f)(\mathcal{A}_q g),$$

$$\mathcal{A}_q^2 f = \alpha \mathrm{U}_2 \mathcal{D}_q^2 f + \mathrm{U}_1 \mathcal{A}_q \mathcal{D}_q f + f,$$

$$\mathcal{D}_q \mathcal{A}_q f = \alpha \mathcal{A}_q \mathcal{D}_q f + \mathrm{U}_1 \mathcal{D}_q^2 f,$$

for polynomials f and g, where $\mathrm{U}_1(x) = (\alpha^2-1)x$, $\mathrm{U}_2(x) = (\alpha^2-1)(x^2-1)$, and $2\alpha = q^{1/2}+q^{-1/2}$. We leave it to the reader to verify by induction that

$$\mathcal{D}_q^k(xf(x)) = \gamma_k \mathcal{A}_q \mathcal{D}_q^{k-1} f(x) + \frac{q^{k/2}+q^{-k/2}}{2} x \mathcal{D}_q^k f(x).$$

where

$$\gamma_k = \frac{q^{k/2}-q^{-k/2}}{q^{1/2}-q^{-1/2}}.$$

Set $P_n^{[k]} = \gamma_n!/\gamma_{n+k}!\, \mathcal{D}_q^k P_{n+k}$, and so $P_n^{[k]} = \gamma_{n+1}^{-1} \mathcal{D}_q P_{n+1}^{[k-1]}$. Since $(P_n)_{n\geq 0}$ and $(P_n^{[k]})_{n\geq 0}$, for a certain fixed k, are sequences of (monic) orthogonal polynomials, any three consecutive elements of these sequences satisfy

$$xP_{n+k}(x) = P_{n+k+1}(x) + B_{n+k}P_{n+k}(x) + C_{n+k}P_{n+k-1}(x), \tag{4}$$

$$xP_{n-1}^{[k]}(x) = P_n^{[k]}(x) + B_{n-1}^{[k]} P_{n-1}^{[k]}(x) + C_{n-1}^{[k]} P_{n-2}^{[k]}(x), \tag{5}$$

with $C_{n+k}\neq 0$ and $C_{n-1}^{[k]}\neq 0$. We apply $\mathcal{D}_q^k$ to (4) to get

$$\gamma_k\mathcal{A}_qP_n^{[k-1]}(x)+\frac{q^{k/2}+q^{-k/2}}{2}x\mathcal{D}_qP_n^{[k-1]}(x)$$
$$=\frac{\gamma_{n+k}}{\gamma_{n+1}}\mathcal{D}_qP_{n+1}^{[k-1]}(x)+B_{n+k-1}\mathcal{D}_qP_n^{[k-1]}(x)+\frac{\gamma_n}{\gamma_{n+k-1}}C_{n+k-1}\mathcal{D}_qP_{n-1}^{[k-1]}(x). \tag{6}$$

From (5) we have

$$\frac{1}{\gamma_n}x\mathcal{D}_qP_n^{[k-1]}(x)$$
$$=\frac{1}{\gamma_{n+1}}\mathcal{D}_qP_{n+1}^{[k-1]}(x)+\frac{1}{\gamma_n}B_{n-1}^{[k]}\mathcal{D}_qP_n^{[k-1]}(x)+\frac{1}{\gamma_{n-1}}C_{n-1}^{[k]}\mathcal{D}_qP_{n-1}^{[k-1]}(x). \tag{7}$$

We now apply $\mathcal{A}_q$ to (6) and (7) and, by combining the resulting equations, we can eliminate $\mathcal{A}_q\mathcal{D}_qP_{n-1}^{[k-1]}(x)$, and obtain the equation

$$D_n(x)\mathcal{A}_q\mathcal{D}_qP_n^{[k-1]}(x)+E_n\mathrm{U}_2(x)\mathcal{D}_q^2P_n^{[k-1]}(x)+\frac{\gamma_k}{\gamma_{n-1}}C_{n-1}^{[k]}P_n^{[k-1]}(x)$$
$$=F_n\mathcal{A}_q\mathcal{D}_qP_{n+1}^{[k-1]}(x), \tag{8}$$

where

$$D_n(x)=\left(\frac{q^{(k+1)/2}+q^{-(k+1)/2}}{2}\frac{1}{\gamma_{n-1}}C_{n-1}^{[k]}-\frac{\alpha}{\gamma_{n+k-1}}C_{n+k-1}\right)x-\frac{1}{\gamma_{n-1}}B_{n+k-1}C_{n-1}^{[k]}$$
$$+\frac{1}{\gamma_{n+k-1}}B_{n-1}^{[k]}C_{n+k-1},$$

$$E_n=\frac{\gamma_{k+1}}{\gamma_{n-1}}C_{n-1}^{[k]}-\frac{1}{\gamma_{n+k-1}}C_{n+k-1},$$

$$F_n=\frac{\gamma_{n+k}}{\gamma_{n+1}\gamma_{n-1}}C_{n-1}^{[k]}-\frac{\gamma_n}{\gamma_{n+1}\gamma_{n+k-1}}C_{n+k-1}.$$

Similarly, we can eliminate $\mathcal{A}_q\mathcal{D}_qP_{n+1}^{[k-1]}(x)$, and shift n to $n+1$ to obtain the equation

$$\widetilde{D}_n(x)\mathcal{A}_q\mathcal{D}_qP_{n+1}^{[k-1]}(x)-\widetilde{E}_n\mathrm{U}_2(x)\mathcal{D}_q^2P_{n+1}^{[k-1]}(x)-\frac{\gamma_k}{\gamma_{n+2}}P_{n+1}^{[k-1]}(x)$$
$$=F_{n+1}\mathcal{A}_q\mathcal{D}_qP_n^{[k-1]}(x), \tag{9}$$

where

$$\gamma_{n+2}\widetilde{D}_n(x)=\frac{q^{k/2}+q^{-k/2}}{2}\frac{\gamma_n}{\gamma_{n+1}}x+B_{n+k}-\frac{\gamma_{n+k+1}}{\gamma_{n+1}}B_n^{[k]},$$

$$\widetilde{E}_n=\frac{\gamma_n\gamma_k}{\gamma_{n+1}\gamma_{n+2}}.$$

We now apply $\mathcal{D}_q$ to (6) and (7) and, by combining the resulting equations, we can eliminate $\mathcal{D}_q^2P_{n-1}^{[k-1]}(x)$, and obtain the equation

$$D_n(x)\mathcal{D}_q^2P_n^{[k-1]}(x)+E_n\mathcal{A}_q\mathcal{D}_qP_n^{[k-1]}(x)=F_n\mathcal{D}_q^2P_{n+1}^{[k-1]}(x). \tag{10}$$

Similarly, we can eliminate $\mathcal{D}_q^2 P_{n+1}^{[k-1]}(x)$, and shift n to $n+1$ to obtain the equation

$$\widetilde{D}_n(x)\mathcal{D}_q^2 P_{n+1}^{[k-1]}(x) - \widetilde{E}_n \mathcal{A}_q \mathcal{D}_q P_{n+1}^{[k-1]}(x) = F_{n+1}\mathcal{D}_q^2 P_n^{[k-1]}(x). \tag{11}$$

Note that if $\mathcal{A}_q P_n^{[k]}(x(s_1)) = 0$ we have $P_n^{[k]}(x(s_1+1/2)) = -P_n^{[k]}(x(s_1-1/2))$. Suppose that $\mathcal{D}_q P_n^{[k]}(x(s_1)) = 0$. So

$$P_n^{[k]}(x(s_1+1/2)) = P_n^{[k]}(x(s_1-1/2)) = 0,$$

which is impossible. Thus $\mathcal{D}_q P_n^{[k]}(x)$ and $\mathcal{A}_q P_n^{[k]}(x)$ have no common zeros. After shifting n to $n-1$ in (11), to obtain a contradiction, suppose that $F_n = 0$, i.e.

$$\widetilde{D}_{n-1}(x)\mathcal{D}_q P_n^{[k]}(x) = \widetilde{E}_{n-1}\mathcal{A}_q P_n^{[k]}(x).$$

Since $\mathcal{D}_q P_n^{[k]}(x)$ and $\mathcal{A}_q P_n^{[k]}(x)$ have no common zeros, we have $\widetilde{E}_{n-1} = 0$, which is impossible. (We can also conclude that $F_{n+1} \neq 0$.) Multiplying (11) by F_n and using (10) and (8), we get

$$f_n(x)\mathcal{D}_q^2 P_n^{[k-1]}(x) + g_n(x)\mathcal{A}_q\mathcal{D}_q P_n^{[k-1]}(x) + \frac{\gamma_k}{\gamma_{n-1}}\widetilde{E}_n C_{n-1}^{[k]} P_n^{[k-1]}(x) = 0, \tag{12}$$

where

$$f_n(x) = E_n\widetilde{E}_n \mathrm{U}_2 - D_n(x)\widetilde{D}_n(x) + F_nF_{n+1},$$

$$g_n(x) = \widetilde{E}_n D_n(x) - E_n\widetilde{D}_n(x).$$

We next claim that there exist nonzero numbers r_n and two polynomials $f(x)$ and $g(x)$ of degree at most two and one, respectively, not simultaneously zero, such that

$$f_n(x) = r_n\, f(x), \quad g_n(x) = r_n\, g(x).$$

Indeed, multiplying (10) by F_{n+1} and (11) by $D_n(x)$, we can eliminate $\mathcal{D}_q^2 P_n^{[k-1]}(x)$, and obtain, using (9) and shifting n to $n-1$, the equation

$$f_{n-1}(x)\mathcal{D}_q^2 P_n^{[k-1]}(x) + g_{n-1}(x)\mathcal{A}_q\mathcal{D}_q P_n^{[k-1]}(x) + \frac{\gamma_k}{\gamma_{n+1}} E_{n-1} P_n^{[k-1]}(x) = 0. \tag{13}$$

(If $D_n(x) = 0$, we combine directly (10) and (9) to obtain (13).) Suppose that $E_{n-1} = 0$, i.e.

$$f_{n-1}(x)\mathcal{D}_q P_{n-1}^{[k]}(x) = -g_{n-1}(x)\mathcal{A}_q P_{n-1}^{[k]}(x).$$

Since $\mathcal{D}_q P_{n-1}^{[k]}(x)$ and $\mathcal{A}_q P_{n-1}^{[k]}(x)$ have no common zeros, we have $f_{n-1}(x) = g_{n-1}(x) = 0$, which is imposible according to (12) after shifting n to $n-1$. Thus, by combining (12) and (13), we can eliminate $P_n^{[k-1]}(x)$, and obtain the equation

$$\left(f_n(x) - \frac{\gamma_{n+1}}{\gamma_{n-1}}\frac{\widetilde{E}_n}{E_{n-1}} C_{n-1}^{[k]} f_{n-1}(x)\right)\mathcal{D}_q P_n^{[k]}(x)$$

$$= -\left(g_n(x) - \frac{\gamma_{n+1}}{\gamma_{n-1}}\frac{\widetilde{E}_n}{E_{n-1}} C_{n-1}^{[k]} g_{n-1}(x)\right)\mathcal{A}_q P_n^{[k]}(x).$$

Again, since $\mathcal{D}_q P_n^{[k]}(x)$ and $\mathcal{A}_q P_n^{[k]}(x)$ have no common zeros, the desired conclusion follows. This allows us to rewrite (12) as

$$f(x)\mathcal{D}_q^2 P_n^{[k-1]}(x) + g(x)\mathcal{A}_q\mathcal{D}_q P_n^{[k-1]}(x) + \lambda_n\, P_n^{[k-1]}(x) = 0, \tag{14}$$

where $f(x)$ and $g(x)$ are polynomials of degree at most two and one, respectively, not simultaneously zero, and $\lambda_n = \gamma_k/(r_n\gamma_{n-1})\widetilde{E}_n C_{n-1}^{[k]} \neq 0$. Thus, by Theorem 1.1, $(P_n^{[k-1]})_{n\geq 0}$ are multiples of the Askey-Wilson polynomials, or special or limiting cases of them. Since now $(P_n)_{n\geq 0}$ and $(P_n^{[k-1]})_{n\geq 0}$ are two sequences of orthogonal polynomials, repeating the previous argument we conclude, for each $j = k-2, \ldots, 0$, that $(P_n^{[j]})_{n\geq 0}$ are multiples of the Askey-Wilson polynomials, or special or limiting cases of them. The rest of the proof is trivial.

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