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THE NUMERICAL RANGE OF 2-DIMENSIONAL KREIN SPACES OPERATORS

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ABSTRACT: The tracial numerical range of operators on a 2-dimensional Krein space is investigated. Results in the vein of those obtained in the context of Hilbert spaces are stated.

KEYWORDS: numerical range, generalized numerical range, indefinite inner product space.
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1. Introduction and main results.

For $J = I_r \oplus (-I_s)$, consider $\mathbb{C}^{r+s}$ endowed with an indefinite inner product $[\cdot,\cdot]$ defined by $[\xi,\eta] = \langle J\xi,\eta\rangle$. Denote by $U(r,s)$ the group of complex matrices U such that $UJU^* = J$. For $n\times n$ complex matrices A, C, consider the *J-tracial numerical range*

$$W_C^J(A) = \{\mathrm{tr}(CUAU^{-1}) : U \in U(r,s)\}. \tag{1}$$

The theory of numerical ranges on an indefinite inner product space has been investigated by some authors (cf. [4, 5, 6, 1, 2] and the references therein). Our aim is the characterization of (1) for $n = 2$ and $r = s = 1$. By adding appropriate multiples of I to the non-scalar matrices A, C, without loss of generality these matrices may be assumed to have rank one. If C is a rank one operator on $\mathbb{C}^{r+s}$, there exist non-zero vectors η,ζ satisfying

$$C\xi = [\xi,\eta]\zeta, \ \ \forall\ \xi \in \mathbb{C}^{r+s}. \tag{2}$$

Considering an orthonormal basis $\{\xi_1,\ldots,\xi_n\}$ with respect to the definite inner product $\langle\cdot,\cdot\rangle$, we get $\mathrm{tr}(CUAU^{-1}) = [UAU^{-1}\zeta, J\eta]$, and so (1) is written as

$$W_C^J(A) = \{[U^{-1}AU\zeta,\eta] : U \in U(r,s)\}.$$

In the sequel, we consider the case $n = 2$ and $r = s = 1$ and we assume that A is a rank one operator defined by

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$$A\xi = [\xi, \kappa]\tau, \ \ \forall\, \xi \in \mathbb{C}^{r+s}. \tag{3}$$

Simple calculations yield $[U^{-1}AU\zeta, \eta] = [U\zeta, \kappa]\overline{[U\eta, \tau]}$, where U runs over the 3-dimensional simple Lie group $SU(1,1)$, that is, the subgroup of $U(1,1)$ constituted by the matrices with determinant 1 (cf. [3]). So, we are concerned with the description of the set of the complex plane

$$W_C^J(A) = \{[U\zeta, \kappa]\overline{[U\eta, \tau]} : U \in SU(1,1)\}. \tag{4}$$

In this paper we treat the case $\eta, \zeta, \kappa, \tau$ are non-neutral vectors, that is, vectors with nonvanishing norm relatively to the indefinite inner product. The results obtained in this case, or the methods developed in its analysis, will be useful to solve the problem when some of the vectors $\eta, \zeta, \kappa, \tau$ are neutral.

Throughout, we use the notation $a = \mathrm{tr}(A)$, $a' = \mathrm{tr}(AJA^*J)$, $c = \mathrm{tr}(C)$, $c' = \mathrm{tr}(CJC^*J)$.

Our main result is the following.

Theorem 1.1 *Let C, A be 2×2 complex rank one matrices and let $\eta, \zeta, \kappa, \tau$ be non-neutral vectors. The following cases may occur:*

(i) If $c'a' < 0$, then $W_C^J(A) = \mathbb{C}$.

(ii) If $c' > 0$ and $a' > 0$, then $W_C^J(A)$ is a closed connected region limited by one branch of a hyperbola, or, in the case of degeneracy of this conic, a closed half-plane.

For $c' < 0$ and $a' < 0$, $W_C^J(A)$ is given as follows:

(iii) For every $\xi \in \mathbb{C}^2$, if either $[C\xi, C\xi] \geq 0$ and $[A\xi, A\xi] \geq 0$, or if $[C\xi, C\xi] \leq 0$ and $[A\xi, A\xi] \leq 0$, then $W_C^J(A) = \mathbb{C}$;

(iv) For every $\xi \in \mathbb{C}^2$, if either $[C\xi, C\xi] \geq 0$ and $[A\xi, A\xi] \leq 0$, or if $[C\xi, C\xi] \leq 0$ and $[A\xi, A\xi] \geq 0$, then $W_C^J(A)$ is an unbounded closed connected region limited by an ellipse.

Moreover, the above conics are expressed as

$$\frac{(x\cos\theta + y\sin\theta - \lambda_0)^2}{\alpha^2} + \epsilon\frac{(-x\sin\theta + y\cos\theta)^2}{\beta^2} = 1, \tag{5}$$

where $\lambda_0 = (ca)/2$, $\theta = \arg\lambda_0$,

$$\alpha = (a'c')^{1/2}/2 + \epsilon/2[(|c|^2 - c')^{1/2}(|a|^2 - a')^{1/2}], \tag{6}$$

$$\beta = (1/2)[|c'|^{1/2}(|a|^2 - a')^{1/2} + |a'|^{1/2}(|c|^2 - c')^{1/2}], \tag{7}$$

being $\epsilon = -1$ and $\epsilon = 1$ for (ii) and (iv), respectively.

2. Preliminary results.

Proposition 2.1. *Let C, A be non-zero rank one operators given by (2) and (3), respectively. If the vectors $\eta, \zeta, \kappa, \tau$ are non-neutral, then $W_C^J(A)$ is a closed subset of $\mathbb{C}$.*

Proof. For $\alpha, \beta \in \mathbb{C}$, consider the matrix $U \in SU(1,1)$ whose first and second rows are $(\alpha, \bar{\beta})$ and $(\beta, \bar{\alpha})$, respectively. By similarity transformations performed by matrices of $SU(1,1)$, the non-neutral vectors ζ, τ may be replaced by vectors with one vanishing component. So, we may consider that $\zeta = (1,0)^T$ or $\zeta = (0,1)^T$, and $\tau = (1,0)^T$ or $\tau = (0,1)^T$. We show that if $|\alpha| \to \infty$, then $|[U\zeta, \kappa]| \to \infty$. Indeed, since κ is non-neutral, its components (κ_1, κ_2) satisfy $|\kappa_1| \neq |\kappa_2|$. If $\zeta = (1,0)^T$, then $[U\zeta, \kappa] = \overline{\kappa_1}\alpha - \overline{\kappa_2}\beta$. Analogously, if $\zeta = (0,1)^T$, then $[U\zeta, \kappa] = \overline{\beta\kappa_1} - \overline{\alpha\kappa_2}$. In either case, recalling that $|\alpha|^2 - |\beta|^2 = 1$, we conclude that $|[U\zeta, \kappa]| \to \infty$ as $|\alpha| \to \infty$. It may be similarly shown that $|[U\eta, \tau]| = |[\eta, U^{-1}\tau]| \to \infty$ as $|\alpha| \to \infty$.

We prove the closedness of $W_C^J(A)$. Let $z_n = \mathrm{tr}(CU_nAU_n^{-1})$, $U_n \in U(1,1)$, $n \in \mathbb{N}$, be an arbitrary sequence of points in $W_C^J(A)$ converging to a (finite) complex number $z_0 = \mathrm{tr}(CU_0AU_0^{-1})$. Let $U_0 = \lim_{n\to+\infty} U_n$. By the first part of the proof we may conclude that $U_0 \in U(1,1)$, and so the closedness follows. $\square$

By replacing the inner product $[\xi_1, \xi_2]$ by $-[\xi_1, \xi_2]$, we may consider that either (i) $\zeta = \tau = (1,0)^T$ or (ii) $\zeta = (0,1)^T$, $\tau = (1,0)^T$ holds. Substituting C by $U\,C\,U^{-1}$ for some $U = \mathrm{diag}(1, \exp(i\,\theta))$, $\theta \in \mathbb{R}$, we may assume that the components of $\eta = (\eta_1, \eta_2)$ have a real ratio. Similarly, the components of $\kappa = (\kappa_1, \kappa_2)$ may be also assumed to have a real ratio. The following possibilities may occur:

(i-1) $\zeta = \tau = (1,0)^T$, $|\eta_1| > |\eta_2|$, $|\kappa_1| > |\kappa_2|$,
(i-2) $\zeta = \tau = (1,0)^T$, $|\eta_1| > |\eta_2|$, $|\kappa_2| > |\kappa_1|$,
(i-3) $\zeta = \tau = (1,0)^T$, $|\eta_2| > |\eta_1|$, $|\kappa_1| > |\kappa_2|$,
(i-4) $\zeta = \tau = (1,0)^T$, $|\eta_2| > |\eta_1|$, $|\kappa_2| > |\kappa_1|$,
(ii-1) $\zeta = (0,1)^T$, $\tau = (1,0)^T$,$|\eta_1| > |\eta_2|$, $|\kappa_2| > |\kappa_1|$,
(ii-2) $\zeta = (0,1)^T$, $\tau = (1,0)^T$,$|\eta_1| > |\eta_2|$, $|\kappa_1| > |\kappa_2|$,
(ii-3) $\zeta = (0,1)^T$, $\tau = (1,0)^T$, $|\eta_2| > |\eta_1|$, $|\kappa_2| > |\kappa_1|$,
(ii-4) $\zeta = (0,1)^T$, $\tau = (1,0)^T$, $|\eta_2| > |\eta_1|$, $|\kappa_1| > |\kappa_2|$.

Proposition 2.2 treats the case ii-1, since we may assume that $\eta_1 = 1$, $-1 < q = \eta_2 < 1$, $\kappa_2 = -1$, $-1 < k = \kappa_1 < 1$.

Proposition 2.2 *Let A and C be 2×2 rank one complex matrices such that $c' < 0$ and $a' < 0$. If $[A\xi, A\xi] \geq 0$ and $[C\xi, C\xi] \leq 0$ for every $\xi \in \mathbb{C}^2$, then $W_C^J(A)$ is an unbounded closed connected region limited by an ellipse with center $(ac)/2$, whose major axis is contained in the line $z = (ac)t$, $t \in \mathbb{R}$, being the length of the semi-major axis α and of the semi-minor axis β given, respectively, by (6) with $\epsilon = 1$ and (7).*

Proposition 2.3 treats the case i-1. In this case, we may assume that $\eta_1 = 1$, $-1 < q = \eta_2 < 1$, $\kappa_1 = 1$, $-1 < k = -\kappa_2 < 1$. Moreover, we may take $0 \leq k, q < 1$.

Proposition 2.3 *Let A and C be 2×2 rank one complex matrices such that $a' > 0$ and $c' > 0$. If $[A\xi, A\xi] \geq 0$ and $[C\xi, C\xi] \geq 0$ for every $\xi \in \mathbb{C}^2$, then $W_C^J(A)$ is an unbounded closed connected region limited by one branch of the hyperbola with center $(ac)/2$ whose principal axis is contained in the line $z = (ac)t, t \in \mathbb{R}$, being the length of the semi-principal axis α and of the semi-transverse axis β given, respectively, by (6) with $\epsilon = -1$ and (7).*

Moreover, if $\alpha > 0$ then $(ac)/2 \notin W_C^J(A)$ and $(ac) \in W_C^J(A)$. On the other hand, if $\alpha \leq 0$ then $(ac)/2, (ac) \in W_C^J(A)$. If $\alpha = 0$, then

$$W_C^J(A) = \{z \in \mathbb{C} : \Re([z - (ac)/2]\overline{ac}) \geq 0\}.$$

Proposition 2.4 deals with the case ii-4. In this case, we may consider $\eta_2 = 1$, $-1 < q = \eta_1 < 1$, $\kappa_1 = 1$, $-1 < k = -\kappa_2 < 1$.

Proposition 2.4 *Let A and C be 2×2 rank one complex matrices such that $c' > 0$ and $a' > 0$. If $[A\xi, A\xi] \geq 0$ and $[C\xi, C\xi] \leq 0$ for every $\xi \in \mathbb{C}^2$, then $W_C^J(A)$ is an unbounded closed connected region limited by one branch of the hyperbola centered at $(ac)/2$, whose principal axis is contained in the line $z = (ac)t, t \in \mathbb{R}$, being the length of the semi-principal axis α and of the semi-transverse axis β given, respectively, by (6) with $\epsilon = -1$ and* (7).

Moreover, if $\alpha > 0$ then $(ac)/2 \notin W_C^J(A)$ and $(ac) \in W_C^J(A)$. If $\alpha < 0$ then $(ac)/2, (ac) \in W_C^J(A)$. If $\alpha = 0$, then $W_C^J(A) = \{z \in \mathbb{C} : \Re([z - (ac)/2]\overline{ac}) \geq 0\}$.

If one of the conditions (i-2), (i-3), (ii-2), (ii-3) holds, the condition (a) of Proposition 2.5 holds. If the condition (i-4) is satisfyed, then the condition (b) holds. Under these conditions, $W_C^J(A)$ is the whole complex plane.

Proposition 2.5 *Let A and C be 2×2 complex matrices with rank one. If either (a) $(a'c') < 0$ or (b) $a' < 0$, $c' < 0$ and $[A\xi, A\xi] \geq 0$, $[C\xi, C\xi] \geq 0$ for every $\xi \in \mathbb{C}^2$, then $W_C^J(A) = \mathbb{C}$.*

3. Proof of Proposition 2.2.

Under the hypothesis of Proposition 2.2, we have

$$C = \begin{pmatrix} 0 & 0 \\ 1 & -q \end{pmatrix}, \quad A = \begin{pmatrix} k & 1 \\ 0 & 0 \end{pmatrix}, \quad 0 \leq q, k < 1$$

and

$$W_C^J(A) = \{(\alpha + q\,\beta)(\alpha + k\overline{\beta}) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\} \tag{8}$$

$$= \{(\cosh t \exp(i\theta) + k \sinh t)(\cosh t \exp(i\phi) + q \sinh t) : 0 \leq t < +\infty, 0 \leq \theta, \phi \leq 2\pi\}.$$

For fixed t, θ, the set of points $(\cosh t \exp(i\theta) + k \sinh t)(\cosh t \exp(i\phi) + q \sinh t)$, $0 \leq \phi \leq 2\pi$, is a circle with center

$$z(t, \theta) = q \sinh t\,(\cosh t \exp(i\theta) + k \sinh t) \tag{9}$$

and radius $r(t, \theta)$ satisfying

$$R = r(t, \theta)^2 = \cosh^2 t\,(k^2 \sinh^2 t + 2k \sinh t \cosh t \cos\theta + \cosh^2 t). \tag{10}$$

For $x_0 = \Re(z(t, \theta))$ and $y_0 = \Im(z(t, \theta))$, we easily obtain

$$x_0 = k\,q \sinh^2 t + q \sinh t \cosh t \cos\theta, \; y_0 = q \sinh t \cosh t \sin\theta. \tag{11}$$

If $q = 0$, then $z(t, \theta) = 0$. The continuous function $r(t, \theta)$ defined on $[0, +\infty) \times [-\pi, +\pi]$ ranges over $[m_0, +\infty)$, m_0 being the minimum of the function $r(t, -\pi) = \frac{1}{2}(1 + \cosh(2t) - |k| \sinh(2t))$. The minimum of $r(t, -\pi)$, $t \geq 0$, is $(1/2)(1 + \sqrt{1 - k^2})$, because the minimum of $f(t) = \cosh t - |k| \sinh t$, $t \geq 0$, is attained at $0 < t_0 = \operatorname{arc\,tanh}(|k|)$, and equals $\sqrt{1 - k^2}$. Thus, Proposition 2.2 follows for $q = 0$.

Now, we assume $q \neq 0$ and, by changing the roles of k and q, we also assume that $k \neq 0$.

Proposition 3.1 *For $-1 < k, q < 1$, $kq \neq 0$, let*

$$C = \begin{pmatrix} 0 & 0 \\ 1 & -q \end{pmatrix}, \quad A = \begin{pmatrix} k & 1 \\ 0 & 0 \end{pmatrix}.$$

Then:

i) The set $W_C^J(A)$ is connected, symmetric with respect to the real axis and also symmetric to the line $\Re(z) = -k\,q/2$.

ii) The inclusion relation $W_C^J(A) \subset \{z \in \mathbb{C} : |z + k\,q/2| \geq \sqrt{1-k^2}/2\}$ holds.

iii) $\partial W_C^J(A)$ has no isolated points.

Proof. We *claim* that the set described by $z(t,\theta)$ in (9), $t \geq 0, 0 \leq \theta \leq 2\pi$, is the complex plane. For $x_0 = \Re(z(t,\theta))$, $y_0 = \Im(z(t,\theta))$ there exist $t \in [0, +\infty)$ and $\theta \in [0, 2\pi]$ such that $x_0 = kq \sinh^2 t + q \sinh t \cosh t \cos\theta$ and $y_0 = q \sinh t \cosh t \sin\theta$. Indeed, consider the polynomial function in $\sinh t$ given by $f(t) = (x_0 - kq\sinh^2 t)^2 + y_0^2 - q^2 \sinh^2 t \cosh^2 t$. Since $f(0) \geq 0$ and $\lim_{t\to\infty} f(t) = -\infty$, a value of t may be found for which $f(t) = 0$. Then, an appropriate value of θ is easily determined and the *claim* follows.

Taking into account (8), (9), (10) and (11), $W_C^J(A)$ may be expressed as

$$W_C^J(A) = \{(x_0 + i\,y_0) + \sqrt{R(x_0,y_0)}\exp(i\theta) : (x_0,y_0) \in \mathbb{R}^2, 0 \leq \theta \leq 2\pi\}. \tag{12}$$

We *claim* that $R = R(x_0, y_0)$ is given by

$$R = \frac{1}{2q^4}(q^4 + 2k\,q^3\,x_0 + 2q^2x_0^2 + 2q^2y_0^2 + \tag{13}$$

$$|q^3|\sqrt{4(x_0 + k\,q/2)^2 + 4(1-k^2)y_0^2 + (1-k^2)q^2}).$$

Indeed, eliminating t and θ in (8), (9), (10) we get

$$H(x_0, y_0, r) = q^4\,R^2 + (-q^4 - 2k\,q^3\,x_0 - 2q^2\,x_0^2 - 2q^2\,y_0^2)\,R$$

$$+(k^2q^2x_0^2 + 2k\,q\,x_0^3 + x_0^4 + k^2\,q^2\,y_0^2 + 2k\,q\,x_0\,y_0^2 + 2x_0^2\,y_0^2 + y_0^4) = 0.$$

Setting $\tilde{R} = \frac{1}{2q^4}(q^4 + 2k\,q^3\,x_0 + 2q^2x_0^2 + 2q^2y_0^2)$, we find

$$R = \tilde{R} \pm \frac{1}{2q^4}|q^3|\sqrt{4(x_0 + k\,q/2)^2 + 4(1-k^2)y_0^2 + (1-k^2)q^2}.$$

Since $\tilde{R} = \frac{1}{q^2}((x_0 + k\,q/2)^2 + y_0^2) + \frac{2-k^2}{4}$, performing some calculations we get the relation

$$R - \tilde{R} \geq \frac{1}{2}(\cosh(2t) - \sinh(2t)) > 0,$$

and the *claim* follows.

The connectedness of $W_C^J(A)$ and the symmetries of $W_C^J(A)$ with respect to $\Im(z) = 0$ and to $\Re(z) = -k\,q/2$ follow from (12) taking into account that R is invariant under the two transformations $y_0 \to -y_0$ and $x_0 + kq/2 \to -x_0 - kq/2$.

ii) We have

$$\begin{aligned} &4q^4\,[R(x_0, y_0) - ((x_0 + k\,q/2)^2 + y_0^2)] \\ &\quad \geq q^4\,(2 - k^2) + 4q^2\,(1 - q^2)((x_0 + k\,q/2)^2 + 4q^2\,(1 - q^2)\,y_0^2) \end{aligned}$$

and the right hand side approaches $+\infty$ as $(x_0 + k\,q/2)^2 + y_0^2 \to +\infty$. Moreover,

$$R(x_0, y_0) - \{(x_0 + k\,q/2)^2 + y_0^2\} \geq \frac{1 - k^2}{4} + \sqrt{1 - k^2}\sqrt{(x_0 + k\,q/2)^2 + y_0^2}.$$

Hence

$$r(x_0, y_0) \geq \sqrt{(x_0 + k\,q/2)^2 + y_0^2} + \sqrt{1 - k^2}/2,$$

and so ii) follows.

iii) By Tarski-Seidenberg's theorem, $\partial W_C^J(A)$ is a semi-algebraic set and lies on an algebraic curve. It consists of some arcs and isolated points. By Proposition 2.1, the set $W_C^J(A)$ is closed in $\mathbb{C}$. If z is an isolated point of $\partial W_C^J(A)$, then z is also an isolated point of $W_C^J(A)$, contradicting the connectedness of $W_C^J(A)$. Thus, $\partial W_C^J(A)$ has no isolated points. $\square$

Next, we determine $\partial W_C^J(A)$. Since it is the image of the algebraic set $SU(1,1)$ under the quadratic map $U \mapsto [U\zeta, \kappa]\overline{[U\eta, \tau]}$, for $\eta, \zeta, \kappa, \tau$ non-neutral vectors, by Tarski-Seidenberg's theorem we may conclude that $\partial W_C^J(A)$ lies on an algebraic curve. By Proposition 3.1 iii), $\partial W_C^J(A)$ has no isolated points. The number of non-smooth boundary points of $W_C^J(A)$ is finite. Every non-smooth boundary point is the limit of a sequence of smooth boundary points of $W_C^J(A)$. Suppose that Z_0 is a smooth point of $\partial W_C^J(A)$ at which the tangent of $\partial W_C^J(A)$ is given by $\Re(z\,\exp(-i\,\theta_0)) = \Re(Z_0\,\exp(-i\,\theta_0))$ for some $\theta_0 \in \mathbb{R}$. We assume that $Z_0 + t\,\exp(i\,(\theta_0)) \notin W_C^J(A)$ for $0 < t < \epsilon$, where ϵ is a sufficiently small positive number. By Proposition 3.1 iii), there exists a complex number $x_0 + i\, y_0$ for which either i) or ii) holds:

(i) $Z_0 = x_0 + i\, y_0 + r(x_0, y_0)\exp(i\,(\theta_0))$ and $\left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}\right)\Big|_{(x_0,y_0)} = -(\cos(\theta_0), \sin(\theta_0))$,

(ii) $Z_0 = x_0 + i\, y_0 - r(x_0, y_0)\exp(i\,(\theta_0))$ and $\left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}\right)\Big|_{(x_0,y_0)} = (\cos(\theta_0), \sin(\theta_0))$.

In either case, we have

$$(\mathrm{grad}\, r)^2\big|_{(x_0,y_0)} = \left(\frac{\partial r}{\partial x}\right)^2\Big|_{(x_0,y_0)} + \left(\frac{\partial r}{\partial y}\right)^2\Big|_{(x_0,y_0)} = 1. \tag{14}$$

We determine the set of points $x_0 + i\, y_0$ for which (14) holds. By the implicit function theorem, we get $\frac{\partial r}{\partial x} = -\frac{(\partial H/\partial x)}{(\partial H/\partial r)}$, $\frac{\partial r}{\partial y} = -\frac{(\partial H/\partial y)}{(\partial H/\partial r)}$. If (x_0, y_0) satisfies the condition (14), then it verifies

$$L(x, y, r) = (\partial H/\partial x)^2 + (\partial H/\partial y)^2 - (\partial H/\partial r)^2 = 0.$$

Eliminating r from the equations $H(x_0, y_0, r) = 0$ and $L(x_0, y_0, r) = 0$, and setting $M(x_0, y_0) = 4(1-q^2)x_0^2 + 4k\,q(1-q^2)x_0 + 4(1-k^2)(1-q^2)\,y_0^2 + k^2\,q^2 - q^4$ we obtain

$$(x_0^2 + y_0^2)(x_0^2 + 2k\,q\,x_0 + k^2\,q^2 + y_0^2) \times M(x_0, y_0)^2 = 0.$$

Since $\partial W_C^J(A)$ has no isolated points, we may disregard the two candidates $(x_0, y_0) = (0, 0)$ and $(x_0, y_0) = (k\,q, 0)$. So, we concentrate on the remaining candidate, the ellipse $M(x_0, y_0) = 0$. We examine the gradient of the function $r(x_0, y_0)$ on this ellipse. By the implicit function theorem, we conclude that $\partial W_C^J(A)$ is contained in

$$\Upsilon = \{x + i\, y = \left(x_0 - \frac{(2x_0 + k\,q)(x_0^2 + y_0^2 + k\,q\,x_0 - q^2\,R)}{2q^2\,(2x_0^2 + 2k\,q\,x_0 + 2y_0^2 - 2q^2\,R + q^2)}\right)$$
$$+ i\left(y_0 - \frac{y_0(2x_0^2 + 2k\,q\,x_0 + 2y_0^2 - 2q^2\,R + k^2\,q^2)}{q^2(2x_0^2 + 2k\,qx_0 + 2y_0^2 - 2q^2\,R + q^2)}\right) : (x_0, y_0) \in \mathbb{R}^2, M(x_0, y_0) = 0\}.$$

The ellipse $M(x_0, y_0) = 0$ has the parametric representation

$$x_0 = -\frac{k\,q}{2} + \frac{\sqrt{1-k^2}q^2\cos\phi}{2\sqrt{1-q^2}},\ \ y_0 = q^2\frac{\sin\phi}{2\sqrt{1-q^2}},\ \ \phi \in [0, 2\pi]$$

and $R = \frac{2-k^2-q^2+2\sqrt{1-k^2}\sqrt{1-q^2}+k^2 q^2 \sin^2\phi}{4(1-q^2)}$. By direct computations, it may be seen that $(x, y) \in \Upsilon$ satisfies the equation

$$\frac{(x + k\,q/2)^2}{[(1/2)(1+\sqrt{1-k^2}\sqrt{1-q^2})]^2} + \frac{y^2}{[(1/2)(\sqrt{1-k^2}+\sqrt{1-q^2})]^2} = 1. \quad (15)$$

By Proposition 3.1, $\partial W_C^J(A)$ is non-empty and every continuous arc in $\mathbb{C}$ joining the point $k\,q/2$ and a sufficiently large number L has a common point with $\partial W_C^J(A)$. Thus, $\partial W_C^J(A)$ is given by (15). Since $L \in W_C^J(A)$ and $k\,q/2 \notin W_C^J(A)$, then $W_C^J(A)$ is the unbounded set limited by the above ellipse and Proposition 2.2 follows. $\square$

4. Proof of Proposition 2.3.

By the hypothesis, we get

$$C = \begin{pmatrix} 1 & -q \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & k \\ 0 & 0 \end{pmatrix}, \quad -1 < q, k < 1,$$

and so

$$W_C^J(A) = \{(\alpha + k\beta)(\overline{\alpha} + q\beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\} \quad (16)$$

$$= \{(\cosh t + k\,\sinh t\,\exp(-i\,\theta))(\cosh t + q\,\sinh t\,\exp(i\,\psi)) : \\ 0 \le t < \infty, 0 \le \theta, \psi \le 2\pi\}.$$

Thus, $W_C^J(A)$ is expressed as the union of the family of circles with center

$$z = x + i\,y = \cosh^2 t + k\,\sinh t\,\cosh t \exp(-i\,\theta), \quad (17)$$

and radius r such that

$$R = r^2 = q^2\,\sinh^2 t\,(\cosh^2 t + k^2 \sinh^2 t + 2k \sinh t \cosh t \cos\theta). \quad (18)$$

The functions $x = x(t, \theta)$, $y = y(t, \theta)$, $r = r(t, \theta)$ satisfy the relation

$$H(x, y, r) = R^2 - 2q^2\,(x^2 - x + y^2)R - k^2\,q^2\,R + \\ +q^4(x^4 - 2x^3 + x^2 + y^4 + 2x^2y^2 - 2x\,y^2 + y^2) = 0. \quad (19)$$

We prove that the locus of the points $z = x(t,\theta) + i\,y(t,\theta)$, $t \geq 0$, $0 \leq \theta \leq 2\pi$, is given by

$$\{(x,y) \in \mathbb{R}^2 : x > 1/2,\ \frac{(x-(1/2))^2}{[(1/2)(\sqrt{1-k^2})]^2} - \frac{y^2}{(k/2)^2} \geq 1\}. \tag{20}$$

This reduces to show that

$$\begin{aligned} &\{\cosh t + k\ \sinh t \exp(-i\theta) : 0 \leq t < \infty, 0 \leq \theta \leq 2\pi\} \\ &= \{(x,y) \in \mathbb{R}^2 : \frac{x^2}{1-k^2} - \frac{y^2}{k^2} \geq 1\}, \end{aligned} \tag{21}$$

$-1 < k < 1, k \neq 0$. We consider the map from the cylinder $[0,+\infty) \times \mathbb{T}^1$ into $\mathbb{C} \cong \mathbb{R}^2$ defined by $z(t,\theta) = x + i\,y = \cosh t + k \sinh t \cos\theta - i \sinh t \sin\theta$. The set

$$\Gamma = \{\cosh t + k\ \sinh t \exp(-i\theta) : 0 \leq t < +\infty, 0 \leq \theta \leq 2\pi\} \tag{22}$$

is closed in $\mathbb{C}$ and its boundary has no isolated points. Its boundary points are necessarily critical values of the differential map $z = z(t,\theta)$. The Jacobian of the map is given by $\frac{\partial(x,y)}{\partial(t,\theta)} = \frac{\partial x}{\partial t}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial t} = -k \sinh t\,(k \cosh t + \sinh t \cos\theta)$. Thus, the critical points (t,θ) satisfy $k\cosh t + \sinh t \cos\theta = 0$ or $t = 0$. The critical points $(0,\theta)$ correspond to the critical value 1. Now, we determine the critical values corresponding to $k \cosh t + \sinh t \cos\theta = 0$. Substituting $x = \cosh t + k \sinh t \cos\theta$, $y = -\sinh t \sin\theta$ in the polynomial $F(x,y) = \frac{x^2}{1-k^2} - \frac{y^2}{k^2} - 1$, we get

$$F(x,y) = \frac{(k\cosh t + k \sinh t \cos\theta)^2}{1-k^2} \geq 0.$$

Having in mind that $\Re(\cosh t + k\ \sinh t \exp(-i\theta)) = \cosh t + k \sinh t \cos\theta > 0$, we conclude that

$$\partial\Gamma \subset \{(x,y) \in \mathbb{R}^2 : x > 0, \frac{x^2}{1-k^2} - \frac{y^2}{k^2} = 1\}$$

and

$$\Gamma \subset \{(x,y) \in \mathbb{R}^2 : x > 0, \frac{x^2}{1-k^2} - \frac{y^2}{k^2} \geq 1\}.$$

Since 1 is a critical value not belonging to the hyperbola $x^2/(1-k^2) - y^2/k^2 = 1$, the connectedness of (22) implies (21).

Following the proof of Proposition 2.2, we consider the polynomial

$$L(x,y,r) = [(\partial H/\partial x)^2 + (\partial H/\partial y)^2 - (\partial H/\partial r)^2]/4.$$

By eliminating r from the equations $H(x,y,r) = 0$ and $L(x,y,r) = 0$, we obtain

$$(x^2+y^2)((x-1)^2+y^2)\{4(1-q^2)\,(x-1/2)^2/(1-k^2) - 4(1-q^2)y^2/k^2 - 1\} = 0.$$

It can be easily seen that the right branch of the hyperbola

$$4(1-q^2)\,(x-1/2)^2/(1-k^2) - 4(1-q^2)y^2/k^2 = 1$$

is contained in the interior of (20), while its left branch lies on the exterior of the set (20).

We may disregard the two points $(x,y) = (0,0)$ and $(x,y) = (1,0)$. The equation $H(x,y,r) = 0$ has two solutions in $R = r^2$, say $R_j(x_0,y_0)$, $j = 1,2$, $R_1 > R_2$. We evaluate the envelopes of the two families of circles $(x-x_0)^2 + (y-y_0)^2 = R_j(x_0,y_0)$, where (x_0,y_0) satisfy

$$4(1-q^2)\,(x_0-1/2)^2/(1-k^2) - 4(1-q^2)y_0^2/k^2 - 1 = 0, \;\; x_0 > 1/2.$$

We assume that

$$\sqrt{1-k^2}\sqrt{1-q^2} - k\,q \neq 0, \tag{23}$$

$$k\sqrt{1-q^2} - q\,\sqrt{1-k^2} \neq 0, \tag{24}$$

or equivalently, $q \neq k$, $q \neq \sqrt{1-k^2}$. In the case $\sqrt{1-k^2}\sqrt{1-q^2} - k\,q > 0$, the envelope corresponding to the bigger radius lies on the right branch of the hyperbola

$$\frac{(x-1/2)^2}{[(1/2)(\sqrt{1-k^2}\sqrt{1-q^2} - k\,q)]^2} - \frac{y^2}{[(1/2)(k\sqrt{1-q^2} + q\,\sqrt{1-k^2})]^2} = 1. \tag{25}$$

In the case $\sqrt{1-k^2}\sqrt{1-q^2} - k\,q < 0$, the envelope lies on the left branch of the hyperbola. The envelope corresponding to the smaller radius lies on the right branch of the hyperbola

$$\frac{(x-1/2)^2}{[(1/2)(\sqrt{1-k^2}\sqrt{1-q^2} + k\,q)]^2} - \frac{y^2}{[(1/2)(k\sqrt{1-q^2} - q\,\sqrt{1-k^2})]^2} = 1. \tag{26}$$

In the case $q = \sqrt{1-k^2}$, the hyperbola (25) reduces to the straight line $x = 1/2$, being the hyperbola (26) replaced by the straight line $y = 0$. To conclude the proof, we notice that

$$W_C^J(A) \subset \{r\exp(i\phi) : 0 \le r < \infty, -\pi < \phi < \pi, |\phi| \le \arcsin(k) + \arcsin(q)\}.$$

This is a consequence of (16) and of the set inclusion

$$\begin{aligned}&\{(\cosh t + k \sinh t \exp(-i\,\theta) : 0 \le t < \infty, 0 \le \theta \le 2\pi\} \\ &\quad \subset \{r + k\,z : 0 \le r < +\infty, |z| \le r\} \\ &\quad = \{r\exp(i\phi) : 0 \le r < \infty, -\pi/2 < \phi < \pi/2, |\sin\phi| \le k\}.\end{aligned}$$

Thus, $-1 \notin W_C^J(A)$. Since $1 \in W_C^J(A)$, Proposition 2.3 follows. □

5. Proof of Proposition 2.4.

Under the conditions of Proposition 2.4, we have

$$W_C^J(A) = \{(\overline{\beta} + k\overline{\alpha})(\beta + q\overline{\alpha}) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\}. \tag{27}$$

If $q = 0$ and $k \ne 0$, recalling Proposition 2.3 it may be easily seen that

$$\begin{aligned}W_C^J(A) &= \{\sinh^2 t + k \sinh t \cosh t \exp(i\phi) : 0 \le t < \infty, 0 \le \phi \le 2\pi\} \\ &= \{(x,y) \in \mathbb{R}^2 : x > -1/2, \frac{(x+1/2)^2}{[(1/2)\sqrt{1-k^2}]^2} - \frac{y^2}{(k/2)^2} \ge 1\}.\end{aligned}$$

In the case $k = 0$, $q \ne 0$, a similar result holds. So, we may assume that $q \ne 0$, $k \ne 0$. In this case, $W_C^J(A)$ is the union of the family of circles with center $z(t,\phi) = \cosh^2 t + k \sinh t \cosh t \exp(i\phi) - 1$, and radius r such that $R(t,\phi) = r^2 = q^2 \cosh^2 t\,(\sinh^2 t + k^2 \cosh^2 t + 2k \sinh t \cosh t \cos\phi)$. The following relation holds

$$\begin{aligned}H(x,y,r) &= R^2 - 2q^2(x^2 + x + y^2)R - k^2\,q^2\,R \\ &+q^4\,(x^4 + 2x^3 + x^2 + y^4 + 2x^2\,y^2 + 2x\,y^2 + y^2) = 0.\end{aligned} \tag{28}$$

Using the new variable $x_1 = x + (1/2)$, this polynomial is written

$$\begin{aligned}&R^2 - 2q^2(x_1^2 + y^2 - (1/4))R - k^2\,q^2\,R \\ &q^4\,(x_1^4 - (1/2)x_1^2 + y^4 + 2x_1^2\,y^2 + (1/2)y^2 + (1/16)),\end{aligned} \tag{29}$$

and it can be easily seen that the centers $z(t,\phi)$ describe the set

$$\{(x,y)\in\mathbb{R}^2 : x_1>0, \frac{x_1^2}{[(1/2)\sqrt{1-k^2}]^2}-\frac{y^2}{(k/2)^2}\geq 1\}. \tag{30}$$

The polynomial H in (29), expressed in the new variable $\hat{x}=x-(1/2)$, coincides with (20). Moreover, since the sets (20) and (30) coincide, the result follows as in Proposition 2.3. $\square$

6. Proof of Proposition 2.5.

In the case (i-2), we may assume that $\eta=1$, $-1<q=\eta_2<1$, $\kappa_2=-1$, $-1<k=\kappa_1<1$. Under these assumptions, we have

$$W_C^J(A)=\{(\beta+k\alpha)(\overline{\alpha}+q\beta):\alpha,\beta\in\mathbb{C},|\alpha|^2-|\beta|^2=1\}. \tag{31}$$

In the case (i-3), we may assume that $\eta_2=1$, $-1<q=\eta_1<1$, $\kappa_1=1$, $-1<k=-\kappa_2<1$, and so

$$W_C^J(A)=\{(\alpha+k\beta)(\beta+q\overline{\alpha}):\alpha,\beta\in\mathbb{C},|\alpha|^2-|\beta|^2=1\}. \tag{32}$$

In the case (i-4), we may assume that $\eta_2=1$, $-1<q=\eta_1<1$, $\kappa_2=-1$, $-1<k=\kappa_1<1$, and we get

$$W_C^J(A)=\{(\beta+k\alpha)(\beta+q\overline{\alpha}):\alpha,\beta\in\mathbb{C},|\alpha|^2-|\beta|^2=1\}. \tag{33}$$

By (31), (32), we observe that the cases (i-2), (i-3) are identified under the exchange of the roles of k and α by q and $\overline{\alpha}$, respectively.

We prove the proposition in the case (i-4). If $q=0$, then

$$W_C^J(A)=\{\sinh^2 t\,\exp(i\,\theta)+k\,\sinh t\,\cosh t\,\exp(i\,\phi):0\leq t<\infty,0\leq\theta,\phi\leq 2\pi\}.$$

This set is invariant under the multiplication $z\mapsto z\exp(i\,\psi)$. The continuous function $f:\mathbb{R}\to[0,+\infty)$ defined by $f(t)=\sinh^2 t+k\,\sinh t\,\cosh t$, satisfies $f(0)=0$ and $\lim_{t\to+\infty}f(t)=+\infty$. By the intermediate value theorem, for every positive real number r, there exists $t\in[0,+\infty)$ such that $f(t)=r$. Hence, $W_C^J(A)=\mathbb{C}$.

Let $-1<k,q<1$ and $k\neq 0$, $q\neq 0$. Then

$$W_C^J(A) = \{(\sinh t \exp(i\,\theta) + k\cosh t)(\sinh t \exp(i\,\phi) + q\cosh t) : \\ 0 \le t < +\infty, 0 \le \theta, \phi \le 2\pi\}.$$

Thus, $W_C^J(A)$ is the union of the family of circles centered at

$$z = x + i\,y = q\,k\,\cosh^2 t + q\,\sinh t\,\cosh t \exp(i\,\theta), \tag{34}$$

and with radius r such that

$$R = r^2 = \sinh^2 t\,(\sinh^2 t + k^2\cosh^2 t + 2k\sinh t\cosh t\cos\theta). \tag{35}$$

The functions $x = x(t,\theta)$, $y = y(t,\theta)$, $r = r(t,\theta)$ satisfy the relation

$$H(x,y,r) = q^4\,r^4 + q^2(-2x^2 - 2y^2 + 2k\,q\,x - q^2)r^2 + (x^2+y^2)(x^2+y^2-2k\,q\,x+k^2\,q^2) = 0. \tag{36}$$

The points $z = x(t,\theta) + i\,y(t,\theta)$, $t \ge 0$, $0 \le \theta \le 2\pi$, form the family of circles with center at $q\,k\,\cosh^2 t$ and radius $q\,\sinh t\,\cosh t$. Since

$$\{k\,\cosh^2 t + \sinh t\,\cosh t \exp(i\,\theta) : 0 \le t < +\infty, 0 \le \theta \le 2\pi\}$$

$$= \{k/2 + k/2\,\cosh(2t) + (1/2)\sinh(2t)\exp(i\,\theta) : 0 \le t < +\infty, 0 \le \theta \le 2\pi\} = \mathbb{C},$$

the points $z(t,\theta)$ describe the whole complex plane. By eliminating R from the equations $L(x,y,r) = [(\partial H/\partial x)^2 + (\partial H/\partial y)^2 - (\partial H/\partial r)^2]/4 = 0$ and $L(x,y,r) = 0$, and setting $E = \{4(1-q^2)\,x^2 + 4(1-k^2)(1-q^2)y^2 - 4k\,q(1-q^2)\,x + k^2\,q^2 - q^4\}$, we obtain

$$K(x,y) = (x^2+y^2)((x-k\,q)^2+y^2)E = 0. \tag{37}$$

We may disregard the two points $(x,y) = (0,0)$ and $(x,y) = (k\,q,0)$. Rewriting the ellipse $E = 0$ in (37) using the new variable $x_1 = x - k\,q/2$, the ellipse is parametrized as

$$x_1 = \frac{q^2\sqrt{1-k^2}}{2\sqrt{1-q^2}}\cos(\phi),\;\; y = \frac{q^2}{2\sqrt{1-q^2}}\sin(\phi),\;\; 0 \le \phi \le 2\pi. \tag{38}$$

On this ellipse the solutions of the equation $H(x,y,r) = 0$, are

$$r_1^2 = \frac{1}{8(1-q^2)}\{(2-k^2)(2-q^2) + 4\sqrt{(1-k^2)(1-q^2)} - k^2\,q^2\cos(2\phi)\},$$

$$r_2^2 = \frac{1}{8(1-q^2)}\{(2-k^2)(2-q^2) - 4\sqrt{(1-k^2)(1-q^2)} - k^2\,q^2\cos(2\phi)\}.$$

The boundary of $W_C^J(A)$ is contained in the curves with parametric equations

$$(X_1(\phi), Y_1(\phi)) = (x_1 - \frac{\partial r_1}{\partial x_1}, y - \frac{\partial r_1}{\partial y}),\ \ (X_2(\phi), Y_2(\phi)) = (x_1 - \frac{\partial r_2}{\partial x_1}, y - \frac{\partial r_2}{\partial y}).$$

We show that these curves do not pass through the origin $(x_1, y) = (0, 0)$. Indeed, suppose that $(x_1 - \frac{\partial r_j}{\partial x_1}, y - \frac{\partial r_j}{\partial y}) = (0, 0),\ \ j = 1, 2$. By the implicit function theorem, we get

$$-\frac{\partial r_j}{\partial x_1} = -\frac{x_1(k^2\,q^2 + 4q^2\,R_j - 4x_1^2 - 4y^2)}{q^2 r_j(-2q^2 + k^2\,q^2 + 4q^2\,r_j - 4x_1^2 - 4y^2)},$$

$$-\frac{\partial r_j}{\partial y} = -\frac{y(-k^2 q^2 + 4q^2 R_j - 4x_1^2 - 4y^2)}{q^2 r_j(-2q^2 + k^2\,q^2 + 4q^2\,r_j - 4x_1^2 - 4y^2)}.$$

Since these partial derivatives are proportional to x_1 and y, respectively, we necessarily have

$$k^2\,q^2 + 4q^2\,r_j^2 - 4x_1^2 - 4y^2 = -k^2 q^2 + 4q^2 r_j^2 - 4x_1^2 - 4y^2.$$

This is impossible, since $k \neq 0$ and $q \neq 0$. Therefore, there exist positive numbers $0 < \epsilon < L_0$ for which

$$\partial W_C^J(A) \subset \{x + i\,y : \epsilon^2 \leq (x - k\,q/2)^2 + y^2 \leq L_0^2\}. \tag{39}$$

In order to show that $W_C^J(A) = \mathbb{C}$, we consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = (\sinh t + k\cosh t)(\sinh t + q\cosh t)$. This function satisfies $g(t) \in W_C^J(A)$ for every $t \in \mathbb{R}$ and $g(0) = k\,q$. Since $\sinh t_1 + k\cosh t_1 = 0$ for some $t_1 \in \mathbb{R}$, we have $g(t_1) = 0$ for certain $t_1 \in \mathbb{R}$. The intermediate value theorem implies that $g(t_2) = (k\,q)/2$ for some $t_2 \in \mathbb{R}$. If $t \to +\infty$, then $g(t) \to +\infty$. Then (38) implies that $W_C^J(A) = \mathbb{C}$.

Next, we prove the proposition in the case (i-2). If $q = 0$, we clearly have

$$W_C^J(A) = \{k\cosh^2 t + \sinh t\cosh t\ \exp(i\theta) : 0 \leq t < \infty, 0 \leq \theta \leq 2\pi\} = \mathbb{C}.$$

If $q \neq 0$, $W_C^J(A)$ is the union of the family of circles with center $z = x + i\,y = k\cosh^2 t + \sinh t\cosh t\ \exp(i\theta)$ and r satisfying

$$R = r^2 = q^2\,\sinh^2 t\,(\sinh^2 t + k^2\cosh^2 t + 2k\sinh t\cosh t\ \cos\theta).$$

The functions x, y, R satisfy the relation

$$H(x,y,r) = R^2 - q^2(2x^2 - 2k\,x + 2y^2 + 1)R + q^4(x^2+y^2)(x^2 - 2k\,x + k^2 + y^2) = 0. \tag{40}$$

By eliminating R from the equations $H(x,y,r) = 0$ and

$$L(x,y,r) = (\partial H/\partial x)^2 + (\partial H/\partial y)^2 - (\partial H/\partial r)^2] = 0,$$

we obtain

$$K(x,y) = (x^2+y^2)((x-k)^2+y^2)\{4(1-q^2)\,(x-k/2)^2 + 4(1-k^2)(1-q^2)y^2 + 1 - k^2\} = 0,$$

and so $\partial W_C^J(A)$ is empty. Since $W_C^J(A)$ is not empty, then $W_C^J(A) = \mathbb{C}$.

If (ii-2) occurs, we may assume that $\eta_1 = 1$, $-1 < q = \eta_2 < 1$, $\kappa_1 = 1$, $-1 < k = -\kappa_2 < 1$, and so

$$W_C^J(A) = \{(\overline{\beta} + k\overline{\alpha})(\overline{\alpha} + q\beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\}. \tag{41}$$

On the other hand, if (ii-3) holds we may consider $\eta_2 = 1$, $-1 < q = \eta_1 < 1$, $\kappa_2 = -1$, $-1 < k = \kappa_1 < 1$, and we get

$$W_C^J(A) = \{(\overline{\alpha} + k\overline{\beta})(\beta + q\overline{\alpha}) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\}. \tag{42}$$

By (41), (42), the cases (ii-2), (ii-3) are identical under exchanging the roles of k and β by q and $\overline{\beta}$, respectively.

If $k = 0$, then

$$W_C^J(A) = \{q\sinh^2 t + \sinh t\cosh t\,\exp(i\theta) : 0 \le t < \infty, 0 \le \theta \le 2\pi\} = \mathbb{C}.$$

If $k \neq 0$, then $W_C^J(A)$ is expressed as the union of the family of circles centered at $z = x + i\,y = q\sinh^2 t + \sinh t\cosh t\,\exp(i\theta)$ and with radius r such that

$$R = r^2 = k^2\,\cosh^2 t\,(\cosh^2 t + q^2\sinh^2 t + 2q\sinh t\cosh t\,\cos\theta).$$

The following relation holds

$$R^2 - k^2(2x^2 + 2q\,x + 2y^2 + 1)\,R + k^4\,(x^2+y^2)(x^2 + 2q\,x + k^2 + y^2) = 0.$$

Replacing k and q by $-q$ and $-k$, respectively, this equation reduces to (40). Therefore, $\partial W_C^J(A)$ is empty and so $W_C^J(A) = \mathbb{C}$. $\square$

Final remarks. It may be easily seen that Propositions 2.1-2.4 are unified in Theorem 1.1.

In the case (ii), if both C and A are J-Hermitian rank one matrices, then $W_C^J(A)$ reduces to a closed half-line. In the case (iv), if at least C or A is a non-zero nilpotent matrix, then $\partial W_C^J(A)$ is a circle. The proofs of the results here presented are rather involved. It would be challenging to obtain easier proofs, as well as to characterize $W_C^J(A)$ for neutral vectors.

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