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ORDERS AND ACTIONS OF BRANCHED COVERINGS OF HYPERBOLIC LINKS

ANTÓNIO SALGUEIRO

ABSTRACT: Let M, M' be compact oriented 3-manifolds and L' a link in M' whose exterior has positive Gromov norm. We prove that the topological types of M and (M', L') determine the degree of a strongly cyclic covering $p : M \to M'$, branched over L'. Moreover, if M' is an homology sphere then these topological types determine also the covering up to conjugacy.

KEYWORDS: Group actions, strongly branched coverings, orbifolds, hyperbolic volume.
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1. Introduction

Let M, M' be compact, oriented 3-manifolds and G a group that acts orientation-preservingly on M, with $M/G \cong M'$. Suppose that M and M' are hyperbolic. If G acts freely then the natural projection $p : M \to M'$ is a covering and the order of G is given by $|G| = \mathrm{vol}(M)/\mathrm{vol}(M')$, where vol is the hyperbolic volume. A similar reasoning applies to manifolds which are not necessarily hyperbolic but whose JSJ decompositions [11, 12] contain hyperbolic pieces; it suffices to take the ratio between the sums of volumes of hyperbolic pieces of M and M'.

In this paper we consider the uniqueness problem of $|G|$ and of the conjugacy class of G in $\mathrm{Diff}^+(M)$ when the action is not free. Let $L' \subset M'$ be a nonempty link and G a group that acts orientation-preservingly on M, such that the natural projection $p : M \to M/G \cong M'$ is a branched covering, with branch set L'. The pre-image of L' in M is a link L. The link $L' \subset M'$ is *prime* if every embedded sphere in M' that cuts L' transversally in two points bounds a submanifold that intersects L' in an unknotted arc.

If $p : M \to M'$ is a branched covering over L' with covering group G, and the stabiliser of each point $x \in L = p^{-1}(L')$ equals G then p is called a covering of M' *strongly branched* over L'. When $p : M \to M'$ is a strongly branched covering, G is a cyclic group and the branching order of every component of L' is the same and equal to $d = |G|$. We will call p an $(L', d)-$*covering* and

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note $\mathsf{O} = (M', L', d)$ its quotient orbifold, that is, the orbifold with underlying manifold M', singular set L' and branching order d along every component of L'.
The main results of this paper are the following theorems.

Theorem 1. *Let M and M' be compact, orientable, 3-manifolds, whose boundary is a (possibly empty) disjoint union of tori. Let $L' \subset M'$ be a prime link. If the exterior of L' in M' is irreducible and its JSJ decomposition contains an hyperbolic piece, then there exists at most one positive integer d for which M is a d-fold covering of M', strongly branched over L'.*

Theorem 2. *Let M' be an integral homology sphere and $L' \subset M'$ an hyperbolic link. Then, for each manifold M, any two cyclic coverings $p_1, p_2 : M \to M'$ branched over L', of prime degrees, are conjugated.*

For completeness sake, in Theorem 1 we allow d to be 1, that is, we show that M' is not a self-covering strongly branched over a link. In [19] we consider the analogous problem for links whose exterior is a graph manifold.
To prove Theorem 1 we define the volume of an orbifold in section 2, and show that, under certain conditions, this volume increases with the branching order of the orbifold, a result that is interesting on its own. In section 3 we deduce Theorem 1 and in section 4 we prove Theorem 2.
This paper contains results obtained in [18], under the supervision of Professor Michel Boileau. I deeply thank him for his endless support.

2. Volumes of orbifolds

Since the exterior E' of L' is irreducible and L' is a prime link, it follows that O is irreducible. By [3], there is a finite family T of essential toric 2-suborbifolds, disjoints, embedded in O such that each connected component of the complement $M' - \mathsf{N}(\mathsf{T})$ of a regular neighbourhood $\mathsf{N}(\mathsf{T})$ of T is either atoroidal or a Seifert orbifold. Moreover, such a minimal family is unique up to isotopy. We call it *the JSJ family* of O. By Thurston's Orbifold Theorem [2], the interior of the atoroidal pieces is either hyperbolic, euclidean, or admits a Seifert fibration. If O is not atoroidal, it may also admit a geometry modelled on the Lie group Sol. We define the *volume* of O as the sum of the hyperbolic volumes of all the hyperbolic pieces of the JSJ decomposition of O, and note it $\mathrm{vol}(\mathsf{O})$. By Mostow's rigidity theorem [1, 16], these hyperbolic volumes are topological invariants, which implies that the volume of O is well-defined.

In this section, we establish a result relating the volume of an orbifold with its branching order. We make use of the following result of Souto [22] to compare the volume of a manifold with the volume of a metric defined on it with sectional curvature bounded below.

Proposition 3. *Let M be a closed, orientable, irreducible, geometrisable 3-manifold. Let g be a metric on M with sectional curvature bounded below by -1. Then* $\mathrm{vol}(M) \leq \mathrm{vol}(M, g)$.

In Theorem 6 we will consider cone-manifold structures on a manifold, so we derive from this proposition the following result.

Proposition 4. *Let M be a closed orientable, irreducible, geometrisable 3-manifold and C an hyperbolic cone-manifold structure on M. Then* $\mathrm{vol}(M) \leq \mathrm{vol}(C)$.

Proof. To apply the previous proposition we need to replace the singular metric g_C defined on M by a smooth metric. For each $\varepsilon > 0$, we will obtain a smooth metric g_ε on M, with sectional curvature bounded below by -1, such that

$$\mathrm{vol}(M, g_\varepsilon) < u(\varepsilon)\,\mathrm{vol}(M, g_C),$$

where u is a function such that $\lim_{\varepsilon \to 0} u(\varepsilon) = 1$.

Let V be a tubular neighbourhood of the singular set L of C, with radius r_0. The cone hyperbolic metric g_C is given on V by

$$ds^2 = dr^2 + f^2(r)\,d\theta^2 + g^2(r)\,dz^2$$

where $f(r) = (d_1/d_2)\sinh r$ and $g(r) = \cosh r$. For $\delta > 0$ sufficiently small, we replace the singular metric g_C in V by a metric g_ε^* given by

$$ds^2 = dr^2 + \varphi^2(r)\,d\theta^2 + \gamma^2(r)\,dz^2$$

where $\varphi, \gamma : [0, r_0 - \delta] \to [0, +\infty)$ are smooth functions such that:

(1) in a neighbourhood of 0, $\varphi(r) = r$ and $\gamma(r)$ is constant;
(2) in a neighbourhood of $r_0 - \delta$, $\varphi(r) = f(r + \delta)$ and $\gamma(r) = g(r + \delta)$;
(3) $\forall r \in [0, r_0 - \delta]$, $\dfrac{\varphi''(r)}{\varphi(r)} \leq 1 + \varepsilon$, $\dfrac{\gamma''(r)}{\gamma(r)} \leq 1 + \varepsilon$ and $\dfrac{\varphi'(r)\gamma'(r)}{\varphi(r)\gamma(r)} \leq 1 + \varepsilon$.

Note $f_\delta(r) = f(r + \delta)$ and $g_\delta(r) = g(r + \delta)$. Let r_1 be the smallest positive solution of the equation $f_\delta(r) = r$. Set $\varphi = f_\delta$ in $[2r_1, r_0 - \delta]$. To define φ in $[0, 2r_1]$, set $\varphi = \mathrm{id}$ in a small neighbourhood of 0 and extend smoothly with $\varphi'' < 0 < \varphi'$ until r_1, $\varphi(r_1) \approx f_\delta(r_1)$ and $\varphi'(r_1) \approx f'_\delta(r_1)$. Then extend

φ smoothly to $[r_1, 2r_1]$ so that φ'' becomes rapidly close to f''_δ and finally maintain φ, φ' and φ'' close to f_δ, f'_δ, f''_δ (see Figure 1).
Now we construct γ. Set $\gamma = g_\delta$ in a small neighbourhood of $r_0 - \delta$. Extend γ smoothly to $[2r_1, r_0 - \delta]$ putting $\gamma'' < g''_\delta$, $\gamma' < g'_\delta$ except in a small neighbourhood of $2r_1$ and $\gamma''(2r_1) = \gamma'(2r_1) = 0$. Finally extend γ to $[0, 2r_1]$ keeping it constant. Choosing a small δ (hence a small r_1) gives $\gamma \approx g_\delta$, $\gamma' \approx g'_\delta$ in $[2r_1, r_0 - \delta]$.

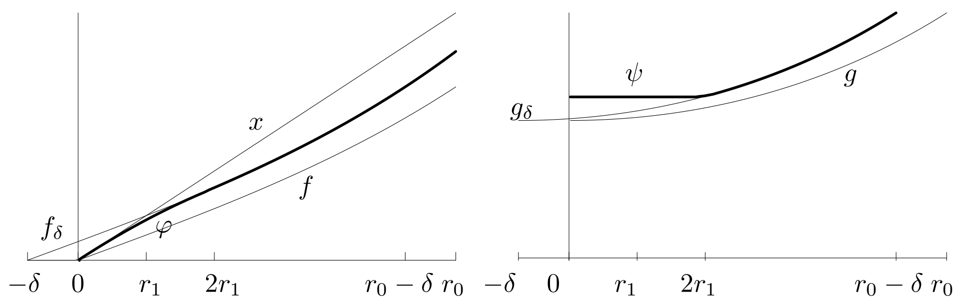

FIGURE 1. Graphs of φ and ψ

By construction, the quotients given in (3) (which are symmetrical to the sectional curvatures of g^*_ε), are all bounded above by $1 + \varepsilon$: in $[0, 2r_1]$, the latter two quotients are zero and the bound on the first quotient comes from the closeness of φ'' and f''_δ; in $[2r_1, r_0 - \delta]$, the bound comes from $\varphi = f_\delta$ and the closeness of γ and g_δ and of their derivatives.
Condition (2) insures that we may glue the metric g^*_ε in V to the metric g_C in $C - V$ along ∂V. The smooth riemannian metric obtained in M', which we note also g^*_ε, has sectional curvature bounded below by $-1 - \varepsilon$.
When $\delta \to 0$, the functions φ and γ approach respectively f and g so that volume $\mathrm{vol}(V, g^*_\varepsilon)$ approaches $\mathrm{vol}(V, g_C)$. Then, for $\delta > 0$ sufficiently small, g^* is a smooth metric on M', with sectional curvature bounded below by $-1 - \varepsilon$, such that $\mathrm{vol}(V, g^*_\varepsilon) < (1 + \varepsilon) \mathrm{vol}(V, g_C)$. Then $g_\varepsilon = (1 + \varepsilon)^{1/2} g^*$ is a metric on M' with sectional curvature bounded below by -1 and

$$\mathrm{vol}(M, g_\varepsilon) < (1 + \varepsilon)^{3/2} \mathrm{vol}(M, g^*) < (1 + \varepsilon)^{5/2} \mathrm{vol}(M, g_C)$$

Since by Proposition 3, $\mathrm{vol}(M) \leq \mathrm{vol}(M, g_\varepsilon)$, for every $\varepsilon > 0$, it follows that $\mathrm{vol}(M) \leq \mathrm{vol}(C)$. □

The following proposition will be useful to classify the non-hyperbolic orbifolds that may appear.

Proposition 5. *Let $\mathsf{O} = (M', L', d)$ be a geometric 3-orbifold where L' is nonempty and $d \geq 3$. If O is not hyperbolic, then either $\mathsf{O} - L'$ admits a Seifert fibration which induces a Seifert fibration on O, or $\mathsf{O} = (\mathbb{S}^3, L', 3)$ where L' is the figure-eight knot.*

Proof. Since O is geometric but not hyperbolic, it is a spherical, euclidean, Seifert or Sol orbifold. We study each case separately.
Suppose that O is a Seifert orbifold. Since $d > 2$, every component of L' is a fibre of O and $\mathsf{O} - L'$ admits a Seifert fibration.
If O is a Sol orbifold, there is a non free action of a group G of isometries of Sol such that $\mathrm{Sol}/G \cong \mathsf{O}$. Let $x \in \mathrm{Sol}$ be a singular point of this action. Since the stabiliser of x is the dihedral group D_4 [21], then $d = 2$, which contradicts the hypothesis that $d \geq 3$.
Suppose now that O is a spherical orbifold that doesn't admit Seifert fibrations. By [7], the singular set of O contains vertices, which contradicts the hypothesis that L' is a link.
If O is an euclidean orbifold that doesn't admit Seifert fibrations and its singular set is a link, the classification of crystallographic groups [6] shows that the underlying space of O is the sphere $\mathbb{S}^3$ and its singular set is the figure-eight knot with branching order $d = 3$. □

Now we use Propositions 4 and 5 to obtain the main result of this section.

Theorem 6. *Let M' be a compact, orientable 3-manifold whose boundary is a (possibly empty) disjoint union of tori and $L' \subset M'$ a nonempty link. For $i = 1, 2$, let $\mathsf{O}_i = (M, L', d_i)$, with $2 \leq d_1 < d_2$. If O_1 and O_2 are irreducible then* $\mathrm{vol}(\mathsf{O}_1) \leq \mathrm{vol}(\mathsf{O}_2)$.

Proof. For $i = 1, 2$, consider the minimal families F_i of toric incompressible non parallel suborbifolds that decompose O_i in geometric pieces. Since, by definition, vol is additive with respect to these families, we may suppose that F_1 and F_2 do not contain tori.
Since the branching degree of O_i is constant, F_i is either empty, a union of euclidean turnovers $\mathbb{S}^2(3,3,3)$, or a union of pillows $\mathbb{S}^2(2,2,2,2)$. The last case is not possible for F_2, since $d_2 > 2$. Furthermore, if F_2 contained an euclidean turnover, then O_1 would contain a spherical turnover $\mathbb{S}^2(2,2,2)$. Since O_1 is irreducible, this spherical turnover bounds a discal orbifold whose singular set is a graph, which contradicts the hypothesis that L' is a link. Then F_2 is empty and O_2 is a geometric orbifold.
Suppose O_2 is not hyperbolic. By Proposition 5, either $\mathsf{O}_2 - L'$ admits a

Seifert fibration which induces a Seifert fibration on O_2, or $\mathsf{O}_2 = (\mathbb{S}^3, L', 3)$ where L' is the figure-eight knot. In the first case, both O_1 and O_2 are Seifert-fibred and $\mathrm{vol}(\mathsf{O}_1) = \mathrm{vol}(\mathsf{O}_2) = 0$. In the latter case, $\mathsf{O}_1 = (\mathbb{S}^3, L', 2)$ is a spherical orbifold since its double covering is a lens space [20]. We have again $\mathrm{vol}(\mathsf{O}_1) = \mathrm{vol}(\mathsf{O}_2) = 0$.
There remains the case where O_2 is an hyperbolic orbifold.
Suppose that ∂M is a nonempty union of tori $T_1, \ldots, T_n$. By Thurston's Hyperbolic Surgery Theorem [23, 8, 2], there exist pairs of integers $(a_i, b_i)_{i=1,\ldots,n}$ such that the closed orbifold $\overline{\mathsf{O}}_2$ obtained by Dehn filling of ratio (a_i, b_i) along the tori $T_i \subset \partial\mathsf{O}_2$ verify

$$\mathrm{vol}(\mathsf{O}_2) - \varepsilon < \mathrm{vol}(\overline{\mathsf{O}}_2) < \mathrm{vol}(\mathsf{O}_2).$$

Since O_1 is not necessarily hyperbolic, Thurston's Hyperbolic Surgery Theorem does not apply directly to it. Consider the JSJ decomposition of O_1

$$\mathsf{O}_1 = \bigcup_j \mathsf{O}_1^j.$$

Each torus T_i belongs to the boundary of a piece O_1^j of this decomposition. If O_1^j is hyperbolic, we may apply Thurston's Hyperbolic Surgery Theorem to obtain a pair (a_i, b_i) such that the closed orbifold $\overline{\mathsf{O}}_1^j$ obtained from O_1^j by Dehn filling of ratio (a_i, b_i) along T_i is hyperbolic and its hyperbolic volume is close to $\mathrm{vol}(\mathsf{O}_1^j)$. If O_1^j is a Seifert orbifold, we may choose the pair (a_i, b_i) such that $\overline{\mathsf{O}}_1^j$ is still a Seifert orbifold. For that, it suffices that the fibres of O_1^j are not meridians of the glued solid torus. Then $\mathrm{vol}(\overline{\mathsf{O}}_1^j) = \mathrm{vol}(\mathsf{O}_1^j) = 0$, that is, we choose a Dehn filling on this Seifert piece that doesn't change its volume.
Since the pieces of the JSJ decomposition of the closed orbifold $\overline{\mathsf{O}}_1$ are the orbifolds $\overline{\mathsf{O}}_1^j$, we have that $\mathrm{vol}(\overline{\mathsf{O}}_1)$ and $\mathrm{vol}(\mathsf{O}_1)$ are close.
Since $\overline{\mathsf{O}}_1$ is a very good orbifold [14], there exists a manifold M and a finite group $G \subset \mathrm{Diff}^+(M)$ such that $M/G \cong \overline{\mathsf{O}}_1$. Lift the hyperbolic metric of $\overline{\mathsf{O}}_2$ by the projection $p : M \to \overline{\mathsf{O}}_1$ induced by the action of G, to obtain an hyperbolic cone-manifold structure C on M. Its singular set is the singular set L of the G-action with cone-angle $2\pi(d_1/d_2) < 2\pi$. By Proposition 4, $\mathrm{vol}(M) \leq \mathrm{vol}(C)$. Since the volumes of M and C are d_1 times the volumes of $\overline{\mathsf{O}}_1$ and $\overline{\mathsf{O}}_2$, respectively, then $\mathrm{vol}(\overline{\mathsf{O}}_1) \leq \mathrm{vol}(\overline{\mathsf{O}}_2)$. It follows from the above

construction that $\mathrm{vol}(\mathsf{O}_1) < \mathrm{vol}(\mathsf{O}_2) + \varepsilon$, for every $\varepsilon > 0$, which shows that $\mathrm{vol}(\mathsf{O}_1) \leq \mathrm{vol}(\mathsf{O}_2)$. □

We note that for $d_2 \geq d_1 \geq 3$ we could prove Theorem 6 without using the method of Besson-Courtois-Gallot involved in the proof of Proposition 3. If $d \geq 4$, we can assume that both O_1 and O_2 are closed hyperbolic orbifolds. We can deform the cone hyperbolic metric of O_1 to the cone hyperbolic metric of O_2. By [2], when we increase the cone-angles from 0, no degeneration occurs, and the Schläfli formula [10, 13, 15] shows that the hyperbolic volume decreases.

In the case $d_1 = 3$, it still possible to use the same reasoning but we need to cut O_1 along the euclidean turnovers $\mathbb{S}^2(3,3,3)$ of the JSJ family of O_1 and cut O_2 along the corresponding family of hyperbolic turnovers $\mathbb{S}^2(d_2, d_2, d_2)$. This family is isotopic to a family of totally geodesic turnovers [2]. Now we must use the theory of deformations of hyperbolic manifolds with totally geodesic boundary [5] to conclude that the hyperbolic volume also decreases in this case.

3. Uniqueness of the degree

In this section we use Theorem 6 to prove Theorem 1.

Proof of Theorem 1. For $i = 1, 2$, let $p_i : M \to M'$ be an (L, d_i)-covering, and suppose $d_1 < d_2$.

For the case $d_1 = 1$, we consider the Gromov simplicial volumes [9, 23] of the pairs $(M, \partial M)$ and $(M', \partial M')$. Since $p_1 : (M, \partial M) \to (M', \partial M')$ is a diffeomorphism, we have $\|(M, \partial M)\| = \|(M', \partial M')\|$. On the other hand $p_2 : (M, \partial M) \to (M', \partial M')$ has degree $d_2 > 1$, which gives

$$\|(M, \partial M)\| \geq d_2 \|(M', \partial M')\| > \|(M', \partial M')\|,$$

a contradiction.

We suppose now that $2 \leq d_1 < d_2$. For $i = 1, 2$, let O_i be the quotient orbifold of M by p_i. By Theorem 6, $\mathrm{vol}(\mathsf{O}_1) \leq \mathrm{vol}(\mathsf{O}_2)$. Lifting the geometric decomposition of O_i to M, we get a G_i-invariant geometric decomposition of M. Hence,

$$0 \leq \frac{\mathrm{vol}(M)}{d_1} = \mathrm{vol}(\mathsf{O}_1) \leq \mathrm{vol}(\mathsf{O}_2) = \frac{\mathrm{vol}(M)}{d_2} < \infty.$$

Since $d_1 < d_2$, this inequality shows that M has null volume. Therefore both orbifolds O_i contain no hyperbolic pieces.

Let E'_0 be an hyperbolic piece of the geometric decomposition of the exterior E' of L' (whose existence is granted by hypothesis) and L'_0 the union of the connected components of L' touching E'_0. Since $\mathrm{vol}(\mathsf{O}_i) = 0$, the link L'_0 is nonempty. We denote O_i^0 the correspondent geometric suborbifold of O_i, $i = 1, 2$. Proposition 5 shows that $\mathsf{O}_2^0 = (\mathbb{S}^3, L', 3)$, where L' is the figure-eight knot. Hence $\mathsf{O}_2^0 = \mathsf{O}_2$ and M' is an euclidean manifold. Again, $\mathsf{O}_1 = (\mathbb{S}^3, L', 2)$ is a spherical orbifold. Therefore M admits both an euclidean and a spherical metric, which is impossible [21]. □

We end this section with an easy corollary.

Corollary 7. *Let $L \subset \mathbb{S}^3$ be a prime link. If L is not a an iterated cable then all cyclic coverings of L are nonhomeomorphic.*

Proof. This follows from the fact that the JSJ decomposition of the exterior of a link has no hyperbolic piece iff the link is an iterated cable. □

4. Uniqueness of the action

In this section we prove Theorem 2. Let $p_1, p_2 : M \to M'$ be two branched coverings over an hyperbolic link L' with prime degrees. Then p_1, p_2 are strongly branched coverings. Theorem 1 allows us to suppose that the degrees of p_1 and p_2 are the same. Then Theorem 2 is a consequence of the following theorem.

Theorem 8. *Let M be a closed orientable manifold, M' is a $\mathbb{Z}_d$-homology sphere where d is prime, and $L' \subset M'$ an hyperbolic link. Then any two $(L', d)-$coverings $p_1, p_2 : M \to M'$ are conjugated.*

To prove this, we consider first the easier cases ($d = 2$, L' is a knot, M is not hyperbolic) and in Proposition 13 we prove the remaining case.

For $i = 1, 2$, let G_i be the covering group of the $(L', d)-$covering $p_i : M \to M'$. Denote E', E_i the exteriors of L' and $L = p_i^{-1}(L')$, respectively. Since the branched covering $p_i : M \to M'$ induces a cyclic covering $p_{i|E_i} : E_i \to E'$, we have the exact sequences

$$1 \longrightarrow \pi_1(E_i) \xrightarrow{p_{i_*}} \pi_1(E') \xrightarrow{\rho_i} \mathbb{Z}_d \longrightarrow 1.$$

Since $\mathbb{Z}_d$ is abelian, the representation $\rho_i : \pi_1(E') \to \mathbb{Z}_d$ factors

$$\begin{array}{ccc} \pi_1(E') & \xrightarrow{\rho_i} & \mathbb{Z}_d \\ & \searrow \quad \nearrow & \\ & H_1(E') & \end{array} .$$

We call the homomorphism $\rho_i : H_1(E') \to \mathbb{Z}_d$ the holonomy of the covering p_i. The image $\rho(\mu)$ of the meridian of each component of L' by the holonomy of the covering is nontrivial.
Since M' is a $\mathbb{Z}_d-$homology sphere, the Mayer-Vietoris sequence of the triple (M', E', V'), where V' is a tubular neighbourhood of L', shows that $H_1(E'; \mathbb{Z}_d) \cong \mathbb{Z}_d^n$, where n is the number of components of L'.

Proposition 9. *Let $p_1, p_2 : M \to M'$ be two (L', d)-coverings and its holonomies $\rho_1, \rho_2 : H_1(E') \to \mathbb{Z}_d$. If $\ker(\rho_1) = \ker(\rho_2)$, then p_1 and p_2 are conjugated.*

Proof. Since $\ker(\rho_1) = \ker(\rho_2)$, the coverings $p_{1|E_1}$ et $p_{2|E_2}$ are equivalent, that is, the following diagram is commutative,

$$\begin{array}{ccc} E_1 & \xrightarrow{\rho_i} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ E' & = & E' \end{array}$$

where $h : E_1 \to E_2$ is a diffeomorphism and $=$ represents the identity. Then, if we note $G_{i|E_i}$ the group of restrictions to E_i of the diffeomorphisms of G_i, we have

$$hG_{2|E_2}h^{-1} = G_{1|E_1}.$$

Now we want to extend $h : E_1 \to E_2$ to a diffeomorphism $M \to M$. The inverse image by p_i of d times the meridian μ of each component of L' is a meridian μ_i of a component of L_i. Then $h(\mu_1) = \mu_2$, that is, the meridian of each component of L_1 is sent by h over the meridian of a component of L_2. This shows that h can be extended to M. □

Corollary 10. *Theorem 8 is true for $d = 2$.*

Proof. There is a single homomorphism $\rho : H_1(E') \to \mathbb{Z}_2$ such that the image of the meridians of each connected component of L' is non trivial. □

Now we suppose that the singular set L' of M' is a knot and, with this condition, we prove that G_1 and G_2 are conjugated in $\text{Diff}^+(M)$.

Proposition 11. *If M' is a $\mathbb{Z}_d-$homology sphere and L' is a knot in M', then any two $(L', d)-$coverings $p_1, p_2 : M \to M'$ are conjugated.*

Proof. Since L' is a knot, $H_1(E') \cong \mathbb{Z}_d$. Therefore, a nontrivial homomorphism $\rho : H_1(E') \to \mathbb{Z}_d$ is unique, up to right compositions by an automorphism of G_i, for d is prime. Then the kernels of the holonomies ρ_1 et ρ_2 are the same for the two actions. By Proposition 9, p_1 and p_2 are conjugated.

Corollary 12. *Theorem 8 is true when $d \geq 3$ and M is not an hyperbolic manifold.*

Proof. Since $d \geq 3$ and M' is a $\mathbb{Z}_d$-homology sphere, Thurston's Orbifold Theorem shows that O is a geometric orbifold. Since O is not hyperbolic and the exterior E' of L' is hyperbolic, Proposition 5 shows that $\mathsf{O} = (\mathbb{S}^3, L', 3)$, where L' is the figure-eight knot. The conclusion that the two $\mathbb{Z}_3$-actions on M are conjugated follows from Proposition 11. □

We now prove Theorem 8 when M is an hyperbolic manifold, L' is disconnected and $d \geq 3$.

Proposition 13. *Let M be a closed hyperbolic manifold. Let G_1 and G_2 two nonfree actions of the cyclic group $\mathbb{Z}_d$ on M, with $d \geq 3$ prime. Then the actions of G_1 and G_2 are conjugated if and only if the quotient orbifolds are diffeomorphic.*

The orbifold theorem shows that the actions of G_1 and G_2 are conjugated to isometric actions. We may then suppose that G_1 and G_2 are isometry groups of M. Note

$$L_i = \mathrm{Fix}(G_i),$$

the set of fixed points of G_i. Since G_i is a cyclic group of prime order, the covering of M' by M is strongly branched. Therefore both links L_i contain the same number of connected components as L'.

Lemma 14. *The group G_i is the group of isometries of M that fix L_i pointwise.*

Proof. Let K be a component of L_i and x an isometry of M such that $K \subseteq \mathrm{Fix}(x)$. Let $\widetilde{K}$ be a component of the covering of K in $\mathbb{H}^3$. Since x and the generator g_i of G_i fix K pointwise, there are isometries $\widetilde{x}$ and $\widetilde{g_i}$ of $\mathbb{H}^3$ that project respectively over x and g_i and fix $\widetilde{K}$ pointwise. Then $\widetilde{x}$ and $\widetilde{g_i}$ are rotations around the hyperbolic line $\widetilde{K}$, thus commuting in $\mathrm{PSL}_2(\mathbb{C})$. Then x and g_i commute in $\mathrm{Isom}(M)$. This shows that x projects by p_i over an isometry x' of O that fixes L' pointwise. Since M' is a $\mathbb{Z}_d-$homology sphere and L' is disconnected, it follows that the isometry x' is trivial [4], and therefore $x \in G_i$. □

Since M is an hyperbolic manifold, the isometry group of M is finite. Since d is a prime number, $\mathrm{Isom}(M)$ contains a Sylow d-group S. After conjugating by an isometry, we may suppose that G_1 and G_2 are in S. We will prove that G_1 and G_2 are the same. Note $N_i = N_S(G_i)$ the normaliser of G_i in S.

Lemma 15. *The group N_i is the group of isometries of M that fixes L_i setwise.*

Proof. Let $x \in \mathrm{Isom}(M)$ be such that $x(L_i) = L_i$. Then $\mathrm{Fix}(x^{-1}G_i x) = x(\mathrm{Fix}\, G_i) = x(L_i) = L_i$. Lemma 14 shows then that $x^{-1}G_i x \subseteq G_i$, and therefore $x \in N_i$. The reciprocal inclusion is immediate. □

The following proposition was proved in [17] in a more general form, where the degree is not necessarily prime.

Proposition 16. *If $G_1 \neq G_2$, then L' has d components.*

Proof. By Sylow theory, either $N_1 = S$ or N_1 contains a subgroup $xG_1x^{-1} \neq G_1$, where $x \in S - N_1$.
In the first case, we have $G_2 \subset N_1$. Then $G_2' = p_1(G_2)$ is a nontrivial subgroup of $\mathrm{Isom}^+(\mathsf{O})$. Since $\mathrm{Fix}(G_2')$ is nonempty and M' is a $\mathbb{Z}_d-$homology sphere, then $\mathrm{Fix}(G_2')$ is a knot. Then the number of components of $L_2 = \mathrm{Fix}(G_2)$ divides $|G_1| = d$. Since d is prime, the link L_2 (and therefore L') has d components.
In the second case, let $y \in xG_1x^{-1} - G_1$. Then $\mathrm{Fix}(y) = \mathrm{Fix}(xg_1x^{-1}) = x(\mathrm{Fix}(g_1)) = x(L_1)$, which is a link with the same number of components as L_1. Since $y \in N_1 - G_1$, then $y' = p_1(y)$ is a nontrivial isometry of O. Since $\mathrm{Fix}(y') \neq \emptyset$ and M' is a $\mathbb{Z}_d-$homology sphere, then $\mathrm{Fix}(y')$ is a knot. Then the number of components of $\mathrm{Fix}(y)$ divides $|G_1| = d$ and as before L' has d components. □

Since d is prime, an element of N_i either preserves each component of L_i, or it permutes cyclically the components of L_i. We consider both cases in Propositions 17 and 18.

Proposition 17. *If every element of N_1 (or N_2) preserves each component of L_1 (respectively L_2), then $G_1 = G_2$.*

Proof. By hypothesis, N_1 contains only hyperbolic transformations which keeps invariant a tubular neighbourhood of each component of L_1. They act on this tubular neighbourhood as rotations along and around its axis with order a power of d. Then N_1 is a subgroup of $\mathbb{Z}_{d^r} \oplus \mathbb{Z}_{d^s}$, where the first factor corresponds to rotations around the components of L_1 and the second corresponds to rotations along these components.
The group N_1 induces a group of isometries N_1' of the quotient orbifold O, which is a subgroup of $\mathbb{Z}_{d^{r-1}} \oplus \mathbb{Z}_{d^s}$. We will prove that N_1' is cyclic.
First notice that N_1' cannot contain a nontrivial element $(a, 0)$, since this

isometry of O would act on a tubular neighbourhood of L' as a rotation around the components of L'. Since M' is a $\mathbb{Z}_d$−homology sphere, this is impossible by Smith theory. Then N_1' cannot contain two distinct elements (a_1, b) and (a_2, b), since it would contain also the nontrivial element $(a_1 - a_2, 0)$. Therefore N_1' is generated by the unique element $\eta = (a, b)$ with minimal positive second coordinate.

Let k be the smallest integer such that $A' = \mathrm{Fix}(\eta^{d^k})$ is nonempty. Since M' is a $\mathbb{Z}_d$−homology sphere, A' is a knot. For every element x' of N_1', we have either $\mathrm{Fix}(x') = \emptyset$, or $\mathrm{Fix}(x') = A'$, since N_1' is cyclic. Then, for every element x of $N_1 - G_1$, we have $\mathrm{Fix}(x) \subseteq p_1^{-1}(A')$. Since d is a prime number,

$$A = p_1^{-1}(A')$$

is either a knot, or a link with d components. Now consider an element $x \in N_1$ such that $\mathrm{Fix}(x)$ is a link with d components. Then $\mathrm{Fix}\, x = L_1$ or $\mathrm{Fix}\, x = A$, according to if $x \in G_1$ or not.

Now we want to prove that $N_S(N_1) = N_1$. Choose any element $y \in N_S(N_1)$. The preceding argument shows that either $y(L_1) = L_1$, or $y(L_1) = A$. In the second case, we have $y^2(L_1) = L_1$. Then, the isometry y has even order, which contradicts the hypothesis that d is prime and greater than 2. It follows that $y(L_1) = L_1$. Lemma 15 shows that $y \in N_1$, for every $y \in N_S(N_1)$. Then $N_S(N_1) = N_1$. Sylow theory shows then that $N_1 = S$.

We have proven that S is commutative. Then G_1 and G_2 commute and therefore $G_1' = p_2(G_1)$ is a subgroup of $\mathrm{Isom}^+(\mathsf{O})$. Since M' is a $\mathbb{Z}_d$−homology sphere, if G_1' was nontrivial, $\mathrm{Fix}\, G_1'$ would be a knot. Then G_2 would permute the components of L_1, which contradicts the hypothesis, and therefore G_1' is trivial, and $G_1 = G_2$. □

To conclude the proof of Theorem 13, there remains to prove the case where N_1 and N_2 contain both elements that permute cyclically the d components of L_1, respectively L_2.

Proposition 18. *If, for $i = 1, 2$, N_i contains an element x_i that permutes cyclically the d components of L_i, then $G_1 = G_2$.*

Proof. Let $x_i \in N_i$ be such an element. We have $x_i g_i x_i^{-1} \in G_i$, and therefore it exists an integer $k_i \in \{0, 1, \ldots, d-1\}$ such that

$$x_i g_i x_i^{-1} = g_i^{k_i}.$$

Then $x_i^2 g_i x_i^{-2} = x_i g_i^{k_i} x_i^{-1} = \left(x_i g_i x_i^{-1}\right)^{k_i} = g_i^{k_i^2}$ and, more generally,

$$x_i^l g_i = g_i^{k_i^l} x_i^l.$$

Then, for $l = d$, we obtain $x_i^d g_i = g_i^{k_i^d} x_i^d$. Since x_i^d and g_i keep invariant each component of L_i, they commute. It follows that

$$g_i = g_i^{k_i^d}.$$

Since d is prime, we obtain from the Fermat's little theorem the congruence $k_i^d \equiv k_i (\operatorname{mod} d)$, and therefore $g_i = g_i^{k_i}$ and $k_i = 1$. Then x_i commutes with g_i.

Then g_i acts locally as a rotation around each component of L_i with the same angle of rotation. Then, the arrow

$$H_1(M' - L'; \mathbb{Z}_d) \cong \mathbb{Z}_d \oplus \cdots \oplus \mathbb{Z}_d \to G_i$$

sends each meridian to the same power of g_i, up to automorphisms of G_i. Then the kernels of the holonomies associated to G_1 and G_2 are the same. By Proposition 9, G_1 and G_2 are conjugated. □

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António Salgueiro
Departamento de Matemática, Universidade de Coimbra, Portugal
E-mail address: ams@mat.uc.pt