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ORDERS OF BRANCHED COVERING OF LINKS

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Abstract: Let M, M' be compact oriented 3-manifolds and L' a link in M'. We prove that, under certain conditions, the topological types of M and (M', L') determine the degree of a cyclic covering $p : M \to M'$, branched over L'.

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1. Introduction

In [3, problem 3.16] we are asked which manifolds M' have the property that homeomorphic coverings of M' have always the same degree. This question, due to Thurston, was completely solved by Wang, Wu and Yu, for those manifolds admitting a geometric decomposition. They have proved [10, 11] that if M' admits a geometric decomposition and has no covering of type (surface)$\times\mathbb{S}^1$ or a torus bundle over $\mathbb{S}^1$, then there exists at most one number d for which M is a covering of M', of degree d. This paper focuses on Thurston's question in the case of branched cyclic coverings, that is, to know when is the order of an 3-manifold orientable cyclic covering of M' by M of prime order branched over L' determined by the topological types of M and (M', L').

The main result is the following theorem:

Main Theorem. *Let M and M' be compact orientable 3-manifolds and L' a prime nontrivial 1-submanifold of M'. If the JSJ graph of the exterior of L' is a tree then there exists at most one prime number $d \geq 2$ for which M is a cyclic covering of M', of degree d, branched over L'.*

This has the following corollary, concerning links in a rational homology sphere.

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Corollary. *Let M' be a rational homology sphere and $L' \subset M'$ be a prime nontrivial link, whose exterior is irreducible. Then two cyclic coverings of M' branched over L' having different prime degrees are not homeomorphic.*

We conjecture that the hypothesis that the JSJ graph of the exterior E' of L' is a tree is not necessary. In fact, Proposition 5.15 shows that the Main Theorem also holds in the "opposite" case, that is, when the JSJ graph of E' is complete.

In the proof of the Main Theorem, we distinguish four cases. In section 3 we consider the case where E' is reducible. For irreducible exteriors, we consider the cases where the JSJ decomposition of E' contains an hyperbolic piece, E' is a Seifert manifold or E' is a nontrivial graph manifold. The hyperbolic case was proved in [6], and in this case the Main Theorem is true even with the slightly weaker condition that the cyclic branched covering is strongly branched, instead of having prime degree. For the remaining cases, we prove the following versions of the Main Theorem in sections 4 and 5.

Theorem 1 *Let M and M' be compact orientable 3-manifolds and $L' \subset M'$ a prime nontrivial link, whose exterior is a Seifert fibred space. Then there is at most one prime number d for which M is a cyclic covering of M', of degree d, branched over L'.*

Theorem 2 *Let M and M' be compact orientable 3-manifolds and $L' \subset M'$ a nontrivial prime link, whose exterior E' is a nontrivial graph manifold. If the JSJ graph of E' is a tree, then there exists at most one prime number d for which M is a cyclic covering of M', of degree d, branched over L'.*

In the case where E' is a Seifert manifold we give examples in paragraph 4.4 where the degree of a cyclic branched covering is not determined, when it is not prime.

In the final section we give examples showing that there is no uniqueness of the action of a cyclic branched covering of prime degree for manifolds whose JSJ decomposition contains a Seifert piece, even when M' is a integral homology sphere.

2. Preliminaries

Let M, M' be compact connected orientable 3-manifolds and L' a disjoint union of closed curves and arcs properly embedded in M'. Let $p : M \to (M', L')$ denote a cyclic covering of M' by M branched over L'. We say that p is *strongly branched* if the stabiliser of each point of the singular set $p^{-1}(L')$ is the whole group of covering transformations. A regular covering of prime degree is strongly branched.

Consider two strongly branched coverings $p_i : M \to (M', L')$ with covering transformation groups $G_i \cong \mathbb{Z}_{d_i}$, $i = 1, 2$. These coverings induce strongly branched coverings $p_{i|} : \partial M \to (\partial M', \partial L')$ on the boundaries with the same degree d_i, $i = 1, 2$. An Euler characteristic argument shows that if $d_1 \neq d_2$, ∂M and $\partial M'$ have the same number of spheres (each sphere of $\partial M'$ containing exactly two points of L'), a certain number of tori (not necessarily the same for M and M' and each torus of $\partial M'$ containing no points of L') and no surface of negative Euler characteristic. By gluing a discal orbifold to each spherical connected component of ∂M and $(\partial M', \partial L')$, we obtain a new manifold $\overline{M}$ and a new pair $(\overline{M'}, \overline{L'})$, called the *closures* of M and (M', L').

Since p_i extends to a strongly branched covering $\overline{p_i} : \overline{M} \to (\overline{M'}, \overline{L'})$ with the same degree, we will suppose, from now on, that the boundaries ∂M and $\partial M'$ are (possibly empty) unions of tori and that L' is a link, that is, L' is a disjoint union of closed curves in the interior of M'. We denote E' the exterior of L' in M'.

Definition. We say that L' is:

(i) *trivial* if it is connected and E' is either a solid torus or a product $T^2 \times I$;
(ii) *prime* if every sphere embedded in M' that cuts L' transversally in two points bounds a submanifold that cuts L' in an unknotted arc.

It is easily seen that a link L' embedded in M' is trivial if an only if M' is either a solid torus or a lens space and L' is (one of) its axis. For trivial links L' there are strongly branched coverings with different degrees. The simplest example is the self covering of the solid torus branched over its axis, which can have any degree. Lens spaces have a similar property, which is proven in Proposition 4.9.

Consider the decompositions of E', M and the quotient orbifold $\mathsf{O}_i = M/p_i$ in nontrivial irreducible manifolds (orbifolds) by essential spherical 2-manifolds (2-orbifolds) [1]. Since d_i is prime, the spherical orbifolds for O_i may be spheres or footballs $\mathbb{S}^2(d_i, d_i)$. If L' is a prime link, then only spheres may appear and a decomposing family of E' also decomposes O_i. The case where E' is reducible is treated in the following section. When E' is irreducible, the O_i are irreducible. Consider the geometric decompositions of E' and O_i. The geometric decompositions of O_i lift to G_i-invariant geometric decompositions of M. If a hyperbolic piece appears in the geometric decomposition of E' then the Main Theorem follows from [6]. If not, then E' is either a Seifert fibred space or a graphed manifold. These cases are treated in the sections 4 and 5.

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3. The exterior is reducible

Proposition 3.1. *Let M and M' be compact connected orientable 3-manifolds and $L' \subset M'$ a prime link with irreducible exterior. Then there is at most one prime number d for which M is a cyclic covering of M', of degree d, branched over L'.*

Proof. Let $p_1, p_2 : M \to (M', L')$ be two cyclic branched coverings with prime degrees d_1, d_2. Let $\mathsf{F}_{E'}$ be a family of spheres that decomposes E' in nontrivial irreducible pieces. Since L' is a prime link, $\mathsf{F}_{E'}$ also decomposes O_i in nontrivial irreducible pieces. The family $\mathsf{F}_{E'}$ is not unique and we choose it in such a way that it exists a connected component E'_0 of $E' - \mathsf{F}_{E'}$ whose boundary contains a torus corresponding to a connected component of L' and, moreover, a copy of every sphere of $\mathsf{F}_{E'}$. Note $\mathsf{F}_i = p_i^{-1}(\mathsf{F}_{E'})$, and O_i^0 the piece of O_i corresponding to E'_0, $i = 1, 2$.

The decomposition of M as connected sum of nontrivial prime manifolds is unique up to permutation of factors. We will analyse the contribution to this decomposition of $p_i^{-1}(S)$ for each sphere S of $\mathsf{F}_{E'} \subset \mathsf{O}_i$.

If S is non-separating, $p_i^{-1}(S)$ is always a collection of d_i non-separating spheres of F_i. This collection contributes with d_i factors $\mathbb{S}^2 \times \mathbb{S}^1$ to the decomposition of M as a connected sum.

If S is separating $p_i^{-1}(S)$ is either a collection of d_i separating spheres of F_i, or a collection of d_i non-separating spheres of F_i. Note E'_S the connected component of $E' - S$ that doesn't contain E'_0 and O_i^S the corresponding piece of O_i. The closures $\overline{E'_S}$ and $\overline{\mathsf{O}_i^S}$ are irreducible and nontrivial by the construction of $\mathsf{F}_{E'}$.

Suppose that $p_i^{-1}(S)$ is made of d_i separating spheres. Then $M - p_i^{-1}(S)$ contains $d_i + 1$ components, among which d_i are homeomorphic to $E'_S = \mathsf{O}_i^S$. Then $p_i^{-1}(S)$ contributes with d_i nontrivial irreducible factors (homeomorphic to $\overline{E'_S}$) for the connected sum decomposition of M.

If $p_i^{-1}(S)$ is made of d_i non-separating spheres, then $M - p_i^{-1}(S)$ contains 2 components, one of which is a d_i-sheeted covering of O_i^S. The closure of this connected component is a d_i-sheeted covering of $\overline{\mathsf{O}_i^S}$ and is therefore irreducible (although it maybe trivial). Therefore $p_i^{-1}(S)$ contributes for the connected sum decomposition of M with an irreducible factor (maybe trivial) and $d_i - 1$ factors $\mathbb{S}^2 \times \mathbb{S}^1$ (when we cut successively along these d_i spheres, the last sphere, just before the cutting, is already separating).

Thus, for every sphere S of $\mathsf{F}_{E'}$, $p_2^{-1}(S)$ contributes with at least as many nontrivial prime factors for the connected sum decomposition of M as $p_1^{-1}(S)$, since $d_2 - 1 \geq d_1$.

It remains to analyse $p_1^{-1}(\overline{\mathsf{O}_i^0})$ and $p_2^{-1}(\overline{\mathsf{O}_2^0})$. The piece $p_2^{-1}(\overline{\mathsf{O}_2^0})$ is prime in M. If this piece is a 3-sphere, then Smith theory shows that O_i^0 is an orbifold whose underlying space is a punctured sphere and the singular set is the trivial knot; in this case, $p_1^{-1}(\overline{\mathsf{O}_i^0})$ is also a 3-sphere.

By the uniqueness of the pieces of the prime decomposition of M, it follows that, for each sphere S of $\mathsf{F}_{E'}$, $p_1^{-1}(S)$ and $p_2^{-1}(S)$ induce the same number of nontrivial prime factors. It follows from the previous discussion that $d_1 = d_2 - 1$, that each sphere S of $\mathsf{F}_{E'}$ is separating and that, for each one of these spheres, $p_2^{-1}(S)$ is a collection of d_2 non-separating spheres. Moreover, $p_2^{-1}(S)$ induces a connected sum of $d_2 - 1 = d_1$ factors $\mathbb{S}^2 \times \mathbb{S}^1$, that is, $p_2^{-1}(\overline{\mathsf{O}_2^S})$ is a 3-sphere. The covering $p_1^{-1}(\overline{\mathsf{O}_1^S})$ is either a connected nontrivial prime manifold or a collection of d_1 copies of $\overline{\mathsf{O}_i^S}$ (which in this case is a nontrivial prime manifold).

Suppose that O_i^S contains components of L'. By the same reasoning as before, it follows that $\overline{\mathsf{O}_i^S}$ is an orbifold whose underlying space is a 3-sphere and the singular set is the trivial knot and that $p_1^{-1}(\overline{\mathsf{O}_1^S})$ is also $\mathbb{S}^3$, which is a contradiction.

Therefore O_i^S contains no components of L', which shows that the fundamental group of $\overline{\mathsf{O}_1^S} = \overline{\mathsf{O}_2^S}$ is $\mathbb{Z}_{d_2}$ and that $p_1^{-1}(\overline{\mathsf{O}_1^S})$ is a collection of either one or d_1 spaces whose fundamental group is $\mathbb{Z}_d$.

The uniqueness of the prime decomposition of M shows that M is the connected sum of a prime manifold N with d_1 factors $\mathbb{S}^2 \times \mathbb{S}^1$ and that $E' = E'_0 \# N'$, where $\pi_1(N') = \mathbb{Z}_{d_2}$. Since $p_1^{-1}(N') = N$, it follows that $\pi_1(N)$ is a finite group with d_2/d_1 elements, which is impossible by $d_2/d_1 = 1+1/d_1 \notin \mathbb{N}$. $\square$

4. Proof of Theorem 1

In this section we prove Theorem 1. In paragraph 4.3 we prove that in certain cases the branching degree is unique without supposing that d is prime.

Let $\mathsf{B}M$ be the space of fibres of a Seifert fibration of M with its orbifold structure. It's underlying space is topologically a surface F and its singular set is a finite set of points corresponding to the singular fibres of M. A Seifert fibration of M may be described by the finite set of invariants

$$(g, n|e_0; \beta_1/\alpha_1, \ldots, \beta_m/\alpha_m)$$

where n is the number of components of ∂F, g is the genus of the closed surface obtained by gluing disks to these components (we write $g < 0$ to indicate a non orientable surface), $e_0 \in \mathbb{Q}$ is the rational Euler number of the Seifert fibration, and $\beta_i/\alpha_i \in \mathbb{Q}/\mathbb{Z}$ are the Seifert invariants of the singular fibres of M. If $n = 0$ we omit it from the notation; if $n \neq 0$, then e_0 is not defined, and is omitted too. Then $\mathsf{B}M$ is given by $\mathsf{B}M \cong F(\alpha_1, \ldots, \alpha_m)$.

Definition. A *Seifert fibration* of (M', L') is a Seifert fibration of M' such that L' is a collection of fibres.

Let O be an orbifold whose topological type (M', L') is Seifert fibred. To point out the fibres of L' in this Seifert fibration (and the corresponding points of $\mathsf{B}(M', L')$), we represent their Seifert invariants in **bold**.

Consider a Seifert fibration of (M', L') given by $(g, n|e_0; \beta_1/\alpha_1, \ldots, \beta_m/\alpha_m)$, where $\beta_i/\alpha_i \in \mathbb{Q}/\mathbb{Z}, i = 1, \ldots, m$ are the Seifert invariants of all singular fibres of M' and also of fibres of L' (these fibres may be regular, thus it is possible that $\beta_i/\alpha_i = 0$). This fibration induces a induces a Seifert fibration on O given by the same partition in circles, denoted by

$$(g, n|e_0; (\beta_1/\alpha_1)_{a_1}, \dots, (\beta_m/\alpha_m)_{a_m}),$$

where a_i is the branching index of the i-th fibre in O (indices a_i equal to 1 are omitted).

Proposition 4.1. *If L' is a nontrivial prime link then a Seifert fibration of the exterior $E' \subset M'$ extends to (M', L').*

Proof. To extend a Seifert fibration of E' to the solid torus component V_i' of $M' - E'$, it suffices that the fibres of the torus $\partial V_i'$ are not meridians of V_i'. Suppose that this condition is false for a torus V_i'. Since L' is nontrivial, it follows that $\mathsf{B}E'$ is not a disk nor an annulus. There exists an essential arc on $\mathsf{B}E'$ whose endpoints lie on $\mathsf{B}(\partial V_i')$ which is not homotopic to an arc of this circle. The union of the fibres that project on this arc is an essential annulus A' in E'. Each fibre of $\partial A'$ bounds a meridian disk of V_i'. It follows that the union of A' with these two disks is a sphere in M' that cuts L' transversally in exactly two points. Since L' is a prime link, this sphere bounds a ball whose intersection with L' is an unknotted arc. Therefore A' is parallel to $\partial E'$, therefore is not essential. □

It is well known that some Seifert fibred spaces admit nonequivalent Seifert fibrations, as it is stated in the following proposition.

Proposition 4.2 ([9])**.** *The only compact connected orientable manifolds M which admit nonequivalent Seifert fibrations are the following:*

(i) *the solid torus has the Seifert fibrations* $(0, 1|\beta/\alpha)$*;*
(ii) *the lens space $L_{a,b}$ has the Seifert fibrations of the form* $(0|e_0; \beta_1/\alpha_1, \beta_2/\alpha_2)$, *where $\alpha_i \geq 1$ and $\alpha_1\alpha_2 e_0 = -a$;*
(iii) *a prism space has exactly two Seifert fibrations, namely* $(0|e_0; 1/2, 1/2, \beta/\alpha)$ *and* $(-1| - 1/e_0; -\alpha/\beta)$*;*
(iv) *twisted I-bundle over the Klein bottle, has exactly two Seifert fibrations which are* $(-1, 1|)$ *and* $(0, 1|1/2, 1/2)$*;*
(v) *the double of the twisted I-bundle over the Klein bottle has exactly two Seifert fibrations which are* $(-1|0;)$ *and* $(0|0; 1/2, 1/2, 1/2, 1/2)$.

The following proposition generalises this result to Seifert fibrations of (M', L').

Proposition 4.3. *Let M' be a compact connected orientable manifold and $L' \subset M'$ a link. Then (M', L') admits nonequivalent Seifert fibrations if and only if it verifies one of the following conditions:*

(i) *M' is a solid torus and L' is its axis;*
(ii) *M' is a lens space and L' is formed by one or both axis;*
(iii) *M' is a prism space and L' is a fibre whose exterior is the twisted I-bundle over the Klein bottle.*

Proof. Let (M', L') be a pair admitting nonequivalent Seifert fibrations, and suppose that L' is connected..

If $M' - \mathsf{N}(L')$ has only one Seifert fibration, then M' has at least two Seifert fibrations whose Seifert invariants differ only for the fibre L'. It follows from the previous proposition that M' is either a solid torus or a lens space and L' is an axis of M'. The second case cannot happen since $M' - \mathsf{N}(L')$ would be a solid torus, which contradicts the uniqueness of the Seifert fibration of $M' - \mathsf{N}(L')$. Therefore M' is a solid torus and L' is its axis.

If $M' - \mathsf{N}(L')$ has nonequivalent Seifert fibrations, then by the previous proposition, $M' - \mathsf{N}(L')$ is either a solid torus or the twisted I-bundle over the Klein bottle. The first case is included in ii. and the second case corresponds to iii.

Now suppose that L' is not connected and let L'_i be the components of L'. Then each pair (M', L'_i) has nonequivalent Seifert fibrations. Therefore M' is either a solid torus, a lens space, or a prism space. If M' is a solid torus, each L'_i is an axis of M'. Therefore M' is trivially Seifert fibred. If M' is a prism space, each L'_i is the fibre of M' whose exterior is the twisted I-bundle over the Klein bottle. Therefore, in both cases, L' would be connected. It follows that M' is a lens space and L' is formed by its two axes. □

Corollary 4.4. *Under the conditions of Proposition 4.3, (M', L') admits nonequivalent Seifert fibrations if and only if $\chi(\mathsf{B}E') \geq 0$ for a Seifert fibration of E'.*

Proof. In cases i. and iii. of the proposition, $\chi(\mathsf{B}E') = 0$. In case ii., $\chi(\mathsf{B}E')$ is positive or zero according to whether L' contains one or two components.

Reciprocally, if $\chi(\mathsf{B}E') \geq 0$ for a certain Seifert fibration of E', this base is diffeomorphic to $\mathbb{D}^2$, $\mathbb{D}^2(a)$, $\mathbb{D}^2(2,2)$, the Möbius strip M^2 or to an annulus $A^2 \cong \mathbb{S}^1 \times I$. The two first bases correspond to case ii., the following two bases to case iii. and the last base corresponds to cases i. or ii., according to L' contains one or two components. □

A Seifert fibration of (M', L') lifts to a Seifert fibration of M preserved by the covering transformations and induces a Seifert fibration on O_d. Therefore $p : M \to (M', L')$ induces an orbifold covering

$$\varphi : \mathsf{B}M \to \mathsf{BO}_d.$$

Definition. Let $V'_1, \ldots, V'_n$ be a nonempty family of fibred solid tori in M', for which $X' = M' - \mathsf{N}(L') - \cup \operatorname{int} V'_i$ and $X = \pi^{-1}(X')$ have only regular fibres. Put

$$u = \deg \varphi$$
$$v = \deg p_{|\text{fibre.}}$$

Then u is the number of fibres of X that cover each fibre of X'. In this situation we say that p is of *type* (u, v).

Since d is prime and $d = uv$, there are only two types of cyclic branched coverings to consider, namely $(u, v) = (1, d)$ and $(u, v) = (d, 1)$.

Lemma 4.5. *Let M and M' be Seifert fibred spaces and $p : M \to M'$ a (branched) covering of type (u, v). Let h'_0 be a fibre of M' and h_0 a connected component of $p^{-1}(h'_0)$. Note β/α and β'/α' the Seifert invariants of h_0 and h'_0, respectively. If V and V' are respectively saturated tubular neighbourhoods of h_0 and h'_0, and $p : V \to V'$ is a covering of type (u_V, v) of branching order $k \geq 1$ over h'_0, then*

$$\beta'/\alpha' = (v/u_V)(\beta/\alpha) \qquad \text{and} \qquad k = \gcd(\alpha u_V, \beta v).$$

Proof. Let $m' \sim \alpha' q' + \beta' h'$ be a meridian of V', where q' is a section of the Seifert fibration of V'. Let q be a connected component of the pre-image of q' in V. Since the branching order over h'_0 is k, the image by p of the meridian $m \sim \alpha q + \beta h$ of V is homologous to km' in $\partial V'$. Then

$$\alpha p(q) + \beta p(h) \sim k\alpha' q' + k\beta' h'.$$

Since $p(q) = u_V q'$ and $p(h) = vh'$, it follows that $\alpha' = \alpha u_V / k$ and $\beta' = \beta v/k$. Therefore, $\beta'/\alpha' = (v/u_V)(\beta/\alpha)$ and $k = \gcd(\alpha u_V, \beta v)$. □

Adding the Seifert invariants of all the fibres yields $e_0(M') = (v/u)\, e_0(M)$. Since a regular fibre is not fixed by an action of type $(1, d)$, this lemma has the following corollary:

Corollary 4.6. *Let $p : M \to (M', L')$ be a cyclic branched covering of type $(1, d)$ with d prime. If $M \cong (g, n|e_0; \beta_1/\alpha_1, \ldots, \beta_m/\alpha_m)$, then the quotient orbifold $\mathsf{O} = M/p$ has Seifert invariant*

$$\left(g, n|de_0; (d\beta_1/\alpha_1)_{\gcd(\alpha_1,d)}, \ldots, (d\beta_m/\alpha_m)_{\gcd(\alpha_m,d)}\right).$$

To prove Theorem 1, we consider two cases, according to whether (M', L') has nonequivalent Seifert fibrations or not.

4.1. (M', L') admits nonequivalent Seifert fibrations.

Lemma 4.7. *If (M', L') admits nonequivalent Seifert fibrations, then M is a solid torus, a lens space or a prism space.*

Proof. For each Seifert fibration of (M', L') we obtain a Seifert fibration of O_d. By Corollary 4.4, $\chi(\mathsf{B}E') \geq 0$ and this implies that $\chi(\mathsf{BO}_d) > 0$, for every Seifert fibration. Therefore also $\chi(\mathsf{B}M) > 0$ for every Seifert fibration.

Suppose that M has a single Seifert fibration. Then $\mathsf{B}M$ is either $\mathbb{S}^2(2, 3, 3)$, $\mathbb{S}^2(2, 3, 4)$ or $\mathbb{S}^2(2, 3, 5)$. We will show that this implies that (M', L') has a single Seifert fibration, in contradiction with the hypothesis.

If $\mathsf{B}M \cong \mathbb{S}^2(2, 3, 5)$, p is of type $(1, d)$, because $\mathsf{B}M$ admits no nontrivial cyclic action. Therefore p is of type $(1, d)$. It follows from Corollary 4.6 that $d = 2$, 3 or 5, and L' is composed of a single regular fibre of M' which is the image of a single fibre of M.

Suppose for example that $d = 5$ (we reason analogously for the other degrees). Then $\mathsf{B}M' \cong \mathbb{S}^2(2, 3)$, M' is a lens space and the knot L' if not an axis of M'. By Proposition 4.3, it follows that (M', L') has a single Seifert fibration.

If $\mathsf{B}M \cong \mathbb{S}^2(2, 3, 4)$, then p is also of type $(1, d)$ and either $d = 2$ or $d = 3$. For $d = 2$, it follows that M' is a lens space with $\mathsf{B}M' \cong \mathbb{S}^2(3, 2)$ and L' is composed of an axis of M' and a regular fibre. Again (M', L') has a single Seifert fibration. The case $d = 3$ is analogous.

If $\mathsf{B}M \cong \mathbb{S}^2(2, 3, 3)$, p may be of type $(2, 1)$. In this case, M' is a lens space with $\mathsf{B}M' \cong \mathbb{S}^2(4, 3)$ and L' is a regular fibre of M'. Again (M', L') has a single Seifert fibration.

In all other cases a similar situation occurs: M' is a lens space and L' contains a component that is not an axis of M', therefore (M', L') has a single Seifert fibration. Since $\chi(\mathsf{B}M) > 0$, M is a solid torus, a lens space or a prism space. □

This proposition and Proposition 4.2 show that if (M', L') admits nonequivalent Seifert fibrations, then M is either a solid torus, a lens space or a prism space. For solid tori, the situation is simple. For any cyclic branched covering $M \to (M', L')$, M' is a solid torus and L' is its axis. Therefore L' is a trivial link.

In the following proposition, we prove Theorem 1 for lens spaces by comparing the Seifert invariants of the different Seifert fibrations of M and (M', L'). Moreover, we deduce from this proposition that when L' is the trivial link composed by one of the axis of $M' \cong L_{a,b'}$, $M \cong L_{a,b}$ is a branched cyclic covering of (M', L') of degree d, for every prime number d such that $db' \equiv b (\mathrm{mod}\, a)$.

Proposition 4.8. *Theorem 1 is true when M is a lens space.*

Proof. By Proposition 4.2, a lens space $L_{a,b}$ has several Seifert fibrations given by $L_{a,b} \cong (0|e_0; \beta_1/\alpha_1, \beta_2/\alpha_2)$, where $\alpha_i \geq 1$ and $\alpha_1\alpha_2 e_0 = -a$. Then a Seifert fibred space is a lens space if and only its base has the form $\mathbb{S}^2(\alpha_1, \alpha_2)$, where $\alpha_1, \alpha_2 \geq 1$.

Fix a Seifert fibration on (M', L') and lift it by the cyclic branched covering $p : M \to (M', L')$ to a Seifert fibration

$$M \cong (0|e_0; \beta_1/\alpha_1, \beta_2/\alpha_2).$$

Suppose first that p is of type $(1, d)$. By Corollary 4.6,

$$\mathsf{O} \cong \left(0|de_0; (d\beta_1/\alpha_1)_{\gcd(\alpha_1,d)}, (d\beta_2/\alpha_2)_{\gcd(\alpha_2,d)}\right).$$

Then M' is a lens space and L' is contained in the union of both axes of M'. Since L' is nontrivial, it contains both axes of M' and d divides simultaneously α_1 and α_2. Then

$$(M', L') \cong \left(0 \;\middle|\; de_0; \frac{\boldsymbol{\beta_1}}{\boldsymbol{\alpha_1/d}}, \frac{\boldsymbol{\beta_2}}{\boldsymbol{\alpha_2/d}}\right).$$

Therefore $M' \cong L_{a',b'}$ where $a' = -de_0(\alpha_1/d)(\alpha_2/d) = a/d$. The degree is then determined by

$$d = a/a'.$$

Suppose now that p is of type $(d, 1)$. Then the covering $\varphi : \mathsf{B}M \to \mathsf{BO}$ is of prime degree d.

If $\mathsf{B}M$ has two singular points and φ identifies these two points, then $d = 2$, $M \cong (0|e_0; \beta/\alpha, \beta/\alpha)$ and

$$(M', L') \cong (0|e_0/2; \beta/\alpha, \mathbf{0}, \mathbf{0}).$$

In this case, L' has one fibre that is not an axis of M'. By Proposition 4.3, (M', L') has a unique Seifert fibration. In fact, this is the only case where we find uniqueness of the Seifert fibration of (M', L').

If each singular point of $\mathsf{B}M$ is fixed by φ, then, by Lemma 4.5,

$$\mathsf{O} \cong \left(0 \,\middle|\, \frac{e_0}{d}; \left(\frac{\beta_1}{d\alpha_1}\right)_{\gcd(d,\beta_1)}, \left(\frac{\beta_2}{d\alpha_2}\right)_{\gcd(d,\beta_2)}\right).$$

As before, the non triviality of L' assures that d divides simultaneously β_1 and β_2. The Seifert fibration of (M', L') is then given by

$$\left(0 \,\middle|\, e_0/d; \frac{\boldsymbol{\beta_1/d}}{\boldsymbol{\alpha_1}}, \frac{\boldsymbol{\beta_2/d}}{\boldsymbol{\alpha_2}}\right).$$

Therefore $M' \cong L_{a',b'}$, where $a' = -(e_0/d)\alpha_1\alpha_2 = a/d$. We obtain as before the degree

$$d = a/a'.$$

Since $\pi_1(M) \cong \pi_1(L_{a,b}) \cong \mathbb{Z}_a$ and $\pi_1(M') \cong \pi_1(L_{a',b'}) \cong \mathbb{Z}_{a'}$, then the degree of the cyclic covering $p : M \to (M', L')$ is given by

$$d = \frac{|\pi_1(M)|}{|\pi_1(M')|},$$

that does not depend on the Seifert fibrations of M and M'. □

Proposition 4.9. *There are cyclic coverings $p_i : L_{a,b} \to L_{a,b'}$ branched over an axis of $L_{a,b'}$, with different degrees, if a, b and b' are positive integers such that* $\gcd(a, b) = \gcd(a, b') = 1$.

Consider again the case from the previous proposition where $p : M \to (M', L')$ was of type $(1, d)$ and L' was trivial. Suppose, without loss of generality, that $d|\alpha_1$ and $d \nmid \alpha_2$. The Seifert fibration of O is then described by

$$\left(0|de_0; \left(\frac{\beta_1}{\alpha_1/d}\right)_d, \frac{d\beta_2}{\alpha_2}\right).$$

Therefore $M' \cong L_{a',b'}$, where $a' = -(de_0)(\alpha_1/d)\alpha_2 = a$. To compute b', recall that

$$b = \overline{\alpha}_1\widetilde{\beta}_2 + \alpha_2\overline{\beta}_1,$$

where $\overline{\alpha}_1, \overline{\beta}_1$ are integers such that $\alpha_1\overline{\beta}_1 - \overline{\alpha}_1\widetilde{\beta}_1 = 1$. Since $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ must satisfy the equality $e_0 = -\widetilde{\beta}_1/\alpha_1 - \widetilde{\beta}_2/\alpha_2$, and similarly for M', we can choose

$$\widetilde{\beta}_1 = \beta_1 \qquad\qquad \widetilde{\beta}_2 = -\frac{\alpha_2}{\alpha_1}(e_0\alpha_1 + \beta_1)$$

$$\widetilde{\beta}_1{}' = \beta_1 = \widetilde{\beta}_1 \qquad\qquad \widetilde{\beta}_2{}' = -\frac{\alpha_2}{\alpha_1/d}(de_0(\alpha_1/d) + \beta_1) = d\widetilde{\beta}_2$$

There remains to compute $\overline{\alpha}_1'$ and $\overline{\beta}_1'$. These integers are chosen such that $\alpha_1'\overline{\beta}_1' - \overline{\alpha}_1'\widetilde{\beta}_1' = 1$. Since $\alpha_1' = \alpha_1/d$ and $\beta_1' = \beta_1$, we choose

$$\overline{\alpha}_1' = \overline{\alpha}_1 \qquad \text{and} \qquad \overline{\beta}_1' = d\overline{\beta}_1.$$

We obtain therefore the value

$$b' = \overline{\alpha}_1'\widetilde{\beta}_2' + \alpha_2'\overline{\beta}_1' = \overline{\alpha}_1(d\widetilde{\beta}_2) + \alpha_2 d\overline{\beta}_1 = db.$$

Finally, consider positive integers a, b and b' such that $\gcd(a, b) = \gcd(a, b') = 1$. Choose any prime number d such that $db \equiv b' (\mathrm{mod}\, a)$. Then, by the argument above, there is a cyclic branched covering $p : L_{a,b} \to (L_{a,b'}, L')$ of order d, where L' is an axis of $L_{a,b'}$. □

Proposition 4.10. *Theorem 1 is true when M is a prism space.*

Let M be a prism space. By Proposition 4.2, M has exactly two nonequivalent Seifert fibrations whose bases are $\mathbb{S}^2(2, 2, \alpha)$ and $P^2(\beta)$, namely the fibrations

$$(0|e_0; 1/2, -1/2, \beta/\alpha) \qquad \text{and} \qquad (-1| - 1/e_0; -\alpha/\beta).$$

Reciprocally, if M has fibration with one of these two bases, M is necessarily a prism space.

Fix a Seifert fibration of (M', L') and consider its reciprocal image by the cyclic branched covering $p : M \to (M', L')$. These fibrations induce orbifolds $\mathsf{B}M$, $\mathsf{B}M'$ and BO. The orbifold BO is a quotient of $\mathsf{B}M$ (either by the trivial group or by the cyclic group of order d), and $\mathsf{B}M'$ is obtained from BO by dividing d by the multiplicities of the points corresponding to L'.

Suppose first that $\alpha \neq 2$. If $\mathsf{B}M \cong \mathbb{S}^2(2, 2, \alpha)$, BO is either $\mathbb{S}^2(2, 2, \alpha)$ or $\mathbb{S}^2(2, 2, 2\alpha)$. If $\mathsf{B}M \cong P^2(\beta)$, BO has the form $P^2(\beta')$. Therefore $\mathsf{B}M'$ is an orbifold of one of the following forms: $\mathbb{S}^2(\alpha')$, $\mathbb{S}^2(2, \alpha')$, $\mathbb{S}^2(2, 2, \alpha')$ or $P^2(\beta')$.

The degree of the cyclic branched covering is $d = 2$ for the two first forms (for which M' is a lens space). Consider then the two remaining forms (for which M' is a prism space). For each of the oriented prism spaces M and M', one of the Seifert fibrations has positive rational Euler number e_0^+ and the other has negative rational Euler number e_0^-. Since p is orientation-preserving, it preserves also the sign of the rational Euler number of this fibration, then there exists at most two fibred applications $p^+, p^- : M \to (M', L')$. The degree $d^\pm$ of the cyclic branched covering $p^\pm$ is the unique integer such that

$$d^\pm \in \left\{ \frac{e_0^\pm(M)}{e_0^\pm(M')}, \frac{e_0^\pm(M')}{e_0^\pm(M)} \right\},$$

according to the type of the covering. This set is the same for both signs, since for each oriented prism space M and M' the two rational Euler numbers verify $e_0^- = -1/e_0^+$. Thus the degree is uniquely determined (in fact, it is given by $d = \alpha'\beta'/\alpha\beta$).

Suppose now that $\alpha = 2$. If $\mathsf{B}M \cong \mathbb{S}^2(2, 2, 2)$, BO is either $\mathbb{S}^2(2, 2, 2)$, $\mathbb{S}^2(2, 2, 4)$ or $\mathbb{S}^2(2, 3, 3)$. If $\mathsf{B}M \cong P^2(\beta)$, l'orbifold BO is of the form $P^2(\beta')$. Then $\mathsf{B}M'$ is an orbifold of one of the following forms: $\mathbb{S}^2$, $\mathbb{S}^2(2, 4)$, $\mathbb{S}^2(2)$ or $P^2(\beta')$. In the three first cases the Seifert fibration of (M', L') and the degree of the cyclic branched covering are unique. In the first two cases $d = 2$ and in the third case $d = 3$. In the last case, d is again given by $d = \alpha'\beta'/\alpha\beta$. □

4.2. (M', L') admits a unique Seifert fibration.

Proposition 4.11. *Theorem 1 is true when (M', L') admits a unique Seifert fibration.*

Proof. We suppose that M is not a solid torus, a lens space or a prism space, since these cases were already treated. Then $\chi(\mathsf{B}M)$ is well defined, since in the remaining cases where M admits nonequivalent Seifert fibrations, they all have bases of zero Euler characteristic, by Proposition 4.2.

The Euler characteristic of BO_d is not determined because it depends on d. However, $\chi(\mathsf{BO}_d)$ and $\chi(\mathsf{B}M)$ are related by

$$\chi(\mathsf{B}M) = u\chi(\mathsf{BO}_d), \tag{1}$$

since $\varphi : \mathsf{B}M \to \mathsf{BO}_d$ is an orbifold covering.

On the other hand, BO_d may be obtained from $\mathsf{B}M'$ by multiplying by d the multiplicities of the points corresponding to the components of L'. If L' has n components and the multiplicities of the corresponding points of $\mathsf{B}M'$ are $a_1, a_2, \ldots, a_n$, then, putting $r = \sum_{i=1}^n 1/a_i$ yields

$$\chi(\mathsf{BO}_d) = \chi(\mathsf{B}M') - r + r/d, \tag{2}$$

since the cyclic covering p is strongly branched. From (1) and (2) we obtain

$$\chi(\mathsf{B}M) = u\left(\chi(\mathsf{B}M') - r + r/d\right). \tag{3}$$

Since d is prime, u is either 1 or d. Since $r > 0$, an indetermination for d may occur only when $u = d$ and $r = \chi(\mathsf{B}M) = \chi(\mathsf{B}M')$. Then $\chi(\mathsf{B}E') = \chi(\mathsf{B}M') - r = 0$. By Corollary 4.4, (M', L') admits nonequivalent Seifert fibrations. Therefore each value of u induces at most one solution d of (3). Therefore (3) has at most two solutions (if $\chi(\mathsf{B}M) = 0$ there is only one). We note these solutions d_1 and d_2, according to $u = 1$ or $u = d$.

Suppose that there are two cyclic branched coverings $p_1, p_2 : M \to (M', L')$ of prime degree, that p_1 is of type $(1, d_1)$ and that p_2 is of type $(d_2, 1)$. Denote the quotient Seifert fibred orbifolds O_1 and O_2. Since there is a covering with $u = 1$, it follows that

$$|\mathsf{B}M| \cong |\mathsf{BO}_1| \cong |\mathsf{BO}_2|.$$

On the other hand, p_2 induces a covering $\varphi_2 : \mathsf{B}M \xrightarrow{d_2} \mathsf{BO}_2$, where the number over the arrow represents the covering degree. This covering induces a cyclic branched covering $\overline{\varphi_2} : |\mathsf{B}M| \xrightarrow{d_2} |\mathsf{B}M|$. The only surfaces having a cyclic branched covering over themselves are the sphere (with two branch points), the projective plane and the disk (each with a single branch point), thus $|\mathsf{B}M| \cong \mathbb{S}^2$, $\mathbb{D}^2$ or $\mathbb{P}^2$.

Let $f = 1, 2$ be the number of branch points of $\overline{\varphi_2}$, k the number of singular points of $\mathsf{B}M$ (and BO_1) and k' be the number of singular points of $\mathsf{B}M$ fixed by φ_2. Then

$$k - k' = d_2(k - f), \qquad 0 \leq k' \leq f,$$

since φ_2 does not fix $k - k'$ points of $\mathsf{B}M$ that project over $k - f$ points of BO_2. Then φ_2 may be:

i) $\mathbb{S}^2(a, a, b, b) \xrightarrow{2} \mathbb{S}^2(a, b, 2, 2)$ with $f = 2$, $k = 4$, $k' = 0$, $d_2 = 2$;
ii) $\mathbb{S}^2(a, a, a) \xrightarrow{3} \mathbb{S}^2(a, 3, 3)$ with $f = 2$, $k = 3$, $k' = 0$, $d_2 = 3$;
iii) $\mathbb{S}^2(a, b, b) \xrightarrow{2} \mathbb{S}^2(2a, b, 2)$ with $f = 2$, $k = 3$, $k' = 1$, $d_2 = 2$;
iv) $\mathbb{S}^2(a, b) \xrightarrow{d_2} \mathbb{S}^2(ad_2, bd_2)$ with $f = 2$, $k = 2$, $k' = 2$, d_2 arbitrary.
v) $\mathbb{D}^2(a, a) \xrightarrow{2} \mathbb{D}^2(a, 2)$ with $f = 1$, $k = 2$, $k' = 0$, $d_2 = 2$;
vi) $\mathbb{P}^2(a, a) \xrightarrow{2} \mathbb{P}^2(a, 2)$ with $f = 1$, $k = 2$, $k' = 0$, $d_2 = 2$;
vii) $\mathbb{D}^2(a) \xrightarrow{d_2} \mathbb{D}^2(ad_2)$ with $f = 1$, $k = 1$, $k' = 1$, d_2 arbitrary;
viii) $\mathbb{P}^2(a) \xrightarrow{d_2} \mathbb{P}^2(ad_2)$ with $f = 1$, $k = 1$, $k' = 1$, d_2 arbitrary.

We discard cases iv), vii) and viii), since M' is either a solid torus, a lens space or a prism space and (M', L') has nonequivalent Seifert fibrations.

We finish the proof by finding a contradiction in all remaining 5 cases, by using Seifert invariants.

In case i), for a cyclic branched covering of type $(d_2, 1)$ we obtain

$$\mathsf{B}(M', L') \cong \mathbb{S}^2(a, b, \mathbf{1}, \mathbf{1}),$$

which shows that L' contains two fibres. Therefore, to obtain the same base $\mathsf{B}(M', L')$ for a covering of type $(1, d_1)$, the degree d_1 must divide a or b, but not both. If for example $d_1 | a$, then

$$\mathsf{B}(M', L') \cong \mathbb{S}^2(b, b, \mathbf{1}, \mathbf{1}).$$

To obtain a coincidence we need $a = b$ and $d_1 | b$, which is impossible.

In case ii) there are always three fibres in L' for coverings of type $(1, d_1)$ and at most two for coverings of type $(d_2, 1)$.

In case iii), if there is only one fibre in L', then $d_1|a$ and $d_1 \nmid b$. We obtain for a cyclic branched covering of type $(1, d_1)$ the base

$$\mathsf{B}(M', L') \cong \mathbb{S}^2(\boldsymbol{a/d_1}, b, b)$$

and for a cyclic branched covering of type $(d_2, 1)$ the base

$$\mathsf{B}(M', L') \cong \mathbb{S}^2(2a, b, \mathbf{1}).$$

Therefore $d_1 = a$ and $b = 2a$, which contradicts the fact that $d_1 \nmid b$. If there are two fibres in L', then $d_1 \nmid a$ and $d_1|b$. Then

$$\mathsf{B}(M', L') \cong \mathbb{S}^2(a, \boldsymbol{b/d_1}, \boldsymbol{b/d_1})$$

for a cyclic branched covering of type $(1, d_1)$ and

$$\mathsf{B}(M', L') \cong \mathbb{S}^2(\boldsymbol{a}, b, \mathbf{1})$$

for a cyclic branched covering of type $(d_2, 1)$. As before, the two bases of the Seifert fibration of (M', L') do not coincide.

In cases v) and vi), L' contains always two fibres for a covering of type $(1, d_1)$ and only one fibre for a covering of type $(d_2, 1)$. □

4.3. Uniqueness of the degree when the rational Euler number of M is nonzero. In this section we prove the following theorem.

Theorem 4.12. *Let M, M' be compact closed orientable 3-manifolds and $L' \subset M'$ a nontrivial prime link. If M admits an unique Seifert fibration, with nonzero rational Euler number, then there exists at most one number d for which M is a cyclic covering of M' of degree d strongly branched over L'.*

Proof. If d is prime, this theorem is a consequence of Theorem 1. A simpler proof in this case comes from the fact that the equality

$$e_0(M')/e_0(M) = v/u = d^{\pm 1}$$

determines the integer d uniquely.

Suppose then that d is not prime. By hypothesis, $\chi(\mathsf{B}M)$ is well defined and by Lemma 4.7, $\chi(\mathsf{B}M')$ is also well defined. As in the proof of Proposition 4.11,

$$\chi(\mathsf{B}M) = u\left(\chi(\mathsf{B}M') - r + r/d\right).$$

Since the rational Euler numbers of M and M' are nonzero, their quotient $k = e_0(M)/e_0(M')$ is a well defined nonzero rational number and

$$u = kv.$$

Since $d = uv$, then $d = kv^2$. Therefore,

$$\chi(\mathsf{B}M) = kv\left(\chi(\mathsf{B}M') - r + \frac{r}{kv^2}\right),$$

or equivalently,

$$k(\chi(\mathsf{B}M') - r)v^2 - \chi(\mathsf{B}M)v + r = 0.$$

Since $\chi(\mathsf{B}M)$, $\chi(\mathsf{B}(M', L'))$, k and r are well determined, this equation has at most two real solutions v_1 and v_2. Since (M', L') admits an unique Seifert fibration, it follows from Corollary 4.4 that $\chi(\mathsf{B}M') - r = \chi(\mathsf{B}E') < 0$. Then, the product of the solutions is

$$v_1 v_2 = \frac{r}{k(\chi(\mathsf{B}M') - r)} < 0.$$

At most one of the solutions v_i is positive, hence there exists at most one possible degree $d = kv_i^2$. □

4.4. Examples of non-uniqueness of the degree. By Theorem 1, if the degree d of a branched cyclic covering $p : M \to (M', L')$ is prime, then it is unique. In the following examples, we show that if the hypothesis that the degree d is prime is withdrawn, then d is not in general unique. We give examples where M and M' are closed and the rational Euler numbers of their Seifert fibrations are nonzero. By Theorem 4.12, these cyclic coverings are not strongly branched.

Example 4.13. *Let* $M \cong (0|0; 1/3, 1/3, 1/3)$ *and* $(M', L') \cong (\mathbb{S}^2 \times \mathbb{S}^1, \{x, y, z\} \times \mathbb{S}^1)$. *Then* M *is a cyclic covering of* (M', L'), *of type* $(1, 3n)$, *for every* $n \in \mathbb{N}$.

Proof. In fact, Corollary 4.6 shows that the quotient orbifold has the Seifert fibration $(0|0; 0_3, 0_3, 0_3)$, for every $n \in \mathbb{N}$. Therefore M' is the space $L_{0,1} \cong \mathbb{S}^2 \times \mathbb{S}^1$ and L' is composed of three circles of its product Seifert fibration. □

Example 4.14. *Consider the manifold* $M \cong (0|-4/3; 1/3, 1/3, 1/3, 1/3)$ *and the pair* (M', L') *where* $M' \cong L_{4,1}$ *and* $L' \subset M'$ *is the union of three fibres of the fibration* $(0| - 4;)$ *of* M'. *Then* M *is a branched cyclic covering of* (M', L'), *of type* $(2, 6)$ *and of type* $(4, 12)$.

Proof. In the first case, the action induced on $\mathsf{B}M$ identifies two of its singular points and for both these points $u_V = 1$. The other two singular points remain fixed and for these points $u_V = 2$. By Lemma 4.5, the quotient $\mathsf{O} \cong$ has the Seifert fibration $(0|-4; 0_3, 0_6, 0_6)$.

In the second case, the action induced on $\mathsf{B}M$ identifies all four singular points and $\mathsf{B}M$ contains two regular points that are fixed. Then, for the singular points $u_V = 1$ and for the regular fixed points $u_V = 4$. By Lemma 4.5, the quotient O has the Seifert fibration $(0|-4; 0_3, 0_4, 0_4)$. □

5. Proof of Theorem 2

In this section we prove Theorem 2. To do this, we don't argue on the JSJ decomposition of a 3-manifold M, but on its *decomposition in geometric pieces* (cf. [5]).

To each compact irreducible orientable 3-orbifold O we associate a *JSJ graph* Γ_{O} as follows. Consider the JSJ family T of tori of O and associate a vertex v_i to each connected component O_i of $\mathsf{O} - \mathsf{T}$ and an edge a^j to each torus $T^j \in \mathsf{T}$. If O_{i_1} and O_{i_2} are the two components of $\mathsf{O} - \mathsf{T}$ on both sides of T^j, then v_{i_1} and v_{i_2} are the two endpoints of the edge a^j (it may happen that $i_1 = i_2$, and in this case a^j is a loop in Γ_{O}).

The geometric decomposition is obtained in the following way. If a torus of the JSJ family of M bounds a piece diffeomorphic to the twisted I-bundle over the Klein bottle, we don't cut M along this torus, but instead, we cut it along the corresponding central Klein bottle. Therefore, in the JSJ graph we replace the vertex v that corresponds to a twisted I-bundle over the Klein bottle, by a loop, as depicted in Figure 1.

Figure 1. JSJ graph vs. geometric graph.

We don't consider these loops as cycles in the graph. Therefore, if the JSJ graph of a 3-manifold M is a tree, we say that this new graph, which we call the *geometric graph* of M, is still a tree.

Consider a component E'^j of the geometric decomposition of E'. This component induces a suborbifold O_i^j of O_i whose boundary is an union of tori. Since L' is a prime link, $L' \cap \mathsf{O}_i^j$ is either empty or a prime link. In the first case, $\mathsf{O}_i^j \cong E'^j$ is a Seifert fibred space. In the second case, we can extend the Seifert fibration of E'^j to O_i^j, by Proposition 4.1. Therefore O_i^j is a Seifert fibred orbifold. This shows that O_i is a graphed orbifold and that the geometric graphs of O_i and of E' are isomorphic.

Lifting the geometric decomposition of O_i by the orbifold covering $p_i : M \to \mathsf{O}_i$, it follows that M is also a nontrivial graph manifold. We note the geometric graphs of M and O_i by Γ and Γ' respectively. The coverings p_i induce (branched) coverings $\Gamma \to \Gamma'$ which are also noted p_i.

5.1. Sufficient conditions for the uniqueness of the degree. In this section, we give conditions under which the degree is determined, without supposing that Γ' is a tree.

The first condition is that the coverings are unbranched. The following theorem has been proved in [10, 11].

Theorem 5.1 (Wang, Wu, Yu)**.** *If M and M' are nontrivial graphed manifolds, then there exists at most one number d for which M is a covering of M' of degree d.*

In the case of unbranched coverings, the following proposition is a corollary of Theorem 5.1. We adapt here the proof given in [10] to the case of branched cyclic coverings.

Definition. Let X be a subset of the vertices of the geometric graph Γ of a 3-orbifold O. We note O^X the suborbifold of O whose geometric graph Γ^X is formed by the vertices of X and the edges of Γ which connect vertices of X. If $X = \{s\}$, we write $\mathsf{O}^{\{s\}} = \mathsf{O}^s$.

Proposition 5.2. *If the graphs Γ and Γ' are isomorphic, then there exists at most a number d for which M is a covering of (M', L') of degree d.*

Proof. We sketch the proof, remarking the changes needed for the case of branched coverings.

Suppose that there are two cyclic coverings $p_i : M \to (M', L')$, $i = 1, 2$, of degrees $d_1 < d_2$. Define, for each edge/loop e of Γ, the rational number

$$b(e) = \frac{1}{|\Delta(e)|\ \chi(\mathsf{B}M^v)\chi(\mathsf{B}M^w)},$$

where v and w are the endpoints of e and $\Delta(e)$ is the algebraic intersection number of the fibres of the Seifert fibrations of M^v and M^w along the torus $\partial N(e)$ corresponding to the torus/Klein bottle e. The integer $\Delta(e)$ is nonzero because otherwise the fibration extends and the decomposition is not geometric.

We define similarly, for $i = 1, 2$, the numbers

$$b_i(e') = \frac{1}{|\Delta(e')|\ \chi(\mathsf{BO}_i^{v'})\chi(\mathsf{BO}_i^{w'})}.$$

For each edge/loop e' of Γ', $\chi(\mathsf{BO}_i^{v'})$ and $\chi(\mathsf{BO}_i^{w'})$ are the Euler characteristics of the bases of the Seifert pieces corresponding to v' and w' of the quotient orbifold $\mathsf{O}_i = M/p_i$ of topological type (M', L').

Since $\Gamma = \Gamma'$, both coverings $p_i : \Gamma \to \Gamma'$ are diffeomorphisms. The preimage by p_i of each edge/loop e' of Γ' contains then a single edge/loop e of Γ. A counting argument (Proposition 5.1 of [10]) shows that

$$d_i = \frac{|\Delta(e)|\ \chi(\mathsf{B}M^v)\chi(\mathsf{B}M^w)}{|\Delta(e')|\ \chi(\mathsf{BO}_i^{v'})\chi(\mathsf{BO}_i^{w'})} = \frac{b_i(e')}{b(e)}. \tag{4}$$

In the proof given, we need only to change the Euler characteristics of the base of the piece of the geometric decomposition of the quotient manifold M' by those of the quotient orbifold O_i.

We note that, for the pieces of the quotient orbifold O_i that contain components of L', these Euler characteristics depend on the branching degree. In other words, if one of the endpoints v' of e' contains components of L', $b_1(e') \neq b_2(e')$. In fact, since $d_1 < d_2$, it follows that

$$\chi(\mathsf{BO}_1^{v'}) > \chi(\mathsf{BO}_2^{v'}).$$

Since the sign of the Euler characteristics of the pieces of the geometric decomposition of O_i is always negative, it follows that

$$b_1(e') \geq b_2(e') > 0,$$

for every edge/loop e' of Γ'. Moreover, this inequality is strict for at least one edge e' of Γ', when one of the endpoints contains ramification.

Adding all $b(e)$ and all $b_i(e')$, it follows from (4) that

$$d_1 = \frac{\sum b_1(e')}{\sum b(e)} > \frac{\sum b_2(e')}{\sum b(e)} = d_2,$$

which gives a contradiction. □

Lemma 5.3. *Let $p_i : M \to (M', L')$, $i = 1, 2$, be two cyclic branched coverings of prime degree. If p_1 and p_2 have the same type (u, v) for every Seifert piece of the geometric decomposition of M, then $d_1 = d_2$.*

Proof. Put, for each vertex w' of Γ', $r_{w'} = \sum_l 1/a_l^{w'}$, where $a_l^{w'}$ are the multiplicities of the points of the underlying space $|\mathsf{BO}_1^{w'}| = |\mathsf{BO}_2^{w'}|$ corresponding to the components of L'.

Suppose that $p_i^{-1}(w')$ contains a single vertex w. Then equality (3) of section 4.2 becomes

$$\chi(\mathsf{B}M^w) = u\left(\chi(\mathsf{B}|\mathsf{O}_i^{w'}| - r_{w'} + r_{w'}/d_i\right).$$

Suppose now that there are d_i pieces M^{w_k} of M which are identified, projecting to an underlying space $|\mathsf{O}_i^{w'}|$ of $\mathsf{O}_i^{w'}$. This situation may occur only for coverings of type $(u, v) = (d_i, 1)$. In this case, $r_{w'} = 0$ and

$$\sum_k \chi(\mathsf{B}M^{w_k}) = d_i \chi(\mathsf{B}|\mathsf{O}_i^{w'}|).$$

Since u is constant, adding the previous equalities for every vertex of Γ and of Γ' gives

$$\sum_w \chi(\mathsf{B}M^w) = u \sum_{w'} \left(\chi(\mathsf{B}|\mathsf{O}_i^{w'}|) - r_{w'} + r_{w'}/d_i\right)$$

Since all Euler characteristics that appear in this expression are non positive, an indetermination of type $0/0$ for d cannot happen. Therefore the degree is determined. □

Lemma 5.4. *If Γ' is a graph with exactly two vertices, then there exists at most one prime number d for which M is a cyclic branched covering of (M', L'), of degree d.*

Proof. If Γ contains more than two vertices, then the degree of $p_i : \Gamma \to \Gamma'$ is

$$d_i = (\text{number of vertices of } \Gamma) - 1.$$

In fact, in this case, a single vertex of Γ' contains components of L' and since d_i is prime, the pre-image of the other vertex of Γ' has d_i vertices.

Consider now the case where Γ contains only two vertices v and w. Then the pre-image of each loop of Γ' is made only of loops from Γ. Clearly each loop of Γ projects over a loop of Γ'. Therefore if we remove the loops of Γ and Γ', we obtain still geometric graphs Γ_0 and Γ'_0 of nontrivial graphed manifolds M_0 and M'_0 and the branched coverings $p_i : M \to (M', L')$ restrict to branched coverings $p_{0i} : M_0 \to (M'_0, L')$, with the same degree d_i as p_i.

By the previous lemma, we may suppose that one of the coverings p_{0i} is of type $(1, d)$ over one of the two Seifert pieces of the geometric decomposition of M_0. Therefore the tori incident on one of the vertices of Γ_0, are invariant by the corresponding action, which shows that the number of edges of Γ and Γ' coincide. Therefore $\Gamma_0 = \Gamma'_0$ and the equality $d_1 = d_2$ comes from Proposition 5.2. □

Definition. The *valence* of a vertex v of a graph Γ is the number of edges and of loops of Γ incident on v. We say that the vertex v is:

(i) *terminal* if its valence is 1;
(ii) *interior* if its valence is greater than 1;
(iii) *saturated* if it is connected to all other vertices of Γ.

We note $\mathsf{I}(\Gamma)$ (respectively $\mathsf{E}(\Gamma)$) the set of interior (respectively terminal) vertices of Γ.

A graph Γ is *complete* if every vertex is saturated and if there is only one edge between each pair of vertices. We note K_n the complete graph with n vertices without loops. A graph Γ is a *star* if it is a tree without loops with a single saturated vertex.

FIGURE 2. The complete graph K_5 and the star with 6 vertices.

Lemma 5.5. *If the graphs Γ and Γ' contain respectively vertices v and v', such that $p_1^{-1}(v') = \{v\} = p_2^{-1}(v')$, then $d_1 = d_2$.*

Proof. If the Seifert piece $\mathsf{O}_i^{v'}$ of the geometric decomposition of O_i contains components of L', the cyclic branched coverings $p_i : M \to (M', L')$ restrict to cyclic branched coverings

$$p_{i|} : M^v \to (|\mathsf{O}_i^{v'}|, \mathsf{O}_i^{v'} \cap L')$$

having the same degree d_i as p_i. It follows from Theorem 1 that $d_1 = d_2$.

If $\mathsf{O}_i^{v'} \cap L' = \emptyset$, the cyclic branched coverings $p_i : M \to (M', L')$ restrict to cyclic unbranched coverings

$$p_{i|} : M^v \to |\mathsf{O}_i^{v'}|.$$

If $\chi(\mathsf{B}M^v) \neq \chi(\mathsf{B}\mathsf{O}_i^{v'})$, then

$$d_1 = \frac{\chi(\mathsf{B}M^v)}{\chi(\mathsf{B}\mathsf{O}_i^{v'})} = d_2,$$

since $\mathsf{O}_1^{v'} = \mathsf{O}_2^{v'}$.

If $\chi(\mathsf{B}M^v) = \chi(\mathsf{B}\mathsf{O}_i^{v'})$ it follows that both coverings $p_{i|}$ have type $(1, d)$. Then G_i preserves the fibres of M, thus this group leaves each torus JSJ of M invariant. Therefore, every vertex of Γ connected to v is fixed by both groups G_1 and G_2. This shows that the star E centred in v is sent by both coverings p_1 and p_2 to the star E' centred in v'. Then the cyclic branched coverings $p_i : M \to (M', L')$ restrict to cyclic coverings (which may be branched)

$$p_{i|} : M^E \to |\mathsf{O}_i^{E'}|.$$

Since the stars E and E' are isomorphic, it follows from Proposition 5.2 that $d_1 = d_2$. □

5.2. Reduction to the case of trees without loops. Definition. Let M be a compact connected orientable graphed 3-manifold and Γ its geometric graph. The *complexity* of M is a triple of nonnegative integers

$$c(M) = (n_v(\Gamma), n_e(\Gamma), n_l(\Gamma)),$$

where $n_v(\Gamma)$, $n_e(\Gamma)$ and $n_l(\Gamma)$ are respectively the number of vertices, edges and loops of Γ.

Definition. Let $\{M_i : i \in I\}$ be a set of compact connected orientable 3-manifolds. We say that the complexity of a manifold M_i is *minimal* if, for the lexicographical order, $c(M_i) \leq c(M_j), \forall j \in I$.

Definition. Let v and w be two vertices of a connected graph Γ. A *path* γ in Γ between v and w is a sequence of edges

$$\overline{v_0 v_1}, \overline{v_1 v_2}, \ldots, \overline{v_{k-1} v_k}$$

with $v_0 = v$ and $v_k = w$. The *length* of γ is the number

$$\operatorname{len}(\gamma) = k$$

of edges of γ. The *distance* between two vertices $v \neq w$ of Γ is the integer

$$d(v,w) = \min\{\operatorname{len}(\gamma) : \gamma \text{ is a path of } \Gamma \text{ between } v \text{ and } w\}.$$

If $v = w$, the distance between v and w is zero, by definition.

The following lemma shows that if the group G_i of the covering p_i fixes a vertex v of Γ, then p_i preserves the distance of the vertices of Γ to v.

Lemma 5.6. *If a vertex v of Γ is fixed by G_i, then*

$$d(v,w) = d(p_i(v), p_i(w)),$$

for every vertex w of Γ.

Proof. Let $v' = p_i(v)$ and $w' = p_i(w)$. Clearly $d(v', w') \leq d(v,w)$. We show by recurrence on the distance $d(v', w')$ that the equality always holds. If $d(v', w') = 0$, the hypothesis that v is fixed by G_i is equivalent to $d(v, w) = 0$.

Now suppose that the equality is valid for every vertex whose distance to v' is at most k and we take a vertex $w \in \Gamma$ for which $d(v', w') = k+1$. There exists a vertex u' of Γ' connected to w', whose distance to v' is k. The edge between w' and u' lifts by p_i to an edge between w and a vertex u of Γ such that $p_i(u) = u'$. By the recurrence hypothesis, $d(v,w) \leq d(v,u)+1 = k+1 = d(v', w')$. □

Lemma 5.7. *If (M, M', L') is a triple that contradicts the statement of Theorem 2 such that the complexity of M is minimal, then Γ and Γ' don't contain loops.*

Proof. Suppose that there is a vertex v' of Γ' and a loop attached to v' whose pre-image by one of the coverings p_i contains a edge a. Since an odd covering of a Klein bottle is formed only by Klein bottles, it follows that the degree d_i of p_i is even. Therefore $d_i = 2$, since it is a prime number. Then $i = 1$ and the endpoints of a are two vertices v_1 and v_2 such that $p_1(v_1) = p_1(v_2) = v'$. Since L' is nonempty, there exists a vertex $w' \in \Gamma'$ that contains components of L'. Note $w = p_1^{-1}(w')$. The minimal path $v'w'$ has two pre-images v_iw that start at v_1 and v_2 (these two paths may have common vertices). Since w is fixed by G_1, the distance to w is preserved by p_1, by the previous lemma. Therefore the path $c = v_1wv_2v_1$ is a cycle of odd length in Γ.

Since the length of c is odd and Γ' is a tree, the other projection $p_2(c)$ contains loops, which contradicts the fact that p_2 is a branched covering of prime degree $d_2 \geq 3$.

Thus the pre-image of a loop by each projection p_i is formed only by loops. Clearly each loop of Γ projects over a loop of Γ'. Therefore if we remove the loops of Γ and Γ', we obtain again geometric graphs of nontrivial graphed manifolds that contradict the statement of the theorem. It follows from the minimality of Γ that the both graphs Γ and Γ' don't contain loops. □

5.3. Γ is a tree. In the proof of Theorem 2 when Γ is a tree, we use the following proposition, which is valid even if Γ is not a tree.

Proposition 5.8. *If (M, M', L') is a triple that contradicts the statement of Theorem 2 for which the complexity of M is minimal, then Γ' is not a star.*

Proof. Suppose that Γ' is a star. We will show that the covering degree is unique, which contradicts the hypothesis.

Let s' be the saturated vertex of Γ' and consider its two pre-images $p_i^{-1}(s')$. Since d_i is prime, $p_i^{-1}(s')$ contains 1 or d_i vertices.

If each of these two sets contains multiple vertices, the pre-image by p_i of each edge of Γ' is formed by d_i edges. Then both actions are of type $(d, 1)$ for each Seifert piece of the geometric decomposition of M. It follows from Lemma 5.3 that $d_1 = d_2$.

Suppose for example that $p_1^{-1}(s')$ contains a single vertex s. Then s is a saturated vertex of Γ. Since the image of a saturated vertex of Γ is a saturated vertex of Γ', it follows that $p_2(s) = s'$. By Lemma 5.5, the degree is unique. □

Lemma 5.9. *If* Γ *is a tree and one of the two pre-images of a terminal vertex of* Γ' *contains an interior vertex of* Γ*, then* $d_1 = d_2$*.*

Proof. Suppose that $p_1^{-1}(v')$ contains an interior vertex, for $v' \in \mathsf{E}(\Gamma')$. Since d_1 is prime, $p_1^{-1}(v')$ contains either 1 or d_1 vertices, which are all interior.

If $p_1^{-1}(v')$ contains d_1 vertices, the valence of each is a number $a > 1$ and each one of the $d_1 a$ edges incident to the vertices of $p_1^{-1}(v')$ project over the only edge incident to v'. But that contradicts the hypothesis that the degree of p_1 is d_1.

Therefore $p_1^{-1}(v')$ contains a single vertex v_1 which is therefore fixed by G_1. If Γ contains another vertex w fixed by G_1, then by taking two distinct paths which connect v_1 and w, we obtain a cycle in Γ, which is impossible because Γ is a tree. We note that this choice of paths is possible since v_1 is an interior vertex.

Therefore, v_1 is the only vertex of Γ fixed by G_1 and consequently $\mathsf{O}_i^{v'}$ contains all components of L'. Therefore $p_2^{-1}(v')$ contains also a single vertex v_2. Since d_1 is prime and Γ is a tree, $\Gamma - v_1$ is the union of d_1 components isomorphic to the tree $\Gamma' - v'$.

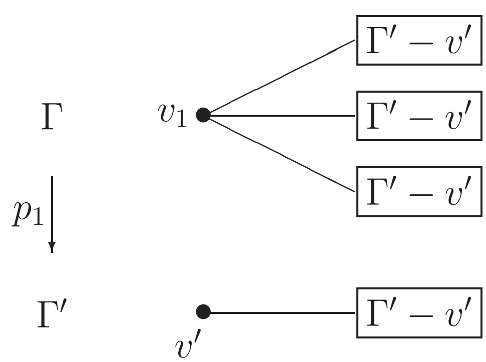

FIGURE 3. The pre-image of $\Gamma' - v'$.

Suppose that $v_1 \neq v_2$ and consider a connected component of $\Gamma - v_1$ which doesn't contain v_2. Since v_1 is fixed by G_1, by Lemma 5.6, we may choose in this connected component a vertex w such that $d(w, v_1) = l$, where

$$l = \max_{w' \in \Gamma'} d(v', w').$$

Every path from w to v_2 passes in v_1, therefore

$$d(w, v_2) = d(w, v_1) + d(v_1, v_2) > l.$$

Lemma 5.6 shows also that the distances of the vertices of Γ to v_2 are preserved by p_2. Therefore $d(p_2(w), v') > l$, which contradicts the definition of l. Thus $v_1 = v_2$, and Lemma 5.5 implies that $d_1 = d_2$. □

Proposition 5.10. *Theorem 2 is true when* Γ *is a tree.*

Proof. Suppose that there is a triple (M, M', L') which contradicts the statement of Theorem 2 such that Γ is a tree and consider one for which the complexity of M is minimal. By Lemma 5.7, Γ has no loops.

It follows from the previous lemma that $p_i^{-1}(\mathsf{E}(\Gamma')) \subseteq \mathsf{E}(\Gamma)$, for $i = 1, 2$ and the inclusions $p_i(\mathsf{E}(\Gamma)) \subseteq \mathsf{E}(\Gamma')$, for $i = 1, 2$ are clear.

If $\mathsf{I}(\Gamma') = \emptyset$, then Γ' contains only two vertices and Lemma 5.4 shows that $d_1 = d_2$. If $\mathsf{I}(\Gamma')$ contains a single vertex, then Γ' is a star, which is impossible by Proposition 5.8. If $\mathsf{I}(\Gamma')$ contains multiple vertices, then $M^{\mathsf{I}(\Gamma)}$ and $|\mathsf{O}_i^{\mathsf{I}(\Gamma')}|$ are nonempty graphed manifolds whose geometric graphs are still trees. The cyclic branched coverings $p_i : M \to (M', L')$ restrict to cyclic coverings

$$p_{i|} : M^{\mathsf{I}(\Gamma)} \to |\mathsf{O}_i^{\mathsf{I}(\Gamma')}|$$

(branched if $\mathsf{O}_i^{\mathsf{I}(\Gamma')} \cap L' \neq \emptyset$), with $p_{i|}$ of same degree d_i that p_i.

By the minimality of the geometric graph of M, it follows that $\mathsf{O}_i^{\mathsf{I}(\Gamma')} \cap L' = \emptyset$. It follows from Theorem 5.1, applied to these coverings, that $d_1 = d_2$, which is impossible. □

5.4. Γ is not a tree.

Lemma 5.11. *Suppose that* (M, M', L') *is a triple which contradicts the statement of Theorem 2 such that the complexity of* M *is minimal and that* Γ *is not a tree. Then* Γ *contains no vertex* v *that is fixed by* G_1 *and* G_2.

Proof. Suppose first that $p_i(v)$ is a terminal vertex of Γ'. Since Γ' is a tree, $\Gamma' - p_i(v)$ is also a tree. Since d_i is prime and v is fixed by G_i, the complement $\Gamma - v = p_i^{-1}(\Gamma' - p_i(v))$ is either connectd or contains d_i components diffeomorphic to $\Gamma' - p_i(v)$. In this last case, Γ would be a tree and $d_1 = d_2$, which would contradict the hypothesis. It follows that $\Gamma - v$ is connected.

Suppose now that $p_i(v) \in \mathsf{I}(\Gamma')$. Since Γ' is a tree, then $\Gamma' - p_i(v)$ has multiple components and $\Gamma - v$ contains at least the same number of components as $\Gamma' - p_i(v)$.

Thus $\Gamma - v$ contains either one or multiple components according to $p_i(v)$ is terminal or interior. Therefore the vertices $p_i(v)$, $i = 1, 2$ are both terminal or both interior. Let

$$\omega(v) = \{w \in \Gamma : d(v, w) \geq d(v, z), \forall z \in \Gamma \text{ adjacent to } w\}$$

the set of vertices of Γ at maximal distance of v and define similarly $\omega'(p_i(v))$. Note

$$\mathsf{W} = \begin{cases} \omega(v), & \text{if } p_i(v) \in \mathsf{I}(\Gamma') \\ \omega(v) \cup \{v\}, & \text{if } p_i(v) \in \mathsf{E}(\Gamma') \end{cases}$$

$$\mathsf{W}'_i = \begin{cases} \omega'(p_i(v)), & \text{if } p_i(v) \in \mathsf{I}(\Gamma') \\ \omega(p_i(v)) \cup \{p_i(v)\}, & \text{if } p_i(v) \in \mathsf{E}(\Gamma'). \end{cases}$$

By Lemma 5.6, the distances to v are preserved by p_1 and p_2, therefore $p_i(\mathsf{W}) = \mathsf{W}'_i$ and $p_i^{-1}(\mathsf{W}'_i) = \mathsf{W}$. Since Γ' is a tree, $\mathsf{W}'_i = \mathsf{E}(\Gamma')$. Since $\mathsf{E}(\Gamma')$ and W don't disconnect, $\mathsf{I}(\Gamma')$ and $\Gamma - \mathsf{W} = p_i^{-1}(\mathsf{I}(\Gamma'))$ are connected graphs.

Then $p_i : M \to (M', L')$ restrict to cyclic coverings (branched if $L' \cap M'_{\mathsf{I}(\Gamma')} \neq \emptyset$)

$$p_i : M^{\Gamma - \mathsf{W}} \to (M'_{\mathsf{I}(\Gamma')}, L' \cap M'_{\mathsf{I}(\Gamma')}).$$

As before, the minimality of the geometric graph of M and Theorem 5.1 shows that $d_1 = d_2$, which is impossible. □

Proposition 5.12. *Theorem 2 is true when Γ is not a tree.*

Proof. Suppose that there exists a triple (M, M', L') which contradicts the statement of Theorem 2 with Γ not a tree. Then there exists a triple such that M is of minimal complexity. By Lemma 5.7, for this triple, Γ and Γ' don't have loops.

Let c be a cycle of minimal length in Γ. Then $p_1(c)$ is a subtree of Γ'. The cycle c cannot contain an edge connecting two vertices with the same image, because Γ' contains no loops. Then c contains a vertex v_1 such that both vertices x and y of c adjacent to v_1 have the same image, otherwise Γ' would contain a cycle.

If $\text{len}(c) = 2$, both vertices x and y coincide. Since Γ' is a tree, both edges of c project over the same edge of Γ'. Since Γ' has no loops, the covering degree is then given by the number of edges between v_1 and $x = y$, from which $d_1 = d_2$, which is impossible.

Claim 1. The vertex v_1 is fixed by G_1.

Proof. Suppose that v_1 is not fixed by $G_1 =< g_1 >$. Since d_1 is prime, $g_1^k(v_1) \neq v_1$, for every $k \in [1, d_1]$. Then none of the pre-images $g_1^k(x)g_1^k(v_1)$ of the edge xv_1 coincides with yv_1. Therefore $p_1(xv_1)$ and $p_1(v_1y)$ are two distinct edges of Γ' connecting $p_1(v_1)$ and $p_1(x) = p_1(y)$, and consequently they define a cycle in Γ', which is impossible. This shows that v_1 is fixed by G_1.

Analogously, the cycle c possesses, in the other end, a vertex v_2 also fixed by G_1. Since c is a minimal cycle, v_1 and v_2 are the only vertices of c fixed by G_1. Since d_1 is prime, the cycle c is the union of two paths among the d_1 minimal paths connecting v_1 à v_2.

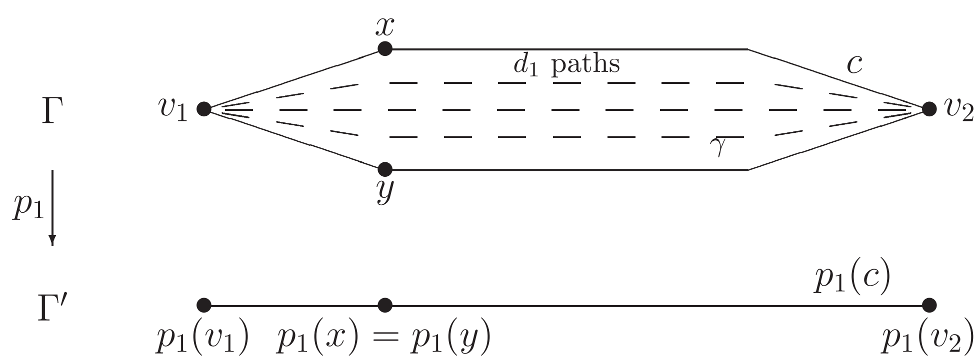

Figure 4. The minimal paths connecting v_1 and v_2.

It follows from a similar argument that c contains exactly two vertices w_1 and w_2 fixed by G_2 and that this cycle is the union of two paths among the d_2 minimal paths connecting w_1 to w_2. Since the complexity of M is minimal, Lemma 5.11 shows that v_1 and v_2 are not fixed by G_2 and that w_1 and w_2 are not fixed by G_1.

Claim 2. $p_2(v_1) = p_2(v_2)$ and $p_1(w_1) = p_1(w_2)$.

Proof. Suppose that $d(v_1, w_1) \neq d(v_1, w_2)$ and, for example, that v_1 is closer to w_1 than w_2. Consider a path γ among the $d_1 - 2$ minimal paths connecting v_1 and v_2 and that are not contained in c (see figure 4). Since Γ' is a tree, the interior of γ contains at least a vertex z whose image $p_2(z)$ belongs to the closed segment $[p_2(v_1), p_2(v_2)] \subset p_2(c)$. Therefore $z \in p_2^{-1}(p_2(c))$ and the minimal path β from z to w_1 is also contained in $p_2^{-1}(p_2(c))$ (see figure 5).

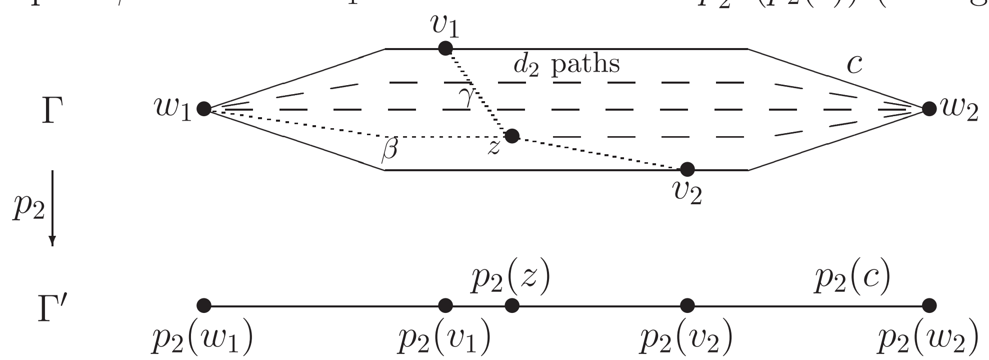

FIGURE 5. A cycle shorter than c.

Since $d(z, w_1) < d(z, v_2) + d(v_2, w_1)$, then

$$\mathrm{len}(v_1 z w_1 v_1) < \mathrm{len}(\gamma) + \mathrm{len}(v_2 w_1 v_1) = 2\,\mathrm{len}(\gamma) = \mathrm{len}(c).$$

Then $v_1 z w_1 v_1$ is a shorter cycle than c, which contradicts the minimality of c. Thus $d(v_1, w_1) = d(v_1, w_2)$ which implies that $d(v_2, w_1) = d(v_2, w_2)$. By Lemma 5.6, $p_2(v_1) = p_2(v_2)$ and it follows similarly that $p_1(w_1) = p_1(w_2)$.

Claims 1 and 2 allow the notation

$$p_1^{-1}(p_1(w_1)) = \{w_j\}_{j=1,\dots,d_1}$$
$$p_2^{-1}(p_2(v_1)) = \{v_i\}_{i=1,\dots,d_2}$$

By a similar reasoning of the proof of Claim 1, we deduce that the vertices v_i are fixed by G_1 and that the vertices w_j are fixed by G_2.

Claim 3. $g_1^k(g_2^l(x)) = g_2^l(g_1^k(x))$, for every vertex $x \in c$.

Proof. If $x = v_1$, then $g_1^k(g_2^l(x)) = g_2^l(x) = g_2^l(g_1^k(x))$ and similarly for $x = v_2$, w_1 and w_2.

Suppose now, without loss of generality, that x belongs to the interior of the sub-path $v_1 w_1$ of c of length $\mathrm{len}(c)/4$. Then the vertex $g_1^k(g_2^l(x))$ belongs to the interior of the path $g_1^k(g_2^l(v_1))g_1^k(g_2^l(w_1)) = g_2^l(v_1)g_1^k(w_1)$, and the same for the vertex $g_2^l(g_1^k(x))$. Since $g_2^l(v_1)$ is fixed by G_1, we have

$$d\left(g_1^k(g_2^l(x)), g_2^l(v_1)\right) = d\left(g_1^k(g_2^l(x)), g_1^k(g_2^l(v_1))\right) = d(x, v_1).$$

Analogously $d\left(g_2^l(g_1^k(x)), g_2^l(v_1)\right) = d(x, v_1)$. Since $g_1^k(g_2^l(x))$ and $g_2^l(g_1^k(x))$ belong to the same path, they coincide.

Consider the graph $C \subseteq \Gamma$ associated with the families of vertices $\{v_i\}_{i=1,\ldots,d_2}$ and $\{w_j\}_{j=1,\ldots,d_1}$, given by

$$C = \bigcup_{\substack{k=1,\ldots,d_1 \\ l=1,\ldots,d_2}} g_1^k(g_2^l(c)) = \bigcup_{\substack{k=1,\ldots,d_1 \\ l=1,\ldots,d_2}} g_2^l(g_1^k(c)).$$

The graph C is invariant by G_1 and G_2 and has the following properties. A vertex $x \in C - \{v_1, \ldots, v_{d_2}, w_1, \ldots, w_{d_1}\}$ belongs to the unique minimal path $v_i w_j$. This path and one of its images $p_1(x)$ or $p_2(x)$ determine x.

For any vertex x in C consider the union Υ_x of all components of $\Gamma - x$ that don't contain $C - x$.

Claim 4. Υ_x is a tree, for every $x \in C$.

Proof. Suppose that x is not fixed by G_1. Then $g_1(\Upsilon_x) = \Upsilon_{g_1(x)}$, because C is invariant by G_1. Therefore Υ_x is isomorphic to $p_1(\Upsilon_x)$. Since Γ' is a tree, Υ_x is also a tree (maybe disconnected and maybe empty).

If x is fixed by G_1, that is, if $x \in \{v_1, \ldots, v_{d_2}\}$, then x is not fixed by G_2. Therefore, the same reasoning by using G_2 shows that Υ_x is a tree.

Claim 5. $\Gamma = C \cup \bigcup_{x \in C} \Upsilon_x$.

Proof. Suppose there were paths in $\Gamma - C$, which connect two different vertices of C. Among these, take a path γ of minimal length and note x_1 and x_2 its endpoints. Then $p_i(\gamma)$ is a path which is not contained in $p_i(C)$ and that connects two vertices of $p_i(C)$. Since Γ' has no cycles, these two vertices are identified, therefore $p_i(x_1) = p_i(x_2)$, $i = 1, 2$.

The vertices x_1 and x_2 belong to unique minimal paths $v_{i_1} w_{j_1}$ and $v_{i_2} w_{j_2}$ in C, respectively. Since $p_1(x_1) = p_1(x_2)$, Lemma 5.6 shows that $v_{i_1} = v_{i_2}$ and $d(x_1, v_{i_1}) = d(x_2, v_{i_2})$. Since $p_2(x_1) = p_2(x_2)$, we obtain analogously $w_{j_1} = w_{j_2}$. Therefore, $x_1 = x_2$, which contradicts the definition of γ.

It follows that each vertex of $\Gamma - C$ is in a Υ_x.

Claim 5 shows that the position of C in Γ is that represented in Figure 6.

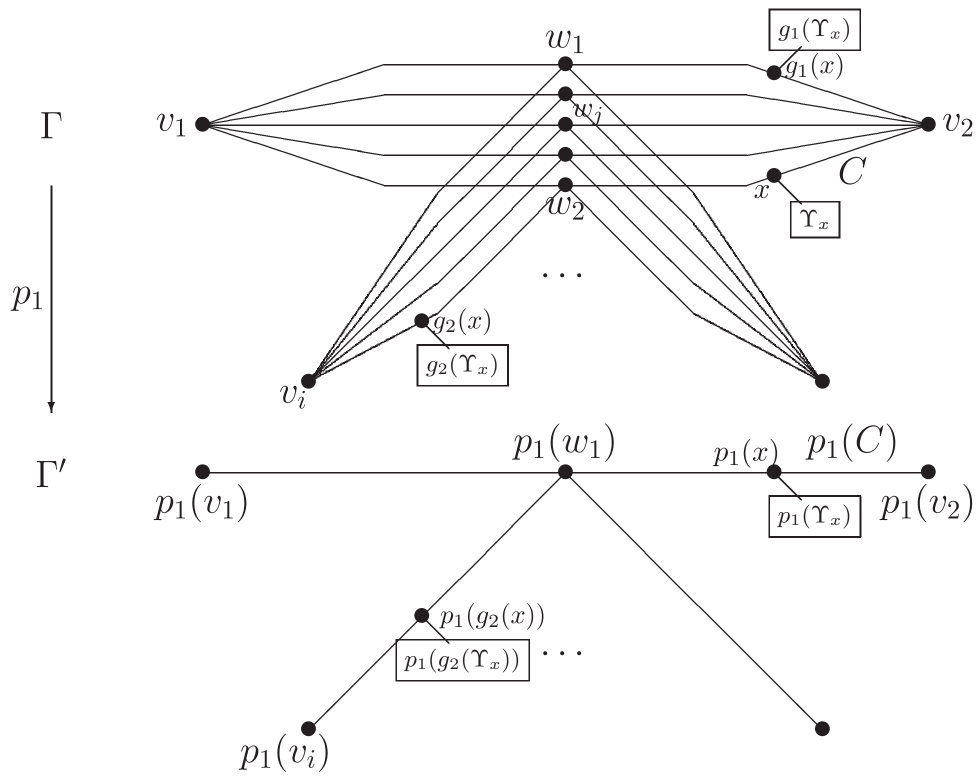

FIGURE 6. Position of C in Γ.

We associate now a notion of depth to each vertex of the tree Γ'.

Definition. We say that the *depth* of a vertex v of Γ' is:

- 0 if v is a terminal vertex of $\Gamma_0 = \Gamma'$;
- n if v is a terminal vertex of $\Gamma'_n = \mathsf{I}(\Gamma'_{n-1})$.

The *depth* of a sub-graph X of Γ' is the maximal depth of the vertices of X in Γ'.

The fact that both vertices are adjacent doesn't mean that their depths are consecutive integers, as is illustrated in Figure 7.

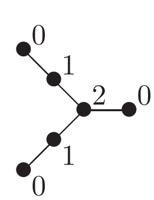

FIGURE 7. The depth of the vertices.

The depth of a vertex of a tree is related to its distance to the terminal vertices. Let x be a vertex of Γ' and X a subset of Γ'. Note

$$r_x(X) = \max_{y \in X} d(x, y),$$

the radius of X relative to x. If $X = \emptyset$, we write $r_x(X) = 0$. The relation between the depth and the distance is given by the following property.

Property. Let x be an interior vertex of Γ' and $\Gamma'^1, \ldots, \Gamma'^k$ the components of $\Gamma' - x$, ordered such that $r_x(\Gamma'^1) \geq \ldots \geq r_x(\Gamma'^k)$. Then,

(i) $\mathrm{depth}(x) = r_x(\Gamma'^2)$;

(ii) if $r_x(\Gamma'^1) = \ldots = r_x(\Gamma'^s) > r_x(\Gamma'^{s+1})$, with $s > 1$, then x is the deepest vertex of Γ' and its valence in $\Gamma'_{\mathrm{depth}(\Gamma')-1}$ is s.

Proof. For each $n \leq \mathrm{depth}(x)$, consider the radii $r_x(\Gamma'^i \cap \Gamma'_n)$. These are related by

$$r_x(\Gamma'^i \cap \Gamma'_n) = r_x(\Gamma'^i \cap \Gamma'_{n-1}) - 1,$$

when $\Gamma'^i \cap \Gamma'_{n-1} \neq \emptyset$. The integer $r_x(\Gamma'^2 \cap \Gamma'_n)$ is positive if and only if $x \in \Gamma'_{n+1}$. By definition of depth, $r_x(\Gamma'^2)$ coincides with $\mathrm{depth}(x)$.

Suppose now that $r_x(\Gamma'^1) = \ldots = r_x(\Gamma'^s) = a > r_x(\Gamma'^{s+1})$. Then Γ'_{a-1} is a star with $s+1$ vertices centred in x.

Claim 6. The graph $\Gamma'_{\mathrm{depth}(\Gamma')-1}$ is a star whose saturated vertex is $p_1(w_1) = p_2(v_1)$ with valence $d_1 = d_2$.

Proof. For each $x \in C - \{v_1, \ldots, v_{d_2}\}$, the tree Υ_x is isomorphic to the trees $p_1(\Upsilon_x)$ and $p_1(g_2(\Upsilon_x))$, which shows that

$$r_{p_1(x)}(p_1(\Upsilon_x)) = r_{p_1(g_2(x))}(p_1(g_2(\Upsilon_x))).$$

On the other hand, a tree Υ_{v_i} is not necessarily isomorphic to $p_1(\Upsilon_{v_i})$. Nevertheless, Lemma 5.6 shows that the distances to v_i are preserved by p_1, from which it follows the same equality for $x = v_i$.

Therefore, the d_2 components of $\Gamma' - p_1(w_1)$ which contain the vertices $p_1(v_i)$ have the same radius a relative to $p_1(w_1)$. We will show that the radius relative to $p_1(w_1)$ of the remaining components of $\Gamma' - p_1(w_1)$ (those in $p_1(\Upsilon_{w_1})$) is smaller than a. The previous claim shows then that $p_1(w_1)$ is the deepest vertex of Γ' and that the valence of this vertex in $\Gamma'_{\mathrm{depth}(\Gamma')-1}$ is d_2. the same reasoning for p_2 shows that $p_2(v_1)$ is also the deepest vertex of Γ' and that its valence in $\Gamma'_{\mathrm{depth}(\Gamma')-1}$ is d_1.

Let X be the connected component of $p_1(\Upsilon_{w_1})$ of maximal radius relative to $p_1(w_1)$. Suppose that $r_{p_1(w_1)}(X) = b \geq a$. Then the diameter of Γ' (that is, the maximal distance between two points of Γ') is at most $2b$. Let u and v be two distinct pre-images by p_1 of the vertex at distance b from $p_1(w_1)$. We may choose, for example, $u \in \Upsilon_{w_1}$ and $v \in \Upsilon_{w_2}$. Then one of the minimal paths γ between u and v passes successively by w_1, v_1 and w_2. The projection $p_2(\gamma)$ is a minimal path between $p_2(u)$ and $p_2(v)$ and passes successively by $p_2(w_1)$, $p_2(v_1)$ and $p_2(w_2)$. The same reasoning of the first paragraph shows that the length of each sub-path uw_1, w_1v_1, v_1w_2 and w_2v is not changed by the projection p_2. It follows that $\mathrm{len}(\gamma) = \mathrm{len}(p_2(\gamma))$ (see Figure 6 for the projection p_2). Then,

$$d(p_2(u), p_2(v)) = d(p_2(u), p_2(v_1)) + d(p_2(v_1), p_2(v)) > 2b,$$

which is impossible. □

5.5. Γ' is a complete graph. In this section we show that the degree is determined also in certain cases when Γ' is not a tree.

Suppose, as in the previous sections, that there exist graphed manifolds M and M', a prime link $L' \subset M'$ and two cyclic branched coverings $p_i : M \to (M', L')$, $i = 1, 2$, of degrees $d_1 < d_2$.

Notation. We note S the (complete) subgraph determined by the saturated vertices of Γ and by the edges that connect them. We define analogously S'.

The following lemma is valid for every graph.

Lemma 5.13. *If $d_1 \neq d_2$, the number of vertices of S and S' is the same.*

Proof. By Lemma 5.3, the action of one of the groups, say G_1, is of type $(1, d)$ for one of the Seifert pieces M^v of the geometric decomposition of M. Then the vertex v, the edges incident on v and their endpoints (that is, the star E_v centred in v) are fixed by G_1. Therefore

$$p_1(E_v) = E'_{v'},$$

where $E'_{v'} \cong E_v$ is the star in Γ' centred in $v' = p_1(v)$. Since the saturated vertices of Γ' belong to $E'_{v'}$, for each vertex w' of S' there is a vertex $w \in \Gamma$ such that $p_1^{-1}(w') = \{w\}$. Therefore $w \in \mathsf{S}$. Reciprocally, if $w \in \mathsf{S}$, then $p_1(w) \in \mathsf{S}'$. Then $p_{1|\mathsf{S}}$ is a bijection between S and S'. □

The following corollary generalises Proposition 5.8.

Corollary 5.14. *If Γ' contains a single saturated vertex, then $d_1 = d_2$.*

Proof. Suppose that $d_1 \neq d_2$. By the previous Lemma, Γ contains also a single saturated vertex. Since the projection of a saturated vertex is a saturated vertex, Lemma 5.5 shows that $d_1 = d_2$. □

Proposition 5.15. *If the exterior E' of L' is a nontrivial graphed manifold whose geometric graph is complete, then there exists at most one prime number d for which M is a cyclic branched covering of (M', L'), of degree d.*

Proof. By the proof of Lemma 5.13, we see that $p_{1|\mathsf{S}}$ is a bijection between S and S', from which $p_1(\Gamma - \mathsf{S}) = \Gamma' - \mathsf{S}'$. Since $\Gamma' = \mathsf{S}'$, it follows that $\Gamma = \Gamma' = K_n$. It follows from Proposition 5.2 that $d_1 = d_2$. □

6. Uniqueness of the action

In the case where E' is hyperbolic, it was shown in [6] that the action is unique when M' is an integral homology sphere. However, when the JSJ decomposition of E' contains a Seifert manifold, there is no uniqueness of the action. Examples are given at the end of the section.

Let M and E' be Seifert fibred spaces with a unique Seifert fibration. Then, G_1 and G_2 are conjugated to groups which preserve the Seifert fibration of M. It follows from the proof of the equality of the order of G_1 and G_2, that both actions have the same type. We note G_i the subgroup of $\mathrm{Diff}^+(\mathsf{B}M)$ induced by the action of G_i.

Consider first the case where both actions are of type $(1, d)$. The branching order of a fibre with Seifert invariant (α, β) is $k = \gcd(\alpha, d)$. Therefore, a fibre of M with Seifert invariant (α, β) is pointwise fixed by G_i if and only if $d|\alpha$. Therefore $\mathrm{Fix}(G_1) = \mathrm{Fix}(G_2)$, that is, $L_1 = L_2$.

Both groups G_1 and G_2 induce actions on the common exterior E of L_i. These actions are also of type $(1, d)$ and have no fixed points. Therefore they are included in the Seifert action of the circle on E. Since they have the same order, they are conjugated over E. Therefore G_1 and G_2 are conjugated on M [6], that is the action is unique in this case.

Suppose now that both actions are of type $(d, 1)$. Then $\mathsf{G}_i \subset \mathrm{Diff}^+(\mathsf{B}M)$ are both cyclic groups of order d.

Proposition 6.1. *If the actions of G_1 and G_2 are conjugated in* $\mathrm{Diff}^+(M)$ *then the actions of* G_1 *and* G_2 *are conjugated in* $\mathrm{Diff}^+(\mathsf{B}M)$.

Proof. Let $f \in \mathrm{Diff}^+(M)$ such that $fG_2f^{-1} = G_1$. Then $f(L_1) = L_2$, therefore f restrict to a diffeomorphism $f : E_1 \to E_2$ that conjugates $G_{1|E_1}$ and $G_{2|E_2}$. We obtain the following commutative diagram,

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ E' & \xrightarrow{f'} & E' \end{array}$$

where f' is an automorphism of the exterior E' of L'.

Since E' is a manifold with boundary and the Seifert fibration of $M' - L'$ is unique, f' is isotopic to a fibred diffeomorphism $g' : E' \to E'$ [9]. This isotopy lifts to an isotopy from f to a fibred diffeomorphism $g : E_1 \to E_2$ that conjugates $G_{1|E_1}$ and $G_{2|E_2}$.

Therefore g projects over a diffeomorphism $\mathfrak{g} : \mathsf{B}E_1 \to \mathsf{B}E_2$ that conjugates $\mathsf{G}_{1|\mathsf{B}E_1}$ and $\mathsf{G}_{2|\mathsf{B}E_2}$. Since the actions of G_i on the neighbourhoods of points of $\mathsf{B}M$ corresponding to the fibres of L_i are rotations of order d around these points, the diffeomorphism $\mathfrak{g} : \mathsf{B}E_1 \to \mathsf{B}E_2$ extends to a diffeomorphism $\mathfrak{g} : \mathsf{B}M \to \mathsf{B}M$ that conjugates G_1 and G_2. □

Therefore, to study the uniqueness of the action we need to consider a similar problem on surfaces.

Question. *Let* S, S' *be two 2-orbifolds and let G_1 and G_2 be two subgroups of* $\mathrm{Diff}^+(\mathsf{S})$ *of order d such that* $\mathsf{S}/G_1 \cong \mathsf{S}/G_2 \cong \mathsf{S}'$. *Are the actions of G_1 and G_2 conjugated in* $\mathrm{Diff}^+(\mathsf{S})$*?*

In [2] Bogopol'skiĭ has given several examples of orbifolds S and S' for which this question has a negative answer. In the case of a cyclic group of prime order, it is shown, for example, that if S is diffeomorphic to the closed orientable surface T_4 of genus 4 and S' is the orbifold $\mathbb{S}^2(5,5,5,5)$, then there are three pairwise non conjugated actions of $\mathbb{Z}_5$ on $\mathbb{S}$ with quotient S'.

Proposition 6.2. *Let $M \cong (4|-5/42; 5/2, 10/3, 30/7)$ and M' be the Brieskorn sphere $\Sigma(2,3,7)$. Then there are three cyclic actions of order 5 on M, pairwise non conjugated, whose quotient is M' with branching set L' formed by the tree singular fibres and one regular fibre of M'.*

Proof. The base of the Seifert fibration of M is $T_4(2,3,7)$. Each of the three non conjugated actions of $\mathbb{Z}_5$ over T_4 given by Bogopol'skiĭ has four fixed points. Therefore they induce three non conjugated actions $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}_3$ over $T_4(2,3,7)$ that fix the three singular points and a regular point. The quotient of each one of these actions is $\mathbb{S}^2(5,10,15,35)$.

Let T_4^* be the surface T_4 with 4 punctures. The diffeomorphisms $\mathfrak{g}_i : T_4^* \to T_4^*$ induce fibred diffeomorphisms $g_i : E \to E$, where $E \cong T_4^* \times \mathbb{S}^1$ is the exterior of $L \subset M$ formed by the three singular fibres and a regular fibre. Since the meridian of each connected component of L is of the form $\mu = \alpha q + \beta h$, where $5|\beta$, the actions $g_i : E \to E$ of type $(5,1)$ preserve these meridians. Then they extend to actions of $\mathbb{Z}_5$ over M that preserve the Seifert fibration, which are of type $(5,1)$, and that fix the three singular fibres and a regular fibre of M.

By Proposition 6.1, these actions are not conjugated. By Lemma 4.5, the quotient is O with the Seifert fibration

$$\left(0 \,\middle|\, \frac{-1}{42}; \left(\frac{1}{2}\right)_5, \left(\frac{2}{3}\right)_5, \left(\frac{6}{7}\right)_5, \left(\frac{0}{1}\right)_5 \right).$$

The underlying space of O is then the Brieskorn sphere $\Sigma(2,3,7)$ [4] and its singular set is formed by the three singular fibres and a regular fibre of $\Sigma(2,3,7)$. The orbifold O has then the topological type (M', L') described in the statement of the proposition. □

Finally we show that the examples of non uniqueness of the action in the case of Seifert manifolds induce examples of non uniqueness of the action when the decomposition of M en geometric pieces is nontrivial and contains a Seifert piece.

Let M be a Seifert manifold and (M', L') a pair for which there exists two non conjugated cyclic actions G_1 and G_2 on M such that $p_{G_i} : M \to (M', L')$ are cyclic branched coverings. Take a regular fibre h' in $M' - L'$ and let

$h_1^i, \dots, h_d^i$ be the fibres in $p_{G_i}^{-1}(h')$. We remark that $p_{G_1}^{-1}(h')$ and $p_{G_2}^{-1}(h')$ have the same number of components by section 4.

Note V' a tubular saturated neighbourhood of h' and $E' = M' - \operatorname{int} V'$ the exterior of h' in M'. Then $p_{G_1}^{-1}(E') \cong p_{G_2}^{-1}(E')$, since every fibre h_j^i is regular. Note this pre-image E.

Glue to $\partial V' \subseteq \partial E'$ any manifold N by a diffeomorphism $\varphi' : \partial V' \to T \subseteq \partial N$ in such a way that the manifold $\overline{M'} = E' \cup_\varphi N$ is not a Seifert manifold. Similarly, glue to each connected component of $p_i^{-1}(\partial V')$ a copy N_i of N by using the diffeomorphism defined by $\varphi_i(x) = \varphi'(p_i(x))$. Both manifolds $E \cup_{\varphi_i} (N_1 \cup \cdots \cup N_d)$ that we obtain are diffeomorphic. We obtain therefore a manifold $\overline{M}$ which is not Seifert.

Then the actions of G_1 and G_2 extend to $\overline{M}$ and it follows that $\overline{M}$ is a cyclic covering of $\overline{M'}$ branched over L' in two non conjugated ways. $\square$

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