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A NOTE ON MODULAR CLASSES OF LIE ALGEBROIDS

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Abstract: We introduce the modular class of a twisted Jacobi structure on a Lie algebroid and establish a relation between this modular class and the modular class of the underlying Lie algebroid.

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1. Introduction

The *modular class* of a Poisson manifold is one of the most basic invariants in Poisson geometry. It can be described as the obstruction to the existence of an invariant density with respect to all Hamiltonian vector fields. Moreover, unlike other invariants, it can be computed readily in many examples. The modular class was first defined by Weinstein as a classical counterpart of the modular automorphisms group of a von-Neumman algebra [16], although the notion of *modular vector field*, which depends on a choice of density, had already been introduced by Kozsul in [11] and used by Dufour and Haraki [2] to classify quadratic Poisson structures.

The relevance of the modular class in Poisson geometry led to generalizations by several authors to many different geometric settings: Even, Lu and Weinstein [3] defined the modular class of a Lie algebroid; Kosmann-Schwarzbach and Gengoux [9] the modular class of a twisted Poisson structure; Grabowsky, Marmo and Michor [5] the relative modular class of a Lie algebroid morphism; Vaisman [15] and Iglesias, Lopez, Marrero and Padron [7] the modular class of a Jacobi structure.

The aim of this paper is to introduce the modular class of a twisted Jacobi structure and to relate it with the notion of modular class of a Lie algebroid in the sense of Evans, Lu and Weinstein. In this way, we are able to unify the concept of modular class of Poisson, twisted Poisson and Jacobi structures. We also show how to generalize to the Jacobi setting the relationship between the modular class of a twisted Poisson structure and the

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modular class of the Lie algebroid morphism associated to it, discovered by Kosmann-Schwarzbach and Weinstein [10].

The paper is organized as follows. In Section 2, we recall the definition of the modular class of a Lie algebroid due to Evans, Lu and Weinstein and the relative modular class of a Lie algebroid morphism. In Section 3, we present the modular class of a triangular Lie bialgebroid, as a straightforward generalization of the definition of the modular class of a Poisson manifold. Section 4 is devoted to the modular class of twisted Poisson structures on a Lie algebroid and its relation with the modular class of the underlying morphism of Lie algebroids. Section 5 deals with modular classes on Jacobi algebroids: we give the definition of modular class of a Jacobi bivector on a Jacobi algebroid, generalizing the case of Jacobi manifolds, and then we relate it to the modular class of the underlying morphism of algebroids. Finally, in the last section, we generalize all this to twisted Jacobi algebroids.

2. Modular class of a Lie algebroid

Let $(A, [\,,\,], \rho)$ be a Lie algebroid over the manifold M. For simplicity we will assume that both M and A are orientable, so that there exist non-vanishing sections $\eta \in \Gamma(\wedge^{\mathrm{top}})(A)$ and $\mu \in \Omega^{\mathrm{top}}(M)$.

The *modular form* of the Lie algebroid A with respect to $\eta\otimes\mu$ is the A-form $\xi_A^{\eta\otimes\mu} \in \Gamma(A^*)$ defined by

$$\langle \xi_A^{\eta\otimes\mu}, X\rangle \eta\otimes\mu = [X,\eta]\otimes\mu + \eta\otimes\mathcal{L}_{\rho(X)}\mu, \quad X\in\Gamma(A). \tag{1}$$

One checks that this A-form is closed, so defines a 1-cocycle in the Lie algebroid cohomology of A. Moreover, if one makes a different choice of sections η' and μ', so that $\eta'\otimes\mu' = f\eta\otimes\mu$, for some non-vanishing smooth function $f\in C^\infty(M)$, one finds:

$$\xi_A^{\eta'\otimes\mu'} = \xi_A^{\eta\otimes\mu} - \mathrm{d}\log|f|. \tag{2}$$

This means that the cohomology class $[\xi_A^{\eta\otimes\mu}] \in H^1(A)$ is independent of the choice of η and μ. This cohomology class is called the *modular class* of A and we will denote it by $\mathrm{mod}\,A$.

Let $\phi : (A, [\,,\,]_A, \rho_A) \to (B, [\,,\,]_B, \rho_B)$ be a Lie algebroid morphism over the identity. One defines its *relative modular class* (see [5, 10]) as the cohomology class $\mathrm{mod}^\phi(A,B) \in H^1(A)$ given by:

$$\mathrm{mod}^{\,\phi}(A,B) := \mathrm{mod}\,A - \phi^*\mathrm{mod}\,B. \tag{3}$$

The relative modular class of ϕ is represented by the A-form $W_{\eta\otimes\mu}$ defined by

$$W_{\eta\otimes\mu}(X)\eta\otimes\mu = [X,\eta]_A\otimes\mu + \eta\otimes\mathcal{L}^B_{\phi(X)}\mu, \tag{4}$$

where $\eta\otimes\mu\in\Gamma(\wedge^{\mathrm{top}}A)\otimes\Gamma(\wedge^{\mathrm{top}}B^*)$ and $\mathcal{L}^B$ denotes the Lie derivative in B.

3. Modular class of a triangular Lie bialgebroid

Now consider a Lie algebroid $(A,[\,,\,],\rho)$ over a manifold M, equipped with a Poisson bivector P, i.e., a section $P\in\Gamma(\wedge^2 A)$ such that $[P,P]=0$. The bracket on sections of A^* defined by

$$[\alpha,\beta]_P = \mathcal{L}_{P^\sharp\alpha}\beta - \mathcal{L}_{P^\sharp\beta}\alpha - \mathrm{d}\left(P(\alpha,\beta)\right), \quad \alpha,\beta\in\Gamma(A^*),$$

is a Lie bracket and $(A^*,[\,,\,]_P,\rho\circ P^\sharp)$ is a Lie algebroid. The differential of this Lie algebroid is $\mathrm{d}_P=[P,-]$. Since d_P is a derivation of the bracket $[\,,\,]$ and the differential d of A is a derivation of the bracket $[\,,\,]_P$, the pair (A,A^*) is a Lie bialgebroid. In fact it is a special kind of Lie bialgebroid called a triangular Lie bialgebroid.

The definition of the modular class of a Lie algebroid with a Poisson tensor is a straightforward generalization of the same concept in Poisson manifolds. Let us recall it. A non-vanishing top-section of A^*, $\mu\in\Gamma(\wedge^n A^*)$, defines an isomorphism $\Phi:\Gamma(\wedge^k A)\to\Gamma(\wedge^{n-k}A^*)$, given by

$$\Phi(X) = i_X\mu. \tag{5}$$

Associated with such a section, we have a *curl operator* $D_\mu:\Gamma(\wedge^k A)\to\Gamma(\wedge^{k-1}A)$ which is defined by

$$D_\mu = \Phi^{-1}\circ\mathrm{d}\circ\Phi. \tag{6}$$

When $k=1$, the curl of an A-section X is just the *divergence* of X, i.e., the function on M given by:

$$D_\mu X = \frac{\mathcal{L}_X\mu}{\mu}, \quad X\in\Gamma(A), \tag{7}$$

Definition 1. *The* ***modular vector field*** *of a Poisson structure P on a Lie algebroid A is the curl of the bivector field P:*

$$X_\mu = D_\mu P. \tag{8}$$

Equivalently, the modular vector field of the Poisson structure P is the A-section X_μ that satisfies:

$$\langle \alpha, X_\mu \rangle \mu = -\alpha \wedge \mathrm{d} i_P \mu, \quad \alpha \in \Gamma(A^*). \tag{9}$$

Using this expression, it is immediate to check that $\mathrm{d}_P X_\mu = 0$.

The modular vector field X_μ depends on the choice of μ as indicated by the notation. However, if $\mu' \in \Gamma(\wedge^n A^*)$ is another non-vanishing top-section, so that $\mu' = f\mu$ for some non-vanishing function $f \in C^\infty(M)$, one can check that

$$X_{\mu'} = X_\mu + \mathrm{d}_P \ln |f|. \tag{10}$$

It follows that the A^*-cohomology class $[X_\mu]$ is independent of the choice of top-section. The class

$$\mathrm{mod}\, P := [X_\mu] \in H^1(A^*)$$

is called the *modular class of the Poisson bivector P*.

Example 2. *Let us take a Poisson bivector on $A = TM$. Then M is a Poisson manifold with a Poisson tensor P,* $\mathrm{mod}\, P$ *is the modular class of (M, P), as defined by Weinstein* [16]*, and we have the following simple relation between the modular classes:*

$$\mathrm{mod}\, T^*M = 2\, \mathrm{mod}\, P. \tag{11}$$

Relation (11) does not hold for general triangular Lie bialgebroids. As it was first explained by Kosmann-Schwarzbach and Weinstein in [10], one needs to take into account the relative modular class of the Lie algebroid morphism $P^\sharp : A^* \to A$. This will be discussed in the next section in the more general setting of Lie algebroids with a twisted-Poisson bivector.

We finish this section with the following simple remark. The relative modular class of the Lie algebroid morphism $P^\sharp : A^* \to A$ is given by

$$\mathrm{mod}^{P^\sharp}(A^*, A) = \mathrm{mod}\, A^* - (P^\sharp)^* \mathrm{mod}\, A = \mathrm{mod}\, A^* + P^\sharp\, \mathrm{mod}\, A. \tag{12}$$

It follows from (4) that a representative of this modular class is the A-section W_η defined by

$$\langle \alpha, W_\eta \rangle \eta \otimes \eta = [\alpha, \eta]_P \otimes \eta + \eta \otimes \mathcal{L}_{P^\sharp(\alpha)} \eta, \quad \alpha \in \Gamma(A^*), \tag{13}$$

where $\eta \in \Gamma(\wedge^{\mathrm{top}} A^*)$.

4. Modular class of a twisted Poisson structure

Twisted Poisson structures on manifolds were first studied by Ševera and Weinstein [14] and then extended to Lie algebroids by Roytenberg [13]. We explain now how one can define its modular class.

Given any bivector field P on a Lie algebroid $(A, [\,,\,], \rho)$, the associated morphism $P^\sharp : \Gamma(A^*) \to \Gamma(A)$ can be extended to a homomorphism $\wedge^k P^\sharp$: $\Gamma(\wedge^k A^*) \to \Gamma(\wedge^k A)$, by setting:

$$\wedge^0 P^\sharp(f) = f \quad \text{and} \quad \wedge^k P^\sharp \eta(\alpha_1, \ldots, \alpha_k) = (-1)^k \eta(P^\sharp(\alpha_1), \ldots, P^\sharp(\alpha_k)), \tag{14}$$

for $f \in C^\infty(M)$, $\eta \in \Gamma(\wedge^k A^*)$ and $\alpha_1, \ldots, \alpha_k \in \Gamma(A^*)$.

A *twisted Poisson structure* on a Lie algebroid $(A, [\,,\,], \rho)$ is a pair (P, ψ), where P is a section of $\wedge^2 A$ (a bivector) and ψ is a section of $\wedge^3 A^*$ (a 3-form), such that

$$\frac{1}{2}[P, P] = (\wedge^3 P^\sharp)\psi \quad \text{and} \quad \mathrm{d}\psi = 0.$$

It induces a structure of Lie algebroid on A^* with anchor $\rho_* = \rho \circ P^\sharp$ and Lie bracket given by

$$[\alpha, \beta]_{P,\psi} = \mathcal{L}_{P^\sharp\alpha}\beta - \mathcal{L}_{P^\sharp\beta}\alpha - \mathrm{d}(P(\alpha, \beta)) + \psi(P^\sharp\alpha, P^\sharp\beta, -), \quad \alpha, \beta \in \Gamma(A^*). \tag{15}$$

The differential of this Lie algebroid is given by

$$\mathrm{d}_{P,\psi} f = [P, f] = -P^\sharp(\mathrm{d}f) \quad \text{and} \quad \mathrm{d}_{P,\psi} X = [P, X]_A + \psi(P^\sharp -, P^\sharp -, X), \tag{16}$$

for all $f \in C^\infty(M)$ and $X \in \Gamma(A)$.

Remark 3. *The pair (A, A^*) is a special kind of quasi-Lie bialgebroids, called a* triangular quasi-Lie bialgebroid *(see [13]).*

The modular vector field of a twisted Poisson structure was defined in [9] using generators of the Gerstanhaber algebra $(\Gamma(\wedge^\bullet A^*), [\,,\,]_P)$ (see also [8]). Fix a global section η of $\wedge^{\text{top}} A^*$. The *modular vector field of the twisted Poisson structure* (P, ψ) is the section of A given by

$$Z_\eta = X_\eta + P^\sharp i_P \psi, \tag{17}$$

where X_η is defined by

$$\langle \alpha, X_\eta \rangle \eta = -\alpha \wedge \mathrm{d}i_P\eta, \quad \alpha \in \Gamma(A^*).$$

Now, as before, one checks that the A^*-cohomology class of Z_η is independent of the choice of top-section.

Definition 4. *The* ***modular class of the twisted Poisson structure*** (P, ψ) *is the cohomology class*

$$\mathrm{mod}\,(P, \psi) := [Z_\eta] \in H^1(A^*).$$

The general relationship between the modular class of a (twisted) Poisson structure and the underlying Lie algebroid, generalizing Example 2, is given by the following proposition:

Proposition 5 ([10]). *Let (A, P, ψ) be a Lie algebroid with a twisted Poisson structure (P, ψ). Then*

$$2\,\mathrm{mod}\,(P, \psi) = \mathrm{mod}^{\,P^\sharp}(A^*, A). \tag{18}$$

5. Modular class of a Jacobi algebroid

In this section we will generalize the concept of modular class to twisted Jacobi structures. First we start with some background about Jacobi algebroids.

A *Jacobi algebroid* [4] or *generalized Lie algebroid* [6] is a pair (A, ϕ_0) where $A = (A, [\,,\,], \rho)$ is a Lie algebroid over a manifold M and $\phi_0 \in \Gamma(A^*)$ is a 1-cocycle in the Lie algebroid cohomology with trivial coefficients, $\mathrm{d}\phi_0 = 0$.

Associated with a Jacobi algebroid we have the following geometric data:

- The *Schouten-Jacobi bracket* on the graded algebra $\Gamma(\wedge^\bullet A)$ of multivector fields on A given by

$$[P, Q]^{\phi_0} = [P, Q] + (p-1) P \wedge i_{\phi_0} Q - (-1)^{p-1}(q-1) i_{\phi_0} P \wedge Q, \tag{19}$$

  for $P \in \Gamma(\wedge^p A), Q \in \Gamma(\wedge^q A)$.
- The ϕ_0-*differential* of A given by

$$\mathrm{d}^{\phi_0}\omega = \mathrm{d}\omega + \phi_0 \wedge \omega, \quad \omega \in \Gamma(\wedge^\bullet A^*). \tag{20}$$

  d^{ϕ_0} is the cohomology operator associated with the representation of the Lie algebra $\Gamma(A)$ on $C^\infty(M)$ given by $\rho^{\phi_0}(X) f = \rho(X) f + f \langle \phi_0, X\rangle$.
- The ϕ_0-*Lie derivative*:

$$\mathcal{L}^{\phi_0}_X \omega = i_X\, \mathrm{d}^{\phi_0}\omega + (-1)^{p-1} \mathrm{d}^{\phi_0}\, i_X \omega, \quad X \in \Gamma(\wedge^p A), \omega \in \Gamma(\wedge^\bullet A^*). \tag{21}$$

Consider the vector bundle $\hat{A} = A \times \mathbb{R}$ over $M \times \mathbb{R}$. Sections of this bundle may be seen as time-dependent sections of A and they are generated as a $C^\infty(M \times \mathbb{R})$-module by the sections of A (which are simply the time-independent sections of $\hat{A}$). If one introduces an anchor map on $\hat{A}$ by

$$\hat{\rho}(X) = \rho(X) + \langle \phi_0, X\rangle \frac{\partial}{\partial t}, \quad X \in \Gamma(A), \tag{22}$$

and a graded Lie bracket on time independent multivectors by

$$[X, Y]_{\hat{A}} = [X, Y], \quad X, Y \in \Gamma(\wedge^\bullet A), \tag{23}$$

together they define a Lie algebroid structure on $\hat{A}$, called the *induced Lie algebroid structure from A by ϕ_0*. Notice that, if $\hat{\mathrm{d}}$ is the differential in $\hat{A}$, we have $\phi_0 = \hat{\mathrm{d}}\, t$, which means that the 1-cocycle ϕ_0 can be seen as an exact 1-form on $\hat{A}$.

The Jacobi-Schouten bracket (19) on A and the Schouten bracket (23) on $\hat{A}$ are related by:

$$\left[\tilde{X}, \tilde{Y}\right]_{\hat{A}} = \widetilde{[X, Y]^{\phi_0}}, \tag{24}$$

where $X \to \tilde{X}$ denotes the gauging on $\Gamma(\wedge^\bullet A)$ defined by

$$\tilde{X} = e^{-(p-1)t} X, \quad X \in \Gamma(\wedge^p A). \tag{25}$$

Definition 6. *A **Jacobi bivector** on A is a bivector $P \in \Gamma(\wedge^2 A)$ such that*

$$[P, P]^{\phi_0} = 0. \tag{26}$$

A Jacobi bivector P on A determines a Poisson bivector $\tilde{P}$ on $\hat{A}$, by gauging $\tilde{P} = e^{-t}P$, and hence a Lie algebroid structure on $\hat{A}^*$ over $M \times \mathbb{R}$. The anchor on $\hat{A}^*$ is given by:

$$\hat{\rho}_*(\alpha) = \hat{\rho} \circ \tilde{P}^\sharp(\alpha) \tag{27}$$

while the Lie bracket is given by:

$$[\alpha, \beta]_{\tilde{P}} = \widehat{\mathcal{L}}_{\tilde{P}^\sharp\alpha}\beta - \widehat{\mathcal{L}}_{\tilde{P}^\sharp\beta}\alpha - \hat{\mathrm{d}}\tilde{P}(\alpha, \beta), \tag{28}$$

where $\alpha, \beta \in \Gamma(\hat{A}^*)$ and $\widehat{\mathcal{L}}$ is the Lie derivative in $\hat{A}$.

In particular, if α, $\beta \in \Gamma(A^*)$ are time-independent sections, we find that

$$\left[e^t\alpha, e^t\beta\right]_{\tilde{P}} = e^t(\mathcal{L}^{\phi_0}_{P^\sharp\alpha}\beta - \mathcal{L}^{\phi_0}_{P^\sharp\beta}\alpha - \mathrm{d}^{\phi_0}P(\alpha, \beta)). \tag{29}$$

This shows that the Lie bracket

$$[\alpha, \beta]_P = \mathcal{L}^{\phi_0}_{P^\sharp\alpha}\beta - \mathcal{L}^{\phi_0}_{P^\sharp\beta}\alpha - \mathrm{d}^{\phi_0}P(\alpha, \beta), \tag{30}$$

together with the anchor

$$\rho_* = \rho \circ P^\sharp, \tag{31}$$

endows A^* with a Lie algebroid structure over M. Finally, the section of A defined by

$$X_0 := -P^\sharp(\phi_0),$$

is a 1-cocycle of A^*. We conclude that (A^*, X_0) is also a Jacobi algebroid.

Since $\mathrm{d}_*^{X_0}$ is a derivation of $(\Gamma(\wedge^\bullet A), [\,,\,]^{\phi_0})$ or, equivalently, d^{ϕ_0} is a derivation of $(\Gamma(\wedge^\bullet A^*), [\,,\,]_*^{X_0})$, the pair $((A, \phi_0), (A^*, X_0))$ is a special kind of Jacobi bialgebroid called *triangular Jacobi bialgebroid* and we will denote it by (A, ϕ_0, P).

Now, the gauging (25) can be extended to multisections of A^* if we set:

$$\hat{\omega} = e^{pt}\omega, \quad \omega \in \Gamma(\wedge^p A^*), \tag{32}$$

and one checks that:

Proposition 7. *Let α, β be multisections of A^*. Then*

$$\left[\hat{\alpha}, \hat{\beta}\right]_{\tilde{P}} = \widehat{[\alpha, \beta]}_P \tag{33}$$

and $P^\sharp : A^* \to A$ is a Lie algebroid morphism over identity.

Let A have rank n and ν be a section of $\wedge^n A^*$. From (9), the modular vector field of the triangular Lie bialgebroid $(\hat{A}, \tilde{P})$ with respect to $\hat{\nu} = e^{nt}\nu$ is the section $\hat{X}^{\hat{\nu}} \in \Gamma(\hat{A})$ defined by:

$$\langle \alpha, \hat{X}^{\hat{\nu}} \rangle \hat{\nu} = -\alpha \wedge \hat{\mathrm{d}}\, i_{\tilde{P}} \hat{\nu}, \quad \alpha \in \Gamma(\hat{A}^*)$$

The following lemma follows from a straightforward computation:

Lemma 8. *The modular vector field of $(\hat{A}, \tilde{P})$ is given by:*

$$\hat{X}^{\hat{\nu}}(\alpha)\hat{\nu} = e^{-t}\alpha \wedge ((1-n)\phi_0 \wedge i_P\hat{\nu} - \mathrm{d} i_P \hat{\nu}), \quad \alpha \in \Gamma(A^*). \tag{34}$$

This motivates the definition of the *modular vector field of the Jacobi structure* (P, ϕ_0): it is the A-section $X^\nu_{(A,\phi_0,P)}$ defined by

$$\begin{aligned}\langle \alpha, X^\nu_{(A,\phi_0,P)} \rangle \nu &= -\alpha \wedge ((n-1)\phi_0 \wedge i_P\nu + \mathrm{d} i_P\nu) \\ &= (n-1)\langle \alpha, X_0 \rangle \nu - \alpha \wedge \mathrm{d} i_P \nu.\end{aligned}$$

This definition generalizes the definition of modular vector field of a Jacobi manifold given by Vaisman [15], which one recovers when we set $A = TM$. Again, one checks that it is 1-cocycle in the A^*-cohomology, and that its class is independent of the choice of top-section.

Definition 9 ([7]). *The **modular class of the triangular Jacobi algebroid** is the cohomology class:*

$$\mathrm{mod}\,(A, P, \phi_0) := [X^{\nu}_{(A,\phi_0,P)}] \in H^1(A^*).$$

Just like in the Poisson case, we have the following relation between the modular class of the Jacobi structure and the relative modular class of the morphism $P^\sharp$.

Proposition 10. *Let (A, P, ϕ_0) be a triangular Jacobi algebroid of rank n. Then*

$$2\,\mathrm{mod}\,(A, P, \phi_0) = \mathrm{mod}\,^{P^\sharp}(A^*, A) + n\,[X_0]\,. \tag{35}$$

Proof: The relation follows immediately from the definition of $X^{\nu}_{(A,\phi_0,P)}$ and the following relation obtained in [1]:

$$\xi^{\nu\otimes\mu}_{A^*} + P^\sharp(\xi^{\eta\otimes\mu}_{A}) = (n-2)X_0 - 2\langle \alpha \wedge \mathrm{d}i_P\nu, \eta\rangle,$$

where $\eta \in \Gamma(\wedge^n A)$ and $\nu \in \Gamma(\wedge^n A^*)$ are such that $\langle \nu, \eta\rangle = 1$. ■

6. Modular classes of twisted Jacobi algebroids

A *twisted Jacobi structure* on a Jacobi algebroid $A = (A, \phi_0)$ (also called a *twisted Jacobi algebroid* [12]), is a pair (P, ψ), where P is a bivector and ψ is a 3-form on A, such that

$$\frac{1}{2}\,[P, P]^{\phi_0} = \wedge^3 P^\sharp(\psi) \quad \text{and} \quad \mathrm{d}\psi = \psi \wedge \phi_0. \tag{36}$$

Proposition 11. *The Lie algebroid $\hat{A} = A \times \mathbb{R}$ is a twisted Poisson Lie algebroid.*

Proof: The twisted Poisson structure on $\hat{A}$ is given by the gauged bivector $\tilde{P} = e^{-t}P$ and the 3-form $\overline{\psi} = e^t\psi$. Relation (24) and (36) imply that

$$\frac{1}{2}\left[\tilde{P}, \tilde{P}\right] = \frac{e^{-2t}}{2}\,[P, P]^{\phi_0} = e^{-2t} \wedge^3 P^\sharp(\psi) = \wedge^3 \tilde{P}^\sharp(\overline{\psi})$$

and, since $\mathrm{d}\psi = \psi \wedge \phi_0$, we have

$$\hat{\mathrm{d}}\overline{\psi} = \hat{\mathrm{d}}(e^t\psi) = e^t(\phi_0 \wedge \psi + \mathrm{d}\psi) = 0.$$

■

The twisted Poisson structure on $\hat{A}$ induces a Lie algebroid structure on $\hat{A}^*$ (see (15)), and this defines a Lie algebroid structure on A^* with anchor $\rho_* = \rho \circ P^\sharp$ and Lie bracket

$$[\alpha, \beta]_{P,\psi} = \mathcal{L}^{\phi_0}_{P^\sharp\alpha} - \mathcal{L}^{\phi_0}_{P^\sharp\beta} - \mathrm{d}^{\phi_0}(P(\alpha, \beta)) + \psi(P^\sharp\alpha, P^\sharp\beta, -), \quad \alpha, \beta \in \Gamma(A^*). \tag{37}$$

Several properties of the untwisted case still remain valid:

Proposition 12. *Let α and β be multisections on A^*.Then*

(1) $\left[\hat{\alpha}, \hat{\beta}\right]_{\tilde{P},\overline{\psi}} = \widehat{[\alpha, \beta]_{P,\psi}}$.
(2) $P^\sharp : A^* \to A$ *is a Lie algebroid morphism over the identity:* $P^\sharp [\alpha, \beta]_{P,\psi} = \left[P^\sharp\alpha, P^\sharp\beta\right]$.
(3) *The A-section $X_0 = -P^\sharp(\phi_0)$ is a 1-cocycle on A^*.*

Given a section $\nu \in \Gamma(\wedge^{\mathrm{top}} A^*)$, we have the induced section $\hat{\nu} = e^{nt}\nu$ of $\hat{A}^*$. We can use this section to represent the modular class of the twisted Poisson structure $(\tilde{P}, \overline{\psi})$ by the modular vector field associated with $\hat{\nu}$ (see (17)):

$$Z^{\hat{\nu}} = X^{\hat{\nu}} + \tilde{P}^\sharp i_{\tilde{P}}\overline{\psi}, \tag{38}$$

where $X^{\hat{\nu}}$ is given by

$$\begin{aligned}
\langle \alpha, X^{\hat{\nu}} \rangle \hat{\nu} &= -\alpha \wedge \hat{\mathrm{d}}\, i_{\tilde{P}}\hat{\nu} \\
&= -e^{-(n+1)t}\alpha \wedge ((n-1)\phi_0 \wedge i_P\nu + \mathrm{d}i_P\nu) \\
&= e^{-t}\left((n-1)\langle \alpha, X_0 \rangle \hat{\nu} - \alpha \wedge \mathrm{d}i_P\hat{\nu}\right).
\end{aligned}$$

Since $\tilde{P}^\sharp i_{\tilde{P}}\overline{\psi} = e^{-t}P^\sharp i_P\psi$, we see that $e^t Z_{\hat{\nu}}$ is a time independent section of $\hat{A}$, and so it is determined by a section $X^\nu_{\phi_0,P,\psi}$ of A. One checks that $X^\nu_{\phi_0,P,\psi}$ is a 1-cocycle for the A^*-cohomology and that its cohomology class is independent of the choice of top-form ν. This motivates the following definition:

Definition 13. *The **modular vector field of the twisted Jacobi structure** (ϕ_0, P, ψ) is the A^*-cohomology class:*

$$\mathrm{mod}\,(\phi_0, P, \psi) := [X^\nu_{\phi_0,P,\psi}] \in H^1(A^*).$$

The natural question arises about its relation with the modular class of the Lie algebroid A^*.

Proposition 14. *Let (A, ϕ_0) be a Jacobi algebroid with a twisted Jacobi structure (P, ψ). Then*

$$2\,\mathrm{mod}\,(\phi_0, P, \psi) = \mathrm{mod}^{\,P^\sharp}(A^*, A) + n\,[X_0]\,. \tag{39}$$

Proof: Let $\nu \in \Gamma(\wedge^n A^*)$ be a top-form on A, then $\hat{\nu} = e^{nt}\nu$ is a top-form on $\hat{A}$. The relative modular class of $\tilde{P}^\sharp$ is represented by the $\hat{A}$-section $W^{\hat{\nu}}$ (see (12)) defined by:

$$\begin{aligned}
\langle \hat{\alpha}, W^{\hat{\nu}} \rangle \hat{\nu} \otimes \hat{\nu} &= [\hat{\alpha}, \hat{\nu}]_{\tilde{P}} \otimes \hat{\nu} + \hat{\nu} \otimes \widehat{\mathcal{L}}_{\tilde{P}^\sharp \hat{\alpha}} \hat{\nu} \\
&= e^{nt}\,[\alpha, \nu]_P \otimes \hat{\nu} + \hat{\nu} \otimes \widehat{\mathcal{L}}_{P^\sharp \alpha}(e^{nt}\nu) \\
&= e^{nt}\,[\alpha, \nu]_P \otimes \hat{\nu} + \hat{\nu} \otimes e^{nt}(n\langle \phi_0, P^\sharp \alpha \rangle \nu + \mathcal{L}_{P^\sharp \alpha} \nu) \\
&= \langle \alpha, nX_0 + W_\nu \rangle \hat{\nu} \otimes \hat{\nu},
\end{aligned}$$

where W_ν is a representative of the relative modular class of the morphism $P^\sharp$. Since $W^{\hat{\nu}} = 2Z^{\hat{\nu}}$ (see, also, [10]) we conclude that

$$\langle \alpha, nX_0 + W_\nu \rangle = \langle \hat{\alpha}, W^{\hat{\nu}} \rangle = 2\langle \hat{\alpha}, Z^{\hat{\nu}} \rangle = 2\langle \alpha, e^t Z^{\hat{\nu}} \rangle = 2\langle \alpha, X^\nu_{\phi_0, P, \psi} \rangle,$$

and $2\mathrm{mod}\,(\phi_0, P, \psi) = \mathrm{mod}^{\,P^\sharp}(A^*, A) + nX_0$. ∎

Remark 15. *This result includes as special cases the (twisted) Poisson structures and the Jacobi structures: when ϕ_0 is zero, P is a twisted Poisson structure and we recover (18); when ψ is zero, then P is a Jacobi structure and we recover relation (35).*

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