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ON A DOUBLY NONLINEAR DIFFUSION MODEL OF CHEMOTAXIS WITH PREVENTION OF OVERCROWDING

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Abstract: This paper addresses the existence and regularity of weak solutions for a fully parabolic model of chemotaxis, with prevention of overcrowding, that degenerates in a two-sided fashion, including an extra nonlinearity represented by a p-Laplacian diffusion term. To prove the existence of weak solutions, a Schauder fixed-point argument is applied to a regularized problem and the compactness method is used to pass to the limit. The local Hölder regularity of weak solutions is established using the method of intrinsic scaling. The results are a contribution to showing, qualitatively, to what extent the properties of the classical Keller-Segel chemotaxis models are preserved in a more general setting. Some numerical examples illustrate the model.

Keywords: Chemotaxis, reaction-diffusion equations, degenerate PDE, parabolic p-Laplacian, doubly nonlinear, intrinsic scaling.
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1. Introduction

1.1. Scope. It is the purpose of this paper to study the existence and regularity of weak solutions of the following parabolic system, which is a generalization of the well-known Keller-Segel model [1, 2, 3] of chemotaxis:

$$\partial_t u - \operatorname{div}\left(|\nabla A(u)|^{p-2}\nabla A(u)\right) + \operatorname{div}\left(\chi u f(u)\nabla v\right) = 0 \quad \text{in } Q_T, \tag{1.1a}$$

$$\partial_t v - d\Delta v = g(u,v) \quad \text{in } Q_T, \tag{1.1b}$$

$$|\nabla A(u)|^{p-2} a(u)\frac{\partial u}{\partial \eta} = 0, \quad \frac{\partial v}{\partial \eta} = 0 \quad \text{on } \Sigma_T := \partial\Omega \times (0,T), \tag{1.1c}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad \text{on } \Omega, \tag{1.1d}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, with a sufficiently smooth boundary $\partial\Omega$ and outer unit normal η, and $Q_T := \Omega \times (0,T)$, for some $T > 0$. Equation (1.1a) is *doubly nonlinear*, since we apply the p-Laplacian diffusion

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operator, where we assume $2 \leq p < \infty$, to the integrated diffusion function $A(u) := \int_0^u a(s)\,ds$, where $a(\cdot)$ is a non-negative integrable function with support on the interval $[0,1]$.

In the biological phenomenon described by (1.1), the quantity $u = u(x,t)$ is the density of organisms, such as bacteria or cells. The conservation PDE (1.1a) incorporates two competing mechanisms, namely the density-dependent diffusive motion of the cells, described by the doubly nonlinear diffusion term, and a motion in response to and towards the gradient ∇v of the concentration $v = v(x,t)$ of a substance called *chemoattractant*. The movement in response to ∇v also involves the density-dependent probability $f(u(x,t))$ for a cell located at (x,t) to find space in a neighboring location, and a constant χ describing chemotactic sensitivity. On the other hand, the PDE (1.1b) describes the diffusion of the chemoattractant, where $d > 0$ is a diffusion constant and the function $g(u,v)$ describes the rates of production and degradation of the chemoattractant; we here adopt the common choice

$$g(u,v) = \alpha u - \beta v, \quad \alpha, \beta \geq 0. \tag{1.2}$$

We assume that there exists a maximal population density of cells u_{m} such that $f(u_{\mathrm{m}}) = 0$. This corresponds to a switch to repulsion at high densities, known as prevention of overcrowding, volume-filling effect or density control (see [4]). It means that cells stop to accumulate at a given point of Ω after their density attains a certain threshold value, and the chemotactic cross-diffusion term $\chi u f(u)$ vanishes identically when $u \geq u_{\mathrm{m}}$. We also assume that the diffusion coefficient $a(u)$ vanishes at 0 and u_{m}, so that (1.1a) degenerates for $u = 0$ and $u = u_{\mathrm{m}}$, while $a(u) > 0$ for $0 < u < u_{\mathrm{m}}$. A typical example is $a(u) = \epsilon u(1 - u_{\mathrm{m}})$, $\epsilon > 0$. Normalizing variables by $\tilde{u} = u/u_{\mathrm{m}}$, $\tilde{v} = v$ and $\tilde{f}(\tilde{u}) = f(\tilde{u}u_{\mathrm{m}})$, we have $\tilde{u}_{\mathrm{m}} = 1$; in the sequel we will omit tildes in the notation.

The main intention of the present work is to address the question of the regularity of weak solutions, which is a delicate analytical issue since the structure of equation (1.1a) combines a degeneracy of p-Laplacian type with a two-sided point degeneracy in the diffusive term. We prove the local Hölder continuity of the weak solutions of (1.1) using the method of intrinsic scaling (see [5, 6]). The novelty lies in tackling the two types of degeneracy simultaneously and finding the right geometric setting for the concrete structure of the PDE. The resulting analysis combines the technique used by Urbano

[7] to study the case of a diffusion coefficient $a(u)$ that decays like a power at both degeneracy points (with $p = 2$) with the technique by Porzio and Vespri [8] to study the p-Laplacian, with $a(u)$ degenerating at only one side. We recover both results as particular cases of the one studied here. To our knowledge, the p-Laplacian is a new ingredient in chemotaxis models, so we also include a few numerical examples that illustrate the behavior of solutions of (1.1) for $p > 2$, compared with solutions to the standard case $p = 2$, but including nonlinear diffusion.

1.2. Related work. To put this paper in the proper perspective, we recall that the Keller-Segel model is a widely studied topic, see e.g. Murray [3] for a general background and Horstmann [1] for a fairly complete survey on the Keller-Segel model and the variants that have been proposed. Nonlinear diffusion equations for biological populations that degenerate at least for $u = 0$ were proposed in the 1970s by Gurney and Nisbet [9] and Gurtin and McCamy [10]; more recent works include those by Witelski [11], Dkhil [12], Burger et al. [13] and Bendahmane et al. [4]. Furthermore, well-posedness results for these kinds of models include, for example, the existence of radial solutions exhibiting chemotactic collapse [14], the local-in-time existence, uniqueness and positivity of classical solutions, and results on their blow-up behavior [15], and existence and uniqueness using the abstract theory developed in [16], see [17]. Burger et al. [13] prove the global existence and uniqueness of the Cauchy problem in $\mathbb{R}^N$ for linear and nonlinear diffusion with prevention of overcrowding. The model proposed herein exhibits an even higher degree of nonlinearity, and offers further possibilities to describe chemotactic movement; for example, one could imagine that the cells or bacteria are actually placed in a medium with a non-Newtonian rheology. In fact, the evolution p-Laplacian equation $u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, is also called non-Newtonian filtration equation, see [18] and [19, Chapter 2] for surveys. Coming back to the Keller-Segel model, we also mention that another effort to endow this model with a more general diffusion mechanism has recently been made by Biler and Wu [20], who consider fractional diffusion.

Various results on the Hölder regularity of weak solutions to quasilinear parabolic systems are based on the work of DiBenedetto [5]; the present article also contributes to this direction. Specifically for a chemotaxis model, Bendahmane, Karlsen, and Urbano [4] proved the existence and Hölder regularity of weak solutions for a version of (1.1) for $p = 2$. For a detailed description of the intrinsic scaling method and some applications we refer to the books [5, 6].

Concerning uniqueness of solution, the presence of a nonlinear degenerate diffusion term and a nonlinear transport term represents a disadvantage and we could not obtain the uniqueness of a weak solution. This contrasts with the results by Burger et al. [13], where the authors prove uniqueness of solutions for a degenerate parabolic-elliptic system set in an unbounded domain, using a method which relies on a continuous dependence estimate from [21], that does not apply to our problem because it is difficult to bound Δv in $L^\infty(Q_T)$ due to the parabolic nature of (1.1b).

1.3. Weak solutions and statement of main results. Before stating our main results, we give the definition of a weak solution to (1.1), and recall the notion of certain functional spaces. We denote by p' the conjugate exponent of p (we will restrict ourselves to the degenerate case $p \geq 2$): $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, $C_w(0, T, L^2(\Omega))$ denotes the space of continuous functions with values in (a closed ball of) $L^2(\Omega)$ endowed with the weak topology, and $\langle \cdot, \cdot \rangle$ is the duality pairing between $W^{1,p}(\Omega)$ and its dual $(W^{1,p}(\Omega))'$.

Definition 1.1. *A weak solution of (1.1) is a pair (u, v) of functions satisfying the following conditions:*

$$
\begin{aligned}
&0 \leq u(x,t) \leq 1 \text{ and } v(x,t) \geq 0 \text{ for a.e. } (x,t) \in Q_T,\\
&u \in C_w\big(0, T, L^2(\Omega)\big), \quad \partial_t u \in L^{p'}\big(0, T; (W^{1,p}(\Omega))'\big), \quad u(0) = u_0,\\
&A(u) = \int_0^u a(s)\, ds \in L^p\big(0, T; W^{1,p}(\Omega)\big),\\
&v \in L^\infty(Q_T) \cap L^r\big(0, T; W^{1,r}(\Omega)\big) \cap C\big(0, T, L^r(\Omega)\big) \quad \text{for all } r > 1,\\
&\partial_t v \in L^2\big(0, T; (H^1(\Omega))'\big), \quad v(0) = v_0,
\end{aligned}
$$

and, for all $\varphi \in L^p(0, T; W^{1,p}(\Omega))$ and $\psi \in L^2(0, T; H^1(\Omega))$,

$$
\int_0^T \langle \partial_t u, \varphi \rangle \, dt + \iint_{Q_T} \Big\{ |\nabla A(u)|^{p-2} \nabla A(u) - \chi u f(u) \nabla v \Big\} \cdot \nabla \varphi \, dx\, dt = 0,
$$

$$
\int_0^T \langle \partial_t v, \psi \rangle \, dt + d \iint_{Q_T} \nabla v \cdot \nabla \psi \, dx\, dt = \iint_{Q_T} g(u,v) \psi \, dx\, dt.
$$

To ensure, in particular, that all terms and coefficients are sufficiently smooth for this definition to make sense, we require that $f \in C^1[0,1]$ and $f(1) = 0$, and assume that the diffusion coefficient $a(\cdot)$ has the following properties: $a \in C^1[0,1]$, $a(0) = a(1) = 0$, and $a(s) > 0$ for $0 < s < 1$. Moreover, we assume that there exist constants $\delta \in (0, 1/2)$ and $\gamma_2 \geq \gamma_1 > 1$ such that

$$\begin{aligned} &\gamma_1 \phi(s) \leq a(s) \leq \gamma_2 \phi(s) \quad \text{for } s \in [0, \delta] \\ &\gamma_1 \psi(1-s) \leq a(s) \leq \gamma_2 \psi(1-s) \quad \text{for } s \in [1-\delta, 1], \end{aligned} \tag{1.3}$$

where we define the functions $\phi(s) := s^{\beta_1/(p-1)}$ and $\psi(s) := s^{\beta_2/(p-1)}$ for $\beta_2 > \beta_1 > 0$.

Our first main result is the following existence theorem for weak solutions.

Theorem 1.1. *If $u_0, v_0 \in L^\infty(\Omega)$ with $0 \leq u_0 \leq 1$ and $v_0 \geq 0$ a.e. in Ω, then there exists a weak solution to the degenerate system (1.1) in the sense of Definition 1.1.*

In Section 2, we first prove the existence of solutions to a regularized version of (1.1) by applying the Schauder fixed-point theorem. The regularization basically consists in replacing the degenerate diffusion coefficient $a(u)$ by the regularized, strictly positive diffusion coefficient $a_\varepsilon(u) := a(u) + \varepsilon$, where $\varepsilon > 0$ is the regularization parameter. Once the regularized problem is solved, we send the regularization parameter ε to zero to produce a weak solution of the original system (1.1) as the limit of a sequence of such approximate solutions. Convergence is proved by means of *a priori* estimates and compactness arguments.

We denote by $\partial_t Q_T$ the parabolic boundary of Q_T, define $\tilde{M} := \|u\|_{\infty, Q_T}$, and recall the definition of the intrinsic parabolic p-distance from a compact set $K \subset Q_T$ to $\partial_t Q_T$ as

$$p\text{-dist}(K; \partial_t Q_T) := \inf_{(x,t) \in K,\ (y,s) \in \partial_t Q_T} \left(|x - y| + \tilde{M}^{(p-2)/p} |t - s|^{1/p} \right).$$

Our second main result is the interior local Hölder regularity of weak solutions.

Theorem 1.2. *Let u be a bounded local weak solution of (1.1) in the sense of Definition 1.1, and $\tilde{M} = \|u\|_{\infty, Q_T}$. Then u is locally Hölder continuous in*

Q_T, i.e., there exist constants $\gamma > 1$ and $\alpha \in (0,1)$, depending only on the data, such that, for every compact $K \subset Q_T$,

$$\big|u(x_1,t_1) - u(x_2,t_2)\big| \le \gamma \tilde{M} \left\{ \frac{|x_1 - x_2| + \tilde{M}^{(p-2)/p} |t_2 - t_1|^{1/p}}{p\text{-dist}(K; \partial_t Q_T)} \right\}^{\alpha},$$

$$\forall (x_1,t_1), (x_2,t_2) \in K.$$

In Section 3, we prove Theorem 1.2 using the method of intrinsic scaling. This technique is based on analyzing the underlying PDE in a geometry dictated by its own degenerate structure, that amounts, roughly speaking, to accommodate its degeneracies. This is achieved by rescaling the standard parabolic cylinders by a factor that depends on the particular form of the degeneracies and on the oscillation of the solution, and which allows for a recovery of homogeneity. The crucial point is the proper choice of the intrinsic geometry which, in the case studied here, needs to take into account the p-Laplacian structure of the diffusion term, as well as the fact that the diffusion coefficient $a(u)$ vanishes at $u = 0$ and $u = 1$. At the core of the proof is the study of an alternative, now a standard type of argument [5]. In either case the conclusion is that when going from a rescaled cylinder into a smaller one, the oscillation of the solution decreases in a way that can be quantified.

In the statement of Theorem 1.2 and its proof, we focus on the interior regularity of u; that of v follows from classical theory of parabolic PDEs [22]. Moreover, standard adaptations of the method are sufficient to extend the results to the parabolic boundary, see [5, 23].

1.4. Outline. The remainder of the paper is organized as follows: Section 2 deals with the general proof of our first main result (Theorem 1.1). Section 2.1 is devoted to the detailed proof of existence of solutions to a non-degenerate problem; in Section 2.2 we state and prove a fixed-point-type lemma, and the conclusion of the proof of Theorem 1.1 is contained in Section 2.3. In Section 3 we use the method of intrinsic scaling to prove Theorem 1.2, establishing the Hölder continuity of weak solutions to (1.1). Finally, in Section 4 we present two numerical examples showing the effects of prevention of overcrowding and of including the p-Laplacian term, and in the Appendix we give further details about the numerical method used to treat the examples.

2. Existence of solutions

We first prove the existence of solutions to a non-degenerate, regularized version of problem (1.1), using the Schauder fixed-point theorem, and our approach closely follows that of [4]. We define the following closed subset of the Banach space $L^p(Q_T)$:

$$\mathcal{K} := \left\{ u \in L^p(Q_T) \,:\, 0 \le u(x,t) \le 1 \text{ for a.e. } (x,t) \in Q_T \right\}.$$

2.1. Weak solution to a non-degenerate problem. We define the new diffusion term $A_\varepsilon(s) := A(s) + \varepsilon s$, with $a_\varepsilon(s) = a(s) + \varepsilon$, and consider, for each fixed $\varepsilon > 0$, the non-degenerate problem

$$\partial_t u_\varepsilon - \operatorname{div}\left(|\nabla A_\varepsilon(u_\varepsilon)|^{p-2}\nabla A_\varepsilon(u_\varepsilon)\right) + \operatorname{div}\left(\chi f(u_\varepsilon)\nabla v_\varepsilon\right) = 0 \quad \text{in } Q_T, \tag{2.1a}$$

$$\partial_t v_\varepsilon - d\Delta v_\varepsilon = g(u_\varepsilon, v_\varepsilon) \quad \text{in } Q_T, \tag{2.1b}$$

$$|\nabla A_\varepsilon(u_\varepsilon)|^{p-2} a_\varepsilon(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial \eta} = 0, \quad \frac{\partial v_\varepsilon}{\partial \eta} = 0 \quad \text{on } \Sigma_T, \tag{2.1c}$$

$$u_\varepsilon(x,0) = u_0(x), \quad v_\varepsilon(x,0) = v_0(x) \quad \text{for } x \in \Omega. \tag{2.1d}$$

With $\bar{u} \in \mathcal{K}$ fixed, let v_ε be the unique solution of the problem

$$\partial_t v_\varepsilon - d\Delta v_\varepsilon = g(\bar{u}, v_\varepsilon) \quad \text{in } Q_T, \tag{2.2a}$$

$$\frac{\partial v_\varepsilon}{\partial \eta} = 0 \quad \text{on } \Sigma_T, \quad v_\varepsilon(x,0) = v_0(x) \quad \text{for } x \in \Omega. \tag{2.2b}$$

Given the function v_ε, let u_ε be the unique solution of the following quasilinear parabolic problem:

$$\partial_t u_\varepsilon - \operatorname{div}\left(|\nabla A_\varepsilon(u_\varepsilon)|^{p-2}\nabla A_\varepsilon(u_\varepsilon)\right) + \operatorname{div}\left(\chi u_\varepsilon f(u_\varepsilon)\nabla v_\varepsilon\right) = 0 \quad \text{in } Q_T, \tag{2.3a}$$

$$|\nabla A_\varepsilon(u_\varepsilon)|^{p-2} a_\varepsilon(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial \eta} = 0 \quad \text{on } \Sigma_T, \quad u_\varepsilon(x,0) = u_0(x) \quad \text{for } x \in \Omega. \tag{2.3b}$$

Here v_0 and u_0 are functions satisfying the assumptions of Theorem 1.1.

Since for any fixed $\bar{u} \in \mathcal{K}$, (2.2a) is uniformly parabolic, standard theory for parabolic equations [22] immediately leads to the following lemma.

Lemma 2.1. *If $v_0 \in L^\infty(\Omega)$, then problem (2.2) has a unique weak solution $v_\varepsilon \in L^\infty(Q_T) \cap L^r(0,T;W^{2,r}(\Omega)) \cap C(0,T;L^r(\Omega))$, for all $r > 1$, satisfying in* *particular*

$$\|v_\varepsilon\|_{L^\infty(Q_T)} + \|v_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \le C, \qquad \|v_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le C, \\ \|\partial_t v_\varepsilon\|_{L^2(Q_T)} \le C, \tag{2.4}$$

where $C > 0$ *is a constant depending only on* $\|v_0\|_{L^\infty(\Omega)}$, α, β, *and* $\mathrm{meas}(Q_T)$.

The following lemma (see [22]) holds for the quasilinear problem (2.3).

Lemma 2.2. *If* $u_0 \in L^\infty(\Omega)$, *then, for any* $\varepsilon > 0$, *there exists a unique weak solution* $u_\varepsilon \in L^\infty(Q_T) \cap L^p(0,T;W^{1,p}(\Omega))$ *to problem* (2.3).

2.2. The fixed-point method. We define a map $\Theta : \mathcal{K} \to \mathcal{K}$ such that $\Theta(\bar{u}) = u_\varepsilon$, where u_ε solves (2.3), i.e., Θ is the solution operator of (2.3) associated with the coefficient $\bar{u}$ and the solution v_ε coming from (2.2). By using the Schauder fixed-point theorem, we now prove that Θ has a fixed point. First, we need to show that Θ is continuous. Let $\{\bar{u}_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}$ and $\bar{u} \in \mathcal{K}$ be such that $\bar{u}_n \to \bar{u}$ in $L^p(Q_T)$ as $n \to \infty$. Define $u_{\varepsilon n} := \Theta(\bar{u}_n)$, i.e., $u_{\varepsilon n}$ is the solution of (2.3) associated with $\bar{u}_n$ and the solution $v_{\varepsilon n}$ of (2.2). To show that $u_{\varepsilon n} \to \Theta(\bar{u})$ in $L^p(Q_T)$, we start with the following lemma.

Lemma 2.3. *The solutions* $u_{\varepsilon n}$ *to problem* (2.3) *satisfy*

(i) $0 \le u_{\varepsilon n}(x,t) \le 1$ *for a.e.* $(x,t) \in Q_T$.
(ii) *The sequence* $\{u_{\varepsilon n}\}_n$ *is bounded in* $L^p(0,T;W^{1,p}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.
(iii) *The sequence* $\{u_{\varepsilon n}\}_n$ *is relatively compact in* $L^p(Q_T)$.

Proof: The proof follows from that of Lemma 2.3 in [4] if we take into account that $\{\partial_t u_{\varepsilon n}\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^{p'}(0,T;(W^{1,p}(\Omega))')$. ■

The following lemma contains a classical result (see [22]).

Lemma 2.4. *There exists a function* $v_\varepsilon \in L^2(0,T;H^1(\Omega))$ *such that the sequence* $\{v_{\varepsilon n}\}_{n\in\mathbb{N}}$ *converges strongly to* v *in* $L^2(0,T;H^1(\Omega))$.

Lemmas 2.2–2.4 imply that there exist $u_\varepsilon \in L^p(0,T;W^{1,p}(\Omega))$ and $v_\varepsilon \in L^2(0,T;H^1(\Omega))$ such that, up to extracting subsequences if necessary, $u_{\varepsilon n} \to u_\varepsilon$ strongly in $L^p(Q_T)$ and $v_{\varepsilon n} \to v_\varepsilon$ strongly in $L^2(0,T;H^1(\Omega))$ as $n \to \infty$, so Θ is indeed continuous on $\mathcal{K}$. Moreover, due to Lemma 2.3, $\Theta(\mathcal{K})$ is bounded in the set

$$\mathcal{W} := \left\{ u \in L^p\left(0,T;W^{1,p}(\Omega)\right) \,:\, \partial_t u \in L^{p'}\left(0,T;(W^{1,p}(\Omega))'\right) \right\}.$$

Similarly to the results of [24], it can be shown that $\mathcal{W} \hookrightarrow L^p(Q_T)$ is compact, and thus Θ is compact. Now, by the Schauder fixed point theorem, the operator Θ has a fixed point u_ε such that $\Theta(u_\varepsilon) = u_\varepsilon$. This implies that there exists a solution $(u_\varepsilon, v_\varepsilon)$ of

$$\int_0^T \langle \partial_t u_\varepsilon, \varphi \rangle \, dt + \iint_{Q_T} \Big\{ |\nabla A_\varepsilon(u_\varepsilon)|^{p-2} \nabla A_\varepsilon(u_\varepsilon) - \chi u_\varepsilon f(u_\varepsilon) \nabla v_\varepsilon \Big\} \cdot \nabla \varphi \, dx \, dt = 0,$$

$$\int_0^T \langle \partial_t v_\varepsilon, \psi \rangle \, dt + d \iint_{Q_T} \nabla v_\varepsilon \cdot \nabla \psi \, dx \, dt = \iint_{Q_T} g(u_\varepsilon, v_\varepsilon) \psi \, dx \, dt, \qquad (2.5)$$

$$\forall \varphi \in L^p(0,T; W^{1,p}(\Omega)) \text{ and } \forall \psi \in L^2(0,T; H^1(\Omega)).$$

2.3. Existence of weak solutions. We now pass to the limit $\varepsilon \to 0$ in solutions $(u_\varepsilon, v_\varepsilon)$ to obtain weak solutions of the original system (1.1). From the previous lemmas and considering (2.1b), we obtain the following result.

Lemma 2.5. *For each fixed $\varepsilon > 0$, the weak solution $(u_\varepsilon, v_\varepsilon)$ to (2.1) satisfies the maximum principle*

$$0 \le u_\varepsilon(x,t) \le 1 \quad \text{and} \quad v_\varepsilon(x,t) \ge 0 \quad \text{for a.e. } (x,t) \in Q_T. \qquad (2.6)$$

Moreover, the first two estimates of (2.4) in Lemma 2.1 are independent of ε.

Lemma 2.5 implies that there exists a constant $C > 0$, which does not depend on ε, such that

$$\|v_\varepsilon\|_{L^\infty(Q_T)} + \|v_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \le C, \quad \|v_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le C. \qquad (2.7)$$

Notice that, from (2.6) and (2.7), the term $g(u_\varepsilon, v_\varepsilon)$ is bounded. Thus, in light of classical results on L^r regularity, there exists another constant $C > 0$, which is independent of ε, such that

$$\|\partial_t v_\varepsilon\|_{L^r(Q_T)} + \|v_\varepsilon\|_{L^r(0,T;W^{1,r}(\Omega))} \le C \text{ for all } r > 1.$$

Taking $\varphi = A_\varepsilon(u_\varepsilon)$ as a test function in (2.5) yields

$$\int_0^T \langle \partial_t u_\varepsilon, A(u_\varepsilon) \rangle \, dt + \varepsilon \int_0^T \langle \partial_t u_\varepsilon, u_\varepsilon \rangle \, dt + \iint_{Q_T} |\nabla A_\varepsilon(u_\varepsilon)|^p \, dx \, dt$$

$$- \iint_{Q_T} \chi f(u_\varepsilon) \nabla v_\varepsilon \cdot \nabla A_\varepsilon(u_\varepsilon) \, dx \, dt = 0;$$

then, using (2.7), the uniform L^∞ bound on u_ε, an application of Young's inequality to treat the term $\nabla v_\varepsilon \cdot \nabla A_\varepsilon(u_\varepsilon)$, and defining $\mathcal{A}_\varepsilon(s) := \int_0^s A_\varepsilon(r)\,dr$, we obtain

$$\sup_{0\le t\le T}\int_\Omega \mathcal{A}_\varepsilon(u_\varepsilon)(x,t)\,dx + \varepsilon \sup_{0\le t\le T}\int_\Omega \frac{|u_\varepsilon(x,t)|^2}{2}\,dx + \iint_{Q_T} |\nabla A_\varepsilon(u_\varepsilon)|^p\,dx\,dt \le C \tag{2.8}$$

for some constant $C>0$ independent of ε.

Let $\varphi \in L^p(0,T;W^{1,p}(\Omega))$. Using the weak formulation (2.5), (2.7) and (2.8), we may follow the reasoning in [4] to deduce the bound

$$\|\partial_t u_\varepsilon\|_{L^{p'}(0,T;(W^{1,p}(\Omega))')} \le C. \tag{2.9}$$

Therefore, from (2.7)–(2.9) and standard compactness results (see [24]), we can extract subsequences, which we do not relabel, such that, as $\varepsilon \to 0$,

$$\begin{cases} A_\varepsilon(u_\varepsilon) \to A(u) \text{ strongly in } L^p(Q_T) \text{ and a.e.,}\\ u_\varepsilon \to u \text{ strongly in } L^q(Q_T) \text{ for all } q\ge 1,\\ v_\varepsilon \to v \text{ strongly in } L^2(Q_T),\\ \nabla v_\varepsilon \to \nabla v \text{ weakly in } L^2(Q_T),\\ \nabla A_\varepsilon(u_\varepsilon) \to \nabla A(u) \text{ weakly in } L^p(Q_T),\\ |\nabla A_\varepsilon(u_\varepsilon)|^{p-2}\nabla A_\varepsilon(u_\varepsilon) \to \Gamma_1 \text{ weakly in } L^{p'}(Q_T),\\ v_\varepsilon \to v \text{ weakly in } L^2(0,T;H^1(\Omega)),\\ \partial_t u_\varepsilon \to \partial_t u \text{ weakly in } L^{p'}(0,T;(W^{1,p}(\Omega))'),\\ \partial_t v_\varepsilon \to \partial_t v \text{ weakly in } L^2(0,T;(H^1(\Omega))'). \end{cases} \tag{2.10}$$

To establish the second convergence in (2.10), we have applied the dominated convergence theorem to $u_\varepsilon = A_\varepsilon^{-1}(A_\varepsilon(u_\varepsilon))$ (recall that A is monotone) and the weak-$\star$ convergence of u_ε to u in $L^\infty(Q_T)$. We also have the following lemma, see [4] for its proof.

Lemma 2.6. *The functions v_ε converge strongly to v in $L^2(0,T;H^1(\Omega))$ as $\varepsilon \to 0$.*

Next, we identify Γ_1 as $|\nabla A(u)|^{p-2}\nabla A(u)$ when passing to the limit $\varepsilon \to 0$ in (2.5). Due to this particular nonlinearity, we cannot employ the monotonicity argument used in [4]; rather, we will utilize a Minty-type argument [25] and make repeated use of the following "weak chain rule" (see e.g. [26] for a proof).

Lemma 2.7. *Let $b : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous and nondecreasing. Assume $u \in L^\infty(Q_T)$ is such that*

$$\partial_t u \in L^{p'}(0,T;(W^{1,p}(\Omega))'), \quad b(u) \in L^p(0,T;W^{1,p}(\Omega))$$

and $u(x,0) = u_0(x)$ a.e. on Ω, with $u_0 \in L^\infty(\Omega)$. If we define

$$B(u) = \int_0^u b(\xi) d\xi,$$

then

$$-\int_0^s \langle \partial_t u, b(u)\phi \rangle \, dt = \int_0^s \int_\Omega B(u)\partial_t \phi \, dx \, dt + \int_\Omega B(u_0)\phi(x,0) \, dx - \int_\Omega B(u(x,s))\phi(x,s) \, dx$$

holds for all $\phi \in \mathcal{D}([0,T] \times \Omega)$ and for any $s \in (0,T)$.

Lemma 2.8. *There hold $\Gamma_1 = |\nabla A(u)|^{p-2}\nabla A(u)$ and $\nabla A_\varepsilon(u_\varepsilon) \to \nabla A(u)$ strongly in $L^p(Q_T)$.*

Proof: We define $\mathcal{Q}_T := \{(t,s,x) : (x,s) \in Q_t, \ t \in [0,T]\}$. The first step will be to show that, for all $\sigma \in L^p(0,T;W^{1,p}(\Omega))$,

$$\iiint_{\mathcal{Q}_T} \big(\Gamma_1 - |\nabla \sigma|^{p-2}\nabla\sigma\big) \cdot \big(\nabla A(u) - \nabla \sigma\big) \, dx \, ds \, dt \geq 0. \tag{2.11}$$

For all fixed $\varepsilon > 0$, we have the decomposition

$$\iiint_{\mathcal{Q}_T} \big(|\nabla A_\varepsilon(u_\varepsilon)|^{p-2}\nabla A_\varepsilon(u_\varepsilon) - |\nabla \sigma|^{p-2}\nabla\sigma\big) \cdot \big(\nabla A(u) - \nabla \sigma\big) \, dx \, ds \, dt$$
$$= I_1 + I_2 + I_3,$$
$$I_1 := \iiint_{\mathcal{Q}_T} |\nabla A_\varepsilon(u_\varepsilon)|^{p-2}\nabla A_\varepsilon(u_\varepsilon) \cdot \big(\nabla A(u) - \nabla A_\varepsilon(u_\varepsilon)\big) \, dx \, ds \, dt,$$
$$I_2 := \iiint_{\mathcal{Q}_T} \big(|\nabla A_\varepsilon(u_\varepsilon)|^{p-2}\nabla A_\varepsilon(u_\varepsilon) - |\nabla \sigma|^{p-2}\nabla\sigma\big) \cdot \big(\nabla A_\varepsilon(u_\varepsilon) - \nabla \sigma\big) \, dx \, ds \, dt,$$
$$I_3 := \iiint_{\mathcal{Q}_T} |\nabla \sigma|^{p-2}\nabla\sigma \cdot \big(\nabla A_\varepsilon(u_\varepsilon) - \nabla A(u)\big) \, dx \, ds \, dt.$$

Clearly, $I_2 \geq 0$ and from (2.10) we deduce that $I_3 \to 0$ as $\varepsilon \to 0$. For I_1, if we multiply (2.1a) by $\phi \in L^p(0,T;W^{1,p}(\Omega))$ and integrate over $\mathcal{Q}_T$, we obtain

$$\int_0^T \int_0^t \langle \partial_t u_\varepsilon, \phi \rangle \, ds \, dt - \iiint_{\mathcal{Q}_T} \chi u_\varepsilon f(u_\varepsilon) \nabla v_\varepsilon \cdot \nabla \phi \, dx \, ds \, dt$$
$$+ \iiint_{\mathcal{Q}_T} |\nabla A_\varepsilon(u_\varepsilon)|^{p-2} \nabla A_\varepsilon(u_\varepsilon) \cdot \nabla \phi \, dx \, ds \, dt = 0.$$

Now, if we take $\phi = A(u) - A_\varepsilon(u_\varepsilon) \in L^p(0,T;W^{1,p}(\Omega))$ and use Lemma 2.7, we obtain

$$\begin{aligned} I_1 = &- \int_0^T \int_0^t \langle \partial_t u_\varepsilon, A(u) \rangle \, ds \, dt + \int_0^T \int_0^t \langle \partial_t u_\varepsilon, A_\varepsilon(u_\varepsilon) \rangle \, ds \, dt \\ &+ \iiint_{\mathcal{Q}_T} \chi u_\varepsilon f(u_\varepsilon) \nabla v_\varepsilon \cdot \big(\nabla A(u) - \nabla A_\varepsilon(u_\varepsilon) \big) \, dx \, ds \, dt \\ = &- \int_0^T \int_0^t \langle \partial_t u_\varepsilon, A(u) \rangle \, ds \, dt + \iint_{Q_T} \mathcal{A}_\varepsilon(u_\varepsilon) \, dx \, dt - T \int_\Omega \mathcal{A}_\varepsilon(u_0) \, dx \\ &+ \iiint_{\mathcal{Q}_T} \chi u_\varepsilon f(u_\varepsilon) \nabla v_\varepsilon \cdot \big(\nabla A(u) - \nabla A_\varepsilon(u_\varepsilon) \big) \, dx \, ds \, dt. \end{aligned}$$

Therefore, using (2.10) and Lemma 2.6 and defining $\mathcal{A}(u) := \int_0^u A(s) \, ds$, we conclude that

$$\lim_{\varepsilon \to 0} I_1 = - \int_0^T \int_0^t \langle \partial_t u, A(u) \rangle \, ds \, dt + \int_0^T \int_\Omega \mathcal{A}(u(x,t)) \, dx \, dt - T \int_\Omega \mathcal{A}(u_0(x)) \, dx,$$

and from Lemma 2.7, this yields $I_1 \to 0$ as $\varepsilon \to 0$. Consequently, we have shown that

$$\lim_{\varepsilon \to 0} \iiint_{\mathcal{Q}_T} \big(|\nabla A_\varepsilon(u_\varepsilon)|^{p-2} \nabla A_\varepsilon(u_\varepsilon) - |\nabla \sigma|^{p-2} \nabla \sigma \big) \cdot \big(\nabla A(u) - \nabla \sigma \big) \, dx \, ds \, dt \geq 0,$$

which proves (2.11).

Choosing $\sigma = A(u) - \lambda \xi$ with $\lambda \in \mathbb{R}$ and $\xi \in L^p(0,T;W^{1,p}(\Omega))$ and combining the two inequalities arising from $\lambda > 0$ and $\lambda < 0$, we obtain the first assertion of the lemma. The second assertion directly follows from (2.11). ■

With the above convergences we are now able to pass to the limit $\varepsilon \to 0$, and we can identify the limit (u, v) as a (weak) solution of (1.1). In fact, if $\varphi \in L^p(0,T;W^{1,p}(\Omega))$ is a test function for (2.5), then by (2.10) it is now clear that, as $\varepsilon \to 0$,

$$\int_0^T \langle \partial_t u_\varepsilon, \varphi \rangle \, dt \to \int_0^T \langle \partial_t u, \varphi \rangle \, dt,$$

$$\iint_{Q_T} |\nabla A_\varepsilon(u_\varepsilon)|^{p-2} \nabla A_\varepsilon(u_\varepsilon) \cdot \nabla \varphi \, dx \, dt \to \iint_{Q_T} |\nabla A(u)|^{p-2} \nabla A(u) \cdot \nabla \varphi \, dx \, dt.$$

Since $h(u_\varepsilon) = u_\varepsilon f(u_\varepsilon)$ is bounded in $L^\infty(Q_T)$ and by Lemma 2.6, $v_\varepsilon \to v$ in $L^2(0,T;H^1(\Omega))$, it follows that

$$\iint_{Q_T} \chi u_\varepsilon f(u_\varepsilon) \nabla v_\varepsilon \cdot \nabla \varphi \, dx \, dt \to \iint_{Q_T} \chi u f(u) \nabla v \cdot \nabla \varphi \, dx \, dt \quad \text{as } \varepsilon \to 0.$$

We have thus identified u as the first component of a solution of (1.1). Using a similar argument, we can identify v as the second component of a solution.

3. Hölder continuity of weak solutions

3.1. Preliminaries. We start by recasting Definition 1.1 in a form that involves the Steklov average, defined for a function $w \in L^1(Q_T)$ and $0 < h < T$ by

$$w_h := \begin{cases} \dfrac{1}{h} \displaystyle\int_t^{t+h} w(\cdot, \tau) \, d\tau & \text{if } t \in (0, T-h], \\ 0 & \text{if } t \in (T-h, T]. \end{cases}$$

Definition 3.1. *A local weak solution for (1.1) is a measurable function u such that, for every compact $K \subset \Omega$ and for all $0 < t < T-h$,*

$$\int_{K \times \{t\}} \Big\{ \partial_t(u_h)\varphi + \big(|\nabla A(u)|^{p-2} \nabla A(u)\big)_h \cdot \nabla \varphi - \big(\chi u f(u) \nabla v\big)_h \cdot \nabla \varphi \Big\} \, dx = 0,$$
$$\forall \varphi \in W_0^{1,p}(K). \quad (3.1)$$

The following technical lemma on the geometric convergence of sequences (see e.g., [27, Lemma 4.2, Ch. I]) will be used later.

Lemma 3.1. *Let $\{X_n\}$ and $\{Z_n\}$, $n \in \mathbb{N}_0$, be sequences of positive real numbers satisfying*

$$X_{n+1} \le C b^n \big(X_n^{1+\alpha} + X_n^\alpha Z_n^{1+\kappa} \big), \quad Z_{n+1} \le C b^n \big(X_n + Z_n^{1+\kappa} \big),$$

where $C > 1$, $b > 1$, $\alpha > 0$ and $\kappa > 0$ are given constants. Then $X_n, Z_n \to 0$ as $n \to \infty$ provided that

$$X_0 + Z_0^{1+\kappa} \leq (2C)^{-(1+\kappa)/\sigma} b^{-(1+\kappa)/\sigma^2}, \quad \text{with } \sigma = \min\{\alpha, \kappa\}.$$

3.2. The rescaled cylinders. Let $B_\rho(x_0)$ denote the ball of radius ρ centered at x_0. Then, for a point $(x_0, t_0) \in \mathbb{R}^{n+1}$, we denote the cylinder of radius ρ and height τ by

$$(x_0, t_0) + Q(\tau, \rho) := B_\rho(x_0) \times (t_0 - \tau, t_0).$$

Intrinsic scaling is based on measuring the oscillation of a solution in a family of nested and shrinking cylinders whose dimensions are related to the degeneracy of the underlying PDE. To implement this, we fix $(x_0, t_0) \in Q_T$; after a translation, we may assume that $(x_0, t_0) = (0, 0)$. Then let $\varepsilon > 0$ and let $R > 0$ be small enough so that $Q(R^{p-\varepsilon}, 2R) \subset Q_T$, and define

$$\mu^+ := \operatorname*{ess\,sup}_{Q(R^{p-\varepsilon},2R)} u, \qquad \mu^- := \operatorname*{ess\,inf}_{Q(R^{p-\varepsilon},2R)} u, \qquad \omega := \operatorname*{ess\,osc}_{Q(R^{p-\varepsilon},2R)} u \equiv \mu^+ - \mu^-.$$

Now construct the cylinder $Q(a_0 R^p, R)$, where

$$a_0 = \left(\frac{\omega}{2}\right)^{2-p} \frac{1}{\phi(\omega/2^m)^{p-1}},$$

with m to be chosen later. To ensure that $Q(a_0 R^p, R) \subset Q(R^{p-\varepsilon}, 2R)$, we assume that

$$\frac{1}{a_0} = \left(\frac{\omega}{2}\right)^{p-2} \phi\left(\frac{\omega}{2^m}\right)^{p-1} > R^\varepsilon, \tag{3.2}$$

and therefore the relation

$$\operatorname*{ess\,osc}_{Q(a_0 R^p,R)} u \leq \omega \tag{3.3}$$

holds. Otherwise, the result is trivial as the oscillation is comparable to the radius. We mention that for ω small and for $m > 1$, the cylinder $Q(a_0 R^p, R)$ is long enough in the $t-$direction, so that we can *accommodate* the degeneracies of the problem. Without loss of generality, we will assume $\omega < \delta < 1/2$.

Consider now, inside $Q(a_0 R^p, R)$, smaller subcylinders of the form

$$Q_R^{t^*} \equiv (0, t^*) + Q(dR^p, R), \qquad d = \left(\frac{\omega}{2}\right)^{2-p} \frac{1}{[\psi(\omega/4)]^{p-1}}, \qquad t^* < 0.$$

These are contained in $Q(a_0R^p, R)$ if $a_0R^p \geq -t^* + dR^p$, which holds whenever $\phi(\omega/2^m) \leq \psi(\omega/4)$ and

$$t^* \in \left(\frac{(\omega/2)^{2-p}R^p}{\psi(\omega/4)^{p-1}} - \frac{(\omega/2)^{p-2}R^p}{\phi(\omega/2^m)^{p-1}}, 0 \right).$$

These particular definitions of a_0 and of d turn out to be the natural extensions to the case $p > 2$ of their counterparts in [7]. Notice that for $p = 2$ and $a(u) \equiv 1$, we recover the standard parabolic cylinders.

The structure of the proof will be based on the analysis of the following alternative: either there is a cylinder $Q_R^{t^*}$ where u is essentially away from its infimum, or such a cylinder can not be found and thus u is essentially away from its supremum in all cylinders of that type. Both cases lead to the conclusion that the essential oscillation of u within a smaller cylinder decreases by a factor that can be quantified, and which does *not* depend on ω.

Remark 3.1. *(See* [8, Remark 4.2]*) Let us introduce quantities of the type* $B_iR^\theta\omega^{-b_i}$, *where* B_i *and* $b_i > 0$ *are constants that can be determined a priori from the data, independently of* ω *and* R, *and* θ *depending only on* N *and* p. *We assume without loss of generality, that*

$$B_iR^\theta\omega^{-b_i} \leq 1.$$

If this was not valid, then we would have $\omega \leq CR^\varepsilon$ *for the choices* $C = \max_i B_i^{1/b}$ *and* $\varepsilon = \theta/\min_i b_i$, *and the result would be trivial.*

3.3. The first alternative.

Lemma 3.2. *There exists* $\nu_0 \in (0,1)$, *independent of* ω *and* R, *such that if*

$$\left|\left\{(x,t) \in Q_R^{t^*} \,:\, u(x,t) > 1 - \omega/2\right\}\right| \leq \nu_0\left|Q_R^{t^*}\right| \tag{3.4}$$

for some cylinder of the type $Q_R^{t^*}$, *then* $u(x,t) < 1 - \omega/4$ *a.e. in* $Q_{R/2}^{t^*}$.

Proof: Let $u_\omega := \min\{u, 1 - \omega/4\}$, take the cylinder for which (3.4) holds, define

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}}, \quad n \in \mathbb{N}_0,$$

and construct the family

$$Q_{R_n}^{t^*} := (0, t^*) + Q(dR_n^p, R_n) = B_{R_n} \times (\tau_n, t^*), \quad \tau_n := t^* - dR_n^p, \quad n \in \mathbb{N}_0;$$

note that $Q^{t^*}_{R_n} \to Q^{t^*}_{R/2}$ as $n \to \infty$. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of piecewise smooth cutoff functions satisfying

$$\begin{cases} \xi_n = 1 \text{ in } Q^{t^*}_{R_{n+1}}, \quad \xi_n = 0 \text{ on the parabolic boundary of } Q^{t^*}_{R_n}, \\ |\nabla \xi_n| \le \dfrac{2^{n+1}}{R}, \quad 0 \le \partial_t \xi_n \le \dfrac{2^{p(n+1)}}{dR^p}, \quad |\Delta \xi_n| \le \dfrac{2^{p(n+1)}}{R^p}, \end{cases} \tag{3.5}$$

and define

$$k_n := 1 - \frac{\omega}{4} - \frac{\omega}{2^{n+2}}, \quad n \in \mathbb{N}_0.$$

Now take $\varphi = [(u_\omega)_h - k_n]^+ \xi_n^p$, $K = B_{R_n}$ in (3.1) and integrate in time over (τ_n, t) for $t \in (\tau_n, t^*)$. Applying integration by parts to the first term gives

$$\begin{aligned} F_1 &:= \int_{\tau_n}^t \int_{B_{R_n}} \partial_s u_h [(u_\omega)_h - k_n]^+ \xi_n^p \,dx\,ds \\ &= \frac{1}{2} \int_{\tau_n}^t \int_{B_{R_n}} \partial_s \Big(\big([(u_\omega)_h - k_n]^+ \big)^2 \Big) \xi_n^p \,dx\,ds \\ &\quad + \Big(1 - \frac{\omega}{4} - k_n\Big) \int_{\tau_n}^t \int_{B_{R_n}} \partial_s \bigg(\bigg(\Big[u - \Big(1 - \frac{\omega}{4}\Big)\Big]^+ \bigg)_h \bigg) \xi_n^p \,dx\,ds \\ &= \frac{1}{2} \int_{B_{R_n}\times\{t\}} \big([u_\omega - k_n]_h^+ \big)^2 \xi_n^p \,dx\,ds - \frac{1}{2} \int_{B_{R_n}\times\{\tau_n\}} \big([u_\omega - k_n]_h^+ \big)^2 \xi_n^p \,dx\,ds \\ &\quad - \frac{p}{2} \int_{\tau_n}^t \int_{B_{R_n}} \big([u_\omega - k_n]_h^+ \big)^2 \xi_n^{p-1} \partial_s \xi_n \,dx\,ds \\ &\quad + \Big(1 - \frac{\omega}{4} - k_n\Big) \int_{\tau_n}^t \int_{B_{R_n}} \partial_s \bigg(\bigg(\Big[u - \Big(1 - \frac{\omega}{4}\Big)\Big]^+ \bigg)_h \bigg) \xi_n^p \,dx\,ds. \end{aligned}$$

In light of standard convergence properties of the Steklov average, we obtain

$$\begin{aligned} F_1 \to F_1^* := &\frac{1}{2} \int_{B_{R_n}\times\{t\}} \big([u_\omega - k_n]^+ \big)^2 \xi_n^p \,dx\,ds \\ &- \frac{p}{2} \int_{\tau_n}^t \int_{B_{R_n}} \big([u_\omega - k_n]^+ \big)^2 \xi_n^{p-1} \partial_s \xi_n \,dx\,ds \\ &+ \Big(1 - \frac{\omega}{4} - k_n\Big) \bigg(\int_{B_{R_n}\times\{t\}} \Big[u - \Big(1 - \frac{\omega}{4}\Big)\Big]^+ \xi_n^p \,dx\,ds \end{aligned}$$

$$- p \int_{B_{R_n} \times \{\tau_n\}} \left[u - \left(1 - \frac{\omega}{4}\right)\right]^+ \xi_n^{p-1} \partial_s \xi_n \, dx \, ds \Bigg) \quad \text{as } h \to 0.$$

Using (3.5) and the nonnegativity of the third term, we arrive at

$$\begin{aligned} F_1^* &\geq \frac{1}{2} \int_{B_{R_n} \times \{t\}} \left([u_\omega - k_n]^+\right)^2 \xi_n^p \, dx - \frac{p}{2d} \left(\frac{\omega}{4}\right)^2 \frac{2^{p(n+1)}}{R^p} \int_{\tau_n}^t \int_{B_{R_n}} \chi_{\{u_\omega \geq k_n\}} \, dx \, ds \\ &\quad - \frac{p}{d} \left(\frac{\omega}{4}\right)^2 \frac{2^{p(n+1)}}{R^p} \int_{\tau_n}^t \int_{B_{R_n}} \chi_{\{u \geq 1 - \omega/4\}} \, dx \, ds \\ &\geq \frac{1}{2} \int_{B_{R_n} \times \{t\}} \left([u_\omega - k_n]^+\right)^2 \xi_n^p \, dx - \frac{3}{2} \frac{p}{d} \left(\frac{\omega}{4}\right)^2 \frac{2^{p(n+1)}}{R^p} \int_{\tau_n}^t \int_{B_{R_n}} \chi_{\{u_\omega \geq k_n\}} \, dx \, ds, \end{aligned}$$

the last inequality coming from $u \geq 1 - \omega/4 \Rightarrow u_\omega \geq k_n$. Since

$$[u_\omega - k_n]^+ \leq \omega/4,$$

we know that

$$\begin{aligned} \left([u_\omega - k_n]^+\right)^2 &= \left([u_\omega - k_n]^+\right)^{2-p} \left([u_\omega - k_n]^+\right)^p \\ &\geq \left(\frac{\omega}{4}\right)^{2-p} \left([u_\omega - k_n]^+\right)^p \\ &\geq \left(\frac{\omega}{2}\right)^{2-p} \left([u_\omega - k_n]^+\right)^p; \end{aligned}$$

therefore, the definition of d implies that

$$\begin{aligned} F_1^* &\geq \frac{1}{2} \left(\frac{\omega}{2}\right)^{2-p} \int_{B_{R_n} \times \{t\}} \left([u_\omega - k_n]^+\right)^p \xi_n^p \, dx \\ &\quad - \frac{3}{2} p 2^{p-2} \left(\frac{\omega}{4}\right)^p \frac{2^{p(n+1)}}{R^p} \psi(\omega/4)^{p-1} \int_{\tau_n}^t \int_{B_{R_n}} \chi_{\{u_\omega \geq k_n\}} \, dx \, ds. \end{aligned} \tag{3.6}$$

We now deal with the diffusive term. The term

$$F_2 := \int_{\tau_n}^t \int_{B_{R_n}} \left(a(u)^{p-1} |\nabla u|^{p-2} \nabla u\right)_h \cdot \nabla \left\{ [(u_\omega)_h - k_n]^+ \xi_n^p \right\} dx \, ds$$

converges, for $h \to 0$, to

$$\begin{aligned} F_2^* := \int_{\tau_n}^t \int_{B_{R_n}} & a(u)^{p-1} |\nabla u|^{p-2} \nabla u \cdot \big(\nabla (u_\omega - k_n)^+ \xi_n^p \\ & + p (u_\omega - k_n)^+ \xi_n^{p-1} \nabla \xi_n \big) \, dx \, ds \end{aligned}$$

$$= \int_{\tau_n}^t \int_{B_{R_n}} a(u)^{p-1} \left| \xi_n \nabla (u_\omega - k_n)^+ \right|^p dx\, ds + \tilde{F}_2^*,$$

where we define

$$\tilde{F}_2^* := p \int_{\tau_n}^t \int_{B_{R_n}} a(u)^{p-1} |\nabla u|^{p-2} \nabla u \cdot \nabla \xi_n (u_\omega - k_n)^+ \xi_n^{p-1}\, dx\, ds.$$

Since $\nabla (u_\omega - k_n)^+$ is nonzero only within the set $\{k_n < u < 1 - \omega/4\}$ and

$$a(u) \geq \gamma_1 \psi(\omega/4) \quad \text{on} \quad \{k_n < u < 1 - \omega/4\},$$

we may estimate the first term of F_2^* from below by

$$\int_{\tau_n}^t \int_{B_{R_n}} a(u)^{p-1} \left| \xi_n \nabla (u_\omega - k_n)^+ \right|^p dx\, ds$$
$$\geq [\gamma_1 \psi(\omega/4)]^{p-1} \int_{\tau_n}^t \int_{B_{R_n}} \left| \xi_n \nabla (u_\omega - k_n)^+ \right|^p dx\, ds. \tag{3.7}$$

Let us now focus on $\tilde{F}_2^*$. Using that $\nabla (u_\omega - k_n)^+$ is nonzero only within the set $\{k_n < u < 1 - \omega/4\}$, integrating by parts, and using (1.3) and (3.5), we obtain

$$\begin{aligned}
\left| \tilde{F}_2^* \right| &\leq p \int_{\tau_n}^t \int_{B_{R_n}} |a(u)|^{p-1} \left| \nabla (u_\omega - k_n)^+ \right|^{p-1} |\nabla \xi_n| (u_\omega - k_n)^+ \xi_n^{p-1}\, dx\, ds \\
&\quad + \left| p \Big(1 - \frac{\omega}{4} - k_n\Big) \int_{\tau_n}^t \int_{B_{R_n}} \xi_n^{p-1} \nabla \xi_n \cdot \nabla \left\{ \frac{1}{p-1} \left(\int_{1-\omega/4}^u a(s)\, ds \right)_+^{p-1} \right\} dx\, ds \right| \\
&\leq p \left[\gamma_2 \psi\left(\omega/2\right) \right]^{p-1} \int_{\tau_n}^t \int_{B_{R_n}} |\nabla \xi_n| (u_\omega - k_n)^+ \left| \xi_n \nabla (u_\omega - k_n)^+ \right|^{p-1} dx\, ds \\
&\quad + p \left(\frac{\omega}{4} \right) \Bigg| - \int_{\tau_n}^t \int_{B_{R_n}} \left(\int_{1-\omega/4}^u a(s)\, ds \right)_+^{p-1} \Big((p-1) \xi_n^{p-2} |\nabla \xi_n|^2 \\
&\qquad + \xi_n^{p-1} \Delta \xi_n \Big)\, dx\, ds \Bigg|.
\end{aligned}$$

Next, we take into account that

$$\left(\int_{1-\omega/4}^u a(s)\, ds \right)^+ \leq \frac{\omega}{4} \psi\left(\omega/4\right),$$

and apply Young's inequality

$$ab \leq \frac{\epsilon^r}{r} a^r + \frac{b^{r'}}{r' \epsilon^{r'}} \quad \text{if } a, b \geq 0, \quad \frac{1}{r} + \frac{1}{r'} = 1, \quad \epsilon > 0, \tag{3.8}$$

for the choices

$$r = p, \quad a = |\nabla \xi_n| (u_\omega - k_n)^+, \quad b = \left| \nabla (u_\omega - k_n)^+ \right|^{p-1}$$

$$\text{and} \quad \epsilon_1^{-p'} = \frac{p'}{p} \frac{(\gamma_1^{p-1} - 1) \psi(\omega/4)^{p-1}}{\gamma_2^{p-1} \psi(\omega/2)^{p-1}} > 0.$$

This leads to

$$\begin{aligned}
\left| \tilde{F}_2^* \right| &\leq \frac{1}{\epsilon_1^p} \left[\gamma_2 \psi\left(\omega/2\right) \right]^{p-1} \left(\frac{\omega}{4} \right)^p \frac{2^{p(n+1)}}{R^p} \int_{\tau_n}^t \int_{B_{R_n}} \chi_{\{u_\omega \geq k_n\}} \, dx \, ds \\
&\quad + (p-1) \epsilon_1^{p'} \left[\gamma_2 \psi(\omega/2) \right]^{p-1} \int_{\tau_n}^t \int_{B_{R_n}} \left| \xi_n \nabla (u_\omega - k_n)^+ \right|^p dx \, ds \\
&\quad + p^2 \left(\frac{\omega}{4} \right)^p \psi\left(\omega/4\right)^{p-1} \frac{2^{p(n+1)}}{R^p} \int_{\tau_n}^t \int_{B_{R_n}} \chi_{\{u_\omega \geq k_n\}} \, dx \, ds \\
&\leq \left\{ \frac{(p-1) \gamma_2^{p-1} \psi\left(\omega/2\right)^{p-1}}{(\gamma_1^{p-1} - 1) \psi\left(\omega/4\right)^{p-1}} \right\}^{p-1} \left[\gamma_2 \psi(\omega/2) \right]^{p-1} \times \\
&\qquad \times \left(\frac{\omega}{4} \right)^p \frac{2^{p(n+1)}}{R^p} \int_{\tau_n}^t \int_{B_{R_n}} \chi_{\{u_\omega \geq k_n\}} \, dx \, ds \\
&\quad + \left(\gamma_1^{p-1} - 1 \right) \psi\left(\omega/4\right)^{p-1} \int_{\tau_n}^t \int_{B_{R_n}} \left| \xi_n \nabla (u_\omega - k_n)^+ \right|^p dx \, ds \\
&\quad + p^2 \left(\frac{\omega}{4} \right)^p \psi\left(\omega/4\right)^{p-1} \frac{2^{p(n+1)}}{R^p} \int_{\tau_n}^t \int_{B_{R_n}} \chi_{\{u_\omega \geq k_n\}} \, dx \, ds.
\end{aligned} \tag{3.9}$$

Hence, from (3.7) and (3.9), and observing that

$$\left[\frac{\psi\left(\omega/2\right)}{\psi\left(\omega/4\right)} \right]^{p(p-1)} = \left(\frac{4}{2} \right)^{p\beta_2} = 2^{p\beta_2},$$

we obtain

$$
\begin{aligned}
F_2^* \geq \psi\,(\omega/4)^{p-1} & \int_{\tau_n}^{t} \int_{B_{R_n}} \left|\xi_n \nabla (u_\omega - k_n)^+\right|^p dx\,ds \\
& - \left\{ p^2 + 2^{p\beta_2} \left[\frac{p'\gamma_2^p}{p(\gamma_1^{p-1}-1)} \right]^{p-1} \right\} \left(\frac{\omega}{4}\right)^p \frac{2^{p(n+1)}}{R^p} \times \\
& \times \psi\,(\omega/4)^{p-1} \int_{\tau_n}^{t} \int_{B_{R_n}} \chi_{\{u_\omega \geq k_n\}}\, dx\,ds.
\end{aligned} \tag{3.10}
$$

Finally, for the lower order term

$$
F_3 := \int_{\tau_n}^{t} \int_{B_{R_n}} \big(\chi u f(u) \nabla v\big)_h \cdot \nabla \big\{ [(u_\omega)_h - k_n]^+ \xi_n^p \big\}\, dx\,ds
$$

we have

$$
\begin{aligned}
F_3 \to F_3^* &:= \int_{\tau_n}^{t} \int_{B_{R_n}} \chi u f(u) \nabla v \cdot \big(\nabla (u_\omega - k_n)^+ \xi_n^p + p (u_\omega - k_n)^+ \xi_n^{p-1} \nabla \xi_n \big)\, dx\,ds \\
&= \int_{\tau_n}^{t} \int_{B_{R_n}} \chi u f(u) \nabla v \cdot \nabla (u_\omega - k_n)^+ \xi_n^p\, dx\,ds \\
&\quad + p \int_{\tau_n}^{t} \int_{B_{R_n}} \chi u f(u) \nabla v \cdot \nabla \xi_n (u_\omega - k_n)^+ \xi_n^{p-1}\, dx\,ds \quad \text{as } h \to 0.
\end{aligned}
$$

Applying Young's inequality (3.8), with

$$
r = p, \quad a = \nabla (u_\omega - k_n)^+ \xi_n, \quad b = \chi u f(u) \xi_n^{p-1} \nabla v
$$

$$
\text{and} \quad \epsilon_2^p = \frac{p}{2} \psi (\omega/4)^{p-1} > 0,
$$

using the fact that $(u_\omega - k_n)^+ \leq \omega/4$ and defining $M := \|\chi u f(u)\|_{L^\infty(Q_T)}$, we may estimate F_3^* as follows:

$$
\begin{aligned}
F_3^* &\leq \frac{\epsilon_2^p}{p} \int_{\tau_n}^{t} \int_{B_{R_n}} \left|\nabla (u_\omega - k_n)^+ \xi_n\right|^p dx\,ds + \frac{M^{p'}}{p'\epsilon_2^{p'}} \int_{\tau_n}^{t} \int_{B_{R_n}} |\nabla v|^{p'} \chi_{\{u_\omega \geq k_n\}}\, dx\,ds \\
&\quad + pM \int_{\tau_n}^{t} \int_{B_{R_n}} |\nabla v| \left(\frac{\omega}{4}\right) |\nabla \xi_n| \chi_{\{u_\omega \geq k_n\}}\, dx\,ds \\
&\leq \frac{1}{2} \psi\,(\omega/4)^{p-1} \int_{\tau_n}^{t} \int_{B_{R_n}} \left|\nabla (u_\omega - k_n)^+ \xi_n\right|^p dx\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{(p/2)^{-p'/p}}{p'} \frac{M^{p'}}{\psi(\omega/4)} \int_{\tau_n}^{t} \int_{B_{R_n}} |\nabla v|^{p'} \chi_{\{u_\omega \geq k_n\}} \, dx \, ds \\
&+ \epsilon_3^p \left(\frac{\omega}{4}\right)^p \int_{\tau_n}^{t} \int_{B_{R_n}} |\nabla \xi_n|^p \chi_{\{u_\omega \geq k_n\}} \, dx \, ds \\
&+ \frac{pM^{p'}}{p' \epsilon_3^{p'}} \int_{\tau_n}^{t} \int_{B_{R_n}} |\nabla v|^{p'} \chi_{\{u_\omega \geq k_n\}} \, dx \, ds,
\end{aligned}
$$

applying again Young's inequality (3.8) to the last term in the right-hand side, this time with

$$
r = p, \quad a = |\nabla \xi_n| \omega/4, \quad b = M|\nabla v|, \quad \epsilon_3^{p'} = \psi\left(\omega/4\right) > 0.
$$

Using (3.5), we obtain

$$
\begin{aligned}
F_3^* \leq F_3^{**} :=& \frac{1}{2} \psi\left(\omega/4\right)^{p-1} \int_{\tau_n}^{t} \int_{B_{R_n}} \left|\nabla (u_\omega - k_n)^+ \xi_n\right|^p \, dx \, ds \\
&+ \frac{M^{p'}}{p' \psi(\omega/4)} \left[\left(\frac{p}{2}\right)^{-p'/p} + p\right] \int_{\tau_n}^{t} \int_{B_{R_n}} |\nabla v|^{p'} \chi_{\{u_\omega \geq k_n\}} \, dx \, ds \\
&+ \left(\frac{\omega}{4}\right)^p \frac{2^{p(n+1)}}{R^p} \psi\left(\omega/4\right)^{p-1} \int_{\tau_n}^{t} \int_{B_{R_n}} \chi_{\{u_\omega \geq k_n\}} \, dx \, ds.
\end{aligned}
$$

Additionally, using Hölder's inequality, we may write

$$
\int_{\tau_n}^{t} \int_{B_{R_n}} |\nabla v|^{p'} \chi_{\{u_\omega \geq k_n\}} \, dx \, ds \leq \|\nabla v\|_{L^{p'p}(Q_T)}^{p'} \left(\int_{\tau_n}^{t} \left|A_{k_n, R_n}^+(\sigma)\right| d\sigma \right)^{1-1/p},
$$

where $|A_{k_n, R_n}^+(\sigma)|$ denotes the measure of the set

$$
A_{k_n, R_n}^+(\sigma) := \left\{ x \in B_{R_n} \, : \, u(x, \sigma) > k_n \right\}.
$$

Thus we obtain

$$
\begin{aligned}
F_3^{**} \leq & \frac{1}{2}\psi(\omega/4)^{p-1} \int_{\tau_n}^{t} \int_{B_{R_n}} \left|\xi_n \nabla (u_\omega - k_n)^+\right|^p \, dx \, ds \\
& + \left(\frac{\omega}{4}\right)^p \frac{2^{p(n+1)}}{R^p} \psi\left(\omega/4\right)^{p-1} \int_{\tau_n}^{t} \int_{B_{R_n}} \chi_{\{u_\omega \geq k_n\}} \, dx \, ds \\
& + \frac{M^{p'}}{p'\psi\left(\omega/4\right)} \left[\left(\frac{p}{2}\right)^{-p'/p} + p\right] \|\nabla v\|^{p'}_{L^{p'p}(Q_T)} \left(\int_{\tau_n}^{t} \left|A^+_{k_n,R_n}(\sigma)\right| d\sigma\right)^{1-1/p}.
\end{aligned} \tag{3.11}
$$

Combining the resulting estimates (3.6), (3.10), (3.11) and multiplying by $2(\omega/2)^{p-2}$ yields

$$
\begin{aligned}
& \operatorname*{ess\,sup}_{\tau_n \leq t \leq t^*} \int_{B_{R_n} \times \{t\}} \left([u_\omega - k_n]^+\right)^p \xi_n^p \, dx \, ds + \frac{2}{d} \int_{\tau_n}^{t^*} \int_{B_{R_n}} \left|\xi_n \nabla (u_\omega - k_n)^+\right|^p \, dx \, ds \\
& \leq \left\{ \frac{3}{2} p 2^{p-2} + p^2 + 2^{p\beta_2} \left[\frac{p'\gamma_2^p}{p(\gamma_1^{p-1} - 1)}\right]^{p-1} \right\} \left(\frac{\omega}{4}\right)^p \frac{2^{p(n+1)}}{R^p} \frac{2}{d} \int_{\tau_n}^{t^*} \int_{B_{R_n}} \chi_{\{u_\omega \geq k_n\}} \, dx \, ds \\
& \quad + 2\frac{(\omega/2)^{p-2} M^{p'}}{p'\psi\left(\omega/4\right)} \left[\left(\frac{p}{2}\right)^{-p'/p} + p\right] \|\nabla v\|^{p'}_{L^{p'p}(Q_T)} \left(\int_{\tau_n}^{t^*} \left|A^+_{k_n,R_n}(\sigma)\right| d\sigma\right)^{1-1/p}.
\end{aligned} \tag{3.12}
$$

Next we perform a change in the time variable putting $\bar{t} := \frac{1}{d}(t - t^*)$, which transforms $Q(dR_n^p, R_n)$ into $Q^{t^*}_{R_n}$. Furthermore, if we define $\bar{u}_\omega(\cdot, \bar{t}) := u_\omega(\cdot, t)$ and $\bar{\xi}_n(\cdot, \bar{t}) = \xi_n(\cdot, t)$, then defining for each n,

$$
A_n := \int_{-R_n^p}^{0} \int_{B_{R_n}} \chi_{\{\bar{u}_\omega \geq k_n\}} \, dx \, d\bar{t} = \frac{1}{d} \int_{\tau_n}^{t} \int_{B_{R_n}} \chi_{\{u_\omega \geq k_n\}} \, dx \, ds
$$

we may rewrite (3.12) more concisely as

$$
\begin{aligned}
& \left\|(\bar{u}_\omega - k_n)^+ \bar{\xi}_n\right\|^p_{V^p(Q^{t^*}_{R_n})} \\
& \leq 2 \left\{ \frac{3}{2} p 2^{p-2} + p^2 + 2^{p\beta_2} \left[\frac{p'\gamma_2^p}{p(\gamma_1^{p-1} - 1)}\right]^{p-1} \right\} \left(\frac{\omega}{4}\right)^p \frac{2^{p(n+1)}}{R^p} A_n \\
& \quad + 2 \left[\left(\frac{p}{2}\right)^{-p'/p} + p\right] \frac{M^{p'}}{p'} \left(\frac{\omega}{2}\right)^{(p-2)/p} \psi\left(\omega/4\right)^{1-p-1/p} \|\nabla v\|^{p'}_{L^{p'p}(Q_T)} A_n^{1-1/p},
\end{aligned} \tag{3.13}
$$

where $V^p(\Omega_T) = L^\infty(0,T;L^p(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega))$ endowed with the obvious norm. Next, observe that by application of a well-known embedding theorem (cf. [5, §I.3]), we get

$$\begin{aligned} \frac{1}{2^{p(n+1)}}\left(\frac{\omega}{4}\right)^p A_{n+1} &= |k_n - k_{n+1}|^p A_{n+1} \\ &\leq \left\|(\bar{u}_\omega - k_n)^+\right\|^p_{p,Q(R^p_{n+1},R_{n+1})} \\ &\leq \left\|(\bar{u}_\omega - k_n)^+\bar{\xi}_n\right\|^p_{p,Q(R^p_n,R_n)} \\ &\leq C\left\|(\bar{u}_\omega - k_n)^+\bar{\xi}_n\right\|^p_{V^p(Q^{t^*}_{R_n})} A_n^{p/(N+p)}. \end{aligned} \tag{3.14}$$

Now, applying (3.13), we get

$$\begin{gathered} \frac{1}{2^{p(n+1)}}\left(\frac{\omega}{4}\right)^p A_{n+1} \\ \leq 2C\left\{\frac{3}{2}p2^{p-2} + p^2 + 2^{p\beta_2}\left[\frac{p'\gamma_2^p}{p(\gamma_1^{p-1}-1)}\right]^{p-1}\right\}\left(\frac{\omega}{4}\right)^p \frac{2^{p(n+1)}}{R^p} A_n^{1+p/(N+p)} \\ +2C\left[\left(\frac{p}{2}\right)^{-q/p} + p\right]\frac{M^{p'}}{p'}\left(\frac{\omega}{2}\right)^{(p-2)/p}\psi(\omega/4)^{1-p-1/p}\|\nabla v\|^{p'}_{L^{p'p}(Q_T)} A_n^{1-1/p+p/(N+p)}. \end{gathered} \tag{3.15}$$

Now let us define

$$X_n := \frac{A_n}{|Q(R^p_n, R_n)|}, \qquad Z_n := \frac{A_n^{1/p}}{|B_{R_n}|}, \qquad n \in \mathbb{N}_0.$$

Dividing (3.15) by $\frac{1}{2^{p(n+1)}}\left(\frac{\omega}{4}\right)^p |Q(R^p_{n+1}, R_{n+1})|$ yields

$$\begin{aligned} X_{n+1} \leq 2^{pn}\Bigg(&2C\left\{\frac{3}{2}p2^{p-2} + p^2 + 2^{p\beta_2}\left[\frac{p'\gamma_2^p}{p(\gamma_1^{p-1}-1)}\right]^{p-1}\right\}X_n^{1+p/(N+p)} \\ &+ 2^{3-2/p+p}C\left[\left(\frac{p}{2}\right)^{-p'/p} + p\right]\frac{M^{p'}}{p'}\left(\frac{\omega}{2}\right)^{p-2}\psi\left(\omega/4\right)^{1-p-1/p} \times \\ &\times R^{N\kappa}\|\nabla v\|^q_{L^{p'p}(Q_T)} X_n^{p/(N+p)} Z_n^{p-1}\Bigg) \\ \leq \gamma 2^{pn}&\left(X_n^{1+\alpha} + X_n^\alpha Z_n^{1+\kappa}\right), \qquad n \in \mathbb{N}_0, \end{aligned}$$

with $\alpha = p/(N+p) > 0$, $\kappa = p-2 > 0$ and

$$\gamma := 2C \max\left\{ \frac{3}{2} p 2^{p-2} + p^2 + 2^{p\beta_2} \left[\frac{p' \gamma_2^p}{p(\gamma_1^{p-1} - 1)} \right]^{p-1}, \right.$$
$$\left. 2^{3-2/p+p} \left[\left(\frac{p}{2}\right)^{-p'/p} + p \right] \frac{M^{p'}}{p'} \left(\frac{\omega}{2}\right)^{p-2} \left[\psi\left(\omega/4\right)\right]^{1-p-1/p} R^{N\kappa} \right\} > 0.$$

(In the choice of κ we need the assumption that p is *strictly* larger than 2.) In the spirit of Remark 3.1, let us assume that

$$\left(\frac{\omega}{2}\right)^{p-2} \left[\psi\left(\omega/4\right)\right]^{1-p-1/p} R^{N\kappa} \leq 1.$$

Therefore, with this assumption we conclude that γ is independent of ω and R.

Reasoning analogously, we obtain

$$Z_{n+1} \leq \gamma 2^{pn} \left(X_n + Z_n^{1+\kappa} \right).$$

Now, let $\sigma = \min\{\alpha, \kappa\}$ and notice that, if we set $\nu_0 := 2\gamma^{-(1+\kappa)/\sigma}(2^p)^{-(1+\kappa)/\sigma^2}$, it follows from (3.4) that

$$X_0 + Z_0^{1+\kappa} \leq 2\gamma^{-(1+\kappa)/\sigma}(2^p)^{-(1+\kappa)/\sigma^2}. \tag{3.16}$$

Then, using Lemma 3.1, we are able to conclude that $X_n, Z_n \to 0$ as $n \to \infty$. Finally, notice that $R_n \to R/2$ and $k_n \to 1 - \omega/4$, and this implies that

$$\left|\left\{(x,t) \in Q\big((R/2)^p, R/2\big) \,:\, \bar{u}_\omega(x,\bar{t}) \geq 1 - \omega/4\right\}\right|$$
$$= \left|\left\{(x,t) \in Q_{R/2}^{t^*} \,:\, u(x,t) > 1 - \omega/4\right\}\right| = 0.$$

This completes the proof. ■

Now we show that the conclusion of Lemma 3.2 is valid in a full cylinder of the type $Q(\tau, \rho)$. To this end, we exploit the fact that at the time level $-\hat{t} := t^* - d(R/2)^p$, the function $x \mapsto u(x,t)$ is strictly below $1 - \omega/4$ in the ball $B_{R/2}$. We use this time level as an initial condition to make the conclusion of the lemma hold up to $t = 0$, eventually shrinking the ball. This requires the use of logarithmic estimates.

Given constants a, b, c with $0 < c < a$, we define the nonnegative function

$$\varrho_{a,b,c}^{\pm}(s) := \left(\ln \frac{a}{a + c - (s-b)|_{\pm}} \right)^{+}$$

$$= \begin{cases} \ln \dfrac{a}{a+c\pm(b-s)} & \text{if } b\pm c \lesseqgtr s \lesseqgtr b\pm(a+c), \\ 0 & \text{if } s \lesseqgtr b\pm c, \end{cases} \tag{3.17}$$

whose first derivative is given by

$$(\varrho^{\pm}_{a,b,c})'(s) = \begin{cases} \dfrac{1}{(b-s)\pm(a+c)} & \text{if } b\pm c \lesseqgtr s \lesseqgtr b\pm(a+c) \\ 0 & \text{if } s \lesseqgtr b\pm c \end{cases} \gtreqless 0,$$

and its second derivative, away from $s = b \pm c$, is

$$\left(\varrho^{\pm}_{a,b,c}\right)'' = \left\{\left(\varrho^{\pm}_{a,b,c}\right)'\right\}^2 \geq 0.$$

Given u bounded in $(x_0,t_0)+Q(\tau,\rho)$ and a number k, define

$$H^{\pm}_{u,k} := \operatorname*{ess\,sup}_{(x_0,t_0)+Q(\tau,\rho)} \left|(u-k)^{\pm}\right|,$$

and the function

$$\Psi^{\pm}\left(H^{\pm}_{u,k},(u-k)^{\pm},c\right) := \varrho^{\pm}_{H^{\pm}_{u,k},k,c}(u), \qquad 0<c<H^{\pm}_{u,k}. \tag{3.18}$$

Lemma 3.3. *For every number $\nu_1 \in (0,1)$, there exists $s_1 \in \mathbb{N}$, independent of ω and R, such that*

$$\left|\left\{x \in B_{R/4} : u(x,t) \geq 1-\omega/2^{s_1}\right\}\right| \leq \nu_1 |B_{R/2}| \quad \text{for all } t \in (-\hat{t},0).$$

Proof: Let $k = 1-\omega/4$ and

$$c = \omega/2^{2+n}, \tag{3.19}$$

with $n \in \mathbb{N}$ to be chosen. Let $0 < \zeta(x) \leq 1$ be a piecewise smooth cutoff function defined on $B_{R/2}$ such that $\zeta = 1$ in $B_{R/4}$ and $|\nabla\zeta| \leq C/R$. Now consider the weak formulation (3.1) with $\varphi = 2\varrho^+(u_h)(\varrho^+)'(u_h)\zeta^p$ for $K = B_{R/2}$, where ϱ^+ is the function defined in (3.17). After an integration in time over $(-\hat{t},t)$, with $t \in (-\hat{t},0)$, we obtain $G_1+G_2-G_3=0$, where we define

$$G_1 := 2\int_{-\hat{t}}^{t}\int_{B_{R/2}} \partial_s\{u_h\}\varrho^+(u_h)(\varrho^+)'(u_h)\zeta^p\,dx\,ds,$$

$$G_2 := 2\int_{-\hat{t}}^{t}\int_{B_{R/2}} \left(|\nabla A(u)|^{p-2}a(u)\nabla u\right)_h \cdot \nabla\left\{\varrho^+(u_h)(\varrho^+)'(u_h)\zeta^p\right\}dx\,ds,$$

$$G_3 := 2\int_{-\hat t}^{t}\int_{B_{R/2}} \big(\chi u f(u)\nabla v\big)_h \cdot \nabla\big\{\varrho^+(u_h)(\varrho^+)'(u_h)\zeta^p\big\}\,dx\,ds.$$

Using the properties of the function ζ, we arrive at

$$\begin{aligned} G_1 &= \int_{-\hat t}^{t}\int_{B_{R/2}} \partial_s\left\{\varrho^+(u_h)\right\}^2\zeta^p\,dx\,ds \\ &= \int_{B_{R/2}\times\{t\}} \left\{\varrho^+(u_h)\right\}^2\zeta^p\,dx - \int_{B_{R/2}\times\{-\hat t\}} \left\{\varrho^+(u_h)\right\}^2\zeta^p\,dx. \end{aligned}$$

Due to Lemma 3.2, at time $-\hat t$, the function $x\mapsto u(x,t)$ is strictly below $1-\omega/4$ in the ball $B_{R/2}$, and therefore $\varrho^+(u(x,-\hat t)) = 0$ for $x\in B_{R/2}$. Consequently,

$$\begin{aligned} G_1 &\to \int_{B_{R/2}\times\{t\}} \left\{\varrho^+(u)\right\}^2\zeta^p\,dx - \int_{B_{R/2}\times\{-\hat t\}} \left\{\varrho^+(u)\right\}^2\zeta^p\,dx \\ &= \int_{B_{R/2}\times\{t\}} \left\{\varrho^+(u)\right\}^2\zeta^p\,dx \quad \text{as } h\to 0. \end{aligned} \tag{3.20}$$

The definition of $H^{\pm}_{u,k}$ implies that

$$u-k \le H^+_{u,k} = \operatorname*{ess\,sup}_{Q(\hat t,R/2)}\left|\left(u-1+\frac{\omega}{4}\right)^+\right| \le \frac{\omega}{4}. \tag{3.21}$$

If $H^+_{u,k}=0$, the result is trivial; so we assume $H^+_{u,k}>0$ and choose n large enough so that

$$0<\frac{\omega}{2^{2+n}}<H^+_{u,k}.$$

Therefore, since $H^+_{u,k}+k-u+c>0$, the function $\varrho^+(u)$ is defined in the whole cylinder $Q(\hat t, R/2)$ by

$$\varrho^{\pm}_{H^+_{u,k},k,c}(u) = \begin{cases} \ln\dfrac{H^+_{u,k}}{H^+_{u,k}+c+k-u} & \text{if } u>k+c,\\ 0 & \text{otherwise.}\end{cases}$$

Relation (3.21) implies that

$$\frac{H^+_{u,k}}{H^+_{u,k}+c+k-u} \le \frac{\frac{\omega}{4}}{2c-\frac{\omega}{4}} = 2^n, \quad \text{and therefore } \varrho^+(u)\le n\ln 2; \tag{3.22}$$

in the nontrivial case $u > k + c$, we also have an estimate for the derivative of the logarithmic function:

$$\big|(\varrho^+)'(u)\big|^{2-p} = \left|\frac{-1}{H^+_{u,k} + c + k - u}\right|^{2-p} \le \left|\frac{1}{c}\right|^{2-p} = \Big(\frac{\omega}{2^{2+n}}\Big)^{p-2}. \tag{3.23}$$

With these estimates at hand, we have for the diffusive term:

$$\begin{aligned} G_2 \to G_2^* &:= 2\int_{-\hat{t}}^{t}\int_{B_{R/2}} a(u)^{p-1}|\nabla u|^{p-2}\nabla u\cdot\nabla\big\{\varrho^+(u)(\varrho^+)'(u)\zeta^p\big\}\,dx\,ds \\ &= \int_{-\hat{t}}^{t}\int_{B_{R/2}} a(u)^{p-1}|\nabla u|^p\big\{2\big(1+\varrho^+(u)\big)\,\big[(\varrho^+)'(u)\big]^2\zeta^p\big\}\,dx\,ds \\ &\quad + \tilde{G}_2^* \quad \text{as } h\to 0, \end{aligned}$$

where we define

$$\tilde{G}_2^* := 2p\int_{-\hat{t}}^{t}\int_{B_{R/2}} a(u)^{p-1}|\nabla u|^{p-2}\nabla u\cdot\nabla\zeta\big\{\varrho^+(u)(\varrho^+)'(u)\zeta^{p-1}\big\}\,dx\,dt.$$

Applying Young's inequality (3.8) with the choices

$$r = p, \quad a = |\nabla u|^{p-1}\zeta^{p-1}\big|(\varrho^+)'(u)\big|^{2/p'}, \quad b = \big|(\varrho^+)'(u)\big|^{1-2/p'}|\nabla\zeta|$$

and $\epsilon_4 = 1$, we obtain

$$\begin{aligned} \big|\tilde{G}_2^*\big| &\le 2p\int_{-\hat{t}}^{t}\int_{B_{R/2}} a(u)^{p-1}|\nabla u|^{p-1}|\nabla\zeta|\varrho^+(u)\big|(\varrho^+)'(u)\big|\zeta^{p-1}\,dx\,ds \\ &= 2p\int_{-\hat{t}}^{t}\int_{B_{R/2}} a(u)^{p-1}\varrho^+(u)|\nabla u|^{p-1}\zeta^{p-1}\big|(\varrho^+)'(u)\big|^{2/p'}\times \\ &\quad \times\big|(\varrho^+)'(u)\big|^{1-2/p'}|\nabla\zeta|\,dx\,ds \\ &\le 2\epsilon_4^p\int_{-\hat{t}}^{t}\int_{B_{R/2}} a(u)^{p-1}\varrho^+(u)|\nabla u|^p\big[(\varrho^+)'(u)\big]^2\zeta^p\,dx\,ds \\ &\quad + \frac{2p}{p'\epsilon_4^q}\int_{-\hat{t}}^{t}\int_{B_{R/2}} a(u)^{p-1}\varrho^+(u)|\nabla\zeta|^p\big|(\varrho^+)'(u)\big|^{2-p}\,dx\,ds \\ &= 2\int_{-\hat{t}}^{t}\int_{B_{R/2}} a(u)^{p-1}\varrho^+(u)|\nabla u|^p\big[(\varrho^+)'(u)\big]^2\zeta^p\,dx\,ds \end{aligned}$$

$$+ 2(p-1) \int_{-\hat{t}}^{t} \int_{B_{R/2}} a(u)^{p-1} \varrho^+(u) |\nabla \zeta|^p \big|(\varrho^+)'(u)\big|^{2-p} \, dx \, ds.$$

In face of this estimate, we obtain

$$\begin{aligned} G_2^* &= 2 \int_{-\hat{t}}^{t} \int_{B_{R/2}} a(u)^{p-1} |\nabla u|^p \big[(\varrho^+)'(u)\big]^2 \zeta^p \, dx \, ds \\ &\quad - 2(p-1) \int_{-\hat{t}}^{t} \int_{B_{R/2}} a(u)^{p-1} \varrho^+(u) |\nabla \zeta|^p \big|(\varrho^+)'(u)\big|^{2-p} \, dx \, ds \\ &\geq 2 \left[\gamma_1 \psi\left(\omega/4\right)\right]^{p-1} \int_{-\hat{t}}^{t} \int_{B_{R/2}} |\nabla u|^p \big[(\varrho^+)'(u)\big]^2 \zeta^p \, dx \, ds \\ &\quad - 2(p-1) \int_{-\hat{t}}^{t} \int_{B_{R/2}} a(u)^{p-1} \varrho^+(u) |\nabla \zeta|^p \big|(\varrho^+)'(u)\big|^{2-p} \, dx \, ds \\ &\geq 2 \left[\gamma_1 \psi\left(\omega/4\right)\right]^{p-1} \int_{-\hat{t}}^{t} \int_{B_{R/2}} |\nabla u|^p \big[(\varrho^+)'(u)\big]^2 \zeta^p \, dx \, ds \\ &\quad - 2(p-1) n \ln 2 \left(\frac{C}{R}\right)^p \left(\frac{\omega}{2^{2+n}}\right)^{p-2} \int_{-\hat{t}}^{t} \int_{B_{R/2}} a(u)^{p-1} \chi_{\{u > 1-\omega/4\}} \, dx \, ds, \end{aligned}$$

and, finally,

$$\begin{aligned} G_2^* &\geq 2 \left[\gamma_1 \psi\left(\omega/4\right)\right]^{p-1} \int_{-\hat{t}}^{t} \int_{B_{R/2}} |\nabla u|^p \big[(\varrho^+)'(u)\big]^2 \zeta^p \, dx \, ds \\ &\quad - 2(p-1) n \ln 2 \left(\frac{C}{R}\right)^p \left(\frac{\omega}{2^{2+n}}\right)^{p-2} \hat{t} |B_{R/2}| \left[\gamma_2 \psi\left(\omega/4\right)\right]^{p-1}, \end{aligned} \tag{3.24}$$

where we have used estimates (3.22), (3.23), the properties of ζ, and the fact that

$$\gamma_1 \psi\left(\omega/4\right) \leq a(u) \leq \gamma_2 \psi\left(\omega/4\right) \quad \text{on the set } \{u > 1 - \omega/4\}.$$

Moreover, from the definition of $\hat{t}$ and our choice of t^* (recall that $t^* \geq dR^p - a_0 R^p$), there holds

$$\hat{t} \leq a_0 R^p = \left(\frac{\omega}{2}\right)^{2-p} \frac{R^p}{\phi\left(\omega/2^m\right)^{p-1}}. \tag{3.25}$$

Taking into account (3.25), we obtain from (3.24) that

$$
\begin{aligned}
G_2^* \geq\ & 2\left[\gamma_1 \psi\left(\omega/4\right)\right]^{p-1} \int_{-\hat{t}}^{t} \int_{B_{R/2}} |\nabla u|^p \left[(\varrho^+)'(u)\right]^2 \zeta^p \, dx \, ds \\
& - 2(p-1) n \ln 2 C^p 2^{(1+n)(2-p)} |B_{R/2}| \left[\gamma_2 \frac{\psi\left(\omega/4\right)}{\phi\left(\omega/2^m\right)}\right]^{p-1}.
\end{aligned} \tag{3.26}
$$

On the other hand, for the lower order term, by passing to the limit $h \to 0$, we have

$$
\begin{aligned}
G_3 \to G_3^* := & \ 2 \int_{-\hat{t}}^{t} \int_{B_{R/2}} \chi u f(u) \nabla v \cdot \nabla u \left\{ \left(1 + \varrho^+(u)\right) \left[(\varrho^+)'(u)\right]^2 \zeta^p \right\} dx \, ds \\
& + 2p \int_{-\hat{t}}^{t} \int_{B_{R/2}} \chi u f(u) \nabla v \cdot \nabla \zeta \left\{ \varrho^+(u) (\varrho^+)'(u) \zeta^{p-1} \right\} dx \, ds \\
\leq & \ 2M \int_{-\hat{t}}^{t} \int_{B_{R/2}} \left(1 + \varrho^+(u)\right) \left[(\varrho^+)'(u)\right]^2 \zeta^p |\nabla u| |\nabla v| \, dx \, ds \\
& + 2pM \int_{-\hat{t}}^{t} \int_{B_{R/2}} \varrho^+(u) \left|(\varrho^+)'(u)\right|^{1-2/p'} |\nabla v| |\nabla \zeta| \times \\
& \times \left|(\varrho^+)'(u)\right|^{2/p'} \zeta^{p-1} \, dx \, ds.
\end{aligned}
$$

Applying Young's inequality (3.8) to the first term on the right-hand side with

$$
r = p, \quad a = |\nabla u|, \quad b = |\nabla v| \quad \text{and} \quad \epsilon_5 = \left(\frac{p\psi\left(\omega/4\right)^{p-1}}{M(1 + n \ln 2)} \right)^{1/p},
$$

and to the second term with

$$
r = p, \quad a = \left|(\varrho^+)'(u)\right|^{1-2/p'}, \quad b = |\nabla v| \left|(\varrho^+)'(u)\right|^{2/p'} \zeta^{p-1} \quad \text{and} \quad \epsilon_6 = 1,
$$

we obtain

$$
\begin{aligned}
G_3^* \leq & \ 2\psi\left(\omega/4\right)^{p-1} \int_{-\hat{t}}^{t} \int_{B_{R/2}} |\nabla u|^p \left[(\varrho^+)'(u)\right]^2 \zeta^p \, dx \, ds \\
& + 2M \int_{-\hat{t}}^{t} \int_{B_{R/2}} \varrho^+(u) |\nabla \zeta| \left[(\varrho^+)'(u)\right]^{2-p} dx \, ds \\
& + 2M \frac{p-1}{p} \left(\frac{p\psi\left(\omega/4\right)^{p-1}}{M(1 + n \ln 2)} \right)^{1/(1-p)} \times
\end{aligned}
$$

$$\times \int_{-\hat{t}}^{t} \int_{B_{R/2}} \big(1+\varrho^+(u)\big)\big[(\varrho^+)'(u)\big]^2 \zeta^p |\nabla v|^{p'}\,dx\,ds$$
$$+ 2M(p-1) \int_{-\hat{t}}^{t} \int_{B_{R/2}} \varrho^+(u)|\nabla\zeta||\nabla v|^{p'}\big[(\varrho^+)'(u)\big]^2 \zeta^p\,dx\,ds.$$

Using the estimates (3.22) and (3.23) and the properties of ζ, we then get

$$\begin{aligned}
G_3^* \le{} & 2\psi(\omega/4)^{p-1} \int_{-\hat{t}}^{t} \int_{B_{R/2}} |\nabla u|^p \big[(\varrho^+)'(u)\big]^2 \zeta^p\,dx\,ds \\
& + 2Mn\ln 2 \frac{C}{R}\left(\frac{\omega}{2^{2+n}}\right)^{p-2} \hat{t}|B_{R/2}| \\
& + 2M\frac{p-1}{p}\left(\frac{p\psi(\omega/4)^{p-1}}{M(1+n\ln 2)}\right)^{1/(1-p)} (1+n\ln 2)\left(\frac{\omega}{2^{2+n}}\right)^{-2} \times \\
& \quad \times \int_{-\hat{t}}^{t} \int_{B_{R/2}} |\nabla v|^{p'} \chi_{\{u>1-\omega/4\}}\,dx\,ds \\
& + 2M(p-1)n\ln 2 \frac{C}{R}\left(\frac{\omega}{2^{2+n}}\right)^{-2} \int_{-\hat{t}}^{t} \int_{B_{R/2}} |\nabla v|^{p'} \chi_{\{u>1-\omega/4\}}\,dx\,ds.
\end{aligned}$$

Then, applying Hölder's inequality and recalling the definition of $\hat{t}$, we get

$$\begin{aligned}
G_3^* \le{} & 2\psi(\omega/4)^{p-1} \int_{-\hat{t}}^{t} \int_{B_{R/2}} |\nabla u|^p \big[(\varrho^+)'(u)\big]^2 \zeta^p\,dx\,ds \\
& + 2MCn\ln 2\, 2^{(1+n)(2-p)} \phi(\omega/2^m)^{1-p} |B_{R/2}| R^{p-1} \\
& + 2M(p-1)\left\{\left(\frac{p\psi\,(\omega/4)^{p-1}}{M(1+n\ln 2)}\right)^{1/(1-p)} \frac{1+n\ln 2}{p} + \frac{C}{R} n\ln 2\right\} \times \\
& \quad \times \left(\frac{\omega}{2^{2+n}}\right)^{-2} \|\nabla v\|^{p'}_{L^{p'p}(Q_T)} \big(a_0 R^p |B_{R/2}|\big)^{1-1/p}.
\end{aligned}$$

In addition, thanks to Remark 3.1, we may estimate

$$\begin{aligned}
& \left(\frac{\omega}{2^{2+n}}\right)^{-2} \left(\frac{p^{-p'}\psi\,(\omega/4)^{p-1}}{M(1+n\ln 2)}\right)^{1/(1-p)} a_0^{1-1/p} R^{p-1} \le 1, \\
& C\left(\frac{\omega}{2^{2+n}}\right)^{-2} a_0^{1-1/p} R^{p-2} \le 1, \quad \phi\left(\frac{\omega}{2^m}\right)^{1-p} R^{p-1} \le 1,
\end{aligned}$$

and this finally gives

$$
\begin{aligned}
G_3^* \leq\ & 2\psi\,(\omega/4)^{p-1} \int_{-\hat{t}}^{t} \int_{B_{R/2}} |\nabla u|^p \big[(\varrho^+)'(u)\big]^2 \zeta^p \, dx\, ds \\
& + 2MCn \ln 2\, 2^{(1+n)(2-p)} |B_{R/2}| \\
& + 2M(p-1)Cn \ln 2 \|\nabla v\|^{p'}_{L^{p'p}(Q_T)} |B_{R/2}|^{1-1/p}.
\end{aligned} \tag{3.27}
$$

Combining estimates (3.20), (3.26) and (3.27) yields

$$
\begin{aligned}
& \int_{B_{R/2}\times\{t\}} \big\{\varrho^+(u)\big\}^2 \zeta^p \, dx\, ds \\
& \leq 2M(p-1)Cn \ln 2 \|\nabla v\|^{p'}_{L^{p'p}(Q_T)} |B_{R/2}|^{1-1/p} \\
& \quad + (1-\gamma_1^{p-1}) 2\, [\psi\,(\omega/4)]^{p-1} \int_{-\hat{t}}^{t} \int_{B_{R/2}} |\nabla u|^p \big[(\varrho^+)'(u)\big]^2 \zeta^p \, dx\, ds \\
& \quad + 2n \ln 2\, 2^{(1+n)(2-p)} |B_{R/2}| \left\{ MC + (p-1) C^p \gamma_2^{p-1} \left[\frac{\psi\,(\omega/4)}{\phi\,(\omega/2^m)} \right]^{p-1} \right\},
\end{aligned}
$$

and since $\gamma_1 > 1$ and $n > 0$, this implies

$$
\begin{aligned}
& \sup_{-\hat{t}\leq t\leq 0} \int_{B_{R/2}\times\{t\}} \big\{\varrho^+(u)\big\}^2 \zeta^p \, dx \\
& \leq 2M(p-1)Cn \ln 2 \|\nabla v\|^{p'}_{L^{p'p}(Q_T)} |B_{R/2}|^{1-\frac{1}{p}} \\
& \quad + 2n \ln 2\, 2^{2-p} |B_{R/2}| \left\{ MC + (p-1) C^p \gamma_2^{p-1} \left[\frac{\psi\,(\omega/4)}{\phi\,(\omega/2^m)} \right]^{p-1} \right\}.
\end{aligned} \tag{3.28}
$$

Since the integrand in the left-hand side of (3.28) is nonnegative, the integral can be estimated from below by integrating over the smaller set $S = \{x \in B_{R/2} : u(x,t) \geq 1 - \omega/2^{2+n}\} \subset B_{R/2}$. Thus, noticing that

$$
\zeta = 1 \quad \text{and} \quad \big\{\varrho^+(u)\big\}^2 \geq \big(\ln(2^{n-1})\big)^2 = (n-1)^2 (\ln 2)^2 \quad \text{on } S,
$$

we obtain that (3.28) reads

$$
\begin{aligned}
& \big|\big\{x \in B_{R/2} : u(x,t) \geq 1 - \omega/2^{2+n}\big\}\big| \\
& \quad \leq \frac{2Cn|B_{R/4}|}{(n-1)^2 \ln 2} \left\{ 2^{2-p} \left[MC + (p-1) C^p \gamma_2^{p-1} \left[\frac{\psi\,(\omega/4)}{\phi\,(\omega/2^m)} \right]^{p-1} \right] \right.
\end{aligned}
$$

$$+ M(p-1)\|\nabla v\|^{p'}_{L^{p'p}(Q_T)}\Big\}$$

for all $t \in (-\hat{t}, 0)$. To prove the lemma we just need to choose s_1 depending on ν_1 such that $s_1 = 2 + n$ with

$$n > 1 + \frac{2C}{\nu_1 \ln 2}\Bigg\{2^{2-p}\Bigg[MC + (p-1)C^p\gamma_2^{p-1}\left[\frac{\psi(\omega/4)}{\phi(\omega/2^m)}\right]^{p-1}\Bigg]$$

$$+M(p-1)\|\nabla v\|^{p'}_{L^{p'p}(Q_T)}\Bigg\},$$

since if $n \geq 1 + 2/\alpha$ then $n/(n-1)^2 \leq \alpha$, $\alpha > 0$. Furthermore, s_1 is independent of ω because

$$\left[\frac{\psi(\omega/4)}{\phi(\omega/2^m)}\right]^{p-1} = \left[\frac{(\omega/4)^{\beta_2/(p-1)}}{(\omega/2^m)^{\beta_1/(p-1)}}\right]^{(p-1)} = \omega^{\beta_2-\beta_1}2^{m\beta_1-2\beta_2} \leq 2^{m\beta_1-2\beta_2}.$$

The last inequality holds since $\beta_2 > \beta_1$. ∎

Now, the first alternative is established by the following proposition.

Proposition 3.1. *The numbers $\nu_1 \in (0,1)$ and $s_1 \gg 1$ can be chosen a priori independently of ω and R, such that if (3.4) holds, then*

$$u(x,t) < \frac{\omega}{2^{s_1+1}} \quad \text{a.e. in } Q(\hat{t}, R/8).$$

We omit the proof of Proposition 3.1 because it is based on the argument of [5, Lemma 3.3] and [7], and we may use for the extension the same technique applied in the proof of Lemma 3.2.

Corollary 3.1. *There exist numbers $\nu_0, \sigma_0 \in (0,1)$ independent of ω and R such that if (3.4) holds, then*

$$\operatorname*{ess\,osc}_{Q(\hat{t},R/8)} u \leq \sigma_0\omega.$$

Proof: In light of Proposition 3.1, we know that there exists a number s_1 such that

$$\operatorname*{ess\,sup}_{Q(\hat{t},R/8)} u \leq 1 - \frac{\omega}{2^{s_1+1}},$$

and this yields

$$\operatorname*{ess\,osc}_{Q(\hat{t},R/8)} u = \operatorname*{ess\,sup}_{Q(\hat{t},R/8)} u - \operatorname*{ess\,inf}_{Q(\hat{t},R/8)} u \leq \left(1 - \frac{1}{2^{s_1+1}}\right)\omega.$$

In this way, choosing $\sigma_0 = 1-1/2^{s_1+1}$, which is independent of ω, we complete the proof. ∎

3.4. The second alternative. Let us suppose now that (3.4) does not hold. Then the complementary case is valid and for every cylinder $Q_R^{t^*}$ we have

$$\left|\left\{(x,t) \in Q_R^{t^*} \,:\, u(x,t) < \omega/2\right\}\right| \leq (1-\nu_0)\left|Q_R^{t^*}\right|. \tag{3.29}$$

Following an analogous analysis to the performed in the case in which the solution is near its degeneracy at one, a similar conclusion is obtained for the second alternative (cf. [4] and [7]). Specifically, we first use logarithmic estimates to extend the result to a full cylinder and then we conclude that the solution is essentially away from 0 in a cylinder $Q(\tau,\rho)$. In this way we prove the following corollary.

Corollary 3.2. *Let $\tilde{t}$ denote the second-alternative-counterpart of $\hat{t}$. Then there exists $\sigma_1 \in (0,1)$, depending only on the data, such that*

$$\operatorname*{ess\,osc}_{Q(\tilde{t},R/8)} u \leq \sigma_1\omega.$$

Since (3.4) or (3.29) must be valid, the conclusion of Corollary 3.1 or 3.2 must hold. Thus, choosing $\sigma = \max\{\sigma_0,\sigma_1\}$ and $t^\diamond = \min\{\hat{t},\tilde{t}\}$, we obtain the following proposition.

Proposition 3.2. *There exists a constant $\sigma \in (0,1)$, depending only on the data, such that*

$$\operatorname*{ess\,osc}_{Q(t^\diamond,R/8)} u \leq \sigma\omega.$$

The local Hölder continuity of u in Q_T now follows (see, e.g., [5], [6], or the proof of [23, Th. 2]).

4. Numerical examples

In this section, we provide two numerical examples to illustrate how the approximate solutions of the chemotaxis model (1.1) vary when changing the parameter p from standard nonlinear diffusion ($p = 2$) to doubly nonlinear

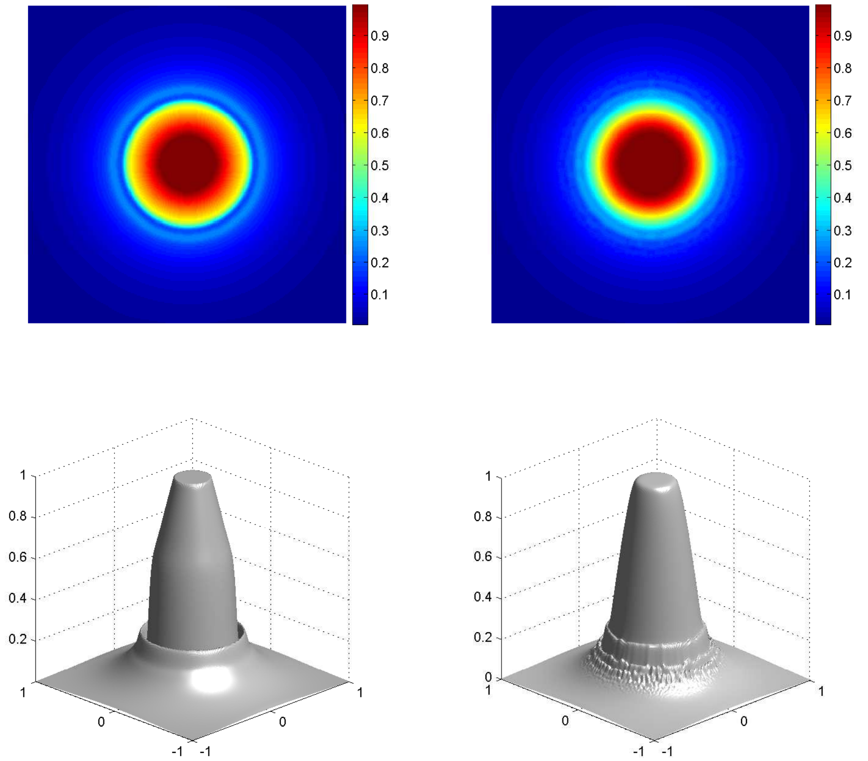

FIGURE 1. Example 1: Numerical solution for species u, at $t = 1.0$ for $p = 2$ (left), and $p = 6$ (right).

diffusion ($p > 2$). For the discretization of both examples, a standard first order finite volume method (see the Appendix for details on the numerical scheme) on a regular mesh of 262144 control volumes is used. We choose a simple square domain $\Omega = [-1, 1]^2$ and use the functions $a(u) = \epsilon u(1 - u)$, $f(u) = (1 - u)^2$ and $g(u, v) = \alpha u - \beta v$, along with parameters that are indicated separately for each case.

4.1. Example 1. For the first example, we choose $\epsilon = 0.01$, $\alpha = 40$, $\beta = 160$, $\chi = 0.2$ and $d = 0.05$. The initial condition for the species density is given by

$$u_0(x) = \begin{cases} 1 & \text{for } \|x\| \leq 0.2, \\ 0 & \text{otherwise,} \end{cases}$$

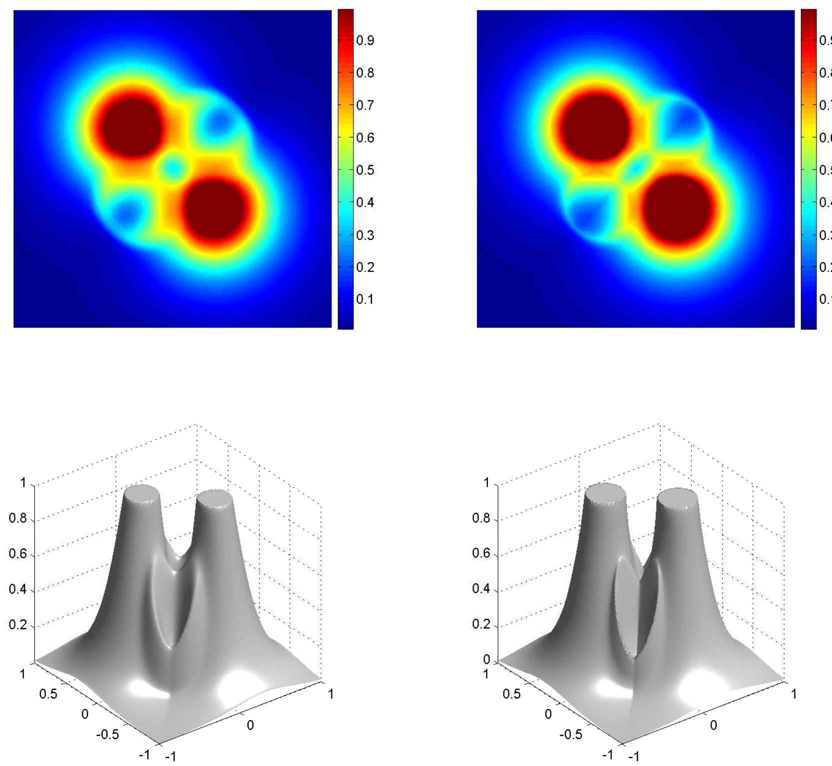

FIGURE 2. Example 2: Numerical solution for species u, at $t = 0.1$ for $p = 2$ (left), and $p = 6$ (right).

and the chemoattractant is assumed to have the uniform concentration $v_0(x) = 4.5$. In a first simulation, we consider the simple case of $p = 2$ and we compare the result with an analogous experiment with $p = 6$. We evolve the system until $t = 1.0$, and show in Figure 1 a snapshot of the cell density at this instant for both cases.

4.2. Example 2. We now choose the parameters $\epsilon = 0.5$, $\alpha = 5$, $\beta = 0.5$, $\chi = 1$ and $d = 0.25$. The initial condition for the species density is given by

$$u_0(x) = \begin{cases} 1 & \text{for } \|x - (-0.25, 0.25)\| \leq 0.2 \text{ or } \|x - (0.25, -0.25)\| \leq 0.2 \\ 0 & \text{otherwise,} \end{cases}$$

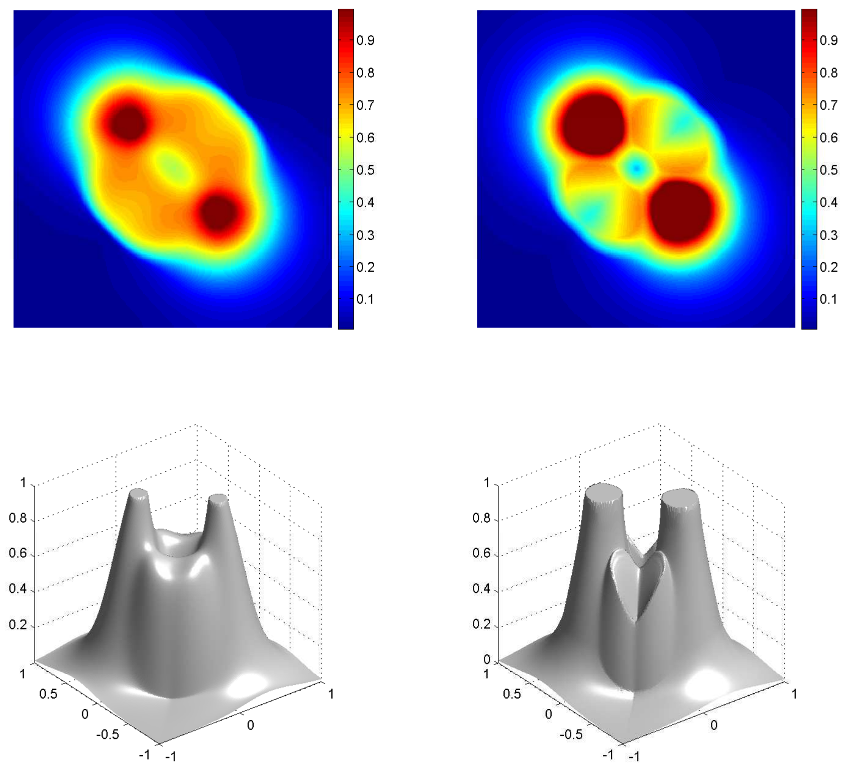

FIGURE 3. Example 2: Numerical solution for species u, at $t = 0.5$ for $p = 2$ (left), and $p = 6$ (right).

and for the chemoattractant

$$v_0(x) = \begin{cases} 4.5 & \text{for } \|x - (0.25, 0.25)\| \leq 0.2 \text{ or } \|x + (0.25, 0.25)\| \leq 0.2 \\ 0 & \text{otherwise.} \end{cases}$$

The behavior of the system for the cases $p = 2$ and $p = 6$ at different times is presented in Figures 2, 3 and 4.

4.3. Concluding remarks. W first mention that, from the previous examples, one observes that even though the numerical solutions obtained with $p = 2$ differ from those obtained with $p > 2$, the qualitative structure of the solutions remains unchanged. We also stress that the numerical examples illustrate the effectiveness of the mechanism of prevention of overcrowding, or volume filling effect, since all solutions assume values between zero and one only. In particular, all examples exhibit plateau-like structures where

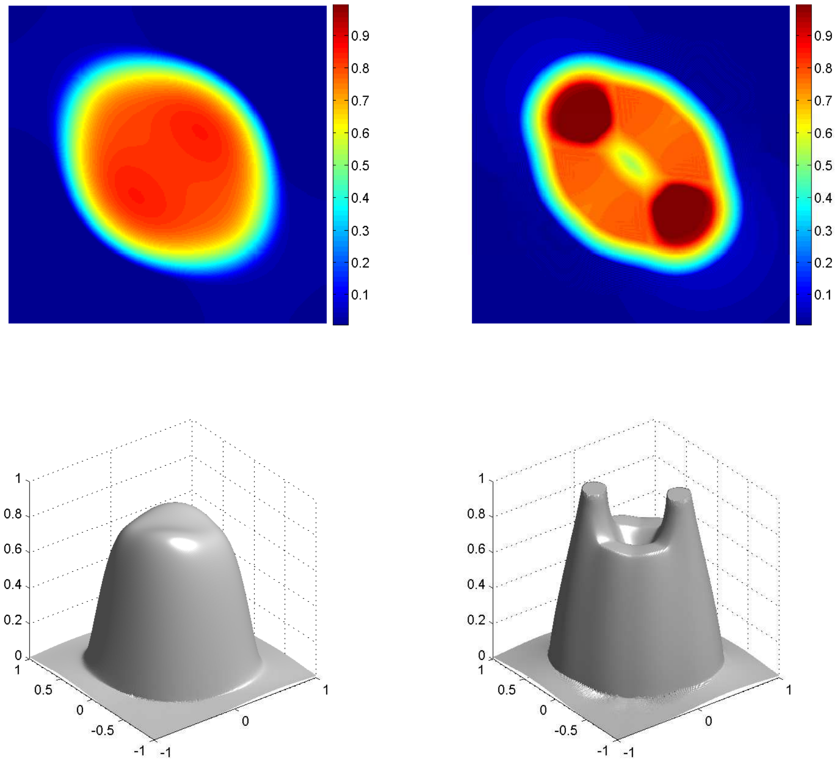

FIGURE 4. Example 2: Numerical solution for species u, at $t = 2.5$ for $p = 2$ (left), and $p = 6$ (right).

$u = u_{\mathrm{m}} = 1$, at least for small times, which diffuse very slowly, illustrating that the diffusion coefficient vanishes at $u = 1$ (recall the special form of the functions $a(u)$ and $f(u)$: they include the factor $(1 - u)$, and therefore the species diffusion and chemotactical cross diffusion terms vanish at $u = 0$ and $u = u_{\mathrm{m}} = 1$).

In Example 2, the solution for $p = 2$ has a smoother shape than the one for $p = 6$, which exhibits sharp edges. These sharp edges do not only appear for $u = 0$ and $u = u_{\mathrm{m}}$, where one expects them, due to the degeneracy of the diffusion term and the choice of initial data, but also for intermediate solution values, as is illustrated by the plots for $p = 6$ of Figures 2 and 3.

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Appendix

The definition of the finite volume method is based on the framework of [28]. An admissible mesh for Ω is given by a family $\mathcal{T}$ of control volumes of maximum diameter h, a family of edges $\mathcal{E}$ and a family of points $(x_K)_{K\in\mathcal{T}}$. For $K \in \mathcal{T}$, x_K is the center of K, $\mathcal{E}_{\rm int}(K)$ is the set of edges σ of K in the interior of $\mathcal{T}$, and $\mathcal{E}_{\rm ext}(K)$ the set of edges of K on the boundary $\partial\Omega$. For all $\sigma \in \mathcal{E}$, the transmissibility coefficient is

$$\tau_\sigma = \begin{cases} \dfrac{|\sigma|}{d(x_K, x_L)} & \text{for } \sigma \in \mathcal{E}_{\rm int}(K),\ \sigma = K|L, \\ \dfrac{|\sigma|}{d(x_K, \sigma)} & \text{for } \sigma \in \mathcal{E}_{\rm ext}(K), \end{cases}$$

where $K|L$ denotes the common edge of neighboring finite volumes K and L. For $K \in \mathcal{T}$ and $\sigma = K|L \in \mathcal{E}(K)$ with common vertexes $(a_{\ell,K,L})_{1\le\ell\le I}$ with $I \in \mathbb{N}\backslash\{0\}$, let T_σ ($T^{\rm ext}_{K,\sigma}$ for $\sigma \in \mathcal{E}_{\rm ext}(K)$, respectively) be the open and convex polygon built by the convex envelope with vertices (x_K, x_L) (x_K, respectively) and $(a_{\ell,K,L})_{1\le\ell\le I}$. The domain Ω can be decomposed into

$$\overline{\Omega} = \cup_{K\in\mathcal{T}}\big((\cup_{L\in N(K)}\overline{T}_{K,L}) \cup (\cup_{\sigma\in\mathcal{E}_{\rm ext}(K)}\overline{T}^{\rm ext}_{K,\sigma})\big).$$

For all $K \in \mathcal{T}$, the approximation $\nabla_h u_{K,\sigma}$ of ∇u is defined by

$$\nabla_h u^n_{K,\sigma} := \begin{cases} u^n_L - u^n_K & \text{if } \sigma = K|L \in \mathcal{E}_{\rm int}(K), \\ 0 & \text{if } \sigma \in \mathcal{E}_{\rm ext}(K). \end{cases}$$

To discretize (1.1), we choose an admissible mesh of Ω and a time step size $\Delta t > 0$. If $M_T > 0$ is the smallest integer such that $M_T\Delta t \ge T$, then $t^n := n\Delta t$ for $n \in \{0, \dots, M_T\}$.

We define cell averages of the unknowns $A(u)$, $f(u)$ and $g(u,v)$ over $K \in \mathcal{T}$:

$$A_K^{n+1} := \frac{1}{\Delta t|K|}\int_{t^n}^{t^{n+1}}\int_K A\big(u(x,t)\big)\,dx\,dt,$$

$$g_K^{n+1} := \frac{1}{\Delta t|K|}\int_{t^n}^{t^{n+1}}\int_K g\big(u(x,t),v(x,t)\big)\,dx\,dt,$$

$$f_K^{n+1} := \frac{1}{\Delta t|K|}\int_{t^n}^{t^{n+1}}\int_K f\big(u(x,t)\big)\,dx\,dt,$$

and the initial conditions are discretized by

$$u_K^0 = \frac{1}{|K|}\int_K u_0(x)\,dx, \quad v_K^0 = \frac{1}{|K|}\int_K v_0(x)\,dx.$$

We now give the finite volume scheme employed to advance the numerical solution from t^n to t^{n+1}, which is based on a simple explicit Euler time discretization. Assuming that at $t = t^n$, the pairs (u_K^n, v_K^n) are known for all $K \in \mathcal{T}$, we compute (u_K^{n+1}, v_K^{n+1}) from

$$\begin{aligned}|K|\frac{u_K^{n+1}-u_K^n}{\Delta t} &= \sum_{\sigma\in\mathcal{E}(K)} \tau_\sigma \left|\nabla_h A_{K,\sigma}^n\right|_h^{p-2} \nabla_h A_{K,\sigma}^n \\ &\quad + \chi \sum_{\sigma\in\mathcal{E}(K)} \tau_\sigma \left[\left(\nabla_h v_{K,\sigma}^n\right)^+ u_K^n f_K^n - \left(\nabla_h v_{K,\sigma}^n\right)^- u_L^n f_L^n\right], \\ |K|\frac{v_K^{n+1}-v_K^n}{\Delta t} &= \sum_{\sigma\in\mathcal{E}(K)} \tau_\sigma \nabla_h v_{K,\sigma}^n + |K| g_K^n.\end{aligned}$$

Here $|\cdot|_h$ denotes the discrete Euclidean norm. The Neumann boundary conditions are taken into account by imposing zero fluxes on the external edges.

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