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THE THIRD COHOMOLOGY GROUP CLASSIFIES DOUBLE CENTRAL EXTENSIONS

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Dedicated to Dominique Bourn on the occasion of his sixtieth birthday

ABSTRACT: We characterise the double central extensions in a semi-abelian category in terms of commutator conditions. We prove that the third cohomology group $H^3(Z,A)$ of an object Z with coefficients in an abelian object A classifies the double central extensions of Z by A.

KEYWORDS: cohomology, categorical Galois theory, semi-abelian category, higher central extension, Baer sum.
AMS SUBJECT CLASSIFICATION (2000): 18G50, 18G60, 20J, 55N.

Introduction

The second cohomology group $H^2(Z,A)$ of a group Z with coefficients in an abelian group A is well-known to classify the central extensions of Z by A in the following manner. Any central extension f of Z by A induces a short exact sequence

$$0 \longrightarrow A \overset{\ker f}{\rightarrowtail} X \overset{f}{\twoheadrightarrow} Z \longrightarrow 0$$

such that $axa^{-1}x^{-1} = 1$ for all $a \in A$ and $x \in X$. The elements of the group $H^2(Z,A)$ are equivalence classes of such central extensions; here two extensions $f\colon X \to Z$ and $f'\colon X' \to Z$ are equivalent if and only if there exists a group (iso)morphism $x\colon X \to X'$ satisfying $f' \circ x = f$ and $x \circ \ker f = \ker f'$. The group structure on $H^2(Z,A)$ is given by the classical Baer sum—see for instance [21]. In [14], see also [5] and [8], this construction was extended categorically from the context of groups to semi-abelian categories [18]. Thus a similar interpretation of the second cohomology group also makes sense for, say, Lie algebras over a field, commutative algebras, non-unital rings, or (pre)crossed modules.

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The aim of the present work is to prove a two-dimensional version of this result, at once in a categorical context: we show that the third cohomology group $H^3(Z,A)$ of an object Z with coefficients in an abelian object A of a semi-abelian category $\mathcal{A}$ classifies the double central extensions in $\mathcal{A}$ of Z by A. Thus the connections between two branches of non-abelian (co)homology are made explicit.

On one hand, there is the direction approach to cohomology established by Bourn and Rodelo [3, 4, 5, 9, 23]; here the cohomology groups H^nA of an internal abelian group A are described through direction functors, in such a way that any short exact sequence of internal abelian groups induces a long exact cohomology sequence. This concept of *direction* may be understood as follows. It is well-known that in a Barr exact context, H^1A can be interpreted in terms of A-*torsors*. An A-torsor is a generalised affine space over A: a "group without unit" where any choice of a unit gives back A—its direction. Further borrowing intuition from Affine Geometry, H^1A is described in terms of autonomous Mal'tsev operations with given direction A. On level 2—the level which corresponds to the "third cohomology group" from the title—the direction functor theory is based on that of level 1: now H^2A is described in terms of internal groupoids with given direction A. By means of higher order internal groupoids, the theory is inductively extended to higher levels H^nA.

On the other hand, there is the approach to semi-abelian homology [1, 11] based on categorical Galois theory [2, 15] initiated by Janelidze [16, 17] and further worked out by Everaert, Gran and Van der Linden [10]. Here the basic situation is given by a semi-abelian category $\mathcal{A}$ and a Birkhoff subcategory $\mathcal{B}$ of $\mathcal{A}$: the derived functors of the reflector $I\colon \mathcal{A}\to\mathcal{B}$ are computed in terms of higher Hopf formulae using the induced Galois structures of higher central extensions. In the specific case where $\mathcal{B}$ is the Birkhoff subcategory $\mathsf{Ab}\mathcal{A}$ determined by the abelian objects in $\mathcal{A}$ and $I=\mathsf{ab}$ is the abelianisation functor, we start from the Galois structure

$$\Gamma = (\mathcal{A} \underset{\supset}{\overset{\mathsf{ab}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{Ab}\mathcal{A}, |\mathsf{Ext}\mathcal{A}|, |\mathsf{ExtAb}\mathcal{A}|). \qquad \textbf{(A)}$$

The class of extensions $|\mathsf{Ext}\mathcal{A}|$ (respectively $|\mathsf{ExtAb}\mathcal{A}|$) consists of the regular epimorphisms in $\mathcal{A}$ (in $\mathsf{Ab}\mathcal{A}$) and forms the class of objects of the category $\mathsf{Ext}\mathcal{A}$ (or $\mathsf{ExtAb}\mathcal{A}$) whose morphisms are commutative squares between extensions. The covers with respect to this Galois structure Γ are exactly the central extensions in the sense of commutator theory: an extension $f\colon X\to Z$

is central if and only if $[R[f], \nabla_X] = \Delta_X$, i.e., the commutator of the kernel pair of f with the largest relation ∇_X on X is the smallest relation Δ_X on X. These central extensions, in turn, determine a reflective subcategory $\mathsf{CExt}\mathcal{A}$ of $\mathsf{Ext}\mathcal{A}$; the reflector $\mathsf{centr}\colon \mathsf{Ext}\mathcal{A} \to \mathsf{CExt}\mathcal{A}$ which sends f to the central extension $\mathsf{centr}f\colon X/[R[f], \nabla_X] \to Z$ is the **centralisation functor**. Thus we obtain the Galois structure

$$\Gamma_1 = (\mathsf{Ext}\mathcal{A} \underset{\supset}{\overset{\mathsf{centr}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{CExt}\mathcal{A}, |\mathsf{Ext}^2\mathcal{A}|, |\mathsf{Ext}^2\mathsf{Ab}\mathcal{A}|). \qquad (\mathbf{B})$$

The classes $|\mathsf{Ext}^2\mathcal{A}|$ and $|\mathsf{Ext}^2\mathsf{Ab}\mathcal{A}|$ consist of *double extensions* in $\mathcal{A}$ or in $\mathsf{Ab}\mathcal{A}$. A **double extension** is a commutative square

$$\begin{array}{ccc} X & \xrightarrow{c} & C \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[f]{} & Z \end{array}$$

such that all its maps and the comparison map $(d, c)\colon X \to D \times_Z C$ to the pullback of f with g are regular epimorphisms. The covers with respect to the Galois structure Γ_1 are used in the computation of the third homology functor $H_3(-, \mathsf{ab})\colon \mathcal{A} \to \mathsf{Ab}\mathcal{A}$ (see [10]) and form the main subject of the present paper—they are the “double central extensions” from the title.

We start by recalling the main properties of the Galois structure Γ_1 in Section 1. In Section 2 we characterise the Γ_1-covers in terms of commutators (as Janelidze does in the category of groups [16] and Gran and Rossi do in the context of Mal’tsev varieties [13]) and in terms of internal pregroupoids [20]. Section 3 recalls Bourn and Rodelo’s definition of the third cohomology group in semi-abelian categories. We obtain a natural notion of direction for double extensions and show in Section 4 that the set $\mathsf{Centr}^2(Z, A)$ of equivalence classes of double central extensions of an object Z by an abelian object A carries a canonical abelian group structure. In Section 5 we conclude the paper with the isomorphism $H^3(Z, A) \cong \mathsf{Centr}^2(Z, A)$ between the third cohomology group of an object Z with coefficients in an abelian object A and the group $\mathsf{Centr}^2(Z, A)$.

We conjecture that this result may be generalised to higher degrees, so that also for $n > 2$ the $(n+1)$-st cohomology group $H^{n+1}(Z, A)$ of Z with coefficients in A classifies the n-fold central extensions of Z by A. This will be the subject of future work.

Acknowledgement. Many thanks to Tomas Everaert for important comments and suggestions, and the present, more elegant, proof for Theorem 2.7.

1. Preliminaries

1.1. Semi-abelian categories. The basic context where we shall be working is that of *semi-abelian categories* [18]. Some examples are the categories Gp of all groups, Rng of non-unital rings, $\mathsf{Lie}_{\mathbb{K}}$ of Lie algebras over a field $\mathbb{K}$, XMod of crossed modules, and Loop of loops. We briefly recall the main definitions.

A category is **semi-abelian** when it is pointed, Barr exact and Bourn protomodular and has binary coproducts. A category is **pointed** when it has a zero object 0: a terminal object which is also initial. A **Barr exact** category is **regular**—finitely complete with pullback-stable regular epimorphisms and coequalisers of kernel pairs—and such that every internal equivalence relation is a kernel pair. A pointed and regular category is **Bourn protomodular** when the **(regular) Short Five Lemma** holds: given any commutative diagram of regular epimorphisms with their kernels

$$
\begin{array}{ccccc}
K[f] & \xrightarrow{\ker f} & X & \xrightarrow{f} & Z \\
{\scriptstyle k}\downarrow & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle z} \\
K[f'] & \xrightarrow[\ker f']{} & X' & \xrightarrow[f']{} & Z'
\end{array}
$$

the morphisms k and z being isomorphisms implies that x is an isomorphism.

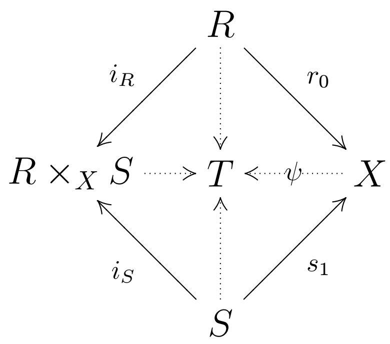
Any semi-abelian category $\mathcal{A}$ is a **Mal'tsev category**: it is finitely complete and, in $\mathcal{A}$, every reflexive relation is an equivalence relation.

1.2. The commutator of equivalence relations. Let $R = (R, r_0, r_1)$ and $S = (S, s_0, s_1)$ be equivalence relations on an object X of a Mal'tsev category $\mathcal{A}$. Let $R \times_X S$ denote the pullback of r_1 and s_0:

$$
\begin{array}{ccc}
R \times_X S & \overset{p_S}{\underset{i_S}{\rightleftarrows}} & S \\
{\scriptstyle i_R}\uparrow\downarrow{\scriptstyle p_R} & & \uparrow\downarrow{\scriptstyle s_0} \\
R & \overset{r_1}{\rightleftarrows} & X.
\end{array}
$$

The object $R \times_X S$ "consists of" triples (α, β, γ) where $\alpha R \beta$ and $\beta S \gamma$. We say that R and S **commute** when there exists a **connector** between R and S: a morphism $p\colon R \times_X S \to X$ which satisfies $p(\alpha, \alpha, \gamma) = \gamma$ and $p(\alpha, \gamma, \gamma) = \alpha$ [6]; see also [1, Definition 2.6.1].

When $\mathcal{A}$ is a semi-abelian category, the **commutator** of R and S [22], denoted by $[R, S]$, is the universal equivalence relation on X which, when divided out, makes them commute. More precisely, $[R, S]$ is the kernel pair $R[\psi]$ of the map ψ in the diagram



where the dotted arrows denote the colimit of the outer square [1, Section 2.8]. The direct images ψR and ψS of R and S along the regular epimorphism ψ commute; hence R and S commute if and only if $[R, S] = \Delta_X$ [6, Proposition 4.2].

An equivalence relation R on an object X is **central** when it commutes with ∇_X—when $[R, \nabla_X] = \Delta_X$.

1.3. Central extensions. Let $\mathcal{A}$ be a semi-abelian category. For any object X of $\mathcal{A}$ we may take the kernel of the X-component of the unit of the adjunction in **A** to obtain a short exact sequence

$$0 \longrightarrow [X] \xrightarrow{\mu_X} X \xrightarrow{\eta_X} \mathsf{ab}X \longrightarrow 0.$$

Thus we acquire a functor $[-]\colon \mathcal{A} \to \mathcal{A}$ together with a natural transformation $\mu\colon [-] \Rightarrow 1_{\mathcal{A}}$.

Lemma 1.4. *The functors* ab *and* $[-]$ *preserve pullbacks of regular epimorphisms along split epimorphisms.*

Proof: It is well-known that the functor ab has this property: see, for instance, [12]. Since kernels commute with pullbacks, it follows that the functor $[-]$ has the same property. ∎

An extension $f\colon X \to Z$ is **central** (with respect to the Galois structure Γ in diagram **A**) if and only if either one of the projections p_0 or p_1 of its kernel pair $(R[f], p_0, p_1)$ is a trivial extension, i.e., a pullback of an extension in $\mathsf{Ab}\mathcal{A}$. It follows that f is central if and only if the right hand side square in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & [R[f]] & \xrightarrow{\mu_{R[f]}} & R[f] & \xrightarrow{\eta_{R[f]}} & \mathsf{ab}R[f] & \longrightarrow & 0 \\
 & & {\scriptstyle [p_0]}\downarrow & & \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle \mathsf{ab}p_0} & & \\
0 & \longrightarrow & [X] & \xrightarrow[\mu_X]{} & X & \xrightarrow[\eta_X]{} & \mathsf{ab}X & \longrightarrow & 0
\end{array}$$

is a pullback or, equivalently, $[p_0]$ is an isomorphism. Hence the kernel of $[p_0]$, which is denoted by $[f]_1$, is zero if and only if f is central. Considering $[f]_1$ as a normal subobject of X, the **centralisation functor** $\mathsf{centr}\colon \mathsf{Ext}\mathcal{A} \to \mathsf{CExt}\mathcal{A}$, from the Galois structure Γ_1 in diagram **B**, takes the extension $f\colon X \to Z$ and maps it to the quotient $\mathsf{centr}f\colon X/[f]_1 \to Z$ of $f\colon X \to Z$ by the extension $[f]_1 \to 0$.

This notion of centrality for extensions is compatible with the above-mentioned notion of centrality for equivalence relations. Indeed, an extension f is central if and only if so is its kernel pair $R[f]$; see [12].

Given an object Z and an abelian object A, a **central extension of** Z **by** A is a central extension $f\colon X \to Z$ with kernel $K[f] = A$. The group of isomorphism classes of central extensions of Z by A is denoted $\mathsf{Centr}^1(Z, A)$. Recall the following result from [14]:

Proposition 1.5. *If $\mathcal{A}$ is a semi-abelian category and Z is an object of $\mathcal{A}$ then the functor $\mathsf{Centr}^1(Z, -)\colon \mathsf{Ab}\mathcal{A} \to \mathsf{Ab}$ preserves finite products.*

Proof: We shall only repeat the main point of the construction behind [14, Proposition 6.1]. Let $a\colon A \to B$ be a morphism of abelian objects in $\mathcal{A}$ and $f\colon X \to Z$ a central extension of Z by A. Let $A \oplus B$ denote the biproduct of A with B in $\mathsf{Ab}\mathcal{A}$. The functor $\mathsf{Centr}^1(Z, -)$ maps the equivalence class of f to the equivalence class of the central extension f' in the diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A \oplus B & \xrightarrow{\ker f \times 1_B} & X \times B & \xrightarrow{f \circ \mathrm{pr}_X} & Z & \longrightarrow & 0 \\
 & & {\scriptstyle [a, 1_B]}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & B & \longrightarrow & X' & \xrightarrow[f']{} & Z & \longrightarrow & 0.
\end{array} \qquad \textbf{(C)}$$

The extension f' is central as a quotient of the central extension $f \circ \mathrm{pr}_X$. ■

1.6. Double extensions as spans. Recall that a **double extension (of an object** Z**)** in a semi-abelian category is a commutative square

$$\begin{array}{ccc} X & \xrightarrow{c} & C \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[f]{} & Z \end{array} \qquad (\mathbf{D})$$

such that all its maps and the comparison map $(d, c)\colon X \to D \times_Z C$ to the pullback of f with g are regular epimorphisms. Double extensions may be characterised in terms of spans in a slice category as follows.

Definition 1.7. A span (X, d, c)

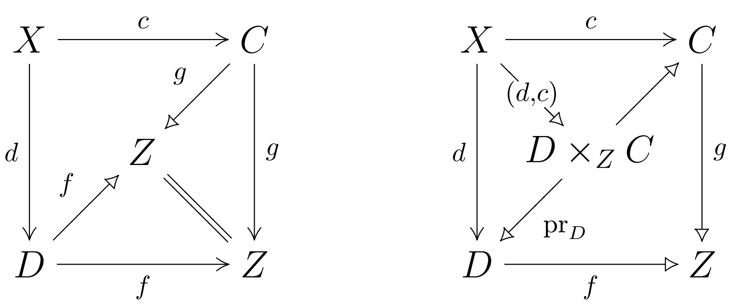
$$\begin{array}{ccc} & X & \\ {\scriptstyle d}\swarrow & & \searrow{\scriptstyle c} \\ D & & C \end{array} \qquad (\mathbf{E})$$

in a regular category $\mathcal{A}$

(1) **has global support** when $!_D\colon D \to 1$ and $!_C\colon C \to 1$ are regular epimorphisms;
(2) is **aspherical** when also $(d, c)\colon X \to D \times C$ is a regular epimorphism.

Proposition 1.8. *Let* $\mathcal{A}$ *be a semi-abelian category. A commutative square* $\mathbf{D}$ *in* $\mathcal{A}$ *is a double extension if and only if* (X, d, c) *is an aspherical span in* $\mathcal{A} \downarrow Z$.

Proof: Since the terminal object of $\mathcal{A} \downarrow Z$ is $1_Z\colon Z \to Z$, (X, d, c) has global support whenever f and g are regular epimorphisms in $\mathcal{A}$.



But the product of f and g in $\mathcal{A} \downarrow Z$ is the map $f \circ \mathrm{pr}_D\colon D \times_Z C \to Z$ starting from the pullback of f and g in $\mathcal{A}$, hence this span is aspherical if and only if also the map $(d,c)\colon X \to D \times_Z C$ is a regular epimorphism in $\mathcal{A}$—which means that the square square $\mathbf{D}$ is a double extension. ■

1.9. Double central extensions. Let $\mathcal{A}$ be a semi-abelian category. By definition, a double extension is **central** when it is a cover with respect to the Galois structure Γ_1. Hence the double extension $\mathbf{D}$, considered as a map $(c,f)\colon d \to g$ in the category $\mathsf{Ext}\mathcal{A}$, is central if and only if the first projection

$$\begin{array}{ccc} R[c] & \xrightarrow{p_0} & X \\ {\scriptstyle R[(c,f)]}\downarrow & & \downarrow{\scriptstyle d} \\ R[f] & \xrightarrow[p_0]{} & D \end{array} \qquad\qquad \begin{array}{ccc} R[c] & \xrightarrow{p_0} & X \\ \downarrow & & \downarrow \\ R[c]/[R[(c,f)]]_1 & \longrightarrow & X/[d]_1 \end{array}$$

of its kernel pair—the left hand side square—is a trivial extension with respect to Γ_1. (Alternatively, one could use the square of second projections.) This means that the comparison map to its reflection into $\mathsf{CExt}\mathcal{A}$—the right hand side square—is a pullback. For this to happen, the natural map $[R[(c,f)]]_1 \to [d]_1$ must be an isomorphism. This, in turn, is equivalent to the square

$$\begin{array}{ccc} [R[d]\square R[c]] & \xrightarrow{[p_0]} & [R[d]] \\ {\scriptstyle [p_0]}\downarrow & & \downarrow{\scriptstyle [p_0]} \\ [R[c]] & \xrightarrow[{[p_0]}]{} & [X] \end{array}$$

being a pullback, because $[R[(c,f)]]_1$ and $[d]_1$ are the kernels of the vertical maps above. Here $(R[d]\square R[c], p_0, p_1)$ denotes the kernel pair of $R[(c,f)]$; it consists of all quadruples $(\alpha,\beta,\gamma,\delta) \in X^4$ in the following configuration:

$$\begin{bmatrix} \alpha & c & \beta \\ d & & d \\ \delta & c & \gamma \end{bmatrix}$$

$d(\alpha) = d(\delta)$, $c(\alpha) = c(\beta)$, $c(\gamma) = c(\delta)$ and $d(\gamma) = d(\beta)$.

2. Characterisation of double central extensions in terms of commutators

In this section we characterise the covers with respect to the Galois structure Γ_1 in terms of *internal pregroupoids* in the sense of Kock [20]. This characterisation turns out to be equivalent to the conditions given by Janelidze in [16] and Gran and Rossi in [13]—and thus we prove a categorical version of the next result.

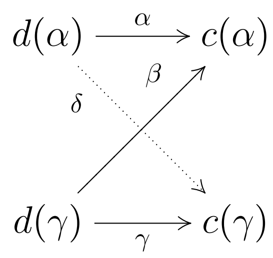
Proposition 2.1. [13, 16] *Let $\mathcal{A}$ be a Mal'tsev variety. A double extension* $\mathbf{D}$ *in $\mathcal{A}$ is central if and only if* $[R[d], R[c]] = \Delta_X = [R[d] \cap R[c], \nabla_X]$. ∎

The concept of internal pregroupoid generalises internal groupoids in the following manner: in a pregroupoid, the domain and codomain of a map may live in different objects, and no identities need exist.

Definition 2.2. [19, 20] Let $\mathcal{A}$ be a finitely complete category. A **pregroupoid** (also called a **herdoid**) (X, d, c, p) in $\mathcal{A}$ is a span $\mathbf{E}$ with a partial ternary operation p on X satisfying:

(1) $p(\alpha, \beta, \gamma)$ is defined if and only if $c(\alpha) = c(\beta)$ and $d(\gamma) = d(\beta)$;
(2) $dp(\alpha, \beta, \gamma) = d(\alpha)$ and $cp(\alpha, \beta, \gamma) = c(\gamma)$ if $p(\alpha, \beta, \gamma)$ is defined;
(3) $p(\alpha, \alpha, \gamma) = \gamma$ if $p(\alpha, \alpha, \gamma)$ is defined, and $p(\alpha, \gamma, \gamma) = \alpha$ if $p(\alpha, \gamma, \gamma)$ is defined;
(4) $p(\alpha, \beta, p(\gamma, \delta, \epsilon)) = p(p(\alpha, \beta, \gamma), \delta, \epsilon)$ if either side is defined.

An "element" α of X should be interpreted as a map $\alpha\colon d(\alpha) \to c(\alpha)$; its domain $d(\alpha)$ is an element of D, while its codomain $c(\alpha)$ is an element of C. The operation p sends a composable triple



to the dotted diagonal $\delta = p(\alpha, \beta, \gamma)\colon d(\alpha) \to c(\gamma)$. In case the pregroupoid is a groupoid (i.e., when the span is a reflexive graph so that $C = D$ and identities exist), $p(\alpha, \beta, \gamma) = \gamma \circ \beta^{-1} \circ \alpha$.

We denote the category of (pre)groupoids in $\mathcal{A}$ by $(\mathsf{Pre})\mathsf{Gd}\mathcal{A}$.

Definition 2.3. Suppose that $\mathcal{A}$ is regular. A pregroupoid (X, d, c, p) **has global support** or is **aspherical** whenever the span (X, d, c) has global support or is aspherical. This definition applies in the obvious way to internal groupoids.

Suppose that $\mathcal{A}$ is a Mal'tsev category. As explained in the introduction of [6], an internal pregroupoid structure p on a span (X, d, c) is the same thing as a connector between the kernel pairs $R[c]$ and $R[d]$ of c and d. Indeed, using that $\mathcal{A}$ is Mal'tsev, one shows that conditions (2) and (4) of Definition 2.2 are automatically satisfied: see Proposition 2.6.11 in [1] or Proposition 4.1 in [6]. Two equivalence relations admit at most one connector; hence, if it exists, a pregroupoid structure p on a span (X, d, c) is necessarily unique. In this case we shall say that the span (X, d, c) **is** a pregroupoid and drop the structure p from the notation.

Because of Proposition 1.8 which exhibits the close connection between double extensions in $\mathcal{A}$ and spans in a slice category $\mathcal{A} \downarrow Z$, we are also mostly interested in pregroupoids in slice categories. Asking that a span (X, d, c) is a pregroupoid in $\mathcal{A} \downarrow Z$ amounts to asking that (X, d, c) is a pregroupoid in $\mathcal{A}$: when $\mathcal{A}$ is semi-abelian, this happens precisely when the first equality $[R[d], R[c]] = \Delta_X$ of Proposition 2.1 holds.

Definition 2.4. Suppose that $\mathcal{A}$ is semi-abelian and let Z be an object of $\mathcal{A}$. An aspherical (pre)groupoid (X, d, c) in $\mathcal{A} \downarrow Z$ is **central** when $(d, c)\colon X \to D \times_Z C$ is a central extension in $\mathcal{A}$.

Since $R[d] \cap R[c] = R[(d, c)\colon X \to D \times_Z C]$, this makes the centrality of the aspherical pregroupoid (X, d, c) equivalent to the second equality $[R[d] \cap R[c], \nabla_X] = \Delta_X$ of Proposition 2.1. And thus we proved:

Proposition 2.5. *Let $\mathcal{A}$ be a semi-abelian category. A double extension* $\mathbf{D}$ *in $\mathcal{A}$ satisfies*

$$[R[d], R[c]] = \Delta_X = [R[d] \cap R[c], \nabla_X] \tag{$\mathbf{F}$}$$

if and only if the span (X, d, c) is a central pregroupoid in the slice category $\mathcal{A} \downarrow Z$. ■

Proposition 2.6. *In a semi-abelian category, condition* $\mathbf{F}$ *is preserved and reflected by pullbacks of double extensions along double extensions.*

Proof: The proof given in Section 4 of [13] in the context of Mal'tsev varieties is still valid in the present situation. ■

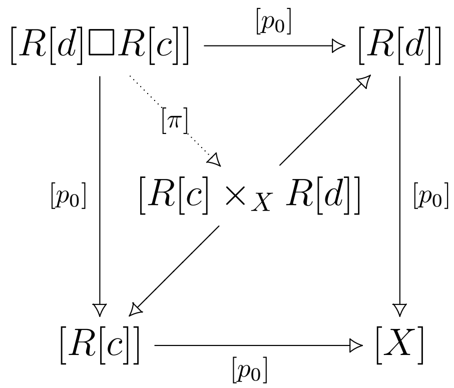
Theorem 2.7. *Consider a double extension* $\mathbf{D}$ *in a semi-abelian category $\mathcal{A}$. The following are equivalent:*

(1) $\mathbf{D}$ *is a double central extension;*
(2) *(X, d, c) is a central pregroupoid in $\mathcal{A} \downarrow Z$;*

(3) $[R[d], R[c]] = \Delta_X = [R[d] \cap R[c], \nabla_X]$.

Proof: By Proposition 2.5 we already know that (2) and (3) are equivalent. To see that (1) implies (3), suppose that $\mathbf{D}$ is a double central extension. Then either one of the projections of its kernel pair is trivial with respect to Γ_1, meaning that it is a pullback of a double extension between central extensions (i.e., a morphism of the category $\mathsf{CExt}\mathcal{A}$). This latter double extension satisfies the condition corresponding to $\mathbf{F}$; hence applying Proposition 2.6 twice shows that (3) holds.

Now we prove that (2) implies (1). The pregroupoid structure of (X, d, c) is a connector $p\colon R[c] \times_X R[d] \to X$. As explained in Subsection 1.9, we are to show that the outer square in the diagram



is a pullback. Here $\pi\colon R[d]\Box R[c] \to R[c] \times_X R[d]$ is defined by

$$\begin{bmatrix} \alpha & c & \beta \\ d & & d \\ \delta & c & \gamma \end{bmatrix} \mapsto (\alpha, \beta, \gamma).$$

By Lemma 1.4 we know that the inner quadrangle is a pullback, hence it suffices that $[\pi]$ is an isomorphism. The left hand side square

$$\begin{array}{ccc} R[d]\Box R[c] & \xrightarrow{\pi} & R[c] \times_X R[d] \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ R[d] \cap R[c] & \xrightarrow[p_0]{} & X \end{array} \qquad \begin{array}{ccc} [R[d]\Box R[c]] & \xrightarrow{[\pi]} & [R[c] \times_X R[d]] \\ {\scriptstyle [q]}\downarrow & & \downarrow{\scriptstyle [p]} \\ [R[d] \cap R[c]] & \xrightarrow[{[p_0]}]{} & [X], \end{array}$$

where q is defined by

$$\begin{bmatrix} \alpha & c & \beta \\ d & & d \\ \delta & c & \gamma \end{bmatrix} \mapsto (p(\alpha, \beta, \gamma), \delta),$$

is a pullback. Since p_0 is a split epimorphism we may again use Lemma 1.4 to show that also the right hand side square above is a pullback. It follows that $[\pi]$ is an isomorphism if and only if $[p_0]$ is an isomorphism, so that the internal pregroupoid (X, d, c) is central if and only if $\mathbf{D}$ is a double central extension. ■

3. The third cohomology group

In this section we translate the description of the *second order direction functor* and its associated cohomology groups, developed in [23] for Barr exact categories, to the context of semi-abelian categories. A similar translation was made in [23] for *Moore categories* (i.e., strongly protomodular semi-abelian categories) where the connection with n-fold crossed extensions is explored. Note that what we call the *third* cohomology group here is actually the *second* cohomology group in [23]; the dimension shift is there for historical reasons, in order to comply with the "non-abelian" numbering used in classical cohomology of groups. From now on, $\mathcal{A}$ will denote a semi-abelian category and Z a fixed object of $\mathcal{A}$.

An aspherical (abelian) groupoid in $\mathcal{A} \downarrow Z$ consists of a commutative diagram

$$\begin{array}{ccc} X & \begin{array}{c} \xrightarrow{\;d\;} \\ \xleftarrow{\;i\;} \\ \xrightarrow[\;c\;]{} \end{array} & Y \\ & {\scriptstyle f\circ d = f\circ c} \searrow \quad \swarrow {\scriptstyle f} & \\ & Z & \end{array} \qquad (\mathbf{G})$$

such that the top line is a groupoid in $\mathcal{A}$, and both the morphisms f and $(d, c)\colon X \to R[f]$ are regular epimorphisms. Such an internal groupoid has an underlying double extension

$$\begin{array}{ccc} X & \xrightarrow{\;c\;} & Y \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\;f\;]{} & Z. \end{array} \qquad (\mathbf{H})$$

We denote by $\mathsf{Asph}(\mathcal{A} \downarrow Z)$ the category of aspherical groupoids in $\mathcal{A} \downarrow Z$.

The category $\mathsf{Mod}_Z\mathcal{A}$ of Z**-modules** is the category $\mathsf{Ab}(\mathcal{A}\downarrow Z)$ of abelian groups in $\mathcal{A}\downarrow Z$. So, a Z-module gives us a split exact sequence

$$0 \longrightarrow A \xrightarrow{\ker p} P \underset{s}{\overset{p}{\rightleftarrows}} Z \longrightarrow 0$$

where A is an abelian object and p is a split epimorphism (equipped with an additional structure making it an abelian group in $\mathcal{A}\downarrow Z$). Using the equivalence between split epimorphisms and internal actions [7], we can replace P with a semi-direct product $Z \ltimes (A,\xi)$. For simplicity, we denote a Z-module just by its induced Z-algebra (A,ξ).

In the context of semi-abelian categories, the direction functor from [23, Definition 3.7] determines a functor $\mathsf{d}_Z\colon \mathsf{Asph}(\mathcal{A}\downarrow Z)\to \mathsf{Mod}_Z\mathcal{A}$ mapping an aspherical internal groupoid $\mathbf{G}$ to the Z-module $\mathsf{d}_Z(\mathbf{G}) = (A,\xi)$ defined by the downward pullback/upward pushout

$$\begin{array}{ccc} R[(d,c)] & \longrightarrow & Z\ltimes(A,\xi) \\ {\scriptstyle (1_X,1_X)}\uparrow\downarrow{\scriptstyle p_0} & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle p} \\ X & \xrightarrow[f\circ d]{} & Z. \end{array} \tag{I}$$

More precisely, the pair $(p,s)\colon Z\ltimes(A,\xi)\rightleftarrows Z$ arises as a pushout of $(1_X,1_X)$ along $f\circ d$ but, using the properties of $\mathbf{G}$, one may show that the square of downward arrows in $\mathbf{I}$ is a pullback [4]. Thus we see that $A = K[p] = K[p_0] = K[(d,c)] = K[d]\cap K[c]$.

Remark 3.1. Suppose $(\mathcal{C},\otimes,E)$ is a symmetric monoidal category such that the following property holds:

$$\forall C\in\mathcal{C}, \exists \overline{C}\in\mathcal{C}\colon C\otimes\overline{C}\sim E, \tag{J}$$

where $\sim$ means "is connected to (by a zigzag)". Then it is easy to check that the monoidal structure of $\mathcal{C}$ induces an abelian group structure on the set $\pi_0\mathcal{C}$ of its connected components (equivalence classes wits respect to $\sim$). The addition is defined by $[C_1]+[C_2] = [C_1\otimes C_2]$, the unit is $[E]$ and $-[C] = [\overline{C}]$.

It is shown in [4] that the fibres of d_Z are symmetric monoidal categories with property $\mathbf{J}$. The tensor product is called the **Baer sum** since it gives the Baer sum of (2-fold) extensions in the classical examples. So, for any Z-module (A,ξ), $\pi_0\mathsf{d}_Z^{-1}(A,\xi)$ is an abelian group.

Definition 3.2. [23] Let (A,ξ) be a Z-module. The **third cohomology group** $H^3(Z,(A,\xi))$ **of** Z **with coefficients in** (A,ξ) is the abelian group

$\pi_0 \mathsf{d}_Z^{-1}(A,\xi)$ of equivalence classes of aspherical internal groupoids in $\mathcal{A}\downarrow Z$ with direction (A,ξ). This defines an additive functor

$$H^3(Z,-)\colon \mathsf{Mod}_Z\mathcal{A} \to \mathsf{Ab}.$$

We are especially interested in the case of trivial Z-modules (A,τ), i.e., abelian objects A with the trivial Z-action τ. In this situation we write $H^3(Z,A)$ for $H^3(Z,(A,\tau))$. The functor $H^3(Z,-)$ restricts to an additive functor $\mathsf{Ab}\mathcal{A} \to \mathsf{Ab}$.

Proposition 3.3. *The direction of an aspherical groupoid* $\mathbf{G}$ *in* $\mathcal{A}\downarrow Z$ *is a trivial* Z*-module* (A,τ) *in* $\mathcal{A}$ *if and only if* $\mathbf{G}$ *is a central groupoid.*

Proof: Let us first suppose that $\mathsf{d}_Z(\mathbf{G}) = (A,\tau)$. Then, $\mathsf{d}_Z(\mathbf{G})$, defined by $(p,s)\colon Z\ltimes(A,\tau) \rightleftarrows Z$ in diagram **I**, is the product projection with its canonical inclusion $(\mathrm{pr}_Z,(1_Z,0))\colon Z\times A \rightleftarrows Z$. It follows that the pullback $(p_0,(1_X,1_X))\colon R[(d,c)] \rightleftarrows X$ is also a product projection with its canonical inclusion, namely $(\mathrm{pr}_X,(1_X,0))\colon X\times A \rightleftarrows X$. In particular, the splitting $(1_X,1_X)$ is a normal monomorphism in $\mathcal{A}$, which by Theorem 5.2 in [6] (see also Corollary 6.1.8 in [1]) means that $R[(d,c)] = R[d]\cap R[c]$ is central. Hence $[R[d]\cap R[c],\nabla_X] = \Delta_X$ and the groupoid is central.

Conversely, suppose that $\mathbf{G}$ is a central groupoid in $\mathcal{A}\downarrow Z$. By the same arguments as above we see that $(p_0,(1_X,1_X))$ and hence (p,s) are product projections with their canonical inclusions. It follows that A has a trivial Z-action τ. ■

Corollary 3.4. *Let* $\mathbf{G}$ *be an aspherical groupoid in* $\mathcal{A}\downarrow Z$ *and let* $\mathbf{H}$ *be the corresponding double extension. Then* $\mathbf{H}$ *is a double central extension if and only if* $\mathsf{d}_Z(\mathbf{G})$ *is a trivial* Z*-module* (A,τ) *in* $\mathcal{A}$. ■

Thus we see that the direction of a central internal groupoid $\mathbf{G}$ is just the intersection $A = K[d]\cap K[c]$ of the kernels of d and c; indeed, this object A is always abelian as the kernel of the central extension (d,c). In view of this fact we can extend the concept of direction to double central extensions.

4. The group of equivalence classes of double central extensions

Definition 4.1. The **direction** of a double central extension $\mathbf{D}$ is the abelian object $K[d]\cap K[c]$. This defines a functor

$$\mathsf{D}_Z\colon \mathsf{CExt}^2_Z\mathcal{A} \to \mathsf{Ab}\mathcal{A},$$

where $\mathsf{CExt}^2_Z\mathcal{A}$ denotes the category of double central extensions of the object Z of $\mathcal{A}$.

The fibre $\mathsf{D}_Z^{-1}A$ of this functor over an abelian object A is the category of **double central extensions of** Z **by** A. Two double central extensions of Z by A which are connected by a zigzag in $\mathsf{D}_Z^{-1}A$ are called **equivalent**. The equivalence classes form the set $\mathsf{Centr}^2(Z,A) = \pi_0\mathsf{D}_Z^{-1}A$ of connected components of this category.

Remark 4.2. Depending on the context it might not be clear whether $\mathsf{Centr}^2(Z,A)$ is indeed a set (rather than a proper class) but in any case Theorem 5.3 implies that $\mathsf{Centr}^2(Z,A)$ is only as large as is $H^3(Z,A)$.

Remark 4.3. The double central extension $\mathbf{D}$ induces a 3×3 diagram

$$\begin{array}{ccccc} A & \longrightarrow & K[d] & \longrightarrow & K[g] \\ \downarrow & & \downarrow & & \downarrow \\ K[c] & \longrightarrow & X & \xrightarrow{c} & C \\ \downarrow & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle g} \\ K[f] & \longrightarrow & D & \xrightarrow[f]{} & Z \end{array}$$

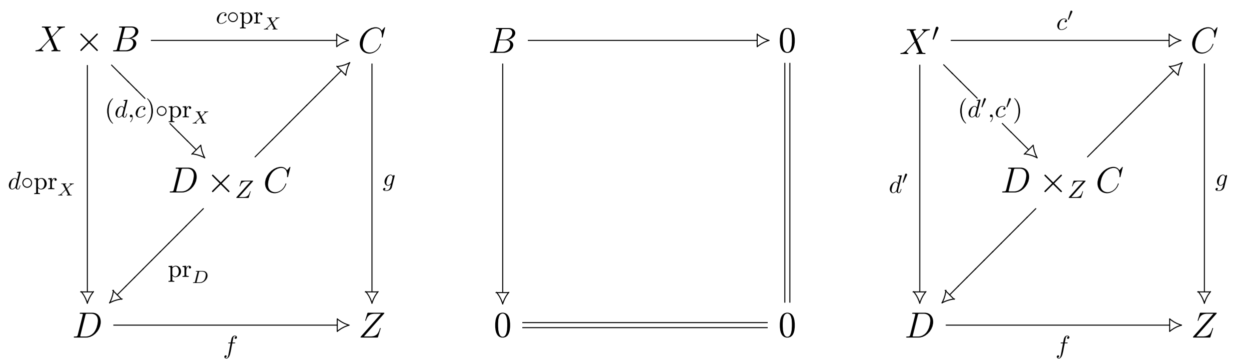
and the object A in this diagram is the direction of $\mathbf{D}$.

We now show that $\mathsf{Centr}^2(Z,A)$ carries a canonical abelian group structure.

Proposition 4.4. *Let $\mathcal{A}$ be a semi-abelian category and let Z be an object of $\mathcal{A}$. Mapping an abelian object A of $\mathcal{A}$ to the set $\mathsf{Centr}^2(Z,A)$ of equivalence classes of double central extensions of Z by A gives a finite product-preserving functor $\mathsf{Centr}^2(Z,-)\colon \mathsf{Ab}\mathcal{A} \to \mathsf{Set}$.*

Proof: Let $a\colon A \to B$ be a morphism of abelian objects in $\mathcal{A}$ and $\mathbf{D}$ a double central extension of Z by A. Then $(d,c)\colon X \to D\times_Z C$ is a central extension of $D\times_Z C$ by A, and the construction of Proposition 1.5 yields a central extension (d',c') of $D\times_Z C$ by B. The morphism $\mathsf{Centr}^2(Z,a)$ now maps the equivalence class of $\mathbf{D}$ to the class of the right hand side square below.

Indeed, since the left hand side square



—which arises from the regular epimorphism in the top sequence in **C**—is a double central extension as the product of **D** with the middle double central extension, so is its right hand side quotient. The functoriality of $\mathsf{Centr}^2(Z, -)$ now follows from the functoriality of $\mathsf{Centr}^1(Z, -)$.

It is clear that $\mathsf{Centr}^2(Z, -)$ preserves the terminal object: any double central extension with direction 0 is connected to

$$\begin{array}{ccc} Z & = & Z \\ \| & & \| \\ Z & = & Z. \end{array}$$

To show that $\mathsf{Centr}^2(Z, -)$ also preserves binary products, we must provide an inverse to the map

$$(\mathsf{Centr}^2(Z, \mathrm{pr}_A), \mathsf{Centr}^2(Z, \mathrm{pr}_B))\colon \mathsf{Centr}^2(Z, A \times B) \to \mathsf{Centr}^2(Z, A) \times \mathsf{Centr}^2(Z, B).$$

This inverse is given by the product in the category $\mathsf{CExt}^2_Z\mathcal{A}$ of double central extensions of Z. Let indeed the two squares

$$\begin{array}{ccc} X & \xrightarrow{c} & C \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[f]{} & Z \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X' & \xrightarrow{c'} & C' \\ {\scriptstyle d'}\downarrow & & \downarrow{\scriptstyle g'} \\ D' & \xrightarrow[f']{} & Z \end{array}$$

be double central extensions of Z by A and B, respectively. Then their product in $\mathsf{CExt}^2_Z\mathcal{A}$ is the square

$$\begin{array}{ccc} X \times_Z X' & \xrightarrow{c\times_Z c'} & C \times_Z C' \\ {\scriptstyle d\times_Z d'}\downarrow & & \downarrow{\scriptstyle g\circ \mathrm{pr}_C} \\ D \times_Z D' & \xrightarrow[f\circ \mathrm{pr}_D]{} & Z. \end{array}$$

In fact, this square represents a pregroupoid in $\mathcal{A} \downarrow Z$ as a product of two such pregroupoids, and the comparison map $(d \times_Z d', c \times_Z c')$ to the pullback is a central extension as a pullback of the central extension $(d, c) \times (d', c')$. Finally, the direction of this double central extension is the kernel of $(d \times_Z d', c \times_Z c')$, which is nothing but $A \times B$. ■

Corollary 4.5. *The functor* $\mathsf{Centr}^2(Z, -)$ *uniquely factors over the forgetful functor* $\mathsf{Ab} \to \mathsf{Set}$ *to yield a functor* $\mathsf{Centr}^2(Z, -)\colon \mathsf{Ab}\mathcal{A} \to \mathsf{Ab}$.

Proof: Any abelian object of $\mathcal{A}$ carries a canonical internal abelian group structure; we just showed that the functor $\mathsf{Centr}^2(Z, -)$ preserves such structures. See also Remark 5.5. ■

5. $H^3(Z, A)$ and $\mathsf{Centr}^2(Z, A)$ are isomorphic

Proposition 5.1. *Let* $\mathcal{A}$ *be a finitely complete category. The forgetful embedding* $\mathsf{Gd}\mathcal{A} \hookrightarrow \mathsf{PreGd}\mathcal{A}$ *has a right adjoint* $\mathsf{gd}\colon \mathsf{PreGd}\mathcal{A} \to \mathsf{Gd}\mathcal{A}$. *Moreover, when* $\mathcal{A}$ *is semi-abelian,* Z *is an object of* $\mathcal{A}$ *and* A *is an abelian object of* $\mathcal{A}$, *this adjunction restricts to the fibres of the direction functors* d_Z *and* D_Z

$$\mathsf{d}_Z^{-1}(A, \tau) \underset{\mathsf{gd}}{\overset{\subset}{\underset{\longleftarrow}{\overset{\longrightarrow}{\scriptstyle\perp}}}} \mathsf{D}_Z^{-1}A. \qquad (\mathbf{K})$$

Proof: Given an internal pregroupoid (X, d, c), the associated internal groupoid $\mathsf{gd}(X, d, c)$ has as underlying reflexive graph

$$R[c] \times_X R[d] \underset{\mathrm{cod}}{\overset{\mathrm{dom}}{\rightrightarrows}} X, \quad (\text{with } \mathrm{id}\colon X \to R[c] \times_X R[d])$$

where dom and cod are the first and third projections and id is the diagonal. This reflexive graph is a groupoid: the composition maps a pair $(\alpha R[c]\beta R[d]\gamma, \gamma R[c]\delta R[d]\epsilon)$ to the triple $(\alpha, p(\delta, \gamma, \beta), \epsilon)$, where p is the pregroupoid structure of (X, d, c). The (X, d, c)-component of the counit of the adjunction is defined by the map

$$(p,d,c)\colon (R[c]\times_X R[d],\mathrm{dom},\mathrm{cod})\to (X,d,c)$$

in $\mathsf{PreGd}\mathcal{A}$; and given an internal groupoid

$$v \circlearrowright X \underset{c}{\overset{d}{\rightleftarrows}} Y \quad (\text{with } i\colon Y\to X)$$

with inversion map v, the associated unit component is

$$\begin{array}{ccc} X & \underset{c}{\overset{d}{\rightleftarrows}}\ (i) & Y, \\ {\scriptstyle (i\circ d, v, i\circ c)}\downarrow & & \downarrow{\scriptstyle i} \\ R[c]\times_X R[d] & \underset{\mathrm{cod}}{\overset{\mathrm{dom}}{\rightleftarrows}}\ (\mathrm{id}) & X : \end{array} \qquad (\mathbf{L})$$

one easily checks that the triangular identities hold.

Corollary 3.4 implies that the embedding $\mathsf{Gd}\mathcal{A}\hookrightarrow \mathsf{PreGd}\mathcal{A}$ restricts to the fibres of d_Z and D_Z. Now suppose that $\mathbf{D}\in \mathsf{D}_Z^{-1}A$; then (X,d,c) is a central pregroupoid in $\mathcal{A}\downarrow Z$ by Theorem 2.7, and $A=K[(d,c)]$. Using that the square

$$\begin{array}{ccc} R[c]\times_X R[d] & \xrightarrow{p} & X \\ {\scriptstyle (\mathrm{dom},\mathrm{cod})}\downarrow & & \downarrow{\scriptstyle (d,c)} \\ X\times_Z X & \xrightarrow[d\times_Z c]{} & D\times_Z C \end{array} \qquad (\mathbf{M})$$

is a pullback, we see that $(\mathrm{dom},\mathrm{cod})$ is a central extension and that $A=K[(\mathrm{dom},\mathrm{cod})]$. Hence the groupoid $\mathsf{gd}(X,d,c)$ in $\mathcal{A}\downarrow Z$ is central, which by Proposition 3.3 means that it has direction (A,τ), that is, it is in the fibre $\mathsf{d}_Z^{-1}(A,\tau)$—so the functor gd also restricts to the fibres of the direction functors d_Z and D_Z.

To see that these restrictions are still adjoint to each other, it suffices to prove that the components of the unit and the counit are in the fibre of $1_{(A,\tau)}$ (respectively 1_A). This is the case, because both the square $\mathbf{M}$ and the similar square corresponding to $\mathbf{L}$ are pullbacks. ∎

Remark 5.2. Consider an adjunction

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}.$$

(1) The functors F and G induce functions $\varphi\colon \pi_0\mathcal{C} \to \pi_0\mathcal{D}$, defined by $\varphi[C] = [FC]$, and $\gamma\colon \pi_0\mathcal{D} \to \pi_0\mathcal{C}$, defined by $\gamma[D] = [GD]$, respectively.
(2) F being left adjoint to G implies that $\varphi^{-1} = \gamma$, i.e., $\pi_0\mathcal{C} \cong \pi_0\mathcal{D}$. In fact, $(\varphi \circ \gamma)[D] = [FGD] = [D]$, for any object D of $\mathcal{D}$, since FGD is connected to D by the D-component of the counit of the adjunction; thus $\varphi \circ \gamma = 1_{\pi_0\mathcal{D}}$. Similarly $\gamma \circ \varphi = 1_{\pi_0\mathcal{C}}$, using the unit of the adjunction instead.

Now suppose that the category $\mathcal{C}$ carries a symmetric monoidal structure $(\mathcal{C}, \otimes, E)$ as in Remark 3.1.

(3) $\pi_0\mathcal{C}$ is an abelian group.
(4) $\pi_0\mathcal{D}$ is an abelian group with addition given by

$$[D_1] + [D_2] = [F(\, GD_1 \otimes GD_2 \,)],$$

unit $[FE]$ and $-[D] = [F(\, \overline{GD} \,)]$.
(5) The function φ is a group isomorphism with inverse γ.

Theorem 5.3. *In any semi-abelian category, the third cohomology group $H^3(Z, A)$ of an object Z with coefficients in an abelian object A is isomorphic to the group $\mathsf{Centr}^2(Z, A)$ of equivalence classes of double central extensions of Z by A.*

Proof: By the unicity in Corollary 4.5, to show that the functors $H^3(Z, -)$ and $\mathsf{Centr}^2(Z, -)$ are isomorphic as functors $\mathsf{Ab}\mathcal{A} \to \mathsf{Ab}$, it suffices to give a bijection between the underlying sets $H^3(Z, A)$ and $\mathsf{Centr}^2(Z, A)$, natural in A. Through Remark 5.2, the adjunction $\mathbf{K}$ from Proposition 5.1 induces the needed isomorphisms

$$\varphi\colon H^3(Z, A) \to \mathsf{Centr}^2(Z, A)$$

and $\gamma\colon \mathsf{Centr}^2(Z, A) \to H^3(Z, A)$. ■

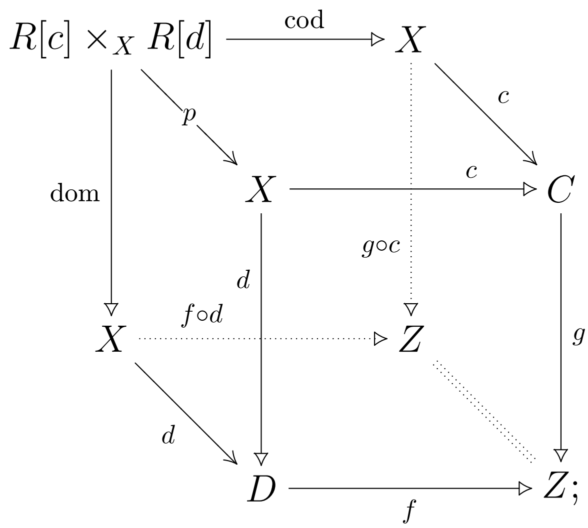
Remark 5.4. We have $\varphi\colon H^3(Z, A) \to \mathsf{Centr}^2(Z, A)\colon\ [\mathbf{G}] \mapsto [\mathbf{H}]$ and

$$\gamma\colon \mathsf{Centr}^2(Z, A) \to H^3(Z, A)\colon\ [\mathbf{D}] \mapsto [\mathsf{gd}(\mathbf{D})]\,,$$

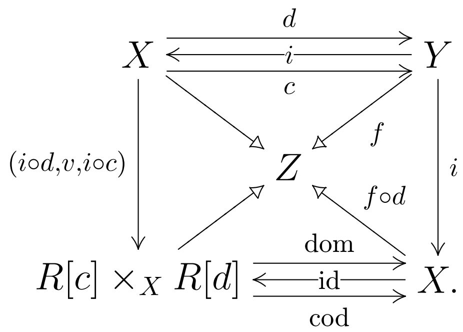
where

$$\mathsf{gd}(\mathbf{D}) = \begin{array}{c} R[c] \times_X R[d] \underset{\text{cod}}{\overset{\text{dom}}{\underset{\longleftarrow\text{id}\longrightarrow}{\rightrightarrows}}} X \\ \searrow \qquad \swarrow{\scriptstyle f\circ d = g\circ c} \\ Z \end{array}$$

such that $\varphi \circ \gamma = 1_{\mathsf{Centr}^2(Z,A)}$, because for any double central extension $\mathbf{D}$ of Z by A, $(\varphi \circ \gamma)[\mathbf{D}]$ is equal to $[\mathbf{D}]$ through (p, d, c), the $\mathbf{D}$-component of the counit of the adjunction $\mathbf{K}$



and $\gamma \circ \varphi = 1_{H^3(Z,A)}$, since for any central internal groupoid $\mathbf{G}$, with inversion map v and direction (A, τ), $(\gamma \circ \varphi)[\mathbf{G}]$ is equal to $[\mathbf{G}]$ through $((i \circ d, v, i \circ c), i)$, the $\mathbf{G}$-component of the unit of the adjunction $\mathbf{K}$



Remark 5.5. We know that $\mathsf{d}_Z^{-1}(A, \tau)$ is a symmetric monoidal category with property $\mathbf{J}$ by Remark 3.1. The arguments in Remark 5.2 show how the addition on $H^3(Z, A)$ is transported to an abelian group structure on

$\mathsf{Centr}^2(Z, A)$ as described in Remark 5.2, (4). This makes the connection between the canonical abelian group structure from Proposition 4.4 and Corollary 4.5 and the Baer sum on $\mathsf{d}_Z^{-1}(A, \tau)$ explicit.

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