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Graded Lie-Rinehart algebras ☆

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ABSTRACT

The present work introduces the class of graded Lie-Rinehart algebras as a natural generalization of graded Lie algebras. It is demonstrated that a tight G-graded Lie-Rinehart algebra L over a commutative and associative G-graded algebra A, where G is an abelian group, can be decomposed into the orthogonal direct sums $L = \bigoplus_{i \in I} I_i$ and $A = \bigoplus_{j \in J} A_j$, where each I_i and A_j is a non-zero ideal of L and A, respectively. Additionally, both decompositions satisfy that for any $i \in I$, there exists a unique $j \in J$ such that $A_j I_i \neq 0$ and that any I_i is a graded Lie-Rinehart algebra over A_j. In the case of maximal length, the aforementioned decompositions of L and A are through indecomposable (graded) ideals, and the (graded) simplicity of any I_i and any A_j are also characterized.

1. Introduction and first definitions

On one hand, Lie-Rinehart algebras were introduced by Herz in [17] and subsequently their theory was developed along the papers [29,30]. It is noteworthy that this concept can be regarded as a Lie $\mathbb{F}$-algebra which is simultaneously an A-module, where A is an associative and commutative $\mathbb{F}$-algebra, and both structures are well-related. Let us note that one can find also the notion of Lie-Rinehart algebra in Jacobson's work when studying certain field extensions but also this notion can be found, under different names, in differential geometry and differential Galois theory, see for instance [18]. In particular, along Huebschmann's work Lie-Rinehart algebras were considered to be the algebraic counterpart of Lie algebroids defined over smooth manifolds and this work has been developed through a series of articles (see [20–22]).

Along the last years, Lie-Rinehart algebras have been considered, in general, in many areas of Mathematics, from a geometric viewpoint (see for instance [26]) and of course from an algebraic viewpoint [12,13,24]. In particular, many authors

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have studied thoroughly Lie-Rinehart structures in connection with symplectic geometry, Poisson structures, Lie groupoids and algebroids and other types of quantizations [17,25–27,32,33]. Throughout the last years have been studied also some generalizations of these algebras such as restricted Lie-Rinehart algebras [14] or 3-Lie-Rinehart superalgebras [7]. Consequently, one can assert that Lie-Rinehart algebras constitutes a fundamental tool in many topics of research in mathematics.

On the other hand, gradings appear elsewhere in the theory of Lie algebras, for example in the Cartan decomposition of a finite-dimensional complex semisimple Lie algebra (see for instance [1,6,10,15,16,19,23]). Also, graded modules have attracted the attention of many researchers in the last years (see [2,4,5,8,28,31]). The concept of graded Lie-Rinehart algebra will allow us to study, under a unique structure, a graded Lie algebra which is also a module over a graded associative algebra.

In the current paper, for G an abelian grading group, we introduce the class of graded Lie-Rinehart algebras (L, A) being A an associative G-graded algebra. We study the structure of the aforementioned algebras which can be considered to be a natural extension of graded Lie algebras. Let us note that our techniques consist mainly of introducing notions of connections over the elements of G. Additionally, for the maximal length case, we show that the decompositions of L and A obtained are by indecomposable (graded) ideals. Likewise, the (graded) simplicity of such ideals is also characterized.

Our paper is organized as follows. Firstly, in Section 2 we develop connection techniques within the context of Lie-Rinehart algebras (L, A) and after, we apply all of these techniques to the study of the inner structure of L. Secondly, along Section 3 we get a decomposition of A as direct sum of adequate ideals. In Section 4, we relate the results obtained in Section 2 and 3 on L and A to prove our above mentioned main results. The final section is centered in the case in which all of the non-zero homogeneous spaces are one-dimensional, that is, we are dealing with graded Lie-Rinehart algebras of maximal length. For this class of algebras we prove that the given decompositions of L and A are by means of the family of their indecomposable (graded) ideals and the simplicity is characterizated.

We begin by recalling the definition of Lie-Rinehart algebra. First, look at the next observation.

Remark 1.1. Let $\mathbb{F}$ be an arbitrary base field and A a commutative and associative $\mathbb{F}$-algebra. A *derivation* on A is a $\mathbb{F}$-linear map $D : A \to A$ which satisfies

$$D(ab) = D(a)b + aD(b) \quad \textit{(Leibniz's law)} \tag{1}$$

for all $a, b \in A$. The set $\mathrm{Der}(A)$ of all derivations of A is a Lie $\mathbb{F}$-algebra with Lie bracket $[D, D'] = DD' - D'D$ and an A-module simultaneously. These two structures are related by the following identity

$$[D, aD'] = a[D, D'] + D(a)D', \text{ for all } a \in A \text{ and } D, D' \in \mathrm{Der}(A).$$

Definition 1.2. A *Lie-Rinehart algebra* over an associative and commutative $\mathbb{F}$-algebra A (whose product is denoted by juxtaposition) is a Lie $\mathbb{F}$-algebra L (with product $[\cdot, \cdot]$) endowed with an A-module structure and with a map (called *anchor*)

$$\rho : L \to \mathrm{Der}(A),$$

which is simultaneously an A-module and a Lie algebra homomorphism, and such that the following relation holds

$$[v, aw] = a[v, w] + \rho(v)(a)w, \tag{2}$$

for any $v, w \in L$ and $a \in A$. We denote it by (L, A) or just by L if there is no possible confusion.

Example 1.3. Any Lie algebra L is a Lie-Rinehart algebra over $A := \mathbb{F}$ as a consequence of $\mathrm{Der}(\mathbb{F}) = 0$.

Example 1.4. Any associative and commutative $\mathbb{F}$-algebra A gives rise to a Lie-Rinehart algebra by taking $L := \mathrm{Der}(A)$ and $\rho := Id_{\mathrm{Der}(A)}$.

A *subalgebra* (S, A) of (L, A), S for short, is a Lie subalgebra of L such that $AS \subset S$ and satisfying that S acts on A via the composition

$$S \hookrightarrow L \xrightarrow{\rho} \mathrm{Der}(A).$$

A subalgebra (I, A) of (L, A), I for short, is called an *ideal* if I is a Lie ideal of L and satisfies

$$\rho(I)(A)L \subset I. \tag{3}$$

Next, we present the class of graded algebras within the context of Lie-Rinehart algebras starting by recalling the definition of a graded algebra.

Definition 1.5. Let G be an abelian multiplicative group with neutral element 1. An algebra A over an arbitrary base field $\mathbb{F}$ is *G-graded* or just *graded* if $A = \oplus_{g\in G} A_g$ satisfying $A_g A_{g'} \subset A_{gg'}$, for $g, g' \in G$, where we are denoting by juxtaposition the respective products on A and G.

To introduce the concept of graded Lie-Rinehart algebra look at the Remark 1.1 and consider a graded associative algebra $A = \bigoplus_{g\in G} A_g$. Then the Lie algebra $\mathrm{Der}(A)$ is naturally graded as

$$\mathrm{Der}(A) = \bigoplus_{h\in G} (\mathrm{Der}(A))_h$$

where $(\mathrm{Der}(A))_h := \{D \in \mathrm{Der}(A) : D(A_g) \subset A_{gh} \text{ for all } g \in G\}$. Then Remark 1.1 and Example 1.4 give us that $L := \mathrm{Der}(A) = \oplus_{g\in G}(\mathrm{Der}(A))_g$ is a Lie-Rinehart algebra over $A = \oplus_{g\in G} A_g$ satisfying $A_h L_g \subset L_{hg}$ and $\rho(L_g)(A_h) \subset A_{gh}$.

Definition 1.6. Let G be an abelian grading group whose product is denoted by juxtaposition. We say that (L, A) is a *G-graded Lie-Rinehart algebra*, or just a *graded Lie-Rinehart algebra*, if L is a G-graded Lie $\mathbb{F}$-algebra and A is a G-graded (associative and commutative) $\mathbb{F}$-algebra satisfying

$$A_h L_g \subset L_{hg}, \tag{4}$$

$$\rho(L_g)(A_h) \subset A_{gh}, \tag{5}$$

for any $g, h \in G$.

Note that split Lie algebras, graded Lie algebras and split Lie-Rinehart algebras are examples of graded Lie-Rinehart algebras. Therefore, the present paper generalizes the results obtained in [3,9,10].

Example 1.7. Consider the $\mathbb{F}$-algebra of dual numbers

$$A := \mathbb{F}[\xi] = \mathbb{F}/\langle \xi^2 \rangle = \{k_1 + k_2\xi : k_1, k_2 \in \mathbb{F}, \xi^2 = 0\}.$$

This is a $\mathbb{Z}_2$-graded commutative and associative algebra. Indeed, $A = A_{\bar{0}} \oplus A_{\bar{1}}$ with $A_{\bar{0}} = \mathbb{F}1$, $A_{\bar{1}} = \mathbb{F}\xi$. We can endow to A with the Lie algebra structure L given by the bracket

$$[k_1 + k_2\xi, k_1' + k_2'\xi] := (k_1 k_2' - k_2 k_1')\xi$$

for $k_1 + k_2\xi, k_1' + k_2'\xi \in A$, which is also $\mathbb{Z}_2$-graded as $L = L_{\bar{0}} \oplus L_{\bar{1}}$ where $L_{\bar{0}} = \mathbb{F}1$, $L_{\bar{1}} = \mathbb{F}\xi$. Then L is a Lie-Rinehart algebra over A with anchor map $\rho : L \to \mathrm{Der}(A)$ given by $\rho(k_1 + k_2\xi) := \mathrm{ad}_{k_1}$ being $\mathrm{ad}_{k_1}(k_1' + k_2'\xi) := [k_1, k_1' + k_2'\xi]$ for $k_1 + k_2\xi \in L, k_1' + k_2'\xi \in A$.

We denote by Σ_G and Λ_G the corresponding *G-supports* of the grading in L and A respectively, being

$$\Sigma_G := \{g \in G \setminus \{1\} : L_g \neq 0\} \qquad \Lambda_G := \{h \in G \setminus \{1\} : A_h \neq 0\}$$

Thus, they can be expressed as follows

$$L = L_1 \oplus \Big(\bigoplus_{g\in\Sigma_G} L_g\Big) \qquad A = A_1 \oplus \Big(\bigoplus_{h\in\Lambda_G} A_h\Big).$$

Along this paper (L, A) is a G-graded Lie-Rinehart algebra with restrictions neither on the dimension of L, A nor the abelian group G, nor on the base field $\mathbb{F}$.

2. Connections in Σ_G. Decompositions

Let G be an abelian grading group and (L, A) a G-graded Lie-Rinehart algebra. We define $\Sigma_G^{-1} := \{g^{-1} : g \in \Sigma_G\}$. Likewise, we define $\Lambda_G^{-1} := \{g^{-1} : g \in \Lambda_G\}$. Furthermore, the above definitions allow us denote

$$\Sigma_G^{\pm} := \Sigma_G \cup \Sigma_G^{-1} \quad \text{and} \quad \Lambda_G^{\pm} := \Lambda_G \cup \Lambda_G^{-1}.$$

Definition 2.1. Let $g, g' \in \Sigma_G$. g is said to be *Σ_G-connected* to g' if there exists $\{g_1, g_2, \ldots, g_n\} \subset \Sigma_G^{\pm} \cup \Lambda_G^{\pm}$ such that

i. $g_1 = g$.
ii. $\{g_1, g_1 g_2, \ldots, g_1 g_2 \cdots g_{n-1}\} \subset \Sigma_G^{\pm}$.
iii. $g_1 g_2 \cdots g_n \in \{g', (g')^{-1}\}$.

It is said that $\{g_1,\dots,g_n\}$ is a Σ_G-*connection* from g to g'.

The next result shows that the Σ_G-connection relation is of equivalence. Although its proof is quite similar to that of [10, Proposition 2.1], we add an outline of it.

Proposition 2.2. *The relation $\sim$ defined in Σ_G by $g\sim g'$ if and only if g is Σ_G-connected to g, is of equivalence.*

Proof. $\{g\}$ is a Σ_G-connection from g to itself and therefore $g\sim g$.

If $g\sim g'$ and $\{g_1,\dots,g_n\}$ is a Σ_G-connection from g to g', then

$$\{g_1\cdots g_n, g_n^{-1}, g_{n-1}^{-1},\dots,g_2^{-1}\}\subset \Sigma_G^{\pm}\cup\Lambda_G^{\pm}$$

is a Σ_G-connection from g' to g in the case of $g_1\cdots g_n=g'$, and

$$\{g_1^{-1}\cdots g_n^{-1}, g_n, g_{n-1},\dots,g_2\}\subset \Sigma_G^{\pm}\cup\Lambda_G^{\pm}$$

in the case of $g_1\cdots g_n=(g')^{-1}$. Therefore $g'\sim g$.

Finally, suppose $g\sim g'$ and $g'\sim g''$, and set $\{g_1,\dots,g_n\}$ for a Σ_G-connection from g to g' and $\{g'_1,\dots,g'_m\}$ for a Σ_G-connection from g' to g''. If $m>1$, then $\{g_1,\dots,g_n,g'_2,\dots,g'_m\}$ is a Σ_G-connection from g to g'' in the case of $g_1\cdots g_n=g'$, and $\{g_1,\dots,g_n,{g'}_2^{-1},\dots,{g'}_m^{-1}\}$ in the case of $g_1\cdots g_n=(g')^{-1}$. If $m=1$, then $g''\in\{g',(g')^{-1}\}$ and so $\{g_1,\dots,g_n\}$ is a Σ_G-connection from g to g''. Therefore $g\sim g''$ and $\sim$ is of equivalence. □

On account of Proposition 2.2 the Σ_G-connection relation defined in Σ_G is of equivalence, hence we can consider the quotient set

$$\Sigma_G/\sim:=\{[g]:g\in\Sigma_G\},$$

becoming $[g]$ the set of elements in Σ_G which are Σ_G-connected to g. Thus, our purpose now is to associate an (adequate) ideal $I_{[g]}$ of the Lie-Rinehart algebra (L,A) to each $[g]$.

Lemma 2.3. *If $g'\in[g]$ and $g'', g'g''\in\Sigma_G$, then $g'', g'g''\in[g]$.*

Proof. Analogous to the proof of [10, Lemma 2.1]. □

For $[g]$, with $g\in\Sigma_G$, we define

$$L_{[g],1}:=\Big(\sum_{g'\in[g]\cap\Lambda_G} A_{(g')^{-1}}L_{g'}+\sum_{g'\in[g]}[L_{(g')^{-1}},L_{g'}]\Big)\subset L_1$$

and

$$V_{[g]}:=\bigoplus_{g'\in[g]}L_{g'}.$$

Thus, we can consider the following (graded) subspace of L,

$$I_{[g]}:=L_{[g],1}\oplus V_{[g]}.$$

Proposition 2.4. *For any $[g]\in\Sigma_G/\sim$, the following assertions hold.*

i) $[I_{[g]},I_{[g]}]\subset I_{[g]}$.
ii) $AI_{[g]}\subset I_{[g]}$.

Proof. i) We have

$$\begin{aligned}[I_{[g]},I_{[g]}]&=[L_{[g],1}\oplus V_{[g]},L_{[g],1}\oplus V_{[g]}]\\&\subset[L_{[g],1},L_{[g],1}]+[L_{[g],1},V_{[g]}]+[V_{[g]},L_{[g],1}]+[V_{[g]},V_{[g]}].\end{aligned}\tag{6}$$

Let start by taking into account the second summand in (6). If there exist $g'\in[g]$ such that $A_{(g')^{-1}}L_{g'}$ is non-zero, Equation (4) leads to $A_{(g')^{-1}}L_{g'}\subset L_{(g')^{-1}g'}\subset L_1$. If some $g'\in[g]$ satisfies $[L_{(g')^{-1}},L_{g'}]\neq 0$, we get as in the previous case $[L_{(g')^{-1}},L_{g'}]\subset L_1$. We have $[L_{[g],1},L_{g'}]\subset L_{g'}\subset V_{[g]}$. Analogously, we get $[V_{[g]},L_{[g],1}]\subset V_{[g]}$ and we conclude for the second and third summand in Equation (6) that

$$[L_{[g],1}, V_{[g]}] + [V_{[g]}, L_{[g],1}] \subset V_{[g]}.$$

Let consider now the fourth summand in (6). Given $g', g'' \in [g]$ we get $[L_{g'}, L_{g''}] \subset L_{g'g''}$. If $g'g'' = 1$ we have $[L_{g'}, L_{g''}] \subset L_{[g],1}$. Suppose $g'g'' \in \Sigma_G$, then by Lemma 2.3 we have $[L_{g'}, L_{g''}] \subset L_{g'g''} \subset V_{[g]}$.

Finally, for $g', g'' \in [g]$ the first summand of (6) remains

$$\begin{aligned}
&\Big[\sum_{g'\in[g]\cap\Lambda_G} A_{(g')^{-1}}L_{g'} + \sum_{g'\in[g]}[L_{g'}, L_{(g')^{-1}}], \sum_{g''\in[g]\cap\Lambda_G} A_{(g'')^{-1}}L_{g''} + \sum_{g''\in[g]}[L_{g''}, L_{(g'')^{-1}}]\Big] \subset \\
&\sum_{g',g''\in[g]\cap\Lambda_G}[A_{(g')^{-1}}L_{g'}, A_{(g'')^{-1}}L_{g''}] + \sum_{g'\in[g]\cap\Lambda_G, g''\in[g]}\Big[A_{(g')^{-1}}L_{g'}, [L_{g''}, L_{(g'')^{-1}}]\Big] \\
&+ \sum_{g'\in[g], g''\in[g]\cap\Lambda_G}\Big[[L_{g'}, L_{(g')^{-1}}], A_{(g'')^{-1}}L_{g''}\Big] + \sum_{g',g''\in[g]}\Big[[L_{g'}, L_{(g')^{-1}}], [L_{g''}, L_{(g'')^{-1}}]\Big] \qquad (7)
\end{aligned}$$

For the first summand in (7), if there exist $g', g'' \in [g] \cap \Lambda_G$ such that $[A_{(g')^{-1}}L_{g'}, A_{(g'')^{-1}}L_{g''}] \neq 0$, on account of Equation (2) and Equation (5) we get

$$\begin{aligned}
[L_{(g')^{-1}g'}, A_{(g'')^{-1}}L_{g''}] &\subset A_{(g'')^{-1}}[L_{(g')^{-1}g'}, L_{g''}] + \rho(L_{(g')^{-1}g'})(A_{(g'')^{-1}})L_{g''} \\
&\subset A_{(g'')^{-1}}L_{g''} \subset L_{[g],1}
\end{aligned}$$

If there exist $g' \in [g] \cap \Lambda_G, g'' \in [g]$ such that the second summand of (7) is non-zero, Equation (2) and Equation (5) allow us to assert the following expression

$$\begin{aligned}
\big[A_{(g')^{-1}}L_{g'}, [L_{g''}, L_{(g'')^{-1}}]\big] &= A_{(g')^{-1}}\big[[L_{g''}, L_{(g'')^{-1}}], L_{g'}\big] + \rho([L_{g''}, L_{(g'')^{-1}}])(A_{(g')^{-1}})L_{g'} \\
&\subset A_{(g')^{-1}}L_{g'} \subset L_{[g],1}
\end{aligned}$$

Similarly, the proof of the third summand of (7) can be obtained.

Let consider now the fourth summand in (7), the bilinearity of the product and Jacobi identity lead to

$$\begin{aligned}
&\sum_{g',g''\in[g]}\big[[L_{g'}, L_{(g')^{-1}}], [L_{g''}, L_{(g'')^{-1}}]\big] \subset \\
&\sum_{g',g''\in[g]}\Big(\Big[L_{g'}, \big[L_{(g')^{-1}}, [L_{g''}, L_{(g'')^{-1}}]\big]\Big] + \Big[L_{(g')^{-1}}, \big[L_{g'}, [L_{g''}, L_{(g'')^{-1}}]\big]\Big]\Big) \subset \\
&\sum_{g'\in[g]}\big([L_{g'}, L_{(g')^{-1}}] + [L_{(g')^{-1}}, L_{g'}]\big) \subset \sum_{g'\in[g]}[L_{g'}, L_{(g')^{-1}}] = L_{[g],1}.
\end{aligned}$$

Therefore all summands in (7) are contained in $L_{[g],1}$, and therefore all summands in (6) are included in $I_{[g]}$ as desired.

ii) Note that

$$AI_{[g]} = \Big(A_1 \oplus \big(\bigoplus_{k\in\Lambda_G} A_k\big)\Big)\Big(\Big(\sum_{g'\in[g]\cap\Lambda_G} A_{(g')^{-1}}L_{g'} + \sum_{g'\in[g]}[L_{(g')^{-1}}, L_{g'}]\Big) \oplus \bigoplus_{g'\in[g]} L_{g'}\Big).$$

Next, we divide our study into six cases:

- For $g' \in [g] \cap \Lambda_G$, since L is an A-module we get

$$A_1(A_{(g')^{-1}}L_{g'}) = (A_1A_{(g')^{-1}})L_{g'} \subset A_{(g')^{-1}}L_{g'} \subset L_{[g],1}. \qquad (8)$$

- Suppose $g' \in [g]$, by Equation (2) we get

$$A_1[L_{(g')^{-1}}, L_{g'}] \subset [L_{(g')^{-1}}, A_1L_{g'}] + \rho(L_{(g')^{-1}})(A_1)L_{g'}.$$

Since $A_1L_{g'} \subset L_{g'}$ we get $[L_{(g')^{-1}}, A_1L_{g'}] \subset [L_{(g')^{-1}}, L_{g'}]$. Also, taking into account Equation (5) we obtain $\rho(L_{(g')^{-1}})(A_1) \subset A_{(g')^{-1}}$. If $A_{(g')^{-1}} \neq 0$ (otherwise is trivial), $(g')^{-1} \in \Lambda_G$ therefore $\rho(L_{(g')^{-1}})(A_1)L_{g'} \subset A_{(g')^{-1}}L_{g'}$ with $g' \in [g] \cap \Lambda_G$. Hence,

$$A_1[L_{(g')^{-1}}, L_{g'}] \subset L_{[g],1}. \qquad (9)$$

- Set $g' \in [g]$, from the action of A over L it follows

$$A_1L_{g'} \subset L_{g'} \subset V_{[g]}. \qquad (10)$$

- Consider $k \in \Lambda_G, g' \in [g] \cap \Lambda_G$, using that A is commutative and L is an A-module,

$$A_k(A_{(g')^{-1}}L_{g'}) = (A_kA_{(g')^{-1}})L_{g'} = A_{(g')^{-1}}(A_kL_{g'}) \subset A_{(g')^{-1}}L_{kg'} \subset L_k$$

If $L_k \neq 0$ (otherwise is trivial), we have $k \in \Sigma_G$ and then with the Σ_G-connection $\{g', k, (g')^{-1}\}$ we have $k \in [g]$. That is,

$$L_k \subset V_{[g]}. \tag{11}$$

- For $k \in \Lambda_G, g' \in [g]$ using Equation (2) and Equation (5) we obtain

$$\begin{aligned} A_k[L_{(g')^{-1}}, L_{g'}] &\subset [L_{(g')^{-1}}, A_k L_{g'}] + \rho(L_{(g')^{-1}})(A_k)L_{g'} \\ &\subset [L_{(g')^{-1}}, L_{kg'}] + A_{k(g')^{-1}} L_{g'} \subset L_k \end{aligned}$$

As in the previous case, if $L_k \neq 0$ we get $k \in \Sigma_G$ and $k \in [g]$. That is,

$$A_k[L_{(g')^{-1}}, L_{g'}] \subset V_{[g]}. \tag{12}$$

- For $k \in \Lambda_G, g' \in [g]$ we obtain $A_k L_{g'} \subset L_{kg'}$. If $kg' \in \Sigma_G$, by using the Σ_G-connection $\{g', k\}$ we get $g' \sim kg'$, and by transitivity $kg' \in [g]$, which allows us to assert

$$A_k L_{g'} \subset V_{[g]}. \tag{13}$$

From Equations (8)-(13), assertion ii) is proved. □

Proposition 2.5. *Let $[g], [h] \in \Sigma_G/\sim$ with $[g] \neq [h]$. Then $[I_{[g]}, I_{[h]}] = 0$.*

Proof. We have

$$\begin{aligned} [I_{[g]}, I_{[h]}] &= [L_{[g],1} \oplus V_{[g]}, L_{[h],1} \oplus V_{[h]}] \\ &\subset [L_{[g],1}, L_{[h],1}] + [L_{[g],1} V_{[h]}] + [V_{[g]}, L_{[h],1}] + [V_{[g]}, V_{[h]}]. \end{aligned} \tag{14}$$

Consider the above fourth summand $[V_{[g]}, V_{[h]}]$ and suppose there exist $g' \in [g]$ and $h' \in [h]$ such that $[L_{g'}, L_{h'}] \neq 0$. As $g' \neq (h')^{-1}$ necessarily, then $g'h' \in \Sigma_G$. Since $g \sim g'$ and $g'h' \in \Sigma_G$, by Lemma 2.3 we obtain $g \sim g'h'$. In a similar way we can prove $h \sim g'h'$, which leads to $g \sim h$, obtaining then a contradiction. Hence $[L_{g'}, L_{h'}] = 0$ and so

$$[V_{[g]}, V_{[h]}] = 0. \tag{15}$$

Consider now second summand in (14) and suppose there exist $g' \in [g] \cap \Lambda_G$ and $h' \in [h]$ such that $\big[A_{(g')^{-1}} L_{g'} + [L_{(g')^{-1}}, L_{g'}], L_{h'}\big] \neq 0$. Then, $[A_{(g')^{-1}} L_{g'}, L_{h'}]$ or $\big[[L_{(g')^{-1}}, L_{g'}], L_{h'}\big]$ is non-zero. In the first case, by Equation (2) we have the following

$$\begin{aligned} [A_{(g')^{-1}} L_{g'}, L_{h'}] &= A_{(g')^{-1}}[L_{h'}, L_{g'}] + \rho(L_{h'})(A_{(g')^{-1}})L_{g'} \\ &\subset A_{(g')^{-1}} L_{h'g'} + A_{h'(g')^{-1}} L_{g'}. \end{aligned}$$

If either $L_{h'g'} \neq 0$ or $A_{h'(g')^{-1}} \neq 0$ with the Σ_G-connections $\{g', h', (g')^{-1}\}$ or $\{g', h'(g')^{-1}\}$, respectively, we conclude $g' \sim h'$, that is, $[g] = [h]$, which constitutes a contradiction. In the latter case, by using Jacobi identity we can assert the following

$$\big[[L_{(g')^{-1}}, L_{g'}], L_{h'}\big] = \big[[L_{g'}, L_{h'}], L_{(g')^{-1}}\big] + \big[[L_{h'}, L_{(g')^{-1}}], L_{g'}\big]$$

and by Equation (15) we get

$$[L_{g'}, L_{h'}] = [L_{h'}, L_{(g')^{-1}}] = 0.$$

Hence $\big[[L_{(g')^{-1}}, L_{g'}], L_{h'}\big] = 0$ and we show

$$[L_{[g],1}, V_{[h]}] = 0. \tag{16}$$

Analogously can be proved for the third summand $[V_{[g]}, L_{[h],1}] = 0$.

Finally, the first summand $[L_{[g],1}, L_{[h],1}]$ in (14) is

$$\begin{aligned} &\Big[\sum_{g' \in [g] \cap \Lambda_G} A_{(g')^{-1}} L_{g'} + \sum_{g' \in [g]} [L_{(g')^{-1}}, L_{g'}], \sum_{h' \in [h] \cap \Lambda_G} A_{(h')^{-1}} L_{h'} + \sum_{h' \in [h]} [L_{(h')^{-1}}, L_{h'}]\Big] \subset \\ &\sum_{g' \in [g] \cap \Lambda_G, h' \in [h] \cap \Lambda_G} [A_{(g')^{-1}} L_{g'}, A_{(h')^{-1}} L_{h'}] + \sum_{g' \in [g] \cap \Lambda_G, h' \in [h]} \Big[A_{(g')^{-1}} L_{g'}, [L_{(h')^{-1}}, L_{h'}]\Big] \\ &+ \sum_{g' \in [g], h' \in [h] \cap \Lambda_G} \Big[[L_{(g')^{-1}}, L_{g'}], A_{(h')^{-1}} L_{h'}\Big] + \sum_{g' \in [g], h' \in [h]} \Big[[L_{(g')^{-1}}, L_{g'}], [L_{(h')^{-1}}, L_{h'}]\Big] \end{aligned} \tag{17}$$

For the first summand in (17), if there exist $g' \in [g] \cap \Lambda_G, h' \in [h] \cap \Lambda_G$ such that $[A_{(g')^{-1}}L_{g'}, A_{(h')^{-1}}L_{h'}] \neq 0$, then Equation (2) and Equation (5) lead to

$$\begin{aligned}
[A_{(g')^{-1}}L_{g'}, A_{(h')^{-1}}L_{h'}] &\subset A_{(h')^{-1}}[A_{(g')^{-1}}L_{g'}, L_{h'}] + \rho(A_{(g')^{-1}}L_{g'})(A_{(h')^{-1}})L_{h'} \\
&\subset A_{(h')^{-1}}\Big(A_{(g')^{-1}}[L_{h'}, L_{g'}] + \rho(L_{h'})(A_{(g')^{-1}})L_{g'}\Big) \\
&+ \rho(A_{(g')^{-1}}L_{g'})(A_{(h')^{-1}})L_{h'} \\
&\subset A_{(h')^{-1}}\Big(A_{(g')^{-1}}L_{h'g'} + A_{h'(g')^{-1}}L_{g'}\Big) + A_{(g')^{-1}}A_{g'(h')^{-1}}L_{h'} \\
&\subset A_{(h')^{-1}}A_{(g')^{-1}}L_{h'g'} + A_{(h')^{-1}}A_{h'(g')^{-1}}L_{g'} + A_{(g')^{-1}}A_{g'(h')^{-1}}L_{h'}
\end{aligned}$$

If $A_{(h')^{-1}}A_{(g')^{-1}}L_{h'g'}$ is non-zero, considering then the connection $\{g', h', (g')^{-1}\}$ we conclude $[g] = [h]$, which constitutes a contradiction. Similarly, if $A_{(h')^{-1}}A_{h'(g')^{-1}}L_{g'}$ or $A_{(g')^{-1}}A_{g'(h')^{-1}}L_{h'}$ is non-zero, then by means of the Σ_G-connection $\{g', h'(g')^{-1}\}$ or $\{h', g'(h')^{-1}\}$, respectively, the same contradiction is obtained.

For $g' \in [g], h' \in [h] \cap \Lambda_G$ in the third summand of (17) using Equation (2) we obtain two summands. For the first summand we use Jacobi identity and Equation (15), and for the latter we apply Equation (5) and that ρ is a Lie algebra homomorphism, having then

$$\begin{aligned}
\Big[[L_{(g')^{-1}}, L_{g'}], A_{(h')^{-1}}L_{h'}\Big] &= A_{(h')^{-1}}\Big[[L_{(g')^{-1}}, L_{g'}], L_{h'}\Big] + \rho([L_{(g')^{-1}}, L_{g'}])(A_{(h')^{-1}})L_{h'} \\
&\subset A_{(h')^{-1}}\Big(\Big[[L_{(g')^{-1}}, L_{h'}], L_{g'}\Big] + \Big[[L_{g'}, L_{h'}], L_{(g')^{-1}}\Big]\Big) \\
&\quad + \rho([L_{(g')^{-1}}, L_{g'}])(A_{(h')^{-1}})L_{h'} \\
&\subset \big[\rho(L_{(g')^{-1}}), \rho(L_{g'})\big](A_{(h')^{-1}})L_{h'} \\
&\subset \Big(\rho(L_{(g')^{-1}})\Big(\rho(L_{g'})(A_{(h')^{-1}})\Big)\Big)L_{h'} \\
&\quad + \Big(\rho(L_{g'})\Big(\rho(L_{(g')^{-1}})(A_{(h')^{-1}})\Big)\Big)L_{h'} \\
&\subset \Big(\rho(L_{(g')^{-1}})(A_{g'(h')^{-1}})\Big)L_{h'} + \Big(\rho(L_{g'})(A_{(g')^{-1}(h')^{-1}})\Big)L_{h'}
\end{aligned}$$

If $\big(\rho(L_{(g')^{-1}})(A_{g'(h')^{-1}})\big)L_{h'}$ or $\big(\rho(L_{g'})(A_{(g')^{-1}(h')^{-1}})\big)L_{h'}$ is non-zero considering the Σ_G-connection $\{(g')^{-1}, g'(h')^{-1}\}$ or $\{g', (g')^{-1}(h')^{-1}\}$, respectively, we get $[g] = [h]$.

The proof for the second summand in (17) is analogous.

Consider now the fourth summand in (17), on account of the bilinearity of the product and Jacobi identity we obtain

$$\begin{aligned}
&\sum_{g' \in [g], h' \in [h]} [[L_{(g')^{-1}}, L_{g'}], [L_{(h')^{-1}}, L_{h'}]] \subset \\
&\sum_{g' \in [g], h' \in [h]} \Big(\Big[L_{(g')^{-1}}, \big[L_{g'}, [L_{(h')^{-1}}, L_{h'}]\big]\Big] + \Big[L_{g'}, \big[L_{(g')^{-1}}, [L_{(h')^{-1}}, L_{h'}]\big]\Big]\Big) \subset \\
&\sum_{g' \in [g]} \Big(\Big[L_{(g')^{-1}}, \big[L_{g'}, L_{[h],1}\big]\Big] + \Big[L_{g'}, \big[L_{(g')^{-1}}, L_{[h],1}\big]\Big]\Big)
\end{aligned}$$

and by Equation (16) we also conclude that this fourth summand is zero. Therefore, Equation (17) vanishes and then

$$[L_{[g],1}, L_{[h],1}] = 0. \tag{18}$$

In conclusion, from Equations (14)-(18) we get $[I_{[g]}, I_{[h]}] = 0$. □

Let us note that we consider the usual regularity concepts in the graded sense. Thus, given a generic G-graded algebra $\mathfrak{A} = \oplus_{g \in G}\mathfrak{A}_g$ an ideal I of $\mathfrak{A}$ is said to be a *graded ideal* if it splits as $I = \oplus_{g \in G} I_g$, where $I_g := I \cap \mathfrak{A}_g$, and satisfies $[I_g, I_{g'}] \subset I_{gg'}$, for $g, g' \in G$. We also say that a Lie-Rinehart algebra (L, A) is *gr-simple* if $[L, L] \neq 0$, $AA \neq 0$, $AL \neq 0$ and its only graded ideals are $\{0\}$, L and $\mathrm{Ker}\rho$.

Theorem 2.6. *The following assertions are verified.*

i) *For all $[g] \in \Sigma_G/\sim$, the linear space $I_{[g]} = L_{[g],1} \oplus V_{[g]}$ is a graded ideal of L.*

ii) *If L is gr-simple then all the elements of Σ_G are Σ_G-connected. Furthermore,*

$$L_1 = \Big(\sum_{g\in\Sigma_G\cap\Lambda_G} A_{g^{-1}}L_g\Big) + \Big(\sum_{g\in\Sigma_G}[L_{g^{-1}}, L_g]\Big).$$

Proof. i) We get $[I_{[g]}, L_1] \subset I_{[g]}$ and Propositions 2.4-i) and 2.5 allow us to assert

$$[I_{[g]}, L] = \Big[I_{[g]}, L_1 \oplus \big(\bigoplus_{g'\in[g]} L_{g'}\big) \oplus \big(\bigoplus_{h\notin[g]} L_h\big)\Big] \subset I_{[g]},$$

so $I_{[g]}$ is a Lie ideal of L. It is easy to check that by Proposition 2.4-ii) we also have that $I_{[g]}$ is an A-module. Finally, Equation (2) leads to

$$\rho(I_{[g]})(A)L \subset [I_{[g]}, AL] + A[I_{[g]}, L] \subset [I_{[g]}, L] + AI_{[g]} \subset I_{[g]},$$

and we conclude $I_{[g]}$ is an ideal of L. Since by construction $I_{[g]}$ is graded, we obtain the required result.

ii) The gr-simplicity of L implies $I_{[g]} \in \{\mathrm{Ker}\rho, L\}$ for any $g \in \Sigma_G$. If some $g \in \Sigma_G$ is such that $I_{[g]} = L$, then $[g] = \Sigma_G$. Otherwise, if $I_{[g]} = \mathrm{Ker}\rho$ for all $g \in \Sigma_G$ is $[g] = [h]$ for any $g, h \in \Sigma_G$, and again $[g] = \Sigma_G$. Therefore in any case Σ_G has all its elements Σ_G-connected and $L_1 = \big(\sum_{g\in\Sigma_G\cap\Lambda_G} A_{g^{-1}}L_g\big) + \big(\sum_{g\in\Sigma_G}[L_{g^{-1}}, L_g]\big)$ which concludes the proof. □

Theorem 2.7. *Let (L, A) be a graded Lie-Rinehart algebra. Then*

$$L = U + \sum_{[g]\in\Sigma_G/\sim} I_{[g]},$$

where U is a linear complement of $\big(\sum_{g\in\Sigma_G\cap\Lambda_G} A_{g^{-1}}L_g\big) + \big(\sum_{g\in\Sigma_G}[L_{g^{-1}}, L_g]\big)$ in L_1, and any $I_{[g]} \subset L$ is one of the graded ideals described in Theorem 2.6-i). Moreover, $[I_{[g]}, I_{[h]}] = 0$ provided that $[g] \neq [h]$.

Proof. We have $I_{[g]}$ is well-defined and, by Theorem 2.6-i), an ideal of L, being clear that

$$L = L_1 \oplus \big(\bigoplus_{g\in\Sigma_G} L_g\big) = U + \sum_{[g]\in\Sigma_G/\sim} I_{[g]}.$$

Finally, Proposition 2.5 gives $[I_{[g]}, I_{[h]}] = 0$ if $[g] \neq [h]$. □

For a Lie-Rinehart algebra L, we denote by $\mathcal{Z}(L) := \big\{v \in L : [v, L] = \rho(v) = 0\big\}$ the *center* of L as in reference [34].

Corollary 2.8. *If $\mathcal{Z}(L) = 0$ and $L_1 = \big(\sum_{g\in\Sigma_G\cap\Lambda_G} A_{g^{-1}}L_g\big) + \big(\sum_{g\in\Sigma_G}[L_{g^{-1}}, L_g]\big)$ then L is the direct sum of the graded ideals given in Theorem 2.6-i),*

$$L = \bigoplus_{[g]\in\Sigma_G/\sim} I_{[g]}.$$

(Observe that $[I_{[g]}, I_{[h]}] = 0$ provided that $[g] \neq [h]$.)

Proof. Since $L_1 = \big(\sum_{g\in\Sigma_G\cap\Lambda_G} A_{g^{-1}}L_g\big) + \big(\sum_{g\in\Sigma_G}[L_{g^{-1}}, L_g]\big)$ we get

$$L = \sum_{[g]\in\Sigma_G/\sim} I_{[g]}.$$

In order to verify the direct character of the sum, take some $v \in I_{[g]} \cap \big(\sum_{[h]\in\Sigma_G/\sim, [h]\neq[g]} I_{[h]}\big)$. Since $v \in I_{[g]}$, the fact that $\big[I_{[g]}, I_{[h]}\big] = 0$ provided that $[g] \neq [h]$, gives us

$$\Big[v, \sum_{[h]\in\Sigma_G/\sim, [h]\neq[g]} I_{[h]}\Big] = 0.$$

Analogously, as $v \in \sum_{[h]\in\Sigma_G/\sim, [h]\neq[g]} I_{[h]}$ we get $[v, I_{[g]}] = 0$. Therefore $[v, L] = 0$. Now, Equation (2) allows us to conclude $\rho(v) = 0$. That is, $v \in \mathcal{Z}(L) = 0$. □

3. Connections in the G-support of A. Decompositions of A

Along this section we introduce an adequate notion of connection among the elements of the G-support Λ_G for a commutative and associative $\mathbb{F}$-algebra A associated with a G-graded Lie-Rinehart $\mathbb{F}$-algebra L (see Definition 1.6). Let us recall that A admits a group graduation as

$$A = A_1 \oplus \Big(\bigoplus_{g \in \Lambda_G} A_g \Big),$$

being $\Lambda_G = \{g \in G \setminus \{1\} : A_g \neq 0\}$. Let us note also that we will continue considering the sets $\Sigma_G^{\pm}$, $\Lambda_G^{\pm}$ as in the previous section.

Definition 3.1. Let $g, g' \in \Lambda_G$. g is said to be Λ_G-*connected* to g' if there exists $\{k_1, k_2, \dots, k_n\} \subset \Lambda_G^{\pm} \cup \Sigma_G^{\pm}$ such that

i. $k_1 = g$.
ii. $\{k_1, k_1k_2, \dots, k_1k_2 \cdots k_{n-1}\} \subset \Lambda_G^{\pm} \cup \Sigma_G^{\pm}$.
iii. $k_1k_2 \cdots k_n \in \{g', (g')^{-1}\}$.

It is said that $\{k_1, \dots, k_n\}$ is a Λ_G-*connection* from g to g'.

Similarly to Section 2 we can prove the next results.

Proposition 3.2. *The relation $\approx$ in Λ_G, defined by $g \approx g'$ if and only if g is Λ_G-connected to g', is an equivalence relation.*

Remark 3.3. Let $g, g' \in \Lambda_G$ such that $g \approx g'$. If $g'h \in \Lambda_G$, for $h \in \Lambda_G$, then $g \approx g'h$.

By means of Proposition 3.2 we can consider the quotient set

$$\Lambda_G/\approx := \{[g] : g \in \Lambda_G\},$$

becoming $[g]$ the set of elements of Λ_G which are Λ_G-connected to g. Thus, our purpose now in this section is to associate an (adequate) ideal $\mathcal{A}_{[g]}$ of the algebra A to any $[g] \in \Lambda_G/\approx$. Once having fixed $g \in \Lambda_G$, we start by defining the sets

$$A_{[g],1} := \Big(\sum_{g' \in [g] \cap \Sigma_G} \rho(L_{(g')^{-1}})(A_{g'}) \Big) + \Big(\sum_{g' \in [g]} A_{(g')^{-1}} A_{g'} \Big) \subset A_1$$

and

$$A_{[g]} := \bigoplus_{g' \in [g]} A_{g'}.$$

Note that we denote by $\mathcal{A}_{[g]}$ the direct sum of the two graded subspaces above, i.e.

$$\mathcal{A}_{[g]} := A_{[g],1} \oplus A_{[g]}.$$

Proposition 3.4. *For any $[g] \in \Lambda_G/\approx$ we have $\mathcal{A}_{[g]}\mathcal{A}_{[g]} \subset \mathcal{A}_{[g]}$.*

Proof. Since the algebra A is commutative we have

$$\mathcal{A}_{[g]}\mathcal{A}_{[g]} = \Big(A_{[g],1} \oplus A_{[g]}\Big)\Big(A_{[g],1} \oplus A_{[g]}\Big) \subset A_{[g],1}A_{[g],1} + A_{[g],1}A_{[g]} + A_{[g]}A_{[g]}. \tag{19}$$

Let us consider the second summand in Equation (19). Thus, given $g' \in [g]$ we have $A_{[g],1}A_{g'} \subset A_1 A_{g'} \subset A_{g'}$, and therefore

$$A_{[g],1}A_{g'} \subset A_{[g]}. \tag{20}$$

Taking into account now the third summand in (19), let be $g', g'' \in [g]$ such that $0 \neq A_{g'}A_{g''} \subset A_{g'g''}$. If $g'g'' = 1$ we have $A_{(g')^{-1}}A_{g'} \subset A_1$, and so $A_{(g')^{-1}}A_{g'} \subset A_{[g],1}$. Suppose $g'g'' \in \Lambda_G$, then by Remark 3.3 we have $g'g'' \in [g]$ and so $A_{g'}A_{g''} \subset A_{g'g''} \subset A_{[g]}$. Hence $A_{[g]}A_{[g]} = (\bigoplus_{g' \in [g]} A_{g'})(\bigoplus_{g'' \in [g]} A_{g''}) \subset A_{[g],1} \oplus A_{[g]}$, i.e.

$$A_{[g]}A_{[g]} \subset \mathcal{A}_{[g]}. \tag{21}$$

Finally we consider the first summand $A_{[g],1}A_{[g],1}$ in (19) and suppose there exist $g', g'' \in [g] \cap \Sigma_G$ such that

$$\Big(\rho(L_{(g')^{-1}})(A_{g'}) + A_{(g')^{-1}}A_{g'}\Big)\Big(\rho(L_{(g'')^{-1}})(A_{g''}) + A_{(g'')^{-1}}A_{g''}\Big) \neq 0,$$

so

$$\rho(L_{(g')^{-1}})(A_{g'})\rho(L_{(g'')^{-1}})(A_{g''}) + \rho(L_{(g')^{-1}})(A_{g'})(A_{(g'')^{-1}}A_{g''}) + (A_{(g')^{-1}}A_{g'})\rho(L_{(g'')^{-1}})(A_{g''}) + (A_{(g')^{-1}}A_{g'})(A_{(g'')^{-1}}A_{g''}) \neq 0 \tag{22}$$

For the latter summand in Equation (22), in the case of $g'' \neq (g')^{-1}$, the commutativity and associativity of A allow us to assert

$$(A_{(g')^{-1}}A_{g'})(A_{(g'')^{-1}}A_{g''}) = (A_{(g')^{-1}}A_{(g'')^{-1}})(A_{g'}A_{g''}) \subset A_{(g'g'')^{-1}}A_{g'g''} \subset A_{[g],1}$$

because by Remark 3.3 we get $g'g'' \in [g]$. In the case of $g'' = (g')^{-1}$, it follows

$$(A_{(g')^{-1}}A_{g'})(A_{g'}A_{(g')^{-1}}) = A_{(g')^{-1}}(A_{g'}A_{g'}A_{(g')^{-1}}) \subset A_{(g')^{-1}}A_{g'} \subset A_{[g],1}.$$

For the second summand in Equation (22) using Equation (5) we get

$$\Big(\rho(L_{(g')^{-1}})(A_{g'})\Big)\Big(A_{(g'')^{-1}}A_{g''}\Big) \subset A_1\big(A_{(g'')^{-1}}A_{g''}\big) \subset A_{(g'')^{-1}}A_{g''} \subset A_{[g],1}.$$

Analogously, it can be checked the third summand in Equation (22).

Finally, for the first summand in (22), as with the second summand, by Equation (5) we have

$$\big(\rho(L_{(g')^{-1}}(A_{g'})\big)\big(\rho(L_{(g'')^{-1}})(A_{g''})\big) \subset A_1\big(\rho(L_{(g'')^{-1}})(A_{g''})\big) \subset A_{[g],1}.$$

That is, by considering Equation (22) we have shown

$$A_{[g],1}A_{[g],1} \subset A_{[g],1}. \tag{23}$$

From Equations (19)-(21) and (23) we get $\mathcal{A}_{[g]}\mathcal{A}_{[g]} \subset \mathcal{A}_{[g]}$. □

Proposition 3.5. *For any $[g], [h] \in \Lambda_G/\approx$ such that $[g] \neq [h]$ we have $\mathcal{A}_{[g]}\mathcal{A}_{[h]} = 0$.*

Proof. We have

$$\Big(A_{[g],1} \oplus A_{[g]}\Big)\Big(A_{[h],1} \oplus A_{[h]}\Big) \subset A_{[g],1}A_{[h],1} + A_{[g],1}A_{[h]} + A_{[g]}A_{[h],1} + A_{[g]}A_{[h]}. \tag{24}$$

Consider the fourth summand $A_{[g]}A_{[h]}$ and suppose there exist $g' \in [g]$, $h' \in [h]$ such that $A_{g'}A_{h'} \neq 0$, so $A_{g'h'} \neq 0$. Observe that necessarily $h' \neq (g')^{-1}$, then $g'h' \in \Lambda_G$. By Remark 3.3 we obtain $g' \approx g'h'$, meaning that $g'h' \in [g]$. Similarly, $g'h' \in [h]$, so $[g] = [h]$, a contradiction. Hence $A_{g'}A_{h'} = 0$ and so

$$A_{[g]}A_{[h]} = 0. \tag{25}$$

Consider now the second summand $A_{[g],1}A_{[h]}$ in Equation (24). We take $g' \in [g] \cap \Sigma_G$ and $h' \in [h]$ such that

$$\Big(\rho(L_{(g')^{-1}})(A_{g'}) + A_{(g')^{-1}}A_{g'}\Big)A_{h'} \neq 0.$$

Suppose $(A_{(g')^{-1}}A_{g'})A_{h'} \neq 0$. By using associativity of A we get $A_{(g')^{-1}}(A_{g'}A_{h'}) \neq 0$, so $A_{g'h'} \neq 0$ and then $g'h' \in \Lambda_G$. Arguing as above $g \approx h$, a contradiction. If the another summand $\rho(L_{(g')^{-1}})(A_{g'})A_{h'} \neq 0$, since $\rho(L_{(g')^{-1}})$ is a derivation then $\rho(L_{(g')^{-1}})(A_{g'}A_{h'})$ or $A_{g'}\rho(L_{(g')^{-1}})(A_{h'})$ is non-zero, but in any case we argue similarly as above to get $g \approx h$, a contradiction. From here

$$A_{[g],1}A_{[h]} = 0. \tag{26}$$

By commutativity, for the third summand also $A_{[g]}A_{[h],1} = 0$.

Finally, let us prove $A_{[g],1}A_{[h],1} = 0$. Suppose there exist $g' \in [g] \cap \Sigma_G, h' \in [h] \cap \Sigma_G$ such that

$$\rho(L_{(g')^{-1}})(A_{g'})\rho(L_{(h')^{-1}})(A_{h'}) + \rho(L_{(g')^{-1}})(A_{g'})(A_{(h')^{-1}}A_{h'}) + (A_{(g')^{-1}}A_{g'})\rho(L_{(h')^{-1}})(A_{h'}) + (A_{(g')^{-1}}A_{g'})(A_{(h')^{-1}}A_{h'}) \neq 0. \tag{27}$$

If the last summand in Equation (27) is non-zero, by the commutativity and associativity of A, since $h' \neq (g')^{-1}$, we have

$$(A_{(g')^{-1}}A_{g'})(A_{(h')^{-1}}A_{h'}) = (A_{(g')^{-1}}A_{(h')^{-1}})(A_{g'}A_{h'}) \subset A_{(g'h')^{-1}}A_{g'h'}$$

and by Remark 3.3 we get $g'h' \in [g]$ as well $g'h' \in [h]$, a contradiction.

If for the second summand in Equation (27) there exist $g' \in [g] \cap \Sigma_G, h' \in [h]$, since $\rho(L_{(g')^{-1}})$ is a derivation we get

$$\Big(\rho(L_{(g')^{-1}})(A_{g'})\Big)\Big(A_{(h')^{-1}}A_{h'}\Big) \subset \Big(\rho(L_{(g')^{-1}})(A_{g'})A_{(h')^{-1}}\Big)A_{h'}$$
$$\subset \rho(L_{(g')^{-1}})\Big(A_{g'}A_{(h')^{-1}}\Big)A_{h'} + \Big(A_{g'}\rho(L_{(g')^{-1}})(A_{(h')^{-1}})\Big)A_{h'}$$

If the first summand or the second one is nonzero we get as in the previous cases that $[g]=[h]$, a contradiction. Similarly, can be proven the third summand in Equation (27).

Finally, if for the first summand in (27) there exist $g' \in [g] \cap \Sigma_G$, $h' \in [h] \cap \Sigma_G$ we have

$$\Big(\rho(L_{(g')^{-1}})(A_{g'})\Big)\Big(\rho(L_{(h')^{-1}})(A_{h'})\Big) \subset \rho(L_{(g')^{-1}})\Big((A_{g'})\rho(L_{(h')^{-1}})(A_{h'})\Big)$$
$$+ A_{g'}\rho(L_{(g')^{-1}})\Big(\rho(L_{(h')^{-1}})(A_{h'})\Big)$$
$$\subset \rho(L_{(g')^{-1}})\Big(\rho(L_{(h')^{-1}})(A_{g'}A_{h'}) + \rho(L_{(h')^{-1}})(A_{g'})(A_{h'})\Big)$$
$$+ A_{g'}\rho(L_{(h')^{-1}})\Big(\rho(L_{(g')^{-1}})(A_{h'})\Big) + A_{g'}\rho\Big([L_{(h')^{-1}}, L_{(g')^{-1}}]\Big)A_{h'}$$

and if some summand is non-zero arguing as above we obtain again the contradiction $[g]=[h]$.

Since Equation (27) vanishes we assert

$$A_{[g],1}A_{[h],1} = 0, \tag{28}$$

and from (24)-(26) and (28) we conclude $\mathcal{A}_{[g]}\mathcal{A}_{[h]} = 0$. □

We recall that a subspace I of a commutative and associative algebra A is an *ideal of* A if $AI \subset I$. In the case of A being a G-graded algebra, we say that an ideal $I \subset A$ is *graded* if it splits as $I = \oplus_{g \in G} I_g$, where $I_g := I \cap A_g$, and satisfies $I_g I_{g'} \subset I_{gg'}$, for $g, g' \in G$. We say that A is *gr-simple* if $AA \neq 0$ and it contains no proper graded ideals.

Theorem 3.6. *Let A be a commutative and associative $\mathbb{F}$-algebra associated with a G-graded Lie-Rinehart $\mathbb{F}$-algebra L. Then the following assertions hold.*

i) *For any $[g] \in \Lambda_G/\approx$, the linear space*

$$\mathcal{A}_{[g]} = A_{[g],1} \oplus A_{[g]}$$

is a graded ideal of A.

ii) *If A is gr-simple then all elements of Λ_G are Λ_G-connected. Furthermore,*

$$A_1 = \sum_{g \in \Lambda_G \cap \Sigma_G} \rho(L_{g^{-1}})(A_g) + \sum_{g \in \Lambda_G} A_{g^{-1}}A_g$$

Proof. i) Since $\mathcal{A}_{[g]}A_1 \subset \mathcal{A}_{[g]}$, Propositions 3.4 and 3.5 allow us to assert

$$\mathcal{A}_{[g]}A = \mathcal{A}_{[g]}\Big(A_1 \oplus (\bigoplus_{g' \in [g]} A_{g'}) \oplus (\bigoplus_{h \notin [g]} A_h)\Big) \subset \mathcal{A}_{[g]}.$$

We conclude $\mathcal{A}_{[g]}$ is an ideal of A and, since by construction is G-graded, is a graded ideal.

ii) The gr-simplicity of A implies $\mathcal{A}_{[g]} = A$, for $g \in \Lambda_G$. From here, it is clear that $[g] = \Lambda_G$ and $A_1 = \sum\limits_{g \in \Lambda_G \cap \Sigma_G} \rho(L_{g^{-1}})(A_g)$ $+ \sum\limits_{g \in \Lambda_G} A_{g^{-1}}A_g$. □

Theorem 3.7. *Let A be a commutative and associative $\mathbb{F}$-algebra associated with a G-graded Lie-Rinehart $\mathbb{F}$-algebra L. Then*

$$A = V + \sum_{[g] \in \Lambda_G/\approx} \mathcal{A}_{[g]},$$

where V is a linear complement in A_1 of $\sum_{g \in \Lambda_G \cap \Sigma_G} \rho(L_{g^{-1}})(A_g) + \sum_{g \in \Lambda_G} A_{g^{-1}}A_g$ and any $\mathcal{A}_{[g]}$ is one of the graded ideals of A described in Theorem 3.6-i). Furthermore, $\mathcal{A}_{[g]}\mathcal{A}_{[h]} = 0$ when $[g] \neq [h]$.

Proof. We know that $\mathcal{A}_{[g]}$ is well-defined and, by Theorem 3.6-i), a graded ideal of A, being clear that

$$A = A_1 \oplus (\bigoplus_{g \in \Lambda_G} A_g) = V + \sum_{[g] \in \Lambda_G/\approx} \mathcal{A}_{[g]}.$$

Finally, Proposition 3.5 gives $\mathcal{A}_{[g]}\mathcal{A}_{[h]} = 0$ if $[g] \neq [h]$. □

Let us denote by $\mathcal{A}nn(A) := \{a \in A : aA = 0\}$ the *annihilator* of the commutative and associative algebra A.

Corollary 3.8. *Let A be a commutative and associative $\mathbb{F}$-algebra associated with a G-graded Lie-Rinehart $\mathbb{F}$-algebra L. If $\mathcal{A}nn(A) = 0$ and*

$$A_1 = \sum_{g \in \Lambda_G \cap \Sigma_G} \rho(L_{g^{-1}})(A_g) + \sum_{g \in \Lambda_G} A_{g^{-1}} A_g,$$

then A is the direct sum of the graded ideals given in Theorem 3.6-i),

$$A = \bigoplus_{[g] \in \Sigma_A/\approx} \mathcal{A}_{[g]}.$$

Furthermore, $\mathcal{A}_{[g]}\mathcal{A}_{[h]} = 0$ provided that $[g] \neq [h]$.

Proof. Since $A_1 = \sum\limits_{g \in \Lambda_G \cap \Sigma_G} \rho(L_{g^{-1}})(A_g) + \sum\limits_{g \in \Lambda_G} A_{g^{-1}} A_g$ we obtain $A = \sum_{[g] \in \Lambda_G/\approx} \mathcal{A}_{[g]}$. In order to verify the direct character of the sum, take some

$$a \in \mathcal{A}_{[g]} \cap \Big(\sum_{[h] \in \Lambda_G/\approx, [h] \neq [g]} \mathcal{A}_{[h]} \Big).$$

Since $a \in \mathcal{A}_{[g]}$, the fact that $\mathcal{A}_{[g]}\mathcal{A}_{[h]} = 0$ provided that $[g] \neq [h]$ gives us

$$a \Big(\sum_{[h] \in \Lambda_G/\approx, [h] \neq [g]} \mathcal{A}_{[h]} \Big) = 0.$$

In a similar way, since $a \in \sum_{[h] \in \Sigma_A/\approx, [h] \neq [g]} \mathcal{A}_{[h]}$ we get $a\mathcal{A}_{[g]} = 0$. That is, $a \in \mathcal{A}nn(A) = 0$. □

4. Relating the decompositions of L and A

The aim of this section is to show that the decompositions of L and A as direct sum of ideals, given in Sections 2 and 3 respectively, are closely related.

Thus, given a graded Lie-Rinehart algebra (L, A) we call the *annihilator of A in L* (that is, using the structure of L as A-module) the set

$$\mathcal{A}nn_L(A) := \{v \in L : Av = 0\}.$$

Obviously, $\mathcal{A}nn_L(A)$ is an ideal of L. In fact, $A\mathcal{A}nn_L(A) = 0$, also $[L, \mathcal{A}nn_L(A)] \subset \mathcal{A}nn_L(A)$ since

$$A[L, \mathcal{A}nn_L(A)] = [L, A\mathcal{A}nn_L(A)] + \rho(L)(A)\mathcal{A}nn_L(A) = 0.$$

Also, it verifies Equation (3)

$$A\big(\rho(\mathcal{A}nn_L(A))(A)L\big) = \big(A\rho(\mathcal{A}nn_L(A))(A)\big)L = \rho(A\mathcal{A}nn_L(A))(A)L = \rho(0)(A)L = 0.$$

Definition 4.1. A G-graded Lie-Rinehart algebra (L, A) is *tight* if $\mathcal{Z}(L) = \mathcal{A}nn_L(A) = \mathcal{A}nn(A) = \{0\}$, $AA = A$, $AL = L$ and

$$L_1 = \Big(\sum_{g \in \Sigma_G \cap \Lambda_G} A_{g^{-1}} L_g \Big) + \Big(\sum_{g \in \Sigma_G} [L_{g^{-1}}, L_g] \Big),$$

$$A_1 = \Big(\sum_{g \in \Lambda_G \cap \Sigma_G} \rho(L_{g^{-1}})(A_g) \Big) + \Big(\sum_{g \in \Lambda_G} A_{g^{-1}} A_g \Big).$$

If (L, A) is tight then Corollary 2.8 and Corollary 3.8 say that

$$L = \bigoplus_{[g]\in\Sigma_G/\sim} I_{[g]} \quad \text{and} \quad A = \bigoplus_{[g]\in\Lambda_G/\approx} \mathcal{A}_{[g]},$$

with any $I_{[g]}$ a graded ideal of L verifying $[I_{[g]}, I_{[h]}] = 0$ if $[g] \neq [h]$, and any $\mathcal{A}_{[g]}$ a graded ideal of A satisfying $\mathcal{A}_{[g]}\mathcal{A}_{[h]} = 0$ if $[g] \neq [h]$.

Proposition 4.2. *Let (L, A) be a tight G-graded Lie-Rinehart algebra. Then for $[g] \in \Sigma_G/\sim$ there exists a unique $[h] \in \Lambda_G/\approx$ such that $\mathcal{A}_{[h]}I_{[g]} \neq 0$.*

Proof. First we prove the existence. Given $[g] \in \Sigma_G/\sim$, let us suppose that $AI_{[g]} = 0$. Since $I_{[g]}$ is a graded ideal it follows

$$[I_{[g]}, AL] = \Big[I_{[g]}, \bigoplus_{h\in\Sigma_G/\sim} AI_{[h]}\Big] = [I_{[g]}, AI_{[g]}] = 0.$$

By hypothesis $AL = L$, then $I_{[g]} \subset \mathcal{Z}(L) = \{0\}$, which constitutes a contradiction. Since $A = \bigoplus_{[g]\in\Lambda_G/\approx} \mathcal{A}_{[g]}$, there exists $[h] \in \Lambda_G/\approx$ such that $\mathcal{A}_{[h]}I_{[g]} \neq 0$.

Now we prove that $[h]$ is unique. Suppose that m is another element of G which satisfies $\mathcal{A}_{[m]}I_{[g]} \neq 0$. From $\mathcal{A}_{[h]}I_{[g]} \neq 0$ and $\mathcal{A}_{[m]}I_{[g]} \neq 0$ we can take $h' \in [h], m' \in [m]$ and $g', g'' \in [g]$ such that $A_{h'}L_{g'} \neq 0$ and $A_{m'}L_{g''} \neq 0$. Since $g', g'' \in [g]$, we can fix a Σ_G-connection

$$\{g', g_2, \ldots, g_n\} \subset \Sigma_G^{\pm} \cup \Lambda_G^{\pm}$$

from g' to g''.

We have to distinguish four cases:

- If $h'g' \neq 1$ and $m'g'' \neq 1$, then $h'g', m'g'' \in \Sigma_G$, and so $h' \approx m'$. Indeed, in the case $g'g_2 \cdots g_n = g''$, the Λ_G-connection from h' to m' is

  $$\{h', g', (h')^{-1}, g_2, \ldots, g_n, m', (g'')^{-1}\},$$

  and in the case $g'g_2 \cdots g_n = (g'')^{-1}$ is

  $$\{h', g', (h')^{-1}, g_2, \ldots, g_n, (m')^{-1}, g''\}.$$

  From here $[h] = [m]$.
- If $h'g' = 1$ and $m'g'' \neq 1$, we get $h' = (g')^{-1}$, $m'g'' \in \Sigma_G$ and then

  $$\{(g')^{-1}, g_2^{-1}, \ldots, g_n^{-1}, (m')^{-1}, g''\}$$

  is a Λ_G-connection from h' to m' in the case $g'g_1 \cdots g_n = g''$, while

  $$\{(g')^{-1}, g_2^{-1}, \ldots, g_n^{-1}, m', (g'')^{-1}\}$$

  is a Λ_G-connection in the case $g'g_1 \cdots g_n = (g'')^{-1}$. From here, $[h] = [m]$.
- Suppose $h'g' \neq 1$ and $m'g'' = 1$. We can argue as in the previous item to get $[h] = [m]$.
- Finally, we consider $h'g' = m'g'' = 1$. Hence $h' = (g')^{-1}, m' = (g'')^{-1}$. Then

  $$\{(g')^{-1}, g_2^{-1}, \ldots, g_n^{-1}\}$$

  is a Λ_G-connection between h' and m' which implies $[h] = [m]$.

We conclude $[h]$ is the unique element in $\Lambda_G/\approx$ such that $\mathcal{A}_{[h]}I_{[g]} \neq 0$ for the given $[g] \in \Sigma_G/\sim$. □

Observe that the above proposition shows that $I_{[g]}$ is an $\mathcal{A}_{[h]}$-module. Hence we can assert the following result.

Theorem 4.3. *Let (L, A) be a tight graded Lie-Rinehart $\mathbb{F}$-algebra over an associative and commutative $\mathbb{F}$-algebra A. Then*

$$L = \bigoplus_{i\in I} I_i \quad \text{and} \quad A = \bigoplus_{j\in J} A_j$$

where any I_i is a non-zero graded ideal of L satisfying $[I_i, I_h] = 0$ when $i \neq h$, and any A_j is a non-zero graded ideal of A such that $A_jA_k = 0$ when $j \neq k$. Moreover, both decompositions satisfy that for any $r \in I$ there exists a unique $\tilde{r} \in J$ such that

$$A_{\tilde{r}}I_r \neq 0.$$

Furthermore, any I_r is a graded Lie-Rinehart algebra over $A_{\tilde{r}}$.

5. Graded Lie-Rinehart algebras of maximal length

In this last section we are going to show that for Lie-Rinehart algebras of maximal length the decomposition of a graded Lie-Rinehart algebra (L, A) given in Theorem 4.3 can be obtained by means of the families of the indecomposable (graded) ideals of L and of indecomposable (graded) ideals of A. Also, we will characterize the (graded) simplicity of the ideals in the decompositions of L and A given in Theorem 4.3. At following we suppose that Σ_G is *symmetrical*, meaning that, if $g \in \Sigma_G$ then $g^{-1} \in \Sigma_G$, and also that Λ_G is symmetrical in the same sense.

Next we introduce the concepts of indecomposable and maximal length in the framework of graded Lie-Rinehart algebras in a similar way to the ones for other classes of graded Lie algebras and graded Leibniz algebras (see [10,11] for these notions and examples).

Definition 5.1. Let $A = \oplus_{g\in G} A_g$ be a G-graded algebra. It is said that A is *(graded)-decomposable* if $A = J \oplus K$ with J, K two $(G$-graded)-ideals of A. Otherwise, A is called *(graded)-indecomposable* or just *indecomposable* for short.

Definition 5.2. A graded Lie-Rinehart algebra (L, A) is *of maximal length* if $\dim L_g = 1$ for $g \in \Sigma_G$, as well, $\dim A_k = 1$ for $k \in \Lambda_G$.

Theorem 5.3. *Let (L, A) be a tight graded Lie-Rinehart $\mathbb{F}$-algebra of maximal length over an associative and commutative $\mathbb{F}$-algebra A. If Σ_G, Λ_G are symmetrical then*

$$L = \bigoplus_{i\in I} I_i \qquad \textit{and} \qquad A = \bigoplus_{j\in J} A_j$$

where any I_i is a non-zero (graded)-indecomposible ideal of L satisfying $[I_i, I_h] = 0$ when $i \neq h$, and any A_j is a non-zero (graded)-indecomposible ideal of A such that $A_j A_k = 0$ when $j \neq k$. Moreover, both decompositions satisfy that for any $r \in I$ there exists a unique $\tilde{r} \in J$ such that

$$A_{\tilde{r}} I_r \neq 0.$$

Furthermore, any I_r is a graded Lie-Rinehart algebra over $A_{\tilde{r}}$.

Proof. We just have to prove the indecomposability of any I_i and any A_j in Theorem 4.3. Consider some $I_i = I_{[g]}$ for $[g] \in \Sigma_G/\sim$. Suppose $I_{[g]} = J \oplus K$ with $J, K \subset L$ ideals of L satisfying $J = \oplus_{j\in[g]\cap\mathcal{J}} L_j$, $K = \oplus_{k\in[g]\cap\mathcal{K}} L_k$ being $\mathcal{J} \cup \mathcal{K} = [g]$. Given $j \in [g] \cap \mathcal{J}$ and $k \in [g] \cap \mathcal{K}$, since $j \sim k$, and using maximal length of L there exists a Σ_G-connection $\{g_1, \dots, g_n\} \subset \Sigma_G^+ \cup \Lambda_G^+$ being $g_1 = j$ such that

$$\{\{\cdots\{\{L_j, K_{g_2}\}, K_{g_3}\}, \cdots\}, K_{g_n}\} = L_k,$$

where K_g is either L_g or A_g, depending if either $g \in \Sigma_G^+$ or $g \in \Lambda_G^+$, respectively, and $\{\cdot,\cdot\}$ may represent either the product $[\cdot,\cdot]$ in L or the A-module structure, depending the considered factors. Then $L_k \in J$, a contradiction. Therefore L is indecomposable.

In a similar way we conclude A is indecomposable. Consider some $A_j = I_{[g]}$ for $[g] \in \Lambda_G/\sim$. Suppose $I_{[g]} = J \oplus K$ with $J, K \subset A$ ideals of A satisfying $J = \oplus_{j\in[g]\cap\mathcal{J}} A_j$, $K = \oplus_{k\in[g]\cap\mathcal{K}} A_k$ being $\mathcal{J} \cup \mathcal{K} = [g]$. Given $j \in [g] \cap \mathcal{J}$ and $k \in [g] \cap \mathcal{K}$, since $j \sim k$, and using maximal length of A there exists a Λ_G-connection $\{g_1, \dots, g_n\} \subset \Sigma_G^+ \cup \Lambda_G^+$ being $g_1 = j$ such that

$$\{\{\cdots\{\{A_j, K_{g_2}\}, K_{g_3}\}, \cdots\}, K_{g_n}\} = A_k,$$

where K_g is either L_g or A_g, depending if either $g \in \Sigma_G^+$ or $g \in \Lambda_G^+$, respectively, and $\{\cdot,\cdot\}$ may represent either the product $[\cdot,\cdot]$ in L or the juxtaposition product in A or the A-module structure or the anchor ρ, depending the considered factors. Then $A_k \in J$, a contradiction. Therefore, A is indecomposable. □

We recall that an element x of an arbitrary algebra $\mathfrak{A}$ is said an *element of division* if for any $y \in \mathfrak{A}$ such that $0 \neq xy = s$ (resp. $0 \neq yx = t$) there exist $v, w \in \mathfrak{A}$ satisfying $y = sv$ and $x = ws$ (resp. $y = vw$ and $x = sw$). The algebra $\mathfrak{A}$ is called *of division* if all non-zero element of $\mathfrak{A}$ is an element of division.

Given a graded algebra $\mathfrak{A} = \oplus_{g\in G}\mathfrak{A}_g$, for each homogenous component $\mathfrak{A}_g$ we denote by $I(\mathfrak{A}_g)$ to the minimal graded ideal that contains $\mathfrak{A}_g$.

Following the previous ideas we introduce the next definition.

Definition 5.4. Let $\mathfrak{A} = \oplus_{g\in\Sigma_G}\mathfrak{A}_g$ be a graded algebra. We say Σ_G is *of weak division* if for any $g, h \in \Sigma_G$ such that $\mathfrak{A}_g\mathfrak{A}_h \neq 0$ we get $\mathfrak{A}_g\mathfrak{A}_h \subset I(\mathfrak{A}_{gh})$.

Theorem 5.5. *Let (L, A) be a tight graded Lie-Rinehart $\mathbb{F}$-algebra of maximal length over an associative and commutative $\mathbb{F}$-algebra A. If Σ_G, Λ_G are symmetrical then*

$$L = \bigoplus_{i \in I} I_i \qquad \text{and} \qquad A = \bigoplus_{j \in J} A_j$$

where any I_i (resp. A_j) is a non-zero gr-simple ideal if and only if Σ_G (resp. Λ_G) is of weak division.

Proof. Suppose any I_i is gr-simple. Given $g, h \in \Sigma_G$ such that $[L_g, L_h] \neq 0$, by maximal length is $[L_g, L_h] = L_{gh}$. If we consider $I(L_{gh}) = I_{[g]}$ then $L_g, L_h \subset I(L_{gh})$. Analogously, suppose A_j is gr-simple. Given $g, h \in \Lambda_G$ such that $A_g A_h \neq 0$, by maximal length is $A_g A_h = A_{gh}$. If we consider $I(A_{gh}) = A_{[g]}$ then $A_g, A_h \subset I(A_{gh})$.

To prove the another implication, if Σ_G is of weak division, let us consider a non-zero graded ideal I of $I_{[g]}$. By maximal length $I = \oplus_{h \in \mathcal{I}} L_h$ with $\mathcal{I} \subset \Sigma_G$. Given any $h \in [g]$ we get $h \sim g$, so there exists a Σ_G-connection $\{g_1, \ldots, g_n\}$ with $g_1 = h$ satisfying by maximal length that

$$[[[L_h, L_{g_2}], L_{g_3}], \cdots, L_{g_n}] = L_g.$$

Then $L_g \subset I$, so we conclude $I = I_{[g]}$. That is, $I_{[g]}$ is a gr-simple.

Similarly, we can prove that any $A_{[g]}$ is gr-simple. If Λ_G is of weak division, let us consider a non-zero graded ideal I of $A_{[g]}$. By maximal length $I = \oplus_{h \in \mathcal{I}} A_h$ with $\mathcal{I} \subset \Lambda_G$. Given any $h \in [g]$ we get $h \sim g$, so there exists a Λ_G-connection $\{g_1, \ldots, g_n\}$ with $g_1 = h$ satisfying by maximal length that

$$(((A_h A_{g_2}) A_{g_3}) \cdots A_{g_n}) = A_g.$$

Then $A_g \subset I$, so we conclude $I = A_{[g]}$. That is, $A_{[g]}$ is a gr-simple. □

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