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An Intrinsic Version of the k-Harmonic Equation

Lígia Abrunheiro [1,2,†] and Margarida Camarinha [3,*,†]

1 Aveiro Institute of Accounting and Administration of the University of Aveiro (ISCA-UA), 3810-500 Aveiro, Portugal; abrunheiroligia@ua.pt
2 Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
3 CMUC, University of Coimbra, Department of Mathematics, 3000-143 Coimbra, Portugal
* Correspondence: mmlsc@mat.uc.pt
† These authors contributed equally to this work.

Abstract: The notion of k-harmonic curves is associated with the kth-order variational problem defined by the k-energy functional. The present paper gives a geometric formulation of this higher-order variational problem on a Riemannian manifold M and describes a generalized Legendre transformation defined from the kth-order tangent bundle T^kM to the cotangent bundle $T^*T^{k-1}M$. The intrinsic version of the Euler–Lagrange equation and the corresponding Hamiltonian equation obtained via the Legendre transformation are achieved. Geodesic and cubic polynomial interpolation is covered by this study, being explored here as harmonic and biharmonic curves. The relationship of the variational problem with the optimal control problem is also presented for the case of biharmonic curves.

Keywords: k-harmonic curves; Riemannian manifolds; Lagrangian and Hamiltonian formalism; Legendre transformation

MSC: 70H50; 70H03; 70H05; 58E20; 53B21; 58E25

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1. Introduction

Polyharmonic curves of order k in Riemannian manifolds are the critical points of the k-energy functional

$$J_k(\gamma) = \frac{1}{2}\int_0^T \left\langle \frac{D^k\gamma}{dt^k}, \frac{D^k\gamma}{dt^k} \right\rangle dt \tag{1}$$

and are described by the Euler–Lagrange equation

$$\frac{D^{2k}\gamma}{dt^{2k}} + \sum_{j=2}^{k}(-1)^j R\left(\frac{D^{2k-j}\gamma}{dt^{2k-j}}, \frac{D^{j-1}\gamma}{dt^{j-1}}\right)\frac{d\gamma}{dt} = 0. \tag{2}$$

Notice that the functional (1) is considered a higher-order version of the energy functional

$$J_1(\gamma) = \frac{1}{2}\int_0^T \left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right\rangle dt \tag{3}$$

and, in this sense, k-harmonic curves, also referred to as polyharmonic curves, higher-order geodesics, or Riemannian polynomials, are seen as a natural generalization of geodesic curves, the extremal curves of the functional (3).

The study of polyharmonic curves fits into the more general theory of polyharmonic maps between Riemannian manifolds, just as the theory of geodesics falls under that of harmonic applications. Polyharmonic maps have only recently became a subject of interest (see [1] and references therein), but biharmonic maps and, in particular, biharmonic submanifolds and curves have been extensively studied in the last decades (see, for instance, [2–6]).

There is a strong relationship between optimal control problems and variational problems, particularly concerning the variational problem associated with the k-energy functional (1). The main topic of this subject is the study of the dynamic interpolation problem, where the goal is to find the curves that minimize the 2-energy functional and satisfy some interpolation conditions. Applications to motion planning and tracking problems for nonlinear systems were the special motivation for the analysis of this second-order problem. The first steps in this direction were given by L. Noakes, G. Heinzinger, and B. Paden in [7] and by P. Crouch and F. Silva Leite in [8], where the authors obtained the necessary optimality conditions for the problem and called Riemannian cubic splines to the curves under these conditions. In the context of robotic motion planning, a natural extension of the dynamic interpolation problem to higher orders has also been developed, giving rise to the notion of higher-order splines in Riemannian manifolds [9,10].

Applications of polyharmonic curves to trajectory planning problems in robotics and computational anatomy, especially when the configuration space is a Lie group, also brought the subject to the field of geometric mechanics. The Hamiltonian structure and symmetry reductions of the polyharmonic equation have deserved special attention and have also been extended to the study of optimal control problems for mechanical control systems [11–17].

Polyharmonic curves depend on the choice of the parametrization, as happens with geodesics. From the point of view of differential geometry, these curves are studied by considering arclength parametrization. When they are seen as motion trajectories, arclength parametrization is not always possible (when motion reaches zero velocity) and is thought of as a constraint.

In this work, we present an intrinsic version of the k-harmonic equation based on the symplectic formalism for higher-order regular Lagrangians given in [18]. More specifically, we consider a geometric formulation of the kth-order variational problem on a Riemannian manifold using the framework of symplectic geomety and define a generalized Legendre transformation involving higher-order tangent and cotangent bundles. The corresponding Hamiltonian equation obtained via this Legendre transformation is also explained. This study covers some research topics of interest, such as the interpolation theories involving geodesics and cubic splines. In fact, these cases are explored in the present work as being free harmonic and biharmonic curves (without any constraints on the parameter). The relationship of the variational problem with the optimal control problem is also an interesting field of research and is presented for the case of biharmonic curves, always with emphasis on the intrinsic approach.

The structure of the paper is as follows. In Section 2, we recall some important notions from the geometry of higher-order tangent bundles. The variational problem associated with the k-harmonic curves is studied in Section 3. We begin by showing that the kth-order Lagrangian is regular and then adapt the Lagrangian formalism of higher order to the problem being studied. A higher-order Legendre transformation that allows relating the Lagrangian and the Hamiltonian formalisms is described. Section 4 is devoted to the first-order case, which corresponds to the classical geodesic problem. In Section 5, the formalism for biharmonic curves is explored in more depth and, in this case, the associated optimal control problem is also exposed.

2. Higher-Order Tangent Bundles

Let M be a differentiable manifold of finite dimension n. Consider a local coordinate system $(U, x^1, \dots, x^n)$ on M, simply denoted by (x^i). Throughout this paper, we use similar abbreviations for the coordinate notations. Let k be an integer greater than or equal to 1.

In this work, we are interested in the formalism of higher-order tangent bundles. In order to introduce the geometry of those bundles (see [18] for further details), we consider the well-defined equivalence relationship on the set of smooth curves in M, as follows:

> We say that two smooth curves in M, γ_1 and γ_2, defined on an interval $(-a, a)$ with $a \in \mathbb{R}$, have a contact of order k at 0 if $\gamma_1(0) = \gamma_2(0) = x$, and for a local

coordinate system (U, φ) on M around x, the derivatives of $\varphi \circ \gamma_1$ and $\varphi \circ \gamma_2$ up to order k, included, coincide at 0.

The equivalence class determined by a curve γ is represented by $[\gamma]_0^k$ and is called *k-jet* or *k-velocity*.

Definition 1. *The tangent bundle of order k of M is the set of all equivalence classes of curves in M that have contact of order k and is denoted by T^kM.*

The following characteristics of the tangent bundle T^kM should be emphasized:

- T^kM is a $(k+1)n$-dimensional manifold and a fibered manifold over M with projection $\pi_k : T^kM \to M,\ [\gamma]_0^k \mapsto \gamma(0) = x$.
- T^kM has natural local coordinates $\left(\pi_k^{-1}(U), x_0^i; x_1^i; x_2^i; \ldots; x_k^i\right)$ induced by (x^i), where

$$x_l^i : \pi_k^{-1}(U) \subset T^kM \to \mathbb{R},\ [\gamma]_0^k \mapsto \frac{d^l}{dt^l}\left(x^i \circ \gamma\right)(t)\bigg|_{t=0},$$

  for $l = 0, \ldots, k$ and $i = 1, \ldots, n$.
- If $k = 0$, T^0M is identified with the manifold M and for $k = 1$, T^1M is just the tangent bundle of M, TM.
- There are canonical projections $\tau_k^l : T^kM \to T^lM,\ [\gamma]_0^k \mapsto [\gamma]_0^l,\ l = 0, \ldots, k$, which define several different fibered structures on T^kM. Locally,

$$\tau_k^l(x_0^i; x_1^i; x_2^i; \ldots; x_k^i) = (x_0^i; x_1^i; x_2^i; \ldots; x_l^i). \tag{4}$$

  Note that $\tau_k^0 = \pi_k$. The tangent applications $T\tau_k^l : T(T^kM) \to T(T^lM)$ are defined by

$$\left(T\tau_k^l\right)(X) = \sum_{i=1}^{n} \sum_{j=0}^{l} X_i^j \frac{\partial}{\partial x_j^i}, \tag{5}$$

  for each $X = \sum_{i=1}^n \sum_{j=0}^k X_i^j (\partial/\partial x_j^i) \in T_{[\gamma]_0^k} T^kM$, with $[\gamma]_0^k \in T^kM$.

Definition 2. *Let γ be a smooth curve in M. The lift to T^kM of γ is a smooth curve in T^kM denoted by γ_k and defined by $\gamma_k(t) = [\gamma_t]_0^k$, where $\gamma_t(s) = \gamma(t+s)$.*

If γ is locally given by (x^i), then $(x^i; dx^i/dt; \ldots; d^kx^i/dt^k)$ locally represents γ_k.

2.1. The Liouville Vector Field of Higher Order

In order to introduce the notion of a Liouville vector field of higher order, we begin by defining k vertical bundles of T^kM determined by foliations of type (4) of T^kM. Let $r = 1, \ldots, k$.

Definition 3. *The vertical bundle of T^kM over $T^{r-1}M$, denoted by $V^{\tau_k^{r-1}}(T^kM)$, is the set of all tangent vectors to T^kM that are projected onto zero by $T\tau_k^{r-1}$.*

According to (4) and (5), if $[\gamma]_0^k \in T^kM$ and X is an element of $V^{\tau_k^{r-1}}(T^kM)$ at $[\gamma]_0^k$, then X is locally written as

$$X = \sum_{i=1}^{n} \sum_{j=r}^{k} X_i^j (\partial/\partial x_j^i).$$

Remark 1. *In the particular case when $k = 1$ and $r = 1$, the projection τ_k^{r-1} is just the canonical projection of the tangent bundle TM, $\tau_1^0 = \pi_M : TM \to M$. The only vertical bundle is TM over*

M, usually denoted by VTM, whose elements are tangent vectors of TM and which are projected onto zero for $T\pi_M$.

Now consider:

- The canonical applications

$$j_r: \begin{array}{ccc} T^kM & \longrightarrow & T(T^{r-1}M) \\ [\gamma]_0^k & \longmapsto & [\gamma_{r-1}]_0^1 \end{array}. \tag{6}$$

- The vector bundle isomorphisms over T^kM

$$h_r \ : \ T^kM \times_{T^{r-1}M} T(T^{r-1}M) \ \longrightarrow \ V^{\tau_k^{k-r}}(T^kM)$$

locally defined by

$$\begin{aligned} & h_r\left(x_0^i;\ldots;x_k^i,x_0^i;\ldots;x_{r-1}^i,X_i^0,\ldots,X_i^{r-1}\right) \\ = & \sum_{i=1}^n\sum_{j=1}^r \frac{(k-r+j)!}{(j-1)!}X_i^{j-1}\frac{\partial}{\partial x_{k-r+j}^i}, \end{aligned} \tag{7}$$

where $T^kM \times_{T^{r-1}M} T(T^{r-1}M)$ is the induced bundle of $T(T^{r-1}M)$ via τ_k^{r-1}.

Remark 2. *If* $k=1$*, we have just one vectorial bundle isomorphism over* TM*,*

$$h_1 \ : \ TM \times_M TM \ \longrightarrow \ V(TM),$$

which is locally given by $h_1(x_0,x_1,x_0,X^0) = (x_0,x_1,0,X^0)$.

If $k=2$*, we can define two vectorial bundle isomorphisms over* T^2M*,*

$$h_1 \ : \ T^2M \times_M TM \ \longrightarrow \ V^{\tau_2^1}(T^2M)$$

$$h_2 \ : \ T^2M \times_{TM} T(TM) \ \longrightarrow \ V^{\tau_2^0}(T^2M),$$

which is locally defined by

$$h_1\left(x_0,x_1,x_2,x_0,X^0\right) = \left(x_0,x_1,x_2,0,0,2X^0\right)$$

$$h_2\left(x_0,x_1,x_2,x_0,x_1,X^0,X^1\right) = \left(x_0,x_1,x_2,0,X^0,2X^1\right).$$

Definition 4. *The canonical vector field of order r on* T^kM *is the vector field*

$$C_r : T^kM \longrightarrow V^{\tau_k^{r-1}}(T^kM) \subset T(T^kM)$$

defined by the composition $C_r = h_{k-r+1} \circ (Id \times j_{k-r+1})$

$$T^kM \xrightarrow{Id\times j_{k-r+1}} T^kM \times_{T^{k-r}M} T(T^{k-r}M) \xrightarrow{h_{k-r+1}} V^{\tau_k^{r-1}}(T^kM),$$

where Id is the identity map in T^kM*. The Liouville vector field of order k is the canonical vector field of order* 1 *on* T^kM*,* C_1*.*

Locally, we have

$$C_r = \sum_{i=1}^n \sum_{j=1}^{k-r+1} \frac{(r+j-1)!}{(j-1)!}x_j^i\frac{\partial}{\partial x_{r+j-1}^i}. \tag{8}$$

For $r = 1$, $C_1 = \sum_{i=1}^{n} \sum_{j=1}^{k} j x_j^i (\partial / \partial x_j^i)$.

Remark 3. *If $k = 1$, then $j_1 : TM \to TM$ is the identity map, and we have just the Liouville vector field on TM, $C_1 : TM \to V(TM) \subset T(TM)$:*

$$C_1 = \sum_{i=1}^{n} x_1^i \frac{\partial}{\partial x_1^i}.$$

If $k = 2$, we have two canonical vector fields on T^2M, the Liouville vector fields $C_1 : T^2M \to V^{\tau_2^0}(T^2M) \subset T(T^2M)$ and $C_2 : T^2M \to V^{\tau_2^1}(T^2M) \subset T(T^2M)$, which are locally given by

$$C_1 = \sum_{i=1}^{n} \left(x_1^i \frac{\partial}{\partial x_1^i} + 2x_2^i \frac{\partial}{\partial x_2^i} \right) \quad \textit{and} \quad C_2 = \sum_{i=1}^{n} 2x_1^i \frac{\partial}{\partial x_2^i}.$$

2.2. The Canonical Almost-Tangent Structure of Higher Order

We now generalize to higher order the notion of canonical almost-tangent structures. For $r = 1, \dots, k$, consider the following:

- The vector bundle isomorphisms over T^kM defined by (7), h_{k-r+1}.
- The canonical inclusions $i_{k-r+1} : V^{\tau_k^{r-1}}(T^kM) \to T(T^kM)$.
- The vectorial bundle homomorphisms over T^kM given by

$$\begin{array}{rccc} s_r : & T(T^kM) & \longrightarrow & T^kM \times_{T^{k-r}M} T(T^{k-r}M) \\ & X & \longmapsto & \left(\pi_{T^kM}(X), T\tau_k^{k-r}(X) \right), \end{array}$$

where $\pi_{T^kM} : T(T^kM) \to T^kM$ is the canonical projection. Note that

$$Ker\,(s_r) = V^{\tau_k^{k-r}}(T^kM).$$

Definition 5. *The endomorphism $J_r : T(T^kM) \to T(T^kM)$ defined by*

$$J_r = i_{k-r+1} \circ h_{k-r+1} \circ s_r,$$

is called the vertical endomorphism of order r of $T(T^kM)$,

$$T(T^kM) \xrightarrow{s_r} T^kM \times_{T^{k-r}M} T(T^{k-r}M) \xrightarrow{h_{k-r+1}} V^{\tau_k^{r-1}}(T^kM) \xrightarrow{i_{k-r+1}} T(T^kM) \cdot$$

The vertical endomorphism J_1 is called a canonical almost-tangent structure of order k on T^kM.

Locally, we have

$$J_r = \sum_{i=1}^{n} \sum_{j=1}^{k-r+1} \frac{(r+j-1)!}{(j-1)!} \frac{\partial}{\partial x_{r+j-1}^i} \otimes dx_{j-1}^i.$$

Proposition 1. *The vertical endomorphism J_r of order r of $T(T^kM)$ has a constant rank equal to $(k-r+1)n$ and satisfies*

$$(J_r)^s = \begin{cases} 0 & \textit{if} \quad rs \geq k+1 \\ J_{rs} & \textit{if} \quad rs < k+1. \end{cases}$$

According to the above proposition, J_1 is an almost-tangent structure on T^kM since $(J_1)^{k+1} = 0$ and $rank\, J_1 = kn$.

Remark 4. *If $k = 2$, we have two vertical endomorphisms of $T(T^2M)$, J_1 and J_2, whose matrix representations are, respectively, given by*

$$J_1 = \begin{bmatrix} 0 & 0 & 0 \\ I_n & 0 & 0 \\ 0 & 2I_n & 0 \end{bmatrix} \quad \text{and} \quad J_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2I_n & 0 & 0 \end{bmatrix},$$

where I_n and 0 are the identity matrix and the null matrix of order n, respectively. In this case, $(J_1)^3 = 0$ and rank $J_1 = 2n$, so J_1 determines an almost-tangent structure of order 2 on T^2M, the so-called canonical almost-tangent structure of order 2 on T^2M. Notice that $(J_1)^2 = J_2$.

Proposition 2. *Let J_i be the vertical endomorphism of order i of $T(T^kM)$ and let C_i be the canonical vector field of order i on T^kM (i = r, s). The following relationships are satisfied:*

$$J_rC_s = \begin{cases} 0 & \text{if} \quad r+s \geq k+1 \\ C_{r+s} & \text{if} \quad r+s < k+1 \end{cases}$$

$$[C_r, J_s] = \begin{cases} 0 & \text{if} \quad r+s > k+1 \\ -sJ_{r+s-1} & \text{if} \quad r+s \leq k+1 \end{cases}$$

$$[J_r, J_s] = 0,$$

with $r, s = 1, \ldots, k$.

Notice that, on T^2M, we have $C_2 = J_1C_1$.

Definition 6. *The vertical differentiation of order r on the exterior algebra of T^kM, denoted by $d_{J_r} : \bigwedge^p(T^kM) \to \bigwedge^{p+1}(T^kM)$, is given by the commutator*

$$d_{J_r} = \left[i_{J_r}, d\right] = i_{J_r}d - di_{J_r},$$

where d is the exterior differentiation and i_{J_r} is the inner product of J_r.

Proposition 3. *The vertical differentiation d_{J_r} of order r on the exterior algebra on T^kM satisfies, for each function f on T^kM, the following relationship:*

$$d_{J_r}f = J_r^*(df) \quad \text{and} \quad d_{J_r}df = -d(J_r^*df).$$

Locally,

$$\begin{aligned} d_{J_r}f &= \sum_{i=1}^{n}\sum_{l=r}^{k} \frac{l!}{(l-r)!}\frac{\partial f}{\partial x_l^i}dx_{l-r}^i \\ d_{J_r}(dx_l^i) &= 0, \text{ for } i = 1, \ldots, n \quad \text{and} \quad l = 0, \ldots, k. \end{aligned} \tag{9}$$

Remark 5. *Notice that on TM, we have*

$$d_{J_1}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_1^i}dx_0^i, \quad \text{where } f \text{ is a function on } TM.$$

Moreover, on T^2M, we obtain

$$d_{J_1}f = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_1^i}dx_0^i + 2\frac{\partial f}{\partial x_2^i}dx_1^i\right) \quad \text{and} \quad d_{J_2}f = \sum_{i=1}^{n} 2\frac{\partial f}{\partial x_2^i}dx_0^i,$$

where f is a function on T^2M.

2.3. The Tulczyjew Differential Operator

Definition 7. *The Tulczyjew differential operator or total time derivative operator on T^rM is the operator d_T that maps each function f on $T^{r-1}M$ to a function d_Tf on T^rM such that*

$$d_Tf([\gamma]_0^r) = j_r([\gamma]_0^r)f,$$

for each $[\gamma]_0^r \in T^rM$, with $j_r : T^rM \to T(T^{r-1}M)$ defined in (6).

In local coordinates, we obtain

$$d_T = \sum_{i=1}^{n}\sum_{j=0}^{r-1} x^i_{j+1}\frac{\partial}{\partial x^i_j}. \tag{10}$$

We shall mention that the total time derivative d_T may be naturally extended to an operator that acts on differentiable forms. This operator maps p-forms on T^kM into p-forms on $T^{k+1}M$. Moreover, we have $d_Td = dd_T$, where d is the exterior differentiation defined on the exterior algebra on T^kM.

Definition 8. *Let X be a vector field along a curve γ in M. The kth-order lift X_k of X is a vector field along the lifted curve γ_k, $X_k : T^kM \to T(T^kM)$, satisfying $(d/dt)\circ X_{k-1} = X_k \circ d_T$.*

Note that X_k is obtained by applying repeated lifts to X. Its local coordinate expression is given by

$$X_k(t) = \sum_{i=1}^{n}\sum_{j=0}^{k}\frac{d^jX^i}{dt^j}\frac{\partial}{\partial x^i_j}\bigg|_{\gamma_k(t)},$$

where $X(t) = \sum_{i=1}^{n} X^i\,(\partial/\partial x^i)\big|_{\gamma(t)}$.

3. Higher-Order Variational Problem

From now on, we take M to be a Riemannian manifold with the Riemannian metric $\langle\cdot,\cdot\rangle$. The Levi–Civita connection on M is denoted by ∇. Let DX/dt represent the covariant derivative $\nabla_{(d\gamma/dt)}X$ along the curve γ in M, with X being a vector field along γ. Set $D^j\gamma/dt^j = D(D^{j-1}\gamma/dt^{j-1})/dt$ as the jth-order covariant derivative of γ, where $j \geq 2$ and $D\gamma/dt = d\gamma/dt$. Consider the following sign convention for the curvature tensor field R:

$$R(X,Y) = [\nabla_X,\nabla_Y] - \nabla_{[X,Y]}.$$

See [19] for more details about Riemannian geometry.

Remember that if a curve γ in M is locally represented by (x^i), then γ_k is locally represented by $(x^i; dx^i/dt; \ldots; d^kx^i/dt^k)$. Thus, the velocity vector field along the curve γ is $d\gamma/dt = \sum_{i=1}^{n}(dx^i/dt)\,(\partial/\partial x^i)\big|_{\gamma(t)}$. Moreover, given a vector field $X = \sum_{i=1}^{n} X^i(\partial/\partial x^i)$, the covariant derivative of X along γ is given by

$$\frac{DX}{dt} = \sum_{k=1}^{n}\left(\frac{dX^k}{dt} + \sum_{i,j=1}^{n}\Gamma^k_{ij}\frac{dx^i}{dt}X^j\right)\frac{\partial}{\partial x^k}\bigg|_{\gamma(t)}.$$

In particular, the covariant acceleration of γ can be written as

$$\frac{D^2\gamma}{dt^2} = \sum_{k=1}^{n}\left(\ddot{x}^k + \sum_{i,j=1}^{n}\Gamma^k_{ij}\dot{x}^i\dot{x}^j\right)\frac{\partial}{\partial x^k}\bigg|_{\gamma(t)},$$

where we simplify the notations of the derivatives, using $\dot{x}^i$ for the first derivative dx^i/dt and similar notations for the higher-order derivatives. Here, Γ^k_{ij} are the Christoffel symbols defining the Riemannian connection, which can be obtained using the identity

$$\Gamma_{ij}^{k} = \frac{1}{2}\sum_{l=1}^{n} g^{kl}\left(\frac{\partial g_{jl}}{\partial x^{i}} + \frac{\partial g_{li}}{\partial x^{j}} - \frac{\partial g_{ij}}{\partial x^{l}}\right),$$

where g_{ij} are the components of the Riemannian metric and $[g^{ij}]_{1\leq i,j\leq n}$ is the inverse matrix of the matrix $[g_{ij}]_{1\leq i,j\leq n}$.

Using the Riemannian structure of M, we can also define the bundle morphism A_k from T^kM to TM given by $A_k\left([\gamma]_0^k\right) = \left(D^k\gamma/dt^k\right)(0)$. The morphism A_k can be expressed as follows.

$$A_k\left([\gamma]_0^k\right) = \sum_{r=1}^{n} A_k^r \frac{\partial}{\partial x^r}\bigg|_{\gamma(0)},$$

where

$$A_k^r(x_0,\ldots,x_k) = x_k^r + B_{k-1}^r(x_0,\ldots,x_{k-1}) \tag{11}$$

and

$$B_k^r(x_0,\ldots,x_k) = \sum_{l=0}^{k-1}\sum_{i=1}^{n}\frac{\partial}{\partial x_l^i}B_{k-1}^r(x_0,\ldots,x_{k-1})x_{l+1}^i + \sum_{i,j=1}^{n}\Gamma_{ij}^k x_1^i A_{k-1}^j(x_0,\ldots,x_{k-1}).$$

3.1. The k-Energy Functional

Let $\mathcal{C}_k$ be the class of smooth curves $\gamma : [0,T] \to M$ satisfying the boundary conditions

$$\gamma(0) = x_0, \quad \gamma(T) = x_T, \quad \frac{D^j\gamma}{dt^j}(0) = y_{0j}, \quad \frac{D^j\gamma}{dt^j}(T) = y_{Tj}, \quad j = 1,\ldots,k-1,$$

where $x_0, x_T \in M$, y_{ij} are fixed n-vectors ($i = 0, T$; $j = 1,\ldots,k-1$) and $T \in \mathbb{R}^+$. Consider the kth-order variational problem described by the action functional J_k defined by (1). From the point of view of intrinsic variational calculus, J_k can be written as

$$J_k(\gamma) = \int_0^T \gamma_k^*(L)dt = \int_0^T L(x^i; dx^i/dt; \ldots; d^kx^i/dt^k)dt,$$

where L is the Lagrangian of order k associated with the problem. Therefore, the Lagrangian of the problem, $L : T^kM \to \mathbb{R}$, is defined, for each $[\gamma]_0^k \in T^kM$, by

$$L\left([\gamma]_0^k\right) = \frac{1}{2}\left\langle A_k\left([\gamma]_0^k\right), A_k\left([\gamma]_0^k\right)\right\rangle, \tag{12}$$

where A_k is given by (11). We may remark that (12) may be locally expressed by

$$L(x_0^i; x_1^i; x_2^i; \ldots; x_k^i) = \frac{1}{2}\sum_{i,j=1}^{n} g_{ij}A_k^i A_k^j.$$

Differentiating L, we obtain

$$\frac{\partial L}{\partial x_k^i} = \sum_{j=1}^{n} g_{ij}A_k^j(x_0,\ldots,x_k) = \left\langle A_k(x_0,\ldots,x_k), \frac{\partial}{\partial x^i}\right\rangle.$$

Furthermore, $\left[\left(\partial^2 L/\partial x_k^i \partial x_k^j\right)\right]_{1\leq i,j\leq n} = [g_{ij}]_{1\leq i,j\leq n}$ and, since this is the matrix that represents the Riemannian metric, we have the guarantee that the Lagrangian L is regular.

3.2. Intrinsic Version of the Euler–Lagrange Equation

Given a curve γ in $\mathcal{C}_k$, the tangent space to $\mathcal{C}_k$ at γ, $T_\gamma\mathcal{C}_k$, is constituted by smooth vector fields X along γ such that $X_j(0) = X_j(T) = 0$ for $j = 1,\ldots,k-1$, where X_j is the jth-order lift of X. The variation of the curve γ is given by a smooth 1-parameter family of curves $\gamma_\epsilon \in \mathcal{C}_k$ with $\gamma_0 = \gamma$, and the corresponding variation vector field $X \in T_\gamma\mathcal{C}$ is defined by $X(\gamma(t)) = (d\gamma_\epsilon/d\epsilon)(t)|_{t=0}$. The first-order variation of J_k associated with X takes the form

$$dJ_k(\gamma)(X) = \frac{d}{d\epsilon}J_k(\gamma_\epsilon)\bigg|_{\epsilon=0} = \int_0^T X_k(L)dt.$$

Hamilton's variational principle establishes that a curve $\gamma \in \mathcal{C}_k$ is a critical curve of $J_k : \mathcal{C}_k \to \mathbb{R}$ if, for an arbitrary variation vector $X \in T_\gamma\mathcal{C}_k$, we have $dJ_k(\gamma)(X) = 0$. If the Lagrangian L is regular (which is the case that we are considering), the arbitrariness of the variation vector field X in the condition for the action integral to be stationary,

$$dJ_k(\gamma)(X) = \int_0^T X_k(L)dt = 0, \quad \forall X \in T_\gamma\mathcal{C}_k,$$

gives the geometric version of the Euler–Lagrange equation

$$i_{X_E}\omega_L = dE, \tag{13}$$

where ω_L is the Poincaré–Cartan 2-form on $T^{2k-1}M$ and $E : T^{2k-1}M \to \mathbb{R}$ is the energy function associated with $L : T^kM \to \mathbb{R}$, defined, respectively, by

$$\omega_L = \sum_{r=1}^{k}(-1)^r\frac{1}{r!}d_T^{r-1}dd_{J_r}L \qquad \text{and} \qquad E = \sum_{r=1}^{k}(-1)^{r-1}\frac{1}{r!}d_T^{r-1}(C_rL) - L.$$

Consider the one-form

$$\alpha_L = \sum_{r=1}^{k}(-1)^{r-1}\frac{1}{r!}d_T^{r-1}d_{J_r}L. \tag{14}$$

Proposition 4. *The one-form α_L on $T^{2k-1}M$ is semibasic of type k; that is, $\alpha_L \in Im(J_k^*)$.*

One calls α_L the *Jacobi–Ostrogradsky form* associated with the Lagrangian L. We have $\omega_L = -d\alpha_L$. Locally,

$$\alpha_L = \sum_{i=1}^{n}\sum_{r=0}^{k-1} p_r^i dx_r^i, \quad \omega_L = \sum_{i=1}^{n}\sum_{r=0}^{k-1} dx_r^i \wedge dp_r^i \quad \text{and} \quad E = \sum_{i=1}^{n}\sum_{r=0}^{k-1} p_r^i x_{r+1}^i - L,$$

where $p_r^i = \sum_{l=0}^{k-r-1}(-1)^l d_T^l(\partial L/\partial x_{r+l+1}^i)$, $i = 1, \ldots, n$.

The Euler-Lagrange Equation (13) uniquely defines the vector field X_E on $T^{2k-1}M$ since, due to the regularity of the Lagrangian L, ω_L is sympletic (see [18]). Moreover, since $J_1X_E = C_1$, X_E is a semispray on $T^{2k-1}M$ of type 1, which represents the kth-order differential Equation (2). This means that the integral curves of X_E are lifts to T^kM of the curves in M satisfying the Euler–Lagrange Equation (2).

We also remark that Equation (2) can be rewritten in local coordinates as follows:

$$\sum_{j=0}^{k}(-1)^j\frac{d^j}{dt^j}\left(\frac{\partial L}{\partial x_j^i}\right) = 0, \quad i = 1, \ldots, n.$$

3.3. Generalized Legendre Transformation and the Hamiltonian Approach

Proposition 4 allows us to conclude that the Jacobi–Ostrogradsky form (14), α_L, is semibasic of type k and consequently determines, via the identity

$$\alpha_L = \ll Leg \circ \pi_{T^{2k-1}M}, T\tau_{2k-1}^1 \gg,$$

the Legendre transformation Leg (see [18] for more details),

$$\begin{array}{ccc} T^{2k-1}M & \xrightarrow{\quad Leg \quad} & T^*T^{k-1}M \\ & {\scriptstyle \tau^{k-1}_{2k-1}} \searrow \qquad \swarrow {\scriptstyle \tau_{T^{k-1}M}} & \\ & T^{k-1}M & \end{array} \qquad ,$$

where $\ll .,. \gg$ represents the pairing duality of vectors and covectors on $T^{k-1}M$, and $\pi_{T^{2k-1}M} : TT^{2k-1}M \to T^{2k-1}M$, $\tau^{k-1}_{2k-1} : T^{2k-1}M \to T^{k-1}M$ and $\tau_{T^{k-1}M} : T^*T^{k-1}M \to T^{k-1}M$ are the natural projections. Locally, we have

$$Leg(x^i_0; x^i_1; \ldots; x^i_{2k-2}; x^i_{2k-1}) = (x^i_0; x^i_1; \ldots; x^i_{k-1}; \alpha^0_i; \alpha^1_i; \ldots; \alpha^{k-1}_i),$$

where α^r_i, $r = 0, 1, \ldots, k-1$, are the real functions defined by (14).

When Leg is a diffeomorphism, we say that the Lagrangian L is *hyper-regular*, and we have a symplectomorphism from $(T^{2k-1}M, \omega_L)$ to $(T^*T^{k-1}M, \omega_1)$, where ω_1 is the symplectic canonical form on $T^*T^{k-1}M$. Under the hyper-regularity condition, we can consider the Hamiltonian energy function associated with L given by

$$H_L = E \circ Leg^{-1},$$

and the system $(T^{2k-1}M, \omega_L, E)$ is associated with the Hamiltonian system $(T^*T^{k-1}M, \omega_1, H_L)$. The dynamics of the Hamiltonian system is described by

$$i_{X_{H_L}} \omega_1 = dH_L, \tag{15}$$

and the Hamiltonian vector field X_{H_L} defined by (15) verifies

$$X_{H_L} = (Leg)_* X_E = T(Leg) \circ X_E \circ Leg^{-1}.$$

The Hamiltonian function $H_L : T^*T^{k-1}M \to \mathbb{R}$ is locally given by

$$H_L(x^i_0; x^i_1; \ldots; x^i_{k-1}; p^0_i; p^1_i; \ldots; p^{k-1}_i) = \sum_{i=1}^{n} \sum_{r=0}^{k-1} p^r_i x^i_{r+1} - L(x^i_0; x^i_1; \ldots; x^i_{k-1}).$$

4. First-Order Variational Problem

4.1. The Energy Functional

We consider the first-order variational problem defined by the energy functional (3). The Euler–Lagrange equation is the geodesic equation

$$\frac{D^2\gamma}{dt^2} = 0.$$

If we represent the curve γ locally by (x^i), the geodesic equation can be rewritten in local coordinates as

$$\ddot{x}^k + \sum_{i,j=1}^{n} \Gamma^k_{ij} \dot{x}^i \dot{x}^j = 0, \ k = 1, \ldots, n.$$

The lift of a curve γ to TM, γ_1, is locally represented by $(x^i, \dot{x}^i)$, and the value of the energy functional (3) can be written as

$$J_2(\gamma) = \int_0^T \gamma_2^*(L) dt = \int_0^T L(x^i; \dot{x}^i; \ddot{x}^i) dt,$$

where L is the Lagrangian of order 2 associated with the problem. Therefore, the Lagrangian $L : TM \to \mathbb{R}$ is defined, for each $y \in TM$, by

$$L(y) = \frac{1}{2} \langle y, y \rangle. \tag{16}$$

Using the canonical local coordinates $(x^i; y^i)$ of TM, the Lagrangian L may be locally expressed by

$$L(x^i; y^i) = \frac{1}{2} \sum_{i,j=1}^{n} g_{ij} y^i y^j,$$

4.2. Intrinsic Version of the Geodesic Equation

We may remark that we can associate with a Lagrangian $L : TM \to \mathbb{R}$ the following structures:

- The Poincaré–Cartan 2-form on TM given by

$$\omega_L = -dd_{J_1} L,$$

  where d and d_{J_1} represent, on the exterior algebra of differentiable forms on TM, the usual exterior differentiation and the vertical differentiation of order 1 defined by (9), respectively. In other words, $\omega_L = -d\alpha_L$, where α_L is the Jacobi–Ostrogradsky form on TM associated with L by

$$\alpha_L = d_{J_1} L,$$

  with $d_{J_1} L = \sum_{i=1}^{n} (\partial L / \partial y^i) dx^i$.
- The energy function $E : TM \to \mathbb{R}$ is defined by

$$E = C_1 L - L,$$

  with $C_1 L = \sum_{i=1}^{n} (\partial L / \partial y^i) y^i$, where we have used (8).

The following result allows us to obtain an expression for the Poincaré–Cartan 2-form ω_L associated with the Lagrangian (16).

Proposition 5. *The Jacobi–Ostrogradsky form α_L associated with the Lagrangian (16) is given by*

$$\alpha_L = \sum_{i=1}^{n} \alpha_i^0 dx^i,$$

where α_i^0 are the real functions defined by

$$\alpha_i^0(y) = \left\langle y, \frac{\partial}{\partial x^i} \right\rangle.$$

Corollary 1. *The Poincaré–Cartan 2-form ω_L associated with the Lagrangian (16) is given by*

$$\omega_L = \sum_{i=1}^{n} dx^i \wedge d\alpha_i^0.$$

The expression for the energy function E associated with the Lagrangian (16) is given in the following result.

Proposition 6. *The energy function $E : TM \to \mathbb{R}$ associated with the Lagrangian (16) is expressed by*

$$E(y) = \frac{1}{2} \langle y, y \rangle,$$

and coincides with the Lagrangian (16).

Remark 6. *The energy function E associated with the Lagrangian (16) is written along the lift γ_1 to TM of the curve γ as*

$$\gamma_1^*(E) = \frac{1}{2}\left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right\rangle.$$

The Equation (13) uniquely defines the vector field X_E on TM. X_E is called the *geodesic vector field*. In local coordinates, the geodesic vector field X_E can be written as

$$X_E(x^i; y^i) = \sum_{k=1}^{n}\left(y^k \frac{\partial}{\partial x^k} - \sum_{i,j=1}^{n} \Gamma_{ij}^k y^i y^j \frac{\partial}{\partial y^k} \right).$$

The integral curves of X_E are the lifts of the geodesics in M to TM. The corresponding flow is called the *geodesic flow*.

The classical geodesic equation can be deduced directly from Equation (13) as follows. The equations $i_{X_E}\omega_L(\partial/\partial x^j) = dE(\partial/\partial x^j),\ j = 1, \ldots, n$, are equivalent to

$$\sum_{i=1}^{n} y^i \frac{\partial \alpha_i^0}{\partial x^j} - X_E\left(\alpha_j^0\right) = \frac{\partial E}{\partial x^j},\ j = 1, \ldots, n.$$

Considering this equation along the lift γ_1 to TM of the curve γ, we obtain the equations

$$\frac{\partial L}{\partial x^j} - \frac{d}{dt}\left(\alpha_j^0\right) = 0,\ j = 1, \ldots, n. \tag{17}$$

Along γ_1, we have

$$\frac{\partial L}{\partial x^j} = \left\langle \frac{d\gamma}{dt}, \nabla_{\frac{d\gamma}{dt}} \frac{\partial}{\partial x^j} \right\rangle$$

and

$$\frac{d}{dt}(\alpha_j^0) = \left\langle \frac{d\gamma}{dt}, \nabla_{\frac{d\gamma}{dt}} \frac{\partial}{\partial x^j} \right\rangle - \left\langle \frac{D^2\gamma}{dt^2}, \frac{\partial}{\partial x^j} \right\rangle,\ j = 1, \ldots, n.$$

Consequently, the Equation (17) takes the form

$$\left\langle \frac{D^2\gamma}{dt^2}, \frac{\partial}{\partial x^j} \right\rangle = 0,\ j = 1, \ldots, n.,$$

and the geodesic equation follows.

Observe that Equation (17) can also be rewritten as follows:

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) = 0,\ i = 1, \ldots, n.$$

4.3. The Cogeodesic Flow

The Legendre transformation Leg,

$$\begin{array}{ccc} TM & \xrightarrow{Leg} & T^*M \\ & {\scriptstyle \tau_1^1} \searrow \quad \swarrow {\scriptstyle \tau_M} & \\ & M & \end{array},$$

coincides with the isomorphism $\flat$, with inverse $\sharp$, which are the musical isomorphisms with respect to the Riemannian metric. That is, $Leg(y) = y^\flat$, with

$$y^\flat(z) = \langle y, z\rangle \Leftrightarrow p(z) = \left\langle p^\sharp, z \right\rangle, y, z \in T_xM, p \in T_x^*M, x \in M.$$

Locally, we have

$$Leg(x^i; y^i) = \left(x^i; \sum_{j=1}^{n} g_{ij} y^j \right),$$

with inverse

$$Leg^{-1}(x^i; p_i) = \left(x^i; \sum_{j=1}^{n} g^{ij} p_j\right).$$

We have a symplectomorphism

$$Leg : (TM, \omega_L) \to (T^*M, \omega_1),$$

where ω_1 is the symplectic canonical form on T^*M. This means that

$$\omega_L = Leg^* \omega_1.$$

The Hamiltonian function $H_L : T^*M \to \mathbb{R}$ is given by

$$H_L(p) = \frac{1}{2} p\left(p^\sharp\right).$$

Locally, H_L can be written as follows.

$$H_L(x^i; p_i) = \frac{1}{2} \sum_{i,j=1}^{n} g^{ij} p_i p_j.$$

The dynamics of the Hamiltonian system (T^*M, ω_1, H_L) is described by (15) and is locally expressed by the following equations.

$$\begin{aligned} \dot{x}^i &= \sum_{j=1}^{n} g^{ij} p_j \\ \dot{p}_i &= -\frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial}{\partial x^i} g^{jk} p_j p_k, \end{aligned}$$

for $i = 1, \ldots, n$. The corresponding Hamiltonian flow is called the *cogeodesic flow*.

5. Second-Order Variational Problem

In this section, the attention goes to the tangent bundles T^2M and T^3M. The canonical local coordinates on T^2M and T^3M will be denoted by $(x^i; y^i; u^i)$ and $(x^i; y^i; u^i; v^i)$, respectively.

5.1. The Bienergy Functional

We see now the second-order variational problem associated with the 2-energy functional

$$J_2(\gamma) = \frac{1}{2} \int_0^T \left\langle \frac{D^2\gamma}{dt^2}, \frac{D^2\gamma}{dt^2} \right\rangle dt.$$

The Euler–Lagrange equation for this problem is the fourth-order differential equation

$$\frac{D^4\gamma}{dt^4} + R\left(\frac{D^2\gamma}{dt^2}, \frac{d\gamma}{dt}\right)\frac{d\gamma}{dt} = 0. \tag{18}$$

The solutions of the above equation are biharmonic curves, better known as Riemannian cubic polynomials on M.

The lift γ_2 of a curve to T^2M γ is locally represented by $(x^i; \dot{x}^i; \ddot{x}^i)$. The value of the 2-energy functional J_2 at γ can be written as

$$J_2(\gamma) = \int_0^T \gamma_2^*(L) dt = \int_0^T L(x^i; \dot{x}^i; \ddot{x}^i) dt,$$

with L being the Lagrangian of order 2 associated with the problem. Therefore, the Lagrangian $L : T^2M \to \mathbb{R}$ is defined, for each $[\gamma]_0^2 \in T^2M$, by

$$L\left([\gamma]_0^2\right) = \frac{1}{2}\left\langle A_2\left([\gamma]_0^2\right), A_2\left([\gamma]_0^2\right)\right\rangle, \tag{19}$$

where $A_2 = K \circ j_2$, with $j_2 : T^2M \to TTM$ defined according to (6), locally given by

$$j_2(x^i; y^i; u^i) = \sum_{i=1}^{n}\left(y^i\frac{\partial}{\partial x^i} + u^i\frac{\partial}{\partial y^i}\right),$$

and $K : TTM \to TM$ is the connection application induced by the Levi–Civita connection, locally given by

$$K\left(\sum_{i=1}^{n} X_i\frac{\partial}{\partial x^i} + \sum_{i=1}^{n} X_{i+n}\frac{\partial}{\partial y^i}\right) = \sum_{i=1}^{n}\left(X_{n+i} + \sum_{j,k=1}^{n}\Gamma^i_{jk}y^jX_k\right)\frac{\partial}{\partial x^i}.$$

It is important to note that (19) may be locally expressed by

$$L(x^i; y^i; u^i) = \frac{1}{2}\langle\overline{u}, \overline{u}\rangle, \tag{20}$$

where

$$\overline{u} = \sum_{k=1}^{n}\left(u^k + \sum_{i,j=1}^{n}\Gamma^k_{ij}y^iy^j\right)\frac{\partial}{\partial x^k}\bigg|_x. \tag{21}$$

For the sake of simplicity, we should use $\overline{u}^k$ to represent $u^k + \sum_{i,j=1}^{n}\Gamma^k_{ij}y^iy^j$.

5.2. Intrinsic Version of the Biharmonic Equation

Consider now the Lagrangian (20) of the second-order variational problem. Differentiating L, we obtain

$$\frac{\partial L}{\partial u^i} = \sum_{k=1}^{n} g_{ik}\left(u^k + \sum_{j,l=1}^{n}\Gamma^k_{jl}y^jy^l\right) = \left\langle\overline{u}, \frac{\partial}{\partial x^i}\right\rangle \tag{22}$$

where $\bar{u} \in T_xM$ is defined by (21). As observed in Section 3, $\left[\left(\partial^2L/\partial u^i\partial u^j\right)\right]_{1\le i,j\le n} = [g_{ij}]_{1\le i,j\le n}$ is non-singular, and so we have the guarantee that the Lagrangian L is regular.

We may remark that we can associate with a Lagrangian $L : T^2M \to \mathbb{R}$ the following structures:

- The Poincaré–Cartan 2-form on T^3M given by

$$\omega_L = -dd_{J_1}L + \frac{1}{2}d_Tdd_{J_2}L,$$

where d, d_T, and d_{J_i} represent, on the exterior algebra of differentiable forms on T^2M, the usual exterior differentiation, the total derivation operator defined according to (10), and the vertical differentiation of order i (i = 1, 2) defined by (9), respectively. In other words, $\omega_L = -d\alpha_L$, where α_L is the Jacobi–Ostrogradsky form on T^3M associated with L defined on T^3M by

$$\alpha_L = d_{J_1}L - \frac{1}{2}d_Td_{J_2}L, \tag{23}$$

with

$$d_{J_1}L = \sum_{i=1}^{n}\left(\frac{\partial L}{\partial y^i}dx^i + 2\frac{\partial L}{\partial u^i}dy^i\right) \quad\text{and}\quad d_{J_2}L = 2\sum_{i=1}^{n}\frac{\partial L}{\partial u^i}dx^i.$$

- The energy function $E : T^3M \to \mathbb{R}$ defined by

$$E = C_1L - \frac{1}{2}d_T(C_2L) - L, \tag{24}$$

with

$$C_1L = \sum_{i=1}^{n}\left(\frac{\partial L}{\partial y^i}y^i + 2\frac{\partial L}{\partial u^i}u^i\right) \quad \text{and} \quad C_2L = 2\sum_{i=1}^{n}\frac{\partial L}{\partial u^i}y^i,$$

where we have used (8).

The following result allows us to obtain an expression for the Poincaré–Cartan 2-form ω_L associated with (20).

Proposition 7. *The Jacobi–Ostrogradsky form α_L associated with the Lagrangian (20) is given by*

$$\alpha_L = \sum_{i=1}^{n}\left(\alpha_i^0 dx^i + \alpha_i^1 dy^i\right),$$

where α_i^0 and α_i^1 are the real functions defined by

$$\begin{aligned} \alpha_i^0(x^i; y^i; u^i; v^i) &= \left\langle \overline{u}, \nabla_y \frac{\partial}{\partial x^i}\right\rangle - \left\langle \overline{v}, \frac{\partial}{\partial x^i}\right\rangle \\ \alpha_i^1(x^i; y^i; u^i; v^i) &= \left\langle \overline{u}, \frac{\partial}{\partial x^i}\right\rangle, \end{aligned} \tag{25}$$

with the vectors $\overline{u}$, y, and $\overline{v} \in T_xM$ being, respectively, (21),

$$y = \sum_{k=1}^{n} y^k \frac{\partial}{\partial x^k}\bigg|_x \quad \text{and} \quad \overline{v} = \sum_{k=1}^{n}\overline{v}^k\frac{\partial}{\partial x^k}\bigg|_x, \tag{26}$$

where

$$\overline{v}^k = v^k + 3\sum_{i,j=1}^{n}\Gamma_{ij}^k y^i u^j + \sum_{i,j,l=1}^{n} A_{ijl}^k y^i y^j y^l, \quad \text{with} \quad A_{ijl}^k = \frac{\partial \Gamma_{ij}^k}{\partial x^l} + \sum_{r=1}^{n}\Gamma_{jr}^k\Gamma_{il}^r.$$

Proof. From

$$\frac{\partial L}{\partial y^i} = 2\sum_{k,m,s=1}^{n} g_{mk}\Gamma_{is}^m y^s\left(u^k + \sum_{j,l=1}^{n}\Gamma_{jl}^k y^j y^l\right) = 2\left\langle \overline{u}, \nabla_y\frac{\partial}{\partial x^i}\right\rangle, \tag{27}$$

and (22), we obtain

$$\begin{aligned} d_{J_1}L &= 2\left\langle \overline{u}, \nabla_y \frac{\partial}{\partial x^i}\right\rangle dx^i + 2\left\langle \overline{u}, \frac{\partial}{\partial x^i}\right\rangle dy^i \\ d_{J_2}L &= 2\left\langle \overline{u}, \frac{\partial}{\partial x^i}\right\rangle dx^i \\ d_T d_{J_2}L &= 2\left(\left\langle \overline{v}, \frac{\partial}{\partial x^i}\right\rangle + \left\langle \overline{u}, \nabla_y\frac{\partial}{\partial x^i}\right\rangle\right)dx^i + 2\left\langle \overline{u}, \frac{\partial}{\partial x^i}\right\rangle dy^i. \end{aligned}$$

Now substituting these in (23), the expression of α_L appears immediately. □

Corollary 2. *The Poincaré–Cartan 2-form ω_L associated with the Lagrangian (20) is given by*

$$\omega_L = \sum_{i=1}^{n}\left(dx^i \wedge d\alpha_i^0 + dy^i \wedge d\alpha_i^1\right)$$

where α_i^0 and α_i^1 are the real functions (25).

The expression for the energy function E associated with (20) is given in the following result.

Proposition 8. *The energy function E: $T^3M \to \mathbb{R}$ associated with the Lagrangian (20) is expressed by*

$$E(x^i; y^i; u^i; v^i) = \frac{1}{2}\langle \overline{u}, \overline{u}\rangle - \langle \overline{v}, y\rangle,$$

with the vectors $\overline{u}$, y, and $\overline{v} \in T_xM$ defined, respectively, by (21) and (26).

Proof. Using (22) and (27), we obtain the expressions $C_1L = 2\langle \overline{u}, \overline{u}\rangle$, $C_2L = 2\langle y, \overline{u}\rangle$ and $d_T(C_2L) = 2\langle \overline{u}, \overline{u}\rangle + 2\langle \overline{v}, y\rangle$. Consequently, from (24), the results follow. □

Remark 7. *The energy function E associated with the Lagrangian (20) is written, along the lift of γ_3 to T^3M for the curve γ, as*

$$\gamma_3^*(E) = \frac{1}{2}\left\langle \frac{D^2\gamma}{dt^2}, \frac{D^2\gamma}{dt^2}\right\rangle - \left\langle \frac{D^3\gamma}{dt^3}, \frac{d\gamma}{dt}\right\rangle.$$

The Euler–Lagrange Equation (13) uniquely defines the vector field X_E on T^3M since, due to the regularity of the Lagrangian L, ω_L is sympletic (see [18]). Moreover, since $J_1X_E = C_1$, X_E is a semispray on T^3M of type 1 that represents the fourth-order differential Equation (18). Indeed, as we shall see below, the fourth-order differential Equation (18) is deduced straight from the vector field X_E defined by (13). The vector field X_E can be written as

$$X_E(x^i; y^i; u^i; v^i) = \sum_{i=1}^{n}\left(y^i\frac{\partial}{\partial x^i} + u^i\frac{\partial}{\partial y^i} + v^i\frac{\partial}{\partial u^i} + G^i\frac{\partial}{\partial v^i}\right),$$

for some functions G^i defined on domains of induced local charts. Thus, the equations $i_{X_E}\omega_L(\partial/\partial x^j) = dE(\partial/\partial x^j)$ are equivalent to

$$\sum_{i=1}^{n}\left(y^i\frac{\partial\alpha_i^0}{\partial x^j} + u^i\frac{\partial\alpha_i^1}{\partial x^j}\right) - X_E\left(\alpha_j^0\right) = \frac{\partial E}{\partial x^j},\ j = 1, \ldots, n,$$

where we use Corollary 2. Considering this equation along the lift of γ_3 to T^3M for the curve γ, we obtain the equations

$$\frac{\partial L}{\partial x^j} - \frac{d}{dt}\left(\alpha_j^0\right) = 0,\ j = 1, \ldots, n. \tag{28}$$

Along the canonical prolongation γ_3 of the curve γ, we have

$$\frac{\partial L}{\partial x^j} = \left\langle \frac{D^2\gamma}{dt^2}, \nabla^2_{\frac{d\gamma}{dt}}\frac{\partial}{\partial x^j}\right\rangle + \left\langle R\left(\frac{D^2\gamma}{dt^2}, \frac{d\gamma}{dt}\right)\frac{d\gamma}{dt}, \frac{\partial}{\partial x^j}\right\rangle$$

and

$$\frac{d}{dt}(\alpha_j^0) = \left\langle \frac{D^2\gamma}{dt^2}, \nabla^2_{\frac{d\gamma}{dt}}\frac{\partial}{\partial x^j}\right\rangle - \left\langle \frac{D^4\gamma}{dt^4}, \frac{\partial}{\partial x^j}\right\rangle,\ j = 1, \ldots, n.$$

Consequently, Equation (28) takes the form

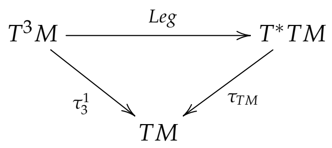
$$\left\langle \frac{D^4\gamma}{dt^4} + R\left(\frac{D^2\gamma}{dt^2}, \frac{d\gamma}{dt}\right)\frac{d\gamma}{dt}, \frac{\partial}{\partial x^j}\right\rangle = 0,\ j = 1, \ldots, n.$$

and the Euler–Lagrange Equation (18) follows.

Thus, the Euler–Lagrange Equation (18) can be interpreted as the semispray of order 3 on T^3M, X_E, whose integral curves project onto the solutions of (18). The flow of the semispray X_E is called the *biharmonic flow*.

5.3. The Cobiharmonic Flow

The Legendre transformation Leg can be represented as follows:



,

where $\ll .,. \gg$ represents the pairing duality of vector and covectors on TM. and $\pi_{T^3M} : TT^3M \to T^3M$, $\tau_3^1 : T^3M \to TM$ and $\tau_{TM} : T^*TM \to TM$ are the natural projections. Locally, we have

$$Leg(x^i; y^i; u^i; v^i) = (x^i; y^i; \alpha_i^0; \alpha_i^1),$$

where α_i^0 and α_i^1 are the real functions defined by (25). In this context, Leg is a global diffeomorphism, since its restriction to each fiber of $\tau_3^1 : T^3M \to TM$ is an isomorphism. We have a symplectomorphism

$$Leg : (T^3M, \omega_L) \to (T^*TM, \omega_1),$$

where ω_1 is the symplectic canonical form on T^*TM. The inverse of the Legendre map is defined by

$$Leg^{-1}(x^i; y^i; p_i; q_i) = (x^i; y^i; u^i; v^i),$$

where

$$\begin{aligned}
u^i &= \sum_{j=1}^{n} g^{ij} q_j - \sum_{j,k=1}^{n} \tau_{jk}^i y^j y^k \\
v^i &= \sum_{j=1}^{n} g^{ij} p_j - \sum_{j,l,r=1}^{n} g^{ij} \Gamma_{jl}^r y^l q_r - 3 \sum_{j,l,r=1}^{n} \Gamma_{rl}^i g^{rj} q_j y^l - \sum_{j,l,m=1}^{n} A_{jlm}^i y^j y^l y^m.
\end{aligned}$$

Consider the Hamiltonian function associated with L; that is,

$$H_L = E \circ Leg^{-1}.$$

The function $H_L : T^*TM \to \mathbb{R}$ is locally given by

$$H_L(x^i; y^i; p_i; q_i) = \sum_{i=1}^{n} p_i y^i + \sum_{i=1}^{n} q_i \left(\frac{1}{2} \sum_{j=1}^{n} g^{ij} q_j - \sum_{j,k=1}^{n} \Gamma_{jk}^i y^j y^k \right).$$

The dynamics of the Hamiltonian system (T^*TM, ω_1, H_L) is described by (15) and locally expressed by the following equations:

$$\begin{aligned}
\dot{x}^i &= y^i \\
\dot{y}^i &= \sum_{j=1}^{n} g^{ij} q_j - \sum_{j,k=1}^{n} \Gamma_{jk}^i y^j y^k \\
\dot{p}_i &= \sum_{j=1}^{n} q_j \left(\sum_{k,l=1}^{n} \frac{\partial \Gamma_{lk}^j}{\partial x^i} y^l y^k - \frac{1}{2} \sum_{k=1}^{n} \frac{\partial g^{jk}}{\partial x^i} q_k \right) \\
\dot{q}_i &= 2 \sum_{j,k=1}^{n} q_j \Gamma_{ik}^j y^k - p_i,
\end{aligned}$$

for $i = 1, \ldots, n$. We call these Hamiltonian equations the *cobiharmonic equations*, and we call the flow of the Hamiltonian vector field X_{H_L} the *cobiharmonic flow*.

Note that the Hamiltonian system (T^*TM, ω_1, H_L) coincides with the Hamiltonian system resulting from the study of the following optimal control problem:

$$\min_u \frac{1}{2} \sum_{i,j=1}^{n} g_{ij}\, u^i u^j \quad \text{subject to} \quad \begin{aligned} \dot{x}^i &= y^i \\ \dot{y}^i &= u^i - \sum_{j,k=1}^{n} \Gamma^i_{jk} y^j y^k, \end{aligned}$$

where $(x^i; y^i)$ are the state variables and u^i represents the control variables, $i = 1, \ldots, n$. This optimal control problem was proposed in 2000 by M. Camarinha, P. Crouch, and F. Silva Leite ([11]). The equivalence between the two formulations, the optimal and the variational formulations, has also been explored by L. Abrunheiro, M. Camarinha, and J. Clemente-Gallardo in [20], with special emphasis on the intrinsic approaches.

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