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Bounds for sine and cosine via eigenvalue estimation

Abstract: Define $n \times n$ tridiagonal matrices $\mathbf{T}$ and $\mathbf{S}$ as follows: All entries of the main diagonal of $\mathbf{T}$ are zero and those of the first super- and subdiagonal are one. The entries of the main diagonal of $\mathbf{S}$ are two except the (n, n) entry one, and those of the first super- and subdiagonal are minus one. Then, denoting by $\lambda(\cdot)$ the largest eigenvalue,

$$\lambda(\mathbf{T}) = 2\cos\frac{\pi}{n+1}, \quad \lambda(\mathbf{S}^{-1}) = \frac{1}{4\cos^2\frac{n\pi}{2n+1}}.$$

Using certain lower bounds for the largest eigenvalue, we provide lower bounds for these expressions and, further, lower bounds for $\sin x$ and $\cos x$ on certain intervals. Also upper bounds can be obtained in this way.

Keywords: eigenvalue bounds, trigonometric inequalities

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1 Introduction

Given $n \geq 2$, let tridiag (a, b) denote the symmetric tridiagonal $n \times n$ matrix with diagonal a and first super- and subdiagonal b. Define

$$\mathbf{T} = (t_{ij}) = \operatorname{tridiag}(0, 1).$$

Also define

$$\mathbf{S} = (s_{ij}) = \operatorname{tridiag}(2, -1) - \mathbf{F},$$

where the entries of $\mathbf{F}$ are zero except the (n, n) entry one. Let $\lambda(\cdot)$ and $\mu(\cdot)$ denote the largest and respectively smallest eigenvalue. Then

$$\lambda(\mathbf{T}) = 2\cos\frac{\pi}{n+1} \tag{1}$$

and

$$\mu(\mathbf{S}) = 4\cos^2\frac{n\pi}{2n+1},$$

due to Rutherford [14, p. 230] (see also [2, 17]). Then

$$\lambda(\mathbf{S}^{-1}) = \frac{1}{4\cos^2\frac{n\pi}{2n+1}}. \tag{2}$$

There are several eigenvalue bounds in the literature. Using them, can we find reasonably good bounds for the right-hand sides of (1) and (2)? Many eigenvalue bounds are too rough for this purpose, but the following bounds have some interest.

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Let $\mathbf{A}$ be a complex Hermitian $n \times n$ matrix and let $\mathbf{0} \neq \mathbf{x} \in \mathbb{C}^n$. Then (see, e.g., [5, Theorem 4.2.2])

$$\lambda(\mathbf{A}) \geq \frac{\mathbf{x}^* \mathbf{A} \mathbf{x}}{\mathbf{x}^* \mathbf{x}} \tag{3}$$

with equality if and only if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda(\mathbf{A})$. In particular, choosing $\mathbf{x} = (1 \ \ldots \ 1)^T = \mathbf{e}$, we obtain

$$\lambda(\mathbf{A}) \geq \frac{\operatorname{su} \mathbf{A}}{n}, \tag{4}$$

where su denotes the sum of entries. Equality holds if and only if $\mathbf{e}$ is an eigenvector corresponding to $\lambda(\mathbf{A})$. If $\mathbf{A}$ is (entrywise) nonnegative, then this bound is often rather good. The explanation is that there is a nonnegative eigenvector $\mathbf{z}$ corresponding to $\lambda(\mathbf{A})$. Since $\mathbf{e}$ is positive, the directions of $\mathbf{e}$ and $\mathbf{z}$ cannot be completely different.

Each row of $\mathbf{A}$ is in $\mathbf{e}$ "with equal weight", but better "weights" may be the row sums of $\mathbf{A}$; denote them by $r_1, \ldots, r_n$. So assume $\mathbf{A} \neq \mathbf{O}$ and substitute $\mathbf{x} = (r_1 \ \ldots \ r_n)^T = \mathbf{A}\mathbf{e}$ in (3). Then

$$\lambda(\mathbf{A}) \geq \frac{\operatorname{su} \mathbf{A}^3}{\operatorname{su} \mathbf{A}^2}. \tag{5}$$

Equality holds if and only if $\mathbf{A}\mathbf{e}$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda(\mathbf{A})$. Usually (5) is better than (4) but not always [8]. For further discussion on this topic, see [6].

We will in Sections 2 and 3 underestimate $\lambda(\mathbf{T})$ and $\lambda(\mathbf{S}^{-1})$, respectively. In studying $\lambda(\mathbf{T})$, we apply (5) because it is better than (4) and easy to compute. In studying $\lambda(\mathbf{S}^{-1})$, we apply (4) because (5) is rather complicated. Using these lower bounds, we will obtain also lower bounds for $\sin x$ and $\cos x$ on certain intervals. We will in Section 4 improve the lower bound for $\lambda(\mathbf{T})$ by a suitable shifting. To see how good our bounds are, we will compare them with certain other bounds in Section 5. Finally, we will outline some further developments in Section 6, and draw conclusions and make remarks in Section 7.

2 Underestimating λ(T)

Assume $n \geq 3$. Since $\mathbf{T}$ is the adjacency matrix of the linear graph $1 - 2 - \cdots - n$, the (i, j) entry of $\mathbf{T}^k$ counts the paths from i to j of length k. So the main diagonal of $\mathbf{T}^2$ is $(1, 2, \ldots, 2, 1)$, the second super- and subdiagonal is $(1, \ldots, 1)$, and the remaining entries are zero. Moreover, the first super- and subdiagonal of $\mathbf{T}^3$ is $(2, 3, \ldots, 3, 2)$, the third super- and subdiagonal is $(1, \ldots, 1)$, and the remaining entries are zero. Hence

$$\operatorname{su} \mathbf{T}^2 = 2 + (n-2) \cdot 2 + 2(n-2) = 4n - 6,$$
$$\operatorname{su} \mathbf{T}^3 = 2[2 \cdot 2 + (n-3) \cdot 3 + n - 3] = 8n - 16.$$

Since $\mathbf{T}\mathbf{e}$ is not an eigenvector corresponding to $\lambda(\mathbf{T})$, we therefore have by (1) and (5)

$$\cos \frac{\pi}{n+1} > \frac{2n-4}{2n-3}, \tag{6}$$

which trivially holds also for $n = 2$. Thus (6) is valid for all integers $n \geq 2$.

We show that in fact

$$\cos \frac{\pi}{x+1} > \frac{2x-4}{2x-3} \tag{7}$$

for all real numbers

$$x > \tfrac{3}{2}. \tag{8}$$

Because

$$\lim_{x \to \infty} \frac{\frac{2x-4}{2x-3}}{\cos \frac{\pi}{x+1}} = 1, \tag{9}$$

the bound (7) is good when x is large.

Since

$$\cos x = \cos \frac{\pi}{\frac{\pi-x}{x}+1}, \qquad \frac{2\,\frac{\pi-x}{x}-4}{2\,\frac{\pi-x}{x}-3} = \frac{2\pi-6x}{2\pi-5x},$$

the claim (7) is equivalent to that in the following

Theorem 1. *If*

$$0 < x < \tfrac{2}{5}\pi, \tag{10}$$

then

$$\cos x > \frac{2\pi-6x}{2\pi-5x}. \tag{11}$$

Proof. Assume (10). Since

$$\frac{2\pi-6x}{2\pi-5x} = 1 - \frac{x}{2\pi-5x} \quad \text{and} \quad \cos x > 1 - \frac{x^2}{2},$$

the claim follows if

$$\frac{x^2}{2} \le \frac{x}{2\pi-5x},$$

i.e.,

$$5x^2 - 2\pi x + 2 \ge 0.$$

This holds, because the discriminant $D = 4\pi^2 - 40 < 0$. □

Corollary 1. *If*

$$\tfrac{\pi}{10} < x < \tfrac{\pi}{2}, \tag{12}$$

then

$$\sin x > \frac{2\pi-12x}{\pi-10x}. \tag{13}$$

Proof. Assume (12); then $\frac{\pi}{2} - x$ satisfies (10). Apply (11) to it. □

By (9), the bound (11) is good when $\frac{\pi-x}{x}$ is large, i.e., $x \approx 0$, and (13) is good when $x \approx \frac{\pi}{2}$.

3 Underestimating $\lambda(\mathbf{S}^{-1})$

Since $\mathbf{S}$ contains negative entries, it is not reasonable to apply (4) in underestimating $\lambda(\mathbf{S})$. Indeed, the bound so obtained appears to be very poor. But

$$\mathbf{S}^{-1} = (\min(i,j))$$

is positive; so let us try (4) to underestimate $\lambda(\mathbf{S}^{-1})$.

For $k = 1, \ldots, n$, denote by $\mathbf{E}_k$ the $k \times k$ matrix with all entries one. For $k = 1, \ldots, n-1$, define the $n \times n$ matrix $\mathbf{F}_k$ by

$$\mathbf{F}_k = \begin{pmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{E}_k \end{pmatrix}.$$

Then

$$\mathbf{S}^{-1} = \mathbf{E}_n + \mathbf{F}_{n-1} + \cdots + \mathbf{F}_1,$$

and so

$$\operatorname{su}\mathbf{S}^{-1} = \operatorname{su}\mathbf{E}_n + \operatorname{su}\mathbf{F}_{n-1} + \cdots + \operatorname{su}\mathbf{F}_1 = n^2 + (n-1)^2 + \cdots + 1^2 = \frac{1}{6}(2n^3 + 3n^2 + n).$$

Since $\mathbf{e}$ is not an eigenvector of $\mathbf{S}^{-1}$ corresponding to $\lambda(\mathbf{S}^{-1})$, we therefore get by (2) and (4)

$$\frac{1}{4\cos^2\frac{n\pi}{2n+1}} > \frac{2n^2+3n+1}{6},$$

which simplifies into

$$\cos\frac{\pi}{2n+1} > \frac{2n^2+3n-2}{2n^2+3n+1}.$$

We show that in fact

$$\cos\frac{\pi}{2x+1} > \frac{2x^2+3x-2}{2x^2+3x+1} \tag{14}$$

for all real numbers x satisfying

$$x < -1 \vee -\tfrac{1}{2} < x < \tfrac{1}{2} \vee x > 1.$$

Because

$$\lim_{x\to\pm\infty}\frac{\frac{2x^2+3x-2}{2x^2+3x+1}}{\cos\frac{\pi}{2x+1}} = 1,$$

the bound (14) is good when $|x|$ is large.

Since

$$\cos x = \cos\frac{\pi}{2\frac{\pi-x}{2x}+1}, \quad \frac{2(\frac{\pi-x}{2x})^2+3\frac{\pi-x}{2x}-2}{2(\frac{\pi-x}{2x})^2+3\frac{\pi-x}{2x}+1} = \frac{\pi^2+\pi x-6x^2}{\pi^2+\pi x},$$

the claim (14) is equivalent to that in the following

Theorem 2. *If*

$$-\pi < x < 0 \vee 0 < x < \tfrac{\pi}{3} \vee x > \tfrac{\pi}{2}, \tag{15}$$

then

$$\cos x > \frac{\pi^2+\pi x-6x^2}{\pi^2+\pi x}. \tag{16}$$

Proof. We divide the proof in three cases.

Case 1. $-\pi < x < 0 \vee 0 < x < \frac{12}{\pi} - \pi$. Then

$$\frac{6x^2}{\pi^2+\pi x} - \frac{x^2}{2} = x^2\,\frac{12-\pi^2-\pi x}{2(\pi^2+\pi x)} > 0,$$

and so

$$\cos x > 1 - \frac{x^2}{2} > 1 - \frac{6x^2}{\pi^2+\pi x} = \frac{\pi^2+\pi x-6x^2}{\pi^2+\pi x}.$$

Case 2. $\frac{12}{\pi} - \pi \le x < \frac{\pi}{3}$. Write the claim (16) as

$$\cos x > \frac{(\pi-2x)(3x+\pi)}{\pi(x+\pi)},$$

equivalently

$$d(x) = \pi(x+\pi)\cos x - (\pi-2x)(3x+\pi) > 0. \tag{17}$$

Denote $x = \frac{\pi}{3} - t$, then

$$0 < t \le 4\big(\tfrac{\pi}{3} - \tfrac{3}{\pi}\big) = 0.369. \tag{18}$$

(This and corresponding equality signs later denote equality in the precision of the number of digits shown.) Since

$$\cos x = \cos\big(\tfrac{\pi}{3} - t\big) = \frac{1}{2}\cos t + \frac{\sqrt{3}}{2}\sin t > \frac{1}{2}\Big(1 - \frac{t^2}{2}\Big) + \frac{\sqrt{3}}{2}\Big(t - \frac{t^3}{6}\Big) = c(t),$$

we have

$$d(\tfrac{\pi}{3}-t) > \pi\left(\tfrac{\pi}{3}-t+\pi\right)c(t) - [\pi - 2(\tfrac{\pi}{3}-t)][3(\tfrac{\pi}{3}-t)+\pi] =$$
$$\frac{\pi}{4\sqrt{3}}t^4 + \left(\frac{\pi}{4} - \frac{\pi^2}{3\sqrt{3}}\right)t^3 + \left(6 - \frac{\pi^2}{3} - \frac{\pi\sqrt{3}}{2}\right)t^2 + \left(\frac{2\pi^2}{\sqrt{3}} - \frac{7\pi}{2}\right)t = g(t).$$

Because the exact coefficients of $g(t)$ are quite involved, we underestimate

$$g(t) > 0.4t^4 - 1.2t^3 - 0.02t^2 + 0.4t = h(t).$$

The zeros of $h(t)$ are $t_1 = -0.5387$, $t_2 = 0$, $t_3 = 0.6405$, $t_4 = 2.898$. Since $h(0.5) = 0.07 > 0$, we have $h(t) > 0$ for all t satisfying $0 < t < t_3$, in particular, under (18). Then also $g(t) > 0$, and (17) follows.

Case 3. $x > \frac{\pi}{2}$. Denote $x = \frac{\pi}{2} + t$, then $t > 0$. Because

$$\cos x = \cos\left(\tfrac{\pi}{2} + t\right) = -\sin t > -t,$$

we have

$$d(\tfrac{\pi}{2}+t) > \pi(\tfrac{\pi}{2}+t+\pi)(-t) - [\pi - 2(\tfrac{\pi}{2}+t)][3(\tfrac{\pi}{2}+t)+\pi] = t[(6-\pi)t + \pi(5 - \tfrac{3}{2}\pi)] > 0.$$

The proof is complete. □

Corollary 2. *If*

$$x < 0 \vee \tfrac{\pi}{6} < x < \tfrac{\pi}{2} \vee \tfrac{\pi}{2} < x < \tfrac{3\pi}{2}, \tag{19}$$

then

$$\sin x > \frac{10\pi x - 12x^2}{3\pi^2 - 2\pi x}. \tag{20}$$

Proof. Assume (19); then $\frac{\pi}{2} - x$ satisfies (15). Apply (16) to it. □

4 Improving (6)

For all real numbers t, we have

$$\lambda(\mathbf{T}) = \lambda(\mathbf{T} + t\mathbf{I}) - t.$$

Since $(\mathbf{T} + t\mathbf{I})\mathbf{e}$ is not an eigenvector of $\mathbf{T} + t\mathbf{I}$, we have by (5)

$$\lambda(\mathbf{T}) > \frac{\mathrm{su}\,(\mathbf{T} + t\mathbf{I})^3}{\mathrm{su}\,(\mathbf{T} + t\mathbf{I})^2} - t = f(t). \tag{21}$$

To improve (6), we try to find $t = t_0$ maximizing the right-hand side of (21). Assuming $n \geq 3$, we have

$$f(t) = \frac{nt^3 + 6(n-1)t^2 + 6(2n-3)t + 8(n-2)}{nt^2 + 4(n-1)t + 2(2n-3)} - t = \frac{2(n-1)t^2 + 4(2n-3)t + 8(n-2)}{nt^2 + 4(n-1)t + 2(2n-3)}.$$

It is straightforward to show that $f'(t) = 0$ if and only if

$$(n-2)t^2 + 2(n-3)t - 2 = 0$$

and that t_0 is the positive root of this equation. Thus

$$t_0 = \frac{3 - n + \sqrt{n^2 - 4n + 5}}{n - 2},$$

which, however, is too complicated. Therefore we replace 5 with 4 there and take

$$t = \frac{3 - n + \sqrt{n^2 - 4n + 4}}{n - 2} = \frac{3 - n + n - 2}{n - 2} = \frac{1}{n - 2}.$$

Substituting in (21), we get

$$\cos \frac{\pi}{n + 1} > \frac{4n^3 - 20n^2 + 35n - 21}{4n^3 - 18n^2 + 29n - 16}. \tag{22}$$

The corresponding equality holds for $n = 2$.

Extending (6) to (7) works under (8), but this condition does not allow extending (22) to

$$\cos \frac{\pi}{x + 1} > \frac{4x^3 - 20x^2 + 35x - 21}{4x^3 - 18x^2 + 29x - 16}. \tag{23}$$

For example, if $x = 3.5$, then the left-hand side is 0.7660, less than the right-hand side 0.7671. To find a condition for (23), we apply ideas of Laguerre developed later in an exchange of letters between Fekete and Pólya, see [10, p. 69] and [4, p. 12]. The following theorem holds actually for Laurent series, but power series are enough to us.

Theorem 3. *Given real numbers $\alpha_0, \alpha_1, \ldots$, not all zero, consider the series*

$$\phi(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots$$

with convergence radius $R > 0$. Let $0 < r < R$, denote by ϕ_r the restriction $\phi|_{]0,r[}$, and let k be a nonnegative integer. The number of sign changes of the sequence $(\beta_0^{(k)}, \beta_1^{(k)}, \beta_2^{(k)}, \ldots)$, defined by

$$\frac{\phi(rx)}{(1 - x)^k} = \beta_0^{(k)} + \beta_1^{(k)} x + \beta_2^{(k)} x^2 + \cdots,$$

is an upper bound for the number of zeros of ϕ_r.

We do not use the full force of this theorem. It is enough that we can conclude: If $\beta_0^{(k)}, \beta_1^{(k)}, \beta_2^{(k)}, \ldots \geq 0$ (not all zero) for some k, then $\phi(x) > 0$ for all x satisfying $0 < x < r$.

Theorem 4. *If*

$$x > \frac{\pi}{0.63} - 1 = 3.98666\ldots, \tag{24}$$

then (23) *holds.*

Proof. Substituting

$$x \mapsto \frac{\pi}{x + 1},$$

the claim (23) reads

$$\cos x + \frac{80x^3 - 87\pi x^2 + 32\pi^2 x - 4\pi^3}{-67x^3 + 77\pi x^2 - 30\pi^2 x + 4\pi^3} = \cos x + \frac{p(x)}{q(x)} > 0 \tag{25}$$

for all x satisfying

$$0 < x < 0.63. \tag{26}$$

Since the discriminant of

$$q'(x) = -201x^2 + 154\pi x - 30\pi^2$$

is $154^2 - 4 \cdot 201 \cdot 30 = -404 < 0$, we have $q'(x) < 0$ for all x.

Assume (26). Since $q(x) > q(0.63) = 16.75 > 0$, an equivalent claim to (25) is

$$q(x) \cos x + p(x) > 0.$$

We prove a stronger claim

$$f(x) = q(x)\Big(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\Big) + p(x) > 0.$$

Let us apply Theorem 3 to $\phi = f$, $r = 0.63$. We find the $\beta_i^{(0)}$'s from

$$\phi(rx) = \alpha_0 + r\alpha_1 x + r^2\alpha_2 x^2 + \cdots = \beta_0^{(0)} + \beta_1^{(0)} x + \beta_2^{(0)} x^2 + \cdots ,$$

so

$$\beta_i^{(0)} = \alpha_i r^i, \quad i = 0, 1, 2, \dots .$$

We construct the $\beta_i^{(k)}$'s recursively. Since

$$\frac{f(rx)}{(1-x)^{k+1}} = \frac{1}{1-x}\frac{f(rx)}{(1-x)^k} = (1 + x + x^2 + \cdots)(\beta_0^{(k)} + \beta_1^{(k)} x + \beta_2^{(k)} x^2 + \cdots) =$$
$$\beta_0^{(k)} + (\beta_0^{(k)} + \beta_1^{(k)})x + (\beta_0^{(k)} + \beta_1^{(k)} + \beta_2^{(k)})x^2 + \cdots = \beta_0^{(k+1)} + \beta_1^{(k+1)} x + \beta_2^{(k+1)} x^2 + \cdots ,$$

we get

$$\beta_i^{(k+1)} = \beta_0^{(k)} + \cdots + \beta_i^{(k)}, \quad i, k = 0, 1, 2, \dots . \tag{27}$$

Now a simple computation yields

$$f(0.63x) = 0.00145481x^9 - 0.00833744x^8 - 0.0937648x^7 + 0.619422x^6 +$$
$$2.10029x^5 - 18.2393x^4 + 40.2686x^3 - 37.0818x^2 + 12.4357x. \tag{28}$$

Therefore $\beta_0^{(0)} = \beta_{10}^{(0)} = \beta_{11}^{(0)} = \cdots = 0$, which implies by (27) that $\beta_0^{(1)} = 0$ and

$$\beta_9^{(1)} = \beta_{10}^{(1)} = \cdots = 0.00145481 - 0.00833744 - 0.0937648 + 0.619422 +$$
$$2.10029 - 18.2393 + 40.2686 - 37.0818 + 12.4357 = 0.0022 > 0.$$

Hence, by (27), $\beta_0^{(k)} = 0$ and $\beta_9^{(k)}, \beta_{10}^{(k)}, \dots > 0$ for all $k \geq 1$.

It remains to show that $\beta_1^{(k)}, \dots, \beta_8^{(k)} \geq 0$ for some k. Let $\mathbf{L}$ be the 8×8 lower triangular matrix with diagonal and lower triangle one, and denote $\mathbf{b}_k = (\beta_1^{(k)} \dots \beta_8^{(k)})^T$. We find $\mathbf{b}_0$ from (28) and obtain

$$\mathbf{b}_3 = \mathbf{L}^3\mathbf{b}_0 = (12.4\ \ 0.225\ \ 3.64\ \ 4.43\ \ 4.71\ \ 5.09\ \ 5.48\ \ 5.88)^T.$$

Now the proof is complete. □

As in the proof of (11) and (13), we can find lower bounds for $\sin x$ and $\cos x$, but they are quite complicated.

Shifting does not improve (4), because

$$\frac{\operatorname{su}(\mathbf{A} + t\mathbf{I})}{n} - t = \frac{\operatorname{su}\mathbf{A}}{n}$$

for all t. Therefore we cannot apply this trick to (14).

5 Comparisons

We compare our bounds for $\sin x$ with certain other bounds. Because our bounds work well near to $\frac{\pi}{2}$, we choose for comparison only such bounds that are defined there. Most of them are improvements of Jordan's inequality

$$\sin x > \frac{2}{\pi}x, \quad 0 < x < \tfrac{\pi}{2}. \tag{29}$$

Kober's inequality

$$\cos x > 1 - \frac{2}{\pi}x, \quad 0 < x < \tfrac{\pi}{2},$$

is equivalent to this (simply substitute $x \mapsto \frac{\pi}{2} - x$ in one of them to get the other), and so brings nothing new to us. There is an extensive literature on refining and extending these inequalities. Qi, Niu and Guo [11] surveyed this topic concerning (29).

We compare our bounds (13) and (20) with each other and with the following bounds:

$$\sin x > \frac{\pi^2 x - x^3}{\pi^2 + x^2}, \quad 0 < x < \pi, \quad \text{(Redheffer [12, 13], Williams [18]);} \tag{30}$$

$$\sin x > \frac{3}{\pi}x - \frac{4}{\pi^3}x^3, \quad 0 < x < \tfrac{\pi}{2}, \quad \text{(Caccia [1]);} \tag{31}$$

$$\sin x > x + \frac{2(2-\pi)}{\pi^2}x^2, \quad 0 < x < \tfrac{\pi}{2}, \quad \text{(Sándor [15]);} \tag{32}$$

$$\sin x > (\sqrt{2} - 1)\Big(\frac{2\sqrt{2}}{\pi}x + 1\Big), \quad \tfrac{\pi}{4} < x < \tfrac{\pi}{2}, \quad \text{(Sándor [16]);} \tag{33}$$

$$\sin x > x + \frac{12 - 4\pi}{\pi^2}x^2 + \frac{4\pi - 16}{\pi^3}x^3, \quad 0 < x < \tfrac{\pi}{2}, \quad \text{(Özban [19]);} \tag{34}$$

$$\sin x > \Big(\frac{9\pi}{80} + \frac{2}{\pi}\Big)x - \frac{1}{2\pi}x^3 + \frac{1}{5\pi^3}x^5, \quad 0 < x < \tfrac{\pi}{2}, \quad \text{(Kuo [7]).} \tag{35}$$

In studying (13), we restrict to $\frac{\pi}{10} < x < \frac{\pi}{2}$, and in studying (20) to $\frac{\pi}{6} < x < \frac{\pi}{2}$. In comparing them with (33), we restrict to $\frac{\pi}{4} < x < \frac{\pi}{2}$.

We list the conditions under which the first-mentioned bound is better than the second.

(13) vs. (20): $\frac{3\pi}{10} < x < \frac{\pi}{3}$.

(13) vs. (30): $x > 0.8622$.

(20) vs. (30): Always.

(13) vs. (31): $0.8579 < x < 1.1181$.

(20) vs. (31): Always.

(13) vs. (32): $x > 0.7449$.

(20) vs. (32): Always.

(13) vs. (33): $x > 0.8505$.

(20) vs. (33): $x > 0.8085$.

(13) vs. (34): $0.9205 < x < 1.0482$.

(20) vs. (34): $x < 1.0526$.

(13) vs. (35): Never.

(20) vs. (35): $x < 0.6815$ or $x > 1.4798$.

6 Further developments

We extend (11). Let $b > a > 0$. We determine $d(\le 1/a)$ so that

$$\cos x > \frac{1 - bx}{1 - ax} \tag{36}$$

for all x satisfying

$$0 < x < d. \tag{37}$$

As in the proof of Theorem 1, we can see that (36) holds if

$$\frac{x^2}{2} \le \frac{(b-a)x}{1-ax}.$$

Under (37), this is equivalent to

$$p(x) = ax^2 - x + 2(b-a) \ge 0. \tag{38}$$

The discriminant $D = 1 - 8a(b-a)$.

Case 1. $D \le 0$, i.e.,

$$b \ge a + \frac{1}{8a}.$$

Then (38) holds for all x. Given $a > 0$, the choice

$$b = a + \frac{1}{8a}$$

is clearly optimal. So we have proved that

$$\cos x > \frac{1 - (a + \frac{1}{8a})x}{1 - ax},$$

assuming (37) with $d = 1/a$. In particular, take $a = \frac{5}{2\pi}$; then

$$\cos x > \frac{1 - (\frac{5}{2\pi} + \frac{\pi}{20})x}{1 - \frac{5}{2\pi}x} = \frac{2\pi - (5 + \frac{\pi^2}{10})x}{2\pi - 5x}$$

for all x satisfying $0 < x < \frac{2\pi}{5}$. This improves (11) slightly.

Case 2. $D > 0$. Since both zeros of $p(x)$ are positive, x must be less than or equal to the smaller zero.

We have now proved the following

Theorem 5. *Let $b > a > 0$. If $D = 1 - 8a(b-a) \le 0$, then*

$$\cos x > \frac{1 - bx}{1 - ax} \tag{39}$$

for all x satisfying

$$0 < x < \frac{1}{a}.$$

If $D > 0$, then (39) *holds for all x satisfying*

$$0 < x \le \frac{1 - \sqrt{1 - 8a(b-a)}}{2a}.$$

The referee suggested that perhaps, by considering certain matrices with complex entries, hyperbolic versions of our bounds can be found. We leave the question concerning such matrices open (see Remark 8) but study what happens in an attempt to find the hyperbolic version of (39) by using power series.

Let $b > a > 0$. We try to find a reasonable condition concerning $x(> 0)$ so that

$$\cosh x > \frac{1 + bx}{1 + ax}.$$

Applying the inequality $\cosh x > 1 + \frac{1}{2}x^2$ and proceeding as above, we obtain a sufficient condition

$$p(x) = ax^2 + x - 2(b-a) \ge 0. \tag{40}$$

Since $p(x)$ has both positive and negative zero, x must be greater than or equal to the positive zero. Thus we have proved the following

Theorem 6. *Let $b > a > 0$. Then*

$$\cosh x > \frac{1 + bx}{1 + ax}$$

for all x satisfying

$$x \geq \frac{-1 + \sqrt{1 + 8a(b - a)}}{2a}.$$

It might be of interest to compare this bound with well-known bounds and to study other related bounds, but we do not pursue this topic further.

7 Conclusions and remarks

We first found by eigenvalue estimation lower bounds for $\cos \frac{\pi}{n+1}$ and $\cos \frac{\pi}{2n+1}$, where n is a positive integer. By using power series, we extended these bounds to hold for $\cos \frac{\pi}{x+1}$ and $\cos \frac{\pi}{2x+1}$, where x is a real number satisfying certain conditions. We reformulated these bounds to work for $\sin x$ and $\cos x$ under certain conditions. We also improved some bounds by shifting and compared our bounds for $\sin x$ with certain other bounds. Finally, we outlined a more general approach. Some remarks follow.

Remark 1 Although our bounds for $\sin x$ managed rather well in some comparisons, their drawback is that they are not valid for all x satisfying $0 < x < \frac{\pi}{2}$ and that they are quite poor for some values of x.

Remark 2 By using well-known trigonometric identities, we can find several other lower bounds for $\sin x$ and $\cos x$ on certain intervals. For example, begin by writing (7) and (14) as

$$\sqrt{\frac{1 + \cos \frac{2\pi}{x+1}}{2}} > \frac{2x - 4}{2x - 3}, \quad \sqrt{\frac{1 + \cos \frac{2\pi}{2x+1}}{2}} > \frac{2x^2 + 3x - 2}{2x^2 + 3x + 1}.$$

Remark 3 Also several upper bounds for $\sin x$ and $\cos x$ can be found. For example, begin by writing (7) and (14) as

$$1 - 2\sin^2 \frac{\pi}{2x + 2} > \frac{2x - 4}{2x - 3}, \quad 1 - 2\sin^2 \frac{\pi}{4x + 2} > \frac{2x^2 + 3x - 2}{2x^2 + 3x + 1}.$$

Remark 4 Graphics shows that (23) holds actually for all $x > \xi$ where $\xi = 3.95528$ is the largest root of the equation corresponding to (23). Without trusting the graphics, we can slightly weaken the condition (24) as follows: Define $\alpha = 0.633989$ by

$$\frac{\pi}{\alpha} - 1 = \xi.$$

Choose a satisfying $\alpha > a > 0.63$. Proceed as in the proof of Theorem 4 (but $k = 3$ may not be enough; then more work is needed) to show that

$$x > \frac{\pi}{a} - 1$$

implies (23).

Remark 5 Also $\mathbf{H} = (2\mathbf{I} - \mathbf{T})^{-1}$ is positive, but su $\mathbf{H}$ does not seem to follow any simple rule. So we cannot proceed with this matrix as we did with $\mathbf{S}^{-1}$.

Remark 6 Fan, Taussky and Todd [3, Theorem 8] used $\mu(\mathbf{S})$ and the complementary of (3) to show that

$$4\sin^2 \frac{\pi}{2(2n - 1)} \sum_{i=2}^{n} x_i^2 \leq \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2$$

for all real numbers $x_1 = 0,\ x_2, \ldots, x_n$. But $\lambda(\mathbf{S})$ and (3) can be similarly applied to show the reverse

$$\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2 \leq 4\cos^2 \frac{\pi}{2n - 1} \sum_{i=2}^{n} x_i^2,$$

due to Milovanović and Milovanović [9, Corollary 2].

Remark 7 Similarly to Section 6, we can study bounds of type

$$\frac{\sin x}{x} > 1 - \frac{x^2}{6} \geq \frac{1 - bx}{1 - ax}$$

and

$$\frac{\sinh x}{x} > 1 + \frac{x^2}{6} \geq \frac{1 + bx}{1 + ax}.$$

Remark 8 As already noted in Section 6, it might be of interest to find a complex Hermitian matrix $\mathbf{A}$ such that $\lambda(\mathbf{A})$ can be expressed by hyperbolic functions and that the bound (3) with smartly chosen $\mathbf{x}$ works well. Analogously to our procedure, bounds for hyperbolic functions can then be obtained.

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