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Beyond the Schwinger boson representation of the $su(2)$-algebra

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With the use of two kinds of boson operators, a new boson representation of the $su(2)$-algebra is proposed. The basic idea comes from the pseudo $su(1,1)$-algebra recently given by the present authors [Y. Tsue et al., Prog. Theor. Exp. Phys. **2013**, 103D04 (2013)]. It forms a striking contrast to the Schwinger boson representation of the $su(2)$-algebra, which is also based on two kinds of bosons. It is proved that this new boson representation obeys the $su(2)$-algebra in a certain subspace in the whole boson space constructed by the Schwinger boson representation of the $su(1,1)$-algebra. This representation may be suitable for describing the time dependence of the system interacting with the external environment in the framework of the thermo field dynamics formalism, i.e., phase space doubling. Further, several deformations related to the $su(2)$-algebra in this boson representation are discussed. On the basis of these deformed algebras, various types of time evolution of a simple boson system are investigated.

Subject Index D50

1. Introduction and summary

It may be hardly necessary to mention, but the $su(2)$-algebra has made a central contribution to the development of microscopic study of nuclear structure. The BCS–Bogoliubov theory may be a typical example of this. We know that the $su(2)$-algebra may be the simplest and the most popular Lie algebra in nuclear structure theory. In many cases, Lie algebras have been treated under the name of boson realizations of Lie algebras. The prototype has been called boson expansion or boson mapping for many-fermion systems, which is related to the $so(2N)$-algebra for a space of N single-particle states [1–3]. With the development of the study of boson expansion, it came to be called the boson realization of the Lie algebra [4]. The simplest case is the boson realization of the $su(2)$-algebra, which is called the Holstein–Primakoff representation [5]. The three generators are expressed in terms of one kind of boson operator for a given value of the magnitude of the $su(2)$-spin, which determines the irreducible representation. Of course, it is positive-definite. We also know another boson representation of the $su(2)$-algebra, i.e., the Schwinger boson representation [7]. The explicit form will be given in the relations (3.1) and (3.2). It consists of two kinds of boson operators. Unlike the Holstein–Primakoff representation, the magnitude of the $su(2)$-spin is expressed in terms of an

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operator, i.e., half of the sum of two boson number operators, which can be seen in the relation (3.2b). Clearly, it is a positive-definite operator. On the other hand, we know the boson representation of the $su(1,1)$-algebra, also initiated by Schwinger [7]. This representation is also constructed in terms of two kinds of boson operators. The explicit form will be presented in the relations (2.1) and (2.3). In contrast with the case of the $su(2)$-algebra, in this representation, the quantum numbers specifying the irreducible representation are the eigenvalues of an operator that is related to half of the difference between the two boson number operators, which can be seen in the relation (2.3b). Therefore, this operator is not positive-definite. By analogy with the magnitude of the $su(2)$-spin, we will call the positive eigenvalue the magnitude of the $su(1,1)$-spin.

The Schwinger boson representation of the $su(1,1)$-algebra may be not familiar to the field of nuclear structure theory mainly treating zero temperature. An advantage of the Schwinger boson representation of the $su(1,1)$-algebra, in which two-kind of bosons are introduced, is to be able to describe the damping and amplification of an isolated harmonic oscillator induced classical-mechanically by the velocity-dependent force quantum-mechanically in conservative form [8–10]. In the background of this description, there exists the idea of the thermo field dynamics formalism based on phase space doubling [11,12]. We should note the following: The above-mentioned isolated oscillator is of one dimension and the Schwinger boson representation is constructed in 2D space, because of the use of two kinds of bosons. Therefore, the idea of phase space doubling is conjectured to be useful in the present problem. The original intrinsic oscillator is isolated and it is expressed in terms of one kind of boson; the external environment is expressed in terms of another kind of boson, in which the frequency is the same as that of the original one. The interaction between both systems is naturally introduced.

If we follow the idea of phase space doubling, the total Hamiltonian for the present system, $\hat{H}$, can be expressed in the form

$$\hat{H} = \hat{H}_0 + \hat{V}_i, \tag{1.1a}$$

$$\hat{H}_0 = \hat{H}_{\rm intr} - \hat{H}_{\rm extr}. \tag{1.1b}$$

Here, $\hat{H}_{\rm intr}$ and $\hat{H}_{\rm extr}$ denote the Hamiltonians of the intrinsic and external environment systems, respectively. Both Hamiltonians are of harmonic oscillator type with the same frequencies. The part $\hat{V}_i$ denotes the interaction between both systems. It should be noted that $\hat{H}$ is not the total energy operator but the operator for the time evolution of the system. As for $\hat{V}_i$, a certain linear combination of the raising and lowering operators of the $su(1,1)$-algebra in the Schwinger boson representation was adopted in Refs. [8–10]. Further, we notice that $\hat{H}_0$ is a constant of motion, because $\hat{H}_0$ is related to the magnitude of the $su(1,1)$-spin. In Refs. [9,10], the Hamiltonian (1.1) was treated in the framework of the time-dependent variational method. Of course, under careful consideration of the magnitude of the $su(1,1)$-spin as a constant of motion in the $su(1,1)$-algebra, the trial state is constructed. Therefore, the expectation value of $\hat{H}_0$ does not depend on time, but the expectation value of $\hat{H}_{\rm intr}$ does. With the use of $\hat{V}_i$ mentioned above, the expectation value becomes a decreasing or increasing function of time. The former and latter show damped and amplified oscillations, respectively. Some variations of the above-mentioned idea were discussed in Ref. [10].

However, we must point out that the $su(1,1)$-algebra in the Schwinger boson representation cannot be applied directly to the many-fermion system. The reason for this is very simple: We cannot find the $su(1,1)$-algebra in the many-fermion system. In response to this situation, the present authors proposed an idea [13]. Hereafter, Ref. [13] will be referred to as (A). The orthogonal set constructed under the Schwinger boson representation of the $su(1,1)$-algebra forms an infinite-dimensional space for a given value of the magnitude of the $su(1, 1)$-spin. However, the orthogonal set for a many-fermion system is of finite dimensions for a given value of the magnitude of the $su(2)$-spin. Then, a possible idea is to define, in the infinite-dimensional space, a certain subspace that is in one-to-one correspondence with the fermion space. Further, we require that any matrix element of the raising and lowering operators in this subspace does not change the form from that given in this algebra. If restricted to this subspace, the algebra becomes deformed from the $su(1, 1)$-algebra. In Ref. [13], we call it the pseudo $su(1, 1)$-algebra. Three generators are expressed in terms of the original $su(1, 1)$-generators with a certain parameter that is closely related to the dimensions of the subspace. Then, we can connect the pseudo $su(1, 1)$-algebra with the $su(2)$-algebra, which governs, e.g., the Cooper pair. The above idea suggests to us that the many-fermion system can be treated by the pseudo $su(1, 1)$-algebra and, in (A), we described a simple fermion system based on the thermo field dynamics formalism. As a result, the periodical dependence of the energy of the intrinsic system on time was shown. This result is in contrast to that in the $su(1, 1)$-algebra. For the time-dependent variational method adopted in (A), we prepare a trial state that contains one complex parameter for the variation and the normalization constant of the trial state must be given. However, in the case of the pseudo $su(1, 1)$-algebra, the explicit form of the normalization constant is too complicated to treat it practically. It is shown in (A).

The main aim of this paper is to present a new boson representation of the $su(2)$-algebra that may be suitable for the application of the idea of phase space doubling. As has already been mentioned, the Schwinger representation is powerless for the use of phase space doubling. In contrast to the case of the Schwinger boson representation, the operator for the magnitude of the $su(2)$-spin in the present case is expressed in terms of the form related to the difference between two boson number operators. Using an idea similar to that for constructing the pseudo $su(1, 1)$-algebra, we can express the $su(2)$-generators as functions of the three $su(1, 1)$-generators and a certain parameter. If the value of this parameter is appropriately chosen, the present representation is reduced to the boson realization of the $su(2)$-algebra governing, e.g., the Cooper pair. Of course, these generators are available in the subspace leading to the pseudo $su(1, 1)$-algebra. We intend to apply the present representation to the Hamiltonian (1.1) in the same manner as in (A). The interaction term $\hat{V}_i$ must be appropriately chosen. Therefore, we prepare a certain trial state for the time-dependent variation so as to calculate the normalization constant easily. Expecting various results, we give several types of deformations from the new boson representation of the $su(2)$-algebra. As has already been mentioned, the algebra discussed in this paper is applied to the Hamiltonian (1.1). As for $\hat{V}_i$, we adopt a certain linear combination of the raising and lowering operators of the algebra obtained by each deformation and we can show that the expectation value of the intrinsic Hamiltonian, $\hat{H}_{\rm intr}$, changes periodically with time. Depending on the choice of deformation, the features of the change show various shapes. In classical mechanics, we know one problem: elastic collision of a simply oscillating light particle with a sufficiently heavy one on a horizontal straight line. We can show that the problem discussed in this paper reduces to the same as the above.

After recapitulating, in Sect. 2, the pseudo $su(1, 1)$-algebra presented in (A) with some new aspects, a new boson representation of the $su(2)$-algebra is proposed in Sect. 3. It is based on the Schwinger boson representation of the $su(1, 1)$-algebra. Section 4 is devoted to the investigation of various deformations from the new boson representation. The normalization constant in the orthogonal set obtained in each deformation is easily calculated with a rather simple approximation. In Sects. 5 and 6, the time evolution is investigated for the boson system characterized by the Hamiltonian (1.1). Depending on the chosen deformation, the change of energy of the intrinsic system with time is periodic, but the behaviors are different from one another. In Sect. 7, it is shown that the new boson representation of the $su(2)$-algebra constructed in this paper certainly satisfies the $su(2)$-algebra in the subspace leading to the pseudo $su(1,1)$-algebra. The subspace is explicitly presented and, in this subspace, the raising and lowering operators for the magnitude of the $su(2)$-spin are discussed. In Sect. 8, the connections of the new boson representation developed in this paper to the other boson representations are discussed. In the last section, two problems are treated. One is related to the algebras in the space orthogonal to the subspace (2.8). The other is concerned with another example of t_m: $t_m = 3t - 1$.

2. Recapitulation of pseudo $su(1,1)$-algebra in the Schwinger boson representation given in (A)

Our preliminary task is to recapitulate the Schwinger boson representation of the $su(1,1)$-algebra in a form suitable for later discussion. The details can be seen in Refs. [7–10]. It starts with the following three operators:

$$\hat{T}_+ = \hat{a}^*\hat{b}^*, \quad \hat{T}_- = \hat{b}\hat{a}, \quad \hat{T}_0 = \tfrac{1}{2}\left(\hat{a}^*\hat{a} + \hat{b}^*\hat{b} + 1\right). \tag{2.1}$$

Here, $\left(\hat{a}, \hat{a}^*\right)$ and $\left(\hat{b}, \hat{b}^*\right)$ denote two kinds of boson operators. The operators $\hat{T}_{\pm,0}$ form the $su(1,1)$-algebra:

$$\hat{T}_-^* = \hat{T}_+, \quad \hat{T}_0^* = \hat{T}_0, \tag{2.2a}$$

$$\left[\hat{T}_+,\ \hat{T}_-\right] = -2\hat{T}_0, \quad \left[\hat{T}_0,\ \hat{T}_\pm\right] = \pm\hat{T}_\pm. \tag{2.2b}$$

The Casimir operator $\hat{\boldsymbol{T}}^2$, together with its properties, is given by

$$\hat{\boldsymbol{T}}^2 = \hat{T}_0^2 - \tfrac{1}{2}\left(\hat{T}_-\hat{T}_+ + \hat{T}_+\hat{T}_-\right) = \hat{T}\left(\hat{T} - 1\right), \tag{2.3a}$$

$$\hat{T} = \tfrac{1}{2}\left(-\hat{a}^*\hat{a} + \hat{b}^*\hat{b} + 1\right), \quad \left[\hat{T},\ \hat{T}_{\pm,0}\right] = 0. \tag{2.3b}$$

The eigenstate of $\hat{T}$ and $\hat{T}_0$ with its eigenvalue t and t_0, respectively, is obtained in the form

$$|t, t_0\rangle = \frac{1}{\sqrt{(t_0 - t)!(t_0 + t - 1)!}}\left(\hat{a}^*\right)^{t_0 - t}\left(\hat{b}^*\right)^{t_0 + t - 1}|0\rangle, \quad \text{(normalized)}$$

$$\hat{a}|0\rangle = \hat{b}|0\rangle = 0. \tag{2.4}$$

Since $t_0 - t = 0, 1, 2, 3, \ldots$ and $t_0 + t - 1 = 0, 1, 2, 3, \ldots$, $|t, t_0\rangle$ is treated separately in the following two cases:

(i) $t = 1/2, 1, 3/2, \ldots, \infty, \quad t_0 = t, t+1, t+2, \ldots, \infty$ for a given t, (2.5a)

(ii) $t = 0, -1/2, -1, \ldots, -\infty, \quad t_0 = -t+1, -t+2, -t+3, \ldots, \infty$ for a given t. (2.5b)

It is noted that, in this paper, we mainly treat case (i). In Sect. 7, we will mention case (ii) briefly in connection with case (i). The state $|t, t_0\rangle$ can be rewritten in the form

$$|t, t_0\rangle = s\frac{1}{\sqrt{(t_0 - t)!(t_0 + t - 1)!}}\left(\hat{T}_+\right)^{t_0 - t}|t\rangle, \quad (|t, t_0 = t\rangle = |t\rangle) \tag{2.6a}$$

$$\text{i.e.} \quad |n; t\rangle = \frac{1}{\sqrt{n!(2t - 1 + n)!}}\left(\hat{T}_+\right)^n|t\rangle. \tag{2.6b}$$

Here, $|t\rangle$ denotes the minimum weight state:

$$|t\rangle = \left(\hat{b}^*\right)^{2t-1}|0\rangle, \quad \hat{T}_-|t\rangle = 0, \quad \hat{T}_0|t\rangle = \hat{T}|t\rangle = t|t\rangle. \tag{2.7}$$

By n time operations of the raising operator $\hat{T}_+$ on $|t\rangle$, $|n;t\rangle$ is obtained. It should be noted that this operation is permitted until infinite time; in other words, a maximum weight state does not exist. Clearly, the states in (2.6) form an orthogonal set with infinite dimensions.

Recently, a possible form of the pseudo $su(1,1)$-algebra was presented by the present authors in Ref. [13], which is referred to as (A). It aims to demonstrate a deformation of the Cooper pair obeying the $su(2)$-algebra in a many-fermion system. In (A), one type of possible deformation of the $su(1,1)$-algebra in the Schwinger boson representation was treated; we called it the pseudo $su(1,1)$-algebra. In this paper, this algebra is formulated in a manner slightly modified from that given in (A). The basic scheme of the pseudo $su(1,1)$-algebra is to construct it in the subspace of the space (2.5a):

$$t = 1/2, 1, 3/2, \ldots, \mu - 1/2, \mu, \quad t_0 = t, t+1, t+2, \ldots, t_m - 1, t_m \text{ for a given } t. \tag{2.8}$$

Here, μ and t_m denote integer or half-integer, where t_m is a function of t. Depending on the model under investigation, the value of μ and the functional form of t_m are chosen appropriately. Later, we will show a possible idea for the determination of μ and t_m.

Let $\hat{\mathcal{T}}_{\pm,0}$ denote three generators of the pseudo $su(1,1)$-algebra. The role of $\hat{\mathcal{T}}_{\pm,0}$ is the same as that of $\hat{T}_{\pm,0}$. One time operation of $\hat{\mathcal{T}}_+$ makes the eigenvalue of $\hat{\mathcal{T}}_0$ increase by one in the state specified by the quantum number (t, t_0). In this algebra, there exist not only a minimum but also a maximum weight state. Further, the following is required for a given t: the minimum and maximum weight states are identical to $|t,t\rangle(=|t\rangle)$ and $|t,t_m\rangle$, respectively, and successive operation of $\hat{\mathcal{T}}_+$ on $|t\rangle$ reduces to the state (2.6). The above requirement is formulated as follows:

$$\hat{\mathcal{T}}_-|t,t\rangle = 0, \quad \hat{\mathcal{T}}_0|t,t\rangle = t|t,t\rangle, \tag{2.9a}$$

$$\hat{\mathcal{T}}_+|t,t_m\rangle = 0, \quad \hat{\mathcal{T}}_0|t,t_m\rangle = t_m|t,t_m\rangle, \tag{2.9b}$$

$$\left(\hat{\mathcal{T}}_+\right)^{t_0-t}|t\rangle = \left(\hat{T}_+\right)^{t_0-t}|t\rangle. \tag{2.9c}$$

As has already been mentioned, the eigenvalue of $\hat{\mathcal{T}}_0$ increases one by one from t to t_m and, then, $\hat{\mathcal{T}}_0$ may be permitted to be of the form

$$\hat{\mathcal{T}}_0 = \hat{T}_0 + f\left(\hat{T}\right). \tag{2.10}$$

The reason for this is very simple. The eigenvalue of $\hat{T}_0$ increases one by one and $f\left(\hat{T}\right)$ does not make the eigenvalue of $\hat{T}_0$ change. Then, by operating $\hat{\mathcal{T}}_0$ on the states $|t,t\rangle$ and $|t,t_m\rangle$, the following relation is obtained:

$$t + f(t) = t, \quad t_m + f(t) = t_m, \quad \text{i.e., } f(t) = 0. \tag{2.11}$$

Therefore, $\hat{\mathcal{T}}_0$ is equal to $\hat{T}_0$:

$$\hat{\mathcal{T}}_0 = \hat{T}_0. \tag{2.12a}$$

If one notices the relation $\hat{T}_-|t\rangle = 0$ and $\sqrt{t_m - \hat{T}_0}\cdot\left(\sqrt{t_m - \hat{T}_0 + \epsilon}\right)^{-1}|t,t_m\rangle = \sqrt{0}/\sqrt{\epsilon}|t,t_m\rangle \to 0$ $(\epsilon \to 0)$, the requirement (2.9) suggests the following form for $\hat{\mathcal{T}}_\pm$:

$$\hat{\mathcal{T}}_+ = \hat{T}_+ \cdot \sqrt{t_m - \hat{T}_0}\cdot\left(\sqrt{t_m - \hat{T}_0 + \epsilon}\right)^{-1}, \quad \hat{\mathcal{T}}_- = \left(\sqrt{t_m - \hat{T}_0 + \epsilon}\right)^{-1}\cdot\sqrt{t_m - \hat{T}_0}\cdot\hat{T}_-. \tag{2.12b}$$

Here, as has already been shown, ϵ denotes a positive infinitesimal parameter. With the use of the commutation relation (2.2b), the arrangement of the operators can be changed.

It has already been mentioned that t_m is a function of t, i.e., $t_m = F_m(t)$. Depending on the problem under investigation, the form of $F_m(t)$ is fixed. If one intends to apply the present algebra to the investigation of such a state as a boson coherent state, the present one must be formulated in the operator form, i.e., quanta such as t_0, t, and t_m do not appear. Then, it may be permissible to define an operator $\hat{T}_m = F_m(\hat{T})$ and, if $\hat{T}$ is replaced with t in $F_m(\hat{T})$, $\hat{T}_m$ becomes $F_m(t) = t_m$. Therefore, $\hat{\mathcal{T}}_{\pm,0}$ can be expressed as

$$\begin{aligned}
\hat{\mathcal{T}}_+ &= \hat{T}_+ \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \left(\sqrt{\hat{T}_m - \hat{T}_0 + \epsilon}\right)^{-1}, \\
\hat{\mathcal{T}}_- &= \left(\sqrt{\hat{T}_m - \hat{T}_0 + \epsilon}\right)^{-1} \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \hat{T}_-, \\
\hat{\mathcal{T}}_0 &= \hat{T}_0.
\end{aligned} \tag{2.13}$$

The operators $\hat{\mathcal{T}}_{\pm,0}$ satisfy

$$\hat{\mathcal{T}}_-^* = \hat{\mathcal{T}}_+, \quad \hat{\mathcal{T}}_0^* = \hat{\mathcal{T}}_0,$$

$$\left[\hat{\mathcal{T}}_+,\ \hat{\mathcal{T}}_-\right] = -2\hat{\mathcal{T}}_0 + \epsilon \frac{\left(\hat{T}_0 + \hat{T}\right)\left(\hat{T}_0 - \hat{T} + 1\right)}{\hat{T}_m - \hat{T}_0 + \epsilon}, \tag{2.14a}$$

$$\left[\hat{\mathcal{T}}_0,\ \hat{\mathcal{T}}_\pm\right] = \pm\hat{\mathcal{T}}_\pm. \tag{2.14b}$$

The operator $\hat{\boldsymbol{\mathcal{T}}}^2$ corresponding to the Casimir operator $\hat{\boldsymbol{T}}^2$ is obtained in the form

$$\begin{aligned}
\hat{\boldsymbol{\mathcal{T}}}^2 &= \hat{\mathcal{T}}_0^2 - \tfrac{1}{2}\left(\hat{\mathcal{T}}_-\hat{\mathcal{T}}_+ + \hat{\mathcal{T}}_+\hat{\mathcal{T}}_-\right) \\
&= \hat{\mathcal{T}}\left(\hat{\mathcal{T}} - 1\right) + \tfrac{1}{2}\epsilon \frac{\left(\hat{T}_0 + \hat{T}\right)\left(\hat{T}_0 - \hat{T} + 1\right)}{\hat{T}_m - \hat{T}_0 + \epsilon}, \quad \hat{\mathcal{T}} = \hat{T}.
\end{aligned} \tag{2.15}$$

If, in the second terms on the right-hand sides of the relations (2.14a) and (2.15) the limiting process $\hat{T}_m \to \infty$ proceeds to replace $\hat{T}_0$ and $\hat{T}$ with the eigenvalues t_0 and t, the above algebra reduces to the $su(1,1)$-algebra. This can be seen in the form of the generators (2.12) directly. For example, in the case $\hat{T}_m = C_m + 1 - \hat{T}$, $\hat{T}_m \to \infty$ if $C_m \to \infty$. Of course, C_m is a parameter. This formula may be understood in the following manner: In the state (2.9c) for a given t, the created bosons consist of $2(t_m - t)$ $(= 2(F_m(t) - t))$ bosons in the a–b pair type and $(2t - 1)$ bosons in the single b-boson type. In the present scheme, creation of a boson in the single a-boson type is forbidden. In the case $t = 1/2$, the created bosons are all in the a–b pair type, i.e., the number of created bosons is $2\,(F_m(1/2) - 1/2)$. The above consideration suggests the following aspect: In the state (2.9c), there exist $(2t - 1)$ vacancies and, if $(2t - 1)$ a-bosons occupy these vacancies, all the created bosons come to be in the a–b pair type. On the basis of the above argument, we require that, independent of the value of t, the number of created bosons is limited in the present system. Then, the following relation may be acceptable:

$$\begin{aligned}
&2(F_m(t) - t) + 2(2t - 1) = 2\left(F_m\left(\tfrac{1}{2}\right) - \tfrac{1}{2}\right)\ (= 2C_m), \\
&\text{i.e.,} \quad t_m = C_m + 1 - t.
\end{aligned} \tag{2.16a}$$

Since $t_m = F_m(t)$ is decreasing for t, there exists a certain point $t = t_c$ satisfying $F_m(t_c) = t_c$ and, thus, t_c is nothing but μ $(t_c = \mu)$. This argument gives us

$$\mu = \tfrac{1}{2}(C_m + 1). \tag{2.16b}$$

With the aid of the $su(1,1)$- and pseudo $su(1,1)$-algebras in the Schwinger boson representation, the role of the phase space doubling mentioned qualitatively in Sect. 1 can be understood more quantitatively. From the relations (2.1) and (2.3b), the following expression for $\hat{b}^*\hat{b}$ is derived:

$$\hat{b}^*\hat{b} = \hat{T}_0 + \hat{T} - 1. \tag{2.17a}$$

Let a time-dependent state vector, which is the eigenstate of $\hat{T}$ with the eigenvalue t, exist. This is that of the superposition of the orthogonal set (2.4) or (2.6) from $t_0 = t$ to $t \to \infty$. Then, the expectation value of $\hat{b}^*\hat{b}$ at the time τ is expressed as

$$\left\langle \hat{b}^*\hat{b}\right\rangle_\tau = t_0(\tau) + t - 1, \quad t_0(\tau) = \left\langle \hat{T}_0\right\rangle_\tau, \quad t_0(0) > t. \tag{2.17b}$$

The damped and amplified harmonic oscillator can be understood by investigating the behavior of $t_0(\tau)$, which is determined by the Hamiltonian adopted in the model under investigation. If $t_0(\tau)$ is monotonically decreasing $(t_0(0) > t_0(\tau) > t)$, $t_0(\tau) \to t$ $(\tau \to \infty)$. Then, $\left\langle \hat{H}_{\rm intr}\right\rangle_\tau = \left\langle \omega\hat{b}^*\hat{b}\right\rangle_\tau$ decreases from the value $\omega(t_0(\tau) + t - 1)$ to $\omega(2t - 1)$ at the limit $\tau \to \infty$. This corresponds to the damped oscillator. On the other hand, if $t_0(\tau)$ is monotonically increasing $(t_0(0) < t_0(\tau))$, $t_0(\tau) \to \infty$ $(\tau \to \infty)$. Then, $\left\langle \omega\hat{b}^*\hat{b}\right\rangle_\tau$ increases from the value $\omega(t_0(0) + t - 1)$ to ∞ at the time $\tau \to \infty$. This case corresponds to the amplified oscillation.

It may be interesting to investigate the pseudo $su(1,1)$-algebra in relation to the boson realization of a many-fermion system. In (A), a possible deformation of the Cooper pair in the frame of this algebra was discussed. Let a time-dependent state, which is the eigenstate of $\hat{T}$ with the eigenvalue t, exist. In this case, the state is that of a superposition of the eigenstates of $\hat{T}_0$, the eigenvalues of which change from $t_0 = t$ to $t_0 = t_m$. Therefore, the expectation value of $\hat{b}^*\hat{b}$ for this state at the time τ is given in the relation (2.17), but $t_0(\tau)$ should obey the inequality

$$t \le t_0(\tau) \le t_m. \tag{2.18}$$

Since, e.g., $\langle \hat{b}^*\hat{b}\rangle_\tau$ changes in the range (2.18), $\langle \hat{b}^*\hat{b}\rangle_\tau$ can change periodically in the range

$$2t - 1 \le \left\langle \hat{b}^*\hat{b}\right\rangle_\tau \le t_m + t - 1. \tag{2.19}$$

Of course, it depends on the Hamiltonian. In (A), an example of the periodical change was shown. In conclusion, in order to make the idea of phase space doubling effective in the $su(1,1)$- and pseudo $su(1,1)$-algebras, the operator $\left(\hat{b}^*\hat{b} - \hat{a}^*\hat{a}\right)$, i.e., $\hat{T}$, should be a constant of motion.

3. A new boson realization of a many-fermion system obeying the $su(2)$-algebra in terms of the $su(1,1)$-algebra

In (A), the pseudo $su(1,1)$-algebra in the fermion space was discussed in terms of a possible deformation of the Cooper pair obeying the $su(2)$-algebra. In this paper, the pseudo $su(1,1)$-algebra is treated from the side of the $su(2)$-algebra in the frame of the boson space constructed by two kinds of bosons $\left(\hat{a}, \hat{a}^*\right)$ and $\left(\hat{b}, \hat{b}^*\right)$. One popular boson representation of the $su(2)$-algebra is presented by Schwinger [7]:

$$\hat{S}_+ = \hat{a}^*\hat{b}, \quad \hat{S}_- = \hat{b}^*\hat{a}, \quad \hat{S}_0 = \tfrac{1}{2}\left(\hat{a}^*\hat{a} - \hat{b}^*\hat{b}\right). \tag{3.1}$$

Here, $\hat{S}_{\pm,0}$ denote the generators. The Casimir operator $\hat{\boldsymbol{S}}^2$ is given by

$$\hat{\boldsymbol{S}}^2 = \hat{S}_0^2 + \tfrac{1}{2}\left(\hat{S}_-\hat{S}_+ + \hat{S}_+\hat{S}_-\right) = \hat{S}\left(\hat{S} + 1\right), \tag{3.2a}$$

$$\hat{S} = \tfrac{1}{2}\left(\hat{a}^*\hat{a} + \hat{b}^*\hat{b}\right), \quad \left[\hat{S},\ \hat{S}_{\pm,0}\right] = 0. \tag{3.2b}$$

We note the relation

$$\hat{S} = \hat{T}_0 - \tfrac{1}{2}, \quad \hat{S}_0 = -\hat{T} + \tfrac{1}{2}. \tag{3.3}$$

Since $\hat{S}$ is not related to $\hat{T}$, the form (3.1) may not be suitable for treating the Hamiltonian (1.1) based on phase space doubling, even if it is expressed in terms of $su(2)$-generators. Rather, it may be better to apply this representation to the problem of the energy transfer between the b- and a-boson systems.

This suggests to us that, in order to obtain an $su(2)$-algebra in the boson representation for phase space doubling, it may be necessary to connect the operator $\hat{S}$ with $\hat{T}$ directly. With this aim, the same idea as in the pseudo $su(1,1)$-algebra (2.9) is adopted:

$$\hat{\mathcal{S}}_-|t,t\rangle = 0, \quad \hat{\mathcal{S}}_0|t,t\rangle = -s|t,t\rangle, \tag{3.4a}$$

$$\hat{\mathcal{S}}_+|t,t_m\rangle = 0, \quad \hat{\mathcal{S}}_0|t,t_m\rangle = s|t,t_m\rangle, \tag{3.4b}$$

$$\left(\hat{\mathcal{S}}_+\right)^{t_0-t}|t\rangle = \sqrt{\frac{(2t-1)!}{(t_0+t-1)!}\cdot\frac{(t_m-t)!}{(t_m-t_0)!}}\left(\hat{T}_+\right)^{t_0-t}|t\rangle, \tag{3.4c}$$

since the eigenvalue of $\hat{\mathcal{S}}_0$ increases one by one from $-s$ to s and, then, $\hat{\mathcal{S}}_0$ may be permitted to set up the relation

$$\hat{\mathcal{S}}_0 = \hat{T}_0 + f\left(\hat{T}\right). \tag{3.5}$$

The above is the same as the case of the pseudo $su(1,1)$-algebra. Operating $\hat{\mathcal{S}}_0$ on the states $|t,t\rangle$ and $|t,t_m\rangle$, the following relation is obtained:

$$t + f(t) = -s, \quad t_m + f(t) = s. \tag{3.6}$$

The relation (3.6) leads us to $f(t) = -(t_m+t)/2$ and $s = (t_m-t)/2$. Therefore, $\hat{\mathcal{S}}_0$ can be expressed as

$$\hat{\mathcal{S}}_0 = \hat{T}_0 - \tfrac{1}{2}\left(\hat{T}_m + \hat{T}\right). \tag{3.7a}$$

Later, we discuss $s = (t_m-t)/2$. If one notices the relations $\hat{T}_-|t\rangle = 0$ and $\sqrt{\hat{T}_m - \hat{T}_0}|t,t_m\rangle = 0$, the requirement (3.4) suggests the following form for $\hat{\mathcal{S}}_\pm$:

$$\hat{\mathcal{S}}_+ = \hat{T}_+\cdot\sqrt{\hat{T}_m-\hat{T}_0}\cdot\left(\sqrt{\hat{T}_0+\hat{T}+\epsilon}\right)^{-1}, \quad \hat{\mathcal{S}}_- = \left(\sqrt{\hat{T}_0+\hat{T}+\epsilon}\right)^{-1}\cdot\sqrt{\hat{T}_m-\hat{T}_0}\cdot\hat{T}_-. \tag{3.7b}$$

In the space obeying the relation (3.4), the operators $\hat{\mathcal{S}}_{\pm,0}$ satisfy

$$\hat{\mathcal{S}}_-^* = \hat{\mathcal{S}}_+, \quad \hat{\mathcal{S}}_0^* = \hat{\mathcal{S}}_0, \tag{3.8a}$$

$$\left[\hat{\mathcal{S}}_+,\ \hat{\mathcal{S}}_-\right] = 2\hat{\mathcal{S}}_0, \quad \left[\hat{\mathcal{S}}_0,\ \hat{\mathcal{S}}_\pm\right] = \pm\hat{\mathcal{S}}_\pm. \tag{3.8b}$$

The proof of the relation (3.8b) will be shown in Sect. 7. Certainly, $\hat{\mathcal{S}}_{\pm,0}$ form the $su(2)$-algebra. The Casimir operator $\hat{\boldsymbol{\mathcal{S}}}^2$ can be expressed as

$$\hat{\boldsymbol{\mathcal{S}}}^2 = \hat{\mathcal{S}}\left(\hat{\mathcal{S}}+1\right), \quad \hat{\mathcal{S}} = \tfrac{1}{2}\left(\hat{T}_m - \hat{T}\right). \tag{3.8c}$$

The operator $\hat{\mathcal{S}}$ is given from the form $s = (t_m - t)/2$ obtained in the relation (3.6). If $\hat{T}_m = C_m + 1 - \hat{T}$, $\hat{\mathcal{S}}_{\pm,0}$ and $\hat{\mathcal{S}}$ can be expressed as follows:

$$\hat{\mathcal{S}}_+ = \hat{a}^*\hat{b}^* \cdot \sqrt{C_m - \hat{b}^*\hat{b}} \cdot \left(\sqrt{\hat{b}^*\hat{b} + 1 + \epsilon}\right)^{-1}, \tag{3.9a}$$

$$\hat{\mathcal{S}}_- = \left(\sqrt{\hat{b}^*\hat{b} + 1 + \epsilon}\right)^{-1} \cdot \sqrt{C_m - \hat{b}^*\hat{b}} \cdot \hat{b}\hat{a}, \tag{3.9b}$$

$$\hat{\mathcal{S}}_0 = \frac{1}{2}\left(\hat{a}^*\hat{a} + \hat{b}^*\hat{b}\right) - \frac{1}{2}C_m, \tag{3.9c}$$

$$\hat{\mathcal{S}} = \frac{1}{2}\left(\hat{a}^*\hat{a} - \hat{b}^*\hat{b}\right) + \frac{1}{2}C_m. \tag{3.10}$$

The above is a new boson representation of the $su(2)$-algebra. The expressions (3.7) and (3.8c) hold in the subspace (2.8) of the space (2.5a). A detailed explanation will be given in Sect. 7. The form (3.10) suggests that the present representation may be suitable for phase space doubling, because $\hat{b}^*\hat{b}$ and $\hat{a}^*\hat{a}$ appear in the relation between the subtraction of the b- and a-boson numbers.

We have developed a new boson representation of the $su(2)$-algebra. Such boson representations have played the role of describing various gross properties of many-fermion systems. In these studies, investigation of the behaviors of individual fermions is of secondary importance. These have been called the boson realization of the Lie algebraic approach to many-fermion problems. The $su(2)$-pairing model is a typical example. In this model, three quantities occupy the central part of the gross properties, i.e., the total number of single-particle states $4\Omega_0$ (if following (A), conventionally 2Ω), the total fermion number N, and the seniority number ν. Therefore, in order to complete the present new boson representation in relation to the $su(2)$-pairing model, it may be inevitable to connect t, t_0, and t_m with ν, N, and Ω_0.

First, the form $\widetilde{S}_0 = (1/2)\widetilde{N} - \Omega_0$ in the relation (A.3.1) is taken up in terms of the eigenvalue s_0 and N:

$$s_0 = \frac{1}{2}N - \Omega_0. \tag{3.11a}$$

In the case $s_0 = -s$, $-s = N_{\min}/2 - \Omega_0$ and $N_{\min} = \nu$, and then, the following relation is derived:

$$s = \Omega_0 - \frac{1}{2}\nu. \tag{3.11b}$$

Further, in the case $s_0 = s$, $s = N_{\max}/2 - \Omega_0 = \Omega_0 - \nu/2$, and then, $N_{\max} = 4\Omega_0 - \nu$. Noticing the relation $\left[\widetilde{N},\ \widetilde{S}_+\right] = 2\widetilde{S}_+$, N can be given as

$$N = \nu,\quad \nu + 2, \ldots, 4\Omega_0 - \nu\ (= \nu + 2(2\Omega_0 - \nu)). \tag{3.12}$$

Since $s \geq 0$, ν is given as

$$\nu = 0, 1, 2, \ldots, 2\Omega_0. \tag{3.13}$$

On the other hand, the relations (3.7a) and (3.8c) lead us to

$$s_0 = t_0 - \frac{1}{2}\big(t_m + t\big), \tag{3.14a}$$

$$s = \frac{1}{2}\big(t_m - t\big). \tag{3.14b}$$

Equating the relations (3.11) and (3.14) with each other, the following is obtained:

$$2t - 1 = \nu + \frac{1}{2}\left((2t_0 - 1) - N\right), \tag{3.15a}$$

$$2t_m - 1 = (4\Omega_0 - \nu) + \frac{1}{2}\left((2t_0 - 1) - N\right). \tag{3.15b}$$

Discussion of the relations (3.15a) and (3.15b) starts in a postulate mentioned below. The boson vacuum $|0\rangle = |t = 1/2, t_0 = 1/2\rangle$ corresponds to the fermion vacuum $|0) = |\nu = 0, N = 0)$. This postulate suggests that the relations (2.8) and (2.13) give us

$$2t - 1 = \nu. \tag{3.16a}$$

The relations (3.15a) and (3.15b) lead us to

$$2t_0 - 1 = N, \tag{3.16b}$$

i.e.,

$$2t_m - 1 = 4\Omega_0 - \nu. \tag{3.16c}$$

Since $2t_m - 1 = 4\Omega_0 - (2t - 1)$, the following relation is obtained:

$$t_m = (2\Omega_0 + 1) - t. \tag{3.17}$$

The relations (3.14b) and (3.17) give

$$s = \left(\Omega_0 + \tfrac{1}{2}\right) - t. \tag{3.18a}$$

The maximum values of t and ν are μ and $2\Omega_0$, respectively, and the relations (3.13) and (3.15a) give us

$$2\mu - 1 = 2\Omega_0, \quad \text{i.e.,} \quad \mu = \Omega_0 + \tfrac{1}{2}. \tag{3.18b}$$

The above argument presents us with the operator form of the $su(2)$-algebra. First, the seniority and the total fermion number operator $\hat{\nu}$ and $\hat{N}$, respectively, are introduced in the form

$$\hat{\nu} = 2\hat{T} - 1 = -\hat{a}^*\hat{a} + \hat{b}^*\hat{b}, \tag{3.19a}$$

$$\hat{N} = 2\hat{T}_0 - 1 = \hat{a}^*\hat{a} + \hat{b}^*\hat{b}. \tag{3.19b}$$

Further, $\hat{T}_m$ is given as

$$\hat{T}_m = (2\Omega_0 + 1) - \hat{T} = 2\Omega_0 + \tfrac{1}{2} + \tfrac{1}{2}\left(\hat{a}^*\hat{a} - \hat{b}^*\hat{b}\right). \tag{3.20}$$

The operators $\hat{\mathcal{S}}_{\pm,0}$ and $\hat{\mathcal{S}}$ can be expressed as

$$\begin{aligned}\hat{\mathcal{S}}_+ &= \hat{T}_+ \cdot \sqrt{(2\Omega_0 + 1) - \left(\hat{T}_0 + \hat{T}\right)} \cdot \left(\sqrt{\hat{T}_0 + \hat{T} + \epsilon}\right)^{-1} \\ &= \hat{a}^*\hat{b}^* \cdot \sqrt{2\Omega_0 - \hat{b}^*\hat{b}} \cdot \left(\sqrt{\hat{b}^*\hat{b} + 1 + \epsilon}\right)^{-1},\end{aligned} \tag{3.21a}$$

$$\begin{aligned}\hat{\mathcal{S}}_- &= \left(\sqrt{\hat{T}_0 + \hat{T} + \epsilon}\right)^{-1} \cdot \sqrt{(2\Omega_0 + 1) - \left(\hat{T}_0 + \hat{T}\right)} \cdot \hat{T}_- \\ &= \left(\sqrt{\hat{b}^*\hat{b} + 1 + \epsilon}\right)^{-1} \cdot \sqrt{2\Omega_0 - \hat{b}^*\hat{b}} \cdot \hat{b}\hat{a},\end{aligned} \tag{3.21b}$$

$$\hat{\mathcal{S}}_0 = \hat{T}_0 - \left(\Omega_0 + \tfrac{1}{2}\right) = \tfrac{1}{2}\left(\hat{a}^*\hat{a} + \hat{b}^*\hat{b}\right) - \Omega_0, \tag{3.21c}$$

$$\hat{\mathcal{S}} = \left(\Omega_0 + \tfrac{1}{2}\right) - \hat{T} = \tfrac{1}{2}\left(\hat{a}^*\hat{a} - \hat{b}^*\hat{b}\right) + \Omega_0. \tag{3.22}$$

The above corresponds to the case $C_m = 2\Omega_0$ in the relations (3.7) and (3.9). Needless to say, the expressions (3.21) and (3.22) hold in the subspace (2.8) of the space (2.5a). The details will be discussed in Sect. 7. The expressions (3.21) and (3.22) form the third boson representation of the $su(2)$-algebra. Of course, the first and second are the Holstein–Primakoff and Schwinger boson representations, respectively [5–7].

4. Various deformations of the $su(2)$-algebra in the third boson representation

In this section, we will investigate various deformations of the $su(2)$-algebra developed in Sect. 3. For this aim, let us rewrite the generators $\hat{\mathcal{S}}_\pm$ shown in the relation (3.7b) as follows:

$$\hat{\mathcal{S}}_+ = \hat{T}_+ \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \sqrt{1 + \left(\hat{T}_0 - \hat{T}\right)} \cdot \left(\sqrt{\hat{T}_-\hat{T}_+ + \epsilon}\right)^{-1}, \tag{4.1a}$$

$$\hat{\mathcal{S}}_- = \left(\sqrt{\hat{T}_-\hat{T}_+ + \epsilon}\right)^{-1} \cdot \sqrt{1 + \left(\hat{T}_0 - \hat{T}\right)} \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \hat{T}_-. \tag{4.1b}$$

The operator $\hat{\mathcal{S}}_0$ is unchanged from the form (3.7a) and $\hat{T}_-\hat{T}_+$ is given as

$$\hat{T}_-\hat{T}_+ = \left(\hat{T}_0 + \hat{T}\right)\left(\hat{T}_0 - \hat{T} + 1\right) = \left(\hat{T}_0 + \hat{T}\right)\left(1 + \left(\hat{T}_0 - \hat{T}\right)\right). \tag{4.2a}$$

If $\hat{T}_0$ is replaced with $\left(\hat{T}_0 - 1\right)$, $\hat{T}_-\hat{T}_+$ becomes $\hat{T}_+\hat{T}_-$:

$$\hat{T}_+\hat{T}_- = \left(\hat{T}_0 - \hat{T}\right)\left(\hat{T}_0 + \hat{T} - 1\right). \tag{4.2b}$$

With the use of the relation (4.2a), it may be easily verified that the form (4.1) is equivalent to the relation (3.7b).

The discussion starts in the introduction of the operator $\hat{\mathcal{R}}_\pm$ deformed from $\hat{\mathcal{S}}_\pm$:

$$\hat{\mathcal{R}}_+ = \hat{T}_+ \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \sqrt{\hat{Q}_p + p\left(\hat{T}_0 - \hat{T}\right)} \cdot \left(\sqrt{\hat{T}_-\hat{T}_+ + \epsilon}\right)^{-1}, \tag{4.3a}$$

$$\hat{\mathcal{R}}_- = \left(\sqrt{\hat{T}_-\hat{T}_+ + \epsilon}\right)^{-1} \cdot \sqrt{\hat{Q}_p + p\left(\hat{T}_0 - \hat{T}\right)} \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \hat{T}_-. \tag{4.3b}$$

Here, $\hat{Q}_p$ is a function of $\hat{T}$:

$$\hat{Q}_p = Q_p(\hat{T}), \quad \hat{Q}_p|t, t_0\rangle = Q_p(t)|t, t_0\rangle, \quad Q_p(t) = q_{p,t}. \tag{4.4}$$

From the outside, we must fix the concrete form of $\hat{Q}_p$ and, corresponding to the form of $\hat{Q}_p$, the deformation is determined. Without loss of generality, the following conditions are added:

$$p = \pm 1, 0 \quad \text{and if} \quad p = 0,\ \hat{Q}_0 = 1, \quad \text{i.e.,} \quad q_{0,t} = 1. \tag{4.5}$$

The operator $\left(\hat{Q}_p + p\left(\hat{T}_0 - \hat{T}\right)\right)$ should be positive-definite and it can be shown in the form

$$\sqrt{\hat{Q}_p + p\left(\hat{T}_0 - \hat{T}\right)}|t, t\rangle = \sqrt{q_{p,t}}|t, t\rangle, \tag{4.6a}$$

$$\sqrt{\hat{Q}_p + p\left(\hat{T}_0 - \hat{T}\right)}|t, t_m\rangle = \sqrt{q_{p,t} + p(t_m - t)}|t, t_m\rangle. \tag{4.6b}$$

The relation (4.6) leads us to

$$q_{p,t} \geq 0, \tag{4.7a}$$

$$q_{p,t} + p(t_m - t) \geq 0, \quad \text{i.e.,} \quad q_{1,t} \geq -(t_m - t), \quad q_{-1,t} \geq t_m - t. \tag{4.7b}$$

If $q_{p,t} = 0$, $\sqrt{\hat{Q}_p + p\left(\hat{T}_0 - \hat{T}\right)}|t, t\rangle = 0$ and, then, $\hat{\mathcal{R}}_+|t, t\rangle = 0$, which is not interesting, because of $\hat{\mathcal{R}}_-|t, t\rangle = 0$. Therefore, in the relation (4.7a), $q_{p,t} \geq 0$ should be changed to $q_{p,t} > 0$. From the above argument, $q_{p,t}$ should obey the condition for the positive-definiteness:

$$q_{1,t} > 0, \quad q_{-1,t} \geq t_m - t. \tag{4.8}$$

The operators $\hat{\mathcal{R}}_\pm$ are reduced to $\hat{\mathcal{S}}_\pm$, shown in the relation (4.1):

$$\text{If} \quad \hat{Q}_1 = 1, \quad \hat{\mathcal{R}}_\pm = \hat{\mathcal{S}}_\pm. \tag{4.9}$$

The products of $\hat{\mathcal{R}}_+$ and $\hat{\mathcal{R}}_-$ are expressed as

$$\hat{\mathcal{R}}_+\hat{\mathcal{R}}_- = \left(\hat{T}_m - \hat{T}_0 + 1\right)\left(\hat{Q}_p + p\left(\hat{T}_0 - \hat{T} - 1\right)\right)\left(1 - \frac{\epsilon}{\hat{T}_+\hat{T}_- + \epsilon}\right), \tag{4.10a}$$

$$\hat{\mathcal{R}}_-\hat{\mathcal{R}}_+ = \left(\hat{T}_m - \hat{T}_0\right)\left(\hat{Q}_p + p\left(\hat{T}_0 - \hat{T}\right)\right)\left(1 - \frac{\epsilon}{\hat{T}_-\hat{T}_+ + \epsilon}\right). \tag{4.10b}$$

In the case $p = \pm 1$, the relation (4.10) gives us the commutation relation

$$\left[\hat{\mathcal{R}}_+, \ \hat{\mathcal{R}}_-\right] = p \cdot 2\hat{\mathcal{R}}_0 - \left(\hat{T}_m - \hat{T} + 1\right)\sum_t |t\rangle\langle t|. \tag{4.11}$$

Here, $\hat{\mathcal{R}}_0$ is defined as

$$\hat{\mathcal{R}}_0 = \hat{T}_0 - \tfrac{1}{2}\left(\hat{T}_m + \hat{T} - p\hat{Q}_p + 1\right), \quad \left[\hat{\mathcal{R}}_0, \ \hat{\mathcal{R}}_\pm\right] = \pm\hat{\mathcal{R}}_\pm. \tag{4.12}$$

Under the condition (4.9), $\hat{\mathcal{R}}_0$ is reduced to $\hat{\mathcal{S}}_0$:

$$\text{If} \quad \hat{Q}_1 = 1, \quad \hat{\mathcal{R}}_0 = \hat{\mathcal{S}}_0. \tag{4.13}$$

The operator $\hat{\boldsymbol{\mathcal{R}}}^2$ corresponding to the Casimir operator $\hat{\boldsymbol{\mathcal{S}}}^2$ is expressed in the form

$$\begin{aligned} \hat{\boldsymbol{\mathcal{R}}}^2 &= \hat{\mathcal{R}}_0^2 + p \cdot \tfrac{1}{2}\left(\hat{\mathcal{R}}_-\hat{\mathcal{R}}_+ + \hat{\mathcal{R}}_+\hat{\mathcal{R}}_-\right) \\ &= F_p(\hat{\mathcal{R}}) + \tfrac{1}{2}\left(\hat{T}_m - \hat{T} + 1\right)\left(\hat{Q}_p - p\right)\sum_t |t\rangle\langle t|. \end{aligned} \tag{4.14}$$

Here, $\hat{\mathcal{R}}$ and $F_p(\hat{\mathcal{R}})$ are given as

$$\text{(i)} \quad \hat{\mathcal{R}} = \pm\tfrac{1}{2}\left(\hat{T}_m - \hat{T} + p\hat{Q}_p \mp 1\right), \quad F_p(\hat{\mathcal{R}}) = \hat{\mathcal{R}}\left(\hat{\mathcal{R}} + 1\right), \tag{4.15a}$$

$$\text{(ii)} \quad \hat{\mathcal{R}} = \pm\tfrac{1}{2}\left(\hat{T}_m - \hat{T} + p\hat{Q}_p \pm 1\right), \quad F_p(\hat{\mathcal{R}}) = \hat{\mathcal{R}}\left(\hat{\mathcal{R}} - 1\right). \tag{4.15b}$$

Under the condition (4.9), the upper sign of $\hat{\mathcal{R}}$ in (i) is reduced to $\hat{\mathcal{S}}$:

$$\text{If} \quad \hat{Q}_1 = 1, \quad \hat{\mathcal{R}} = \hat{\mathcal{S}} \quad \text{and} \quad \hat{\boldsymbol{\mathcal{R}}}^2 = F_1(\hat{\mathcal{R}}) = \hat{\boldsymbol{\mathcal{S}}}^2 = \hat{\mathcal{S}}\left(\hat{\mathcal{S}} + 1\right). \tag{4.16}$$

The above argument suggests that $\hat{\mathcal{R}}$ shown in the relation (4.15) can be regarded as an operator that plays the same role as $\hat{\mathcal{S}}$. Then, it may be permitted to require the condition

$$r_t \geq 0. \quad \left(\hat{\mathcal{R}}|t, t_0\rangle = r_t|t, t_0\rangle\right) \tag{4.17}$$

The above is a formal aspect of the case $p = \pm 1$. Later, we will discuss some concrete examples.

Next, we discuss the case $p = 0$. Under the condition (4.5), the relation (4.10) is reduced to

$$\hat{\mathcal{R}}_+\hat{\mathcal{R}}_- = \hat{T}_m - \hat{T} + 1 - \left(\hat{T}_m - \hat{T} + 1\right)\sum_t |t\rangle\langle t|, \tag{4.18a}$$

$$\hat{\mathcal{R}}_-\hat{\mathcal{R}}_+ = \hat{T}_m - \hat{T}. \tag{4.18b}$$

In this case, $\hat{\mathcal{R}}_0$ cannot be defined, because of the commutation relation

$$\left[\hat{\mathcal{R}}_+, \hat{\mathcal{R}}_-\right] = 1 - \left(\hat{T}_m - \hat{T} + 1\right)\sum_t |t\rangle\langle t|. \tag{4.19}$$

However, it should be noticed that $\hat{T}_0$ plays the same role as $\hat{\mathcal{R}}_0$ in the case $p = \pm 1$:

$$\left[\hat{T}_0, \hat{\mathcal{R}}_\pm\right] = \pm\hat{\mathcal{R}}_\pm. \tag{4.20}$$

Of course, $\hat{\boldsymbol{\mathcal{R}}}^2$ cannot be defined. The operators $\hat{\mathcal{R}}_\pm$ can be expressed as

$$\hat{\mathcal{R}}_+ = \hat{T}_+ \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \left(\sqrt{\hat{T}_-\hat{T}_+ + \epsilon}\right)^{-1}, \tag{4.21a}$$

$$\hat{\mathcal{R}}_- = \left(\sqrt{\hat{T}_-\hat{T}_+ + \epsilon}\right)^{-1} \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \hat{T}_-. \tag{4.21b}$$

Although $\hat{\mathcal{R}}_\pm$ do not form any algebra, $\hat{\mathcal{R}}_+$ plays the role of the raising operator for constructing the orthogonal set. It is easily seen that there exist the relations $\hat{\mathcal{R}}_-|t, t\rangle = 0$ and $\hat{\mathcal{R}}_+|t, t_m\rangle = 0$ and the state $|t, t_0\rangle$ is of the form $\left(\hat{\mathcal{R}}_+\right)^{t_0 - t}|t\rangle$.

As promised, we show concrete examples for the case $p = \pm 1$ and examine the following cases:

$$\text{(i)}\ \ \hat{\mathcal{R}}_0|t, t\rangle = -\hat{\mathcal{R}}|t, t\rangle\ (= -r_t|t, t\rangle), \tag{4.22a}$$

$$\text{(ii)}\ \ \hat{\mathcal{R}}_0|t, t_m\rangle = \hat{\mathcal{R}}|t, t_m\rangle\ (= r_t|t, t_m\rangle), \tag{4.22b}$$

$$\text{(iii)}\ \ \hat{\mathcal{R}}_0|t, t\rangle = \hat{\mathcal{R}}|t, t\rangle\ (= r_t|t, t\rangle). \tag{4.22c}$$

These three may be regarded as cases in which traces of the original $su(2)$- and $su(1,1)$-algebras are left. After rather lengthy consideration, the following results are obtained: In case (i), $\hat{Q}_1 = 1$, i.e., $q_{1,t} = 1$, which leads us to

$$\hat{\mathcal{R}} = \tfrac{1}{2}\left(\hat{T}_m - \hat{T}\right), \quad \hat{\mathcal{R}}_0 = \hat{T}_0 - \tfrac{1}{2}\left(\hat{T}_m + \hat{T}\right), \quad \hat{\boldsymbol{\mathcal{R}}}^2 = \hat{\mathcal{R}}\left(\hat{\mathcal{R}} + 1\right). \tag{4.23a}$$

In case (ii), $q_{1,t} > 0$, which gives

$$\begin{aligned} &\hat{\mathcal{R}} = \tfrac{1}{2}\left(\hat{T}_m - \hat{T} + \hat{Q}_1 - 1\right), \quad \hat{\mathcal{R}}_0 = \hat{T}_0 - \tfrac{1}{2}\left(\hat{T}_m + \hat{T} - \hat{Q}_1 + 1\right), \\ &\hat{\boldsymbol{\mathcal{R}}}^2 = \hat{\mathcal{R}}\left(\hat{\mathcal{R}} + 1\right) - \tfrac{1}{2}\left(\hat{T}_m - \hat{T} + 1\right)\left(\hat{Q}_1 - 1\right)\sum_t |t\rangle\langle t|. \end{aligned} \tag{4.23b}$$

In case (iii), it is impossible to find any case that satisfies the conditions (4.8) and (4.17).

It is important to see that case (i) is included in case (ii). If $\hat{Q}_1 = 1$ in case (ii), it is nothing but case (i) and this case corresponds to the $su(2)$-algebra already discussed. If $\hat{Q}_1 \neq 1$, the relation

$\hat{\mathcal{R}}_0|t,t\rangle = -\hat{\mathcal{R}}|t,t\rangle$ does not exist. For example, if $\hat{Q}_1 = 2\hat{T}$, the operator $\hat{Q}_p + p(\hat{T}_0 - \hat{T})$ for $p = 1$ becomes $(\hat{T} + \hat{T}_0)$. Then, in this case, $\hat{\mathcal{R}}_{\pm,0}$ can be expressed in the form

$$\hat{\mathcal{R}}_+ = \hat{T}_+ \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \left(\sqrt{\hat{T}_0 - \hat{T} + 1 + \epsilon}\right)^{-1}, \tag{4.24a}$$

$$\hat{\mathcal{R}}_- = \left(\sqrt{\hat{T}_0 - \hat{T} + 1 + \epsilon}\right)^{-1} \cdot \sqrt{\hat{T}_m - \hat{T}_0} \cdot \hat{T}_-, \tag{4.24b}$$

$$\hat{\mathcal{R}}_0 = \hat{T}_0 - \tfrac{1}{2}\left(\hat{T}_m - \hat{T} + 1\right). \tag{4.24c}$$

$$\hat{\mathcal{R}} = \tfrac{1}{2}\left(\hat{T}_m - \hat{T} - 1\right), \quad \hat{\boldsymbol{\mathcal{R}}}^2 = \hat{\mathcal{R}}\left(\hat{\mathcal{R}} + 1\right) - \tfrac{1}{2}\left(\hat{T}_m - \hat{T} + 1\right)\left(2\hat{T} - 1\right)\sum_t |t\rangle\langle t|. \tag{4.25}$$

Of course, the following relations are obtained:

$$\left[\hat{\mathcal{R}}_+, \hat{\mathcal{R}}_-\right] = 2\hat{\mathcal{R}}_0 - \left(\hat{T}_m - \hat{T} + 1\right)\left(2\hat{T} - 1\right)\sum_t |t\rangle\langle t|, \tag{4.26}$$

$$\hat{\mathcal{R}}_0|t,t\rangle = -\tfrac{1}{2}(t_m - 3t + 1)|t,t\rangle \neq -\hat{\mathcal{R}}|t,t\rangle. \quad (\text{if } t > 1/2) \tag{4.27}$$

The case $t = 1/2$ is reduced to the $su(2)$-algebra $(\hat{\mathcal{S}}_{\pm,0})$. The above argument may be permitted to call $\hat{Q}_1 \neq 1$ as a pseudo $su(2)$-algebra.

The $(t_0 - t)$ time operation of $\hat{\mathcal{R}}_+$ on the state $|t\rangle$ is given in the form

$$\left(\hat{\mathcal{R}}_+\right)^{t_0 - t}|t\rangle = \sqrt{\frac{(t_m - t)!}{(t_m - t_0)!}}\sqrt{\prod_{k=0}^{t_m - t - 1}\left(q_{p,t} + pk\right)}\sqrt{\frac{(2t-1)!}{(t_0 - t)!(t_0 + t - 1)!}}\left(\hat{T}_+\right)^{t_0 - t}|t\rangle. \tag{4.28}$$

Here, $\prod_{k=0}^{t_m - t - 1}\left(q_{p,t} + pk\right)$ can be expressed in terms of the gamma-function:

$$\prod_{k=0}^{t_m - t - 1}\left(q_{p,t} + pk\right) = p^n \frac{\Gamma\left(q_{p,t}/p + n\right)}{\Gamma\left(q_{p,t}/p\right)}. \quad (n = t_0 - t) \tag{4.29}$$

If we intend to describe the system under investigation exactly, it may be enough to use the orthogonal state $(\hat{T}_+)^n|t\rangle$ $(n = 0, 1, 2, \ldots, t_m - t)$. However, if we adopt an approximation such as in (A), the above idea of the deformation becomes useful. We will consider the case on the following state:

$$\left|\phi_{p,t}\right\rangle = \frac{1}{\sqrt{\Gamma_{p,t}}}\exp\left(z\hat{\mathcal{R}}_+\right)|t\rangle. \quad \left(\left\langle\phi_{p,t}\middle|\phi_{p,t}\right\rangle = 1\right) \tag{4.30}$$

Here, z and $\Gamma_{p,t}$ denote a complex parameter and the normalization constant, respectively. In such a problem, it is indispensable which form is chosen for $\hat{\mathcal{R}}_+$. In (A), as $\hat{\mathcal{R}}_+$, $\hat{\mathcal{T}}_+$, e.g., $\hat{T}_+$ itself was used. The norm of the state $(\hat{\mathcal{R}}_+)^n|t\rangle$ is given by

$$\begin{aligned}\langle t|\left(\hat{\mathcal{R}}_-\right)^n \cdot \left(\hat{\mathcal{R}}_+\right)^n|t\rangle &= \frac{(t_m - t)!}{(t_m - t - n)!}\prod_{k=0}^{n-1}\left(q_{p,t} + pk\right) \cdot \frac{(2t-1)!}{n!(2t-1+n)!}\langle t|\left(\hat{T}_-\right)^n \cdot \left(\hat{T}_+\right)^n|t\rangle \\ &= \frac{(t_m - t)!}{(t_m - t - n)!}\prod_{k=0}^{n-1}\left(q_{p,t} + pk\right),\end{aligned} \tag{4.31}$$

$$\langle t|\left(\hat{T}_-\right)^n \cdot \left(\hat{T}_+\right)^n|t\rangle = \frac{(2t-1+n)!n!}{(2t-1)!}. \tag{4.32}$$

Then, $\Gamma_{p,t}$ is expressed as a function of x in the following:

$$\Gamma_{p,t} = \sum_{n=0}^{t_m-t}(-)^n \binom{t_m-t}{n}\frac{\Gamma(q_{p,t}/p+n)}{n!\Gamma(q_{p,t}/p)}(-px)^n$$
$$= G_{t_m-t}\big(q_{p,t}/p-(t_m-t), 1; -px\big), \tag{4.33}$$
$$x = |z|^2. \tag{4.34}$$

Here, the relation (4.29) was used. The function G_{t_m-t} is a Jacobi polynomial. At the limit $p\to 0$ and $q_{p,t}\to 1$, the expression (4.33) is reduced to

$$\Gamma_{0,t} = \lim_{\substack{p\to 0\\ q_{p,t}\to 1}} \Gamma_{p,t} = \sum_{n=0}^{t_m-t}\frac{(-)^n}{n!}\binom{t_m-t}{n}(-x)^n = L_{t_m-t}(-x). \tag{4.35}$$

The function L_{t_m-t} denotes a Laguerre polynomial. Here, the following relation was used:

$$\lim_{\substack{p\to 0\\ q_{p,t}\to 1}} \frac{\Gamma(q_{p,t}/p+n)}{\Gamma(q_{p,t}/p)}(-px)^n = (-x)^n. \tag{4.36}$$

With the use of the relation (4.29), we can obtain the expression (4.35) directly.

The expression (4.33) is too complicated to be applied to any concrete problem. This indicates that an idea for the approximation must be sought. For this aim, three points for $\Gamma_{p,t}$ must be pointed out. The first is the case $q_{1,t}=1$, which leads us to the simple form:

$$\Gamma_{1,t} = (1+x)^{t_m-t}. \tag{4.37}$$

The second is the maximum power of the polynomial $\Gamma_{p,t}$ for x:

$$\text{the maximum power} = t_m - t. \tag{4.38}$$

It is independent of the choice of $q_{p,t}$. The third is related to the behavior of $\Gamma_{p,t}$ near $x=0$. In the region $x\sim 0$, $\Gamma_{p,t}$ can be expressed as

$$\Gamma_{p,t} = 1+\Gamma_{p,t}^{(1)}x+\tfrac{1}{2}\Gamma_{p,t}^{(2)}x^2+\cdots, \tag{4.39a}$$
$$\Gamma_{p,t}^{(1)} = (t_m-t)q_{p,t}, \quad \Gamma_{p,t}^{(2)} = \tfrac{1}{2}(t_m-t)(t_m-t-1)q_{p,t}\big(q_{p,t}+p\big). \tag{4.39b}$$

Concerning $\Gamma_{p,t}(x)$, the above-mentioned three points suggest the following approximation:

$$\Gamma^a(x) = (1+Cx)^k(1+Dx)^l,$$
$$k,\ l\ :\ \text{positive integers}, \quad k+l=m\ (=t_m-t). \tag{4.40}$$

In order to take into account the difference between C and D, $\Gamma^a(x)$ should be treated in the region

$$1\le l\le m-1. \tag{4.41}$$

Hereafter, in order to avoid unnecessary complication, we will omit the index (p,t) and use the symbol $m(=t_m-t)$. We determine C and D so as to make the coefficients of the terms x and x^2 in the form (4.40) agree with those in the relation (4.39):

$$C = q\left(1 \mp \sqrt{\frac{l}{k} \cdot \frac{m-1}{2} \cdot \zeta}\right), \quad D = q\left(1 \pm \sqrt{\frac{k}{l} \cdot \frac{m-1}{2} \cdot \zeta}\right), \tag{4.42}$$

$$\zeta = 1 - \frac{p}{q}\ (\geq 0). \tag{4.43a}$$

The condition $\zeta \geq 0$ and the relations (4.5) and (4.8) give us the inequality for ζ:

$$0 \leq \zeta \leq 1 + \frac{1}{m},$$

i.e.,

$$0 \leq \zeta < 1\ (p = 1), \quad \zeta = 1\ (p = 0), \quad 1 < \zeta \leq 1 + \frac{1}{m}\ (p = -1). \tag{4.43b}$$

The function $\Gamma(x)(= \Gamma_{p,t}(x))$ is a polynomial, in which the coefficient of each term is positive and, then, $\Gamma(x) = 0$ does not have any root in the range $x \geq 0$, i.e., $\Gamma(x) > 0$ $(x \geq 0)$. Therefore, we require the condition $\Gamma^a(x) > 0$ $(x \geq 0)$, i.e.,

$$C > 0, \quad D > 0. \tag{4.44}$$

The quantities C and D are symmetric with each other for k, l, $\mp$, and $\pm$ in the relation (4.42) and, for D, we can adopt the form with the upper of the sign $\pm$:

$$D = q\left(1 + \sqrt{\frac{k}{l} \cdot \frac{m-1}{2} \cdot \zeta}\right)\ (> 0). \tag{4.45a}$$

Therefore, automatically, C becomes of the form

$$C = q\left(1 - \sqrt{\frac{l}{k} \cdot \frac{m-1}{2} \cdot \zeta}\right)\ (> 0). \tag{4.45b}$$

The expression (4.45a) satisfies $D > 0$. The condition $C > 0$ is reduced to

$$l < \lambda_0, \quad \lambda_0 = \frac{m}{1 + \frac{m-1}{2} \cdot \zeta}. \tag{4.46}$$

It should be noted that l is a positive integer, but λ_0 is generally not an integer. We introduce l_0, which is the nearest integer to λ_0 under the condition $l_0 < \lambda_0$. Then, the following inequality is obtained:

$$0 < \lambda_0 - l_0 \leq 1, \quad l = l_0, \quad l_0 - 1, \ldots, 2, 1. \tag{4.47}$$

With the use of the relation (4.46), the inequality (4.47) can be rewritten in the form

$$\begin{aligned} &\zeta_{\min} \leq \zeta \leq \zeta_{\max}, \\ &\zeta_{\min} = \frac{2(m-1-l_0)}{(m-1)(l_0+1)}, \quad \zeta_{\max} = \frac{2(m-l_0)}{(m-1)l_0}. \end{aligned} \tag{4.48}$$

Some concrete cases of the inequality are summarized in Table 1. We can see that the case $(p = 1,$ $q = 1)$ reduces to $\zeta = 0$ and $C = D = 1$. In this case, the relation (4.40) becomes of the form (4.37) independent of the choice of l. Of course, this case corresponds to the $su(2)$-algebra. The result in any other case depends on the choice of l. So far, we have no idea how to determine the value of

Table 1. Parameter set.

l_0	$\zeta_{\min}$	$\zeta_{\max}$	l
$m-1$	0	$\dfrac{2}{(m-1)^2}$	$m-1, m-2, \ldots, 2, 1$
$m-2$	$\dfrac{2}{(m-1)^2}$	$\dfrac{4}{(m-1)(m-2)}$	$m-2, m-3, \ldots, 2, 1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\dfrac{m+1}{2}$	$\dfrac{2(m-3)}{(m-1)(m+3)}$	$\dfrac{2}{m+1}$	$\dfrac{m+1}{2}, \dfrac{m-1}{2}, \ldots, 2, 1\ \ (m: \text{odd})$
$\dfrac{m}{2}$	$\dfrac{2(m-2)}{(m-1)(m+2)}$	$\dfrac{2}{m-1}$	$\dfrac{m}{2}, \dfrac{m-2}{2}, \ldots, 2, 1\ \ (m: \text{even})$
$\dfrac{m-1}{2}$	$\dfrac{2}{m+1}$	$\dfrac{2(m+1)}{(m-1)^2}$	$\dfrac{m-1}{2}, \dfrac{m-3}{2}, \ldots, 2, 1\ \ (m: \text{odd})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
2	$\dfrac{2(m-3)}{3(m-1)}$	$\dfrac{m-2}{m-1}$	$2, 1$
1	$\dfrac{m-2}{m-1}$	$1+\dfrac{1}{m}$	1

l. In relation to a certain approximation adopted in next section, the case $l = 1$ may be the most reasonable.

The expectation values of $\hat{T}_0$, $\hat{\mathcal{R}}_+$, and $\hat{\mathcal{R}}_-$ for $|\phi\rangle$ given in the relation (4.30) are calculated in the form

$$\left\langle\phi\middle|\hat{T}_0\middle|\phi\right\rangle = t + \Lambda(x), \tag{4.49a}$$

$$\left\langle\phi\middle|\hat{\mathcal{R}}_+\middle|\phi\right\rangle = z^*\frac{\Lambda(x)}{x}, \quad \left\langle\phi\middle|\hat{\mathcal{R}}_-\middle|\phi\right\rangle = z\frac{\Lambda(x)}{x}. \tag{4.49b}$$

Here, $\Lambda(x)$ is defined as

$$\Lambda(x) = \frac{x\frac{d\Gamma(x)}{dx}}{\Gamma(x)}. \tag{4.50}$$

In the case of the approximate form of $\Lambda(x)$, $\Lambda^a(x)$ is given in the form

$$\Lambda^a(x) = \frac{kCx}{1+Cx} + \frac{lDx}{1+Dx} = m\left[1 - \left(\frac{k/m}{1+Cx} + \frac{l/m}{1+Dx}\right)\right]. \tag{4.51}$$

The relation (4.51) will play a central role in next section. The relations (4.34) and (4.35) lead us to the following expression for $\Lambda(x)$:

$$\Lambda(x) = m\left(1 - \frac{G_{m-1}(q/p-m+1, 1, -px)}{G_m(q/p-m, 1, -px)}\right), \tag{4.52}$$

$$\Lambda(x) = m\left(1 - \frac{L_{m-1}(-x)}{L_m(-x)}\right). \tag{4.53}$$

Figure 1 shows the comparison of $\Lambda^a(x)$ for the case $(k = m-1,\ l = 1)$ and $\Lambda(x)$.

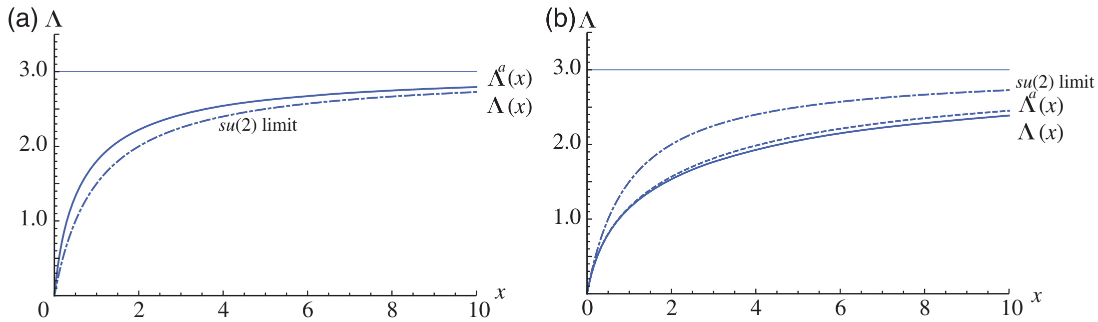

Fig. 1. (a) The comparison of $\Lambda^a(x)$ with $\Lambda(x)$ in Eq. (4.52) for the case $k = m - 1,\ l = 1$ with $m = 3$. Here, $p = 1$ and $q = 2$ are adopted. In this case, $\Lambda^a(x)$ and $\Lambda(x)$ almost overlap one another. (b) The comparison of $\Lambda^a(x)$ (dotted curve) with $\Lambda(x)$ (solid curve) in Eq. (4.53) with $p = 0$ and $q = 1$. In both panels, the dash-dotted curves represent the $su(2)$ limit.

5. A possible application

As mentioned in Sect. 1, the aim of this paper is to formulate a new boson representation of the $su(2)$-algebra and its deformation, in which the idea of phase space doubling is applied straightforwardly. In this section, in order to demonstrate our idea, we will apply the present form to the case of a simple boson model. This model is essentially the same as that discussed in Sect. 7 in (A). We pay attention to the boson Hamiltonian

$$\hat{H}_b = \omega\hat{b}^*\hat{b}. \tag{5.1}$$

This Hamiltonian corresponds to the fermionic one, as discussed in Sect. 7 of (A). The Hamiltonian $\hat{H}_b$ is nothing but $\hat{H}_{\text{intr}}$, introduced in the relation (1.1). Following the idea of phase space doubling, we introduce another boson Hamiltonian $\hat{H}_a = \omega\hat{a}^*\hat{a}$ and set up the form

$$\hat{H}_{ba} = \hat{H}_b - \hat{H}_a = \omega\left(\hat{b}^*\hat{b} - \hat{a}^*\hat{a}\right). \tag{5.2}$$

Of course, $\hat{H}_a$ plays the role of $\hat{H}_{\text{extr}}$ and $\hat{H}_{ba} = \hat{H}_0$. As for the interaction between the two boson systems, $\hat{V}_{ba}$, we adopt the following form:

$$\hat{V}_{ba} = -i\gamma\left(\hat{a}^*\hat{b}^* \cdot f\left(\hat{a}^*\hat{a},\ \hat{b}^*\hat{b}\right) - f\left(\hat{a}^*\hat{a},\ \hat{b}^*\hat{b}\right) \cdot \hat{b}\hat{a}\right). \tag{5.3}$$

Here, γ denotes the interaction strength. For example, the case $f\left(\hat{a}^*\hat{a},\ \hat{b}^*\hat{b}\right) = 1$ corresponds to the $su(1,1)$-algebraic model investigated in Refs. [8–10]. As for $\hat{a}^*\hat{b}^* \cdot f\left(\hat{a}^*\hat{a},\ \hat{b}^*\hat{b}\right)$, we adopt the operator $\hat{\mathcal{R}}_+$, shown in the relation (4.3), and then the Hamiltonian $\hat{H}$ is given by

$$\hat{H} = \hat{H}_{ba} + \hat{V}_{ba} = \omega\left(2\hat{T} - 1\right) - i\gamma\left(\hat{\mathcal{R}}_+ - \hat{\mathcal{R}}_-\right). \tag{5.4}$$

It should be noted that $\hat{H}$ does not mean the total energy. It may be clear that $\hat{T}$ is a constant of motion. The above is our model discussed in this paper.

For treating the Hamiltonian (5.4), we follow the same method as in (A). Regarding $|\phi\rangle$ as a time-dependent variational state, we set up the following variational equation:

$$\delta\int\left\langle\phi\middle|i\partial_\tau - \hat{H}\middle|\phi\right\rangle d\tau = 0. \tag{5.5}$$

Here, in order to avoid confusion between the time variable and the quantum number t, we will use τ for the time variable. If a finite temperature system is considered, the entropy operator $\hat{S}_{\text{ent}}$ should be introduced, the Hamiltonian $\hat{H}$ should be replaced with the free energy $\hat{H} - \hat{S}_{\text{ent}}/\beta$, where

β represents the inverse of temperature, and a mixed state should be introduced, as discussed in Ref. [14]. In this paper, a system at zero temperature is treated. If z and z^* are regarded as time-dependent variational parameters, the variational equation (5.5) leads us to the following equation:

$$\dot{z} = -\gamma\left[1 - \frac{z^2}{x}\left(1 - \frac{\Lambda(x)}{x\frac{d\Lambda(x)}{dx}}\right)\right], \quad \dot{z}^* = -\gamma\left[1 - \frac{z^{*2}}{x}\left(1 - \frac{\Lambda(x)}{x\frac{d\Lambda(x)}{dx}}\right)\right]. \tag{5.6}$$

The expectation value of $\hat{H}$, $\mathcal{H}$, is given in the form

$$\begin{aligned}\mathcal{H} &= \langle\phi|\hat{H}|\phi\rangle = \omega(2t-1) - i\gamma\left(\mathcal{R}_+ - \mathcal{R}_-\right)\\ &= \omega(2t-1) - \gamma i\left(z^* - z\right)\frac{\Lambda(x)}{x}.\end{aligned} \tag{5.7}$$

Here, $\mathcal{R}_\pm$ denote the expectation values of $\hat{\mathcal{R}}_\pm$. The details can be found in (A).

The present system is of two dimensions and, therefore, there exist two constants of motion. One is the quantum number t and the second, which will be denoted by κ, is given through the relation

$$i\left(z^* - z\right)\frac{\Lambda(x)}{x} = 2\kappa. \tag{5.8}$$

This may be self-evident, because $\mathcal{H}$ itself, shown in the relation (5.8), is a constant of motion. If z is expressed in the form $z = u + iv$, we have

$$i\left(z^* - z\right) = 2v. \tag{5.9}$$

In (A), we learned that, instead of x, it may be convenient to adopt the variable y, defined as

$$y = \frac{\Lambda(x)}{x}. \quad \left(x = |z|^2 = u^2 + v^2\right) \tag{5.10}$$

Inversely solving this, x can be expressed as a function of y. Then, v can be expressed in the form

$$v = \frac{\kappa}{y}, \quad \text{i.e.,} \quad yu = \pm\sqrt{xy^2 - \kappa^2}. \tag{5.11}$$

With the use of the relation (5.6), $\dot{x}$ can be given as

$$\dot{x} = -2\gamma\frac{\Lambda(x)}{x\frac{d\Lambda(x)}{dx}}\cdot u. \tag{5.12}$$

The definitions of y and $\dot{x}$, which are given in the relations (5.10) and (5.12), respectively, give us $\dot{y}$ in the form

$$\dot{y} = -\frac{2\gamma}{x + y\frac{dx}{dy}}\cdot\left(\pm\sqrt{xy^2 - \kappa^2}\right). \tag{5.13}$$

Now, let us express x as a function of y. The basic equation for this task is the relation (5.10). As for $\Lambda(x)$, we adopt the approximate form $\Lambda^a(x)$ given in the relation (4.51). For $\Lambda^a(x)$, the relation (5.10) is reduced to the form

$$CDy\cdot x^2 - (mCD - y(C+D))\cdot x + (y - (kC + lD)) = 0. \tag{5.14}$$

A solution of Eq. (5.14) is as follows:

$$\begin{aligned}&x = \frac{m}{2y} - \frac{C+D}{2CD} + \frac{m}{2y}\sqrt{1 + 2Iy + J^2y^2},\\ &I = \left(\frac{k-l}{m^2}\right)\left(\frac{C-D}{CD}\right), \quad J^2 = \frac{1}{m^2}\left(\frac{C-D}{CD}\right)^2. \left(J^2 = \left(\frac{k+l}{k-l}\right)^2 I^2\right)\end{aligned} \tag{5.15}$$

In the case $C = D$, another solution becomes negative and we pick up only the solution (5.15). Next, we consider a possible approximation of $\sqrt{1+2Iy+J^2y^2}$, which, up to the term y^2, is expanded for y:

$$\sqrt{1+2Iy+J^2y^2} = 1 + Iy + \tfrac{1}{2}(J^2 - I^2)y^2. \tag{5.16}$$

Let the following inequality be permitted:

$$y \ll \left|\frac{2I}{J^2 - I^2}\right|. \tag{5.17}$$

Then, we are able to obtain the approximate form

$$\sqrt{1+2Iy+J^2y^2} = 1 + Iy = 1 + \left(\frac{k-l}{m^2}\right)\left(\frac{C-D}{CD}\right)y. \tag{5.18}$$

Later, we will discuss the condition under which the inequality (5.17) is meaningful. Then, we have

$$x = \frac{m}{2y} - \frac{C+D}{2CD} + \frac{m}{2y}\left(1 + \left(\frac{k-l}{m^2}\right)\left(\frac{C-D}{CD}\right)y\right) = \frac{m}{y} - \frac{1}{B}, \tag{5.19}$$

$$\frac{1}{B} = \frac{1}{m}\left(\frac{k}{C} + \frac{l}{D}\right). \tag{5.20}$$

With the use of the relation (5.19), we obtain

$$xy^2 - \kappa^2 = \left(\frac{m^2B}{4} - \kappa^2\right) - \frac{1}{B}\left(y - \frac{mB}{2}\right)^2, \tag{5.21a}$$

$$x + y\frac{dx}{dy} = -\frac{1}{B}. \tag{5.21b}$$

Therefore, $\dot{y}$, shown in the relation (5.13), can be expressed as

$$\dot{y} = \pm 2\gamma B\sqrt{\left(\frac{m^2B}{4} - \kappa^2\right) - \frac{1}{B}\left(y - \frac{mB}{2}\right)^2}. \tag{5.22}$$

By solving Eq. (5.22), y can be expressed as a function of τ. The relation (5.22) can be rewritten in the form

$$\tfrac{1}{2}\dot{y}^2 + \tfrac{1}{2}\cdot 4\gamma^2 B\left(y - \frac{mB}{2}\right)^2 = 2\gamma^2 B\left[\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2\right]. \tag{5.23}$$

The relation (5.23) tells us that the present system is equivalent to a simple harmonic oscillator in classical mechanics. Then, we have

$$y = \frac{mB}{2} + \sqrt{\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2}\cos\left(2\gamma\sqrt{B}\tau + \chi_0\right). \tag{5.24}$$

Here, χ_0 is determined by the initial condition. The quantities x and $\Lambda^a(x)$ can be expressed in the following form:

$$x = \frac{\frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2}\cos\left(2\gamma\sqrt{B}\tau + \chi_0\right)}{\frac{mB}{2} + \sqrt{\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2}\cos\left(2\gamma\sqrt{B}\tau + \chi_0\right)}, \tag{5.25}$$

$$\Lambda^a(x) = \frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2}\cos\left(2\gamma\sqrt{B}\tau + \chi_0\right). \tag{5.26}$$

Thus, we could express x and $\Lambda^a(x)$ as functions of τ. Of course, this is a general solution, in which the initial and boundary conditions are not taken into account. In Sect. 6, we will discuss these conditions.

Finally, we will discuss the inequality (5.17), which leads us to the simple result shown in the relation (5.26). First, we note the maximum value of y, $y_{\max}$, which is expressed as

$$y_{\max} = mB. \tag{5.27}$$

The relation (5.27) is obtained under the condition $\left(\kappa = 0,\ \cos\left(2\gamma\sqrt{B}\tau + \chi_0\right) = 1\right)$ in the relation (5.24). Therefore, we have

$$mB \ll \left|\frac{2I}{J^2 - I^2}\right|. \tag{5.28}$$

After rather lengthy consideration, the following inequality can be derived from the relation (5.28):

$$\zeta_0 \ll \zeta_0(j), \tag{5.29}$$

$$\text{(i)}\ \ \text{for } 0 < j \le \frac{1}{4}, \quad \zeta_0(j) = \frac{2}{m-1}\cdot\frac{1-j}{j}, \tag{5.30a}$$

$$\text{(ii)}\ \ \text{for } \frac{1}{4} < j < \tfrac{1}{2}, \quad \zeta_0(j) = \frac{2}{m-1}\cdot\frac{j(1-j)(1-2j)^2}{\left(1-6j+6j^2\right)^2}, \tag{5.30b}$$

$$\text{(iii)}\ \ \text{for } \tfrac{1}{2} \le j < 1, \quad \zeta_0(j) = \frac{2}{m-1}\cdot\frac{j(1-j)(1-2j)^2}{\left(1-2j+2j^2\right)^2}. \tag{5.30c}$$

Here, j denotes

$$j = \frac{l}{m}.(0 < j < 1) \tag{5.31}$$

On the other hand, we note the inequality (4.48):

$$\zeta < \zeta_{\max}(j_0), \quad \zeta_{\max}(j_0) = \frac{2}{m-1}\cdot\frac{1-j_0}{j_0}. \tag{5.32}$$

Here, j_0 denotes

$$j_0 = \frac{l_0}{m}.\left(0 < j_0 < 1\right) \tag{5.33}$$

Further, we note the relation (4.40), which can be expressed as

$$j_0 \ge j. \tag{5.34}$$

The inequality (5.29) and (5.32) suggest the relation

$$\zeta_{\max}(j_0) \ll \zeta_0(j). \tag{5.35}$$

Figure 2 shows the behavior of ζ_0 and $\zeta_{\max}$ in units of $(2/(m-1))$. From the figure, we can learn the following points: (i) If $j \sim 0$ and $j_0 \sim 1$, the inequality (5.35) is sufficiently satisfied. (ii) If $1/4 \le j < 1/2$ and $j_0 \sim 1$, the inequality (5.35) may be satisfied, but not so sufficiently as in case (i). (iii) If $1/2 < j < 1$ and $1/2 < j_0 < 1$, the inequality (5.35) is not satisfied. The above summary leads us to the following conclusion: If l is rather far from l_0 ($l \ll l_0$), our approximation may be justified. Therefore, the case ($l = 1,\ l_0 = m - 1$) is the most reliable. This point has already been suggested in the previous section.

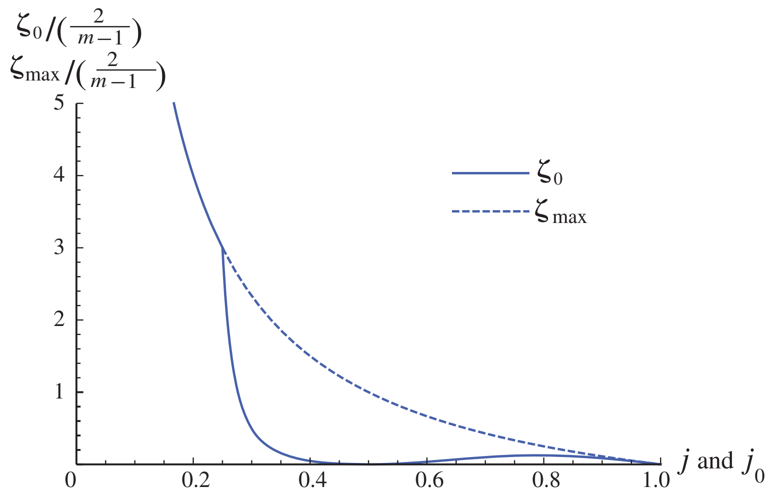

Fig. 2. For the comparison, $\zeta_0(j)$ in Eq. (5.29) and ζ_{max} in Eq. (5.32) are depicted. The vertical axis represents $\zeta_0(j)$ and/or $\zeta_{max}(j_0)$ in units of $2/(m-1)$.

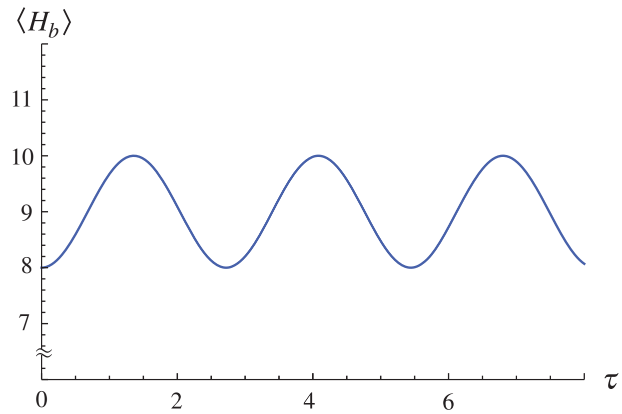

Fig. 3. The time-dependent energy for the b-system depicted as a function of time τ.

As an example of physical systems, let us consider the b-system governed by the Hamiltonian (5.2) considered in this section. Figure 3 shows the energy expectation value for the b-system as a function of time τ, which is depicted by using the approximation in Eq. (5.26). The parameters are taken as $t = 4$, $s = 3/2$, which leads to $t_m = 7$ and $m = 3$. Also, $l = 1$, $p = 1$, $q = 2$, $\gamma = 1$, $\omega = 1$, and $\kappa = 3/2$ are adopted and an initial condition, $\chi_0 = 0$, is given. It is seen that the energy flows into the b-system from the external environment and vice versa. Namely, the behavior of the intrinsic system is not a simple pure damped or amplified oscillator due to the $su(2)$ algebra with $p = 1$ being the compact one, while the simple damped or amplified oscillator appears in the case governed by the $su(1, 1)$-algebra [9].

6. Detailed explanation of the results

First of all, we will examine the formal result for the approximate solution (5.26) closely. For this aim, we first consider the quantity B, defined in the relation (5.20). With the use of the relations

Table 2. Examples for q and B.

q	B
1	1
2	$1+\dfrac{1}{m}$
6	$(6-\sqrt{5})\left(1+\dfrac{1}{m-2\left(1-\sqrt{\frac{3}{5}}\right)}\right)\approx 2.127\left(1+\dfrac{1}{m-0.451}\right)$
9	$3\left(1+\dfrac{1}{m-\frac{1}{2}}\right)$
13	$(13-\sqrt{78})\left(1+\dfrac{1}{m-\left(2-\sqrt{\frac{6}{13}}\right)}\right)\approx 4.168\left(1+\dfrac{1}{m-1.321}\right)$

(4.45a) and (4.45b), B can be expressed in the following form for the case $(k=m-1,\ l=1)$:

$$B=q\left(1-\sqrt{\frac{\zeta}{2}}\right)\left(1+\frac{1}{m-\left(2-\sqrt{\frac{2}{\zeta}}\right)}\right). \tag{6.1}$$

Since $m\geq 2$, B obeys the inequality

$$B\geq q\left(1-\sqrt{\frac{\zeta}{2}}\right). \tag{6.2}$$

We are mostly interested in the case $p=1$, i.e., $\zeta=1-1/q$. Then, we have the relation

$$q\left(1-\sqrt{\frac{\zeta}{2}}\right)-1=\frac{1}{\sqrt{2}}\frac{\sqrt{q-1}(q-2)}{\sqrt{2(q-1)}+\sqrt{q}}\geq 0. \tag{6.3}$$

The inequalities (6.2) and (6.3) lead us to

$$B\geq 1. \tag{6.4}$$

Examples are shown in Table 2. In several places later in the paper, we will use the inequality (6.4).

Now we investigate the general solution (5.26). For this aim, we define two functions:

$$\Lambda^a(\theta)=\frac{mB}{2}-\sqrt{\left(\frac{mB}{2}\right)^2-\left(\sqrt{B}\kappa\right)^2}\cos\theta, \tag{6.5a}$$

$$y(\theta)=\frac{mB}{2}+\sqrt{\left(\frac{mB}{2}\right)^2-\left(\sqrt{B}\kappa\right)^2}\cos\theta. \tag{6.5b}$$

If θ is replaced with $\left(2\gamma\sqrt{B}\tau+\chi_0\right)$, $\Lambda^a(\theta)$ and $y(\theta)$ are reduced to the results (5.26) and (5.24). The present approximate result should obey the following boundary conditions:

$$\text{(i) } y(\theta)\cdot\Lambda^a(\theta)-\kappa^2\geq 0,\quad \text{i.e.,}\quad \left(\frac{mB}{2}\right)^2-\left(\left(\frac{mB}{2}\right)^2-\left(\sqrt{B}\kappa\right)^2\right)\cos^2\theta-\kappa^2\geq 0, \tag{6.6a}$$

$$\text{(ii)}\ \left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2 \geq 0, \tag{6.6b}$$

$$\text{(iii)}\ \Lambda^a(\theta) \leq m, \quad \text{i.e.,} \quad \frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2}\ \cos\theta \leq m. \tag{6.6c}$$

Condition (i) results from the relation (5.11), in which $\left(xy^2 - \kappa^2\right)$ can be expressed in the form $\left(y(\theta) \cdot \Lambda^a(\theta) - \kappa^2\right)$. Condition (ii) may be self-evident. Condition (iii) comes from the relations (4.52) and (4.53). On the basis of conditions (i), (ii), and (iii), we examine our general solution (5.26).

Let us start with condition (i). It is easily verified through the following inequality:

$$y(\theta) \cdot \Lambda^a(\theta) - \kappa^2 = \left(\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2\right) \sin^2\theta + (B-1)\kappa^2 \geq 0. \tag{6.7}$$

Here, we used the conditions (6.4) and (6.6b). It is important to see that condition (i) holds at any value of θ. It may be convenient to treat condition (ii) by classifying it into two cases (a) and (b):

$$\text{(a)}\ \left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2 = 0, \quad \text{i.e.,} \quad |\kappa| = \frac{m}{2}\sqrt{B}, \tag{6.8a}$$

$$\text{(b)}\ \left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2 > 0, \quad \text{i.e.,} \quad |\kappa| < \frac{m}{2}\sqrt{B}. \tag{6.8b}$$

In the present treatment, κ is given as an initial condition and case (a) gives us time-independent $\Lambda^a (= mB/2)$.

The consideration of condition (iii) is rather lengthy; therefore, only the results will be presented. In this case, it may be successful to consider the problem in the following four cases, depicted in Fig. 4: Case (A) is nothing but case (a) given in the relation (6.8a). Since $\Lambda^a(\tau) \leq m$, we have

$$1 \leq B \leq 2, \quad |\kappa| = \frac{m}{2}\sqrt{B}, \quad \Lambda^a(\tau) = \Lambda^a_A(\tau) = \frac{mB}{2}. \tag{6.9}$$

In case (B), the following inequality holds:

$$\frac{mB}{2} + \sqrt{\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2} \leq m. \tag{6.10}$$

By solving the inequality (6.10), we have

$$1 \leq B < 2, \quad m\sqrt{1 - \frac{1}{B}} \leq |\kappa| < \frac{m}{2}\sqrt{B}. \tag{6.11}$$

Here, of course, we used the result of case (b). Then, if at the initial time $\tau = 0$, $\theta = 0$ is chosen, i.e., $\chi_0 = 0$ in the general solution (5.26), we have

$$\Lambda^a(\tau) = \Lambda^a_B(\tau) = \frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2}\ \cos\left(2\gamma\sqrt{B}\tau\right). \tag{6.12}$$

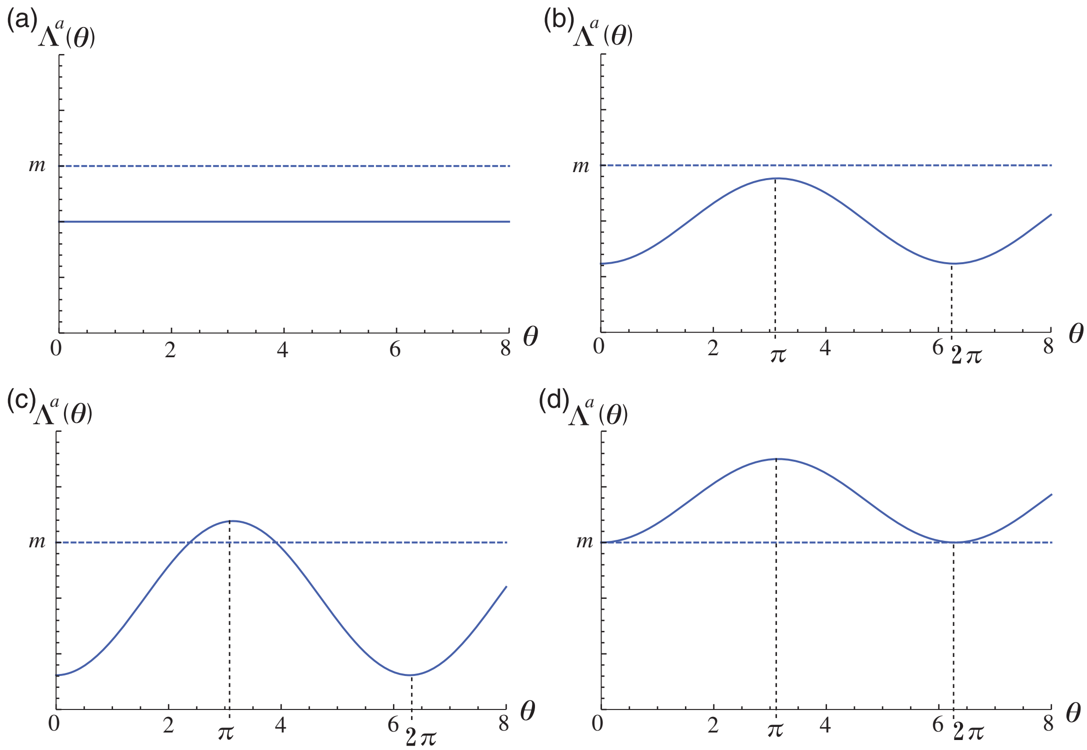

Fig. 4. Schematic depiction of the four cases (a), (b), (c) and (d) under consideration, which lead to Case (A), (B), (C) and (D) under some conditions described in the text.

Case (C) satisfies the inequality

$$\frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2} < m < \frac{mB}{2} + \sqrt{\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2}. \tag{6.13}$$

In this case, we obtain

$$B > 1, \quad |\kappa| < m\sqrt{1 - \frac{1}{B}}. \tag{6.14}$$

We adopt the same initial condition as that in the above, $\theta = 0$, i.e., $\chi_0 = 0$ at $\tau = 0$. As shown in Fig. 4, there exists an angle θ_m and it is given in the form

$$\frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2}\ \cos\theta_m = m,$$
$$\text{i.e., } \cos\theta_m = \frac{m\left(\frac{B}{2} - 1\right)}{\sqrt{\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2}}, \quad (0 < \theta_m < \pi). \tag{6.15}$$

At the time $\tau_m = \theta_m/\left(2\gamma\sqrt{B}\right)$, $\Lambda^a(\theta_m) = m$ and, in the interval $\tau = 0 \to \tau_m$, $\Lambda^a(\tau)$ can be expressed as

$$\Lambda^a(\tau) = \Lambda_0^a(\tau) = \frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2}\ \cos\left(2\gamma\sqrt{B}\tau\right). \tag{6.16}$$

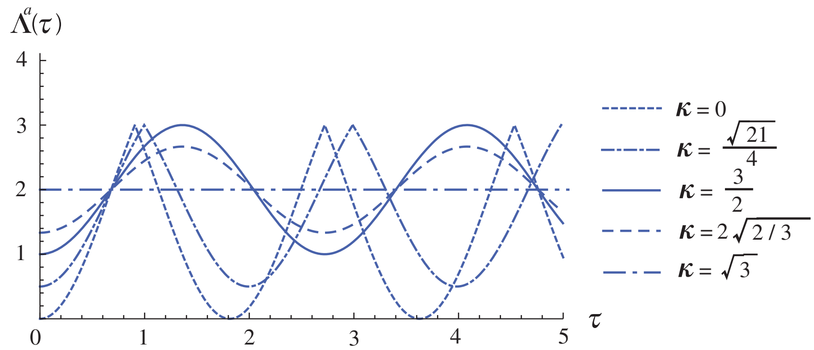

Fig. 5. The behavior of $\Lambda^a(\tau)$ for various values of κ is shown with the same parameters as in Fig. 3, except for κ.

However, after $\tau = \tau_m$, $\Lambda_0^a(\tau)$ cannot be adopted, because, if it is permitted, $\Lambda_0^a(\tau) > m$. Then, we define the following function:

$$\Lambda^a(\tau) = \Lambda_1^a(\tau) = \frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2} \cos\left(2\gamma\sqrt{B}(\tau - 2\tau_m)\right). \tag{6.17}$$

The function $\Lambda_1^a(\tau)$ satisfies $\Lambda_1^a(\tau_m) = \Lambda_0^a(\tau_m) = m$ and, in the interval $\tau = \tau_m \to 3\tau_m$, $\Lambda_1^a(\tau) < m$. Further, in the interval $\tau = 3\tau_m \to 5\tau_m$, we define $\Lambda_2^a(\tau)$ in the form

$$\Lambda^a(\tau) = \Lambda_2^a(\tau) = \frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2} \cos\left(2\gamma\sqrt{B}(\tau - 4\tau_m)\right). \tag{6.18}$$

Certainly, $\Lambda_2^a(3\tau_m) = \Lambda_1^a(3\tau_m) = m$ and $\Lambda_2^a(\tau)$ is useful in the interval $\tau = 5\tau_m \to 7\tau_m$. By proceeding with this task, we arrive at the following solution:

$$\Lambda^a(\tau) = \Lambda_C^a(\tau) = \Lambda_n^a(\tau) = \frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2} \cos\left(2\gamma\sqrt{B}(\tau - 2n\tau_m)\right),$$
$$\text{for} \quad (2n-1)\tau_m \le \tau \le (2n+1)\tau_m. (n = 0, 1, 2, 3, \ldots) \tag{6.19}$$

In case (D), we have the relation

$$\frac{mB}{2} - \sqrt{\left(\frac{mB}{2}\right)^2 - \left(\sqrt{B}\kappa\right)^2} = m. \tag{6.20}$$

The solution of this equation is given as

$$B > 2, \quad |\kappa| = m\sqrt{1 - \frac{1}{B}}, \quad \Lambda^a(\tau) = \Lambda_D^a(\tau) = m. \tag{6.21}$$

Case (D) is regarded as the limit $\theta_m \to 0$ in case (C). The function $\Lambda_D^a(\tau)$ does not depend on τ, but its origin is different from that in case (A). Figure 5 shows the behavior of $\Lambda^a(\tau)$ for various values of κ. The same parameters as those used in Fig. 3 are adopted, except for κ, which is a conserved quantity determined by the initial condition. Under this parameter set, we obtain $B = 4/3$, which is in the range $1 < B < 2$. Then, for various values of κ, the function $\Lambda^a(\tau)$ is turned into $\Lambda_C^a(\tau)$, $\Lambda_B^a(\tau)$, or $\Lambda_A^a(\tau)$ according to Table 3. For $\kappa = 0$ and $\kappa = \sqrt{21}/4 \approx 1.1456$, we take $\Lambda^a(\tau) = \Lambda_C^a(\tau)$. For $\kappa = 3/2$, we adopt $\Lambda^a(\tau) = \Lambda_C^a(\tau) = \Lambda_B^a(\tau)$. For $\kappa = 2\sqrt{2/3} \approx 1.633$, we chose $\Lambda^a(\tau) = \Lambda_B^a(\tau)$. Finally, for $\kappa = \sqrt{3} \approx 1.732$, we adopt $\Lambda^a(\tau) = \Lambda_B^a(\tau) = \Lambda_A^a(\tau)$.

Table 3. For various values of B and κ, $\Lambda^a(\tau)$ is turned into $\Lambda^a_A(\tau)$, $\Lambda^a_B(\tau)$ or $\Lambda^a_C(\tau)$.

B	κ	$\Lambda^a(\tau)$
$B = 1$	$0 \le \lvert\kappa\rvert < \dfrac{m}{2}$	$\Lambda^a_B(\tau)$
	$\lvert\kappa\rvert = \dfrac{m}{2}$	$\Lambda^a_A(\tau)$
$1 < B < 2$	$0 \le \lvert\kappa\rvert < m\sqrt{1-\dfrac{1}{B}}$	$\Lambda^a_C(\tau)$
	$m\sqrt{1-\dfrac{1}{B}} \le \lvert\kappa\rvert < \dfrac{m}{2}\sqrt{B}$	$\Lambda^a_B(\tau)$
	$\lvert\kappa\rvert = \dfrac{m}{2}\sqrt{B}$	$\Lambda^a_A(\tau)$
$B = 2$	$0 \le \lvert\kappa\rvert < \dfrac{m}{\sqrt{2}}$	$\Lambda^a_C(\tau)$
	$\lvert\kappa\rvert = \dfrac{m}{\sqrt{2}}$	$\Lambda^a_A(\tau)$
$B > 2$	$0 \le \lvert\kappa\rvert < m\sqrt{1-\dfrac{1}{B}}$	$\Lambda^a_C(\tau)$
	$\lvert\kappa\rvert = m\sqrt{1-\dfrac{1}{B}}$	$\Lambda^a_D(\tau)$

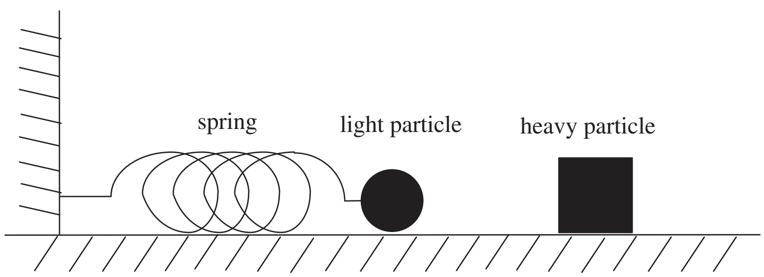

Fig. 6. The elastic collision of a simply oscillating light particle with a sufficiently heavy one.

In classical mechanics, we can find the same problem as that discussed in this section: elastic collision of a simply oscillating light particle with a sufficiently heavy one, which is illustrated in Fig. 6. The results obtained in the above are summarized in Table 3.

So far, we have proposed a new boson representation of the $su(2)$-algebra. The basic idea comes from the pseudo $su(1,1)$-algebra in the Schwinger boson representation. In a certain sense, ours is on the opposite side of the Schwinger representation of the $su(2)$-algebra. In subsequent sections, we will prove that ours satisfies the $su(2)$-algebra in the subspace (2.8) of the whole space (2.5) for the case $t_m = C_m + 1 - t$, and will give connections to the other boson representations.

7. Some theoretical features of the new boson representation

In the same scheme as that in the Schwinger boson representation [7], three generators in our new boson representation are also expressed in terms of two kinds of bosons. Concrete forms can be seen in the relations (3.7) and (3.8c) or (3.9) and (3.10). The operator for the magnitude of the $su(2)$-spin is given in the relation (3.10): $\hat{S} = \left(\hat{a}^*\hat{a} - \hat{b}^*\hat{b}\right)/2 + C_m/2$. Here, C_m denotes a certain constant, which is appropriately chosen. On the other hand, in the Schwinger representation, $\hat{S}$ is given as $\hat{S} = \left(\hat{a}^*\hat{a} + \hat{b}^*\hat{b}\right)/2$, which is seen in the relation (3.2b). This is an essential difference between the Schwinger and our representations. In the previous section, we promised to prove that our new boson representation obeys the $su(2)$-algebra. For this proof, we must consider the subspace (2.8) in the whole space given in the Schwinger boson representation of the $su(1,1)$-algebra. In this subspace, our representation satisfies the $su(2)$-algebra. In the next subsection, Sect. 7.1, it is proved that, in the subspace (2.8), $\hat{\mathcal{S}}_{\pm,0}$, introduced in the relation (3.9), obey the $su(2)$-algebra, and $\hat{\mathcal{S}}$, given in the relation (3.10), plays the role of the magnitude of the $su(2)$-spin. In Sect. 7.2, raising and lowering operators for the magnitude of the $su(2)$-spin are discussed, respectively.

7.1. The boson representation of the $su(2)$-algebra developed in the preceding sections

The main aim of this subsection is to formulate the fact that our representation of the $su(2)$-algebra strictly holds in the space (2.8) as a subspace of the whole space (2.5). We sketch out the space (2.8) and other subspace on the t–t_0 plane. As has already been mentioned, we are interested in the space obeying the condition (2.8), which is a subspace of the whole space specified by the condition (2.5). Figure 7 shows various subspaces for the case $\hat{T}_m = C_m + 1 - \hat{T}$. This formula was given in the range $1/2 \leq t \leq \mu$ $(= (C_m+1)/2)$ for a certain reason, discussed in Sect. 2. We assume that the above formula for $\hat{T}_m$ is also useful in the range $-\infty < t \leq 0$ and, in this paper, the case $\hat{T}_m = C_m + 1 - \hat{T}$ is treated exclusively. In our present scheme, the whole space is divided into five subspaces P, Q, R_p, R_q, and R, which are shown in Fig. 7. Of course, we are mainly interested in P. With the above-mentioned point in mind, the $su(2)$-algebra was formulated in Sect. 3. The expressions for $\hat{\mathcal{S}}_{\pm,0}$ and $\hat{\mathcal{S}}$ are given in the relations (3.7) and (3.8) or (3.8c) and (3.10). In this subsection, we will examine some properties of these expressions in P and Q.

To prepare for the main discussion, we first make a list of the relations. The relations (3.7a) and (3.8c) or (3.9c) and (3.10) are rewritten as

$$\hat{T} = \tfrac{1}{2}\left(C_m + 1\right) - \hat{\mathcal{S}}, \quad \hat{T}_0 = \tfrac{1}{2}\left(C_m + 1\right) + \hat{\mathcal{S}}_0, \tag{7.1}$$

$$\text{i.e.,} \quad t = \tfrac{1}{2}\left(C_m + 1\right) - s, \quad t_0 = \tfrac{1}{2}\left(C_m + 1\right) + s_0. \tag{7.2}$$

The relations (3.9a) and (3.9b) are rewritten as

$$\hat{\mathcal{S}}_+ = \hat{a}^*\hat{b}^* \cdot \left(\sqrt{C_m + 1 - \hat{\mathcal{S}} + \hat{\mathcal{S}}_0 + \epsilon}\right)^{-1} \cdot \sqrt{\hat{\mathcal{S}} - \hat{\mathcal{S}}_0}, \tag{7.3a}$$

$$\hat{\mathcal{S}}_- = \sqrt{\hat{\mathcal{S}} - \hat{\mathcal{S}}_0} \cdot \left(\sqrt{C_m + 1 - \hat{\mathcal{S}} + \hat{\mathcal{S}}_0 + \epsilon}\right)^{-1} \cdot \hat{b}\hat{a}. \tag{7.3b}$$

The minimum weight state denoted by $|s,\sigma\rangle$ obeys the conditions

$$\hat{\mathcal{S}}_-|s,\sigma\rangle = 0, \quad \text{i.e.,} \quad \hat{b}\hat{a}|s,\sigma\rangle = 0, \tag{7.4}$$

$$\hat{\mathcal{S}}|s,\sigma\rangle = s|s,\sigma\rangle, \quad \hat{\mathcal{S}}_0|s,\sigma\rangle = \sigma|s,\sigma\rangle. \tag{7.5}$$

The relations (7.4) and (7.5) with (7.3b) give us

$$P\ ;\ |s,\sigma\rangle = \left(\hat{b}^*\right)^{C_m - 2s}|0\rangle, \quad \sigma = -s, \tag{7.6}$$

$$Q\ ;\ |s,\sigma\rangle = \left(\hat{a}^*\right)^{2s - C_m}|0\rangle, \quad \sigma = s - C_m. \tag{7.7}$$

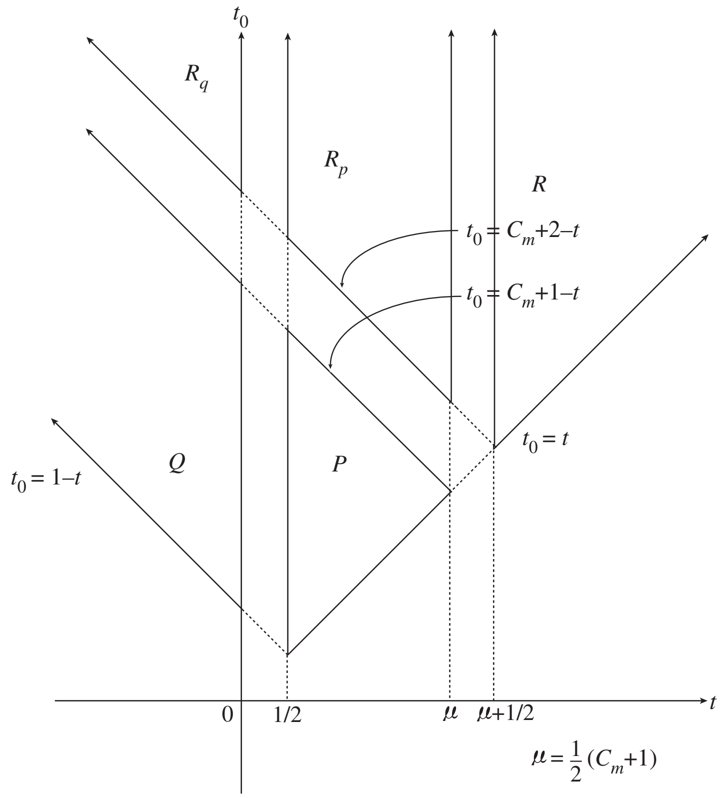

Fig. 7. The subspaces P, Q, R_p, R_q, and R on the t–t_0 plane.

Here, the normalization constants are omitted. Since $C_m - 2s \geq 0$ and $2s - C_m \geq 1$, we have the inequalities

$$P\ ; \quad s \leq \tfrac{1}{2}C_m, \tag{7.8}$$

$$Q\ ; \quad s \geq \tfrac{1}{2}(C_m + 1)\,. \tag{7.9}$$

By operating $\hat{\mathcal{S}}_+$ successively on the state $|s, \sigma\rangle$, we are able to obtain the eigenstate of $\hat{\mathcal{S}}$ and $\hat{\mathcal{S}}_0$ with the eigenvalues s and s_0, respectively, in the form

$$P\ ;\ \Big|s, s_0\Big\rangle = \left(\hat{\mathcal{S}}_+\right)^{s+s_0}\Big|s, -s\Big\rangle, \tag{7.10}$$

$$Q\ ;\ \Big|s, s_0\Big\rangle = \left(\hat{\mathcal{S}}_+\right)^{C_m-s+s_0}\Big|s, s - C_m\Big\rangle. \tag{7.11}$$

Here, the normalization constants are omitted. Since $s + s_0 \geq 0$ and $C_m - s + s_0 \geq 0$, we have

$$P\ ;\ s_0 \geq -s, \tag{7.12}$$

$$Q\ ;\ s_0 \geq s - C_m. \tag{7.13}$$

Our system has the maximum weight state, denoted by $|s, \sigma_m\rangle$. It satisfies

$$\hat{\mathcal{S}}_+|s, \sigma_m\rangle = 0, \quad \text{i.e.,} \quad \sqrt{\hat{\mathcal{S}} - \hat{\mathcal{S}}_0}\ |s, \sigma_m\rangle = 0, \tag{7.14}$$

$$\hat{\mathcal{S}}|s, \sigma_m\rangle = s|s, \sigma_m\rangle, \quad \hat{\mathcal{S}}_0|s, \sigma_m\rangle = \sigma_m|s, \sigma_m\rangle. \tag{7.15}$$

The relations (7.14) and (7.15) give us

$$P \text{ and } Q\ ; \quad \sigma_m = s. \tag{7.16}$$

Then, the quantum number s_0 satisfies

$$P\ ;\ -s \leq s_0 \leq s, \quad \text{i.e.,} \quad s_0 = -s, -s+1, \ldots, s-1, s, \tag{7.17}$$

$$Q\ ;\ s - C_m \leq s_0 \leq s, \quad \text{i.e.,} \quad s_0 = s - C_m,\ s - C_m + 1, \ldots, s-1, s. \tag{7.18}$$

We can see that $\hat{\mathcal{S}}_{\pm,0}$ obey the $su(2)$-algebra in P, but not in Q. The relations (7.8) and (7.9) teach us that upper and lower limits of s in P and Q, respectively, exist. Therefore, it may be interesting to show the lower and upper limits of s in P and Q, respectively. For this task, the first part of the relation (7.2) is useful. As can be seen in Fig. 7, the quantum number t in P and Q obeys

$$P\ ;\ t = \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots, \mu - \tfrac{1}{2},\ \mu, \tag{7.19}$$

$$Q\ ;\ t = 0, -\tfrac{1}{2}, -1, \ldots, -\infty. \tag{7.20}$$

Combining the relations (7.19) and (7.20) with the relation (7.2), we have the following:

$$P\ ;\ s = \tfrac{1}{2}C_m, \quad \tfrac{1}{2}\left(C_m - 1\right), \quad \tfrac{1}{2}\left(C_m - 2\right), \ldots, \tfrac{1}{2}, 0, \tag{7.21}$$

$$Q\ ;\ s = \tfrac{1}{2}\left(C_m + 1\right), \quad \tfrac{1}{2}\left(C_m + 2\right), \quad \tfrac{1}{2}\left(C_m + 3\right), \ldots, \infty. \tag{7.22}$$

In P, there exists a lower limit $s = 0$, but, in Q, the upper limit is ∞.

Next, we investigate the commutation relations for $\hat{\mathcal{S}}_{\pm,0}$ and $\hat{\mathcal{S}}$. For the above discussion, the following played a central role:

$$\left[\hat{\mathcal{S}}_0, \hat{\mathcal{S}}_\pm\right] = \pm\hat{\mathcal{S}}_\pm, \quad \left[\hat{\mathcal{S}}, \hat{\mathcal{S}}_{\pm,0}\right] = 0. \tag{7.23}$$

Of course, the above relation is useful both in P and Q. Our problem is to investigate the relation $[\hat{\mathcal{S}}_+, \hat{\mathcal{S}}_-]$. Direct calculation gives us the form

$$\left[\hat{\mathcal{S}}_+, \hat{\mathcal{S}}_-\right] = 2\left(\hat{\mathcal{S}}_0 + \Delta\hat{\mathcal{S}}_0\right), \tag{7.24}$$

$$\begin{aligned}
\Delta\hat{\mathcal{S}}_0 &= \Delta\hat{\mathcal{S}}_0^{(+)} - \Delta\hat{\mathcal{S}}_0^{(-)}, \\
\Delta\hat{\mathcal{S}}_0^{(+)} &= \frac{\epsilon}{2}\frac{\left(\hat{T}_0 - \hat{T} + 1\right)\left(C_m + 1 - \hat{T}_0 - \hat{T}\right)}{\hat{T}_0 + \hat{T} + \epsilon}, \\
\Delta\hat{\mathcal{S}}_0^{(-)} &= \frac{\epsilon}{2}\frac{\left(\hat{T}_0 - \hat{T}\right)\left(C_m + 2 - \hat{T}_0 - \hat{T}\right)}{\hat{T}_0 + \hat{T} - 1 + \epsilon}.
\end{aligned} \tag{7.25}$$

Evidently, $\left[\hat{\mathcal{S}}_+, \hat{\mathcal{S}}_-\right]$ does not satisfy the commutation relation in the $su(2)$-algebra. As can be seen in Fig. 7, t_0 and t obey the inequality $t_0 + t > 0$ in both P and Q. This indicates that the term $\left(\hat{T}_0 + \hat{T}\right)$ appearing in the denominator of $\Delta\hat{\mathcal{S}}_0^{(+)}$ is positive-definite and, then, we have

$$P \text{ and } Q\ ;\ \Delta\hat{\mathcal{S}}_0^{(+)} \longrightarrow 0, \quad (\epsilon \to 0). \tag{7.26}$$

In the case $\Delta\hat{\mathcal{S}}_0^{(-)}$, it may be necessary to investigate if the operator $\left(\hat{T}_0 + \hat{T} - 1\right)$ appearing in the denominator is positive-definite or not. For this aim, the condition $t_0 + t - 1 = 0$ should be examined: In P, $t_0 = t = 1/2$ and, in Q, $t_0 = -t + 1$ $(t \leq 0)$. Figure 7 gives us these relations. We notice the term $\left(\hat{T}_0 - \hat{T}\right)$ appearing in the numerator of $\Delta\hat{\mathcal{S}}_0^{(-)}$. In this case, $t_0 - t = 1/2 - 1/2 = 0$.

Therefore, the following result is derived:

$$P\ ;\ \Delta\hat{\mathcal{S}}_0^{(-)} \longrightarrow 0, \quad (\epsilon \to 0). \tag{7.27}$$

On the other hand, there does not exist a term that leads to $\Delta\hat{\mathcal{S}}_0^{(-)} \to 0$, $(\epsilon \to 0)$. Therefore, we have

$$Q\ ;\ \Delta\hat{\mathcal{S}}_0^{(-)} \longrightarrow \tfrac{1}{2}\left(\hat{T}_0 - \hat{T}\right)\left(C_m + 2 - \hat{T}_0 - \hat{T}\right)\cdot \hat{Q}_0, \quad (\epsilon \to 0), \tag{7.28}$$

$$\hat{Q}_0 = \sum_{t\leq 0}\big|t, t_0 = -t+1\big\rangle\!\big\langle t, t_0 = -t+1\big| = \sum_{s\geq 1/2\cdot(C_m+1)} |s,\sigma\rangle\langle s,\sigma|. \quad (\sigma = s - C_m) \tag{7.29}$$

For the operator $\hat{\boldsymbol{\mathcal{S}}}^2$, we have

$$\hat{\boldsymbol{\mathcal{S}}}^2 = \hat{\mathcal{S}}\left(\hat{\mathcal{S}}+1\right) - \tfrac{1}{2}\left(\Delta\hat{\mathcal{S}}_0^{(+)} + \Delta\hat{\mathcal{S}}_0^{(-)}\right). \tag{7.30}$$

Therefore, for $\epsilon \to 0$, $\hat{\boldsymbol{\mathcal{S}}}^2$ is expressed in the form

$$P\ ;\ \hat{\boldsymbol{\mathcal{S}}}^2 \longrightarrow \hat{\mathcal{S}}\left(\hat{\mathcal{S}}+1\right), \tag{7.31}$$

$$Q\ ;\ \hat{\boldsymbol{\mathcal{S}}}^2 \longrightarrow \hat{\mathcal{S}}\left(\hat{\mathcal{S}}+1\right) - \tfrac{1}{2}\left(\hat{T}_0 - \hat{T}\right)\left(C_m + 2 - \hat{T}_0 - \hat{T}\right)\cdot\hat{Q}_0. \tag{7.32}$$

The above consideration teaches us that $\hat{\mathcal{S}}_{\pm,0}$ obey the $su(2)$-algebra in P, but they do not obey the $su(2)$-algebra in Q. It may be permitted to call the algebra in Q the pseudo $su(2)$-algebra. With the use of the commutation relation (7.24), we can determine the normalization constants of the states (7.6) and (7.7). For this aim, the following formula is useful:

$$\langle s,\sigma|\left(\hat{\mathcal{S}}_-\right)^n\cdot\left(\hat{\mathcal{S}}_+\right)^n|s,\sigma\rangle = (-)^n n!\frac{\Gamma(2\sigma+n)}{\Gamma(2\sigma)}\langle s,\sigma|s,\sigma\rangle. \tag{7.33}$$

Combining the formula (7.33) with the relations (7.6), (7.7), (7.10), and (7.11), we can determine the norms of the states (7.10) and (7.11) in the following form:

$$P\ ;\ \big\langle s, s_0|s, s_0\big\rangle = \frac{(2s)!(s+s_0)!}{(s-s_0)!}\cdot(C_m - 2s)!, \tag{7.34}$$

$$Q\ ;\ \big\langle s, s_0|s, s_0\big\rangle = (-)^{C_m - s + s_0}\frac{\left(C_m - s + s_0\right)!\,\Gamma\left(s + s_0 - C_m\right)}{\Gamma\left(2\left(s - C_m\right)\right)}\cdot\left(2s - C_m\right)!. \tag{7.35}$$

With the aid of the relations (7.34) and (7.35), we are able to obtain the normalized $|s, s_0\rangle$. Clearly, the relation (7.34) is the same as that in the $su(2)$-algebra. Of course, $\left(C_m - 2s\right)!$ is derived under the minimum weight state (7.6). The above is a supplementary explanation of our boson representation of the $su(2)$-algebra. Certainly, in P, our representation obeys the $su(2)$-algebra. We will discuss the subspaces R_p, R_q, and R in Sect. 9.

7.2. *Raising and lowering operators for the magnitude of the $su(2)$-spin*

In this subsection, we will discuss the role of the original Schwinger representation (3.1) in our present one. It is shown that the operators $\hat{S}_+$ and $\hat{S}_-$ play the role of the raising and lowering operators for the magnitude of the $su(2)$-spin, respectively. First, we notice that the relation between $\left(t, t_0\right)$

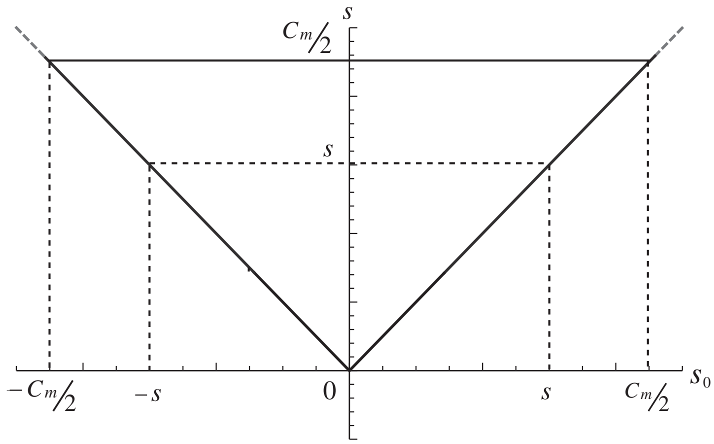

Fig. 8. The condition (7.37) on the s–s_0 plane.

and (s, s_0) is given as

$$t = \tfrac{1}{2}(C_m + 1) - s, \quad t_0 = \tfrac{1}{2}(C_m + 1) + s_0. \tag{7.36}$$

The form (7.36) is derived from the relations (3.7a) and (3.8c) with the relation (2.16a). The space is characterized by the conditions $t \le t_0 \le C_m + 1 - t$ and $1/2 \le t \le \mu$, which leads us to the following conditions for (s, s_0):

$$-s \le s_0 \le s, \tag{7.37a}$$

$$0 \le s \le \tfrac{1}{2}C_m. \tag{7.37b}$$

The condition (7.37) can be shown in Fig. 8. With the use of the relation (7.36), the state $|t, t_0\rangle$ in P can be expressed in the form

$$\begin{aligned} |t, t_0\rangle &= \frac{1}{\sqrt{(t_0 - t)!(t_0 + t - 1)!}} \left(\hat{a}^*\right)^{t_0 - t} \left(\hat{b}^*\right)^{t_0 + t - 1} |0\rangle \\ &= |s, s_0\rangle = \frac{1}{\sqrt{(s + s_0)!(C_m - s + s_0)!}} \left(\hat{a}^*\right)^{s + s_0} \left(\hat{b}^*\right)^{C_m - s + s_0} |0\rangle. \end{aligned} \tag{7.38}$$

The state $\|s, s_0\rangle$ in the Schwinger representation is given as

$$\|s, s_0\rangle = \frac{1}{\sqrt{(s + s_0)!(s - s_0)!}} \left(\hat{a}^*\right)^{s + s_0} \left(\hat{b}^*\right)^{s - s_0} |0\rangle. \tag{7.39}$$

The state (7.38) gives us the relation

$$\hat{S}_+|s, s_0\rangle = \hat{a}^*\hat{b}|s, s_0\rangle = \sqrt{(s + s_0 + 1)(C_m - s + s_0)}|s + 1, s_0\rangle, \tag{7.40a}$$

$$\hat{S}_-|s, s_0\rangle = \hat{b}^*\hat{a}|s, s_0\rangle = \sqrt{(s + s_0)(C_m - s + s_0 + 1)}|s - 1, s_0\rangle, \tag{7.40b}$$

$$\hat{S}_0|s, s_0\rangle = \tfrac{1}{2}\left(\hat{a}^*\hat{a} - \hat{b}^*\hat{b}\right)|s, s_0\rangle = \left(s - \tfrac{1}{2}C_m\right)|s, s_0\rangle. \tag{7.40c}$$

Here, $\hat{S}_{\pm,0}$ denote the generators of the $su(2)$-algebra in the Schwinger representation. In the relations (7.40a) and (7.40b), we can see that $\hat{S}_+$ and $\hat{S}_-$ play the role of the raising and lowering operators, respectively, not for s_0 but for s. However, we must notice the case $s = C_m/2$. Operation of $\hat{S}_+$ on the state $|C_m/2, s_0\rangle$ should vanish, because $C_m/2 + 1$ does not belong to the space P. In order to overcome this trouble, we define the following operators:

$$\hat{\Sigma}_+ = \hat{S}_+ \cdot \sqrt{\tfrac{1}{2}C_m - \hat{\mathcal{S}}} \cdot \left(\sqrt{\tfrac{1}{2}C_m - \hat{\mathcal{S}} + \epsilon}\right)^{-1}, \tag{7.41a}$$

$$\hat{\Sigma}_- = \left(\sqrt{\tfrac{1}{2}C_m - \hat{\mathcal{S}} + \epsilon}\right)^{-1} \cdot \sqrt{\tfrac{1}{2}C_m - \hat{\mathcal{S}}} \cdot \hat{S}_-, \tag{7.41b}$$

$$\hat{\Sigma}_0 = \hat{\mathcal{S}} - \tfrac{1}{2}C_m. \tag{7.41c}$$

Then, we have

$$\hat{\Sigma}_+ |C_m/2, s_0\rangle = 0. \tag{7.42}$$

The above indicates that the state $|C_m/2, s_0\rangle$ is the maximum weight state.

The commutation relations for $\hat{\Sigma}_{\pm,0}$ are given in the form

$$\left[\hat{\Sigma}_0, \hat{\Sigma}_\pm\right] = \pm\hat{\Sigma}_\pm, \tag{7.43}$$

$$\left[\hat{\Sigma}_+, \hat{\Sigma}_-\right] = 2\left(\hat{\Sigma}_0 + \Delta\hat{\Sigma}_0\right), \tag{7.44}$$

$$\Delta\hat{\Sigma}_0 = \frac{\epsilon}{2}\frac{\left(\hat{\mathcal{S}}_0 + \hat{\mathcal{S}} + 1\right)\left(\hat{\mathcal{S}}_0 + C_m - \hat{\mathcal{S}}\right)}{\tfrac{1}{2}C_m - \hat{\mathcal{S}} + \epsilon}. \tag{7.45}$$

The operator $\hat{\boldsymbol{\Sigma}}^2$ corresponding to the Casimir operator of the $su(2)$-algebra is given as

$$\begin{aligned} \hat{\boldsymbol{\Sigma}}^2 &= \hat{\Sigma}_0^2 + \tfrac{1}{2}\left(\hat{\Sigma}_-\hat{\Sigma}_+ + \hat{\Sigma}_+\hat{\Sigma}_-\right) \\ &= \hat{\Sigma}\left(\hat{\Sigma}+1\right) - \left(\Delta\hat{\Sigma}_0^{(+)} + \Delta\hat{\Sigma}_0^{(-)}\right), \end{aligned} \tag{7.46}$$

$$\hat{\Sigma} = \hat{\mathcal{S}}_0 + \tfrac{1}{2}C_m. \tag{7.47}$$

We can see that the set $\hat{\Sigma}_{\pm,0}$ forms a kind of pseudo $su(2)$-algebra.

Thus, we can learn of the existence of raising and lowering operators for the magnitude of the $su(2)$-spin. With the use of the operators $\hat{\Sigma}_+$ and $\hat{\mathcal{S}}_\pm$, we can express the state $|s, s_0\rangle$ in the form

$$|s, s_0\rangle = \left(\hat{\mathcal{S}}_+\right)^{s+s_0} |s, -s\rangle, \tag{7.48a}$$

$$|s, -s\rangle = \left(\hat{\mathcal{S}}_-\hat{\Sigma}_+\right)^{s-\rho} |\rho, -\rho\rangle, \tag{7.48b}$$

$$|\rho, -\rho\rangle = (\hat{b}^*)^{C_m - 2\rho}|0\rangle. \quad \left(\rho = 0 \text{ and } \tfrac{1}{2}\right) \tag{7.48c}$$

In the above relations, we omitted the normalization constants. If $\rho = 0$ and $1/2$, s become integer and half-integer, respectively. However, we must remark that the above idea is not proper to our representation. In the case of the Schwinger representation, we obtain the state $\|s, s_0\rangle$ by replacing $\hat{\mathcal{S}}_\pm$, $\hat{\Sigma}_+$, and $|\rho, -\rho\rangle$ with $\hat{S}_\pm$, $\hat{T}_+$, and $\|\rho, -\rho\rangle = \left(\hat{b}^*\right)^\rho |0\rangle$.

8. Connections to other boson representations

We know two forms of the boson representations of the $su(2)$-algebra. One is, of course, the Schwinger representation [7] and the other is the Holstein–Primakoff representation [5,6]. It may be interesting to investigate the connection between ours and the other two. In this section, the Holstein–Primakoff representation can be derived rather easily from ours. The connection between our new boson representation and the Holstein–Primakoff representation will be seen in Sect. 8.1. However, the relation between the Schwinger representation and ours is rather complicated. One thing that they have common is that they are both formulated in terms of two kinds of bosons. However, ours contains one parameter C_m, which can be seen in the relations (3.9) and (3.10). To understand C_m, the pairing model in the many-fermion system is an instructive example. In this model, we have $2C_m = 4\Omega_0$, the total number of single-particle states, which is shown in Sect. 3. The above example suggests that C_m is regarded as a parameter expressing the size of the system under consideration. In Sect. 8.2, we can show that, as a natural consequence, the magnitude of the $su(2)$-spin s can change in the range $s = 0, 1/2, \ldots, C_m/2 - 1, C_m/2$, which is consistent with the well known formula in the pairing model. In the Schwinger representation, such a restriction does not exist and, thus, $s = 0, 1/2, \ldots, \infty$. Certainly, if $C_m \to \infty$, we can show that ours is reduced to the Schwinger representation. In Sect. 8.2, the above will be discussed.

8.1. Connection to the Holstein–Primakoff representation

It may be interesting to show how the Holstein–Primakoff representation [5,6] can be derived from our representation (3.9). The three $su(2)$-generators (3.9) are rewritten in the following forms:

$$\hat{\mathcal{S}}_+ = \hat{a}^*\hat{\beta}^* \cdot \sqrt{2\hat{\mathcal{S}} - \hat{a}^*\hat{a}}, \quad \hat{\mathcal{S}}_- = \sqrt{2\hat{\mathcal{S}} - \hat{a}^*\hat{a}} \cdot \hat{\beta}\hat{a}, \quad \hat{\mathcal{S}}_0 = \hat{a}^*\hat{a} - \hat{\mathcal{S}}. \tag{8.1}$$

Here, $(\hat{\beta}, \hat{\beta}^*)$ is defined in the form

$$\hat{\beta} = \left(\sqrt{\hat{b}^*\hat{b} + 1 + \epsilon}\right)^{-1} \cdot \hat{b}, \quad \hat{\beta}^* = \hat{b}^* \cdot \left(\sqrt{\hat{b}^*\hat{b} + 1 + \epsilon}\right)^{-1}. \tag{8.2}$$

Further, for the above rewriting, we used the relation

$$C_m - \hat{b}^*\hat{b} = 2\hat{\mathcal{S}} - \hat{a}^*\hat{a}. \tag{8.3}$$

The relation (8.3) comes from the form (3.10). Any of $\left(\hat{\beta}, \hat{\beta}^*\right)$ connect with any of $\left(\hat{a}, \hat{a}^*\right)$, and we have the relation

$$\hat{\beta}\hat{\beta}^* = 1 - \frac{\epsilon}{\hat{b}^*\hat{b} + 1 + \epsilon} \to 1, \quad (\epsilon \to 0) \tag{8.4a}$$

$$\hat{\beta}^*\hat{\beta} = 1 - \frac{\epsilon}{\hat{b}^*\hat{b} + \epsilon}. \tag{8.4b}$$

Therefore, $\left[\hat{\beta}, \hat{\beta}^*\right]$ is given as

$$\left[\hat{\beta}, \hat{\beta}^*\right] = \frac{\epsilon}{\hat{b}^*\hat{b} + \epsilon}. \tag{8.5}$$

The operator $\left[\hat{\beta}, \hat{\beta}^*\right]$ plays the role of projection operator and, for the state (7.38), we have

$$\left[\hat{\beta}, \hat{\beta}^*\right]|s, s_0\rangle = \frac{\epsilon}{C_m - s + s_0 + \epsilon}|s, s_0\rangle = \delta_{s-s_0, C_m}|s, s_0\rangle. \quad (\epsilon \to 0) \tag{8.6}$$

Our boson representation is given in the space P in Fig. 7 and, then, δ_{s-s_0,C_m} takes the value 1 in the case

$$s = s_{\max} = \frac{C_m}{2}, \quad s_0 = -s_{\max} = -\frac{C_m}{2}, \quad \delta_{s-s_0,C_m}|s, s_0\rangle = \left|\frac{C_m}{2}, -\frac{C_m}{2}\right\rangle. \tag{8.7a}$$

In any other case, we have

$$\delta_{s-s_0,C_m}|s, s_0\rangle = 0. \tag{8.7b}$$

With the use of $(\hat{\beta}, \hat{\beta}^*)$, we define the operator $(\hat{c}, \hat{c}^*)$:

$$\hat{c} = \hat{\beta}\hat{a}\ (= \hat{a}\hat{\beta}), \quad \hat{c}^* = \hat{a}^*\hat{\beta}^*\ (= \hat{\beta}^*\hat{a}^*). \tag{8.8}$$

The operator $(\hat{c}, \hat{c}^*)$ satisfies the relation

$$\hat{c}\hat{c}^* = \hat{a}\hat{\beta}\hat{\beta}^*\hat{a}^* = \hat{a}\hat{a}^*, \tag{8.9a}$$

$$\begin{aligned}\hat{c}^*\hat{c} &= \hat{a}^*\hat{\beta}^*\hat{\beta}\hat{a} = \hat{a}^*\hat{a}\cdot\left(\hat{\beta}^*\hat{\beta}\right) = \hat{a}^*\hat{a} - \hat{a}^*\hat{a}\left[\hat{\beta}, \hat{\beta}^*\right] \\ &= \hat{a}^*\hat{a} - \left(\hat{\mathcal{S}} + \hat{\mathcal{S}}_0\right)\left[\hat{\beta}, \hat{\beta}^*\right] \to \hat{a}^*\hat{a}. \quad (\epsilon \to 0)\end{aligned} \tag{8.9b}$$

We notice that $(\hat{\mathcal{S}} + \hat{\mathcal{S}}_0)[\hat{\beta}, \hat{\beta}^*]|C_m/2, -C_m/2\rangle = (C_m/2 - C_m/2)|C_m/2, -C_m/2\rangle$ and, for any other case, $[\hat{\beta}, \hat{\beta}^*]|s, s_0\rangle = 0$. Therefore, $(\hat{c}, \hat{c}^*)$ is a boson operator:

$$\left[\hat{c}, \hat{c}^*\right] = \left[\hat{a}, \hat{a}^*\right] = 1, \quad \hat{c}|0\rangle = \hat{b}|0\rangle = \hat{a}|0\rangle = 0. \tag{8.10}$$

Further, we obtain the relation

$$\left[\hat{c}, \hat{\mathcal{S}}\right] = \left[\hat{c}^*, \hat{\mathcal{S}}\right] = 0. \tag{8.11}$$

With the use of the relations obtained above, $\hat{\mathcal{S}}_{\pm,0}$ can be rewritten in the form

$$\hat{\mathcal{S}}_+ = \hat{c}^*\sqrt{2\hat{\mathcal{S}} - \hat{c}^*\hat{c}}, \quad \hat{\mathcal{S}}_- = \sqrt{2\hat{\mathcal{S}} - \hat{c}^*\hat{c}}\ \hat{c}, \quad \hat{\mathcal{S}}_0 = \hat{c}^*\hat{c} - \hat{\mathcal{S}}. \tag{8.12}$$

By setting $s + s_0 = n$ in the state (7.38), we have

$$f\left(\hat{\mathcal{S}}, \hat{c}^*\hat{c}\right)|s, s_0\rangle = f(s, n)|s, s_0\rangle = f\left(s, \hat{c}^*\hat{c}\right)|s, s_0\rangle. \tag{8.13}$$

Therefore, in the space spanned by $|s, s_0\rangle$ $(s_0 = -s, -s+1, \ldots, s-1, s)$, the expression (8.12) can be regarded as

$$\hat{S}_+ = \hat{c}^*\sqrt{2s - \hat{c}^*\hat{c}}, \quad \hat{S}_- = \sqrt{2s - \hat{c}^*\hat{c}}\ \hat{c}, \quad \hat{S}_0 = \hat{c}^*\hat{c} - s. \tag{8.14}$$

The above is nothing but the Holstein–Primakoff representation. In Ref. [15], we discussed an idea of how to derive the Holstein–Primakoff representation from the Schwinger one. In this case, much more lengthy discussion was necessary. The main reason for this may be attributed to the fact that the operator corresponding to $(\hat{c}, \hat{c}^*)$ does not satisfy the simple boson commutation relation. The state (7.38) can be rewritten as

$$|s, s_0\rangle\ \frac{1}{\sqrt{(s+s_0)!}}\left(\hat{c}^*\right)^{s+s_0}|s, -s\rangle, \tag{8.15}$$

$$|s, -s\rangle = \frac{1}{\sqrt{(C_m - 2s)!}}\left(\hat{b}^*\right)^{C_m-2s}|0\rangle\ (= |\phi\rangle). \tag{8.16}$$

Clearly, $\hat{c}$ satisfies

$$\hat{c}|\phi\rangle = 0. \tag{8.17}$$

Since the state $|\phi\rangle$ is the vacuum of $\hat{c}$, we have the orthogonal set:

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{c}^*\right)^n|\phi\rangle.(n = 0, 1, \ldots, 2s) \tag{8.18}$$

The above is the connection to the Holstein–Primakoff representation.

8.2. Connection to the Schwinger representation

In this subsection, we will discuss how the Schwinger boson representation is connected to the present one. The three generators $\hat{\mathcal{S}}_{\pm,0}$ and $\hat{\mathcal{S}}$ given in the relations (3.9) and (3.10) are rewritten in the form

$$\hat{\mathcal{S}}_+ = \hat{a}^*\hat{\beta}^* \cdot \sqrt{C_m - \hat{b}^*\hat{b}}, \quad \hat{\mathcal{S}}_- = \sqrt{C_m - \hat{b}^*\hat{b}} \cdot \hat{\beta}\hat{a}, \quad \hat{\mathcal{S}}_0 = \tfrac{1}{2}\left(\hat{a}^*\hat{a} - \left(C_m - \hat{b}^*\hat{b}\right)\right), \tag{8.19}$$

$$\hat{\mathcal{S}} = \tfrac{1}{2}\left(\hat{a}^*\hat{a} + \left(C_m - \hat{b}^*\hat{b}\right)\right). \tag{8.20}$$

Here, $\left(\hat{\beta}, \hat{\beta}^*\right)$ is given in the relation (8.2). In order to rewrite the expressions (8.19) and (8.20), we introduce the operator $\left(\breve{b}, \breve{b}^*\right)$ in the form

$$\breve{b} = \hat{\beta}^*\sqrt{C_m - \hat{b}^*\hat{b}}, \quad \breve{b}^* = \sqrt{C_m - \hat{b}^*\hat{b}}\ \hat{\beta}. \tag{8.21}$$

With the use of $\left(\breve{b}, \breve{b}^*\right)$, $\hat{\mathcal{S}}_{\pm,0}$ and $\hat{\mathcal{S}}$ can be expressed as

$$\breve{\mathcal{S}}_+ = \hat{a}^*\breve{b}, \quad \breve{\mathcal{S}}_- = \breve{b}^*\hat{a}, \quad \breve{\mathcal{S}}_0 = \tfrac{1}{2}\left(\hat{a}^*\hat{a} - \breve{b}^*\breve{b}\right), \tag{8.22}$$

$$\breve{\mathcal{S}} = \tfrac{1}{2}\left(\hat{a}^*\hat{a} + \breve{b}^*\breve{b}\right). \tag{8.23}$$

Here, it should be noted that we rewrite $\hat{\mathcal{S}}_{\pm,0}$ and $\hat{\mathcal{S}}$, shown in the relations (8.19) and (8.20), in the forms (8.22) and (8.23) and, then, $\breve{\mathcal{S}}_{\pm,0} = \hat{\mathcal{S}}_{\pm,0}$ and $\breve{\mathcal{S}} = \hat{\mathcal{S}}$. If $\left(\breve{b}, \breve{b}^*\right)$ is a boson operator, the expression (8.22) (and (8.23)) reduces to the Schwinger boson representation (3.1) (and (3.2)). Therefore, it may be interesting to investigate the condition under which $\left(\breve{b}, \breve{b}^*\right)$ can be regarded as a boson operator. For this aim, we must calculate the commutation relation $\left[\breve{b}, \breve{b}^*\right]$, the result of which is as follows:

$$\begin{aligned}\left[\breve{b}, \breve{b}^*\right] &= 1 - \left(C_m + 1 - \hat{b}^*\hat{b}\right)\left[\hat{\beta}, \hat{\beta}^*\right] = 1 - \left(C_m + 1 - \hat{b}^*\hat{b}\right)\frac{\epsilon}{\hat{b}^*\hat{b} + \epsilon} \\ &= 1 - \frac{\epsilon\left(\breve{b}^*\breve{b} + 1\right)}{C_m - \breve{b}^*\breve{b} + \epsilon}.\end{aligned} \tag{8.24}$$

Here, we used the relation

$$\breve{b}\breve{b}^* = \left(C_m + 1 - \breve{b}^*\breve{b}\right)\hat{\beta}^*\hat{\beta}, \quad \breve{b}^*\breve{b} = C_m - \hat{b}^*\hat{b}. \tag{8.25}$$

In the relation (8.24), we can see that if $C_m \to \infty$, $\left(\breve{b}, \breve{b}^*\right)$ may be regarded as a boson operator.

In order to clarify the above-mentioned situation, we consider an orthogonal set constructed by $\breve{b}^*$. First, we introduce the state $|0\rangle\rangle$, defined as

$$|0\rangle\rangle = \left(\hat{\beta}^*\right)^{C_m}|0\rangle = \frac{1}{\sqrt{C_m!}}\left(\hat{b}^*\right)^{C_m}|0\rangle. \tag{8.26}$$

The state $|0\rangle\rangle$ is the vacuum for $\breve{b}$:

$$\breve{b}|0\rangle\rangle = \hat{\beta}^*\sqrt{C_m - \hat{b}^*\hat{b}}\cdot\frac{1}{\sqrt{C_m!}}\left(\hat{b}^*\right)^{C_m}|0\rangle = 0. \tag{8.27}$$

Then, we define the state $|n\rangle\rangle$ in the form

$$|n\rangle\rangle = \left(\hat{\beta}^*\right)^{C_m-n}|0\rangle = \frac{1}{\sqrt{(C_m-n)!}}\left(\hat{b}^*\right)^{C_m-n}|0\rangle. \quad (n = 0, 1, 2, \dots, C_m) \tag{8.28}$$

It can be proved that $|n\rangle\rangle$ is expressed as

$$|n\rangle\rangle = \frac{1}{\sqrt{n!}}\left(\breve{b}^*\right)^n|0\rangle\rangle. \tag{8.29}$$

If $n = C_m$, we have the following relation:

$$\breve{b}^*|C_m\rangle\rangle = \sqrt{C_m - \hat{b}^*\hat{b}}\ \hat{\beta}|0\rangle = 0. \tag{8.30}$$

It may be interesting to see that the operation of $\breve{b}^*$ on $|C_m\rangle\rangle$ vanishes. Therefore, if $C_m \to \infty$, the relation (8.30) becomes meaningless. The commutation relation (8.24) gives us the following:

$$\left[\breve{b}, \breve{b}^*\right]|n\rangle\rangle = |n\rangle\rangle, \quad (n = 0, 1, 2, \dots, C_m - 1) \tag{8.31a}$$

$$\left[\breve{b}, \breve{b}^*\right]|C_m\rangle\rangle = -C_m|C_m\rangle\rangle. \tag{8.31b}$$

In the relation (8.31b), we have $\breve{b}\breve{b}^*|C_m\rangle\rangle = 0$ and $\breve{b}^*\breve{b}|C_m\rangle\rangle = C_m|C_m\rangle\rangle$, which come from the relation (8.30). From the above argument, we can conclude that, if $C_m \to \infty$, $\left(\breve{b}, \breve{b}^*\right)$ can be regarded as a boson operator. In this connection, we mention that, if $C_m = 1$, $\left(\breve{b}, \breve{b}^*\right)$ becomes a fermion operator.

Under the above consideration, we investigate the eigenvalue problem for $\breve{\mathcal{S}}_{\pm,0}$ and $\breve{\mathcal{S}}$. First, we introduce the state $|s\rangle\rangle$ in the form

$$|s\rangle\rangle = \frac{1}{\sqrt{(2s)!}}\left(\breve{b}^*\right)^{2s}|0\rangle\rangle. \tag{8.32}$$

Here, it should be noted that s takes the values

$$s = 0, 1/2, 1, 3/2, \dots, (C_m - 1)/2, C_m/2. \tag{8.33}$$

In the Schwinger representation, $s = 0, 1/2, \dots, \infty$. This difference was mentioned at the beginning of this section. The state $|s\rangle\rangle$ is the minimum weight state satisfying the relation

$$\breve{\mathcal{S}}_-|s\rangle\rangle = 0, \quad \breve{\mathcal{S}}_0|s\rangle\rangle = -s|s\rangle\rangle, \quad \breve{\mathcal{S}}|s\rangle\rangle = s|s\rangle\rangle. \tag{8.34}$$

Then, we define the following state:

$$|s, s_0\rangle\!\rangle = \sqrt{\frac{(s-s_0)!}{(2s)!(s+s_0)!}}\left(\breve{\mathcal{S}}_+\right)^{s+s_0}\|s\rangle. \tag{8.35}$$

Together with the properties, $|s, s_0\rangle\!\rangle$ can be shown in the form

$$|s, s_0\rangle\!\rangle = \frac{1}{\sqrt{(s+s_0)!(s-s_0)!}}\left(\hat{a}^*\right)^{s+s_0}\left(\breve{b}^*\right)^{s-s_0}|0\rangle\!\rangle, \tag{8.36}$$

$$\breve{\mathcal{S}}|s, s_0\rangle\!\rangle = s|s, s_0\rangle\!\rangle, \quad \breve{\mathcal{S}}_0|s, s_0\rangle\!\rangle = s_0|s, s_0\rangle\!\rangle. \tag{8.37}$$

Of course, $s = 0, 1/2, 1, 3/2, \ldots, (C_m - 1)/2, C_m/2$ and, for a given s, $s_0 = -s, -s+1, \ldots, s-1, s$.

The commutation relations among $\breve{\mathcal{S}}_{\pm,0}$ and $\breve{\mathcal{S}}$ are given as follows:

$$\left[\breve{\mathcal{S}}_+, \breve{\mathcal{S}}_-\right] = 2\breve{\mathcal{S}}_0 - \hat{a}^*\hat{a}\left(1 - \left[\breve{b}, \breve{b}^*\right]\right), \tag{8.38a}$$

$$\left[\breve{\mathcal{S}}_0, \breve{\mathcal{S}}_-\right] = -\breve{\mathcal{S}}_- + \tfrac{1}{2}\breve{b}^*\hat{a}\left(1 - \left[\breve{b}, \breve{b}^*\right]\right), \tag{8.38b}$$

$$\left[\breve{\mathcal{S}}, \breve{\mathcal{S}}_-\right] = -\tfrac{1}{2}\breve{b}^*\hat{a}\left(1 - \left[\breve{b}, \breve{b}^*\right]\right). \tag{8.38c}$$

Of course, we have

$$\left[\breve{\mathcal{S}}_0, \breve{\mathcal{S}}_+\right] = \breve{\mathcal{S}}_+ - \tfrac{1}{2}\left(1 - \left[\breve{b}, \breve{b}^*\right]\right)\hat{a}^*\breve{b}, \tag{8.38d}$$

$$\left[\breve{\mathcal{S}}, \breve{\mathcal{S}}_+\right] = \tfrac{1}{2}\left(1 - \left[\breve{b}, \breve{b}^*\right]\right)\hat{a}^*\breve{b}. \tag{8.38e}$$

With the use of the relations (8.28) and (8.31), we have

$$\hat{a}\left(1 - \left[\breve{b}, \breve{b}^*\right]\right)|n\rangle\!\rangle = 0. \quad \left(n = 0, 1, 2, \ldots, C_m - 1, C_m\right) \tag{8.39}$$

Therefore, we can conclude that $\breve{\mathcal{S}}_{\pm,0}$ obey the $su(2)$-algebra and commute with $\breve{\mathcal{S}}$.

In the previous discussion, the connection of the Schwinger representation to ours was clarified. We continue this discussion by introducing a new boson space composed of a boson $\left(\mathring{b}, \mathring{b}^*\right)$. Of course, the orthogonal set is given by

$$|n)) = \frac{1}{\sqrt{n!}}\left(\mathring{b}^*\right)^n|0)). \quad (n = 0, 1, 2, \ldots) \tag{8.40}$$

In this space, the following operator is introduced:

$$\left(\breve{b}\right)_c = \left(\sqrt{C_m - \mathring{b}^*\mathring{b} + \epsilon}\right)^{-1} \cdot \sqrt{C_m - \mathring{b}^*\mathring{b}} \cdot \mathring{b},$$

$$\left(\breve{b}^*\right)_c = \mathring{b}^* \cdot \sqrt{C_m - \mathring{b}^*\mathring{b}} \cdot \left(\sqrt{C_m - \mathring{b}^*\mathring{b} + \epsilon}\right)^{-1}. \tag{8.41}$$

We can easily verify the relation

$$\left(\breve{b}\right)_c|0)) = 0, \tag{8.42}$$

$$\left(\breve{b}^*\right)_c|n)) = \mathring{b}^*|n)), \quad \left(n = 0, 1, 2, \ldots, C_m - 1\right), \left(\breve{b}^*\right)_c|C_m)) = 0. \tag{8.43}$$

Further, we have

$$\left[\left(\breve{b}\right)_c, \left(\breve{b}^*\right)_c\right] = 1 - \frac{\left(\mathring{b}^*\mathring{b}+1\right)\epsilon}{C_m - \mathring{b}^*\mathring{b} + \epsilon} + \frac{\mathring{b}^*\mathring{b}\epsilon}{C_m - \mathring{b}^*\mathring{b} + 1 + \epsilon}. \tag{8.44}$$

This relation leads us to

$$\left[\left(\breve{b}\right)_c, \left(\breve{b}^*\right)_c\right]\Big|n\Big)\Big) = |n)), \quad \Big(n = 0, 1, 2, \ldots, C_m - 1\Big) \tag{8.45a}$$

$$\left[\left(\breve{b}\right)_c, \left(\breve{b}^*\right)_c\right]\Big|C_m\Big)\Big) = -C_m|C_m)). \tag{8.45b}$$

All of the above relations suggest that $\left(\left(\breve{b}\right)_c, \left(\breve{b}^*\right)_c\right)$ may be regarded as the counterpart of $\left(\breve{b}, \breve{b}^*\right)$ in the boson space composed of $\left(\mathring{b}, \mathring{b}^*\right)$.

Next, we consider the counterparts of $\breve{\mathcal{S}}_{\pm,0}$ and $\breve{\mathcal{S}}$ in the spaces composed of $\left(\hat{a}, \hat{a}^*\right)$ and $\left(\mathring{b}, \mathring{b}^*\right)$, which are denoted by $\left(\breve{\mathcal{S}}_{\pm,0}\right)_c$ and $\left(\breve{\mathcal{S}}\right)_c$. In this space, we introduce the following set:

$$|s, s_0)) = \frac{1}{\sqrt{(s+s_0)!(s-s_0)!}}\left(\hat{a}^*\right)^{s+s_0}\left(\mathring{b}^*\right)^{s-s_0}|0)). \quad (s = 0, 1/2, 1, 3/2, \ldots,\ C_m/2) \tag{8.46}$$

Further, $(\breve{\mathcal{S}}_{\pm,0})_c$ and $(\breve{\mathcal{S}})_c$ are introduced in the form

$$\left(\breve{\mathcal{S}}_+\right)_c = \hat{a}^*\left(\breve{b}\right)_c = \left(\sqrt{C_m - \mathring{b}^*\mathring{b} + \epsilon}\right)^{-1} \cdot \sqrt{C_m - \mathring{b}^*\mathring{b}} \cdot \hat{a}^*\mathring{b}, \tag{8.47a}$$

$$\left(\breve{\mathcal{S}}_-\right)_c = \left(\breve{b}^*\right)_c\hat{a} = \mathring{b}^*\hat{a} \cdot \sqrt{C_m - \mathring{b}^*\mathring{b}} \cdot \left(\sqrt{C_m - \mathring{b}^*\mathring{b} + \epsilon}\right)^{-1}, \tag{8.47b}$$

$$\left(\breve{\mathcal{S}}_0\right)_c = \tfrac{1}{2}\left(\hat{a}^*\hat{a} - \left(\breve{b}^*\right)_c\left(\breve{b}\right)_c\right) = \tfrac{1}{2}\left(\hat{a}^*\hat{a} - \mathring{b}^*\mathring{b}\right) + \tfrac{1}{2}\Delta\mathring{\mathcal{S}}, \tag{8.47c}$$

$$\left(\breve{\mathcal{S}}\right)_c = \tfrac{1}{2}\left(\hat{a}^*\hat{a} + \left(\breve{b}^*\right)_c\left(\breve{b}\right)_c\right) = \tfrac{1}{2}\left(\hat{a}^*\hat{a} + \mathring{b}^*\mathring{b}\right) - \tfrac{1}{2}\Delta\mathring{\mathcal{S}}. \tag{8.48}$$

Here, $\Delta\mathring{\mathcal{S}}$ is given as

$$\Delta\mathring{\mathcal{S}} = \frac{\epsilon\mathring{b}^*\mathring{b}}{C_m - \mathring{b}^*\mathring{b} + 1 + \epsilon}. \tag{8.49}$$

Since, in the present space, $\Delta\mathring{\mathcal{S}}$ is positive-definite, it can be omitted and, then, $|s, s_0))$ is the eigenstate of $\left(\breve{\mathcal{S}}\right)_c$ and $\left(\breve{\mathcal{S}}_0\right)_c$ with the eigenvalues s and s_0, respectively. Therefore, the following commutation relations are easily verified:

$$\left[\left(\breve{\mathcal{S}}_0\right)_c, \left(\breve{\mathcal{S}}_\pm\right)_c\right] = \pm\left(\breve{\mathcal{S}}_\pm\right)_c, \quad \left[\left(\breve{\mathcal{S}}\right)_c, \left(\breve{\mathcal{S}}_{\pm,0}\right)_c\right] = 0. \tag{8.50}$$

The commutation relation $\left[\left(\breve{\mathcal{S}}_+\right)_c, \left(\breve{\mathcal{S}}_-\right)_c\right]$ is given by

$$\left[\left(\breve{\mathcal{S}}_+\right)_c, \left(\breve{\mathcal{S}}_-\right)_c\right] = 2\left(\left(\breve{\mathcal{S}}_0\right)_c - \Delta\left(\breve{\mathcal{S}}_0\right)_c\right), \tag{8.51}$$

$$\Delta\left(\breve{\mathcal{S}}_0\right)_c = \tfrac{1}{2}\left(\frac{\epsilon\hat{a}^*\hat{a}\left(\mathring{b}^*\mathring{b}+1\right)}{C_m - \mathring{b}^*\mathring{b} + \epsilon} - \frac{\epsilon\left(\hat{a}^*\hat{a}+1\right)\mathring{b}^*\mathring{b}}{C_m - \mathring{b}^*\mathring{b} + 1 + \epsilon}\right). \tag{8.52}$$

We can show that the operation of $\Delta\left(\breve{\mathcal{S}}_0\right)_c$ on the state $|s, s_0))$ makes the result of the operation vanish. Further, we can prove the relation

$$\left(\breve{\boldsymbol{\mathcal{S}}}^2\right)_c = \left(\breve{\mathcal{S}}_0\right)_c^2 + \frac{1}{2}\left(\left(\breve{\mathcal{S}}_-\right)_c\left(\breve{\mathcal{S}}_+\right)_c + \left(\breve{\mathcal{S}}_+\right)_c\left(\breve{\mathcal{S}}_-\right)_c\right)$$
$$= (\hat{\mathcal{S}})_c\left(\left(\hat{\mathcal{S}}\right)_c + 1\right). \tag{8.53}$$

Thus, we can conclude that $\left(\breve{\mathcal{S}}_{\pm,0}\right)_c$ obey the $su(2)$-algebra with the magnitude of the $su(2)$-spin $\left(\breve{\mathcal{S}}\right)_c$.

9. Concluding remarks

In this paper, we have investigated various theoretical features and an application to a physical system of a new boson representation of the $su(2)$-algebra. As has been stressed in this paper, the essential difference between the Schwinger representation and ours can be found in the expressions of the operators that give the quantum numbers specifying the orthogonal set, (s, s_0). They are completely opposite to each other. Therefore, we must put each representation to its proper use. In this final section, we will mention two points.

The first point is concerned with the promise mentioned at the end of Sect. 7.1. We will examine the subspaces R_p, R_q, and R. The case R is simple: the $su(1,1)$-algebra presented in Sect. 7.1 for the case $t \geq \mu + 1/2$. The cases R_p and R_q are a little bit complicated. As a preliminary argument, we consider the case of the Holstein–Primakoff representation (8.14). Let us investigate the case in which the order of $2s$ and $\hat{c}^*\hat{c}$ in $\hat{S}_\pm$ is changed. In this case, we define the operators $\breve{\mathrm{T}}_{\pm,0}$:

$$\breve{\mathrm{T}}_+ = \hat{c}^* \cdot \sqrt{\hat{c}^*\hat{c} - 2s}, \quad \breve{\mathrm{T}}_- = \sqrt{\hat{c}^*\hat{c} - 2s} \cdot \hat{c}, \quad \breve{\mathrm{T}}_0 = \hat{c}^*\hat{c} - s \ \left(= \hat{S}_0\right). \tag{9.1}$$

The commutation relations are given in the form

$$\left[\breve{\mathrm{T}}_+, \breve{\mathrm{T}}_-\right] = -2\breve{\mathrm{T}}_0, \quad \left[\breve{\mathrm{T}}_0, \breve{\mathrm{T}}_\pm\right] = \pm\breve{\mathrm{T}}_\pm. \tag{9.2}$$

Of course, we have the state

$$|n\rangle = \left(\breve{\mathrm{T}}_+\right)^{n-(2s+1)}|2s+1\rangle = \left(\hat{c}^*\right)^n|0\rangle,$$
$$\breve{\mathrm{T}}_-|2s+1\rangle = 0, \quad n = 2s+1, 2s+2, \ldots. \tag{9.3}$$

The Casimir operator is given as

$$\breve{\mathbf{T}}^2 = (s+1)((s+1)-1). \tag{9.4}$$

Here, the quantity $(s+1)$ indicates the magnitude of the $su(1,1)$-spin. In this case, we obtain the $su(1,1)$-algebra.

Following the above idea, the order of $\hat{T}_m$ and $\hat{T}_0$ in the relations (3.7b) is changed. In this case, we define the operators $\breve{\mathcal{T}}_{\pm,0}$:

$$\breve{\mathcal{T}}_+ = \hat{T}_+ \cdot \sqrt{\hat{T}_0 - \hat{T}_m} \cdot \left(\sqrt{\hat{T}_0 + \hat{T} + \epsilon}\right)^{-1},$$
$$\breve{\mathcal{T}}_- = \left(\sqrt{\hat{T}_0 + \hat{T} + \epsilon}\right)^{-1} \cdot \sqrt{\hat{T}_0 - \hat{T}_m} \cdot \hat{T}_-,$$
$$\breve{\mathcal{T}}_0 = \hat{T}_0 - \frac{1}{2}\left(\hat{T}_m + \hat{T}\right) \ \left(= \hat{\mathcal{S}}_0\right). \tag{9.5}$$

In the case $\hat{T}_m = C_m + 1 - \hat{T}$, $\hat{T}_0 - \hat{T}_m = \hat{T}_0 + \hat{T} - (C_m + 1)$. The commutation relation $\left[\breve{\mathcal{T}}_+, \breve{\mathcal{T}}_-\right]$ is given in the form

$$\left[\breve{\mathcal{T}}_+, \breve{\mathcal{T}}_-\right] = -2\left(\breve{\mathcal{T}}_0 + \Delta\breve{\mathcal{T}}_0\right), \quad \Delta\breve{\mathcal{T}}_0 = \Delta\breve{\mathcal{T}}_0^{(+)} - \Delta\breve{\mathcal{T}}_0^{(-)}, \quad \Delta\breve{\mathcal{T}}_0^{(+)} = \Delta\hat{\mathcal{S}}_0^{(+)},$$
$$\Delta\breve{\mathcal{T}}_0^{(-)} = \Delta\hat{\mathcal{S}}_0^{(-)}. \tag{9.6}$$

Here, $\Delta\hat{\mathcal{S}}_0^{(\pm)}$ is given in the relation (7.25). Since, in the spaces R_p and R_q, we have $t_0 + t > 0$ and $t_0 + t - 1 > 0$, i.e., $\hat{T}_0 + \hat{T}$ and $\hat{T}_0 + \hat{T} - 1$ are positive-definite, then, $\Delta\breve{\mathcal{T}}_0 \to 0$, $(\epsilon \to 0)$. Therefore, together with the other, we have

$$\left[\breve{\mathcal{T}}_+, \breve{\mathcal{T}}_-\right] = -2\breve{\mathcal{T}}_0, \quad \left[\breve{\mathcal{T}}_0, \breve{\mathcal{T}}_\pm\right] = \pm\breve{\mathcal{T}}_\pm. \tag{9.7}$$

The above shows that, in the spaces R_p and R_q, $\breve{\mathcal{T}}_{\pm,0}$ forms the $su(1,1)$-algebra. In the same argument as that of $\Delta\breve{\mathcal{T}}_0$, the Casimir operator is expressed as

$$\breve{\boldsymbol{\mathcal{T}}}^2 = \breve{\mathcal{T}}_0^2 - \tfrac{1}{2}\left(\breve{\mathcal{T}}_-\breve{\mathcal{T}}_+ + \breve{\mathcal{T}}_+\breve{\mathcal{T}}_-\right) = \breve{\mathcal{T}}\left(\breve{\mathcal{T}} - 1\right), \tag{9.8}$$

$$\breve{\mathcal{T}} = \tfrac{1}{2}\left(\hat{T}_m - \hat{T}\right) + 1, \quad \left[\breve{\mathcal{T}}, \breve{\mathcal{T}}_{\pm,0}\right] = 0. \tag{9.9}$$

The orthogonal set should obey the condition $t_0 \geq t_m + 1$ and the minimum weight state is given in the form

$$R_p \text{ and } R_q \ ; \ |t, t_m + 1\rangle\!\rangle, \quad \breve{\mathcal{T}}_-|t, t_m + 1\rangle\!\rangle = 0. \tag{9.10}$$

We can see that the above is in completely the same situation as that in the Holstein–Primakoff representation.

The second point is related to another example of $\hat{T}_m$. In the pseudo $su(1,1)$-algebra, we introduced t_m as the maximum value of t_0 for a given t and, as a possible example, we gave the form $t_m = C_m + 1 - t$ (see (2.16a)). However, this form is not unique and there exist infinite possibilities; here, we give another example: $t_m = 3t - 1$. We consider the form

$$\hat{T}_m = 3\hat{T} - 1. \tag{9.11}$$

In this case, $\hat{\mathcal{S}}_{\pm,0}$ are expressed in the form

$$\hat{\mathcal{S}}_+ = \hat{T}_+ \cdot \sqrt{3\hat{T} - 1 - \hat{T}_0} \cdot \left(\sqrt{\hat{T}_0 + \hat{T} + \epsilon}\right)^{-1} = \hat{a}^*\hat{b}^* \cdot \sqrt{\hat{b}^*\hat{b} - 2\hat{a}^*\hat{a}} \cdot \left(\sqrt{\hat{b}^*\hat{b} + 1 + \epsilon}\right)^{-1}, \tag{9.12a}$$

$$\hat{\mathcal{S}}_- = \left(\sqrt{\hat{T}_0 + \hat{T} + \epsilon}\right)^{-1} \cdot \sqrt{3\hat{T} - 1 - \hat{T}_0} \cdot \hat{T}_- = \left(\sqrt{\hat{b}^*\hat{b} + 1 + \epsilon}\right)^{-1} \cdot \sqrt{\hat{b}^*\hat{b} - 2\hat{a}^*\hat{a}} \cdot \hat{b}\hat{a}, \tag{9.12b}$$

$$\hat{\mathcal{S}}_0 = \hat{T}_0 - 2\hat{T} + \tfrac{1}{2} = \tfrac{1}{2}\left(3\hat{a}^*\hat{a} - \hat{b}^*\hat{b}\right), \tag{9.12c}$$

$$\hat{\mathcal{S}} = \hat{T} - \tfrac{1}{2} = \tfrac{1}{2}\left(-\hat{a}^*\hat{a} + \hat{b}^*\hat{b}\right). \tag{9.13}$$

The expressions (9.12) and (9.13) obey the $su(2)$-algebra in the space P, shown in Fig. 9. The quantum numbers (s, s_0) are related to (t, t_0) in the space P as follows:

$$s = t - \tfrac{1}{2}, \quad s_0 = t_0 - 2t + \tfrac{1}{2}, \tag{9.14}$$

$$t = \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots, \mu - 1, \mu. \tag{9.15}$$

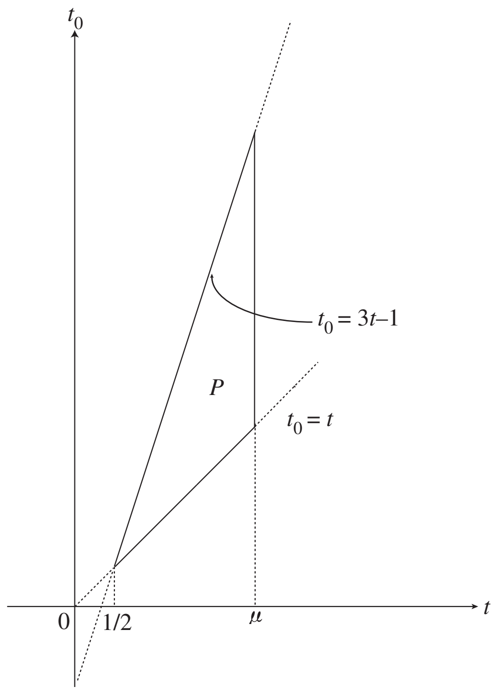

Fig. 9. The space P on the t–t_0 plane.

The eigenstate of $\hat{\mathcal{S}}$ and $\hat{\mathcal{S}}_0$ is given as

$$|s, s_0\rangle = \sqrt{\frac{(2s)!}{(s+s_0)!(3s+s_0)!}}\left(\hat{T}_+\right)^{s+s_0} \cdot \frac{1}{\sqrt{(2s)!}}\left(\hat{b}^*\right)^{2s}|0\rangle$$
$$= \frac{1}{\sqrt{(s+s_0)!(3s+s_0)!}}\left(\hat{a}^*\right)^{s+s_0}\left(\hat{b}^*\right)^{3s+s_0}|0\rangle. \quad (9.16)$$

Evidently, the forms (9.12) and (9.13) are very different from the forms (3.9) and (3.10). However, the form (9.12) is rewritten as

$$\hat{\mathcal{S}}_+ = \hat{a}^*\hat{\beta}^* \cdot \sqrt{2\hat{\mathcal{S}} - \hat{a}^*\hat{a}}, \quad \hat{\mathcal{S}}_- = \sqrt{2\hat{\mathcal{S}} - \hat{a}^*\hat{a}} \cdot \hat{\beta}\hat{a}, \quad \hat{\mathcal{S}}_0 = \hat{a}^*\hat{a} - \hat{\mathcal{S}}. \quad (9.17)$$

The above leads us to the expression (8.12).

Through the above consideration, we may conjecture that there exist various boson representations of the $su(2)$-algebra.

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