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New boson realization of the Lipkin model obeying the $su(2)$-algebra

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A new boson representation of the $su(2)$-algebra proposed by the present authors for describing the damped and amplified oscillator is examined in the Lipkin model as one of the simple many-fermion models. This boson representation is expressed in terms of two kinds of bosons with a certain positive parameter. In order to describe the case of any fermion number, a third boson is introduced. Through this examination, it is concluded that this representation is very workable for the boson realization of the Lipkin model in any fermion number.

Subject Index D50

The study of boson realization for the Lie algebra has a long history. In particular, the boson representation of the $su(2)$-algebra has been applied to simple many-nucleon systems, such as the Lipkin model, and has played an important role in understanding nuclear collective dynamics. In our last paper [1] (hereafter referred to as (I)), we proposed a new boson representation of the $su(2)$-algebra with an aim of describing a harmonic oscillator interacting with the external environment in the frame of the thermo field dynamics formalism [6–9,11]. Actually, in (I), by introducing two kinds of bosons $(\hat{a}, \hat{a}^*)$ and $(\hat{b}, \hat{b}^*)$, the $su(2)$-generators $\hat{\mathcal{S}}_{\pm,0}$ and the operator expressing the magnitude of the $su(2)$-spin $\hat{\mathcal{S}}$ can be expressed in the following form:

$$\hat{\mathcal{S}}_+ = \hat{a}^*\hat{b}^*\sqrt{C_m - \hat{b}^*\hat{b}}\left(\sqrt{\hat{b}^*\hat{b}+1+\epsilon}\right)^{-1}, \quad \hat{\mathcal{S}}_- = \left(\hat{\mathcal{S}}_+\right)^*,$$

$$\hat{\mathcal{S}}_0 = \tfrac{1}{2}\left(\hat{a}^*\hat{a}+\hat{b}^*\hat{b}\right) - \tfrac{1}{2}C_m, \tag{1a}$$

$$\hat{\mathcal{S}} = \tfrac{1}{2}\left(\hat{a}^*\hat{a}-\hat{b}^*\hat{b}\right) + \tfrac{1}{2}C_m, \tag{1b}$$

where ϵ is an infinitesimal parameter that is finally set to 0 when algebraic calculations are carried out. This parameter guarantees the existence of the inverse operator that was discussed in (I). Here, C_m denotes a positive parameter and, depending on the model under investigation, its value is appropriately chosen. As was stressed in (I), representation (1) obeys the $su(2)$-algebra in a certain subspace

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of the whole space constructed by $\hat{a}^*$ and $\hat{b}^*$. On the other hand, as is known well, the Schwinger boson representation [7] is given in the following form:

$$\hat{\mathcal{S}}_+^{\rm Schw} = \hat{a}^*\hat{b}, \quad \hat{\mathcal{S}}_-^{\rm Schw} = \left(\hat{\mathcal{S}}+^{\rm Schw}\right)^*,$$

$$\hat{\mathcal{S}}_0^{\rm Schw} = \frac{1}{2}\left(\hat{a}^*\hat{a} - \hat{b}^*\hat{b}\right), \tag{2a}$$

$$\hat{\mathcal{S}}^{\rm Schw} = \frac{1}{2}\left(\hat{a}^*\hat{a} + \hat{b}^*\hat{b}\right). \tag{2b}$$

Compared with representation (2), the following three points are characteristics of representation (1): (i) $\hat{\mathcal{S}}_\pm$ are of the forms deformed from the boson representation of the $su(1,1)$-algebra presented by Schwinger [7], (ii) it contains the parameter C_m, and (iii) except for C_m, the forms of $\hat{\mathcal{S}}_0$ and $\hat{\mathcal{S}}$ are the opposite of those in representation (2). In (I), we discussed how representation (1) is connected with the many-fermion system in the pairing model (the Cooper pair).

The main purpose of this paper is to demonstrate that representation (1) works quite well for a many-fermion system with the particle–hole pair correlation. The Lipkin model [8] is a simple $su(2)$-algebraic model with the particle–hole pair correlation and it will be shown that our new boson representation (1) is suitable for the boson realization of the Lipkin model. In fact, it is demonstrated that our new boson realization is workable to treat the case with any fermion number in addition to a closed-shell system. Namely, in the usual and conventional cases, the closed-shell system, in which the lower level in the Lipkin model is occupied and the higher level is vacant, is only treated in the context of boson realization. However, in our new boson realization, any fermion number can be considered.

First, we recapitulate the Lipkin model with a certain aspect that has not been investigated explicitly, namely, the role of operators $\widetilde{N}$ and $\widetilde{M}$ in (5). The Lipkin model consists of two single-particle levels, the degeneracies of which are equal to $2\Omega = 2j+1$ (j: half-integer).

The Hamiltonian of the Lipkin mode can be expressed as

$$\widetilde{H} = \sum_m \varepsilon\tilde{c}^*_{p,jm}\tilde{c}_{p,jm} + \sum_m(-\varepsilon)\tilde{c}^*_{h,jm}\tilde{c}_{h,jm} + \chi\sum_{mm'}\left(\tilde{c}^*_{p,jm}\tilde{c}^*_{p,jm'}\tilde{c}_{h,jm}\tilde{c}_{h,jm'} + \tilde{c}^*_{h,jm'}\tilde{c}^*_{h,jm'}\tilde{c}_{p,jm'}\tilde{c}_{p,jm}\right), \tag{3}$$

where p and h represent the upper and lower single particle levels, respectively. Here, ε ($-\varepsilon$) represents the single particle energy of the upper (lower) level and χ represents the strength of the particle–hole interaction. The single-particle states are specified by the quantum numbers (p, jm) and (h, jm). Here, $m = -j, -j+1, \ldots, j-1, j$. The fermion operators are denoted by $\left(\tilde{c}_{p,jm}, \tilde{c}^*_{p,jm}\right)$ and $\left(\tilde{c}_{h,jm}, \tilde{c}^*_{h,jm}\right)$ and, following the conventional treatment, we use the following particle and hole operators:

$$\tilde{c}_{p,jm} = \tilde{a}_m(\text{particle}), \quad (-)^{j-m}\tilde{c}_{h,j-m} = \tilde{b}^*_m(\text{hole}), \tag{4}$$

where $\tilde{a}^*_m$ and $\tilde{b}^*_m$ represent the particle and hole creation operators, respectively. Hereafter, we denote the fermion operators by attaching a symbol $\tilde{}$ (such as $\tilde{c}$ and so on), and boson operators and boson realization by $\hat{}$, respectively. The total fermion number operator $\widetilde{N}$ can be expressed as

$$\widetilde{N} = \sum_m\left(\tilde{c}^*_{p,jm}\tilde{c}_{p,jm} + \tilde{c}^*_{h,jm}\tilde{c}_{h,jm}\right) = \sum_m\left(\tilde{a}^*_m\tilde{a}_m - \tilde{b}^*_m\tilde{b}_m\right) + 2\Omega. \tag{5a}$$

Further, we define the operator $\widetilde{M}$ in the form

$$\widetilde{M} = \sum_m \left(\tilde{c}^*_{p,jm}\tilde{c}_{p,jm} - \tilde{c}^*_{h,jm}\tilde{c}_{h,jm}\right) + 2\Omega = \sum_m \left(\tilde{a}^*_m\tilde{a}_m + \tilde{b}^*_m\tilde{b}_m\right). \tag{5b}$$

The operator $\widetilde{M}$ indicates the addition of the particle and hole number operators. In contrast, the operator $\widetilde{N}$ is related to the subtraction of the hole number operator from the particle one. The difference of signs, $+$ or $-$, in $\widetilde{M}$ and $\widetilde{N}$ makes representation (1) workable for the boson realization of the Lipkin model. With the use of these operators, we define the operators:

$$\widetilde{\mathcal{S}}_+ = \sum_m \tilde{c}^*_{p,jm}\tilde{c}_{h,jm} = \sum_m \tilde{a}^*_m(-)^{j-m}\tilde{b}^*_{-m}, \quad \widetilde{\mathcal{S}}_- = \left(\widetilde{\mathcal{S}}_+\right)^*,$$
$$\widetilde{\mathcal{S}}_0 = \frac{1}{2}\sum_m \left(\tilde{c}^*_{p,jm}\tilde{c}_{p,jm} - \tilde{c}^*_{h,jm}\tilde{c}_{h,jm}\right) = \frac{1}{2}\sum_m \left(\tilde{a}^*_m\tilde{a}_m + \tilde{b}^*_m\tilde{b}_m\right) - \Omega = \frac{1}{2}\widetilde{M} - \Omega. \tag{6a}$$

For $\widetilde{\mathcal{S}}_{\pm,0}$, we have the relations

$$\left[\widetilde{N}, \widetilde{\mathcal{S}}_{\pm,0}\right] = 0, \tag{6b}$$
$$\left[\widetilde{M}, \widetilde{\mathcal{S}}_\pm\right] = \pm 2\widetilde{\mathcal{S}}_\pm, \quad \left[\widetilde{M}, \widetilde{\mathcal{S}}_0\right] = 0, \quad \widetilde{M} = 2\widetilde{\mathcal{S}}_0 + 2\Omega. \tag{6c}$$

The operators $\widetilde{\mathcal{S}}_{\pm,0}$ obey the $su(2)$-algebra:

$$\left[\widetilde{\mathcal{S}}_+, \widetilde{\mathcal{S}}_-\right] = 2\widetilde{\mathcal{S}}_0, \quad \left[\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_\pm\right] = \pm\widetilde{\mathcal{S}}_\pm, \tag{7a}$$
$$\left[\widetilde{\mathcal{S}}_{\pm,0}, \widetilde{\boldsymbol{\mathcal{S}}}^2\right] = 0, \quad \left(\widetilde{\boldsymbol{\mathcal{S}}}^2 = \widetilde{\mathcal{S}}_0^2 + \frac{1}{2}\left(\widetilde{\mathcal{S}}_-\widetilde{\mathcal{S}}_+ + \widetilde{\mathcal{S}}_+\widetilde{\mathcal{S}}_-\right)\right). \tag{7b}$$

Here, $\widetilde{\boldsymbol{\mathcal{S}}}^2$ denotes the Casimir operator. Based on the above discussion, the Lipkin model Hamiltonian (3) is recast into

$$\widetilde{H} = \varepsilon\widetilde{\mathcal{S}}_0 - \chi\left(\widetilde{\mathcal{S}}_+^2 + \widetilde{\mathcal{S}}_-^2\right), \tag{8}$$

in which this model is governed by the $su(2)$-algebra.

It may be important to see that, in the pairing model with one single-particle level, relation (6b) does not exist. The reason is simple: in the pairing model, we have $\widetilde{S}_0 = \left(\widetilde{N} - \Omega\right)/2$. Here, 2Ω denotes the degeneracy of the single-particle level. The Lipkin model has been mainly investigated in a certain case appearing under the condition N (total fermion number) $= 2\Omega$ from the reason schematically induced by the particle–hole pair correlation. However, if intending to give complete description of the Lipkin model as an example of the $su(2)$-algebraic model, it may be necessary to treat all cases including $N \neq 2\Omega$, which requires a new idea. After careful treatment of all the cases, including both the non-closed-shell case and the $N \neq 2\Omega$ case, we will be able to understand why the conventional treatment developed in the extensive previous work on boson realization for the closed-shell system is available and workable.

In connection with $\widetilde{\mathcal{S}}_{\pm,0}$, we can define other new-type operators, which we call the auxiliary $su(2)$-algebra:

$$\widetilde{\Lambda}_+ = \sum_m \tilde{c}^*_{p,jm}\tilde{c}^*_{h,jm} = \sum_m \tilde{a}^*_m(-)^{j-m}\tilde{b}_{-m}, \quad \widetilde{\Lambda}_- = \left(\widetilde{\Lambda}_+\right)^*,$$
$$\widetilde{\Lambda}_0 = \frac{1}{2}\widetilde{N} - \Omega = \frac{1}{2}\sum_m \left(\tilde{a}^*_m\tilde{a}_m - \tilde{b}^*_m\tilde{b}_m\right). \tag{9}$$

They obey

$$\left[\widetilde{\Lambda}_{+}, \widetilde{\Lambda}_{-}\right] = 2\widetilde{\Lambda}_{0}, \quad \left[\widetilde{\Lambda}_{0}, \widetilde{\Lambda}_{\pm}\right] = \pm\widetilde{\Lambda}_{\pm}, \tag{10a}$$

$$\left[\widetilde{\Lambda}_{\pm,0}, \widetilde{\boldsymbol{\Lambda}}^{2}\right] = 0, \quad \left(\widetilde{\boldsymbol{\Lambda}}^{2} = \widetilde{\Lambda}_{0}^{2} + \frac{1}{2}\left(\widetilde{\Lambda}_{-}\widetilde{\Lambda}_{+} + \widetilde{\Lambda}_{+}\widetilde{\Lambda}_{-}\right)\right). \tag{10b}$$

The important relation is as follows:

$$\left[\text{any of } \widetilde{\Lambda}_{\pm,0},\ \text{any of } \widetilde{\mathcal{S}}_{\pm,0}\right] = 0. \tag{11}$$

Relation (11) tells us that the above two algebras are independent of each other. Further, it may be interesting to see that the role of $\widetilde{\Lambda}_0$ is the same as that of $\widetilde{\mathcal{S}}_0$, for which $\widetilde{\Lambda}_+$ and $\widetilde{\Lambda}_-$ are the raising and lowering operators, respectively. As has been repeatedly mentioned, conventionally, the Lipkin model has been investigated in the case of the closed-shell system in which the level h is completely occupied by the fermions ($N = 2\Omega$). However, the auxiliary $su(2)$-algebra enables us to treat the following two cases (A) and (B): (A) $N = 2\Omega$, but with the level p being partially occupied by the fermion $\tilde{c}^{*}_{p,jm}$, and (B) $N \neq 2\Omega$, i.e., the non-closed-shell system.

We can construct the boson representation of the $su(2)$-algebra $\left(\hat{\Lambda}_{\pm,0}\right)$, which satisfies

$$\left[\text{any of } \hat{\Lambda}_{\pm,0},\ \text{any of } \hat{\mathcal{S}}_{\pm,0}\right] = 0. \tag{12}$$

On the basis of the idea of the Holstein–Primakoff representation [9,10], we can set up the form

$$\hat{\Lambda}_{+} = \hat{c}^{*}\sqrt{\left(-\hat{a}^{*}\hat{a} + \hat{b}^{*}\hat{b}\right) - \hat{c}^{*}\hat{c}}, \quad \hat{\Lambda}_{-} = \left(\hat{\Lambda}_{+}\right)^{*},$$
$$\hat{\Lambda}_{0} = \hat{c}^{*}\hat{c} - \tfrac{1}{2}\left(-\hat{a}^{*}\hat{a} + \hat{b}^{*}\hat{b}\right). \tag{13a}$$

Here, $(\hat{c}, \hat{c}^{*})$ denotes the third boson operator adding to $\left(\hat{a}, \hat{a}^{*}\right)$ and $\left(\hat{b}, \hat{b}^{*}\right)$ appearing in (1). It may be important to see that the magnitude of the $su(2)$-spin $\hat{\Lambda}$ can be expressed in the form

$$\hat{\Lambda} = \tfrac{1}{2}\left(-\hat{a}^{*}\hat{a} + \hat{b}^{*}\hat{b}\right) = -\hat{\mathcal{S}} + \tfrac{1}{2}C_{m}. \tag{13b}$$

Let $(\hat{\mathcal{S}}_{\pm,0})$ and $(\hat{\Lambda}_{\pm,0})$ be the counterparts of $(\widetilde{\mathcal{S}}_{\pm,0})$ and $(\widetilde{\Lambda}_{\pm,0})$, respectively:

$$\hat{\mathcal{S}}_{\pm,0} \sim \widetilde{\mathcal{S}}_{\pm,0}, \quad \hat{\Lambda}_{\pm,0} \sim \widetilde{\Lambda}_{\pm,0}. \tag{14}$$

First, we investigate the properties of the fermion and boson vacuums, $|0)$ and $|0\rangle$, respectively:

$$\tilde{a}_{m}|0) = \tilde{b}_{m}|0) = 0 \quad \text{for } m = -j, -j+1, \ldots, j-1, j, \tag{15a}$$

$$\hat{a}|0\rangle = \hat{b}|0\rangle = \hat{c}|0\rangle = 0. \tag{15b}$$

The states $|0)$ and $|0\rangle$ satisfy the relations

$$\widetilde{\mathcal{S}}_{-}|0) = 0, \quad \widetilde{\mathcal{S}}_{0}|0) = -\Omega|0), \quad \widetilde{\Lambda}_{-}|0) = \widetilde{\Lambda}_{0}|0) = 0, \tag{16a}$$

$$\hat{\mathcal{S}}_{-}|0\rangle = 0, \quad \hat{\mathcal{S}}_{0}|0\rangle = -\frac{1}{2}C_{m}|0\rangle, \quad \hat{\Lambda}_{-}|0\rangle = \hat{\Lambda}_{0}|0\rangle = 0. \tag{16b}$$

Clearly, the vacuums $|0)$ and $|0\rangle$ are the minimum weight states of $(\widetilde{\mathcal{S}}_{\pm,0}, \widetilde{\Lambda}_{\pm,0})$ and $\left(\hat{\mathcal{S}}_{\pm,0}, \hat{\Lambda}_{\pm,0}\right)$, respectively. Under the above consideration, it may be permitted to postulate that $|0\rangle$ should correspond to $|0)$:

$$|0\rangle \sim |0). \tag{17}$$

Then, it is natural to regard C_m as 2Ω:

$$C_m = 2\Omega. \tag{18}$$

Further, let us assume the total fermion number operator in the boson space, $\hat{N}$, corresponding to (9) in the following form:

$$\hat{N} = 2\hat{\Lambda}_0 + 2\Omega = \hat{a}^*\hat{a} - \hat{b}^*\hat{b} + 2\hat{c}^*\hat{c} + 2\Omega. \tag{19a}$$

Through the relation (19a), we can treat cases (A) and (B), which have already been mentioned. The operator $\hat{M}$, the counterpart of $\widetilde{M}$ in the boson space, may be permitted to set up from (6a) as

$$\hat{M} = 2\hat{\mathcal{S}}_0 + 2\Omega = \hat{a}^*\hat{a} + \hat{b}^*\hat{b}. \tag{19b}$$

Later, in (25), it will be shown that the operator $\hat{M}$ plays the role of "seniority operator". We can now obtain the orthogonal set of the Lipkin model in the boson realization in each fermion number. The minimum weight state of the $su(2)$-algebra $\left(\hat{\mathcal{S}}_{\pm,0}\right)$ can be expressed in the form

$$\begin{aligned} &|\lambda, \lambda_0\rangle = \left(\hat{\Lambda}_+\right)^{\lambda+\lambda_0} \left(\hat{b}^*\right)^{2\lambda} |0\rangle, \\ &\hat{\mathcal{S}}_-|\lambda, \lambda_0\rangle = 0, \quad \hat{\mathcal{S}}_0|\lambda, \lambda_0\rangle = -s|\lambda, \lambda_0\rangle. \quad (s = \Omega - \lambda) \end{aligned} \tag{20}$$

The relation $s = \Omega - \lambda$ is supported by relation (13b). Since the set $(\hat{\Lambda}_{\pm,0})$ also forms the $su(2)$-algebra, the state $|\lambda, \lambda_0\rangle$ satisfies

$$\hat{\Lambda}|\lambda, \lambda_0\rangle = \lambda|\lambda, \lambda_0\rangle, \quad \hat{\Lambda}_0|\lambda, \lambda_0\rangle = \lambda_0|\lambda, \lambda_0\rangle. \tag{21}$$

Therefore, we have the condition

$$-\lambda \le \lambda_0 \le \lambda. \tag{22}$$

If we notice the relations $s = \Omega - \lambda$ and $N = 2\lambda_0 + 2\Omega$, which come from relations (13b) and (19a), respectively, relation (22) gives the following inequality with respect to s:

$$0 \le s \le \Omega - \left|\Omega - \frac{N}{2}\right|. \tag{23a}$$

Explicitly, relation (23a) can be written as follows:

$$\begin{aligned} &\text{if } N = 0,\ 2, \ldots, 2\Omega - 2, \quad s = N/2, N/2 - 1, \ldots, 1, 0, \\ &\text{if } N = 2\Omega, \quad s = \Omega, \Omega - 1, \ldots, 1, 0 \\ &\text{if } N = 2\Omega + 2, 2\Omega + 4, \ldots, 4\Omega, \quad s = 2\Omega - N/2, 2\Omega - N/2 - 1, \ldots, 1, 0, \\ &\text{if } N = 1, 3, \ldots, 2\Omega - 1, \quad s = N/2, N/2 - 1, \ldots, 3/2, 1/2, \\ &\text{if } N = 2\Omega + 1, 2\Omega + 3, \ldots, 4\Omega - 1, \quad s = 2\Omega - N/2, 2\Omega - N/2 - 1, \ldots, 3/2, 1/2. \end{aligned} \tag{23b}$$

Relation (23) teaches us the typical examples: if $N = 0$, $N = 2\Omega$, and $N = 4\Omega$, we have the results $s = 0$, $0 \le s \le \Omega$, and $s = 0$, respectively. Using state (20), we obtain the state with (s, s_0)

in the form

$$\left(\hat{\mathcal{S}}_+\right)^{s+s_0}|\lambda,\lambda_0\rangle = \left(\hat{\mathcal{S}}_+\right)^{s+s_0}\left(\hat{\Lambda}_+\right)^{\lambda+\lambda_0}(\hat{b}^*)^{2\lambda}|0\rangle = \left(\hat{\mathcal{S}}_+\right)^{s+s_0}\left(\hat{\Lambda}_+\right)^{\frac{N}{2}-s}\left(\hat{b}^*\right)^{2(\Omega-s)}|0\rangle$$
$$= \left(\hat{a}^*\right)^{s+s_0}\left(\hat{b}^*\right)^{2\Omega-s+s_0}\left(\hat{c}^*\right)^{\frac{N}{2}-s}|0\rangle = \Omega, N; s, s_0\rangle. \tag{24}$$

Here, $|\lambda,\lambda_0\rangle$ satisfies

$$\hat{M}|\lambda,\lambda_0\rangle = 2\lambda|\lambda,\lambda_0\rangle = 2(\Omega - s)|\lambda,\lambda_0\rangle. \tag{25}$$

Since $\hat{\mathcal{S}}_-|\lambda,\lambda_0\rangle = 0$, the eigenvalue of $\hat{M}$, 2λ, indicates the particle and hole number that cannot be reduced to $\hat{\mathcal{S}}_0$ and, in some sense, it corresponds to the seniority number in the pairing model. The normalization constant is here omitted in states (20) and (24). Of course, s obeys condition (23). Except for the factor $(\hat{c}^*)^{N/2-s}$, state (24) is identical to state (7.38) in (I) for the pairing model. In the case $s=\Omega$, i.e., $\lambda = 0$, for $N = 2\Omega$, state (24) does not depend on $\hat{c}^*$. This case corresponds to the conventional one, i.e., the case of the closed-shell system. Therefore, the use of the boson $\left(\hat{c},\hat{c}^*\right)$ enables us to describe the cases $s \neq \Omega$ for $N = 2\Omega$ and $N \neq 2\Omega$.

The fact that one can describe both the cases $N = 2\Omega$ and $N \neq 2\Omega$ is supported by the following relation:

$$\hat{S}_\pm|\Omega, N; s, s_0\rangle = \sqrt{(s \mp s_0)(s \pm s_0 + 1)}|\Omega, N; s, s_0 \pm 1\rangle, \tag{26a}$$

$$\hat{S}_0|\Omega, N; s, s_0\rangle = s_0|\Omega, N; s, s_0\rangle, \tag{26b}$$

$$\hat{N}|\Omega, N; s, s_0\rangle = N|\Omega, N; s, s_0\rangle. \tag{26c}$$

Here, $|\Omega, N; s, s_0\rangle$ is normalized. Since we are considering the $su(2)$-algebra, relations (26a) and (26b) are natural results. Relations (26b) and (26c) give us

$$\hat{N}_p|\Omega, N; s, s_0\rangle = \left(\frac{N}{2} + s_0\right)|\Omega, N; s, s_0\rangle, \quad \hat{N}_h|\Omega, N; s, s_0\rangle = \left(\frac{N}{2} - s_0\right)|\Omega, N; s, s_0\rangle. \tag{27}$$

Here, $\hat{N}_p$ and $\hat{N}_h$ denote the counterparts of the fermion number operators in the levels p and h, respectively. If $N = 2\Omega$ and $s_0 = -s = -\Omega$ ($\lambda = 0$), the level h is completely occupied by the fermions and the level p is vacant. The conventional treatment starts in this case for the counterpart of the fermion Hamiltonian:

$$\hat{H} = \varepsilon\hat{S}_0 - \chi\left(\hat{S}_+^2 + \hat{S}_-^2\right). \tag{28}$$

Clearly, N and s are constants of motion.

Hereafter, let us introduce Ω_c in order to discriminate Ω in our present treatment from that in the conventional one. We adopt the symbol Ω_c for the latter. Needless to say, the case ($N = 2\Omega_c$, $s = \Omega_c$) is treated conventionally. Let the result in the case ($N = 2\Omega_c$, $s = \Omega_c$) have been obtained. Noticing $s = \Omega - \lambda$, $N = 2(\Omega + \lambda_0)$ and $\lambda_0 = -\lambda, -\lambda + 1, \ldots, \lambda - 1, \lambda$, we introduce Ω in the form

$$\Omega = \Omega_c + \lambda. \tag{29a}$$

Relation (29a) tells us that we are considering the case $s = \Omega_c$. Then, we have

$$N = 2\Omega_c,\ 2(\Omega_c + 1), \ldots,\ 2(\Omega_c + 2\lambda - 1),\ 2(\Omega_c + 2\lambda). \tag{29b}$$

Relation (29) suggests to us that, if the value of λ is appropriately chosen, the case ($\Omega, N, s = \Omega_c$) obeying relation (29) is reduced to the conventional result for the Hamiltonian (28) in the case ($\Omega = \Omega_c$, $N = 2\Omega_c$, $s = \Omega_c$). In relation (29b), we can find the case ($2\Omega = N = 2(\Omega_c + \lambda)$,

$s = \Omega_c \neq \Omega$). This case corresponds to the closed-shell system, although $s \neq \Omega$. In the conventional treatment, s should be equal to Ω. If the idea presented in this paper is acceptable, the conventional case for the Lipkin model plays a basic role and is a special case. Further, the method developed in this paper is suitable for any case shown in relation (29) and the obtained results are available. This is one of the most important conclusions in our present method.

In this paper, we have investigated a possible boson realization of the Lipkin model, a simple many-fermion model, on the basis of the new boson representation (1). The orthogonal set is specified by Ω, N, s, and s_0 without considering the behavior of individual fermions. With the use of the algebra $\left(\widetilde{\mathcal{S}}_{\pm,0}\right)$ and the auxiliary algebra $\left(\widetilde{\Lambda}_{\pm,0}\right)$, we can give the orthogonal set in the fermion space, in which the behavior of individual fermions is taken into account explicitly. A method to treat the individual fermions may be obtained by appropriate modification of the method given by the present authors, in which the pairing model was discussed [11,12].

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